Advanced calculus for quantitative finance II

Integration, Lebesgue measure and the Black-Scholes model

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Contents of Advanced Calculus II

Contents of Advanced Calculus I

Preface

Advanced calculus occupies the most fundamental position in mathematics training. It is an essential path from elementary numerical calculation to higher-level abstract thinking. Mathematical knowledge and the ability of abstract thinking become more and more important in modern sciences and technology industries. Quantitative finance is a perfect field for showing the power of mathematics.

In our opinion, advanced calculus for students in quantitative finance should not be the same as the one for students in mathematics department since they have different training in mathematics and different kinds of applications in future studies. Based on lecture notes for the advanced calculus courses that the author taught in the National Tsing Hua University of Taiwan for students from the department of quantitative finance, the author produces two books: Advanced calculus for quantitative finance I & II. The goal of these books is to introduce mathematical analysis and provide background behind the Black-Scholes model in options.

The Black-Scholes model and its variants are probably the most common models in finance. Since even an introduction of mathematical Brownian motion is out of the reach of undergraduate mathematics, it is not easy to talk about Ito calculus which the Black-Scholes model lies on. The author takes up the challenge in these books. The goal is to provide deep mathematics for students in quantitative finance and at the same time show them such mathematics is tightly related to their field of studying.

Differences between these books and advanced calculus textbooks for students in mathematics are that they start from a lighter mathematics prerequisites, skip some results such as the inverse function theorem that are not directly related to the study of mathematical finance, introduce probability theory based on Lebesgue integration, provide basic stochastic calculus, give a rather rigor derivation of the Black-Scholes model, introduce Fourier transform to solve the heat equation, and use it to derive a solution for the Black-Scholes equation.

Each of these two books contains 3 midterm exams and 1 final exam, accompany with a practice exam before each examination. Also at the end of each section, there are some exercises for students to get familiar with the materials. Proofs of some more difficult theorems are provided in the appendix of each section.

Main references are

1. Fractals everywhere by Barnsley ([**B**]);
2. Probability theory in finance: a mathematical guide to the Black-Scholes formula by Dineen ([**D**]);
3. Elementary classical analysis by Marsden and Hoffman ([**MH**]);
4. Real mathematical analysis by Pugh ([**P**]);
5. Wikipedia.

Those beautiful pictures at the end of each chapter are free pictures from pixabay.com.

The latex documentclass "elegantbook"(https://github.com/ElegantLaTeX/ElegantBook) is used to edit this book.

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Chapter 6 Higher dimensional Riemann integrals

6.1 Riemann integrals

Definition 6.1 (Rectangle)

A rectangle in $\mathbb{R}^n$ *is a set of the form*

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$$

for some real numbers $a_1 \leq b_1$, $a_2 \leq b_2$, ..., $a_n \leq b_n$. ♣

Example 6.1

1. $[a, a]$ is a rectangle in $\mathbb{R}$ for any $a \in \mathbb{R}$.
2. $[1, 3] \times [\sqrt{2}, \sqrt{3}]$ is a rectangle in $\mathbb{R}^2$.
3. $[1, 4] \times [2, 3] \times [1, 2]$ is a rectangle in $\mathbb{R}^3$.

Definition 6.2 (Partition)

A partition P *of a rectangle* $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ *is a set of points*

$$P = P_1 \times P_2 \times \cdots \times P_n$$

where

$$P_i = \{a_i = t_{i,0} < t_{i,1} < \cdots < t_{i,k_i} = b_i\}$$

is a partition of $[a_i, b_i]$, $k_i \in \mathbb{N}$ *for all* $i = 1, 2, ..., n$. ♣

Example 6.2 Let $R = [1, 2] \times [2, 5]$ be a rectangle in $\mathbb{R}^2$ and

$$P_1 := \{1, 1.1, 1.7, 2\}$$
$$P_2 := \{2, 2.8, 3.9, 5\}$$

Then

$$\begin{aligned} P := P_1 \times P_2 = &\{(1, 2), (1, 2.8), (1, 3.9), (1, 5), (1.1, 2), (1.1, 2.8), (1.1, 3.9), (1.1, 5) \\ &(1.7, 2), (1.7, 2.8), (1.7, 3.9), (1.7, 5), (2, 2), (2, 2.8), (2, 3.9), (2, 5)\} \end{aligned}$$

is a partition of R.

