Solutions

Lecture 1

For later lectures, most of the solutions show how to generate the answer, but for Lecture 1, just the answers are shown below. Remember that it is just as important to hear the problem as to see the problem.

The following mental addition and multiplication problems can be done almost immediately, just by listening to the numbers from left to right.

1. 23 + 5 = 28
2. 23 + 50 = 73
3. 500 + 23 = 523
4. 5000 + 23 = 5023
5. 67 + 8 = 75
6. 67 + 80 = 147
7. 67 + 800 = 867
8. 67 + 8000 = 8067

9. 30 + 6 = 36

10. 300 + 24 = 324

11. 2000 + 25 = 2025

12. 40 + 9 = 49

13. 700 + 84 = 784

14. 140 + 4 = 144

15. 2500 + 20 = 2520

16. 2300 + 58 = 2358

17. 13 × 10 = 130

18. 13 × 100 = 1300

19. 13 × 1000 = 13,000

20. 243 × 10 = 2430

21. 243 × 100 = 24,300

22. 243 × 1000 = 243,000

23. 243 × 1 million = 243 million

24. Fill out the standard 10-by-10 multiplication table as quickly as you can. It's probably easiest to fill it out one row at a time by counting.

×	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

25. Create an 8-by-9 multiplication table in which the rows represent the numbers from 2 to 9 and the columns represent the numbers from 11 to 19. For an extra challenge, fill out the squares in random order.

×	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**
2	22	24	26	28	30	32	34	36	38
3	33	36	39	42	45	48	51	54	57
4	44	48	52	56	60	64	68	72	76
5	55	60	65	70	75	80	85	90	95
6	66	72	78	84	90	96	102	108	114
7	77	84	91	98	105	112	119	126	133
8	88	96	104	112	120	128	136	144	152
9	99	108	117	126	135	144	153	162	171

26. Create the multiplication table in which the rows and columns represent the numbers from 11 to 19. For an extra challenge, fill out the rows in random order. Be sure to use the shortcuts we learned in this lecture, including those for multiplying by 11.

×	11	12	13	14	15	**16**	**17**	**18**	**19**
11	121	132	143	154	165	176	187	198	209
12	132	144	156	168	180	192	204	216	228
13	143	156	169	182	195	208	221	234	247
14	154	168	182	196	210	224	238	252	266
15	165	180	195	210	225	240	255	270	285
16	176	192	208	224	240	256	272	288	304
17	187	204	221	238	255	272	289	306	323
18	198	216	234	252	270	288	306	324	342
19	209	228	247	266	285	304	323	342	361

The following multiplication problems can be done just by listening to the answer from left to right.

27. 41 × 2 = 82

28. 62 × 3 = 186

29. 72 × 4 = 288

30. 52 × 8 = 416

31. 207 × 3 = 621

32. 402 × 9 = 3618

33. 543 × 2 = 1086

Do the following multiplication problems using the shortcut from this lecture.

34. 21 × 11 = 231 (since 2 + 1 = 3, insert 3 between 2 and 1)

35. 17 × 11 = 187

36. 54 × 11 = 594

37. 35 × 11 = 385

38. 66 × 11 = 726 (since 6 + 6 = 12, insert 2 between 6 and 6, then carry the 1)

39. 79 × 11 = 869

40. 37 × 11 = 407

41. 29 × 11 = 319

42. 48 × 11 = 528

43. 93 × 11 = 1023

44. 98 × 11 = 1078

45. 135 × 11 = 1485 (since 1 + 3 = 4 and 3 + 5 = 8)

46. 261 × 11 = 2871

47. $863 \times 11 = 9493$

48. $789 \times 11 = 8679$

49. Quickly write down the squares of all 2-digit numbers that end in 5.

$15^2 = 225$
$25^2 = 625$
$35^2 = 1225$
$45^2 = 2025$
$55^2 = 3025$
$65^2 = 4225$
$75^2 = 5625$
$85^2 = 7225$
$95^2 = 9025$

50. Since you can quickly multiply numbers between 10 and 20, write down the squares of the numbers 105, 115, 125, … 195, 205.

$105^2 = 11{,}025$
$115^2 = 13{,}225$
$125^2 = 15{,}625$
$135^2 = 18{,}225$
$145^2 = 21{,}025$
$155^2 = 24{,}025$
$165^2 = 27{,}225$
$175^2 = 30{,}625$
$185^2 = 34{,}225$
$195^2 = 38{,}025$
$205^2 = 42{,}025$

51. Square 995.

$995^2 = 990{,}025.$

52. Compute 1005^2.

1,010,025 (since 100 × 101 = 10,100; then attach 25)

Exploit the shortcut for multiplying 2-digit numbers that begin with the same digit and whose last digits sum to 10 to do the following problems.

53. 21 × 29 = 609 (using 2 × 3 = 6 and 1 × 9 = 09)

54. 22 × 28 = 616

55. 23 × 27 = 621

56. 24 × 26 = 624

57. 25 × 25 = 625

58. 61 × 69 = 4209

59. 62 × 68 = 4216

60. 63 × 67 = 4221

61. 64 × 66 = 4224

62. 65 × 65 = 4225

Lecture 2

Solve the following mental addition problems by calculating from left to right. For an *added* challenge, look away from the numbers after reading the problem.

1. 52 + 7 = 59

2. 93 + 4 = 97

3. 38 + 9 = 47

4. 77 + 5 = 82

5. 96 + 7 = 103

6. 40 + 36 = 76

7. 60 + 54 = 114

8. 56 + 70 = 126

9. 48 + 60 = 108

10. 53 + 31 = 83 + 1 = 84

11. 24 + 65 = 84 + 5 = 89

12. 45 + 35 = 75 + 5 = 80

13. 56 + 37 = 86 + 7 = 93

14. 75 + 19 = 85 + 9 = 94

15. 85 + 55 = 135 + 5 = 140

16. 27 + 78 = 97 + 8 = 105

17. 74 + 53 = 124 + 3 = 127

18. 86 + 68 = 146 + 8 = 154

19. 72 + 83 = 152 + 3 = 155

Do these 2-digit addition problems in two ways; make sure the second way involves subtraction.

20. 68 + 97 = 158 + 7 = 165
OR 68 + 97 = 68 + 100 – 3 = 168 – 3 = 165

21. 74 + 69 = 134 + 9 = 143
OR 74 + 69 = 74 + 70 – 1 = 144 – 1 = 143

22. 28 + 59 = 78 + 9 = 87
OR 28 + 59 = 28 + 60 – 1 = 88 – 1 = 87

23. 48 + 93 = 138 + 3 = 141
OR 48 + 93 = 48 + 100 – 7 = 148 – 7 = 141
OR 48 + 93 = 93 + 50 – 2 = 143 – 2 = 141

Try these 3-digit addition problems. The problems gradually become more difficult. For the harder problems, it may be helpful to say the problem out loud before starting the calculation.

24. 800 + 300 = 1100

25. 675 + 200 = 875

26. 235 + 800 = 1035

27. 630 + 120 = 730 + 20 = 750

28. 750 + 370 = 1050 + 70 = 1120

29. 470 + 510 = 970 + 10 = 980

30. 980 + 240 = 1180 + 40 = 1220

31. 330 + 890 = 1130 + 90 = 1220

32. 246 + 810 = 1046 + 10 = 1056

33. 960 + 326 = 1260 + 26 = 1286

34. 130 + 579 = 679 + 30 = 709

35. 325 + 625 = 925 + 25 = 950

36. 575 + 675 = 1175 + 75 = 1100 + 150 = 1250

37. 123 + 456 = 523 + 56 = 573 + 6 = 579

38. 205 + 108 = 305 + 8 = 313

39. 745 + 134 = 845 + 34 = 875 + 4 = 879

40. 341 + 191 = 441 + 91 = 531 + 1 = 532
OR 341 + 200 – 9 = 541 – 9 = 532

41. 560 + 803 = 1360 + 3 = 1363

42. 566 + 185 = 666 + 85 = 746 + 5 = 751

43. 764 + 637 = 1364 + 37 = 1394 + 7 = 1401

Do the next few problems in two ways; make sure the second way uses subtraction.

44. 787 + 899 = 1587 + 99 = 1677 + 9 = 1686
OR 787 + 899 = 787 + 900 – 1 = 1687 – 1 = 1686

45. 339 + 989 = 1239 + 89 = 1319 + 9 = 1328
OR 339 + 989 = 339 + 1000 – 11 = 1339 – 11 = 1328

46. 797 + 166 = 897 + 66 = 957 + 6 = 963
OR 797 + 166 = 166 + 800 – 3 = 966 – 3 = 963

47. 474 + 970 = 1374 + 70 = 1444
OR 474 + 970 = 474 + 1000 – 30 = 1474 – 30 = 1444

Do the following subtraction problems from left to right.

48. 97 – 6 = 91

49. 38 – 7 = 31

50. 81 – 6 = 75

51. 54 – 7 = 47

52. 92 – 30 = 62

53. 76 – 15 = 66 – 5 = 61

54. 89 – 55 = 39 – 5 = 34

55. 98 – 24 = 78 – 4 = 74

Do these problems two different ways. For the second way, begin by subtracting too much.

56. 73 – 59 = 23 – 9 = 14
OR 73 – 59 = 73 – (60 – 1) = 13 + 1 = 14

57. 86 – 68 = 26 – 8 = 18
OR = 86 – (70 – 2) = 16 + 2 = 18

58. 74 – 57 = 24 – 7 = 17
OR 74 – 57 = 74 – (60 – 3) = 14 + 3 = 17

59. 62 – 44 = 22 – 4 = 18
OR 62 – (50 – 6) = 12 + 6 = 18

Try these 3-digit subtraction problems, working from left to right.

60. 716 – 505 = 216 – 5 = 211

61. 987 – 654 = 387 – 54 = 337 – 4 = 333

62. 768 – 222 = 568 – 22 = 548 – 2 = 546

63. 645 – 231 = 445 – 31 = 415 – 1 = 414

64. 781 – 416 = 381 – 16 = 371 – 6 = 365
OR 781 – 416 = 381 – 16 = 381 – (20 – 4) = 361 + 4 = 365

Determine the complements of the following numbers, that is, their distance from 100.

65. 100 – 28 = 72

66. 100 – 51 = 49

67. 100 – 34 = 66

68. 100 – 87 = 13

69. 100 – 65 = 35

70. 100 – 70 = 30

71. 100 – 19 = 81

72. 100 – 93 = 07

Use complements to solve these problems.

73. 822 – 593 = 822 – (600 – 7) = 222 + 7 = 229

74. 614 – 372 = 614 – (400 – 28) = 214 + 28 = 242

75. 932 – 766 = 932 – (800 – 34) = 132 + 34 = 166

76. 743 – 385 = 743 – (400 – 15) = 343 + 15 = 358

77. 928 – 262 = 928 – (300 – 38) = 628 + 38 = 666

78. 532 – 182 = 532 – (200 – 18) = 332 + 18 = 350

79. 611 – 345 = 611 – (400 – 55) = 211 + 55 = 226

80. 724 – 476 = 724 – (500 – 24) = 224 + 24 = 248

Determine the complements of these 3-digit numbers.

81. 1000 – 772 = 228

82. 1000 – 695 = 305

83. 1000 – 849 = 151

84. 1000 – 710 = 290

85. 1000 – 128 = 872

86. 1000 – 974 = 026

87. 1000 – 551 = 449

Use complements to determine the correct amount of change.

88. $10 – $2.71 = $7.29

89. $10 – $8.28 = $1.72

90. $10 – $3.24 = $6.76

91. $100 – $54.93 = $45.07

92. $100 – $86.18 = $13.82

93. $20 – $14.36 = $5.64

94. \$20 – \$12.75 = \$7.25

95. \$50 – \$31.41 = \$18.59

The following addition and subtraction problems arise while doing mental multiplication problems and are worth practicing before beginning Lecture 3.

