The Wave Equation, Wave Functions and Applications

A Mathematical Physics Project

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Contents

Abstract:

Wave phenomena is a prominent experience in our daily lives. The sun sends waves of electromagnetic radiation which carries energy to our planet driving life as we know it. The music we enjoy is because of wave phenomena. In fact, all matter in this universe can be explained as behaving as particles and waves. This property of matter is known as wave particle duality. In this project I will use the derivations and problems from *University Physics with Modern Physics* and *Mathematical Methods in the Physical Sciences* to show how the wave equation will be derived using partial differential equations. Solutions to the Wave Equation will be used to create wave functions with the help of the mentioned text books for mechanical and electromagnetic waves as well as applications in quantum mechanics. Using these properties, the speed of sound and light care calculated. Each topic will be followed by known worked examples provided by *University Physics with Modern Physics*. In effort to show the fundamentals of wave phenomena and their applications simplistic models will be used. A conclusion has been written to show how the mentioned topics are related.

Derivation of the wave equation:

To first arrive at the wave equation, we will look at a very simple model. This model has certain conditions to make the derivation as simple as possible. The first limitation is that there is no dissipation in the wave. This means there is no friction force that is dampening the wave. Next the wave keeps the same shape and speed so that there is no dispersion. For the first part of the derivation we will look at the wave moving in the positive x-axis with two dimensions, that of distance and time. The reason for these limiting factors is so we know the dimensions, shape and speed of the wave when it is shifted in any direction.

At time $t = 0$ the shape of the wave is described by a function $F(x, 0) = S(x)$.

After a certain amount of time the wave moves along the +x-axis a unit of distance $l = x - vt$ where v is the speed of which the wave travelled so that:

$$F(x,t) = S(x - vt)$$

Next, we take partial derivatives with respect to position and time:

$$\frac{\partial F}{\partial x} = S'(x - vt) \qquad \frac{\partial^2 F}{\partial x^2} = S"(x - vt)$$

$$\frac{\partial F}{\partial t} = vS'(x - vt) \qquad \frac{\partial^2 F}{\partial t^2} = v^2 S"(x - vt)$$

Using the second partial derivative with respect to time on the left and the second partial derivative with respect to position yields a general form of the wave equation with dimensions of distance in one direction and time:

$$\frac{\partial^2 F}{\partial t^2} = v^2 \frac{\partial^2 S}{\partial x^2}$$

We do not need to limit ourselves to only one direction. A change in the opposite direction will yield the same equation. In fact, from this point it is easy to extend this equation to three-dimensional space.

$$\frac{\partial^2 F}{\partial t^2} = v^2 \left[\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2}\right]$$

That was the easy part. Now we want to use the wave equation to create a wave function for further calculations. For the next step we will revert to only two dimensions, that of one direction and time.

Solutions to the Wave Equation:

To find solutions to the wave equation and apply those solutions to real world problems first we must use deal with second order partial differential equations.

First rearranging terms, we have:

$$\frac{\partial^2 F}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 S}{\partial t^2}$$

We now define our function as two functions. One with respect to distance and the other time:

$$F = S = X(x)T(t)$$

The wave equation then becomes:

$$\frac{\partial^2 (XT)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 (XT)}{\partial t^2}$$

Since these are partial derivatives, we can pull out the constant term. This also allows us to use standard derivative notation instead of partial derivative notation:

$$T\frac{d^2 X}{dx} = \frac{1}{v^2}X\frac{d^2 T}{dt^2}$$

Dividing both sides by $\frac{1}{XT}$

$$\frac{1}{X}\frac{d^2 X}{dx} = \frac{1}{v^2}\frac{1}{T}\frac{d^2 T}{dt^2}$$

Keeping one term constant would result in no change in the other terms so both terms are equal to some constant. We choose $-k^2$ as our constant.

$$\frac{1}{X}\frac{d^2 X}{dx} = \frac{1}{v^2}\frac{1}{T}\frac{d^2 T}{dt^2} = -k^2$$

We can now solve for our two functions $X(x)$ and $T(t)$ separately:

$$\frac{1}{X}\frac{d^2 X}{dx} = -k^2 \quad \frac{1}{v^2}\frac{1}{T}\frac{d^2 T}{dt^2} = -k^2$$