Definition 6.3 (Volume)

The volume of the rectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ *is defined to be*

$$vol(R) := (b_1 - a_1)(b_2 - a_2)\cdots(b_n - a_n)$$

♣

Example 6.3 If $R = [1, 3] \times [2, 5] \times [3, 8]$, $\text{vol}(R) = (3 - 1)(5 - 2)(8 - 3) = 30$.

Definition 6.4 (Lower sum and upper sum)

Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$ *be a rectangle in* $\mathbb{R}^n$ *and* $f : R \to \mathbb{R}$ *be a function. Given a partition*

$$P = P_1 \times P_2 \times \cdots \times P_n$$

of R *where*

$$P_i = \{a_i = t_{i,0} < t_{i,1} < \cdots < t_{i,k_i} = b_i\}$$

Let

$$R_{i_1,i_2,\ldots,i_n} := [t_{1,i_1-1}, t_{1,i_1}] \times [t_{2,i_2-1}, t_{2,i_2}] \times \cdots \times [t_{n,i_n-1}, t_{n,i_n}]$$

$$m_{i_1,\ldots,i_n} = \inf_{x \in R_{i_1,\ldots,i_n}} \{f(x)\}$$

$$M_{i_1,\ldots,i_n} = \sup_{x \in R_{i_1,\ldots,i_n}} \{f(x)\}$$

for $1 \le i_j \le k_j, j = 1, \ldots, n$.

Define the lower sum of f *with respect to* P *to be*

$$L(f, P) := \sum_{i_1=1}^{k_1} \cdots \sum_{i_n=1}^{k_n} m_{i_1,i_2,\ldots,i_n} vol(R_{i_1,i_2\ldots,i_n})$$

and the upper sum of f *with respect to* P *to be*

$$U(f, P) := \sum_{i_1=1}^{k_1} \cdots \sum_{i_n=1}^{k_n} M_{i_1,i_2,\ldots,i_n} vol(R_{i_1,i_2,\ldots,i_n})$$

♣

Example 6.4 Let $R = [1, 2] \times [2, 4]$ and $f : R \to \mathbb{R}$ be defined by

$$f(x, y) = x + y$$

Fix $n \in \mathbb{N}$ and let

$$P = \{1 = t_{1,0} < t_{1,1} < \cdots < t_{1,n} = 2\} \times \{2 = t_{2,0} < t_{2,1} < \cdots < t_{2,n} = 4\}$$

where

$$t_{1,i} = 1 + \frac{i}{n}, \quad , t_{2,j} = 2 + \frac{2j}{n}$$

Then

$$R_{i,j} = [t_{1,i-1}, t_{1,i}] \times [t_{2,j-1}, t_{2,j}] = [1 + \frac{i-1}{n}, 1 + \frac{i}{n}] \times [1 + \frac{2(j-1)}{n}, 1 + \frac{2j}{n}]$$

The volume

$$\text{vol}(R_{i,j}) = \frac{1}{n}\frac{2}{n} = \frac{2}{n^2}$$

We have

$$\inf_{x \in R_{i,j}} \{f(x)\} = (1 + \frac{i-1}{n}) + (2 + \frac{2(j-1)}{n}) = 3 + \frac{i+2j-3}{n}$$

$$\sup_{x \in R_{i,j}} \{f(x)\} = (1 + \frac{i}{n}) + (2 + \frac{2j}{n}) = 3 + \frac{i+2j}{n}$$

and

$$L(f, P) = \sum_{i=1}^{n} \sum_{j=1}^{n} (3 + \frac{i+2j-3}{n})(\frac{2}{n^2})$$

$$U(f, P) = \sum_{i=1}^{n} \sum_{j=1}^{n} (3 + \frac{i+2j}{n})(\frac{2}{n^2})$$

Definition 6.5 (Refinement)

A refinement of a partition P of a rectangle R is a partition P' of R such that $P \subseteq P'$. ♣

Example 6.5 Let $R = [1, 2] \times [3, 4]$ and $P = \{1, 1.2, 1.5, 2\} \times \{3, 4\}$. If

$$P' := P \cup \{(1, 3.5), (1.2, 3.5), (1.5, 3.5), (2, 3.5)\} = \{1, 1.2, 1.5, 2\} \times \{3, 3.5, 4\}$$

P' is a refinement of P.