96. 350 + 35 = 385

97. 720 + 54 = 774

98. 240 + 32 = 272

99. 560 + 56 = 616

100. 4900 + 210 = 5110

101. 1200 + 420 = 1620

102. 1620 + 48 = 1668

103. 7200 + 540 = 7740

104. 3240 + 36 = 3276

105. 2800 + 350 = 3150

106. 2150 + 56 = 2206

107. 800 – 12 = 788

108. 3600 – 63 = 3537

109. 5600 – 28 = 5572

110. 6300 – 108 = 6200 – 8 = 6192

Lecture 3

Calculate the following 2-by-1 multiplication problems in your head using the addition method.

1. $40 \times 8 = 320$

2. $42 \times 8 = 320 + 16 = 336$

3. $20 \times 4 = 80$

4. $28 \times 4 = 80 + 32 = 112$

5. $56 \times 6 = 300 + 36 = 336$

6. $47 \times 5 = 200 + 35 = 235$

7. $45 \times 8 = 320 + 40 = 360$

8. $26 \times 4 = 80 + 24 = 104$

9. $68 \times 7 = 420 + 56 = 476$

10. $79 \times 9 = 630 + 81 = 711$

11. $54 \times 3 = 150 + 12 = 162$

12. $73 \times 2 = 140 + 6 = 146$

13. $75 \times 8 = 560 + 40 = 600$

14. $67 \times 6 = 360 + 42 = 402$

15. $83 \times 7 = 560 + 21 = 581$

16. $74 \times 6 = 420 + 24 = 444$

17. 66 × 3 = 180 + 18 = 198

18. 83 × 9 = 720 + 27 = 747

19. 29 × 9 = 180 + 81 = 261

20. 46 × 7 = 280 + 42 = 322

Calculate the following 2-by-1 multiplication problems in your head using the addition method and the subtraction method.

21. 89 × 9 = 720 + 81 = 801
OR 89 × 9 = (90 – 1) × 9 = 810 – 9 = 801

22. 79 × 7 = 490 + 63 = 553
OR 79 × 7 = (80 – 1) × 7 = 560 – 7 = 553

23. 98 × 3 = 270 + 24 = 294
OR 98 × 3 = (100 – 2) × 3 = 300 – 6 = 294

24. 97 × 6 = 540 + 42 = 582
OR (100 – 3) × 6 = 600 – 18 = 582

25. 48 × 7 = 280 + 56 = 336
OR 48 × 7 = (50 – 2) × 7 = 350 – 14 = 336

The following problems arise while squaring 2-digit numbers or multiplying numbers that are close together. They are essentially 2-by-1 problems with a 0 attached.

26. 20 × 16: 2 × 16 = 20 + 12 = 32, so 20 × 16 = 320

27. 20 × 24: 2 × 24 = 40 + 8 = 48, so 20 × 24 = 480

28. 20 × 25: 2 × 25 = 50, so 20 × 25 = 500

29. 20 × 26: 2 × 26 = 40 + 12 = 52, so 20 × 26 = 520

30. 20 × 28: 2 × 28 = 40 + 16 = 56, so 20 × 2 = 560

31. 20 × 30: 600

32. 30 × 28: 3 × 28 = 60 + 24 = 84, so 30 × 28 = 840

33. 30 × 32: 3 × 32 = 90 + 6 = 96, so 30 × 32 = 960

34. 40 × 32: 4 × 32 = 120 + 8 = 128, so 40 × 32 = 1280

35. 30 × 42: 3 × 42 = 120 + 6 = 126, so 30 × 42 = 1260

36. 40 × 48: 4 × 48 = 160 + 32 = 192, so 40 × 48 = 1920

37. 50 × 44: 5 × 44 = 200 + 20 = 220, so 50 × 44 = 2200

38. 60 × 52: 6 × 52 = 300 + 12 = 312, so 60 × 52 = 3120

39. 60 × 68: 6 × 60 = 360 + 48 = 408, so 60 × 68 = 4080

40. 60 × 69: 6 × 69 = 360 + 54 = 414, so 60 × 69 = 4140

41. 70 × 72: 7 × 72 = 490 + 14 = 504, so 70 × 72 = 5040

42. 70 × 78: 7 × 78 = 490 + 56 = 546, so 70 × 78 = 5460

43. 80 × 84: 8 × 84 = 640 + 32 = 672, so 80 × 84 = 6720

44. 80 × 87: 8 × 87 = 640 + 56 = 696, so 80 × 87 = 6960

45. 90 × 82: 9 × 82 = 720 + 18 = 738, so 90 × 82 = 7380

46. 90 × 96: 9 × 96 = 810 + 54 = 864, so 90 × 96 = 8640

Here are some more problems that arise in the first step of a 2-by-2 multiplication problem.

47. 30 × 23: 3 × 23 = 60 + 9 = 69, so 30 × 23 = 690

48. 60 × 13: 60 × 13 = 60 + 18 = 78, so 60 × 13 = 780

49. 50 × 68: 5 × 68 = 300 + 40 = 340, so 50 × 68 = 3400

50. 90 × 26: 9 × 26 = 180 + 54 = 234, so 90 × 26 = 2340

51. 90 × 47: 9 × 47 = 360 + 63 = 423, so 90 × 47 = 4230

52. 40 × 12: 4 × 12 = 40 + 8 = 48, so 40 × 12 = 480

53. 80 × 41: 8 × 41 = 320 + 8 = 328, so 80 × 41 = 3280

54. 90 × 66: 9 × 66 = 540 + 54 = 594, so 90 × 66 = 5940

55. 40 × 73: 4 × 73 = 280 + 12 = 292, so 40 × 73 = 2920

Calculate the following 3-by-1 problems in your head.

56. 600 × 7 = 4200

57. 402 × 2 = 800 + 4 = 804

58. 360 × 6 = 1800 + 360 = 2160

59. 360 × 7 = 2100 + 420 = 2520

60. 390 × 7 = 2100 + 630 = 2730

61. 711 × 6 = 4200 + 66 = 4266

62. 581 × 2 = 1000 + 160 + 2 = 1162

63. 161 × 2 = 200 + 120 + 2 = 320 + 2 = 322

64. 616 × 7 = 4200 + (70 + 42) = 4200 + 112 = 4312

65. 679 × 5 = 3000 (say it) + (350 + 45) = 3395

66. 747 × 2 = 1400 (say it) + (80 + 14) = 1494

67. 539 × 8 = 4000 (say it) + (240 + 72) = 4312

68. 143 × 4 = 400 + 160 + 12 = 560 + 12 = 572

69. 261 × 8 = 1600 + 480 + 8 = 2080 + 8 = 2088

70. 624 × 6 = 3600 + 120 + 24 = 3720 + 24 = 3744

71. 864 × 2 = 1600 + 120 + 8 = 1720 + 8 = 1728

72. 772 × 6 = 4200 + 420 + 12 = 4620 + 12 = 4632

73. 345 × 6 = 1800 + 240 + 30 = 2040 + 30 = 2070

74. 456 × 6 = 2400 + 300 + 36 = 2700 + 36 = 2736

75. 476 × 4 = 1600 + 280 + 24 = 1880 + 24 = 1904

76. 572 × 9 = 4500 + 630 + 18 = 5130 + 18 = 5148

77. 667 × 3 = 1800 + 180 + 21 = 1980 + 21 = 2001

When squaring 3-digit numbers, the first step is to essentially do a 3-by-1 multiplication problem like the ones below.

78. 404 × 400: 404 × 4 = 1616, so 404 × 400 = 161,600

79. 226 × 200: 226 × 2 = 400 + 52 = 452, so 226 × 200 = 45,200

80. 422 × 400: 422 × 4 = 1600 + 88 = 1688, so 422 × 400 = 168,800

81. 110 × 200: 11 × 2 = 22, so 110 × 200 = 22,000

82. 518 × 500: 518 × 5 = 2500 + 90 = 2590, so 518 × 500 = 259,000

83. 340 × 300: 34 × 3 = 90 + 12 = 102, so 340 × 300 = 102,000

84. 650 × 600: 65 × 6 = 360 + 30 = 390, so 650 × 600 = 390,000

85. 270 × 200: 27 × 2 = 40 + 14 = 54, so 270 × 200 = 54,000

86. 706 × 800: 706 × 8 = 5600 + 48 = 5648, so 706 × 800 = 564,800

87. 162 × 200: 162 × 2 = 200 + 120 + 4 = 320 + 4 = 324, so 162 × 200 = 32,400

88. 454 × 500: 454 × 5 = 2000 (say it) + 250 + 20 = 2000 + 270 = 2270, so 454 × 500 = 227,000

89. 664 × 700: 664 × 7 = 4200 + 420 + 28 = 4620 + 28 = 4648, so 664 × 700 = 464,800

Use the factoring method to multiply these 2-digit numbers together by turning the original problem into a 2-by-1 problem, followed by a 2-by-1 or 3-by-1 problem.

90. 43 × 14 = 43 × 7 × 2 = (280 + 21) × 2 = 301 × 2 = 602
OR 43 × 14 = 43 × 2 × 7 = 86 × 7 = 560 + 42 = 602

91. 64 × 15 = 64 × 5 × 3 = (300 + 20) × 3 = 320 × 3 = 900 + 60 = 960

92. 75 × 16 = 75 × 8 × 2 = (560 + 40) × 2 = 600 × 2 = 1200

93. $54 \times 24 = 54 \times 6 \times 4 = (300 + 24) \times 4 = 324 \times 4 = 1200$ (say it) $+ (24 \times 4)$
$24 \times 4 = 80 + 16 = 96$, so $54 \times 24 = 1296$

94. $89 \times 72 = 89 \times 9 \times 8 = (720 + 81) \times 8 = 801 \times 8 = 6408$

In poker, there are 2,598,960 ways to be dealt 5 cards (from 52 different cards, where order is not important). Calculate the following multiplication problems that arise through counting poker hands.

95. The number of hands that are straights (40 of which are straight flushes) is $10 \times 4^5 = 4 \times 4 \times 4 \times 4 \times 4 \times 10 = 16 \times 4^3 \times 10 = 64 \times 4^2 \times 10 = 256 \times 4 \times 10 = 1024 \times 10 = 10{,}240$

96. The number of hands that are flushes is $(4 \times 13 \times 12 \times 11 \times 10 \times 9)/120 = 13 \times 11 \times 4 \times 9 = 143$ (close together) $\times 4 \times 9 = (400 + 160 + 12) \times 9 = 572 \times 9 = (4500 + 630 + 18) = 5130 + 18 = 5148$

97. The number of hands that are four-of-a-kind is $13 \times 48 = 13 \times 8 \times 6 = (80 + 24) \times 6 = 104 \times 6 = 624$

98. The number of hands that are full houses is $13 \times 12 \times 4 \times 6 = 156$ (close together) $\times 4 \times 6 = (400 + 200 + 24) \times 6 = 624 \times 6 = 3600 + 120 + 24 = 3720 + 24 = 3744$

Lecture 4

Determine which numbers between 2 and 12 divide into each of the numbers below.

1. 4410 is divisible by 2, 3, 5, 6, 7, 9, and 10.

Why? Last digit gives us 2, 5, and 10; digit sum = 9 gives us 3 and 9; divisible by 2 and 3 gives us divisibility by 6. Passes 7 test: $4410 \rightarrow 441 \rightarrow 44 - 2 = 42$ It fails tests for 4 (and, therefore, 8 and 12) and 11.

2. 7062 is divisible by 2, 3, 6, and 11.

Why? Last digit gives us 2; digit sum = 15 gives us 3; 2 and 3 imply 6; alternating sum of digits 7 – 0 + 6 – 2 = 11 gives us 11. Fails other tests.

3. 2744 is divisible by 2, 4, 7, and 8.

Why? 744 is divisible by 8; passes 7 test: 2744 → 274 – 8 = 266 → 26 – 12 = 14. Fails other tests.

4. 33,957 is divisible by 3, 7, 9, and 11.

