Advanced calculus I-1

Metric spaces, topological spaces and sequences

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Version: Second edition

One of the secrets to mathematical problem solving is that one needs to place a high value on partial progress, as being a crucial stepping stone to fully solving the problem. ——- Terry Tao

Contents of Advanced Calculus I-1

Preface

Advanced calculus plays a pivotal role in the realm of mathematics education, serving as the bridge that connects elementary numerical computations to the more intricate world of advanced abstract thinking. It is a subject that often instills trepidation in students as they begin their journey into its complexities. The pages that follow are a product of the author's extensive experience in teaching advanced calculus courses at the mathematics department of National Tsinghua University of Taiwan. The primary objective of these books is to provide mathematics textbooks optimized for mobile phone readability, complete with vibrant visuals to enhance the learning experience. Within these pages, every proof is presented with simplicity in mind, making them accessible, comprehensive, and enriched with graphics and colors. The author's aspiration is to foster greater receptiveness among students, encouraging them to invest more time in this essential course.

Advanced calculus is a rigorous two-semester course, bearing the weight of four credits per semester, with 200 minutes of instruction weekly, accompanied by two hours of recitation, weekly assignments, and four exams per semester. It stands as one of the most challenging classes in the mathematics department, demanding a substantial commitment from students. In the modern era, proficiency in mathematics and abstract thinking is becoming increasingly critical in scientific and technological fields. Mathematics is intrinsically linked to automated processes and big data analysis, making it an indispensable skill. This is why this demanding course continues to attract students from diverse disciplines, including electrical engineering, computer science, financial engineering, management, and medical school.

To cater to the needs of mobile phone users, the book files are designed to be manageable in size. As a result, the content of Advanced Calculus I is divided into two volumes, each aligned with specific exam content: Advanced Calculus I-1 and Advanced Calculus I-2, each encompassing two exams. All exams and practice tests are thoughtfully included within the books, with exercises accompanying each section. Despite being written in English, the language used is intentionally not overly complex. Building upon foundational calculus knowledge, these textbooks can be regarded as self-contained and suitable for self-study.

The main references for Advanced Calculus I-1 are:

1. Real mathematical analysis by Pugh ([**P**]);
2. Elementary classical analysis by Marsden and Hoffman ([**MH**]);
3. Principles of mathematical analysis by Rudin ([**R**]);
4. Wikipedia.

The beautiful pictures found at the end of each chapter are sourced from the generous offerings of pixabay.com.

This book is edited using the LaTeX document class “elegantbook” (https://github.com/ElegantLaTeX/ElegantBook).

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Chapter 1 Set theory and cardinality

1.1 Review

1.1.1 Number systems and logic

The set of natural numbers is

$$\mathbb{N} := \{1, 2, 3, 4, ...\}$$

The set of integers is

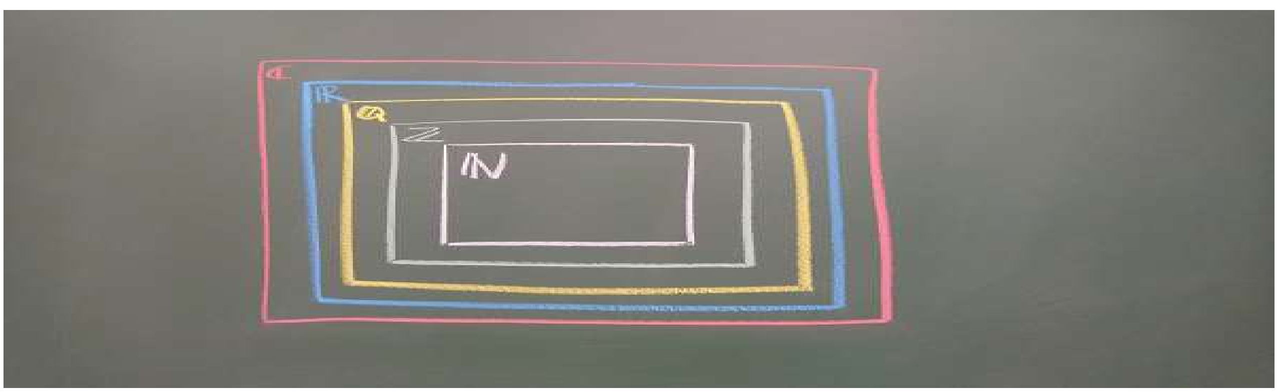
$$\mathbb{Z} := \{..., -2, -1, 0, 1, 2, ...\}$$

The set of rational numbers is

$$\mathbb{Q} := \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$$

The set of real numbers is $\mathbb{R}$. These real numbers $\{\pi, e, \sqrt{2}, -2^{\sqrt{3}}\}$ are not rational numbers. The set of complex numbers is

$$\mathbb{C} := \{a + bi : a, b \in \mathbb{R}\}$$



Let P, Q, R be some statements. There are two equivalent forms which are frequently used in proofs.