First, we will find solutions for the function $X(x)$

$$\frac{1}{X}\frac{d^2 X}{dx} = -k^2$$

Multiplying both sides by the function $X(x)$

$$\frac{d^2 X}{dx} = -k^2 X$$

Then rearranging terms

$$\frac{d^2X}{dx} + k^2X = 0$$

Now we can create a characteristic equation using λ:

$$\lambda^2 + k^2 = 0$$

Solving for λ:

$$\lambda^2 = -k^2$$

$$\lambda = \pm\sqrt{-k^2} = \pm ki = 0 \pm ki$$

We now have two roots to our function $X(x)$ in the form $e^{\alpha}Cos(\mathrm{B})$ where $\alpha = 0$ and $\mathrm{B} = k$ our solutions become:

$$X(x) = \begin{cases} e^0 \cos(kx) \\ e^0 \sin(kx) \end{cases} = \begin{cases} \cos(kx) \\ \sin(kx) \end{cases}$$

Using the same technique for the function $T(t)$ we have the following roots:

$$\lambda = 0 \pm kvi$$

This gives the solutions for $T(t)$ noting that angular velocity is $\omega = kv$:

$$T(t) = \begin{cases} e^0 \cos(kvt) \\ e^0 \sin(kvt) \end{cases} = \begin{cases} \cos(kvt) \\ \sin(kvt) \end{cases}$$

Our final function is then:

$$F(x,t) = X(x)T(t) = Sin(kx)Cos(kvt)$$

For simplicities sake we will apply boundaries stating that both ends of the wave are fixed at the origin to length L. We want to satisfy the condition where $x = L$ and $F = 0$

This means that $\sin(kx) = 0$ and this happens at intervals of $n\pi$, so it follows that:

$$kL = n\pi \text{ or } k = \tfrac{n\pi}{L} \quad \text{and} \quad \omega = kv = \tfrac{n\pi v}{L}$$

(Where ω is the angular velocity and k is a constant of proportionality.)

This yields two possible solutions:

$$F(x,t) = F(x,t) = \sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi vt}{L}\right)$$

substituting $\omega = kv = \tfrac{n\pi v}{L}$

$$F(x,t) = \sin(kx)\cos(\omega t)$$

We can say here that at any position x the sine term containing x will be a constant and dictates the maximum amplitude of the wave notated as A. We can also find the displacement of x at any time t by shifting by a unit of time x/v. This makes t become $t - x/v$. The $\sin(kx)$ is constant at any point x denoted as the amplitude of the wave $\sin(kx) = A$.

Changing our function from $F(x,t)$ to $y(x,t)$ to make it more familiar we arrive at:

$$y(x,t) = Acos(\omega\left(t - \frac{x}{v}\right))$$

Since cosine is an even term we can rewrite as:

$$y(x,t) = Acos\left(\omega\left(\frac{x}{v} - t\right)\right) = Acos(kx - \omega t)$$

This is the wave function of a sinusoidal wave propagating in the positive x-direction derived from the wave equation. This is the wave function used frequently in physics texts. Now we can use this function to look further into applications of the wave equation and wave functions.

Wave Speed:

Since we know our wave function satisfies the wave equation, we can find the speed of the waves we modelled by taking the second partial derivatives of the wave function and solving for v.

$$\frac{\partial y}{\partial t} = -\omega Asin(kx - \omega t) \qquad \frac{\partial^2 y}{\partial t^2} = -\omega^2 Acos(kx - \omega t)$$

$$\frac{\partial y}{\partial x} = -kAsin(kx - \omega t) \qquad \frac{\partial^2 y}{\partial x^2} = -k^2 Acos(kx - \omega t)$$

The wave equation then becomes:

$$-k^2 Acos(kx - \omega t) = \frac{1}{v^2}(-\omega^2 Acos(kx - \omega t))$$

The maximum value of cosine is one and the amplitude cancels. It follows then that:

$$-k^2 = -\frac{\omega^2}{v^2}$$

$$v = \sqrt{\frac{\omega^2}{k^2}} = \frac{\omega}{k}$$

At a period of 2π, $\omega = 2\pi f$ where f is the frequency and

$k = \frac{2\pi}{\lambda}$ where λ is the wavelength. This means wave speed is then:

$$v = \frac{2\pi f}{\frac{2\pi}{\lambda}} = \lambda f$$

This helps understand that the wave speed is the square root of a ratio of restoring force and resisting force that causes the wave to propagate as well as the product of wavelength and frequency. With that conclusion we can then look at wave speed through an ideal gas.

The Speed of Sound Through Air:

To calculate the speed of sound through air we must introduce a few more terms that we must take as true to continue. We must consider the ratio of heat capacities denoted γ, the gas constant $R = 8.314\,\frac{J}{mol*K}$, the temperature T in kelvin and the molar mass M of the atoms that make up air.

