Advanced calculus I-2

Topological properties and function spaces

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In most sciences one generation tears down what another has built, and what one has established, another undoes. In mathematics alone, each generation adds a new story to the old structure.——Hermann Hankel

Contents of Advanced Calculus I-2

Contents of Advanced Calculus I-1

Preface

Advanced calculus, or called mathematical analysis in some universities, is fundamental in mathematical training. It is a two-semester four-credit course in the mathematics department of National Tsing Hua University. This book stems from lecture notes for the classes of advanced calculus I that I taught several times. The goal of this book is to provide rigorous but easy to follow mathematical proofs and a book that is convenient to read on portable digital devices. I try to make this book friendly and, hopefully, readers may find those colorful paragraphs and beautiful pictures of the book attractive. Many students find this course difficult as many abstract concepts are introduced at a rather rapid pace. But being able to think abstractly is probably one of the most important abilities in modern sciences and technologies. Learning mathematics is similar to learning language, we need to have enough vocabulary to express our mathematical ideas and we need to spend enough time on it to get connection of different concepts. I try to cut proofs into small pieces so that readers may verify them easier. Based on some knowledge of basic calculus, this book is self-contained and suitable for self-study. Exercises are provided at the end of each section.

Due to the length and size, materials for advanced calculus I is separated into two books: Advanced Calculus I-1 and Advanced Calculus I-2. We had 3 midterm exams and 1 final exam in the course of advanced calculus I. The book Advanced Calculus I-1 contains materials for midterm exam 1 and 2, and the book Advanced Calculus I-2 contains materials for midterm exam 3 and final exam. Practice exams and exam questions are attached to the books.

Main references of Advanced Calculus I-2 are

1. Fractals everywhere by Barnsley ([1]);
2. Real mathematical analysis by Pugh ([2]);
3. Elementary classical analysis by Marsden and Hoffman ([3]);
4. Wikipedia.

Those beautiful pictures at the end of each chapter are free pictures from pixabay.com.

The latex documentclass “elegantbook"(https://github.com/ElegantLaTeX/ElegantBook) is used to edit this book.

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Chapter 5 Some topological properties

5.1 Connectedness

Definition 5.1. Proper subset

A subset $A \subset X$ is said to be proper if

$$A \neq \emptyset \quad \text{and} \quad A \neq X$$

♣

Definition 5.2. Connectedness

Let X be a topological space. If X has a proper subset A which is both open and closed, then X is said to be disconnected, and we say that A and A^c separate X. If X is not disconnected, then X is connected.

♣

Example 5.1 The set

$$X = \mathbb{R} - \{0\}$$

is disconnected. It is separated by $(-\infty, 0)$ and $(0, \infty)$.

Proposition 5.1

The closed interval $[a, b] \subset \mathbb{R}$ is connected.

♠

Proof We prove by contradiction. Assume that $[a, b]$ is not connected. Then there exists a proper subset $U \subset [a, b]$ which is both open and closed. We have

$$[a, b] = U \coprod U^c$$

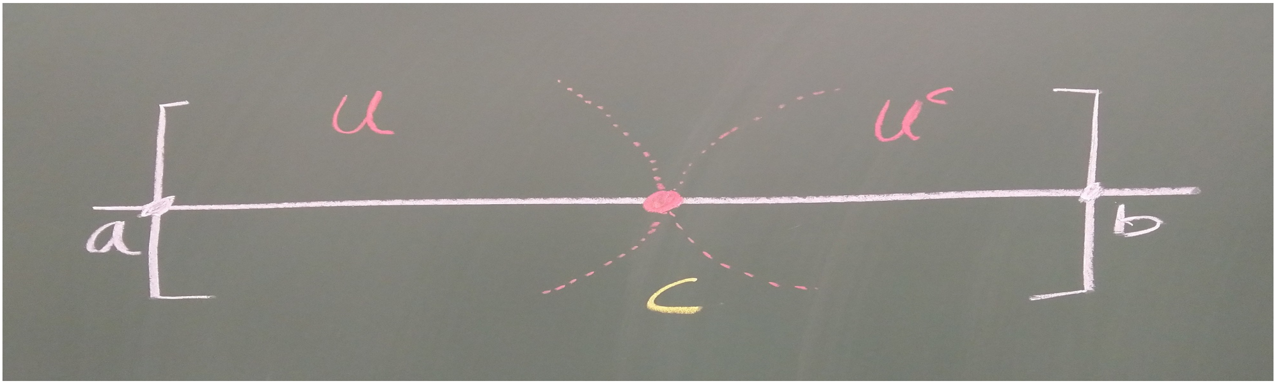
We may assume $b \in U^c$. Since U is nonempty and bounded, by the least upper bound property of $\mathbb{R}$, there exists