Proposition 6.1

If $R \subset \mathbb{R}^n$ is a rectangle and $f : R \to \mathbb{R}$ is a bounded function, then

$$L(f,P) \le L(f,P') \le U(f,P') \le U(f,P)$$

for any partitions $P \subseteq P'$ of R.

Proof Let $R = [a_1, b_1] \times \cdots \times [a_n, b_n]$. Suppose that

$$P = P_1 \times P_2 \times \cdots \times P_n$$

where $P_i = \{a_i = t_{i,0} < t_{i,1} < \cdots < t_{i,k_i} = b_i\}$ and

$$P' = P_1' \times P_2' \times \cdots \times P_n'$$

where $P_i' = \{a_i = t_{i,0}' < t_{i,1}' < \cdots < t_{i,k_i'}' = b_i\}$ Let

$$R_{i_1,\ldots,i_n} = [t_{1,i_1-1}, t_{1,i_1}] \times \cdots \times [t_{n,i_n-1}, t_{n,i_n}] \subset R$$

Since P' is a refinement of P, there is a subset $A_{i_1,\ldots,i_n} \subset P'$ such that

$$R_{i_1,\ldots,i_n} \subset \bigcup_{(i_1',\ldots,i_n') \in A_{i_1,\ldots,i_n}} R_{i_1',\ldots,i_n'}$$

Therefore

$$\begin{aligned}
\sum_{(i_1',\ldots,i_n') \in A} \sup_{x \in R_{i_1',\ldots,i_n'}} \{f(x)\} \text{vol}(R_{i_1',\ldots,i_n'}) &\le \sum_{(i_1',\ldots,i_n') \in A} \sup_{x \in R_{i_1,\ldots,i_n}} \{f(x)\} \text{vol}(R_{i_1',\ldots,i_n'}) \\
&= \sup_{x \in R_{i_1,\ldots,i_n}} \{f(x)\} \sum_{(i_1',\ldots,i_n') \in A_{i_1,\ldots,i_n}} \text{vol}(R_{i_1',\ldots,i_n'}) \\
&= \sup_{x \in R_{i_1,\ldots,i_n}} \{f(x)\} \text{vol}(R_{i_1,\ldots,i_n})
\end{aligned}$$

Note that

$$P' = \bigcup_{(i_1,\ldots,i_n) \in P} A_{i_1,\ldots,i_n}$$

This implies $U(f,P') \le U(f,P)$. Similar to the argument above, we have $L(f,P') \ge L(f,P)$. Since we always have

$$\inf_{x \in R_{i_1',\ldots,i_n'}} \{f(x)\} \le \sup_{x \in R_{i_1',\ldots,i_n'}} \{f(x)\}$$

this gives us

$$L(f, P') \leq U(f, P')$$

Definition 6.6 (Lower integral and upper integral)

Let $R \subset \mathbb{R}^n$ be a rectangle and $f : R \to \mathbb{R}$ be a function. The lower integral of f over R is