The ratio of heat capacities for air at room temperature $T = 293.15\ K$ is 1.40.

Keeping in mind that the speed of a wave is dependent on the ratio of restoring force and resisting force causing propagation we have the equation:

$$v = \sqrt{\frac{\gamma RT}{M}}$$

Plugging in these values we have:

$$v = \sqrt{\frac{(1.4)(8.314\,\frac{J}{mol * k})(293.15k)}{28.8(10^{-3})kg/mol}} = 344\ m/s$$

This is the known speed of sound.

Example Problem 1:

15.3 • Tsunami! On December 26, 2004, a great earthquake occurred off the coast of Sumatra and triggered immense waves (tsunami) that killed some 200,000 people. Satellites observing these waves from space measured 800 km from one wave crest to the next and a period between waves of 1.0 hour. What was the speed of these waves in m/s and in km/h? Does your answer help you understand why the waves caused such devastation?

Young, Hugh D., Roger A. Freedman, and A. Lewis Ford. *University Physics with Modern Physics*. 14th ed. Boston, MA: Pearson Learning Solutions, 2016.

The distance from one wave crest to another is also known as wavelength denoted λ. The period is given as one hour. With this information we can use a relationship we derived earlier to calculate the speed of the tsunami waves.

$$\lambda = 800km \qquad f = \frac{1}{T}$$

$$v = \lambda f = \frac{\lambda}{T} = \frac{800}{1}\frac{km}{hr} = 800\ km/hr \qquad v = \frac{800(10^3)}{3600}\frac{m}{s} = 220\ m/s$$

This tells us the waves are moving incredibly fast which is why it is difficult to have a warning system for tsunami waves.

Derivation of the Electromagnetic Wave Equation:

This derivation is like the derivation for the wave function. However, some new terms and conditions must be met.

First, we must imagine a plane wave. A plane wave is where a plane, in cartesian coordinates in three-space, is perpendicular to the x-axis. One side has no activity and the other side has a uniform magnetic field $\vec{M}$ in the +z-direction and a uniform electric field $\vec{E}$ in the +y-direction. The fields move with the plane in a direction with constant speed. A plane wave is a wave with uniform fields with a perpendicular plane to the direction of propagation.

Now consider letting the magnetic and electric fields vary in their respective directions denoted M_z and E_y along the x-axis. This means they are both a function of position on the x-axis and of time. Containing the field between two planes at x and $x + \Delta x$ means that the wave must satisfy the wave equation.

Wave Equation:

$$\frac{\partial^2 F}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 S}{\partial t^2}$$

Using Faraday's Law and Ampere's Law to show the relationship between magnetic and electric fields.

Faraday's Law:

$$\oint \vec{E}\, d\vec{L} = -Ea = -\frac{d\Phi_M}{dt}$$

Where a is constant of proportionality and Φ_M is magnetic flux.

Evaluating the integral from x to $x + \Delta x$ yields:

$$\oint_x^{x+\Delta x} \vec{E}\, d\vec{L} = -Ea = [E_y(x,t) - E_y(x + \Delta x, t)]a$$

Ampere's Law:

$$\oint \vec{M}\, d\vec{L} = Ma = \mu_0 \epsilon_0 \frac{d\Phi_E}{dt}$$

Where a is constant of proportionality and Φ_E is electric flux.

Evaluating the integral from of Ampere's Law from x to $x + \Delta x$ yields:

$$\oint_x^{x+\Delta x} \vec{M}\, d\vec{L} = Ma = [M_z(x + \Delta x, t) - M_z(x,t)]a$$

Now imagining Δx is very small makes the magnetic flux Φ_M take form:

$$\frac{d\Phi_M}{dt} = \frac{\partial M_z(x,t)}{\partial t} a\Delta x$$

Inserting this information into Faraday's Law yields a relationship between the derivative of magnetic flux and the electric field:

$$\frac{\partial M_z(x,t)}{\partial t} a\Delta x = -[E_y(x,t) - E_y(x+\Delta x,t)]a$$

Simplifying yields:

$$-\frac{d\Phi_M}{dt} = [E_y(x,t) - E_y(x+\Delta x,t)]/\Delta x$$

Allowing $\Delta x \to 0$ shows that:

$$-\frac{\partial M_z}{\partial t} = \frac{\partial E_y}{\partial x}$$

For every component of the magnetic field there must be a component of the electric field.

