On the Economic Mechanics of Warfare

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On the Economic Mechanics of Warfare*

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Abstract

A large literature is concerned with war finance, but there is little by way of understanding how war-related expenditures affect the economically-relevant outcomes of wars, e.g., prevailing side, duration, or destruction and casualties. I present a model of attrition in which I characterize the effects of resources on the outcomes of war for a military conclusion (when one side cannot fight anymore) and a political conclusion (when one side does not want to fight anymore). I discuss the role of GDP for both types of conclusion. I also analyze the mechanics of third-party support to a small country at war with a large one, e.g., Ukraine and Russia. Finally, I show that the model can fit actual battle data.

JEL: E6, H56, N4

Keywords: War; Attrition; Military spending.

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1 Introduction

War may be the costliest activity countries regularly engage in. Naturally, then, there is a large literature concerned with war finance. There is little, however, by way of understanding how expenditures affect the outcomes of war such as duration, destruction (material and human), the prevailing side, and the reason for the conclusion of war. This is what this article is about: A theory of how resources affect the outcomes of war.

Outcomes of war are economically relevant. Duration affects the cost and also has economic, demographic, and political consequences. When, for example, a war reduces fertility (e.g., Caldwell, 2004, Vandenbroucke, 2014), disrupts the marriage market (e.g., Knowles and Vandenbroucke, 2019, Abramitzky et al., 2011), and disrupts the labor market (e.g., Acemoglu et al., 2004, Fernández, 2013, Doepke et al., 2015), the effects are magnified by duration.

The destruction, either material or human (casualties) and, associated to it, the resources remaining at the end of a war, are relevant as well. First, the material destruction affects well-being (e.g., the destruction of housing), productivity (e.g., the destruction of productive capital), and sometimes international relations (e.g., the Marshall Plan). Casualties also matter for well-being and the postwar age and sex composition of the labor force. Second, at the end of a war, a country’s remaining military equipment can be sold or retain value for peacekeeping and/or deterrence purposes. Finally, individuals (troops) can work and be productive after they are demobilized.

The determination of the prevailing side is of interest chiefly because the prevailing side often imposes a payment on its opponent in the form of land (e.g., any war of conquest), treasure (e.g., the treaty of Versailles), and sometimes people (i.e., slaves). Also of interest is the reason why one side ceases to fight. It can be for a lack of resources to fight (e.g., Germany and Japan at the end

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1See, for instance, Keynes (1940), Barro (1979), Lucas and Stokey (1983), and more recently, Ohanian (1997) and McGrattan and Ohanian (2010)
of World War II) or for a lack of political will (e.g., the U.S. in Vietnam).

To analyze the role of resources in a war, I present a model of how a war unfolds and concludes. That is what I mean by “warfare.” It is not a “model of war” explaining why countries go to war, however interesting this question is. Instead, I assume a state of war to exist between two countries. The countries are endowed with initial “weapons” stocks and military technologies that use weapons to destroy their opponent’s weapons. They also receive per-period exogenous flows of weapons (“reinforcements”).

I interpret initial weapons stock as resulting from prewar investments, and reinforcements as resulting from wartime expenditures. These, and the military technologies, are the resources the effects of which I analyze. I interpret weapons as combinations of physical and human capital.

The destruction experienced by countries over the course of a war is mitigated by their reinforcements and the destruction they inflict on their opponents. The model’s outcome is thus the joint dynamics of weapons accumulation and of weapons destruction and casualties. I discuss two possible conclusions: In the first, which I label “military,” a war concludes when a country’s weapons stock reaches an exogenously-determined low threshold and the country cannot fight anymore. In the second, which I label “political,” a war concludes when a country’s casualties reach an exogenously-determined high threshold and political forces request that the fighting ends.

Observations about the theory

Battles v. Attrition Wars are sometime viewed as idiosyncratic events with outcomes largely determined by genius-like generalship and “decisive” battles. I do not take this view. My approach is inspired, instead, by observations from military historians on the one hand, and by Operations Research models on the other hand.

Historians such as O’Brien (2015) and Nolan (2017) argue against a battle-
centric view of wars. O’Brien’s first sentence is “There were no decisive battles in World War II” (p. 1). Nolan insists that attrition, more than battles, is key to understanding the outcomes of conflict such as the Punic wars, the Hundred Years war, the Napoleonic wars, and the two World Wars of the 20th century. In the same vein, Parshall and Tully (2005), in their study of the battle of Midway, argue that, although important, Midway was not decisive. In the last analysis, the notion of a decisive battle is not a useful one. That is first because “decisive” is not well-defined. What is the decided outcome? Is it the prevailing side, the duration, or another outcome? Second, regardless of the definition, assessing whether a battle is decisive requires a counterfactual, which cannot be evaluated without a theory of decision making in war. The latter does not exist to the best of my knowledge. (I discuss decision making below.) For these reasons, the model I present does not have battles in the sense of discrete occurrences of fighting. Instead, I represent war as a continuous process of attrition.

