Mental Math and Paper

Lecture 6

Even if you haven't been balancing your checkbook, you might now want to start. It's a great way to become more comfortable with numbers, and you'll understand exactly what's happening with your money!

In this lecture, we'll learn some techniques to speed up calculations done on paper, along with some interesting ways to check our answers. When doing problems on paper, it's usually best to perform the calculations from right to left, as we were taught in school. It's also helpful to say the running total as you go. To check your addition, add the numbers again, from bottom to top.

When doing subtraction on paper, we can make use of complements. Imagine balancing your checkbook; you start with a balance of \$1235.79, from which you need to subtract \$271.82. First, subtract the cents: 79 – 82. If the problem were 82 – 79, the answer would be 3 cents, but since it's 79 – 82, we take the complement of 3 cents, which is 97 cents. Next, we need to subtract 272, which we do by subtracting 300 (1235 – 300 = 935), then adding back its complement (28): 935 + 28 = 963. The new balance, then, is \$963.97. We can check our work by turning the original subtraction problem into an addition problem.

Cross-multiplication is a fun way to multiply numbers of any length. This method is really just the distributive law at work. For example, the problem 23 × 58 is (20 + 3) × (50 + 8), which has four separate components: 20 × 50, 20 × 8, 3 × 50, and 3 × 8. The 3 × 8 we do at the beginning. The 20 × 50 we do at the end, and the 20 × 8 and 3 × 50 we do in **criss-cross** steps. If we extend this logic, we can do 3-by-3 multiplication or even higher. This method was first described in the book *Liber Abaci*, written in 1202 by Leonardo of Pisa, also known as Fibonacci.

The digit-sum check can be used to check the answer to a multiplication problem. Let's try the problem 314 × 159 = 49,926. We first sum the digits of the answer: 4 + 9 + 9 + 2 + 6 = 30. We reduce 30 to a 2-digit number by adding its digits: 3 + 0 = 3. Thus, the answer reduces to the number 3. Now, we reduce the original numbers: 314 → 3 + 1 + 4 = 8 and 159 → 1 + 5 + 9 = 15, which reduces to 1 + 5 = 6. Multiply the reduced numbers, 8 × 6 = 48, then reduce that number: 4 + 8 = 12, which reduces to 1 + 2 = 3. The reduced numbers for both the answer and the problem match. If all the calculations are correct, then these numbers must match. Note that a match does not mean that your answer is correct, but if the numbers don't match, then you've definitely made a calculation error.

Casting out nines also works for addition and subtraction problems, even those with decimals, and it may be useful for eliminating answers on standardized tests that do not allow calculators.

This method is also known as **casting out nines**, because when you reduce a number by summing its digits, the number you end up with is its remainder when divided by 9. For example, if we add the digits of 67, we get 13, and the digits of 13 add up to 4. If we take 67 ÷ 9, we get 7 with a remainder of 4. Casting out nines also works for addition and subtraction problems, even those with decimals, and it may be useful for eliminating answers on standardized tests that do not allow calculators.

The number 9, because of its simple multiplication table, its divisibility test, and the casting-out-nines process, seems almost magical. In fact, there's even a magical way to divide numbers by 9, using a process called Vedic division. This process is similar to the technique we learned for multiplying by 11 in Lecture 1, because dividing by 9 is the same as multiplying by 0.111111.

The **close-together method** can be used to multiply any two numbers that are near each other. Consider the problem 107 × 111. First, we note how far each number is from 100: 7 and 11. We then add either 107 + 11 or 111 + 7, both of which sum to 118. Next, we multiply 7 × 11, which is 77. Write the numbers down, and that's the answer: 11,877. The algebraic formula for this technique is $(z + a)(z + b) = (z + a + b)z + ab$, where typically, z is an easy number with zeros in it (such as $z = 100$ or $z = 10$) and a and b are the distances from the easy number. This technique also works for numbers below 100, but here, we use negative numbers for the distances from 100. Once you know how to do the close-together method on paper, it's not difficult to do it mentally; we'll try that in the next lecture. ■

Important Terms

casting out nines (also known as the method of digit sums): A method of verifying an addition, subtraction, or multiplication problem by reducing each number in the problem to a 1-digit number obtained by adding the digits. For example, 67 sums to 13, which sums to 4, and 83 sums to 11, which sums to 2. When verifying that $67 + 83 = 150$, we see that 150 sums to 6, which is consistent with $4 + 2 = 6$. When verifying $67 \times 83 = 5561$, we see that 5561 sums to 17 which sums to 8, which is consistent with $4 \times 2 = 8$.

close-together method: A method for multiplying two numbers that are close together. When the close-together method is applied to 23×26, we calculate $(20 \times 29) + (3 \times 6) = 580 + 18 = 598$.

criss-cross method: A quick method for multiplying numbers on paper. The answer is written from right to left, and nothing else is written down.

Suggested Reading

Benjamin and Shermer, *Secrets of Mental Math: The Mathemagician's Guide to Lightning Calculation and Amazing Math Tricks*, chapter 6.

Cutler and McShane, *The Trachtenberg Speed System of Basic Mathematics.*

Flansburg and Hay, *Math Magic: The Human Calculator Shows How to Master Everyday Math Problems in Seconds.*

Handley, *Speed Mathematics: Secrets of Lightning Mental Calculation.*

Julius, *More Rapid Math Tricks and Tips: 30 Days to Number Mastery.*

———, *Rapid Math Tricks and Tips: 30 Days to Number Power.*

Kelly, *Short-Cut Math.*