Now applying the same method using Ampere's Law:

$$-Ea = [E_y(x,t) - E_y(x+\Delta x,t)]a = -\frac{\partial E_y}{\partial t} a\Delta x$$

Plugging this information into Ampere's Law:

$$[M_z(x+\Delta x,t) - M_z(x,t)]a = -\mu_0 \epsilon_0 \frac{\partial E_y}{\partial t} a\Delta x$$

Simplifying and allowing $\Delta x \to 0$ yields:

$$\frac{\partial M_z}{\partial x} = \mu_0 \epsilon_0 \frac{\partial E_y}{\partial t}$$

Taking the partial derivatives of both sides with respect to x and then again with respect to t shows:

$$\frac{\partial^2 M_z}{\partial x^2} = \mu_0 \epsilon_0 \frac{\partial^2 E_y}{\partial t^2}$$

This takes the form of the wave equation:

$$\frac{\partial^2 F}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 S}{\partial t^2}$$

The Speed of Light:

From here we can see that the coefficient terms on the right side are equal:

$$\frac{1}{v^2} = \mu_0 \epsilon_0$$

Solving for wave speed we find:

$$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

Plugging in the known values of permittivity and permeability of space in a vacuum:

$$v = \frac{1}{\sqrt{(\frac{8.85(10^{-12})C^2}{Nm^2})(\frac{4\pi(10^{-17})N}{A^2})}} = 3(10^8)m/s$$

This is the known speed of light.

Example Problem 2:

32.1 • (a) How much time does it take light to travel from the moon to the earth, a distance of 384,000 km? (b) Light from the star Sirius takes 8.61 years to reach the earth. What is the distance from earth to Sirius in kilometers?

Young, Hugh D., Roger A. Freedman, and A. Lewis Ford. *University Physics with Modern Physics*. 14th ed. Boston, MA: Pearson Learning Solutions, 2016.

Using the relationship that velocity *v* is equal to the ratio of distance *d* and time *t*

$$d = 384{,}000\ km = 3.84(10^8)m \qquad v\frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3(10^8)m/s = \frac{d}{t}$$

$$t = \frac{d}{v} = \frac{3.84(10^6)}{3(10^8)}\frac{m}{\frac{m}{s}} = 1.28s$$

$$v = \frac{d}{t} \qquad d = vt$$

$$d = 3(10^8)m/s(\frac{10^{-3}km}{m})(8.61)years(3.154\left(\frac{10^7 seconds}{year}\right) = 8.15(10^{13})km$$

This solution tells us that it only takes light approximately 1.28 seconds to travel 384,000 km from the earth to the moon. If that isn't impressive enough, if we know how long it takes light to travel from the star Sirius we can calculate that it is approximately 81.5 trillion km from our planet!

One Dimensional Particle Waves:

For this section we are going to already assume the relationship proven that $v = \lambda f$. We must also introduce The de Broglie relationships which relate energy to angular velocity ω and momentum p to the wave number.

$$E = hf = \hbar\omega \qquad \text{where } \hbar = h/2\pi \text{ and } \omega = 2\pi f$$

$$p = \frac{h}{\lambda} = \hbar k \qquad \text{where } \hbar = h/2\pi \text{ and } \lambda = \frac{h}{p}$$

Taking these relationships into account and creating a model where we have a particle with no forces acting on it is moving in the +x-direction and neglecting potential energy, we substitute these new relationships into the kinetic energy equation.

$$E = \frac{1}{2}mv^2 = \frac{p^2}{2m} \qquad \text{where } p = mv$$

$$\hbar\omega = \frac{\hbar^2 k^2}{2m}$$

Now for this section we are assuming a wave function. One wave moving in the +x-direction with amplitude A and another wave moving in the +x-direction with Amplitude B.

$$\Psi(x,t) = Acos(kx - \omega t) + Bsin(kx - \omega t)$$

Now and taking first and second partial derivatives of the wave function with respect to position:

$\frac{\partial\Psi}{\partial x} = k[-Asin(kx - \omega t) + Bcos(kx - \omega t)]$ $\qquad$ $\frac{\partial^2\Psi}{\partial x^2} = -k^2[Acos(kx - \omega t) + Bsin(kx - \omega t)]$

It follows then that:

$$\frac{\partial^2\Psi(\mathrm{x,t})}{\partial x^2} = -k^2[Acos(kx - \omega t) + Bsin(kx - \omega t)]$$

$$\frac{\partial^2\Psi(\mathrm{x,t})}{\partial x^2} = -k^2\Psi(x,t)$$

Multiplying both sides by $-\frac{\hbar^2}{2m}$:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(\mathrm{x,t})}{\partial x^2} = \frac{\hbar^2}{2m}k^2\Psi(x,t)$$

