On the Transition to Modern Growth

B. Ravikumar† G. Vandenbroucke‡

September 2023

Abstract

We study a simple model where a single good can be produced using a diminishing-returns technology (Malthus) and a constant-returns technology (Solow). The economy’s output exhibits three stages: (i) stagnation; (ii) transition with increasing growth; (iii) constant growth in the long run. We map the Malthus technology to agriculture and show that the share of employment in agriculture is sufficient to determine both the onset of economic transition and the dynamics of output during the transition. Using 20th century data on agricultural share of employment, we project backward and estimate the onset of transition for the U.S., Western Europe, and India. We then show that the model’s implications are consistent with the observed output dynamics.

JEL codes: O10, O13, O40
Keywords: Economic transition, Agricultural employment, Malthus, Solow

*The views expressed in this article are those of the authors and do not necessarily reflect the views of the Federal Reserve Bank of St. Louis or the Federal Reserve System.
†Research Division, Federal Reserve Bank of St. Louis, P.O. Box 442, St. Louis, MO 63166, USA. Email: b.ravikumar@wustl.edu.
‡Research Division, Federal Reserve Bank of St. Louis, P.O. Box 442, St. Louis, MO 63166, USA. Email: guillaumevdb@gmail.com.
1 Introduction

The standard of living was roughly constant prior to the 19th century despite technological change. Malthus (1798) and Ricardo (1817) accounted for this stagnation with a theory of population and diminishing returns in production. The past two centuries have seen unprecedented growth in living standards. There are several models of transition from stagnation to growth. Our paper belongs to the strand of Hansen and Prescott (2002) and Tamura (2002), where a single final good could be produced with two possible technologies and the economy transitions from one technology to another.\footnote{Other papers on transition include Becker, Murphy, and Tamura (1990) and Galor and Weil (2000) where the economy transitions from stagnation and high fertility to growth and lower fertility. In Goodfriend and McDermott (1995), exogenous population growth allows for increasing returns to specialization, and the economy transitions from household production and stagnation to market production and growth. In Jones (2001), the evolution of population and ideas delivers technological progress that helps the economy transition.}

In Hansen and Prescott (2002), the transition occurs when physical capital and total factor productivity (TFP) reach a threshold. In Tamura (2002), endogenous human capital accumulation eventually lowers trade costs and delivers the transition without relying on any exogenous forces.

We develop a simplified version of their models. Our economy has one good that can be produced using two technologies—Malthus and Solow. The Malthus technology is subject to diminishing returns as land is a fixed factor, and the Solow technology has constant returns. Labor is the only flexible factor of production. TFP in the two technologies and population grow exogenously at possibly different rates.

Similar to Tamura (2002), we map the Malthus technology to agriculture in the data and deliver two results. First, the share of employment in agriculture determines the onset of economic transition; we don’t need real gross domestic product per capita (GDP) data. Specifically, the share declines at a constant rate during the transition, so we can estimate the rate easily. This estimate can then be used to project backward and pin down the onset of economic transition. The reasons why different countries have different shares of agricultural employment are not relevant for this calculation. Second, the share of employment in agriculture is a sufficient statistic for the dynamics of GDP growth during the transition, which increases initially and converges to a constant eventually. In sum, we do not need to know the structural parameters,
TFPs, or population to determine the onset of transition or GDP dynamics.

Our economy uses only the Malthus technology initially, but it transitions to using both the Malthus and Solow technologies when the two TFPs and population reach a threshold. The onset of economic transition is when employment in the Malthus technology starts declining—labor is employed only in the Malthus technology initially, but it is employed in both technologies during the transition and increasingly more in the Solow technology. Quantitatively, using our result that the agricultural share of employment declines at a constant rate during the economic transition, we determine the rate using 20th century data. We then project backward and infer the onset of economic transition. For the United States it is 1882.

In terms of GDP, the economy exhibits three stages: (i) stagnation; (ii) acceleration, where the growth rate increases over time; (iii) constant growth in the long run. Our second result, more specifically, is that, during the transition, GDP growth evolves according to a first-order autoregressive process and that the autoregressive coefficient maps one for one into the rate of decline of the employment share in agriculture. We verify that the mapping is quantitatively consistent with the data for the U.S.