1.

$$(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P) \equiv (\neg P \vee Q)$$

2.

$$(P \Rightarrow Q \vee R) \equiv (P \wedge^{\neg} Q \Rightarrow R)$$

Definition 1.1.1

A predicate $P(x)$ assigns x a statement.

1. *The symbol*

$$(\forall x \in A)P(x)$$

is defined to be

$$x \in A \Rightarrow P(x)$$

2. *The symbol*

$$(\exists x \in A)Q(x)$$

is defined to be

$$(x \in A) \wedge Q(x)$$

♣

Proposition 1.1.1

1.

$$^{\neg}((\forall x \in A)P(x)) \equiv (\exists x \in A)(^{\neg}P(x))$$

2.

$$^{\neg}((\exists x \in A)P(x)) \equiv (\forall x \in A)(^{\neg}P(x))$$

♠

Proof

1.

$$^{\neg}((\forall x \in A)P(x)) \equiv^{\neg} (x \in A \Rightarrow P(x)) \equiv (x \in A) \wedge^{\neg} P(x) \equiv (\exists x \in A)^{\neg}P(x)$$

2.

$$\begin{aligned} ^{\neg}((\exists x \in A)P(x)) &\equiv^{\neg} ((x \in A) \wedge P(x)) \equiv^{\neg} (x \in A) \vee^{\neg} P(x) \\ &\equiv (x \in A) \Rightarrow^{\neg} P(x) \equiv (\forall x \in A)(^{\neg}P(x)) \end{aligned}$$

1.1.2 Set theory

Given sets $A_1, A_2, A_3, \ldots$.. The union of these sets is

$$\bigcup_{i=1}^{\infty} A_i := \{x : x \in A_i \text{ for } \textbf{some } i \in \mathbb{N}\}$$

and the intersection of these sets is

$$\bigcap_{i=1}^{\infty} A_i := \{x : x \in A_i \text{ for } \textbf{each } i \in \mathbb{N}\}$$

More generally, if B is a set and a set A_i is given for each i in B, define the union of sets indexed by B to be

$$\bigcup_{i \in B} A_i := \{x : x \in A_i \text{ for } \textbf{some } i \in B\}$$

and the intersection of sets indexed by B to be

$$\bigcap_{i \in B} A_i := \{x : x \in A_i \text{ for } \textbf{each } i \in B\}$$

In the following, we recall the definition of Cartesian product.

Definition 1.1.2 (Cartesian product)