$$\underline{I}(f) := \sup_p L(f, P)$$

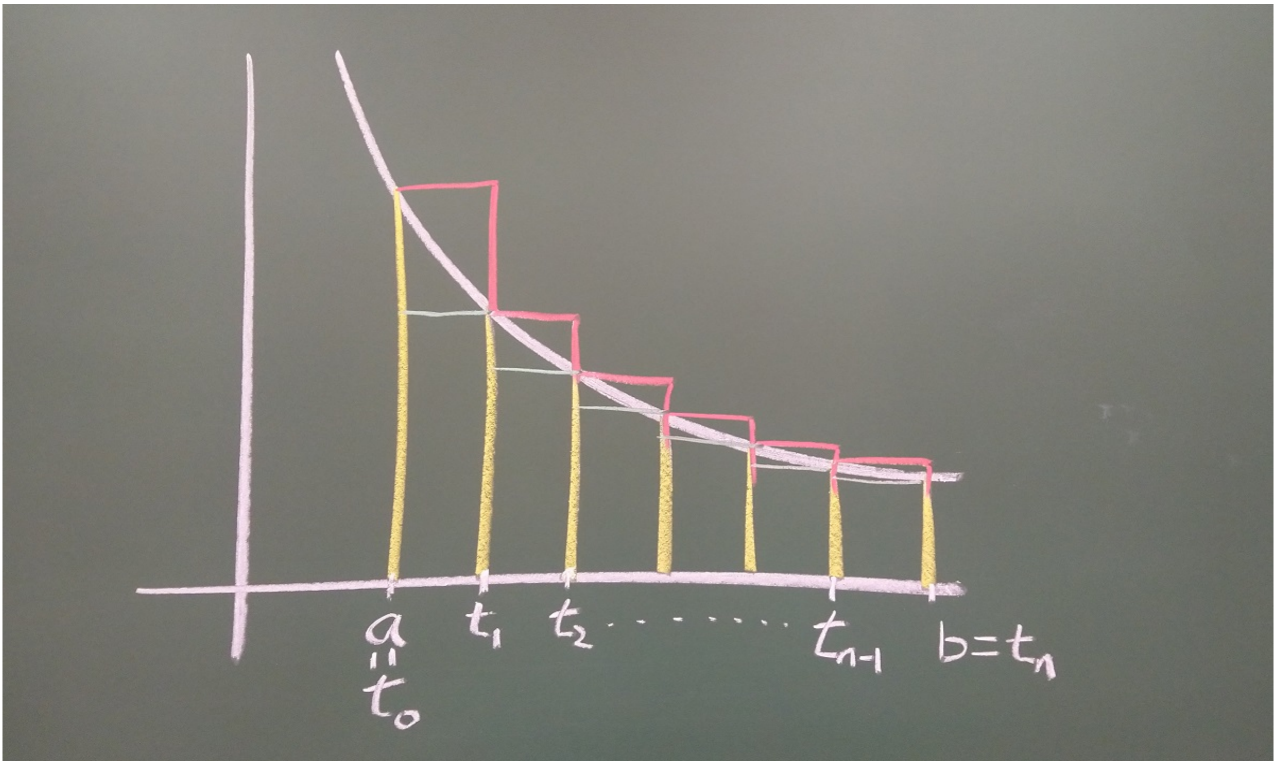
The upper integral of f over R is

$$\overline{I}(f) := \inf_p U(f, P)$$

If $\underline{I}(f) = \overline{I}(f) \in \mathbb{R}$, we say that f is Riemann integrable on R and denote

$$\int_R f dV := \overline{I}(f) = \underline{I}(f)$$

♣



Exercise 6.1

1. Let $R = [1,3] \times [2,5]$ and $f : R \to \mathbb{R}$ be defined by $f(x,y) = xy$. Fix $n \in \mathbb{N}$. Suppose that
$$P = \{1 + \frac{2i}{n} | i = 0, 1, ..., n\} \times \{2 + \frac{3j}{n} | j = 0, ..., n\}$$
Find $L(f,P)$ and $U(f,P)$.
2. Let $f : [0,1] \times [2,3] \to \mathbb{R}$ be defined by
$$f(x,y) = \begin{cases} 1, & \text{if } 0 \leq x \leq y \leq 1 \\ 3, & \text{otherwise} \end{cases}$$
Use the definition of Riemann integral to find
$$\int_{[0,1]\times[2,3]} f dV$$
3. Given a partition $P = P_1 \times \cdots P_n$ of a rectangle $R = [a_1, b_1] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$. Suppose that $P_i = \{a_i = t_{i,0} < t_{i,1} < \cdots < t_{i,k_i} = b_i\}$ for $i = 1, ..., n$. We say that the partition P is an equal partition of R if there is a constant c such that
$$t_{i,j} - t_{i,j-1} = c$$
for all $j = 1, ..., k_i$ and $i = 1, ..., n$.
   (a). Let $R = [1,3] \times [1,4]$ and $P = \{1, 1.2, 3\} \times \{1, 2.7, 4\}$. Find a refinement P' of P such that P' is an equal partition.
   (b). Prove or disprove that for any partition P of a rectangle $[a,b] \subset \mathbb{R}, a < b$, there is a refinement P' of P which is an equal partition.