We repeat the calculations for the United Kingdom, France, the Netherlands, and Sweden. We find the onset of their transitions to be 1808, 1889, 1863, and 1893, respectively. The growth rates of their GDPs are quantitatively consistent with our model’s implications.

Our method to determine the onset of transition and examine GDP dynamics can also be applied to recent emerging economies where historical data are not easily available. We find that the onset of the transition is 1932 for India. Again, we demonstrate quantitatively that the dynamics of GDP are consistent with the employment share in agriculture for India.

This paper differs from the literature in a few ways. First, previous papers are concerned with the “how”—the channels that led from stagnation to growth. They are not concerned with the “when.” In quantitative implementation of these papers, the onset of economic transition is chosen using narratives or historical evidence on GDP and the model parameters are calibrated to deliver the chosen onset of economic transition. For instance, Hansen and Prescott (2002) use this approach for England and Tamura (2002) uses it for Western Europe + Canada + the U.S. Our paper is
concerned with the “when.” We estimate the onset of economic transition without using GDP data. We use GDP data for cross validation. Second, our method does not require knowing (i) (arguably less reliable) data from 17th, 18th, or 19th centuries, or (ii) parameters and dynamics of TFPs and population. Cross-country differences in structural parameters and the processes for TFPs and population yield cross-country differences in rates of decline in the agricultural share of employment, which is a sufficient statistic for the onset of economic transition and the dynamics of GDP. Finally, because we do not rely on historical evidence on GDP, our model is useful for studying the economic transition of recent emerging economies.

Two remarks are in order here. First, in models such as Hansen and Prescott (2002) and Tamura (2002) the onset of economic transition is, by definition, when the Malthusian share of employment starts declining. The details of the model are not relevant to make this observation. In such a scenario, one could make an ad-hoc assumption that the rate of decline is constant and estimate the onset of transition. Our approach has two advantages: (i) the ad-hoc assumption, without more structure, would have no further implications for GDP dynamics during the transition and (ii) we provide a framework where the constant rate of decline is a result, not an assumption.

Second, ours is a model of only economic transition, not demographic transition, as the processes for TFPs and population are exogenous. Endogenizing one or more of these processes could potentially deliver more testable implications, but the fact remains that our simple framework is remarkably consistent with the onset of economic transition and dynamics of GDP during the economic transition. Thus, richer frameworks have to bear the burden of delivering our results.

2 Model

Time is continuous and represented by $t \geq 0$. There are two technologies, denoted $M$ (for Malthus) and $S$ (for Solow), producing a single consumption good. Technology $M$ uses land (fixed and normalized to 1) and labor, and exhibits diminishing returns to labor. Technology $S$ uses only labor and exhibits constant returns to scale. Outputs
from the two technologies are denoted $Y_t^M$ and $Y_t^S$ at date $t$:  

$$Y_t^M = (Z_t^M H_t^M)^{1-\alpha},$$

$$Y_t^S = Z_t^S H_t^S,$$

where $\alpha \in (0, 1)$, $Z_t^M$ and $Z_t^S$ are exogenous labor-augmenting productivity terms, and $H_t^M$ and $H_t^S$ are employment in $M$ and $S$, respectively. Total gross domestic product is  

$$Y_t = Y_t^M + Y_t^S.$$  

The working population is exogenous and denoted by $P_t$. In what follows, we adopt the following two notations. First, we use lowercase letters to denote a variable per unit of population: $x_t \equiv X_t/P_t$. Second, we use a dot notation to denote the growth rate of a variable:  

$$\dot{X}_t \equiv d\ln(X_t)/dt.$$  

We assume that $Z_t^S$, $Z_t^M$, and $P_t$ grow at constant but potentially different rates $\dot{Z}^S$, $\dot{Z}^M$, and $\dot{P}$, respectively. This is for analytical convenience.  

$$Z_t^S = Z_0^S \exp\left(t\dot{Z}^S\right), \quad Z_t^M = Z_0^M \exp\left(t\dot{Z}^M\right), \quad \text{and} \quad P_t = P_0 \exp\left(t\dot{P}\right),$$

where $Z_0^S$, $Z_0^M$, and $P_0$ are initial conditions.