Let X, Y be sets. The Cartesian product of X and Y is the set

$$X \times Y := \{(x, y) | x \in X, y \in Y\}$$

of ordered pairs. The n-th Cartesian product of X is the set

$$X^n := \{(x_1, x_2, \ldots, x_n) | x_i \in X, i = 1, 2, \ldots, n\}$$

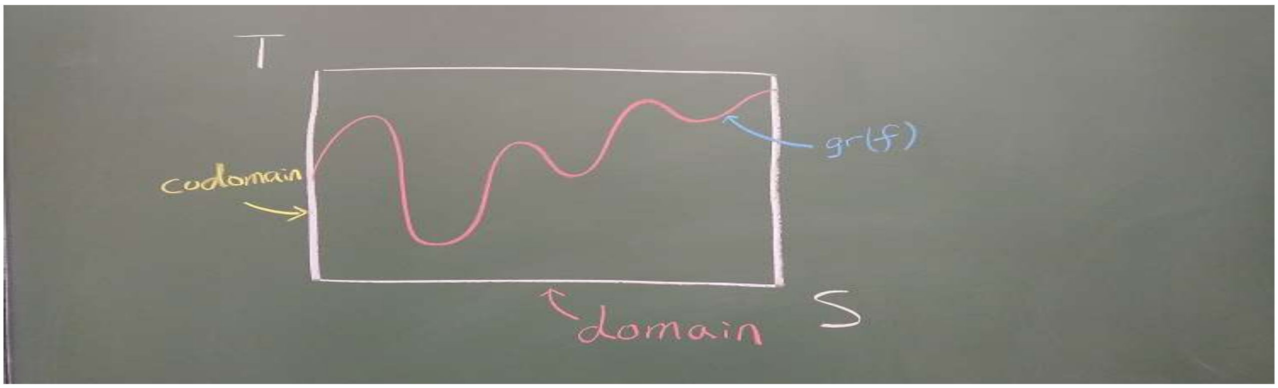
of n-tuples. ♣

Definition 1.1.3

A function $f : S \to T$ from a set S to a set T is a subset

$$gr(f) \subseteq S \times T$$

with the following property: for any $s \in S$, there is a unique $t \in T$ such that $(s, t) \in gr(f)$. Usually we write $f(s) = t$. The set S is called the domain of f and the set T is called the *codomain of f.* ♣



Example 1.1.1 Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^2$$

and $g : \mathbb{R} \to [0, \infty)$ be defined by

$$g(x) = x^2$$

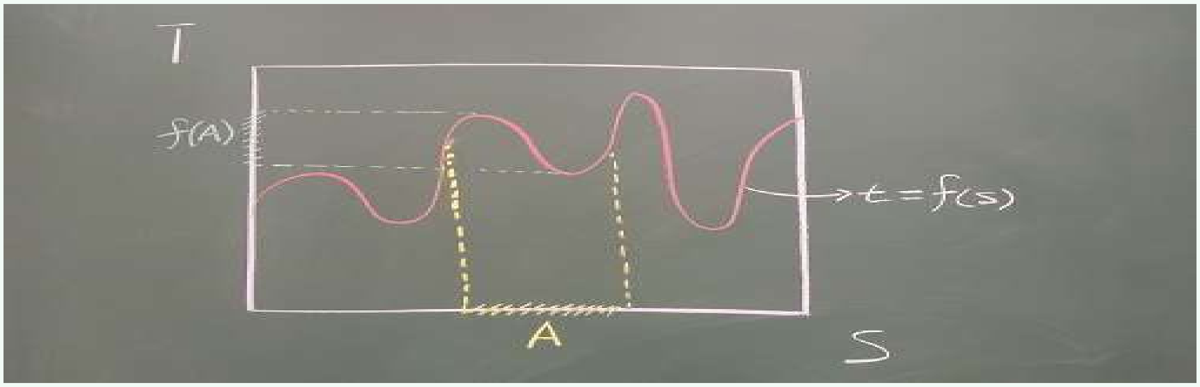
These two function are two distinct functions since they have different codomains.

Definition 1.1.4

If $f : S \to T$ is a function and $A \subseteq S$, we write

$$f(A) := \{f(a)|a \in A\}$$

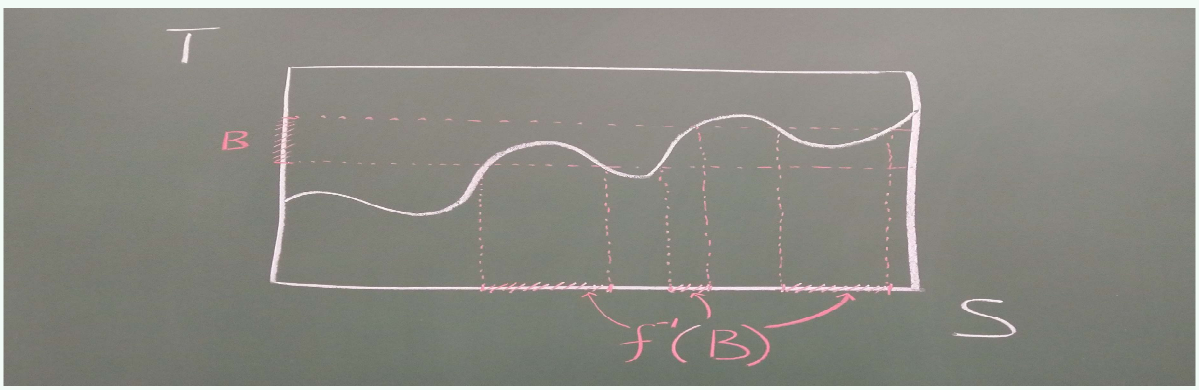
and call $f(A)$ the image of A under f.