Combat models developed in Operations Research since the advent of the so-called Lanchester model (Lanchester, 1916) are systems of differential equations describing the attrition of opposing forces during battles. For the reasons I described above, such emphasis on attrition in Lanchester-type models makes them appealing to think about war as a whole. That is therefore my approach: I use a Lanchester-type model and interpret it as a representation of war.

Decisions I do not model decisions. The model represents the mechanics of attrition in a manner similar to the Solow model’s representation of the mechanics of capital accumulation. Adding decisions would be a natural extension, just like the Ramsey-Cass-Koopmans model was a natural extension of the Solow model. Yet, absent a theory of why there are wars, it is difficult.

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2They write “(...) win or lose at Midway, the vast industrial resources of the United States gave its navy an absolutely irrevocable writ of strategic dominance in the Pacific War.” (p. 424) and “(...) Midway stands as the most important battle of the Pacific War, not because it was decisive in an absolute sense, and not because it won the war in a day, but because of its (...) effects on American military options in the Pacific.” (p. 430).
to assign objective functions to the belligerents. Although interesting, I abstract from these considerations here. Without decisions, there are no tactics and/or strategies. I consider tactical knowledge as one would consider business knowledge in a model featuring production functions: I assume tactical knowledge to be subsumed in the military technology.

The organization of the paper

I develop the model in Section 2. I layout the setup in 2.1 and describe the dynamics of weapons accumulation and casualties in 2.2. In 2.3 I describe the model’s characterization of a military conclusion. Specifically, I describe how resources affect which side prevails, the duration, the remaining resources, and total casualties at the end of a war. In 2.4 I characterize a political conclusion.

In Section 3 I discuss two historical scenarios through the lenses of the model: the Pacific war and the Russo-Ukrainian war. Both cases feature belligerents with large differences in their Gross Domestic Product (GDP). In 3.1 I reinterpret the model in terms of GDP, prewar, and wartime saving rates. I then discuss the role of these economic variable in determining the Pacific war’s outcome. In 3.2 I analyze the mechanics of third-party (e.g., an international coalition) support to a small country at war with a large one, as in the case of Ukraine receiving international support to face Russia.

In Section 4 I replicate an application of the model to the battle of Iwo Jima during World War II. I discuss the data needed for an estimation and show that in the case of the battle of Iwo Jima, the model fits the data well.

Some results

In Section 2.3 I find that, at a military conclusion, casualties on both sides are decreasing with the resources committed by the country obtaining the military victory, and increasing with the resources committed by its opponent. That
is because the country obtaining the military victory can shorten the war by allocating more resources to it, thereby reducing destruction and casualties for both sides.

In Section 3.1, I describe the role of the relative GDP of belligerents. All else equal, a high GDP makes the condition for a military victory more favorable and the condition for suing for peace on political grounds less favorable. The first effect is often mentioned by military historians (e.g., footnote 2). The second effect is less often described: The richer a country, the longer it takes to reach the political threshold for casualties (at which political forces request an end to the fighting) and thus the more time there is for obtaining a favorable military conclusion. I argue that this mechanism yields an interpretation of the U.S. victory in the Pacific war against Japan not often emphasized in the literature on the Pacific war: the Japanese failure to note that the larger U.S. GDP, which was conducive to a Japanese military defeat in the long-run, was not conducive to a Japanese political victory in the short run.

In Section 3.2, I describe the mechanics of third-party support to a small country at war with a large one, e.g., Ukraine v. Russia. When the foreign coalition commits an additional unit of per-period reinforcements, the war becomes shorter. Hence, there is a well-defined cost-minimizing (for the coalition) level of support leading to a military victory for the small country.

2 THE MODEL

2.1 Setup

The model I present is inspired by the so-called Lanchester model (Lanchester, 1916). Time is continuous and there is no uncertainty. There are two countries, Red and Blue, with weapons stocks denoted $K^R_t$ and $K^B_t$, respectively. A country’s weapons stock is an input into its military “production function,” the output of which is a flow of destruction inflicted on the opposite side’s
weapons stock. Let $\theta^R > 0$ denote Red’s attrition coefficient, which is the flow of Blue weapons destroyed per Red weapon at each point in time. Blue’s attrition coefficient, $\theta^B > 0$, has a similar interpretation. Let $X^R \geq 0$ and $X^B \geq 0$ denote constant reinforcement flows at each point in time. I do not model depreciation for simplicity. The laws of motion for $K^R_t$ and $K^B_t$ are

$$\frac{dK^R_t}{dt} = -\theta^B K^B_t + X^R,$$

(1)

$$\frac{dK^B_t}{dt} = -\theta^R K^R_t + X^B,$$

(2)

where the initial conditions $K^R_0$ and $K^B_0$ are given. I use the term “resources” to refer to initial weapons stocks and/or reinforcements.