The working population is allocated between the two technologies,

$$H_t^M + H_t^S = P_t.$$  

Labor is perfectly mobile across the two technologies. Thus, the optimal, i.e., output-maximizing, allocation of labor requires:  

$$Z_t^S \leq (1 - \alpha) \left( Z_t^M \right)^{1-\alpha} \left( H_t^M \right)^{-\alpha},$$

with equality whenever $H_t^M < P_t$. The left-hand side is the marginal product of labor in technology $S$, and the right-hand side is the marginal product of labor in technology $M$. Figure 1 represents the optimal allocation of labor. When $H_t^M < P_t$, the marginal product of labor is equalized across technologies. If $Z_t^S$ is low enough (e.g., $Z_{t,\text{Low}}^S$), however, the marginal product of labor in technology $M$ exceeds that
in technology $S$ even if the entire working population is allocated to technology $M$. In this case Equation (1) holds with a strict inequality.

If initial conditions are such that

$$Z^S_0 < (1 - \alpha) (Z^M_0)^{1-\alpha} (P_0)^{-\alpha},$$

then all labor is allocated to technology $M$ at date 0. If inequality (2), evaluated at $t$, continues to hold, then all labor will continue to be allocated to technology $M$ at $t$. Some labor will be allocated to technology $S$ at some date $t^*$ if Equation (1) is satisfied with equality at $t^*$. For this to occur, it must be the case that

$$\dot{Z}^S > (1 - \alpha) \dot{Z}^M - \alpha \dot{P}.$$ 

We assume conditions (2) and (3) are satisfied for the rest of the paper.
The transition date at which technology \(S\) starts operating satisfies

\[
Z_0^S e^{t^* \dot{Z}^S} = (1 - \alpha) \left( Z_0^M e^{t^* \dot{Z}^M} \right)^{1-\alpha} \left( P_0 e^{t^* \dot{P}} \right)^{-\alpha}.
\]

That is,

\[
t^* = \ln \left[ \frac{(1 - \alpha) (Z_0^S)^{-1} (Z_0^M)^{1-\alpha} (P_0)^{-\alpha}}{\dot{Z}^S - (1 - \alpha) \dot{Z}^M + \alpha \dot{P}} \right]. \tag{4}
\]

2.1 Analysis

It is convenient to analyze the economy in two parts: before and after \(t^*\).

**Before the transition** When \(t \leq t^*\), the analysis is straightforward. The entire working population is employed in technology \(M\) and output per capita is that of technology \(M\). Using our notations defined earlier,

\[
h_t^M = 1, \tag{5}
\]

\[
y_t = y_t^M = \left( Z_t^M \right)^{1-\alpha} (P_t)^{-\alpha}, \tag{6}
\]

\[
y_t = y_t^M = (1 - \alpha) \dot{Z}^M - \alpha \dot{P}. \tag{7}
\]

**During the transition** When both technologies operate, Equation (1) holds with equality and employment in technology \(M\) is

\[
H_t^M = (1 - \alpha)^{1/\alpha} \frac{1}{Z_t^S} \left( \frac{Z_t^M}{Z_t^S} \right)^{(1-\alpha)/\alpha}.
\]

It follows that the *employment share* of technology \(M\) and its growth rate are

\[
h_t^M = (1 - \alpha)^{1/\alpha} \frac{1}{Z_t^S P_t} \left( \frac{Z_t^M}{Z_t^S} \right)^{(1-\alpha)/\alpha}, \tag{8}
\]

\[
\dot{h}_M = \frac{1 - \alpha}{\alpha} \dot{Z}^M - \frac{1}{\alpha} \dot{Z}^S - \dot{P}. \tag{9}
\]

