Advanced calculus for quantitative finance I

Logic, topology and differentiation

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Don't give up easily. Stick to something. Not to the point where it's clearly insane, but be persistent. There is beauty in things that work well–the way a company is run, or the way a theorem comes out——James Simons

Contents of Advanced Calculus I

Preface

Advanced calculus occupies the most fundamental position in mathematics training. It is an essential path from elementary numerical calculation to higher-level abstract thinking. Mathematical knowledge and the ability of abstract thinking become more and more important in modern sciences and technology industries. Quantitative finance is a perfect field for showing the power of mathematics.

In our opinion, advanced calculus for students in quantitative finance should not be the same as the one for students in mathematics department since they have different training in mathematics and different kinds of applications in future studies. Based on lecture notes for the advanced calculus courses that the author taught in the National Tsing Hua University of Taiwan for students from the department of quantitative finance, the author produces two books: Advanced calculus for quantitative finance I & II. The goal of these books is to introduce mathematical analysis and provide background behind the Black-Scholes model in options.

The Black-Scholes model and its variants are probably the most common models in finance. Since even an introduction of mathematical Brownian motion is out of the reach of undergraduate mathematics, it is not easy to talk about Ito calculus which the Black-Scholes model lies on. The author takes up the challenge in these books. The goal is to provide deep mathematics for students in quantitative finance and at the same time show them such mathematics is tightly related to their field of studying.

Differences between these books and advanced calculus textbooks for students in mathematics are that they start from a lighter mathematics prerequisites, skip some results such as the inverse function theorem that are not directly related to the study of mathematical finance, introduce probability theory based on Lebesgue integration, provide basic stochastic calculus, give a rather rigor derivation of the Black-Scholes model, introduce Fourier transform to solve the heat equation, and use it to derive a solution for the Black-Scholes equation.

Each of these two books contains 3 midterm exams and 1 final exam, accompany with a practice exam before each examination. Also at the end of each section, there are some exercises for students to get familiar with the materials. Proofs of some more difficult theorems are provided in the appendix of each section.

Main references are

1. Fractals everywhere by Barnsley ([**B**]);
2. Probability theory in finance: a mathematical guide to the Black-Scholes formula by Dineen ([**D**]);
3. Elementary classical analysis by Marsden and Hoffman ([**MH**]);
4. Real mathematical analysis by Pugh ([**P**]);
5. Wikipedia.

Those beautiful pictures at the end of each chapter are free pictures from pixabay.com.

The latex documentclass "elegantbook"(https://github.com/ElegantLaTeX/ElegantBook) is used to edit this book.

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Chapter 1 Logic and set theory

1.1 Logic

Suppose we are in a system that we can make some arguments following rules in this system. A statement in a system describes relations between elements of the system. A statement is assigned two values, True or False, but not both.

Example 1.1 If we are working in the system of numbers, the assertion "there are infinitely many prime numbers" is a statement in this system. The value of this statement is either True of False, but not both. By some rules of numbers, we will show that the value of this statement is "True". On the other hand, the value of the statement $1 + 1 < 2$ is "False".

Given several statements, we can use logic operations to produce new statements from them. The basic logical operations are "AND", "OR", "IMPLIES", and "NOT".

Example 1.2 Let P be the statement $x > 1$ and Q be the statement $x > 5$. Then we may form some new statements from P and Q:

1. P and Q: "$x > 1$ and $x > 5$" is denoted by
$$P \wedge Q$$
2. P or Q: "$x > 1$ or $x > 5$" is denoted by
$$P \vee Q$$
3. P implies Q: "$x > 1$ implies $x > 5$" is denoted by
$$P \Rightarrow Q$$
4. Not P: "Not $x > 1$" is denoted by
$$\neg P$$

Definition 1.1 (Truth table)

The truth table of a statement P constructed from several statements $P_1, ..., P_k$ is the True or False values of P when the True or False values of $P_1, ..., P_k$ are given. We define the truth table of basic logical operations as following:

1. *The truth table of "AND"*

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

2. *The truth table of "OR"*

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T
F	F	F

3. *The truth table of "NOT"*

P	$\neg P$
T	F
F	T

4. *The truth table of "IMPLIES"*

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

♣

Remark Note that in the truth table of "IMPLIES", whenever the hypothesis is false, no matter what the conclusion is, we assign the truth value of the statement "$P \Rightarrow Q$" to be True.