Why? Digit sum = 27 gives us 3 and 9; passes 7 test: 33,957 → 3395 – 14 = 3381 → 338 – 2 = 336 → 33 – 12 = 21. Passes 11 test: 3 – 3 + 9 – 5 + 7 = 11. Fails other tests.

Use the create-a-zero, kill-a-zero method to test the following.

5. Is 4913 divisible by 17?

Yes, because 4913 → 4913 + 17 = 4930 → 493 → 493 + 17 = 510 → 51 is a multiple of 17.

6. Is 3141 divisible by 59?

No, because 3141 + 59 = 3200 → 320 → 32 is not a multiple of 59.

7. Is 355,113 divisible by 7?

No, because 355,113 + 7 = 355,120 → 35,512 → 35,512 + 28 =355,140→35,514→35,514–14=35,500→3550→355→355+35 = 390 → 39 is not a multiple of 7. Also, it fails the special rule for 7s: 355,113 – 6 = 355,107 → 35,510 – 14 = 35,496 → 3549 – 12 = 3537 → 353 – 14 = 339 → 33 – 18 = 15 is not a multiple of 7.

8. Algebraically, the divisibility rule for 7s says that 10a + b is a multiple of 7 if and only if the number a – 2b is a multiple of 7. Explain why this works.

Suppose 10a + b is a multiple of 7, then it remains a multiple of 7 after we multiply it by –2, so –20a – 2b will still be a multiple of 7. And since 21a is always a multiple of 7 (because it's 7 × 3a), we can add this to get –20a – 2b + 21a, which is a – 2b. So a – 2b is still a multiple of 7.

Conversely, if a – 2b is a multiple of 7, then it remains so after we multiply it by 10, so 10a – 20b is still a multiple of 7. Adding 21b (a multiple of 7) to this tells us that 10a + b is also a multiple of 7.

Mentally do the following 1-digit division problems.

9. 97 ÷ 8

$$\begin{array}{r l} 12 & \text{R } 1 = 12\,1/8 \\ 8\overline{)97} & \\ -80 & \\ \hline 17 & \\ -16 & \\ \hline 1 & \end{array}$$

10. 63 ÷ 4

$$\begin{array}{r l} 15 & \text{R } 3 = 15\,3/4 \\ 4\overline{)63} & \\ -40 & \\ \hline 23 & \\ -20 & \\ \hline 3 & \end{array}$$

11. 159 ÷ 7

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   22 R 5 = 22 5/7
7)159
 -140
   19
  -14
    5
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12. 4668 ÷ 6

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   778
6)4668
 -4200
   468
  -420
    48
   -48
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13. 8763 ÷ 5 = (double both numbers) = 17,526 ÷ 10 = 1752.6

Convert the Fahrenheit temperatures below to Centigrade using the formula C = (F – 32) × 5/9.

14. 80 degrees Fahrenheit: (80 – 32) × 5/9 = 48 × 5/9 = 240 ÷ 9 = 80 ÷ 3 = 26 2/3 degrees Centigrade

15. 65 degrees Fahrenheit: (65 – 32) × 5/9 = 33 × 5/9 = 11 × 5/3 = 55 ÷ 3 = 18 1/3 degrees Centigrade

Mentally do the following 2-digit division problems.

16. 975 ÷ 13

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    75
13)975
  -910
    65
   -65
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17. 259 ÷ 31

$$\begin{array}{r} 8 \text{ R}11 = 8\,11/31 \\ 31\overline{)259} \\ -248 \\ \hline 11 \end{array}$$

18. 490 ÷ 62 (use overshooting): 62 × 8 = 496, so 490 ÷ 62 = 8 R – 6 = 7 R 56

19. 183 ÷ 19 (use overshooting): 19 × 10 = 190, so 183 ÷ 19 = 10 R –7 = 9 R 12

Do the following division problems by first simplifying the problem to an easier division problem.

20. 4200 ÷ 8 = 2100 ÷ 4 = 1050 ÷ 2 = 525

21. 654 ÷ 36 (dividing both by 6) = 109 ÷ 6 = 18 1/6

22. 369 ÷ 45 (doubling) = 738 ÷ 90; 738 ÷ 9 = 82, so the answer is 8.2

23. 812 ÷ 12.5 (doubling) = 1624 ÷ 25 = 3248 ÷ 50 = 6496 ÷ 100 = 64.96

24. Give the decimal expansions for 1/7, 2/7, 3/7, 4/7, 5/7, and 6/7.
1/7 = 0.142857 (repeated)
2/7 = 0.285714 (repeated)
3/7 = 0.428571 (repeated)
4/7 = 0.571428 (repeated)
5/7 = 0.714285 (repeated)
6/7 = 0.857142 (repeated)

25. Give the decimal expansion for 5/16: 50 ÷ 16 = 25 ÷ 8 = 3 1/8 = 3.125, so 5/16 = 0.3125

26. Give the decimal expansion for 12/35: 12/35 = 24 ÷ 70. Given that 24/7 = 3 3/7 = 3.428571…, 12/35 = 0.3428571…

27. When he was growing up, Professor Benjamin's favorite number was 2520. What is so special about that number? It is the smallest positive number divisible by all the numbers from 1 to 10.

Lecture 5

Estimate the following addition and subtraction problems by rounding each number to the nearest thousand, then to the nearest hundred.

1. $3764 + 4668 \approx 4000 + 5000 = 9000$
OR $3764 + 4668 \approx 3800 + 4700 = 8500$

2. $9661 + 7075 \approx 10{,}000 + 7000 = 17{,}000$
OR $9661 + 7075 \approx 9700 + 7100 = 16{,}800$

3. $9613 - 1252 \approx 10{,}000 - 1000 = 9000$
OR $9613 - 1252 \approx 9600 - 1300 = 8300$

4. $5253 - 3741 \approx 5000 - 4000 = 1000$
OR $5253 - 3741 \approx 5300 - 3700 = 1600$

Estimate the grocery total by rounding each number up or down to the nearest half dollar.

5.	**6.**	**7.**
5.24 ≈ 5	0.87 ≈ 1	0.78 ≈ 1
0.42 ≈ 0.5	2.65 ≈ 2.5	1.86 ≈ 2
2.79 ≈ 3	0.20 ≈ 0	0.68 ≈ 0.5
3.15 ≈ 3	1.51 ≈ 1.5	2.73 ≈ 2.5
0.28 ≈ 0.5	0.95 ≈ 1	4.29 ≈ 4.5
0.92 ≈ 1	2.59 ≈ 2.5	3.47 ≈ 3.5
4.39 ≈ 4.5	1.60 ≈ 1.5	2.65 ≈ 2.5
17.5	10.0	16.5

What are the possible numbers of digits in the answers to the following?

8. 5 digits times 3 digits is 7 or 8 digits.

9. 5 digits divided by 3 digits is 2 or 3 digits.

10. 8 digits times 4 digits is 11 or 12 digits.

11. 8 digits divided by 4 digits is 4 or 5 digits.

For the following problems, determine the possible number of digits in the answers. (Some answers may allow two possibilities.) A number written like 3abc represents a 4-digit number with leading digit of 3.

12. 3abc × 7def has 8 digits.

13. 8abc × 1def can have 7 or 8 digits.

14. 2abc × 2def has 7 digits.

15. 9abc ÷ 5de has 2 digits.

16. 1abcdef ÷ 3ghij has 2 digits.

17. 27abcdefg ÷ 26hijk has 4 digits.

18. If a year has about 32 million seconds, then 1 trillion seconds is about how many years?

The number 1 trillion has 13 digits, starting with 1, and 32 million has 8 digits, starting with 3, so 1 trillion divided by 32 million has 5 digits; thus, the answer is approximately 30,000.

19. The government wants to buy a new weapons system costing $11 billion. The U.S. has about 100,000 public schools. If each school decides to hold a bake sale to raise money for the new weapons system, then about how much money does each school need to raise?

The number 11 billion has 11 digits, starting with 11, and 100,000 has 6 digits, starting with 10, so the answer has 11 – 6 + 1 = 6 digits, starting with 1; thus, the answer is about $110,000 per school.

20. If an article is sent to two independent reviewers, and one reviewer finds 40 typos, the other finds 5 typos, and there were 2 typos in common, then estimate the total number of typos in the document.

By Pólya's estimate, the total number of typos in the document is approximately $40 \times 5 \div 2 = 100$.

21. Estimate 6% sales tax on a new car costing $31,500. Adjust your answer for 6.25% sales tax.

$315 \times 6 = 1890$, so the sales tax is about $1900. For an additional 0.25%, increase this amount by $1900 ÷ 24 (since 6/24 = 0.25%), which is about $80; thus, the sales tax with the higher rate is about $1980.

22. To calculate 8.5% tax, you can take 8% tax, then add the tax you just computed divided by what number?

Since 8/16 = 0.5, you divide by 16.

For 8.75% tax, you can take 9% tax, then subtract that tax divided by what number?

To reduce the number by 0.25%, we divide the tax by 36, since 9/36 = 0.25.

23. If money earns interest compounded at a rate of 2% per year, then about how many years would it take for that money to double?

By the Rule of 70, since 70/2 = 35, it will take about 35 years to double.

24. Suppose you borrow \$20,000 to buy a new car, the bank charges an annual interest rate of 3%, and you have 5 years to pay off the loan. Determine an underestimate and overestimate for your monthly payment, then determine the exact monthly payment.

The number of monthly payments is $5 \times 12 = 60$. If no interest were charged, the monthly payment would be $20{,}000/60 \approx \$333$. But since the monthly interest is $3\%/12 = 0.25\%$, then you would owe $\$20{,}000(.25\%) = \$50$ in interest for the first month. The regular monthly payment would be, at most, $\$333 + \$50 = \$383$.

To get the exact monthly payment, we use the interest formula: $P \times i(1 + i)^m/((1 + i)^m - 1)$.

Here, $P = 20{,}000$, $i = 0.0025$, $m = 60$, and our calculator or search engine tells us $(1.0025)^{60} \approx 1.1616$; the monthly payment is about $\$20{,}000(.0025)(1.1616)/(0.1616) \approx \$359.40$/month, which is consistent with our lower bound of \$333 and our upper bound of \$383.

25. Repeat the previous problem, but this time, the bank charges 6% annual interest and gives you 10 years to pay off the loan.

The number of monthly payments is $10 \times 12 = 120$, so the lower estimate is $20{,}000/120 \approx \$167$/month. But since the monthly interest is $6\%/12 = 0.5\%$, then you would owe $\$20{,}000(.5\%) = \$100$ in interest for the first month. Thus, the regular monthly payment would be, at most, $\$167 + \$100 = \$267$. Plugging $P = 20{,}000$, $i = 0.005$, and $m = 120$ into the formula gives us $\$100(1.005)^{120}/((1.005)^{120} - 1) \approx \$181.94/(0.8194) \approx \$222$/month.

26. Use the divide-and-average method to estimate the square root of 27.

If we start with an estimate of 5, $27 \div 5 = 5.4$, and their average is 5.2. (Exact answer begins 5.196…)

27. Use the divide-and-average method to estimate the square root of 153.

If we start with an estimate of 12, 153 ÷ 12 = 12 9/12 = 12.75, and their average is 12.375. (Exact answer begins 12.369…)

28. Speaking of 153, that's the first 3-digit number equal to the sum of the cubes of its digits ($153 = 1^3 + 5^3 + 3^3$). The next number with that property is 370. Can you find the third number with that property?

Since $370 = 3^3 + 7^3 + 0^3$, it follows that $371 = 3^3 + 7^3 + 1^3$.

Lecture 6

Add the following columns of numbers. Check your answers by adding the numbers in reverse order and by casting out nines.