A few observations are worth making here. First, condition (3) implies \(\dot{h}_M < 0\). Equation (9) delivers the result that the employment share of technology \(M\) decreases.
at a constant rate after the onset of transition, as long as the growth rates of the three exogenous variables are constant (see Figure 2). Second, in a world with only population growth, i.e., $\dot{P} > 0$ and $\dot{Z}^M = \dot{Z}^S = 0$, the employment share of technology $M$ declines at the rate of population growth. This is because the additional workers are allocated to technology $S$ where the marginal product of labor is constant. Third, with population growth and TFP growth in technology $S$, i.e., $\dot{P}, \dot{Z}^S > 0$ and $\dot{Z}^M = 0$, the rate of decline of $h^M$ is higher than that with only population growth. This is because the marginal product of labor in technology $S$ is growing, requiring a faster decline of employment in technology $M$ to maintain the equality of marginal products. Finally, if all three exogenous variables are growing, the rate of decline of $h^M$ is less than that with only $\dot{P}, \dot{Z}^S > 0$. This is because the rise in the productivity of technology $M$ allows for a slower decline of employment while still maintaining the equality in the marginal products. Note that, even if the two exogenous TFPs grow at the same rate, $h^M_t$ still declines as long as either $Z_t^S$ or $P_t$ is growing.

Figure 2: Growth rate of employment share in Malthusian technology

*Note:* This is an illustrative path of the growth rate of employment share in the Malthusian technology under the assumption that all of the exogenous variables grow at constant rates.
Figure 3: Growth rate of GDP per capita

Note: This is just an illustrative growth path of GDP under the assumption that all of the exogenous variables grow at constant rates.

The economy’s GDP is

$$ y_t = Z^S_t + \frac{\alpha}{1 - \alpha} Z^S_t h^M_t. $$

The growth rate of $y_t$ is

$$ \dot{y}_t = \frac{(1 - \alpha)(1 - h^M_t)}{1 - \alpha + \alpha h^M_t} \dot{Z}^S + \frac{(1 - \alpha)h^M_t}{1 - \alpha + \alpha h^M_t} \dot{Z}^M - \frac{\alpha h^M_t}{1 - \alpha + \alpha h^M_t} \dot{P}. $$

See Appendix A for the derivation of Equations (10) and (11).

Three remarks about Equation (11) are in order. First, if $h^M_t = 1$, Equation (11) is the same as Equation (7): GDP growth before the transition is that of Malthusian output. That is, even though we derived Equation (11) for $t \geq t^*$, this equation describes the growth rate of GDP for all $t$. Second, as $h^M_t \to 0$, Equation (11) implies $\dot{y} \to \dot{Z}^S$: Asymptotically, the growth rate of GDP is that of TFP in technology $S$. Third, an implication of (3) and (11) is that GDP growth increases during the
transition, even if the TFPs and population grow at constant rates (see Figure 3).

3 QUANTITATIVE ANALYSIS

3.1 Determining the onset of modern growth

The onset of "modern" growth in our model is when technology $S$ starts operating at date $t^*$. While $t^*$ is pinned down by Equation (4), the expression is not quantitatively useful since it depends on unobserved variables. In this section, we derive an expression for $t^*$ that is a function of only employment share in the Malthusian technology. We then map the Malthusian technology to agriculture and use observations on the employment share in agriculture to estimate the onset of the transition.

Under the assumption that $Z^M, Z^S,$ and $P$ grow at constant rates after the economic transition, Equation (9) implies that $\dot{h}^M$ is constant. So, $h^M_t = \exp((t - t^*)\dot{h}^M)$ at any date $t \geq t^*$, or

$$t^* = t - \frac{\ln(h^M_t)}{\dot{h}^M}. \quad (12)$$

Countries could differ in their structural parameters, and levels and growth rates of TFPs and population. In our model these differences manifest themselves in different $h^M_t$ and $\dot{h}^M_t$ (see Equations 8 and 9). What is remarkable about Equation (12) is that the reasons why different countries have different shares and growth rates of agricultural employment do not matter for estimating the onset of modern growth.