6.2 The Riemann-Lebesgue theorem

Definition 6.7 (Measure zero set)

A set $A \subset \mathbb{R}^n$ is said to have measure zero in $\mathbb{R}^n$ if for every $\varepsilon > 0$, there exist countably many rectangles $R_1, R_2, \ldots$ in $\mathbb{R}^n$ such that

$$A \subset \bigcup_{i=1}^{\infty} R_i$$

and

$$\sum_{i=1}^{\infty} vol(R_i) < \varepsilon$$

♣

Example 6.6 $\mathbb{Z}$ has measure zero in $\mathbb{R}$.

Solution *Given $\varepsilon > 0$. Define $\phi : \mathbb{N} \to \mathbb{Z}$ by*

$$\phi(n) := \begin{cases} \dfrac{n-1}{2}, & \text{if } n \text{ is odd ;} \\ -\frac{n}{2}, & \text{if } n \text{ is even} \end{cases}$$

Then ϕ is a bijection. Let $R_n := [\phi(n), \phi(n)] = \{\phi(n)\}$. Then

$$\bigcup_{n=1}^{\infty} R_n = \bigcup_{n=1}^{\infty} \{\phi(n)\} = \mathbb{Z}$$

Since $vol(R_n) = 0$ for all $n \in \mathbb{N}$, we have $\sum_{n=1}^{\infty} vol(R_n) = 0 < \varepsilon$. This shows that $\mathbb{Z}$ has measure zero in $\mathbb{R}$.

Example 6.7 $\mathbb{R} \times \{0\}$ has measure zero in $\mathbb{R}^2$ but $\mathbb{R}$ is not a set of measure zero in $\mathbb{R}$.

Solution *For $n \in \mathbb{N}$, let*

$$R_n := [\phi(n), \phi(n) + 1] \times [0, 0]$$

Then

$$\mathbb{R} \times \{0\} = \bigcup_{n=1}^{\infty} R_n$$

and

$$\sum_{n=1}^{\infty} vol(R_n) = \sum_{n=1}^{\infty} 0 = 0$$

Therefore $\mathbb{R} \times \{0\}$ *has measure zero in* $\mathbb{R}^2$.

To show that $\mathbb{R} \times \{0\}$ *does not have measure zero, we prove by contradiction. Set* $\varepsilon = 1$. *Assume that there are rectangles* $\{S_n\}_{n=1}^{\infty}$ *in* $\mathbb{R}^2$ *that cover* $\mathbb{R} \times \{0\}$ *and* $\sum_{n=1}^{\infty} vol(S_n) < 1$. *Since*

$$[0,1] \subset \mathbb{R} \subset \bigcup_{n=1}^{\infty} S_n$$

we have

$$1 = vol([0,1]) \leq \sum_{n=1}^{\infty} vol(S_n) < 1$$

which is a contradiction. Therefore $\mathbb{R}$ *does not have measure zero in* $\mathbb{R}$.

Example 6.8 $\mathbb{S}^1 \subset \mathbb{R}^2$ has measure zero.

Solution *We consider*

$$X = \{(x,y) \in \mathbb{R}^2 | x \geq 0, y \geq 0\} \cap \mathbb{S}^1$$

the portion of $\mathbb{S}^1$ *in the first quadrant. Given* $\varepsilon > 0$. *We first show that we can cover* X *by rectangles with total volume less that* $\frac{\varepsilon}{4}$. *By symmetry reason, there are similar covers by rectangles of* $\mathbb{S}^1$ *in other 3 quadrants. Take* $n \in \mathbb{N}$ *such that*

$$\frac{1}{\sqrt{n}} < \frac{\varepsilon}{4}$$

Let

$$x_i = \frac{i-1}{n}, \quad y_i = \sqrt{1 - x_i^2} = \sqrt{1 - (\frac{i-1}{n})^2}$$