$$c = \sup(U)$$

Since U is closed in $[a, b]$ and $[a, b]$ is closed in $\mathbb{R}$, by the inheritance principle, Corollary 2.5, U is closed in $\mathbb{R}$, and by Lemma 4.2, $c \in U$. Since $c \notin U^c$, $c \neq b$. The openness of U in $[a, b]$ implies there is $r > 0$ such that

$$[c, c + r) \subset U$$

and hence $c + \frac{r}{2} \in U$ which contradicts to the fact that c is an upper bound of U. Hence $[a, b]$ is connected.



Remark Suppose that X is a connected topological space. If a nonempty subset $A \subset X$ is both open and closed, then $A = X$. This follows directly from the definition.

Definition 5.3. Path-connectedness

Let X be a topological space and $\varphi : [a, b] \to X$ be a continuous function. If

$$\varphi(a) = x_1, \varphi(b) = x_2$$

the function φ is said to be a path joining x_1 and x_2. We say that X is path-connected if for any $x_1, x_2 \in X$, there is a path joining x_1 and x_2. ♣

Example 5.2 The set

$$\mathbb{R}^2 - B_1(0)$$

is path-connected.

Theorem 5.1

All path-connected topological spaces are connected. ♡

Proof Let X be a path-connected topological space. Assume that X is not connected. Then there is a proper, open and closed subset $U \subset X$ such that

$$X = U \coprod U^c$$

Let $x \in U, y \in U^c$. Since X is path-connected, there is a path $\gamma : [a, b] \to X$ such that

$$\gamma(a) = x, \gamma(b) = y$$

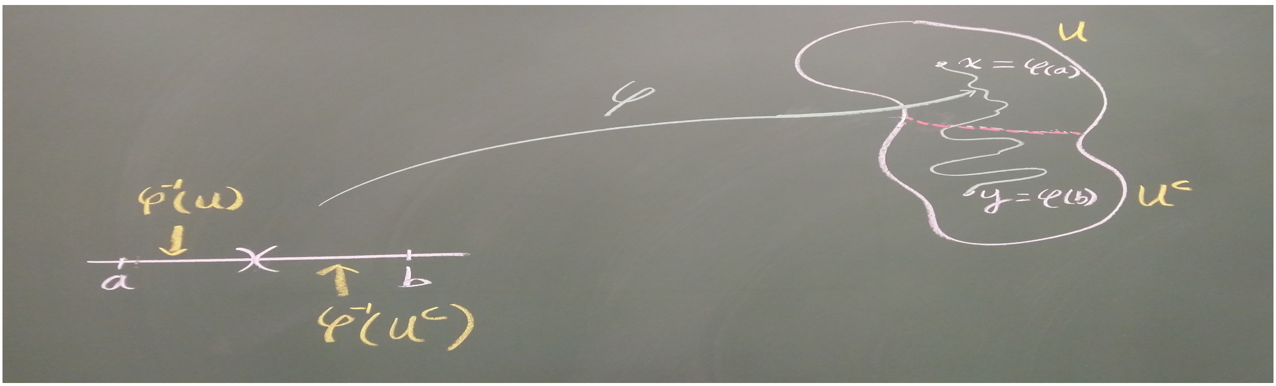
But U, U^c are both open and closed, the continuity of γ implies that