To operationalize Equation (12), consider the specification below for country $i$.

$$\ln(h^M_{t,i}) = \beta_{0,i} + \beta_{1,i} t, \quad (13)$$

which implies $\beta_{1,i} = \dot{h}^M_i$. The transition date is

$$t^*_{i} = -\frac{\beta_{0,i}}{\beta_{1,i}} = -\frac{\ln(h^M_{t,i})}{\dot{h}^M_i} = t - \frac{\ln(h^M_{t,i})}{h^M_i},$$

which is the same as Equation (12). We run the regression with 20th century annual data for the U.S. We find $t^*$ to be 1882. This is illustrated for the U.S. in Figure 4.
Figure 4: Onset of modern growth in the United States

*Source:* Herrendorf, Rogerson, and Valentinyi (2014) and authors’ calculations. The sample period is 1901 to 2000. There are years with missing observations. There are observations for every year starting in 1929.

### 3.2 Dynamics of modern growth

In our model, the growth rate of GDP increases during the transition and converges to a constant in the long run. In this section, we show that the rate of decline of the agricultural share of employment is a sufficient statistic to characterize the dynamics of GDP during the transition.

From Equation (10) it is easy to see that the long-run path of GDP is that of $Z_t^S$. The relative deviation of GDP from its long-run path, which we denote $\hat{y}_t$, is then

$$\hat{y}_t \equiv \frac{y_t - Z_t^S}{Z_t^S} = \frac{\alpha}{1 - \alpha} h_t^M, \quad \text{for} \quad t \geq t^*.$$  \hspace{1cm} (14)

First, Equation (14) implies that $\hat{y}_t$ grows at rate $\dot{h}_M$. As the agricultural share of employment declines, $\hat{y}_t$ approaches zero and the paths of GDP and $Z^S$ converge.
Second, using (14) at two instants $t$ and $t + \omega$, we get

$$\ln(1 + \hat{y}_{t+\omega}) - \ln(1 + \hat{y}_t) + \ln(Z_{t+\omega}^S) - \ln(Z_t^S) = \ln y_{t+\omega} - \ln y_t.$$  

We then approximate $\ln(1 + \hat{y}_{t+\omega}) - \ln(1 + \hat{y}_t) \simeq \hat{y}_{t+\omega} - \hat{y}_t. \text{\footnote{We are not approximating }\ln(1 + \hat{y}_t) \text{ with } \hat{y}_t, \text{ but approximating the difference } \ln(1 + \hat{y}_{t+\omega}) - \ln(1 + \hat{y}_t).}$ This yields

$$\hat{y}_{t+\omega} - \hat{y}_t + \ln(Z_{t+\omega}^S) - \ln(Z_t^S) \simeq \ln y_{t+\omega} - \ln y_t,$$

which implies

$$\exp(\omega \dot{h}_M) \hat{y}_t - \hat{y}_t + \omega \hat{Z}_S \simeq \ln y_{t+\omega} - \ln y_t.$$  

Similarly, at $t + \omega$,

$$\exp(\omega \dot{h}_M) \hat{y}_{t+\omega} - \hat{y}_{t+\omega} + \omega \hat{Z}_S \simeq \ln y_{t+2\omega} - \ln y_{t+\omega}$$  

$$\Rightarrow \exp(\omega \dot{h}_M) \left( \exp(\omega \dot{h}_M) \hat{y}_t - \hat{y}_t \right) + \omega \hat{Z}_S \simeq \ln y_{t+2\omega} - \ln y_{t+\omega}.$$  

Substituting, rearranging, and evaluating at $\omega = 1$, we get

$$\ln y_{t+2} - \ln y_{t+1} \simeq \exp(\dot{h}_M) \left( \ln y_{t+1} - \ln y_t \right) + \hat{Z}_S \left( 1 - \exp(\dot{h}_M) \right). \quad (15)$$

The rate of decline of the agricultural share of employment is thus a sufficient statistic to describe the dynamics of GDP after $t^*$. 

Note that (15) is a result and not just an accounting formula. While, as noted earlier, one could estimate $t^*$ by assuming $\dot{h}_M$ is constant, the ad-hoc assumption would not imply (15). In deriving (15) we have used the model’s optimal allocation of labor across the two technologies and how the model’s parameters map to $h_M$. 

