The next easiest situation is when the 3-digit number can be factored into a 2-digit number × a 1-digit number. For instance, with 47 × 126, 47 is prime, but 126 is 63 × 2; we can multiply 47 × 63, then double that result. For the most difficult problems, we can break the 3-digit number into two parts and apply the distributive law. For a problem like 47 × 283, we multiply 47 × 280 and add 47 × 3.

In our last lecture, we'll see what you can achieve if you become seriously dedicated to calculation, and we'll consider broader benefits from what we've learned that are available to everyone. ■

Suggested Reading

Benjamin and Shermer, *Secrets of Mental Math: The Mathemagician's Guide to Lightning Calculation and Amazing Math Tricks*, chapter 8.

Doerfler, *Dead Reckoning: Calculating Without Instruments.*

Lane, *Mind Games: Amazing Mental Arithmetic Tricks Made Easy.*

Problems

Calculate the following 3-digit squares.

1. 107^2
2. 402^2
3. 213^2
4. 996^2
5. 396^2
6. 411^2
7. 155^2

8. 509^2

9. 320^2

10. 625^2

11. 235^2

12. 753^2

13. 181^2

14. 477^2

15. 682^2

16. 236^2

17. 431^2

Compute these 4-digit squares.

18. 3016^2

19. 1235^2

20. 1845^2

21. 2598^2

22. 4764^2

Raise these 2-digit numbers to the 4^{th} power by squaring the number twice.

23. 20^4

24. 12^4

25. 32^4

26. 55^4

27. 71^4

28. 87^4

29. 98^4

Compute the following 3-digit-by-2-digit multiplication problems.

30. 864 × 20

31. 772 × 60

32. 140 × 23

33. 450 × 56

34. 860 × 84

35. 345 × 12

36. 456 × 18

37. 599 × 74

38. 753 × 56

39. 624 × 38

40. 349 × 97

41. 477 × 71

42. 181 × 86

43. 224×68

44. 241×13

45. 223×53

46. 682×82

Estimate the following 2-digit cubes.

47. 27^3

48. 51^3

49. 72^3

50. 99^3

51. 66^3

BONUS MATERIAL: We can also compute the exact value of a cube with only a little more effort. For example, to cube 42, we use $z = 40$ and $d = 2$. The approximate cube is $40 \times 40 \times 46 = 73{,}600$. To get the exact cube, we can use the following algebra: $(z + d)^3 = z(z(z + 3d) + 3d^2) + d^3$. First, we do $z(z + 3d) + 3d^2 = 40 \times 46 + 12 = 1852$. Then, we multiply this number by z again: $1852 \times 40 = 74{,}080$. Finally, we add $d^3 = 2^3 = 8$ to get 74,088.

Notice that when cubing a 2-digit number, in our first addition step, the value of $3d^2$ can be one of only five numbers: 3, 12, 27, 48, or 75. Specifically, if the number ends in 1 (so $d = 1$) or ends in 9 (so $d = -1$), then $3d^2 = 3$. Similarly, if the last digit is 2 or 8, we add 12; if it's 3 or 7, we add 27; if it's 4 or 6, we add 48; if it's 5, we add 75. Then, in the last step, we will always add or subtract one of five numbers, based on d^3. Here's the pattern:

If last digit is…	1	2	3	4	5	6	7	8	9
Adjust by…	+1	+8	+27	+64	+125	–64	–27	–8	–1

For example, what is the cube of 96? Here, $z = 100$ and $d = -4$. The approximate cube would be $100 \times 100 \times 88 = 880{,}000$. For the exact cube, we first do $100 \times 88 + 48 = 8848$. Then we multiply by 100 and subtract 64: $8848 \times 100 - 64 = 884{,}800 - 64 = 884{,}736$.

Using these examples as a guide, compute the exact values of the following cubes.

52. 13^3

53. 19^3

54. 25^3

55. 59^3

56. 72^3

Solutions for this lecture begin on page 137.