$$\gamma^{-1}(U), \gamma^{-1}(U^c) = (\gamma^{-1}(U))^c$$

are both nonempty, closed and open subsets of $[a, b]$ and

$$[a, b] = \gamma^{-1}(U) \coprod (\gamma^{-1}(U))^c$$

This means that $[a, b]$ is not connected which contradicts to Proposition 5.1. So X is connected.



Theorem 5.2

Suppose that $f : X \to Y$ is a continuous function between topological spaces.

1. *If X is connected, then $f(X)$ is connected.*
2. *If X is path-connected, then $f(X)$ is path-connected.*

♡

Proof

1. Assume that $f(X)$ is not connected. Then there exists a proper, open and closed subset $U \subset f(X)$ such that U and U^c separate $f(X)$. Then $f^{-1}(U)$ and

$$f^{-1}(U^c) = (f^{-1}(U))^c$$

separate X. Therefore X is not connected which is a contradiction.

2. Let $p, q \in f(X)$. Take $x, y \in X$ such that

$$f(x) = p, f(y) = q$$

Since X is path-connected, there is a path $\gamma : [a, b] \to X$ such that

$$\gamma(a) = x, \gamma(b) = y$$

The composition of two continuous functions $f \circ \gamma : [a, b] \to f(X)$ is continuous. Since

$$(f \circ \gamma)(a) = p, (f \circ \gamma)(b) = q$$

$f \circ \gamma$ is a path in $f(X)$ joining p and q which means that $f(X)$ is path-connected.

Example 5.3 The Euclidean space $\mathbb{R}^n$ is clearly path-connected, hence connected. Note that $\mathbb{R}^n - \{0\}$ is path-connected for $n > 1$. For $n \geq 1$, define $f : \mathbb{R}^{n+1} - \{0\} \to S^n$ by

$$f(\mathbb{X}) := \frac{\mathbb{X}}{||\mathbb{X}||}$$

Since f is a surjective continuous function, by the result above, S^n is path-connected.

Example 5.4 Let $n \in \mathbb{N}$. If $GL(n, \mathbb{R})$ is connected, then the image

$$\det\left(GL(n, \mathbb{R})\right) = \mathbb{R} - \{0\}$$

is connected which is a contradiction. Therefore $GL(n, \mathbb{R})$ is disconnected.

Theorem 5.3. Generalized Intermediate Value Theorem

Let X be a connected topological space and $f : X \to \mathbb{R}$ be a continuous function. Suppose that $c \in \mathbb{R}$ and

$$f(x) < c < f(y)$$

for some $x, y \in X$. Then there exits $z \in X$ such that

$$f(z) = c$$

Proof Assume that c is not in the image of f. Let

$$U = f^{-1}((c, \infty))$$

Then

$$U^c = (f^{-1}((-\infty, c)))^c = f^{-1}((-\infty, c)^c) = f^{-1}([c, \infty)) = f^{-1}((c, \infty))$$

By the continuity of f, U and U^c are both open, therefore U is also closed. Since $y \in U$ and $x \notin U$, U is proper. Therefore X is disconnected which is a contradiction.

Recall that a point $c \in X$ is called a fixed point of a function $f : X \to X$ if

$$f(c) = c$$

Corollary 5.1. Brouwer fixed point theorem for dimension 1

Let $f : [0, 1] \to [0, 1]$ be a continuous function. Then f has a fixed point.

Proof

Case 1: If $f(0) = 0$ or $f(1) = 1$, we are done.