1.		**2.**		**3.**	
594 →	9	366 →	6	2.20 →	4
12 →	3	686 →	2	4.62 →	3
511 →	7	469 →	1	1.73 →	2
199 →	1	2010 →	3	32.30 →	8
3982 →	4	62 →	8	3.02 →	5
291 →	3	500 →	5	0.39 →	3
1697 →	5	4196 →	2	5.90 →	5
7286	32	8289	27	50.16	30
|	|	|	|	|	|
5	5	9	9	3	3

Do the following subtraction problems by first mentally computing the cents, then the dollars. Complements will often come in handy. Check your answers with an addition problem and with casting out nines.

4. $1776.65 - 78.95 = 1697.70$ (Verifying, $1697.70 + 78.95 = 1776.65$)

$$\begin{array}{ccccc} | & & | & & | \\ 5 & - & 2 & = & 3 \end{array}$$

5. $5977.31 - 842.78 = 5134.53$ (Verifying, $5134.53 + 842.78 = 5977.31$)

$$\begin{array}{ccccc} | & & | & & | \\ 5 & - & 2 & = & 3 \end{array}$$

6. $761.45 - 80.35 = 681.10$ (Verifying, $681.10 + 80.35 = 761.45$)

$$\begin{array}{ccccc} | & & | & & | \\ 5 & - & 7 + 9 & = & 7 \end{array}$$

Use the criss-cross method to do the following multiplication problems. Verify that your answers are consistent with casting out nines.

7.

$$\begin{array}{rcccr} 29 & \rightarrow & 11 & \rightarrow & 2 \\ \times \underline{82} & \rightarrow & 10 & \rightarrow & \times \underline{1} \\ 2378 & \rightarrow & 20 & \rightarrow & 2 \end{array}$$

8.

$$\begin{array}{rcccr} 764 & \rightarrow & 17 & \rightarrow & 8 \\ \times \underline{514} & \rightarrow & 10 & \rightarrow & \times \underline{1} \\ 392{,}696 & \rightarrow & 35 & \rightarrow & 8 \end{array}$$

9.

$$\begin{array}{rcccr} 5593 & \rightarrow & 22 & \rightarrow & 4 \\ \times \underline{2906} & \rightarrow & 17 & \rightarrow & \times \underline{\ 8} \\ 16{,}253{,}258 & & \rightarrow & & 32 \end{array}$$

10. What is the remainder (not the quotient) when you divide 1,234,567 by 9?

Summing the digits, 1,234,567 → 28 → 10 → 1, so the remainder is 1.

11. What is the remainder (not the quotient) when you divide 12,345,678 by 9?

Summing the digits, 12,345,678 → 36 → 9, so the number is a multiple of 9, so dividing 12,345,678 by 9 yields a remainder of 0.

12. After doing the multiplication problem 1234 × 567,890, you get an answer that looks like 700,7#6,260, but the fifth digit is smudged, and you can't read it. Use casting out nines to determine the value of the smudged number.

Using digit sums, 1234 → 1 and 567,890 → 8, so their product must reduce to 1 × 8 = 8.

Summing the other digits, 7 + 0 + 0 + 7 + 6 + 2 + 6 + 0 = 28 → 1, so the smudged digit must be 7 in order to reach a total of 8.

Use the Vedic method to do the following division problems.

13. 3210 ÷ 9

$$\begin{array}{l} \;\;356 \quad R\,6 \\ 9\overline{)3210} \end{array}$$

14. 20,529 ÷ 9

$$\begin{array}{l} \;\;2279 \quad R18 = 2281\,R\,0 \\ 9\overline{)20529} \end{array}$$

15. 28,306 ÷ 9

$$\begin{array}{l} \;\;1 \\ \;\;2144 \quad R10 = 3145\,R1 \\ 9\overline{)28306} \end{array}$$

16. 942,857 ÷ 9

$$\begin{array}{r@{}l} & \begin{matrix}1 & & 1 & 1 & & \end{matrix} \\ & \begin{matrix}9 & 4 & 6 & 5 & 1\end{matrix}\ \ R\,8 = 104,761\,R\,8 \\ 9 & \overline{)942857} \end{array}$$

Use the close-together method for the following multiplication problems.

17.

$$\begin{array}{rl} 108 & (8) \\ \times\underline{105} & (5) \\ 113 & 40 \end{array}$$

18.

$$\begin{array}{rl} 92 & (-8) \\ \times\underline{95} & (-5) \\ 87 & 40 \end{array}$$

19.

$$\begin{array}{rrl} & 108 & (8) \\ & \times\quad 95 & (-5) \\ \hline 103\times 100 = & 10,300 & \\ 8\times(-5) = & -40 & \\ \hline & 10,260 & \end{array}$$

20.

$$\begin{array}{rrl} & 998 & (-2) \\ & \times\quad 997 & (-3) \\ \hline 995\times 1000 = & 995,000 & \\ (-2)\times(-3) = & +6 & \\ \hline & 995,006 & \end{array}$$

21.

$$\begin{array}{rrl} & 304 & (4) \\ & \times\quad 311 & (11) \\ \hline 315\times 300 = & 94,500 & \\ 4\times 11 = & +44 & \\ \hline & 94,544 & \end{array}$$

Lecture 7

Note: The details of many of the 2-by-1 and 3-by-1 multiplications are provided in the solutions for Lecture 3.

Calculate the following 2-digit squares. Remember to begin by going up or down to the nearest multiple of 10.

1. $14^2 = 10 \times 18 + 4^2 = 180 + 16 = 196$

2. $18^2 = 20 \times 16 + 2^2 = 320 + 4 = 324$

3. $22^2 = 20 \times 24 + 2^2 = 480 + 4 = 484$

4. $23^2 = 20 \times 26 + 3^2 = 520 + 9 = 529$

5. $24^2 = 20 \times 28 + 4^2 = 560 + 16 = 576$

6. $25^2 = 20 \times 30 + 5^2 = 600 + 25 = 625$

7. $29^2 = 30 \times 28 + 1^2 = 840 + 1 = 841$

8. $31^2 = 30 \times 32 + 1^2 = 960 + 1 = 961$

9. $35^2 = 30 \times 40 + 5^2 = 1200 + 25 = 1225$

10. $36^2 = 40 \times 32 + 4^2 = 1280 + 16 = 1296$

11. $41^2 = 40 \times 42 + 1^2 = 1680 + 1 = 1681$

12. $44^2 = 40 \times 48 + 4^2 = 1920 + 16 = 1936$

13. $45^2 = 40 \times 50 + 5^2 = 2000 + 25 = 2025$

14. $47^2 = 50 \times 44 + 3^2 = 2200 + 9 = 2209$

15. $56^2 = 60 \times 52 + 4^2 = 3120 + 16 = 3136$

16. $64^2 = 60 \times 68 + 4^2 = 4080 + 16 = 4096$

17. $71^2 = 70 \times 72 + 1^2 = 5040 + 1 = 5041$

18. $82^2 = 80 \times 84 + 2^2 = 6720 + 4 = 6724$

19. $86^2 = 90 \times 82 + 4^2 = 7380 + 16 = 7396$

20. $93^2 = 90 \times 96 + 3^2 = 8640 + 9 = 8649$

21. $99^2 = 100 \times 98 + 1^2 = 9800 + 1 = 9801$

Do the following 2-digit multiplication problems using the addition method.

22. $31 \times 23 = (30 + 1) \times 23 = (30 \times 23) + (1 \times 23) = 690 + 23 = 713$

23. $61 \times 13 = (60 + 1) \times 13 = (60 \times 13) + (1 \times 13) = 780 + 13 = 793$

24. $52 \times 68 = (50 + 2) \times 68 = (50 \times 68) + (2 \times 68) = 3400 + 136 = 3536$

25. $94 \times 26 = (90 + 4) \times 26 = (90 \times 26) + (4 \times 26) = 2340 + 104 = 2444$

26. $47 \times 91 = 47 \times (90 + 1) = (47 \times 90) + (47 \times 1) = 4230 + 47 = 4277$

Do the following 2-digit multiplication problems using the subtraction method.

27. $39 \times 12 = (40 - 1) \times 12 = 480 - 12 = 468$

28. $79 \times 41 = (80 - 1) \times 41 = 3280 - 41 = 3239$

29. $98 \times 54 = (100 - 2) \times 54 = 5400 - 108 = 5292$

30. $87 \times 66 = (90 - 3) \times 66 = (90 \times 66) - (3 \times 66) = 5940 - 198 = 5742$

31. $38 \times 73 = (40 - 2) \times 73 = (40 \times 73) - (2 \times 73) = 2920 - 146 = 2774$

Do the following 2-digit multiplication problems using the factoring method.

32. $75 \times 56 = 75 \times 8 \times 7 = 600 \times 7 = 4200$

33. $67 \times 12 = 67 \times 6 \times 2 = 402 \times 2 = 804$

34. $83 \times 14 = 83 \times 7 \times 2 = 581 \times 2 = 1162$

35. $79 \times 54 = 79 \times 9 \times 6 = 711 \times 6 = 4266$

36. $45 \times 56 = 45 \times 8 \times 7 = 360 \times 7 = 2520$

37. $68 \times 28 = 68 \times 7 \times 4 = 476 \times 4 = 1904$

Do the following 2-digit multiplication problems using the close-together method.

38. $13 \times 19 = (10 \times 22) + (3 \times 9) = 220 + 27 = 247$

39. $86 \times 84 = (80 \times 90) + (6 \times 4) = 7200 + 24 = 7224$

40. $77 \times 71 = (70 \times 78) + (7 \times 1) = 5460 + 7 = 5467$

41. $81 \times 86 = (80 \times 87) + (1 \times 6) = 6960 + 6 = 6966$

42. $98 \times 93 = (100 \times 91) + (-2 \times -7) = 9100 + 14 = 9114$

43. $67 \times 73 = (70 \times 70) + (-3 \times 3) = 4900 - 9 = 4891$

Do the following 2-digit multiplication problems using more than one method.

44. $14 \times 23 = 23 \times 7 \times 2 = 161 \times 2 = 322$
OR $14 \times 23 = 23 \times 2 \times 7 = 46 \times 7 = 322$
OR $14 \times 23 = (14 \times 20) + (14 \times 3) = 280 + 42 = 322$

45. 35 × 97 = 35 × (100 – 3) = 3500 – 35 × 3 = 3500 – 105 = 3395
OR 35 × 97 = 97 × 7 × 5 = 679 × 5 = 3395

46. 22 × 53 = 53 × 11 × 2 = 583 × 2 = 1166
OR 53 × 22 = (50 + 3) × 22 = 50 × 22 + 3 × 22 = 1100 + 66 = 1166

47. 49 × 88 = (50 – 1) × 88 = (50 × 88) – (1 × 88) = 4400 – 88 = 4312
OR 88 × 49 = 88 × 7 × 7 = 616 × 7 = 4312
OR 49 × 88 = 49 × 11 × 8 = 539 × 8 = 4312

48. 42 × 65 = (40 × 65) + (2 × 65) = 2600 + 130 = 2730
OR 65 × 42 = 65 × 6 × 7 = 390 × 7 = 2730

Lecture 8

Do the following 1-digit division problems on paper using short division.

1. 123,456 ÷ 7

$$\begin{array}{r|l} & \;\;1\;7\;6\;3\;6\ \text{R}4 \\ \cline{2-2} 7 & 1\,2_5 3_4 4_2 5_4 6 \end{array}$$

2. 8648 ÷ 3

$$\begin{array}{r|l} & 2\;8\;8\;2\ \text{R}2 \\ \cline{2-2} 3 & 8_2 6_2 4_0 8 \end{array}$$

3. 426,691 ÷ 8

$$\begin{array}{r|l} & \;\;5\;3\;3\;3\;6\ \text{R}3 \\ \cline{2-2} 8 & 4\,2_2 6_2 6_2 9_5 1 \end{array}$$

4. 21,472 ÷ 4

$$\begin{array}{r|l} & \ \ 5\,3\,6\,8\ \text{R}0 \\ \hline 4 & 21_1 4_2 7_3 2 \end{array}$$

5. 374,476,409 ÷ 6

$$\begin{array}{r|l} & \ \ 6\,2\,4\,1\,2\,7\,3\,4\ \text{R}5 \\ \hline 6 & 37_1 4_2 4_0 7_1 6_4 4_2 0_2 9 \end{array}$$

Do the following 1-digit division problems on paper using short division *and* by the Vedic method.