If $B \subseteq T$, we write

$$f^{-1}(B) := \{s \in S | f(s) \in B\}$$

and call $f^{-1}(B)$ the preimage of B under f.



♣

Definition 1.1.5

Let $f : S \to T$ be a function. We say that

1. *f is injective if whenever $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$;*
2. *f is surjective if for every $t \in T$, there is $s \in S$ such that $f(s) = t$;*
3. *f is bijective if f is both injective and surjective and call such f a bijection.*

If $f : S \to T$ is a bijective function, the function $g : T \to S$ defined by

$$g(t) := f^{-1}(\{t\})$$

is called the inverse function of f. ♣

1.2 Countable and uncountable sets

Definition 1.2.1 (Same cardinality)

We say that two sets A and B have the same cardinality if there is a bijection from A to B. ♣

Note that if A and B have the same cardinality, then B and A also have the same cardinality.

Definition 1.2.2

A set S is said to be finite if there exist a positive integer N and a bijective function

$$f : \{1, 2, ..., N\} \to S$$

Empty set is defined to be a finite set. A set which is not finite is called infinite. An infinite set S is said to be denumerable if there is a bijective function

$$f : \mathbb{N} \to S$$

A set which is either finite or denumerable is said to be countable. A set which is not countable is said to be uncountable. ♣

We show the classifications by the following diagrams:

$$\text{sets} \begin{cases} \text{countable sets} \begin{cases} \text{finite sets} \\ \text{denumerable sets} \end{cases} \\ \text{uncountable sets} \end{cases}$$

$$\text{sets} \begin{cases} \text{finite sets} \\ \text{infinite sets} \begin{cases} \text{denumerable sets} \\ \text{uncountable sets} \end{cases} \end{cases}$$

Example 1.2.2 Show that $\mathbb{Z}$ is denumerable.

Proof List elements of $\mathbb{Z}$ as $\{0, 1, -1, 2, -2, 3, -3, ...\}$. Define $f : \mathbb{N} \to \mathbb{Z}$ by

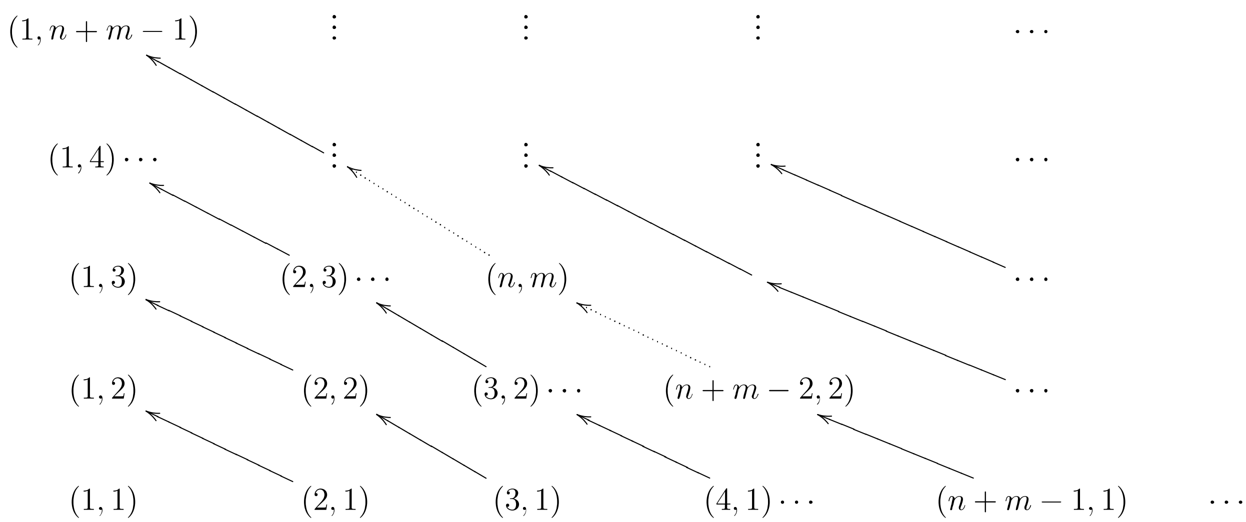
$$f(n) = \begin{cases} \frac{n}{2} & , \quad \text{if } n \text{ is } even \\ -\frac{n-1}{2} & , \quad \text{if } n \text{ is } odd \end{cases}$$

Then f is bijective and hence $\mathbb{Z}$ is denumerable.