I interpret a unit of weapon as a combination of human and material resources, e.g., a soldier with a rifle or an aircraft with a crew, etc. I assume a high degree of complementarity between human and material resources at the weapon level. Thus, I interpret the destruction of a unit of weapon as a combination of casualties and material destruction. (I use these terms interchangeably). I do not distinguish between lethal and non-lethal casualties. It is conceivable that one aircraft with crew effects as much destruction as a large number of soldiers with rifles, suggesting substitutability between different weapon types. In the model there is one aggregate weapon type for simplicity.

I assume no destruction other than that of weapons and, thus, abstract from some form of destruction found in war: First, there is no destruction of the productive capacities of a country, aimed at reducing its ability to reinforce, e.g., no bombing of factories or blockading of ports. Second, there are no civilians. Casualties result only from the destruction of weapons, e.g., no collateral damages or direct targeting of civilians. I present, in Appendix G, a version of the model with civilian casualties and show that the resulting dynamics is isomorphic to that of the model without civilians.

The steady state $(\bar{K}^R, \bar{K}^B)$ of system (1)-(2) is a stalemate where each country’s reinforcements are destroyed by its opponent: $\bar{K}^R = X^B / \theta^R$ and $\bar{K}^B =$
Define $\tilde{K}_R^t = K_t^R - \bar{K}_R^R$ and $\tilde{K}_B^t = K_t^B - \bar{K}_B^B$. Then, (1)-(2) become

$$
\begin{pmatrix}
\frac{d\tilde{K}_R^t}{dt} \\
\frac{d\tilde{K}_B^t}{dt}
\end{pmatrix} =

\begin{pmatrix}
0 & -\theta^B \\
-\bar{\theta}_R & 0
\end{pmatrix}

\begin{pmatrix}
\tilde{K}_R^t \\
\tilde{K}_B^t
\end{pmatrix}.
$$

Let $\lambda_1$ and $\lambda_2$ be the eigenvalues of $\mathcal{M}$ with corresponding eigenvectors $[1, v_1]'$ and $[1, v_2]'$, respectively: $\lambda_1 = -\lambda_2 = -\sqrt{\bar{\theta}_R \theta^B}$ and $v_1 = -v_2 = \sqrt{\bar{\theta}_R / \theta^B}$.

2.2 Dynamics

**Weapons stocks** Standard methods (Appendices A) yield the solution

$$
\tilde{K}_R^t = \frac{1}{2} \left[ e^{t\lambda_1} A - e^{t\lambda_2} B \right] \frac{1}{v_1},
$$

$$
\tilde{K}_B^t = \frac{1}{2} \left[ e^{t\lambda_1} A + e^{t\lambda_2} B \right],
$$

where the constant $A$ and $B$ depend on initial conditions:

$$
A = \tilde{K}_0^B + v_1 \tilde{K}_0^R \quad \text{and} \quad B = \tilde{K}_0^B - v_1 \tilde{K}_0^R.
$$

Note that $\lambda_1 < 0$ and $\lambda_2 > 0$. Thus, the stalemate is a saddle-point in the $(K_t^R, K_t^B)$ state space. There is a stable branch described by $B = 0$ and an unstable branch described by $A = 0$. Note also, from Equation (3), that $dK_t^B/dt > 0$ whenever $K_t^R < \bar{K}_R^R$. That is because when the Red weapons stock is below its stalemate, Red does not offset Blue reinforcements and, thus, the Blue weapons stock increases. Conversely, if the Red weapons stock is above its stalemate, Blue reinforcements do not offset the destruction caused by Red and, thus, the Blue weapons stock decreases: $dK_t^B/dt < 0$ whenever $K_t^R > \bar{K}_R^R$. The same logic applies to the evolution of the Red weapons stock. Figure 1 summarizes these observations in a phase diagram.