6. 112,300 ÷ 9

$$\begin{array}{r|l} & \ \ 1\,2\,4\,7\,7\ \text{R}7 \\ \hline 9 & 1\,1_2 2_4 3_7 0_7 0 \end{array} \qquad \text{Vedic: } \begin{array}{r|l} & 12477\ \ \text{R}7 \\ \hline 9 & 112300 \end{array}$$

7. 43,210 ÷ 9

$$\begin{array}{r|l} & \ \ 4\,8\,0\,1\ \text{R}1 \\ \hline 9 & 43_7 2_0 1_1 0 \end{array} \qquad \text{Vedic: } \begin{array}{r|l} & 47\overset{1}{9}1\ \ \text{R}1 = 4801\ \text{R}1 \\ \hline 9 & 43210 \end{array}$$

8. 47,084 ÷ 9

$$\begin{array}{r|l} & \ \ 5\,2\,3\,1\ \text{R}5 \\ \hline 9 & 4\,7_2 0_2 8_1 4 \end{array} \qquad \text{Vedic: } \begin{array}{r|l} & \overset{1}{4}2\overset{1}{2}1\ \ \text{R}5 = 5231\ \text{R}5 \\ \hline 9 & 47084 \end{array}$$

9. 66,922 ÷ 9

$$\begin{array}{r|l} & \ \ 7\,4\,3\,5\ \ \text{R}7 \\ \hline 9 & 66_3 9_3 2_5 2 \end{array} \qquad \text{Vedic: } \begin{array}{r|l} & \overset{1}{6}\overset{1}{3}35\ \ \text{R}7 = 7435\ \text{R}7 \\ \hline 9 & 66922 \end{array}$$

10. 393,408 ÷ 9

$$\begin{array}{r|l} & \ \ 4\,3\,7\,1\,2\ \ \text{R}0 \\ \hline 9 & 39_3 3_6 4_1 0_1 8 \end{array} \qquad \text{Vedic: } \begin{array}{r|l} & \overset{1}{3}3\overset{1}{6}11\ \ \text{R}9 = 43{,}712\ \text{R}0 \\ \hline 9 & 393408 \end{array}$$

To divide numbers between 11 and 19, short division is very quick, especially if you can rapidly multiply numbers between 11 and 19 by 1-digit numbers. Do the following problems on paper using short division.

11. 159,348 ÷ 11

$$\begin{array}{rl} & 1\ 4\ 4\ 8\ 6\ \text{R}2 \\ 11 & \overline{)\,15_4 9_5 3_9 4_6 8} \end{array}$$

12. 949,977 ÷ 12

$$\begin{array}{rl} & 7\ 9\ 1\ 6\ 4\ \text{R}9 \\ 12 & \overline{)\,9\,4_{10} 9_1 9_7 7_5 7} \end{array}$$

13. 248,814 ÷ 13

$$\begin{array}{rl} & 1\ 9\ 1\ 3\ 9\ \text{R}7 \\ 13 & \overline{)\,2\,4_{11} 8_1 8_5 1_{12} 4} \end{array}$$

14. 116,477 ÷ 14

$$\begin{array}{rl} & 8\ 3\ 1\ 9\ \ \text{R}11 \\ 14 & \overline{)\,1\,1\,6_4 4_2 7_{13} 7} \end{array}$$

15. 864,233 ÷ 15

$$\begin{array}{rl} & 5\ 7\ 6\ 1\ 5\ \text{R}8 \\ 15 & \overline{)\,8\,6_{11} 4_9 2_2 3_8 3} \end{array}$$

16. 120,199 ÷ 16

$$\begin{array}{rl} & 7\ 5\ 1\ 2\ \text{R}7 \\ 16 & \overline{)\,1\,2\,0_8 1_1 9_3 9} \end{array}$$

17. 697,468 ÷ 17

$$\begin{array}{rl} & 4\ 1\ 0\ 2\ 7\ \text{R}9 \\ 17 & \overline{)\,6\,9_1 7_0 4_4 6_{12} 8} \end{array}$$

18. 418,302 ÷ 18

$$\begin{array}{r|l} & \;\;2\;3\;2\;3\;\;9 \text{ R0} \\ \cline{2-2} 18 & 41_5 8_4 3_7 0_{16} 2 \end{array}$$

19. 654,597 ÷ 19

$$\begin{array}{r|l} & \;\;3\;4\;4\;5\;2 \text{ R9} \\ \cline{2-2} 19 & 6\,5_8 4_8 5_9 9_4 7 \end{array}$$

Use the Vedic method on paper for these division problems where the last digit is 9. The last two problems will have carries.

20. 123,456 ÷ 69

$$\begin{array}{r|l} & \;\;1\;7\;8\;9\;\;\text{R15} \\ \cline{2-2} 69 & 12_5 3_5 4_5 5_0 6 \\ +1 & \\ \hline 70 & \end{array}$$

First division step:	12 ÷ 7 = 1 R 5
Second division step:	(53 + 1) ÷ 7 = 7 R 5
Third division step:	(54 + 7) ÷ 7 = 8 R 5
Fourth division step:	(55 + 8) ÷ 7 = 9 R 0
Remainder:	06 + 9 = 15

21. 14,113 ÷ 59

$$\begin{array}{r|l} & \;\;2\;3\;9\;\;\text{R12} \\ \cline{2-2} 59 & 14_2 1_5 1_0 3 \\ +1 & \\ \hline 60 & \end{array}$$

22. 71,840 ÷ 49

$$\begin{array}{r|l} & 1\;4\;6\;6\;\;\text{R6} \\ \cline{2-2} 49 & 7_2 1_2 8_2 4_0 0 \\ +1 & \\ \hline 50 & \end{array}$$

23. 738,704 ÷ 79

$$\begin{array}{l}
\qquad 9\ 3\ 5\ 0 \quad \text{R } 54 \\
79\overline{)\,7 3_1 8_3 7_0 0_5 4} \\
\underline{+1} \\
80
\end{array}$$

24. 308,900 ÷ 89

$$\begin{array}{l}
\qquad 3\ 4\ 7\ 0 \quad \text{R } 70 \\
89\overline{)\,3 0_3 8_5 9_0 0_7 0} \\
\underline{+1} \\
90
\end{array}$$

25. 56,391 ÷ 99

$$\begin{array}{l}
\qquad\quad 5\ 6\ 9 \quad \text{R } 60 \\
\ \ 99\overline{)\,5 6_6 3_8 9_5 1} \\
\underline{\ +1\ } \\
100
\end{array}$$

26. 23,985 ÷ 29

$$\begin{array}{l}
\qquad 1 \\
\qquad 7\ 2\ 6 \quad \text{R } 31 = 826 \text{ R } 31 = 827 \text{ R } 2 \\
29\overline{)\,2 3_2 9_0 8_2 5} \\
\underline{+1} \\
30
\end{array}$$

First division step: 23 ÷ 3 = 7 R 2
Second division step: (29 + 7) ÷ 3 = 12 R 0
Third division step: (08 + 12) ÷ 3 = 6 R 2
Remainder: 25 + 6 = 31

27. 889,892 ÷ 19

$$\begin{array}{l}
\qquad\quad 1\ 1 \\
\qquad 4\ 6\ 7\ 2\ 5 \quad \text{R } 27 = 46{,}835 \text{ R } 27 = 46{,}836 \text{ R } 8 \\
19\overline{)\,8_0 8_0 9_1 8_1 9_1 2} \\
\underline{+1} \\
20
\end{array}$$

First division step: $8 \div 2 = 4 \text{ R } 0$
Second division step: $(08 + 4) \div 2 = 6 \text{ R } 0$
Third division step: $(09 + 6) \div 2 = 7 \text{ R } 1$
Fourth division step: $(18 + 7) \div 2 = 12 \text{ R } 1$
Fifth division step: $(19 + 12) \div 2 = 15 \text{ R } 1$
Remainder: $12 + 15 = 27$

Use the Vedic method for these division problems where the last digit is 8, 7, 6, or 5. Remember that for these problems, the *multiplier* is 2, 3, 4, and 5, respectively.

28. 611,725 ÷ 78

$$\begin{array}{r|l} & 7\ 8\ 4\ 2 \quad \text{R } 49 \\ \hline 78 & 6\ 1_5 1_1 7_1 2_4 5 \\ +2 & \\ \hline 80 & \end{array}$$

First division step: $61 \div 8 = 7 \text{ R } 5$
Second division step: $(51 + 14) \div 8 = 8 \text{ R } 1$
Third division step: $(17 + 16) \div 8 = 4 \text{ R } 1$
Fourth division step: $(12 + 8) \div 8 = 2 \text{ R } 4$
Remainder: $45 + 4 = 49$

29. 415,579 ÷ 38

$$\begin{array}{r|l} & \quad\ \ 1\ 1 \\ & 1\ 0\ 8\ 2\ 5 \quad \text{R } 49 = 10{,}935 \text{ R } 49 = 10{,}936 \text{ R } 11 \\ \hline 38 & 4_0 1_3 5_3 5_3 7_1 9 \\ +2 & \\ \hline 40 & \end{array}$$

First division step: $4 \div 4 = 1$ R 0
Second division step: $(01 + 2) \div 4 = 0$ R 3
Third division step: $(35 + 0) \div 4 = 8$ R 3
Fourth division step: $(35 + 16) \div 4 = 12$ R 3
Fifth division step: $(37 + 24) \div 4 = 15$ R 1
Remainder = $19 + 30 = 49$

30. $650{,}874 \div 87$

$$
\begin{array}{rl}
 & 1 \\
 & 7\ 4\ 7\ 0 \quad \text{R}114 = 7480\ \text{R}114 = 7481\ \text{R}\,27 \\
87 & \overline{\big)\,6\ 5_2 0_5 8_7 7_8 4} \\
+3 & \\
\hline
90 &
\end{array}
$$

31. $821{,}362 \div 47$

$$
\begin{array}{rl}
 & 1\ 7\ 4\ 7\ 5 \quad \text{R}\,37 \\
47 & \overline{\big)\,8_3 2_0 1_2 3_0 6_2 2} \\
+3 & \\
\hline
50 &
\end{array}
$$

32. $740{,}340 \div 96$

$$
\begin{array}{rl}
 & 1\ 1 \\
 & 7\ 6\ 0\ 1 \quad \text{R}\,84 = 7711\ \text{R}\,84 \\
96 & \overline{\big)\,7\,4_4 0_8 3_7 4_4 0} \\
+4 & \\
\hline
100 &
\end{array}
$$

33. $804{,}148 \div 26$

$$
\begin{array}{rl}
 & 1\ 2\ 4 \\
 & 2\ 9\ 6\ 8\ 2 \quad \text{R}176 = 30{,}922\ \text{R}176 = 30{,}928\ \text{R}\,20 \\
26 & \overline{\big)\,8_2 0_1 4_2 1_1 4_0 8} \\
+4 & \\
\hline
30 &
\end{array}
$$

First division step:	$8 \div 3 = 2 \text{ R } 2$
Second division step:	$(20 + 8) \div 3 = 9 \text{ R } 1$
Third division step:	$(14 + 36) \div 3 = 16 \text{ R } 2$
Fourth division step:	$(21 + 64) \div 3 = 28 \text{ R } 1$
Fifth division step:	$(14 + 112) \div 3 = 42 \text{ R } 0$
Remainder:	$08 + 168 = 176$

Note: Problem 33 had many large carries, which can happen when the divisor is larger than the multiplier. Here, the divisor was small (3) and the multiplier was larger (4). Such problems might be better solved using short division.