Remark Most mathematical theorems are in the form "If P, then Q". For example, from calculus, we have a theorem:

If f is differentiable at x_0, then f is continuous at x_0.

Example 1.3 Find the truth table of the following statements:

1.

$$P \vee^{\neg} Q$$

2.

$$^{\neg}P \Rightarrow P \wedge Q$$

Solution

1.

P	Q	$^{\neg}Q$	$P \vee^{\neg} Q$
T	T	F	T
T	F	T	T
F	T	F	F
F	F	T	T

2.

P	Q	$^{\neg}P$	$P \wedge Q$	$^{\neg}P \Rightarrow P \wedge Q$
T	T	F	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	F	F

Exercise 1.1 Find the truth table of the following statements:

1.

$$P \Rightarrow^{\neg} Q$$

2.

$$^{\neg}P \vee Q$$

Definition 1.2

Given two statements P and Q. We define the logical operation "IF AND ONLY IF" by

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Denote this logical operation by

$$P \Leftrightarrow Q$$

♣

Example 1.4 The truth table of "IF AND ONLY IF" is

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Example 1.5 "$x^2 > 1$" if and only if "$x > 1$ or $x < -1$".

Definition 1.3 (Logically equivalent)

Given two statements P, Q which are constructed from several statements $P_1, ..., P_k$. We say that P, Q are logically equivalent if they have the same truth tables. We write

$$P \equiv Q$$

if P and Q are logically equivalent.

♣

Proposition 1.1

We have the following logical equivalences:

1. $$\neg\neg P \equiv P$$

2. $$P \Rightarrow Q \equiv \neg P \vee Q$$

3. $$(P \Rightarrow Q) \equiv (\neg Q \Rightarrow \neg P)$$

♠

Proof We prove only the third logical equivalence:

P	Q	$\neg P$	$\neg Q$	$\neg Q \Rightarrow \neg P$
T	T	F	F	T
T	F	F	T	F
F	T	T	F	T
F	F	T	T	T

Example 1.6 Show that $P \Rightarrow Q$ is not logically equivalent to $\neg P \Rightarrow \neg Q$.

Proof

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$\neg P$	$\neg Q$	$\neg P \Rightarrow \neg Q$
T	T	T	T	F	F	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	T	T

Remark A common mistake is to consider

$$(P \Rightarrow Q) \equiv (\neg P \Rightarrow \neg Q)$$

We give a simple example to show that they are not logically equivalent. Let P be the statement

$$f \text{ is differentiable at } x$$

and Q be the statement

$$f \text{ is continuous at } x$$

From calculus, we have the following result:

$$\text{If } f \text{ is differentiable at } x, \text{ then } f \text{ is continuous at } x$$

This forms the statement

$$P \Rightarrow Q$$

which has value True. But if f is not differentiable at x, for example $f(x) = |x|$, f may still be continuous at x. Thus $\neg P \Rightarrow \neg Q$ has value False.

Example 1.7 In the following, we give some examples to exemplify the third equivalence:

1. "If $x > 1$, then $x + 1 > 2$." is logically equivalent to "If $x + 1 \leq 2$, then $x \leq 1$."

2. "If $a > b$, then $a^2 > b^2 + c^2$." is logically equivalent to "If $a^2 \leq b^2 + c^2$, then $a \leq b$."
3. "If $1 > 2$, then it is raining." is logically equivalent to "If it is not raining, then $1 \leq 2$."

Theorem 1.1 (De Morgan's Laws)

The following logical equivalences are called the De Morgan's Laws:

1. $$\neg(P \vee Q) \equiv \neg P \wedge \neg Q$$
2. $$\neg(P \wedge Q) \equiv \neg P \vee \neg Q$$

♡

Proof We verify the first logical equivalence and the other one is verified similarly.

P	Q	$P \vee Q$	$\neg(P \vee Q)$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

Definition 1.4 (Converse and negation)

Suppose that P, Q are statements. We define the converse of $P \Rightarrow Q$ to be

$$Q \Rightarrow P$$

and the negation of $P \Rightarrow Q$ to be

$$\neg(P \Rightarrow Q)$$

♣

Example 1.8 Show that $P \Rightarrow Q$ is not logically equivalent to $Q \Rightarrow P$.