Case 2: Suppose that $f(0) \neq 0$ and $f(1) \neq 1$. Since the image is contained in $[0, 1]$, we have

$$f(0) > 0 \text{ and } f(1) < 1$$

Let $g : [0, 1] \to \mathbb{R}$ be defined by

$$g(x) = f(x) - x$$

Then g is continuous. Since

$$g(0) = f(0) - 0 = f(0) > 0$$

and

$$g(1) = f(1) - 1 < 0$$

By the Intermediate Value Theorem, there exists $c \in [0, 1]$ such that

$$g(c) = 0$$

and hence

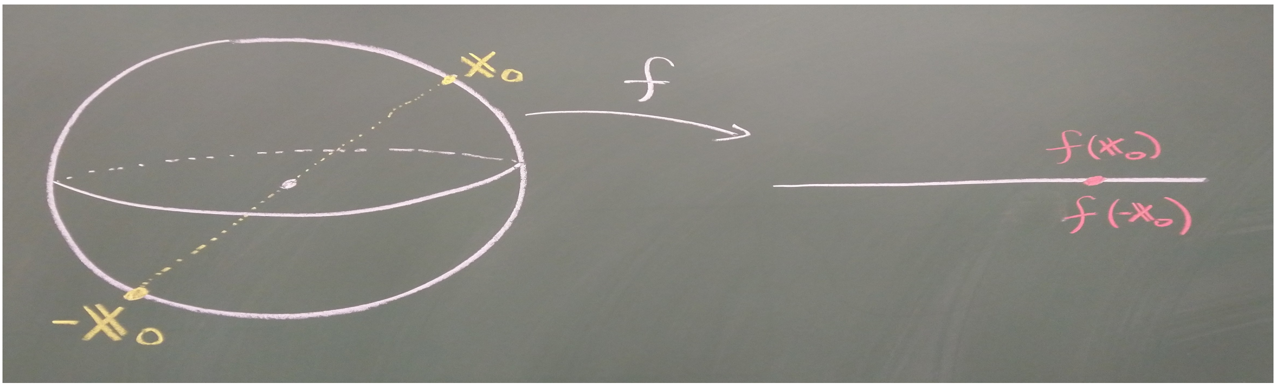
$$f(c) = c$$

Corollary 5.2

Let $f : S^n \to \mathbb{R}$ be a continuous function. There exists $\mathbb{X}_0 \in S^n$ such that

$$f(\mathbb{X}_0) = f(-\mathbb{X}_0)$$

♡



Proof Define $g : S^n \to \mathbb{R}$ by

$$g(\mathbb{X}) := f(\mathbb{X}) - f(-\mathbb{X})$$

Then

$$g(-\mathbb{X}) = f(-\mathbb{X}) - f(\mathbb{X}) = -g(\mathbb{X})$$

If $g(\mathbb{X}) = 0$, $f(\mathbb{X}) = f(-\mathbb{X})$. If $g(\mathbb{X}) \neq 0$, $g(\mathbb{X})$ and $g(-\mathbb{X})$ have different signs, and hence 0 lies between $g(\mathbb{X})$ and $g(-\mathbb{X})$. Since S^n is connected, by the generalized intermediate value theorem

(Theorem 5.3), there is $\mathbb{X}_0 \in S^n$ such that

$$g(\mathbb{X}_0) = 0$$

and thus

$$f(\mathbb{X}_0) = f(-\mathbb{X}_0)$$

Lemma 5.1

Let X be a topological space and $x, y, z \in X$.