34. 380,152 ÷ 35

$$\begin{array}{r|l} & \;\;1\;\;2\;\;3 \\ & \;\;9\;\;6\;\;2\;\;6 \quad \text{R}192 = 10{,}856\,\text{R}192 = 10{,}861\,\text{R}17 \\ \cline{2-2} 35 & 3\,8_2 0_1 1_3 5_1 2 \\ +5 & \\ \hline 40 & \end{array}$$

35. 103,985 ÷ 85

$$\begin{array}{r|l} & \;\;1\;\;2\;\;2\;\;3 \quad \text{R}30 \\ \cline{2-2} 85 & 1\,0_1 3_0 9_1 8_1 5 \\ +5 & \\ \hline 90 & \end{array}$$

36. Do the previous two problems by first doubling both numbers, then using short division.

380,152 ÷ 35 = 760,304 ÷ 70 =

$$\begin{array}{r|l} & 1\;\;0\;\;8\;\;6\;\;1.\;4\;\;6/7 = 10{,}861.48571428\ldots \\ \cline{2-2} 7 & 7_0 6_6 0_4 3_1 0._3 4 \end{array}$$

103,985 ÷ 85 = 207,970 ÷ 170 = 20,797 ÷ 17 =

$$\begin{array}{r|l} & \;\;1\;\;2\;\;2\;\;3\;\;\; 6/17 \\ \cline{2-2} 17 & 2\,0_3 7_3 9_5 7 \end{array}$$

Use the Vedic method for these division problems where the last digit is 1, 2, 3, or 4. Remember that for these problems, the multiplier is –1, –2, –3, and –4, respectively.

37. 113,989 ÷ 21

$$\begin{array}{r|l} & \;\;5\ 4\ 2\ 8 \quad R\,1 \\ \cline{2-2} 21 & 1\ 1\ {}_{1}3_{0}9\ {}_{1}8_{0}9 \\ -1 & \\ \hline 20 & \end{array}$$

38. 338,280 ÷ 51

$$\begin{array}{r|l} & \;\;6\ 6\ 3\ 3 \quad R-3 = 6632\ R\,48 \\ \cline{2-2} 51 & 3\ 3_{3}8_{2}2_{1}8_{0}0 \\ -1 & \\ \hline 50 & \end{array}$$

39. 201,220 ÷ 92

$$\begin{array}{r|l} & \;\;2\ 1\ 8\ 7 \quad R\,16 \\ \cline{2-2} 92 & 2\ 0_{2}1_{8}\ 2_{8}2_{3}0 \\ -2 & \\ \hline 90 & \end{array}$$

40. 633,661 ÷ 42

$$\begin{array}{r|l} & 1\ 5\ 0\ 8\ 7 \quad R\,7 \\ \cline{2-2} 42 & 6_{2}3_{1}3_{3}6_{4}6_{2}1 \\ -2 & \\ \hline 40 & \end{array}$$

Note: In the fourth division step, (36 – 0) ÷ 4 = 9 R 0 = 8 R 4.

41. 932,498 ÷ 83

$$\begin{array}{r|l} & 1\ 1\ 2\ 3\ 5 \quad R-7 = 11{,}234\ R\,76 \\ \cline{2-2} 83 & 9_{1}3_{2}2_{3}4_{4}9_{0}8 \\ -3 & \\ \hline 80 & \end{array}$$

42. 842,298 ÷ 63

$$\begin{array}{r|l} & \;\;1\;\;3\;\;3\;\;6\;\;9 \quad \text{R}\,51 \\ \hline 63 & 8_2 4_3 2_5 2_7 9_7 8 \\ -3 & \\ \hline 60 & \end{array}$$

Note: In the fourth division step, (52 – 9) ÷ 6 = 7 R 1 = 6 R 7.

In the fifth division step, (79 – 18) ÷ 6 = 10 R 1 = 9 R 7.

43. 547,917 ÷ 74

$$\begin{array}{r|l} & \;\;\;7\;\;4\;\;0\;\;4 \quad \text{R}\,21 \\ \hline 74 & 5\,4\,_5 7\,_1 9_3 1\,_3 7 \\ -4 & \\ \hline 70 & \end{array}$$

44. 800,426 ÷ 34

$$\begin{array}{r|l} & \;\;2\;\;3\;\;5\;\;4\;\;1 \quad \text{R}\,32 \\ \hline 34 & 8_2 0_3 0_3 4_2 2_3 6 \\ -4 & \\ \hline 30 & \end{array}$$

Lecture 9

Use the Major system to convert the following words into numbers.

1. News = 20

2. Flash = 856

3. Phonetic = 8217

4. Code = 71

5. Makes = 370

6. Numbers = 23,940

7. Much = 36

8. More = 34

9. Memorable = 33,495

For each of the numbers below, find at least two words for each number. A few suggestions are given, but each number has more possibilities than those listed below.

10. 11 = date, diet, dot, dud, tot, tight, toot

11. 23 = name, Nemo, enemy, gnome, Nome

12. 58 = live, love, laugh, life, leaf, lava, olive

13. 13 = Adam, atom, dime, dome, doom, time, tome, tomb

14. 21 = nut, night, knight, note, ant, aunt, Andy, unit

15. 34 = mare, Homer, Mara, mere, meer, mire, and … more!

16. 55 = lily, Lola, Leila, Lyle, lolly, loyal, LOL

17. 89 = fib, fob, VIP, veep, Phoebe, phobia

Create a mnemonic to remember the years press in 1450.

He put it together using electric DRILLS!
He was TIRELESS in his efforts.

18. Pilgrims arrive at Plymouth Rock in 1620.

When they arrived, the pilgrims conducted a number of TEACH-INS.

A book about their voyage went through several EDITIONS.

19. Captain James Cook arrives in Australia in 1770.

The first animals he spotted were a DUCK and GOOSE.
For exercise, his crew would TAKE WALKS.

20. Russian Revolution takes place in 1917.

In the end, Lenin became TOP DOG, even though he was DIABETIC.

21. First man sets foot on the Moon on July 21, 1969.

The astronauts discovered CANDY (for 7/21) on TOP of their SHIP.
To get to sleep, the astronauts would COUNT DOPEY SHEEP.

Create a mnemonic to remember these phone numbers.

22. The Great Courses (in the U.S.): 800-832-2412

Their OFFICES experience FAMINE when a course is UNWRITTEN.
Their VOICES HAVE MANY a NEW ROUTINE.

23. White House switchboard: 202-456-1414

The president drives a NISSAN while eating RELISH and TARTAR.
The switchboard is run by an INSANE, REALLY SHY TRADER.

24. Create your own personal set of peg words for the numbers 1 through 20.

You'll have to do this one on your own!

25. How could you memorize the fact that the eighth U.S. president was Martin Van Buren?

Imagine a VAN BURNING that was caused by your FOE (named IVY or EVE?).

26. How could you memorize the fact that the Fourth Amendment to the U.S. Constitution prohibits unreasonable searches and seizures?

Imagine a soldier INSPECTING your EAR, which causes a SEIZURE. (Perhaps the solider was dressed like Julius Seizure, and he had gigantic EARs?)

27. How could you memorize the fact that the Sixteenth Amendment to the U.S. Constitution allows the federal government to collect income taxes?

This allowed the government to TOUCH all of our money!

Lecture 10

Here are the year codes for the years 2000 to 2040. The pattern repeats every 28 years (through 2099). For year codes in the 20th century, simply add 1 to the corresponding year code in the 21st century.

2000	2001	2002	2003	2004	2005	2006	2007	2008	2009	2010
0	1	2	3	5	6	0	1	3	4	5
	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020
	6	1	2	3	4	6	0	1	2	4
	2021	2022	2023	2024	2025	2026	2027	2028	2029	2030
	5	6	0	2	3	4	5	0	1	2
	2031	2032	2033	2034	2035	2036	2037	2038	2039	2040
	3	5	6	0	1	3	4	5	6	1

1. Write down the month codes for each month in a leap year. How does the code change when it is not a leap year?

If it is not a leap year, the month codes are (from January to December) 622 503 514 624.
In a leap year, the code for January changes to 5 and February changes to 1.

2. Explain why each year must always have at least one Friday the 13th and can never have more than three Friday the 13ths.

This comes from the fact that in every year (whether or not it's a leap year), all seven month codes, 0 through 6, are used at least once, and no code is used more than three times. For example, if it is not a leap year and the year had three Friday the 13ths, they must have occurred in February, March, and November (all three months have the same month code of 2). In a leap year, this can only happen for the months of January, April, and July (with the same month code of 5).

Determine the days of the week for the following dates. Feel free to use the year codes from the chart.

3. August 3, 2000 = month code + date + year code – multiple of 7
= 1 + 3 + 0 = 4 = Thursday

4. November 29, 2000 = 2 + 29 + 0 – 28 = 3 = Wednesday

5. February 29, 2000 = 1 + 29 + 0 – 28 = 2 = Tuesday

6. December 21, 2012 = 4 + 21 + 1 – 21 = 5 = Friday

7. September 13, 2013 = 4 + 13 + 2 – 14 = 5 = Friday

8. January 6, 2018 = 6 + 6 + 1 = 13 – 7 = 6 = Saturday

Calculate the year codes for the following years using the formula: year + leaps – multiple of 7.

9. 2020: Since leaps = 20 ÷ 4 = 5, the year code is 20 + 5 – 21 = 4.

10. 2033: Since leaps = 33 ÷ 4 = 8 (with remainder 1, which we ignore), the year code is 33 + 8 – 35 = 6.

11. 2047: year code = 47 + 11 – 56 = 2

12. 2074: year code = 74 + 18 – 91 = 1 (or 74 + 18 – 70 – 21 = 1)

13. 2099: year code = 99 + 24 – 119 = 4 (or 99 + 24 – 70 – 49 = 4)

Determine the days of the week for the following dates.

14. May 2, 2002: year code = 2;
month + date + year code – multiple of 7 = 0 + 2 + 2 = 4 = Thursday

15. February 3, 2058: year code = 58 + 14 – 70 = 2;
day = 2 + 3 + 2 – 7 = 0 = Sunday

16. August 8, 2088: year code = 88 + 22 – 105 = 5;
day = 1 + 8 + 5 – 14 = 0 = Sunday

17. June 31, 2016: Ha! This date doesn't exist! But the calculation would produce an answer of 3 + 31 + 6 – 35 = 5 = Friday.

18. December 31, 2099: year code = 4 (above);
day = 4 + 31 + 4 – 35 = 4 = Thursday

19. Determine the date of Mother's Day (second Sunday in May) for 2016.

The year 2016 has year code 6, and May has month code 0. 6 + 0 = 6. To reach Sunday, we must get a total of 7 or 14 or 21. … The first Sunday is May 1 (since 6 + 1 = 7), so the second Sunday is May 8.

20. Determine the date of Thanksgiving (fourth Thursday in November) for 2020.

The year 2020 has year code 4, and November has month code 2: 4 + 2 = 6. To reach Thursday, we must get a day code of 4 or 11 or 18. … Since 6 + 5 = 11, the first Thursday in November will be November 5. Thus, the fourth Thursday in November is November 5 + 21 = November 26.

For years in the 1900s, we use the formula: year + leaps + 1 – multiple of 7. Determine the year codes for the following years.

21. 1902: year code = 2 + 0 + 1 – 0 = 3

22. 1919: year code = 19 + 4 + 1 – 21 = 3

23. 1936: year code = 36 + 9 + 1 – 42 = 4

24. 1948: year code = 48 + 12 + 1 – 56 = 5

25. 1984: year code = 84 + 21 + 1 – 105 = 1

26. 1999: year code = 99 + 24 + 1 – 119 = 5 (This makes sense because the following year, 2000, is a leap year, which has year code 5 + 2 – 7 = 0.)

27. Explain why the calendar repeats itself every 28 years when the years are between 1901 and 2099.

Between 1901 and 2099, a leap year occurs every 4 years, even when it includes the year 2000. Thus any 28 consecutive between 1901 and 2099 will contain exactly 7 leap years. Hence, in a 28-year period, the calendar will shift 28 for each year plus 7 more times for each leap year for a total shifting of 35 days. Because 35 is a multiple of 7, the days of the week stay the same.