Weapons stocks’ trajectories need not be monotonic. The blue arrow starting off in the light-shaded area of Figure 1 represents a case where both weapons
Figure 1: Weapons stock dynamics

stocks are initially below their stalemate values: The Blue weapons stock increases monotonically; the Red weapons stock increases until \( K_t^B = \bar{K}^B \) and then decreases. It is the reverse with the blue arrow starting off in the dark-shaded area: The Red weapons stock decreases monotonically; the Blue weapons stock decreases until \( K_t^R = \bar{K}^R \) and then increases. Finally, when initial conditions are in the northwest quadrant, weapons stocks evolve monotonically: upward for Blue and downward for Red. A similar analysis holds for initial conditions below the stable branch.

**Casualties**  The flow of casualties, at each point in time, are \( d_t^B = \theta^R K_t^R \) for Blue and \( d_t^R = \theta^B K_t^B \) for Red. The dynamics of the flows of casualties transpire thus mimic the weapons stocks dynamics: they need not be monotonic. I show (Equations A.7 and A.9 in Appendix A) that the Red weapons stock is decreasing with the initial Blue weapons stock and with Blue reinforcements.
It follows that \( d_t^B \) is decreasing in \( K_0^B \) and \( X^B \):

\[
\frac{\partial K^R_t}{\partial X^B} < 0 \Rightarrow \frac{\partial d_t^B}{\partial X^B} < 0 \quad \text{and} \quad \frac{\partial K^R_t}{\partial K_0^B} < 0 \Rightarrow \frac{\partial d_t^B}{\partial K_0^B} < 0. \tag{6}
\]

I also show (Equations A.3 and A.5 in Appendix A) that the Blue weapons stock is increasing with \( K_0^B \) and \( X^B \), implying that \( d_t^R \) is increasing as well:

\[
\frac{\partial K^B_t}{\partial X^B} > 0 \Rightarrow \frac{\partial d_t^R}{\partial X^B} > 0 \quad \text{and} \quad \frac{\partial K^B_t}{\partial K_0^B} > 0 \Rightarrow \frac{\partial d_t^R}{\partial K_0^B} > 0. \tag{7}
\]

Symmetric results hold for the effect of Red resources on \( d_t^B \) and \( d_t^R \). Thus, at each point in time, a country’s flow of casualties is decreasing in the resources the country commits to the war and increasing in the resources committed by its opponent.

It is worth emphasizing this result. Suppose Blue commits additional resources to the war, either via \( K_0^B \) or via \( X^B \). First, additional Blue weapons do not imply additional Blue casualties. This is because the military technology (Equations 1 and 2) implies that the destruction inflicted by Red on Blue is proportional to the stock of Red weapons but independent of the stock of Blue weapons (and vice versa).\(^3\) Second, additional Blue weapons imply a higher flow of casualties for Red, which impairs Red’s ability to destroy Blue weapons. The result is a lower flow of Blue casualties.

Let \( D_t^R \) and \( D_t^B \) denote total casualties at \( t \) for Red and Blue, respectively:

\[
D_t^R = \int_0^t b_t^R du \quad \text{and} \quad D_t^B = \int_0^t b_t^B du.
\]

In the remainder of the paper I will refer to \( d_t^B \) and \( d_t^R \) as “flow-casualties” and to \( D_t^B \) and \( D_t^R \) as “casualties.” I show (Equations A.14 and A.15 in Appendix

\(^3\)This technology, labeled “aimed-fire” (e.g., Taylor, 1980), can be opposed to another, “area-fire,” where a country’s casualties are increasing in its own weapons stock.
A) that casualties can be written as

\[ D_t^R = tX^R + K_0^R - K_t^R, \quad (8) \]
\[ D_t^B = tX^B + K_0^B - K_t^B. \quad (9) \]

That is, a country’s casualties at \( t \) are the sum of the initial weapons stock and reinforcements committed until \( t \), net of the remaining weapons stock. Casualties at the end of war depend on how and when the war concludes.

### 2.3 Military conclusion

I adopt the following definition: A “military” conclusion is when a war ends because a belligerent’s weapons stock reaches a critically low, exogenously determined, threshold. I assume the threshold is zero. Not all wars end with a military conclusion, but World War II is an example. The fighting ability of both Germany and Japan was close to nil by the end of the war.