1. *If there is a path $\gamma : [a, b] \to X$ joining x, y, then there is a path $\gamma' : [0, 1] \to X$ joining x, y.*
2. *If there is a path joining x, y and a path joining y, z, then there is a path joining x, z.* ♡

Proof

1. Define $\gamma' : [0, 1] \to X$ by

$$\gamma'(t) := \gamma((1 - t)a + tb)$$

   Then γ' is a composition of the path γ and the continuous function $f : [0, 1] \to [a, b]$ defined by

$$f(t) = (1 - t)a + tb$$

   Thus γ' is a path in X joining

$$\gamma'(0) = \gamma(a) = x \text{ and } \gamma'(1) = \gamma(1) = y$$

2. By (1), there are paths $\gamma_1, \gamma_2 : [0, 1] \to X$ such that

$$\gamma_1(0) = x, \gamma_1(1) = y, \text{ and } \gamma_2(0) = y, \gamma_2(1) = z$$

   Define

$$\gamma(t) = \begin{cases} \widetilde{\gamma}_1(t) = \gamma_1(2t), & \text{if } t \in [0, \frac{1}{2}] \\ \widetilde{\gamma}_2(t) = \gamma_2(2t - 1), & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

Note that on the intersection of the domains

$$[0, \frac{1}{2}] \cap [\frac{1}{2}, 1] = \{\frac{1}{2}\}$$

$$\widetilde{\gamma}_1(\frac{1}{2}) = \widetilde{\gamma}_2(\frac{1}{2}) = y$$

Thus $\gamma : [0, 1] \to X$ is a function. For a closed subset $C \subset X$, note that we have

$$\gamma^{-1}(C) = \widetilde{\gamma}_1^{-1}(C) \bigcup \widetilde{\gamma}_2^{-1}(C)$$

Since $\widetilde{\gamma}_1$ is continuous, $\widetilde{\gamma}_1^{-1}(C)$ is closed in $[0, \frac{1}{2}]$. Furthermore, $[0, \frac{1}{2}]$ is closed in $[0, 1]$, by the inheritance principle, $\widetilde{\gamma}_1^{-1}(C)$ is closed in $[0, 1]$. Similarly, $\widetilde{\gamma}_2^{-1}(C)$ is closed in $[0, 1]$ which implies the union $\gamma^{-1}(C)$ is closed in $[0, 1]$. This means that γ is continuous and thus a path joining x and z.

Proposition 5.2

Let $U \subset \mathbb{R}^n$ be an open set. Then U is connected if and only if U is path-connected. ♠

Proof Suppose that U is connected. We may assume that U is nonempty. Fix $p \in U$. Let

$$V := \{q \in U | p \text{ and } q \text{ are joined by a path}\}$$

If $q \in V$, since U is open in $\mathbb{R}^n$, there is $r > 0$ such that $B_r(q) \subset U$. For $x \in B_r(q)$, the path

$$\gamma(t) := (1 - t)q + tx, t \in [0, 1]$$

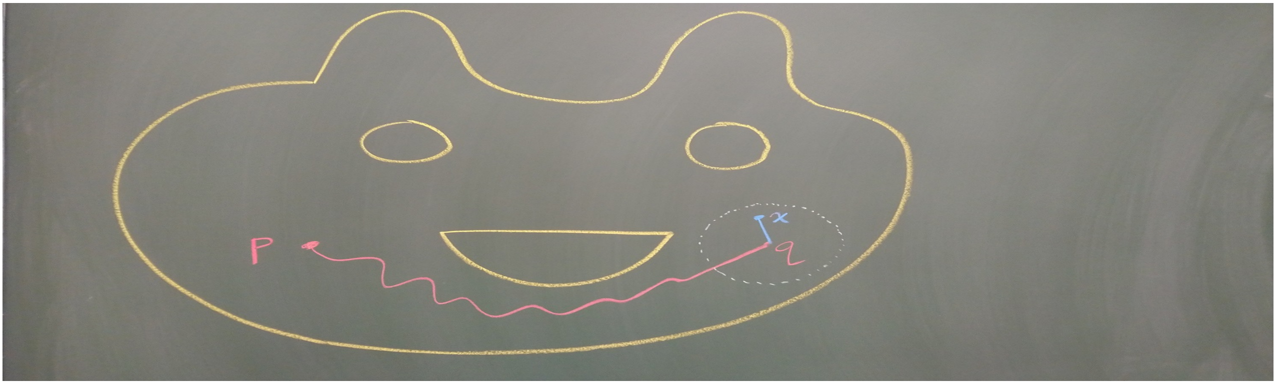
lies in $B_r(q)$ which joins q and x. With the path joining p and q, by Lemma 5.1, we have a path joining x and p. Thus $x \in V$. This means $B_r(q) \subset V$ and hence V is open in U. Let y be a limit point of V in U. There exists a sequence $\{y_n\}_{n=1}^{\infty}$ in V that converges to y. Since U is open in $\mathbb{R}^n$, there is $\delta > 0$ such that $B_\delta(y) \subset U$. For n large enough,