Plugging in known relationship $\hbar\omega = \frac{\hbar^2 k^2}{2m}$:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(\mathrm{x,t})}{\partial x^2} = \hbar\omega[Acos(kx - \omega t) + Bsin(kx - \omega t)]$$

Looking at the first partial derivative with respect to time yields:

$$\frac{\partial\Psi}{\partial t} = \hbar\omega[Asin(kx-\omega t) - Bcos(kx-\omega t)]$$

It is advised in the text to use a “fudge factor” denoted F, to make sure calculations come out correctly:

$$\frac{\partial\Psi}{\partial t} = \hbar\omega[Asin(kx-\omega t) - Bcos(kx-\omega t)]$$

$$= \hbar\omega[FAsin(kx-\omega t) - Fcos(kx-\omega t)]$$

Setting the two right sides of the equations together yields:

$$\hbar\omega[Acos(kx-\omega t) + Bsin(kx-\omega t)] = \hbar\omega[FAsin(kx-\omega t) - FBcos(kx-\omega t)]$$

$$Acos(kx-\omega t) + Bsin(kx-\omega t) = \omega Asin(kx-\omega t) - \omega Bcos(kx-\omega t)$$

The coefficients for the sine and cosine terms must be the same on both sides so it follows then that:

$$A = -FB \qquad B = FA$$

$$A = -F(FA) = -F^2A$$

$$F^2 = -1$$

$$F = \sqrt{-1} = i$$

And $B = iA$

This gives us Schrodinger’s Equation for a one-dimensional free particle:

$$-\frac{\hbar^2}{2m}\frac{\partial^2\Psi(\mathrm{x,t})}{\partial x^2} = i\hbar\frac{\partial\Psi(x,t)}{\partial t}$$

Finding the complex coefficient changes our original wave function to:

$$\Psi(x,t) = Acos(kx-\omega t) + iAsin(kx-\omega t)$$

We can rewrite the formula using Euler’s formula $e^{i\theta} = cos\theta + isin\theta$:

$$\Psi(x,t) = Ae^{i(kx-\omega t)} = Ae^{ikx}e^{-i\omega t}$$

So now if we take the absolute value of the square of the wave function, we can find the most probable position of a particle moving in the +x-direction at any time t within the range of x and $x + dx$. The particle is most likely to be found at the maximum value of $|\Psi|^2 dx$.

Conclusion:

Using the wave equation, we can derive components such as wave functions. These components have proven themselves to be invaluable to science in understanding our reality as well as compensating for these physical phenomena. We have seen that we can calculate the speed of tsunami waves using satellite data. We can also use mechanical wave functions to understand seismic activity such as earthquakes.

We were also able to see, by creating electromagnetic wave functions, that we can determine the speed of light in a vacuum and determine approximately how far away celestial bodies are from us. Understanding electromagnetic waves have helped us create imaging services and allow us to see inside the human body as well as ancient ruins. Understanding electromagnetic radiation helps us harness energy from the sun. Because electromagnetic waves carry energy they have endless applications in our lives. Lasers, radio transmitters and even our microwaves use electromagnetic waves.

On a quantum scale wave functions can be used to track particles by using a probability factor based on wave functions. They have also helped us understand quantum entanglement which paves the way for creating super computers capable of predicting climate patterns and the effects of drugs on the human body without testing on living creatures. From the macro scale of celestial bodies to the quantum universe the wave equation and functions.

Annotated Bibliography:

Young, Hugh D., Roger A. Freedman, and A. Lewis Ford. *University Physics with Modern Physics*. 14th ed. Boston, MA: Pearson Learning Solutions, 2016.

This text was used for the following:

Wave function derivation:

Chapter 15 section 3 Mathematical Description of a Wave pages 473-474

Example Problem 16.4, page 513

Electromagnetic Wave Derivation:

Chapter 32 Sections 1,2 and 3 Electromagnetic Waves pages 1051-1059

Problem 32.1, page 1073

One-Dimensional Quantum Wave Equation:

Chapter 41 section 1 Wave Functions and the One-Dimensional Schrodinger's Equation

Boas, Mary L. *Mathematical Methods in the Physical Sciences*. New Delhi: Wiley, 2015.

This text was used for the following:

Solutions of the Wave Equation:

Chapter 13 section 4 pages 633-637