**The prevailing side** Figure 1 illustrates the condition under which a military conclusion is a victory for Blue or for Red, or is a stalemate. Initial conditions above the stable branch imply that the Red weapons stock eventually reaches 0, while the Blue stock reaches a positive value. This can be seen from Equations (4) and (5) since \( e^{\lambda_1} \) converges to 0 while \( e^{\lambda_2} \) diverges. It follows that, when \( B > 0 \), \( \tilde{K}_t^B \) eventually increases while \( \tilde{K}_t^R \) eventually decreases. Thus,

\[ B > 0 \Rightarrow \text{Blue victory}, \]
\[ B < 0 \Rightarrow \text{Red victory}, \]
\[ B = 0 \Rightarrow \text{stalemate}. \]

The condition for a Blue military victory can be expressed as

\[ \sqrt{\theta^B} K_0^B + X^B / \sqrt{\theta^B} > \sqrt{\theta^R} K_0^R + X^R / \sqrt{\theta^B}. \quad (10) \]
I label the units in Equation (10) “efficiency units” or “fighting strength” (as in the Operations Research literature). I use these terms interchangeably. The general form for efficiency units is

$$\sqrt{\text{attrition coefficient} \times \text{quantity of weapon}}.$$  

The Blue fighting strength, on the left-hand side of Equation (10), is the sum of that arising from the initial weapons stock, $\sqrt{\theta_B K_B^0}$, and that arising from reinforcements, $X_B / \sqrt{\theta_R}$. Note that the latter is also $\sqrt{\theta_R K_R}$, so it is indeed in the same units, and that a higher $\bar{K}_R$ raises the Blue fighting strength. That is because a higher stalemate makes it harder for a country to attrit its opponent. On the right-hand side of Equation (10), the Red fighting strength is defined in the same manner.

Equation (10) is a modified version of the so-called Lanchester Square Law and deserves some comments. Consider the case where $X_B = X_R = 0$. The Blue fighting strength is then $\sqrt{\theta_B K_B^0}$, which increases faster with the weapons stocks than with the attrition coefficient. That is because an additional Blue weapon destroys Red weapons and dilutes Red’s ability to attrit Blue. A higher Blue attrition coefficient serves the first purpose but not the second. (The same logic applies for Red). This dilution argument is an implication of the military technology (Equations 1 and 2): Since the destruction experience by one side is independent of that side’s stock of weapons, fielding more weapons has no adverse effect, only a dilution effect on the other side’s attrition capacity. The property that the size of the weapons stocks is more important than the quality (measured by the attrition coefficient) has often been viewed as a rationalization of the practice of concentrating military units while inducing the enemy to divide its units.

In the general case, when $X_R, X_B \geq 0$, there is an additional benefit from a higher attrition coefficient: It reduces the opposing force’s fighting strength arising from reinforcements. A higher Blue attrition coefficient, for instance, reduces the contribution of Red reinforcements to Red fighting strength. That
is because Blue destroys Red reinforcements with a lower stalemate stock, and, thus, it is easier for Blue to exceed its stalemate and attrit Red.

I assume, for the remainder of this section, that \( B > 0 \), so that Blue is poised to obtain a military victory at \( \tau \), that is

\[
K^R_\tau = 0.
\]

**Duration of war** I show (Equations B.1–B.4 in Appendix B) that

\[
\frac{\partial \tau}{\partial X^B} < 0, \quad \frac{\partial \tau}{\partial X^R} > 0, \quad \frac{\partial \tau}{\partial K^B_0} < 0, \quad \frac{\partial \tau}{\partial K^R_0} > 0. \tag{11}
\]

The duration of war, \( \tau \), is decreasing with the initial stock of Blue weapons and with Blue reinforcements, while it is increasing with the initial stock of Red weapons and with Red reinforcements. More generally, the duration of war before a military conclusion is decreasing in the resources of the country obtaining the military victory, and increasing in the resources of the country being militarily defeated.

The logic behind this result is as follows. When Blue allocates more resources to the war, either via the initial weapons stock or via reinforcements, Blue’s ability to attrit Red is heightened and the Red weapons stock depletes faster. Hence, the war is shorter. If Red allocates more resources to the war and \( B \) remains positive, Blue suffers additional flow-casualties. This slows down Blue’s ability to attrit Red, and, therefore, the war takes longer before the Blue military victory.

It is useful to characterize the level curves of \( \tau \) in the \((K^R, K^B)\) plane. Even though initial weapon stocks are predetermined when the war starts, they are the results of prewar investments. In Section 3 I discuss the role of prewar investments for the outcomes of war and the characterization of the level curves of \( \tau \) becomes useful.
Equations (B.1)–(B.4) imply that the level curves are straight lines:

\[
\left. \frac{dK_0^B}{dK_0^R} \right|_{d\tau=0} = v_1 \left( \frac{-\cosh(\tau\lambda_1)}{\sinh(\tau\lambda_1)} \right),
\]

where \(\cosh\) and \(\sinh\) are the hyperbolic cosine and sine functions, respectively. It follows from (11) that the slopes of the level curves are increasing with \(K_0^B\) and decreasing with \(K_0^R\). Figure 2 represents a set of level curves. Since duration is decreasing in \(K_0^B\), higher curves correspond to shorter wars.