$$d_E(y_n, y) < \delta$$

The path

$$(1 - t)y_n + ty, \quad t \in [0, 1]$$

lies in $B_\delta(y) \subset U$ that joins y_n and y and hence $y \in V$. This means that V is closed in U. Since V is nonempty, both open and closed in U, the connectedness of U implies that $U = V$. Therefore any points in U can be joined by some paths to p and thus U is path connected.



Proposition 5.3

Let X be a metric space and $S \subset X$ be a connected subset. If

$$S \subset T \subset \overline{S}$$

then T is connected. In particular, the closure of a connected set is connected.

Proof Assume that T is not connected. By the definition, there exists a proper subset $A \subset T$ which is both open and closed in T such that A and A^c separate T. By the inheritance principle, $A \cap S$ is open in S. Pick $a \in A$ and $r > 0$ such that $B_r(a) \subset A$. If

$$A \cap S = \emptyset$$

then

$$B_r(a) \cap S = \emptyset$$

By Lemma 2.1, a is not a limit point of S which contradicts to the fact that

$$A \subset T \subset \overline{S}$$

Therefore $A \cap S$ is nonempty. Similarly, $A^c \cap S$ is nonempty. Then $A \cap S$ and $A^c \cap S$ separate S which contradicts to the fact that S is connected. Hence T is connected.

Proposition 5.4

Let X be a topological space and $A \subset X$. Suppose that A and A^c separate X. If $S \subset X$ is connected and $S \cap A \neq \emptyset$, then $S \subset A$. ♠

Proof If S is not contained in A, then the sets

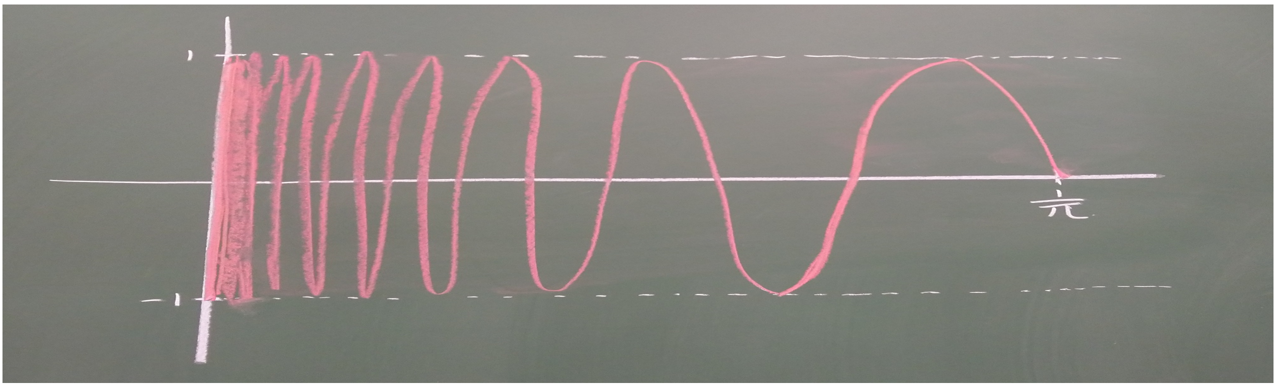
$$U = A \cap S, U^c = A^c \cap S$$

separate S which is a contradiction.