28. Use the 28-year rule to simplify the calculation of the year codes for 1984 and 1999.

For 1984, we subtract 28 × 3 = 84 from 1984. Thus, 1984 has the same year code as 1900, which has year code 1.

For 1999, we subtract 84 to get 1915, which has year code 15 + 3 + 1 – 14 = 5.

Determine the days of the week for the following dates.

29. November 11, 1911: year code = 11 + 2 + 1 – 14 = 0;
day = 2 + 11 + 0 – 7 = 6 = Saturday

30. March 22, 1930: year code = 30 + 7 + 1 – 35 = 3;
day = 2 + 22 + 3 – 21 = 6 = Saturday

31. January 16, 1964: year code = 64 + 16 +1 – 77 = 4;
day = 5 (leap year) + 16 + 4 – 21 = 4 = Thursday

32. August 4, 1984: year code = 1 (above);
day = 1 + 4 + 1 = 6 = Saturday

33. December 31, 1999: year code = 5 (above);
day = 4 + 31 + 5 – 35 = 5 = Friday

For years in the 1800s, the formula for the year code is years + leaps + 3 – multiple of 7. For years in the 1700s, the formula for the year code is years + leaps + 5 – multiple of 7. And for years in the 1600s, the formula for the year code is years + leaps – multiple of 7. Use this knowledge to determine the days of the week for the following dates from the Gregorian calendar.

34. February 12, 1809 (Birthday of Abe Lincoln *and* Charles Darwin):
year code = 9 + 2 + 3 – 14 = 0; day = 2 + 12 + 0 – 14 = 0 = Sunday.

35. March 14, 1879 (Birthday of Albert Einstein):
year code = 79 + 19 + 3 – 98 = 3; day = 2 + 14 + 3 – 14 = 5 = Friday.

36. July 4, 1776 (Signing of the Declaration of Independence):
year code = 76 + 19 + 5 – 98 = 2; day = 5 + 4 + 2 – 7 = 4 = Thursday.

37. April 15, 1707 (Birthday of Leonhard Euler):
year code = 7 + 1 + 5 – 7 = 6; day = 5 + 15 + 6 – 21 = 5 = Friday.

38. April 23, 1616 (Death of Miguel Cervantes):
year code = 16 + 4 – 14 = 6; day = 5 + 23 + 6 – 28 = 6 = Saturday.

39. Explain why the calendar repeats itself every 400 years in the Gregorian calendar. (Hint: how many leap years will occur in a 400-year period?)

In a 400-year period, the number of leap years is 100 – 3 = 97. (Recall that in the next 400 years, 2100, 2200, and 2300 are not leap years, but 2400 is a leap year.) Hence, the calendar will shift 400 times (once for each year) plus 97 more times (for each leap year), for a total of 497 shifts. Because 497 is a multiple of 7 (= 7 × 71), the day of the week will be the same.

40. Determine the day of the week of January 1, 2100.

This day will be the same as January 1, 1700 (not a leap year), which has year code 5; hence, the day of the week will be 6 + 1 + 5 – 7 = 5 = Friday; this is consistent with our earlier calculation that December 31, 2099 is a Thursday.

41. William Shakespeare and Miguel Cervantes both died on April 23, 1616, yet their deaths were 10 days apart. How can that be?

Cervantes was from Spain, which adopted the Gregorian calendar. England, Shakespeare's home, was still on the Julian calendar, which was 10 days "behind" the Gregorian calendar. When Shakespeare died on the Julian date of April 23, 1616, the Gregorian date was May 3, 1616.

Lecture 11

Calculate the following 3-digit squares. Note that most of the 3-by-1 multiplications appear in the problems and solutions to Lecture 3, and most of the 2-digit squares appear in the problems and solutions to Lecture 7.

1. $107^2 = 100 \times 114 + 7^2 = 11{,}400 + 49 = 11{,}449$

2. $402^2 = 400 \times 404 + 2^2 = 161{,}600 + 4 = 161{,}604$

3. $213^2 = 200 \times 226 + 13^2 = 45{,}200 + 169 = 45{,}369$

4. $996^2 = 1000 \times 992 + 4^2 = 992{,}000 + 16 = 992{,}016$

5. $396^2 = 400 \times 392 + 4^2 = 156{,}800 + 16 = 156{,}816$

6. $411^2 = 400 \times 422 + 11^2 = 168{,}800 + 121 = 168{,}921$

7. $155^2 = 200 \times 110 + 45^2 = 22{,}000 + 2025 = 24{,}025$

8. $509^2 = 500 \times 518 + 9^2 = 259{,}000 + 81 = 259{,}081$

9. $320^2 = 300 \times 340 + 20^2 = 102{,}000 + 400 + 102{,}400$

10. $625^2 = 600 \times 650 + 25^2 = 390{,}000 + 625 = 390{,}625$

11. $235^2 = 200 \times 270 + 35^2 = 54{,}000 + 1{,}225 = 55{,}225$

12. $753^2 = 800 \times 706 + 47^2 = 564{,}800 + 2{,}209 = 567{,}009$

13. $181^2 = 200 \times 162 + 19^2 = 32{,}400 + 361 = 32{,}761$

14. $477^2 = 500 \times 454 + 23^2 = 227{,}000 + 529 = 227{,}529$

15. $682^2 = 700 \times 664 + 18^2 = 464{,}800 + 324 = 465{,}124$

16. $236^2 = 200 \times 272 + 36^2 = 54{,}400 + 1{,}296 = 55{,}696$

17. $431^2 = 400 \times 462 + 31^2 = 184{,}800 + 961 = 185{,}761$

Compute these 4-digit squares. Note that all of the required 3-digit squares have been solved in the exercises above. After the first multiplication, you can usually say the millions digit; the displayed word is the phonetic representation of the underlined number. Also, some of these calculations require 4-by-1 multiplications; these are indicated after the solution.

18. $3016^2 = 3000 \times 3032 + 16^2 = 9{,}096{,}000 + 256 = 9{,}096{,}256$

(Note: $3 \times 3032 = 3 \times 3000 + 3 \times 32 = 9000 + 96 = 9096$)

19. $1235^2 = 1000 \times 1470 + 235^2 = 1{,}\underline{470}{,}000$ (ROCKS) + $55{,}\underline{225}$ (NO NAIL) = 1,525,225

20. $1845^2 = 2000 \times 1690 + 155^2 = 3{,}\underline{380}{,}000$ (MOVIES) + $24{,}\underline{025}$ (SNAIL) = 3,404,025

(Note: $2 \times 169 = 200 + 120 + 18 = 320 + 18 = 338$, so $2 \times 1690 = 3{,}380$. Note also that the number 1690 can be found by doubling 1845, giving 3690, which splits into 2000 and 1690.)

21. $2598^2 = 3000 \times 2196 + 402^2 = 6,\underline{588},000$ (LOVE OFF) $+ 161,\underline{604}$ (CHASER) $= 6,749,604$

(Note: $3 \times 2196 = 3 \times 2000 + 3 \times 196 = 6000 + (300 + 270 + 18) = 6000 + (570 + 18) = 6588$. Note also that $2598 \times 2 = 5196 = 3000 + 2196$.)

22. $4764^2 = 5000 \times 4528 + 236^2 = 22,\underline{640},000$ (CHAIRS) $+ 55,\underline{696}$ (SHEEPISH) $= 22,695,696$

(Note: $5 \times 4528 = 5 \times 4500 + 5 \times 28 = 22,500 + 140 = 22,640$. Note also that $4764 \times 2 = 9528 = 5000 + 4528$.)

Raise these two-digit numbers to the 4th power by squaring the number twice.

23. $20^4 = 400^2 = 160,000$

24. $12^4 = 144^2 = 100 \times 188 + 44^2 = 18,800 + 1,936 = 20,736$

25. $32^4 = 1024^2 = 1000 \times 1048 + 24^2 = 1,048,000 + 576 = 1,048,576$

26. $55^4 = 3025^2 = 3000 \times 3050 + 25^2 = 9,150,000 + 625 = 9,150,625$

27. $71^4 = 5041^2 = 5000 \times 5082 + 41^2 = 25,\underline{410},000$ (ROADS) $+ 1,\underline{681}$ (SHIFT) $= 25,411,681$

28. $87^4 = 7569^2 = 8000 \times 7138 + 431^2 = 57,\underline{104},000$ (TEASER) $+ 185,\underline{761}$ (CASHED) $= 57, 289,761$

(Note: $8 \times 7138 = 8 \times 7100 + 8 \times 38 = 56,800 + 304 = 57,104$. Also note that the number 7138 can be obtained by doing $7569 \times 2 = 15,138$ so that the numbers being multiplied are 8000 and 7138.)

29. $98^4 = 9604^2 = 10,000 \times 9208 + 396^2 = 92,\underline{080},000$ (SAVES) $+ 156,\underline{816}$ (FOOTAGE) $= 92,236,816$

(Note: $9604 \times 2 = 19,208 = 10,000 + 9208$.)

Compute the following 3-digit-by-2-digit multiplication problems. Note that many of the 3-by-1 calculations appear in the solutions to Lecture 3, and many of the 2-by-2 calculations appear in the solutions to Lecture 7.

30. $864 \times 20 = 17{,}280$

31. $772 \times 60 = 46{,}320$

32. $140 \times 23 = 23 \times 7 \times 2 \times 10 = 161 \times 2 \times 10 = 322 \times 10 = 3220$

33. $450 \times 56 = 450 \times 8 \times 7 = 3600 \times 7 = 25{,}200$

34. $860 \times 84 = 86 \times 84 \times 10 = 7224 \times 10 = 72{,}240$

35. $345 \times 12 = 345 \times 6 \times 2 = 2070 \times 2 = 4140$

36. $456 \times 18 = 456 \times 6 \times 3 = 2736 \times 3 = 8100 + 108 = 8208$

37. $599 \times 74 = (600 - 1) \times 74 = 44{,}400 - 74 = 44{,}326$

38. $753 \times 56 = 753 \times 8 \times 7 = 6024 \times 7 = 42{,}000 + 168 = 42{,}168$

39. $624 \times 38 = 38 \times 104 \times 6 = (3800 + 152) \times 6 = 3952 \times 6 = 23{,}400 + 312 = 23{,}712$

40. $349 \times 97 = 349 \times (100 - 3) = 34{,}900 - 1047 = 33{,}853$

41. $477 \times 71 = (71 \times 400) + (71 \times 77) = 28{,}400 + 5467 = 33{,}867$

42. $181 \times 86 = (100 \times 86) + (81 \times 86) = 8600 + 6966 = 15{,}566$

43. $224 \times 68 = 68 \times 8 \times 7 \times 4 = 544 \times 7 \times 4 = 3808 \times 4 = 15{,}232$

44. $241 \times 13 = (13 \times 24 \times 10) + (13 \times 1) = 3120 + 13 = 3133$

45. $223 \times 53 = (22 \times 53 \times 10) + (3 \times 53) = 11{,}660 + 159 = 11{,}819$

46. $682 \times 82 = 600 \times 82 + 82^2 = 49{,}200 + 6724 = 55{,}924$

Estimate the following 2-digit cubes.

47. $27^3 \approx 30 \times 30 \times 21 = 30 \times 630 = 18{,}900$

48. $51^3 \approx 50 \times 50 \times 53 = 50 \times 2650 = 132{,}500$

49. $72^3 \approx 70 \times 70 \times 76 = 70 \times 5320 = 372{,}400$

50. $99^3 \approx 100 \times 100 \times 97 = 970{,}000$

51. $66^3 \approx 70 \times 70 \times 58 = 70 \times 4060 = 284{,}200$

BONUS MATERIAL: We can also compute the exact value of a cube with only a little more effort. For example, to cube 42, we use $z = 40$ and $d = 2$. The approximate cube is $40 \times 40 \times 46 = 73{,}600$. To get the exact cube, we can use the following algebra: $(z + d)^3 = z(z(z + 3d) + 3d^2) + d^3$. First, we do $z(z + 3d) + 3d^2 = 40 \times 46 + 12 = 1852$. Then, we multiply this number by z again: $1852 \times 40 = 74{,}080$. Finally, we add $d^3 = 2^3 = 8$ to get 74,088.