**GDP dynamics** Equation (15) represents a testable implication: The autoregressive coefficient of the growth rate of GDP, after $t^*$, is the (exponential of the) growth rate of the share of employment in agriculture. To test the model, we specify the data-generating process for GDP for country $i$ as

$$\ln y_{t,i} = \gamma_{0,i} + \gamma_{1,i} t + \gamma_{2,i} \exp(\gamma_{3,i} t), \quad \text{for } t > t^*_i. \quad (16)$$
Note: Panel A: The trend of GDP is the fitted values of $\ln y_t$ from Equation (16). Panel B: The growth rate of GDP is the corresponding $\dot{y}_t$.

Source: Penn World Tables 10.01, Maddison Project Database 2020, and authors’ calculations. The sample period is 1882 to 2008.

Suppressing the country notation, this functional form has the property that

$$\dot{y}_t \equiv f(t) = \gamma_1 + \gamma_2 \gamma_3 \exp(\gamma_3 t).$$

If $\gamma_2 > 0$ and $\gamma_3 < 0$, $f(t)$ is increasing at a diminishing rate and $\lim_{t \to \infty} f(t) = \gamma_1$. The autoregressive coefficient on GDP growth is $\exp(\gamma_3)$.

We estimate Equation (16) using data from $t^* = 1882$ to 2008 for the U.S. First, our estimates of $\gamma_2$ and $\gamma_3$ satisfy the conditions above, so GDP growth is increasing at a diminishing rate after $t^*$, as in Figure 3. We illustrate the fit of our specification in Figure 5. Second, we find $\exp(\gamma_3) = 0.980$ and $\exp(\dot{h}^M) = 0.967$. We inferred the latter from the estimation of Equation (13). We conclude that the agricultural share of employment describes the dynamics of GDP well.

The onset of transition and dynamics of GDP for the United Kingdom, France, Netherlands, and Sweden are reported in Appendix B.
3.3 Emerging economies

Our method can be applied to recent emerging economies where historical data are not easily available. In this section, we provide the results on the onset of economic transition and GDP dynamics for India.

First we estimate $t^*$ using Equation (13) and data from 1951 to 2018. We find $t^*$ to be 1932 (see Figure 6). Second, we estimate the dynamics of GDP using Equation (16) and find $\exp(\gamma_3) = 0.991$ (see Figure 7). Our estimate of $\exp(\hat{h}^M) = 0.992$.

4 Conclusion

We develop a simple model where a single good can be produced using two technologies: Malthus (diminishing returns) and Solow (constant returns). TFPs and population are exogenous. The economy’s GDP exhibits three stages: (i) stagnation; (ii) transition with increasing growth; (iii) constant growth in the long run. We map the Malthus technology to agriculture and show that the share of employment in agriculture is sufficient to determine both the onset of economic transition and the dynamics of GDP during the transition.

We use 20th century data on agricultural employment to estimate the onset of transition for the U.S., U.K., France, the Netherlands, Sweden, and an emerging economy, India. Our estimate does not rely on GDP data. We then show that the model’s implication for GDP dynamics is quantitatively consistent with the data.
Figure 6: Onset of modern growth in India

Source: World Bank, International Historical Statistics, and authors’ calculations. The sample period is 1951 to 2018. There are years with missing observations. There are observations for every year starting in 1991.

Figure 7: Dynamics of GDP in India

Note: Panel A: The trend of GDP is the fitted values of \( \ln y_t \) from Equation (16). Panel B: The growth rate of GDP is the corresponding \( \dot{y}_t \).

Source: Penn World Tables 10.01, Maddison Project Database 2020, and authors’ calculations. The sample period is 1945 to 2018.
References


A Derivation of Equations (10) and (11)

Using the solution for $H^M_t$ when $t \geq t^*$, the output of technology $M$, when both technologies operate, is

$$Y^M_t = (Z^M_t)^{1-\alpha} \left[ (1 - \alpha) \frac{(Z^M_t)^{1-\alpha}}{Z^S_t} \right]^{(1-\alpha)/\alpha} = (1 - \alpha)^{(1-\alpha)/\alpha} \left( \frac{Z^M_t}{Z^S_t} \right)^{(1-\alpha)/\alpha},$$