Proposition 1.2.1

$\mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$ *is denumerable.* ♠

Solution *We write elements of* $\mathbb{N}^2$ *in the following array and list elements of* $\mathbb{N}^2$ *along the* $45°$ *line from the bottom rows to the left columns:*



The number of elements contained in the triangle below the line through (n, m) *is*

$$1 + 2 + \cdots + (n + m - 2) = \frac{(n + m - 2)(n + m - 1)}{2}$$

Formally, we define a function $f : \mathbb{N}^2 \to \mathbb{N}$ *by*

$$f(n, m) = \frac{(n + m - 2)(n + m - 1)}{2} + m$$

Now we claim that f *is bijective. We show that* f *is injective first. Let* $(n_1, m_1), (n_2, m_2) \in \mathbb{N}^2$. *Suppose that* (n_1, m_1) *and* (n_2, m_2) *are not the same. We need to show that*

$$f(n_1, m_1) \neq f(n_2, m_2)$$

We consider 3 cases:

Case 1: $n_2 + m_2 > n_1 + m_1$

Let

$$n_2 + m_2 = n_1 + m_1 + k, \text{ where } k \geq 1$$

Then

$$
\begin{aligned}
f(n_2, m_2) &= \frac{(n_1+m_1+k-2)(n_1+m_1+k-1)}{2} + m_2 \\
&= \frac{(n_1+m_1)^2 - 3(n_1+m_1) + 2k(n_1+m_1) + (k-1)(k-2)}{2} + m_2 \\
&= \frac{(n_1+m_1)^2 - 3(n_1+m_1) + 2}{2} + k(n_1+m_1) + \frac{(k-1)(k-2)}{2} + m_2 - 1 \\
&> \frac{(n_1+m_1-2)(n_1+m_1-1)}{2} + m_1 \\
&= f(n_1, m_1)
\end{aligned}
$$

Case 2: $n_2 + m_2 = n_1 + m_1$

If $f(n_1, m_1) = f(n_2, m_2)$, then $m_1 = m_2$ and hence $n_1 = n_2$. This contradicts to the assumption that (n_1, m_1) and (n_2, m_2) are different. Therefore $f(n_1, m_1) \neq f(n_2, m_2)$.

Case 3: $n_2 + m_2 < n_1 + m_1$

Interchange the roles of $(n_1, m_1), (n_2, m_2)$ in Case 1, we have

$$f(n_2, m_2) < f(n_1, m_1)$$

This shows that f is injective. To show that f is surjective. Let $M \in \mathbb{N}$. Take $N \in \mathbb{N}$ such that

$$\frac{(N-2)(N-1)}{2} < M \leq \frac{(N-1)N}{2}$$

Let

$$m := M - \frac{(N-2)(N-1)}{2} \text{ and } n := N - m$$

Then $n + m = N$ and

$$
\begin{aligned}
f(n, m) &= \frac{(n+m-2)(n+m-1)}{2} + m \\
&= \frac{(N-2)(N-1)}{2} + M - \frac{(N-2)(N-1)}{2} \\
&= M
\end{aligned}
$$

This shows that f is surjective. Therefore $f : \mathbb{N}^2 \to \mathbb{N}$ is bijective. Let $g : \mathbb{N} \to \mathbb{N}^2$ be the inverse function of f. Since g is bijective, $\mathbb{N}^2$ is denumerable.

Let us recall the well-ordering principle of the natural numbers.

Theorem 1.2.1 (Well-ordering principle)

Every nonempty subset of the natural numbers has a least element. ♡

We have the following result.

Theorem 1.2.2 (Subsets of a countable set are countable)

If $f : A \to S$ is an injective function and S is countable, then A is countable. ♡

Proof Assume that A is not finite. We need to show that A is denumerable. Since f is injective, the function $f : A \to f(A)$ is bijective, hence $f(A)$ is an infinite set. But $f(A) \subset S$, S has to be infinite, and since S is countable, we know that S is denumerable. List elements of S as

$$\{s_1, s_2, s_3, ...\}$$

Let

$$N = \{i | s_i \in f(A)\} \subset \mathbb{N}$$

By the well-ordering principle 1.2.1, there is a smallest positive integer $n_1 \in N$. We have $s_{n_1} \in f(A)$.