Notice that when cubing a 2-digit number, in our first addition step, the value of $3d^2$ can be one of only five numbers: 3, 12, 27, 48, or 75. Specifically, if the number ends in 1 (so $d = 1$) or ends in 9 (so $d = -1$), then $3d^2 = 3$. Similarly, if the last digit is 2 or 8, we add 12; if it's 3 or 7, we add 27; if it's 4 or 6, we add 48; if it's 5, we add 75. Then, in the last step, we will always add or subtract one of five numbers, based on d^3. Here's the pattern:

If last digit is…	1	2	3	4	5	6	7	8	9
Adjust by…	+1	+8	+27	+64	+125	–64	–27	–8	–1

For example, what is the cube of 96? Here, $z = 100$ and $d = -4$. The approximate cube would be $100 \times 100 \times 88 = 880{,}000$. For the exact cube, we first do $100 \times 88 + 48 = 8848$. Then we multiply by 100 and subtract 64: $8848 \times 100 - 64 = 884{,}800 - 64 = 884{,}736$.

Using these examples as a guide, compute the exact values of the following cubes.

52. $13^3 = (10 \times 19 + 27) \times 10 + 3^3 = 2170 + 27 = 2197$

53. $19^3 = (20 \times 17 + 3) \times 20 + (-1)^3 = 343 \times 20 - 1 = 6859$

54. $25^3 = (20 \times 35 + 75) \times 20 + 5^3 = 775 \times 20 + 125 = 15{,}500 + 125 = 15{,}625$

55. $59^3 = (60 \times 57 + 3) \times 60 + (-1)^3 = 3423 \times 60 - 1 = 205{,}379$

(Note: $3423 \times 6 = 3400 \times 6 + 23 \times 6 = 20{,}400 + 138 = 20{,}538$)

56. $72^3 = (70 \times 76 + 12) \times 70 + 2^3 = 5332 \times 70 + 8 = 373{,}248$

(Note: $5332 \times 7 = 5300 \times 7 + 32 \times 7 = 37{,}100 + 224 = 37{,}324$)

Lecture 12

We begin this section with a sample of review problems. Most likely, these problems would have been extremely hard for you to do before this course began, but I hope that now they won't seem so bad.

1. If an item costs \$36.78, how much change would you get from \$100?

Because the dollars sum to 99 and the cents sum to 100, the change is \$63.22.

2. Do the mental subtraction problem: 1618 – 789.

1618 – 789 = 1618 – (800 – 11) – 818 + 11 = 829

Do the following multiplication problems.

3. 13 × 18 = (13 + 8) × 10 + (3 × 8) = 210 + 24 = 234

4. 65 × 65 = 60 × 70 + 5^2 = 4200 + 25 = 4225

5. 997 × 996 = (1000 × 993) + (–3) × (–4) = 993,012

6. Is the number 72,534 a multiple of 11?

Yes, because 7 – 2 + 5 – 3 + 4 = 11.

7. What is the remainder when you divide 72,534 by a multiple of 9?

Because 7 + 2 + 5 + 3 + 4 = 21, which sums to 3, the remainder is 3.

8. Determine 23/7 to 6 decimal places.

23/7 = 3 2/7 = 3.285714 (repeated)

9. If you multiply a 5-digit number beginning with 5 by a 6-digit number beginning with 6, then how many digits will be in the answer?

Just from the number of digits in the problem, you know the answer must be either 11 digits or 10 digits. Then, because the product of the initial digits in this particular problem (5 × 6 = 30), is more than 10, the answer is definitely the longer of the two choices, in this case 11 digits.

10. Estimate the square root of 70.

70 ÷ 8 = 8 3/4 = 8.75. Averaging 8 and 8.75 gives us an estimate of 8.37.

(Exact answer begins 8.366… .)

Do the following problems on paper and just write down the answer.

11. 509 × 325 = 165,425 (by criss-cross method).

12. 21,401 ÷ 9: Using the Vedic method, we get 2 3 7 7 R 8.

13. 34,567 ÷ 89: Using the Vedic method, with divisor 9 and multiplier 1, we get:

$$\begin{array}{r|l} & \;\;3\;\;8\;\;8 \quad R\,35 \\ \hline 89 & 3\;4_7 5_6 6_2 7 \\ +1 & \\ 90 & \end{array}$$

14. Use the phonetic code to memorize the following chemical elements: Aluminum is the 13th element, copper is the 29th element, and lead is the 82nd element.

Aluminum = 13 = DIME or TOMB. An aluminum can filled with DIMEs or maybe a TOMBstone that was “Aluminated”?

Copper = 29 = KNOB or NAP. A doorKNOB made of copper or a COP taking a NAP.

Lead = 82 = VAN or FUN. A VAN filled with lead pipes or maybe being “lead” to a FUN event.

15. What day of the week was March 17, 2000? Day = 2 + 17 + 0 – 14 = 5 = Friday.

16. Compute $212^2 = 200 \times 224 + 12^2 = 44{,}800 + 144 = 44{,}944$

17. Why must the cube root of a 4-, 5-, or 6-digit number be a 2-digit number?

The largest 1-digit cube is $9^3 = 729$, which has 3 digits, and a 3-digit cube must be at least $100^3 = 1{,}000{,}000$, which has 7 digits.

Find the cube roots of the following numbers.

18. 12,167 has cube root 23.

19. 357,911 has cube root 71.

20. 175,616 has cube root 56.

21. 205,379 has cube root 59.

The next few problems will allow us to find the cube root when the original number is the cube of a 3-digit number. We'll first build up some ideas to find the cube root of 17,173,512, which is the cube of a 3-digit number.

22. Why must the first digit of the answer be 2?

$200^3 = 8{,}000{,}000$ and $300^3 = 27{,}000{,}000$, so the answer must be in the 200s.

23. Why must the last digit of the answer be 8?

Because 8 is the only digit that, when cubed, ends in 2.

24. How can we quickly tell that 17,173,512 is a multiple of 9?

By adding its digits, which sum to 27, a multiple of 9.

25. It follows that the 3-digit number must be a multiple of 3 (because if the 3-digit number was not a multiple of 3, then its cube could not be a multiple of 9). What middle digits would result in the number 2_8 being a multiple of 3? There are three possibilities.

For 2_8 to be a multiple of 3, its digits must sum to a multiple of 3. This works only when the middle number is 2, 5, or 8 because the digit sums of 228, 258, and 288 are 12, 15, and 18, respectively.

26. Use estimation to choose which of the three possibilities is most reasonable.

Since 17,000,000 is nearly halfway between 8,000,000 and 27,000,000, the middle choice, 258, seems most reasonable. Indeed, if we approximate the cube of 26 as 30 × 30 × 22 = 19,800, we get 260^3, which is about 20 million, consistent with our answer.

Using the steps above, we can do cube roots of any 3-digit cubes. The first digit can be determined by looking at the millions digits (the numbers before the first comma); the last digit can be determined by looking at the last digit of the cube; the middle digit can be determined through digit sums and estimation. There will always be three or four possibilities for the middle digit; they can be determined using the following observations, which you should verify.

27. Verify that if the digit sum of a number is 3, 6, or 9, then its cube will have digit sum 9.

If the digit sum is 3, 6, or 9, then the number is a multiple of 3, which when cubed will be a multiple of 9; thus, its digits will sum to 9.

28. Verify that if the digit sum of a number is 1, 4, or 7, then its cube will have digit sum 1.

A number with digit sum 1, when cubed, will have a digit sum that can be reduced to $1^3 = 1$. Likewise, $4^3 = 64$ reduces to 1 and $7^3 = 343$ reduces to 1.

29. Verify that if the digit sum of a number is 2, 5, or 8, then its cube will have digit sum 8.

Similarly, a number with digit sum 2, 5, or 8, when cubed, will have the same digit sum as $2^3 = 8$, $5^3 = 125$, and $8^3 = 512$, respectively, all of which have digit sum 8.

Using these ideas, determine the 3-digit number that produces the cubes below.

30. Find the cube root of 212,776,173.

Since $5^3 < 212 < 6^3$, the first digit is 5, and since 7^3 ends in 3, the last digit is 7. Thus, the answer looks like 5_7. The digit sum of 212,776,173 is 36, which is a multiple of 9, so the number 5_7 must be a multiple of 3. Hence, the middle digit must be 0, 3, 6, or 9 (because the digit sums of 507, 537, 567, and 597 are all multiples of 3). Given that 212,000,000 is so close to 600^3 (= 216,000,000), we pick the largest choice: 597.

31. Find the cube root of 374,805,361.

Since $7^3 < 374 < 8^3$, the first digit is 7, and since only 1^3 ends in 1, the last digit is 1. Thus, the answer looks like 7_1. The digit sum of 74,805,361 is 37, which has digit sum 1; by our previous observation, 7_1 must have a digit sum that reduces to 1, 4, or 7. Hence, the middle digit must be 2, 5, or 8 (because 721, 751, and 781 have digit sums 10, 13, and 16, which reduce to 1, 4, and 7, respectively). Given that 374 is much closer to 343 than it is to 512, we choose the smallest possibility, 721. To be on the safe side, we estimate 72^3 as $70 \times 70 \times 76 = 372{,}400$, which means that 720^3 is about 372,000,000; thus, the answer 721 must be correct.

32. Find the cube root of 4,410,944.

Here, $1^3 < 4 < 2^3$, so the first digit is 1, and (by examining the last digit) the last digit must be 4. Hence, the answer looks like 1_4. The digit sum of 4,410,944 is 26, which reduces to 8, so 1_4 must reduce to 2, 5, or 8. Thus, the middle digit must be 0, 3, 6, or 9. Given that 4 is comfortably between 1^3 and 2^3, it must be 134 or 164. Since 163 = 16 × 16 × 16 = 256 × 8 × 2 = 2048 × 2 = 4096, we choose the answer 164.

Compute the following 5-digit squares in your head! (Note that the necessary 2-by-3 and 3-digit square calculations were given in the solutions to Lecture 11.)

33. $11{,}235^2$

11 × 235 × 2 = 2585 × 2 = 5,170. So 11,000 × 235 × 2 = 5,170,000.

We can hold the 5 on our fingers and turn 170 into DUCKS. $11{,}000^2$ = 121,000,000, which when added to 5 million gives us 126 million, which we can say. Next, we have 235^2 = 55,225, which when added to 170,000, gives us the rest of the answer: 225,225. Final answer = 126,225,225.

34. $56{,}753^2$

56,000 × 753 × 2 = 56 × 753 × 2 × 1000 = 42,168 × 2 × 1000 = 84,336,000 = FIRE, MY MATCH.

$56{,}000^2$ = 3,136,000,000, so we can say "3 billion." After adding 136 to 84 (FIRE), we can say "220 million." Then, 753^2 = 567,009, which when added to 336,000 (MY MATCH) gives the rest of the answer, 903,009. Final answer = 3,220,903,009.

35. $82{,}682^2$

82,000 × 682 × 2 = 82 × 682 × 2 × 1000 = 55,924 × 2 × 1000 = 111,848 = DOTTED, VERIFY.

$82{,}000^2$ = 6,724,000,000, so we can say "6 billion," then add 724 to 111 (DOTTED) to get 835 million, but because we see a carry coming (from 848,000 + 682^2), we say "836 million." Next, 682^2 = 465,124 (turning 124 into TENOR, if helpful). Now, 465,000 + 848,000 (VERIFY) = 1,313,000, but we have already taken care of the leading 1, so we can say "313 thousand," followed by (TENOR) 124. Final answer = 6,836,313,124.