Two points deserve commenting at this stage. First, the result in Equation (12) implies that, for \(\tau\) to remain constant, initial conditions have to change as

\[
\sqrt{\theta^B} dK_0^B = \sqrt{\theta^B} dK_0^R \left( \frac{-\cosh(\tau\lambda_1)}{\sinh(\tau\lambda_1)} \right).
\]
Suppose the change in Blue fighting strength was, instead, equal to the change in Red fighting strength, i.e., \( \sqrt{\theta} dK^B_0 = \sqrt{\theta} dK^R_0 \). This would maintain the condition for Blue’s military victory, i.e., \( dB = 0 \). Blue, however, would need extra time to destroy the additional Red weapons and, thus, the war would be longer. To obtain victory in the same time, the increase in Red fighting strength must be met by an increase in Blue fighting strength larger than what is necessary to maintain \( \mathcal{B} \) constant. Hence the second, larger-than-one term on the right-hand side of Equation (12). Second, as the war is shorter, each additional Red weapons must be destroyed faster for the duration of war to remain the same. This is why the level curves are steeper as the duration of war decreases. Similar results obtain when analyzing the level curves of \( \tau \) in the \((X^B, X^R)\) plane.

**End-of-war stocks**  I show (Appendix C) that the Blue end-of-war stock is

\[
K^B_\tau = \bar{K}^B + \sqrt{(v_1 \bar{K}^R)^2 + \mathcal{A} \mathcal{B}},
\]

(13)

and that \( K^B_\tau > K^B_0 \) whenever \( K^R_0 < 2 \bar{K}^R \). Thus, if the initial stock of Red weapons is low enough, Blue reinforcements deployed throughout the war exceed the destruction caused by Red, and the final Blue weapons stock is above its initial value. Panels A and B of Figure 3 represent, with light- and dark-shaded areas, the initial conditions leading to either \( K^B_\tau > K^B_0 \) or \( K^B_\tau < K^B_0 \).

How does \( K^B_\tau \) depend upon resources committed before and during the war? I show (Equations C.2 and C.3 in Appendix C) that

\[
\frac{\partial K^B_\tau}{\partial K^B_0} = \frac{K^B_0 - \bar{K}^B}{\bar{K}_\tau - \bar{K}^B} \leq 0 \quad \text{and} \quad \frac{\partial K^B_\tau}{\partial X^B} = \frac{1}{\theta^B} \frac{K^R_0}{K^B_\tau - \bar{K}^B} > 0.
\]

To understand these effects, note that Equations (1) and (2) imply \( \partial^2 K^B_t / \partial t^2 = \theta^R \theta^B (K^B_t - \bar{K}^B) \): The growth rate of \( K^B_t \) is decreasing below the stalemate and increasing above. Below the stalemate, for instance, Blue does not offset Red reinforcements and, hence, the Red weapons stock grows, impairing the
growth of $K^B_t$.

When $K^B_0 < \bar{K}^B$, an increase in $K^B_0$ implies that the Blue weapons stock starts higher but grows at a decreasing rate until $K^B_t = \bar{K}^B$. Since the war is shorter (Equation 11) the final result is a lower end-of-war weapons stock (trajectories 1 and 2 in Panel A of Figure 3). When $K^B_0 > \bar{K}^B$, the additional $K^B_0$ implies that the Blue weapons stock starts higher and grows at an increasing rate, hence the higher final stock despite the shorter war (trajectories 3 and 4 in Panel A of Figure 3). Blue reinforcements raise the final weapons stock despite the shorter war, regardless of initial conditions.

I also show (Equations C.4 and C.5 in Appendix C) that

$$\frac{\partial K^B}{\partial K^R_0} = -v^2 \frac{K^R_0 - \bar{K}^R}{K^B - \bar{K}^B} \leq 0 \quad \text{and} \quad \frac{\partial K^B}{\partial X^R} = \frac{1}{\theta^B} \frac{K^R - K^B}{K^R - \bar{K}^R} \leq 0.$$ 

If $K^R_0 < \bar{K}^R$, additions to the initial Red stock do not offset Blue reinforcements, and, since the war is longer, the final Blue stock increases (trajectories

**Figure 3:** The determination of the Blue end-of-war weapons stock when $B > 0$
1 and 2 in Panel B of Figure 3). If, however, \( K_R^0 > \bar{K}_R^0 \), the additional Red weapons offset Blue reinforcements in the early stages of the war and the final Blue stock decreases despite the longer war (trajectories 3 and 4 in Panel B of Figure 3). Finally, an increase in Red reinforcement lowers the final Blue stock if \( K^B_\tau < K^B_0 \), that is if \( K^R_0 > 2\bar{K}_R^R \).