Example 5.5 (Topologist's sine curve) Let

$$A = \{(x, \sin \frac{1}{x}) | x \in (0, \frac{1}{\pi}]\} \cup \{(0, y) | y \in [-1, 1]\}.$$

Then A is connected but not path-connected.



Solution *Let*

$$S = \{(x, \sin \frac{1}{x}) | x \in (0, \frac{1}{\pi}]\}$$

It is clear that S is path-connected. If A is not connected, there are two nonempty open subsets U, U^c of A that separate A. We may assume $(0, 0) \in U$. Since $0 \times [-1, 1]$ is connected, by Proposition 5.4, $0 \times [-1, 1] \subset U$. By the openness of U, There is $r > 0$ such that $B_r((0, 0)) \subset U$. Note that for

$n \in \mathbb{N}$ with $\frac{1}{n\pi} < r$,

$$(\frac{1}{n\pi}, 0) \in B_r((0,0))$$

Since S is connected, by Proposition 5.4, $S \subset U$. Therefore $U = A$ which contradicts to the properness of U. So A is connected.

Assume that A is path-connected. Then there is a path $\gamma : [0,1] \to A$ such that

$$\gamma(0) = (\frac{1}{\pi}, 0) \text{ and } \gamma(1) = (0,0)$$

Let

$$c = \inf\{t \in [0,1] | \gamma(t) \in 0 \times [-1,1]\}$$

Then $\gamma([0,c])$ contains at most one point of $0 \times [-1,1]$. Note that for any $p \in S$, $S - \{p\}$ is not path-connected, therefore to reach the y-axis,

$$S \subset \gamma([0,c])$$

Since each point of $0 \times [-1,1]$ is a limit point of S, the closure $\overline{\gamma([0,c])}$ contains all of $0 \times [-1,1]$. Therefore

$$\gamma([0,c]) \neq \overline{\gamma([0,c])}$$

In particular, $\gamma([0,c])$ is not closed and hence not compact. But γ is continuous and $[0,c]$ is compact, $\gamma([0,c])$ is compact which is a contradiction. Therefore, A is not path-connected.

Exercise 5.1

1. Show that a continuous function $f : [0, 1] \to \mathbb{Z}$ is a constant function.
2. Show that $\mathbb{R} - \mathbb{Q}$ and $\mathbb{R} - \mathbb{Z}$ are not homeomorphic.
3. Let $D = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq 1\}$ be the unit closed disc in $\mathbb{R}^2$ and

$$E := \{(x, y) \in D | y = mx \text{ for some } m \in \mathbb{Q}\}$$

   (a). Are $E \cap S^1$ and $[0, 1] \cap \mathbb{Q}$ homeomorphic?

   (b). Are E and $D - E$ homeomorphic?

   (c). Are $\mathbb{R}^2 - E$ and $\mathbb{R}^2 - (D - E)$ homeomorphic?

4. What are connected subsets of $\mathbb{Q}$?
5. Suppose that $A, B \subset \mathbb{R}^2$.

   (a). If A and B are homeomorphic, are A^c and B^c homeomorphic?

   (b). If A and B are connected and homeomorphic, are A^c and B^c homeomorphic?

6. Show that $\mathbb{R}^1$ and $\mathbb{R}^2$ are not homeomorphic. How about $\mathbb{R}^1$ and $\mathbb{R}^n$ for $n \geq 2$?
7. Let $f : \mathbb{R} \to \mathbb{R}$ be a uniformly continuous function. Show that there are constants A, B such that

$$|f(x)| \leq A + B|x|$$

   for all $x \in \mathbb{R}$.

8. Let $U \subset \mathbb{R}^m$ be an open set. If $h : U \to \mathbb{R}^m$ is a homeomorphism and uniformly continuous on U, show that $U = \mathbb{R}^m$.