Suppose that we have chosen $n_1 < n_2 < \cdots < n_k$ such that $s_{n_1}, s_{n_2}, ..., s_{n_k}$ are in $f(A)$. Since $f(A)$ is infinite, $f(A) - \{s_{n_1}, ..., s_{n_k}\}$ is not empty. Let

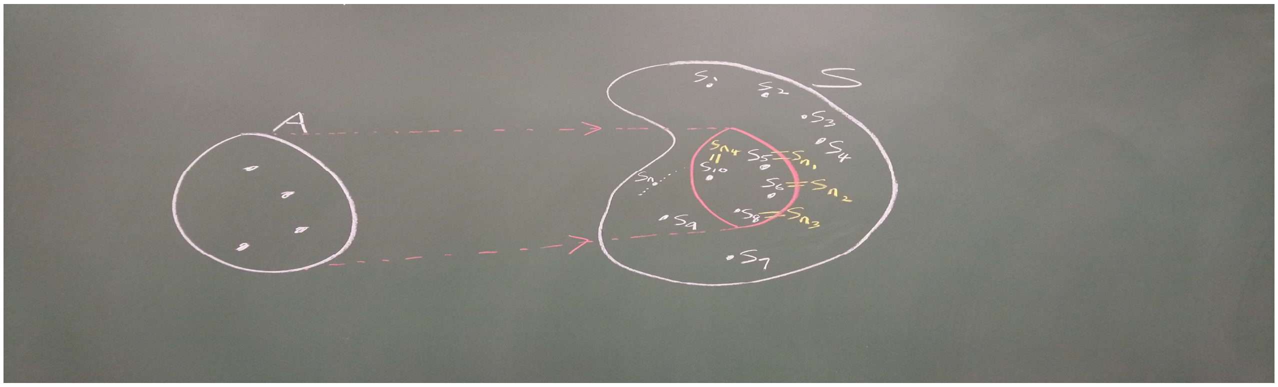
$$M := \{i | s_i \in f(A) - \{s_{n_1}, ..., s_{n_k}\}\} \subset \mathbb{N}$$

By the well-ordering principle, there exists a smallest positive integer $n_{k+1} \in M$. We have $n_{k+1} > n_k$ otherwise $s_{n_{k+1}}$ would have been chosen before. Note that $s_{n_{k+1}} \in f(A)$.

For an arbitrary $s_t \in f(A)$, by at most choosing t times, s_t will be in the list. So all elements in $f(A)$ are listed as

$$\{s_{n_1}, s_{n_2}, s_{n_3}, ...\}$$

Define a function $g : \mathbb{N} \to f(A)$ by mapping k to s_{n_k}. According to the construction, g is bijective and hence $f(A)$ is denumerable. Since $f : A \to f(A)$ is bijective, A is also denumerable.



Corollary 1.2.1

Every infinite subset of a denumerable set is denumerable. ♡

Proof Let S be a denumerable set and $A \subset S$ be an infinite subset. Let $f : A \to S$ be the inclusion map, that is, for $a \in A$, $f(a) = a$. Then f is injective. By the result above, A is countable and by the assumption, A is an infinite set, so A is denumerable.

Remark A very useful method in proving mathematical results is the so-called "proof by contradiction". The strategy is that we assume something that is just opposite to what we want to prove and then deduce by some mathematical arguments to get a conclusion that contradicts to some known fact. This implies that the assumption we made is not true and thus proves what we want. This method will be used throughout this class.

We exemplify the "proof by contradiction" method by proving the following result.

Corollary 1.2.2

Every subset of a countable set is countable. ♡

Proof Let S be a countable set. Assume that there exists a subset $A \subset S$ which is uncountable. By the definition of an uncountable set, A is an infinite set. Since $A \subseteq S$, S is also infinite. Then S is denumerable. By the theorem above, A is countable. This contradicts to our assumption. So there does not exist an uncountable subset of S.