**Casualties** Casualties at \( \tau \) follow from Equations (8) and (9):

\[
D^R_\tau = \tau X^R + K^R_0 \quad \text{and} \quad D^B_\tau = \tau X^B + K^B_0 - K^B_\tau.
\]

(14)

Red loses all the resources it commits to the war. Blue casualties are mitigated by the end-of-war Blue weapons stock given in Equation (13). I show (Appendix D) that

\[
\frac{\partial D^R_\tau}{\partial X^B} < 0, \quad \frac{\partial D^R_\tau}{\partial X^R} > 0, \quad \frac{\partial D^R_\tau}{\partial K^B_0} < 0, \quad \frac{\partial D^R_\tau}{\partial K^R_0} > 0,
\]

(15)

and

\[
\frac{\partial D^B_\tau}{\partial X^B} < 0, \quad \frac{\partial D^B_\tau}{\partial X^R} > 0, \quad \frac{\partial D^B_\tau}{\partial K^B_0} < 0, \quad \frac{\partial D^B_\tau}{\partial K^R_0} > 0.
\]

(16)

That is, in a Blue military victory, Blue casualties are reduced by Blue resources committed to the war and increased by Red resources. Red casualties behave in the same (not symmetric) manner. That is, Red casualties are also reduced by Blue resources and increased by Red resources.

To understand these results, consider the effects of an increase in the initial stock of Blue weapons, from \( K^B_0 \) to \( K^B_{0,\text{new}} \), illustrated in Figure 4. Panel A represents Blue flow-casualties (the solid and dashed lines) and Blue casualties (the areas under the lines). Blue flow-casualties converge to zero because the stock of Red weapons converges to zero in a Blue military victory. Recall that Blue flow-casualties are lower at each point in time when \( K^B_0 \) is higher (Equation 6), implying that the dashed line is below the solid line. Thus, the light-shaded area in Panel A indicates a reduction in Blue casualties, \( D^B_\tau \).

This reduction combines two effects acting in the same direction: the lower
Blue flow-casualties, 
\[d^B_t = \theta^B K^B_0\]

Red flow-casualties, 
\[d^R_t = \theta^R K^R_0\]

Note: In panel A, the vertical axis measures Blue flow-casualties, \(d^B_t\). Thus, the area under a line and up to time \(t\) represents Blue casualties, \(D^B_t = \int_0^t d^B_u du\). The left-pointing arrow under the horizontal axis indicates the shorter duration of war. Panel B reads in the same way.

Panel B represents Red flow-casualties and Red casualties. Two effects operate in opposite directions. First, Red flow-casualties are higher at each point in time (Equation 7). Second, the war is shorter. The dark-shaded area represents the increase in Red casualties due to the first effect. The light-shaded area represents the decrease due to the second effect. In the end, the second effect dominates, as indicated in Equation (15). Thus, in a Blue military victory, Red casualties are reduced when the initial stock of Blue weapons is larger. The effect of an increase in Blue reinforcements, \(X^B\) can be understood in a similar way.

Figure 5 shows the effect of an increase in Red resources, namely \(X^R\). In Panel A, Blue flow-casualties are higher at each point in time because \(X^R\) is higher—
Blue flow-casualties,
\[ d_t^B = \theta^B K_t^B \]

\[ \text{with } X^R \quad \text{with } X_{\text{new}}^R > X^R \]

Red flow-casualties,
\[ d_t^R = \theta^R K_t^R \]

\[ \text{with } X^R \quad \text{with } X_{\text{new}}^R > X^R \]

\[ A \quad \text{— Blue casualties} \quad \text{B} \quad \text{— Red casualties} \]

Figure 5: The effect of \( X^R \) on casualties when \( B > 0 \)

\[ \text{Note: In panel A the vertical axis measures Red flow-casualties, } d_t^R. \text{ Thus, the area under a line and up to time } t \text{ represents Red casualties, } D_t^R = \int_0^t d_u^R du. \text{ The right-pointing arrow under the horizontal axis indicates the longer duration of war. Panel B reads in the same way.} \]

this is the symmetric effect of that described in Equation (7) for \( d_t^R \). Thus, the dark shaded area indicates an increase in Blue casualties resulting from two effects acting in the same direction: the higher flow-casualties and the longer war. Panel B shows Red flow-casualties, which are lower at each point in time—this is the symmetric effect of that described in Equation (6) for \( d_t^B \).

The effect on Red casualties combines two effects acting in opposite directions: The light-shaded area represents a reduction of Red casualties due to fewer flow-casualties. The dark-shaded area represents an increase due to the longer war. In the end, as indicated in Equation (15), the second effect dominates and Red casualties increase as a result of increased Red reinforcements. The effect of the initial Red weapons stock can be understood similarly.