and

$$R_i := [x_i, x_{i+1}] \times [y_{i+1}, y_i]$$

for $i = 1, ..., n$. *Then*

$$
\begin{aligned}
vol(R_i) = (x_{i+1} - x_i)(y_i - y_{i+1}) &= \frac{1}{n}\left(\sqrt{1 - (\frac{i-1}{n})^2} - \sqrt{1 - (\frac{i}{n})^2}\right) \\
&= \frac{(1 - (\frac{i-1}{n})^2) - (1 - (\frac{i}{n})^2)}{n\left(\sqrt{1 - (\frac{i-1}{n})^2} + \sqrt{1 - (\frac{i}{n})^2}\right)} \\
&< \frac{(\frac{i}{n} - \frac{i-1}{n})(\frac{i}{n} + \frac{i-1}{n})}{\sqrt{n^2 - (n-1)^2}} \\
&= \frac{2i - 1}{n^2\sqrt{2n - 1}} \\
&< \frac{2i - 1}{n^2\sqrt{n}}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\sum_{i=1}^{n} vol(R_i) < \sum_{i=1}^{n} \frac{2i-1}{n^2\sqrt{n}} &= \frac{1}{n^2\sqrt{n}}(\sum_{i=1}^{n}(2i - 1)) \\
&= \frac{1}{n^2\sqrt{n}}(2(\frac{n(n+1)}{2}) - n) \\
&= \frac{n^2}{n^2\sqrt{n}} = \frac{1}{\sqrt{n}} < \frac{\varepsilon}{4}
\end{aligned}
$$

This shows that we may cover $\mathbb{S}^1$ *by rectangles with total volume less than* ε *and this means that* $\mathbb{S}^1$ *has measure zero in* $\mathbb{R}^2$.

Proposition 6.2

A countable union of sets of measure zero is a set of measure zero.

Proof Let $\{A_n\}_{n=1}^{\infty}$ be a collection of sets of measure zero in $\mathbb{R}^m$. Given $\varepsilon > 0$. Since A_n has measure zero, there are rectangles $\{R_{n,k}\}_{k=1}^{\infty}$ in $\mathbb{R}^m$ such that

$$A_n \subset \bigcup_{k=1}^{\infty} R_{n,k} \text{ and } \sum_{k=1}^{\infty} \text{vol}(R_{n,k}) < \frac{\varepsilon}{2^n}$$

Recall that a countable union of countable sets is countable, the collection $\{R_{n,k}\}_{n,k=1}^{\infty}$ is countable.

Since

$$\bigcup_{n=1}^{\infty} A_n \subset \bigcup_{n=1}^{\infty} \bigcup_{k=1}^{\infty} R_{n,k}$$

and

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \text{vol}(R_{n,k}) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon \sum_{n=1}^{\infty} \frac{1}{2^n} = \varepsilon$$

Therefore $\bigcup_{n=1}^{\infty} A_n$ is of measure zero.

Corollary 6.1

A countable set $A \subset \mathbb{R}^n$ has measure zero in $\mathbb{R}^n$. ♡

Proof Write $A = \{a_i\}_{i=1}^{\infty}$ with the convention that if A is a finite set, then we set $a_i = a_N$ for $i \geq N$ where $N = |A|$. Since a point is of measure zero, A is a countable union of measure zero sets, by the result above, A has measure zero.

Corollary 6.2

$\mathbb{Q}$ has measure zero in $\mathbb{R}$. ♡

Note that $\mathbb{Q}$ is dense in $\mathbb{R}$, but it has measure zero. The Cantor set C is uncountable, but it is not difficult to show that it also has measure zero.

Definition 6.8 (Set of discontinuities)

Let $R \subset \mathbb{R}^n$ be a rectangle and $f : R \to \mathbb{R}$ be a function. The set

$$Disc(f) := \{x \in R | f \text{ is discontinuous at } x\}$$

is called the set of discontinuities of f. ♣

Example 6.9 For $x \in \mathbb{R}$, let $\lfloor x \rfloor$ denote the smallest integer greater or equal to x. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \lfloor x \rfloor$$

Then

$$Disc(f) = \mathbb{Z}$$