Theorem 1.2.3 (A countable union of countable sets is countable)

Let $\{E_n\}_{n=1}^{\infty}$ be a sequence of countable sets and

$$S = \bigcup_{n=1}^{\infty} E_n$$

Then S is countable. ♡

Proof We list elements of E_n as

$$x_{n1}, x_{n2}, x_{n3}, \ldots.$$

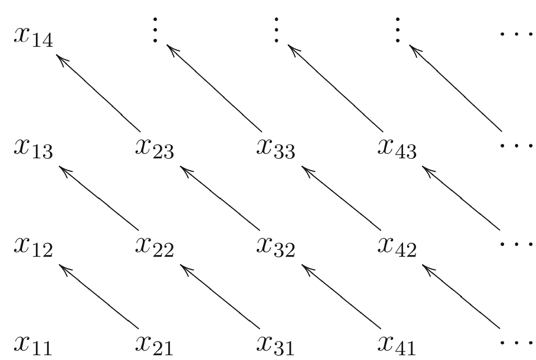
If E_n is a finite set, we list the last element of E_n repeatedly infinitely many times, that is, we let

$$x_{ni} = x_{nk}$$

for all $i \geq k$ where

$$k = |E_n| := \text{ the number of elements of } E_n$$

So we have an array



By Proposition 1.2.1, $\mathbb{N}^2$ is denumerable, so there exists a bijection $f : \mathbb{N} \to \mathbb{N}^2$. Define $g : \mathbb{N} \to S$ by

$$g(n) := x_{f(n)}$$

Then g is surjective but g may not be injective. For $s \in S$, let

$$h(s) := \text{ the smallest number of } g^{-1}(s)$$

where

$$g^{-1}(s) = \{n \in \mathbb{N} | g(n) = s\}$$

is the pre-image of s under g. Then $h : S \to \mathbb{N}$ is injective. Since $\mathbb{N}$ is countable, by the Theorem 1.2.2, S is also countable.

Proposition 1.2.2

$\mathbb{Q}$ is denumerable. ♠

Proof For $n \in \mathbb{N}$, let

$$E_n := \{\frac{m}{n} | m \in \mathbb{Z}\}$$

Then the function $f : \mathbb{Z} \to E_n$ defined by

$$f(m) := \frac{m}{n}$$

is bijective. Therefore E_n is countable. Since

$$\mathbb{Q} = \bigcup_{n=1}^{\infty} E_n$$

By the result above, $\mathbb{Q}$ is countable. Since $\mathbb{Q}$ has infinitely many elements, it is denumerable.

Corollary 1.2.3

Let X be a countable set. If for each $x \in X$, E_x is a countable set, then

$$\bigcup_{x \in X} E_x$$

is countable. ♡

Proof Since X is countable, we may list elements of X as

$$\{x_1, x_2, x_3, ...\}$$

with the convention that if X is finite, we write $x_k = x_n$ for $k > |X|$. Then

$$\bigcup_{x \in X} E_x = \bigcup_{i=1}^{\infty} E_{x_i}$$

The result follows from the above theorem.

Theorem 1.2.4

If X and Y are countable sets, then the Cartesian product

$$X \times Y$$

is countable. ♡

Proof For $x \in X$, let

$$Y_x := \{(x, y) | y \in Y\}$$

Define a function $f : Y \to Y_x$ by

$$f(y) = (x, y)$$

Then f is a bijection and hence Y_x is countable since Y is countable. We observe that

$$X \times Y = \bigcup_{x \in X} Y_x$$

and by Corollary 1.2.3, a countable union of countable sets is countable, we get the result.

Theorem 1.2.5

Let A be a countable set. Then A^n is countable for any $n \in \mathbb{N}$. ♡

Proof We prove by induction. When $n = 1$, $A^1 = A$ is countable. Suppose that A^n is countable. Define $f : A^{n+1} \to A^n \times A$ by

$$f(a_1, ..., a_n, a_{n+1}) = ((a_1, ..., a_n), a_{n+1})$$

Then f is a bijection. This implies that A^{n+1} and $A^n \times A$ have the same cardinality. Since A^n and A are countable, by the theorem above, $A^n \times A$ is countable, therefore, A^{n+1} is countable. By the principle of induction, A^n is countable for any $n \in \mathbb{N}$.

We recall that for two integers p, q, not both zero, if their greatest common divisor(gcd) is 1, then we say that they are relatively prime. Note that

$$\gcd(0, 3) = 3, \quad \gcd(-2, -6) = 2$$