implying output per capita and its growth rates are

$$y^M_t = \frac{1}{1 - \alpha} Z^S_t h^M_t, \quad (A.1)$$

$$\dot{y}^M = \dot{Z}^S + \dot{h}^M. \quad (A.2)$$

For technology $S$, output per capita is

$$y^S_t = Z^S_t (1 - h^M_t) = Z^S_t - (1 - \alpha) y^M_t, \quad (A.3)$$

and its rate of growth is

$$\dot{y}^S_t = \frac{d \ln Z^S_t}{dt} \frac{Z^S_t}{y^S_t} - (1 - \alpha) \frac{d \ln y^M_t}{dt} \frac{y^M_t}{y^S_t} = \frac{1}{1 - h^M_t} \dot{Z}^S - \frac{h^M_t}{1 - h^M_t} \dot{y}^M. \quad (A.4)$$

The economy’s output per capita is $y_t = y^S_t + y^M_t$. Using Equations (A.1) and (A.3), this is also

$$y_t = Z^S_t + \frac{\alpha}{1 - \alpha} Z^S_t h^M_t.$$ 

The economy’s growth rate is

$$\dot{y} = \frac{d \ln y^S_t}{dt} \frac{y^S_t}{y_t} + \frac{d \ln y^M_t}{dt} \frac{y^M_t}{y_t} = \left( 1 - \frac{y^M_t}{y_t} \right) \left( \dot{Z}^S - \frac{1 - h^M_t}{1 - h^M_t} \dot{y}^M + \frac{h^M_t}{1 - h^M_t} \right) + \frac{y^M_t}{y_t} \dot{y}^M.$$

where

$$\frac{y^M_t}{y_t} = \frac{1}{1 - \alpha} Z^S_t h^M_t = \frac{h^M_t}{1 - \alpha + \alpha h^M_t}.$$
It follows that

\[ \frac{d \ln y_t}{dt} = \frac{1 - y_t^M / y_t}{1 - h_t^M} \dot{Z}^S + \left( \frac{y_t^M}{y_t} - \frac{h_t^M}{1 - h_t^M} \left( 1 - \frac{y_t^M}{y_t} \right) \right) \dot{y}^M, \]

\[ = \frac{1 - \alpha}{1 - \alpha + \alpha h_t^M} \dot{Z}^S + \frac{\alpha h_t^M}{1 - \alpha + \alpha h_t^M} \left( \dot{Z}^S + \frac{1 - \alpha}{\alpha} \dot{Z}^M - \frac{1}{\alpha} \dot{Z}^S - \dot{P} \right), \]

\[ = \frac{(1 - \alpha)(1 - h_t^M)}{1 - \alpha + \alpha h_t^M} \dot{Z}^S + \frac{(1 - \alpha)h_t^M}{1 - \alpha + \alpha h_t^M} \dot{Z}^M - \frac{\alpha h_t^M}{1 - \alpha + \alpha h_t^M} \dot{P}. \]
### Western Europe

#### Table 1: Transition and GDP dynamics for Western Europe

<table>
<thead>
<tr>
<th>Country</th>
<th>$t^*$</th>
<th>$\exp(\gamma_3)$</th>
<th>$\exp(\hat{h}^M)$</th>
<th>$h^M$ sample</th>
<th>GDP sample</th>
</tr>
</thead>
<tbody>
<tr>
<td>United Kingdom</td>
<td>1808</td>
<td>0.997</td>
<td>0.979</td>
<td>1901–2000</td>
<td>1808–2008</td>
</tr>
<tr>
<td>France</td>
<td>1889</td>
<td>0.973</td>
<td>0.973</td>
<td>1901–2000</td>
<td>1889–2008</td>
</tr>
<tr>
<td>Netherlands</td>
<td>1863</td>
<td>0.995</td>
<td>0.975</td>
<td>1901–2000</td>
<td>1863–2008</td>
</tr>
<tr>
<td>Sweden</td>
<td>1893</td>
<td>0.895</td>
<td>0.969</td>
<td>1901–2000</td>
<td>1892–2008</td>
</tr>
</tbody>
</table>

*Note:* The estimate of $\gamma_3$ is from regression (16). The estimate of $\hat{h}^M$ is from regression (13).

*Source:* Penn World Tables 10.01, Maddison Project Database 2020, Herrendorf et al. (2014), and authors’ calculations. There are years with missing observations.