Proof

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Proposition 1.2 (Boolean laws for logic)

Let P, Q, R be statements. Then

	OR	*AND*
Commutative laws	$P \vee Q \equiv Q \vee P$	$P \wedge Q \equiv Q \wedge P$
Associate laws	$(P \vee Q) \vee R \equiv P \vee (Q \vee R)$	$(P \wedge Q) \wedge R \equiv P \wedge (Q \wedge R)$
Distribution laws	$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$	$P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$

♠

In the following, we verify the distribution law for OR only. The other results can be verified similarly.

Proof

P	Q	R	$Q \wedge R$	$P \vee (Q \wedge R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \wedge (P \vee R)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

From the truth table, we see that

$$P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$$

Exercise 1.1

1. Let P, Q, R be statements. Show that

$$P \land (Q \lor R) \equiv (P \land Q) \lor (P \land R)$$

2. Let P, Q be statements. Show that

$$\neg(P \land Q) \equiv \neg P \lor \neg Q$$

3. Let P, Q be statements. Show that

$$\neg(P \Rightarrow Q) \equiv P \land \neg Q$$

4. Let P, Q, R be statements. Is

$$(P \Rightarrow Q) \Rightarrow R \equiv P \Rightarrow (Q \Rightarrow R) \quad ?$$

1.2 Set theory

1.2.1 Sets

Set theory is the foundation of mathematics. The idea of sets, collection of some things, is certainly old. It was Cantor who made a systematic study of sets and proved several results that were not welcomed by some people at his time. The idea of sets was not totally flawless. As pointed out by Russell in his famous paradox, we need to make some restriction on sets. Several years after the appearance of Russell's paradox, the ZFC axiomatic set theory finally gives a satisfactory firm foundation of set theory. We are not going to construct a rigor set theory in this book, but take the existence of sets for granted and study mathematics built on it. In this book we will introduce enough set theory for more advanced study.

Definition 1.5 (Element)

A set is a collection of objects. The objects of a set are called elements of the set. We write $x \in S$ if x is an element of S, and $x \notin S$ if x is not an element of S. ♣

Remark(Extensionality axiom)

Two sets are equal if and only if they have the same elements.

Example 1.9 Let $A = \{1, 2, 3\}$ and $B = \{2, 1, 3\}$, then $A = B$.

By extensionality, there can be only one set with no elements.

Definition 1.6 (Empty set)

The set with no elements is called the empty set, denoted by $\emptyset$ or $\{\}$. ♣

Remark

$$\{\{\ \}\} \neq \{\ \}$$

Definition 1.7 (Subset)

Given two sets A and B. If for any x in A, x is in B, then we say that A is a subset of B, written *as*

$$A \subseteq B$$

♣

Example 1.10

$$\{1, 2\} \subset \{1, 2, 3\}$$

Example 1.11 For

$$S = \{\{a\}\}$$

$\{a\} \in S$, but $a \notin S$. a and $\{a\}$ are different.

Definition 1.8 (Power set)

Let S be a set. The power set $\mathscr{P}(S)$ of S is the set consisting of all subsets of S, i.e.

$$\mathscr{P}(S) = \{A | A \subset S\}$$

♣

Example 1.12 Let $S = \{1, 2, 3\}$, then

$$\mathscr{P}(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{1, 3\}, \{1, 2, 3\}\}$$

Since we assume sets are given without any definition, it may seem that any arbitrarily stuff form a set. But as pointed out by Russell, we need to make some restriction on the idea of sets. Let us give some examples.

Example 1.13 Let

$$T = \{\text{sets with at least two elements}\}$$

Because $A = \{1, 2\}$ and $B = \{1, 2, 3\}$ are contained in T, T has at least two elements. So $T \in T$.

Example 1.14Russell's Paradox

Let

$$R = \{A | A \notin A, \text{ for any sets } A\}$$

Question: Is $R \in R$?

Case 1: If $R \in R$, R satisfies $A \notin A$ therefore $R \notin R$.

Case 2: If $R \notin R$, by the definition of R, $R \in R$.

No matter which case, contradiction happens.

To get rid of Russell's Paradox, when we are dealing with sets, we require elements of all sets belong to some universal set $\mathcal{U}$. Then for the set R in Russell's Paradox, we need to rewrite $R = \{A \subseteq \mathcal{U} | A \notin A\}$, but this is equal to $\mathscr{P}(\mathcal{U})$, the set of all subsets of $\mathcal{U}$ since in some fixed universal set, $A \notin A$. Note that

$$\mathcal{U} \neq \mathscr{P}(\mathcal{U})$$

1.2.2 Operations on sets

Definition 1.9

Suppose that A, B are subsets of $\mathcal{U}$.

1. *The union of A and B is defined to be*
$$A \cup B := \{x | (x \in A) \vee (x \in B)\}.$$
2. *The intersection of A and B is defined to be*
$$A \cap B := \{x | (x \in A) \wedge (x \in B)\}.$$
3. *The complement of A in $\mathcal{U}$ is defined to be*
$$A^c := \{x \in \mathcal{U} | x \notin A\}.$$
4. *The difference of A and B is defined to be*
$$A - B := \{x | (x \in A) \wedge (x \notin B)\}.$$

♣

Definition 1.10 (Disjoint)

We say that two sets A and B are disjoint if

$$A \cap B = \emptyset$$

♣

Remark To show that two sets A, B are equal, we need to show that $A \subseteq B$ and $B \subseteq A$.

Example 1.15 Let $A \subseteq \mathcal{U}$. Show that

$$(A^c)^c = A$$

Proof Note that

$$(x \notin A^c) \equiv^{\neg} (x \in A^c) \equiv^{\neg} (x \notin A) \equiv^{\neg} (^{\neg}(x \in A)) \equiv x \in A$$

For $x \in (A^c)^c$, $x \notin A^c$. By the observation above, we have $x \in A$. Therefore

$$(A^c)^c \subseteq A$$

On the other hand, if $x \in A$, then $x \notin A^c$. Hence $x \in (A^c)^c$. Therefore

$$A \subseteq (A^c)^c$$

This proves the result.

Proposition 1.3 (Boolean laws for sets)

	Union	*Intersection*
Commutative Laws	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associate Laws	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
Distribution Laws	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
Identity Laws	$A \cup \emptyset = A$	$A \cap \emptyset = \emptyset$
Complement Laws	$A \cup A^c = U$	$A \cap A^c = \emptyset$

♠

In the following, we verify the distribution law for union only.

Proof We want to show

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

$$\begin{aligned} x \in A \cup (B \cap C) &\Leftrightarrow (x \in A) \vee x \in (B \cap C) \\ &\Leftrightarrow (x \in A) \vee ((x \in B) \wedge (x \in C)) \\ &\Leftrightarrow ((x \in A) \vee (x \in B)) \wedge ((x \in A) \vee (x \in C)) \\ &\Leftrightarrow (x \in A \cup B) \wedge (x \in A \cup C) \\ &\Leftrightarrow x \in (A \cup B) \cap (A \cup C). \end{aligned}$$

Definition 1.11

Given sets $A_1, A_2, A_3, \ldots.$ The union of these sets is

$$\bigcup_{i=1}^{\infty} A_i := \{x : x \in A_i \text{ for some } i \in \mathbb{N}\}$$

and the intersection of these sets is

$$\bigcap_{i=1}^{\infty} A_i := \{x : x \in A_i \text{ for each } i \in \mathbb{N}\}$$

More generally, if B is a set and a set A_i is given for each i in B, define the union of sets indexed by B to be

$$\bigcup_{i \in B} A_i := \{x : x \in A_i \text{ for some } i \in B\}$$

and the intersection of sets indexed by B to be

$$\bigcap_{i \in B} A_i := \{x : x \in A_i \text{ for each } i \in B\}$$

♣

Proposition 1.4

Let X be a set and $A_i \subset X$ for $i \in I$ where I is some index set. Then

1. $$(\bigcup_{i \in I} A_i)^c = \bigcap_{i \in I} A_i^c$$

2. $$(\bigcap_{i \in I} A_i)^c = \bigcup_{i \in I} A_i^c$$

♠