The results in Equations (15) and (16) can be stated more generally as the
following: At a military conclusion, **both** sides’ casualties decrease with the resources of the belligerent obtaining the military victory and increase with the resources of the belligerent being militarily defeated.

### 2.4 Political conclusion

I assume there are exogenously-determined threshold levels of casualties beyond which a country decides to sue for peace. Let $\bar{D}_R$ and $\bar{D}_B$ denote these thresholds for Red and Blue, respectively. I label such conclusion “political,” as opposed to “military,” because the fighting strength of the country suing for peace need not be lower than its opponent’s. For example, Blue can sue for peace when $B > 0$.

The Vietnam war is an example of a conflict that did not reach a military conclusion in the sense of Section 2.3. No belligerent was militarily incapable of fighting when the war concluded. Instead, political forces in the United States compelled decision makers to reduce the U.S. involvement in the war. The peak of U.S. troops in Vietnam was in April 1969 (Anderson, 2002, p. 187), and evidence of political discontent with the war is numerous, e.g., the anti-war movement or the repeal of the Tonkin Gulf Resolution in 1970.

World War I is another example, although more controversial. Allied forces were not in Germany when the war ended, and Douglas Haig, the commander of the British Expeditionary Force, said of the November 1918 armistice: “Germany is not broken in a military sense” (Liddell Hart, 2012, ch. 13).

**Duration of war** Blue sues for peace at $\tau_B < \tau$ if casualties reach their threshold value and Red has not yet sued for peace, i.e., if $D_{\tau_B}^B = \bar{D}^B$ or, using Equation (9),

\[
K_{\tau_B}^B = \tau^B X^B + K_0^B - \bar{D}^B.
\]

Note that if $\tau^B > 0$ exists, it is unique because the slope of the left-hand side (with respect to time) is less than that of the right-hand side. Similarly, Red
sues for peace at \( \tau^R < \tau \) if Blue has not done so already and

\[
K^{R \tau}_R = \tau^R X^R + K^R_0 - \tilde{D}^R.
\]

Recall that casualties are cumulative so that \( D^R_t \) and \( D^B_t \) are monotonically increasing over time. It follows that, if \( B > 0 \),

the conclusion is

\[
\begin{cases} 
\text{military at } \tau & \text{if } \bar{D}^R \geq D^R_\tau \text{ and } \bar{D}^B \geq D^B_\tau, \\
\text{political at } \min \{ \tau^B, \tau^R \} & \text{otherwise,}
\end{cases}
\]

where \( D^R_\tau \) and \( D^B_\tau \) are given by Equation (14).\footnote{If \( B < 0 \) the same rule applies, but the date \( \tau \) of a military conclusion is different from that discussed in Section 2.3 and end-of-war casualties, \( D^B_\tau \) and \( D^R_\tau \), are different from those given in Equation (14).} Figure 6 represents weapons stocks trajectories and the determination of \( \tau, \tau^B, \) and \( \tau^R \) in an example where \( B > 0 \) and Blue’s political threshold, \( \bar{D}^B \), is low enough that \( \tau^B \) is prior to the date at which Blue would obtain a favorable military conclusion, \( \tau \), and prior to the date at which Red would sue for peace, \( \tau^R \). Thus, Blue sues for peace at \( \tau^B \) even though it is poised to obtain a military victory.

How do resources allocated to war by either belligerent affect the date at which one of them initiates a political conclusion? I show (Equations E.1-E.8 in Appendix E) that

\[
\frac{\partial \tau^B}{\partial X^B} > 0, \quad \frac{\partial \tau^B}{\partial X^R} < 0, \quad \frac{\partial \tau^B}{\partial K^B_0} > 0, \quad \frac{\partial \tau^B}{\partial K^R_0} < 0,
\]

and

\[
\frac{\partial \tau^R}{\partial X^B} < 0, \quad \frac{\partial \tau^R}{\partial X^R} > 0, \quad \frac{\partial \tau^R}{\partial K^B_0} < 0, \quad \frac{\partial \tau^R}{\partial K^R_0} > 0.
\]

The date at which a country reaches its political threshold is increasing with the resources committed by the country and decreasing with the resources committed by its opponent. These results are independent of the sign of \( B \). They hold regardless of which country is poised to obtain a military victory.

Suppose Blue commits additional resources to the war, either via \( K^B_0 \) or via...
Figure 6: Weapons stock trajectories and political thresholds when $B > 0$

$X^B$. This reduces the flow of Blue casualties at each point in time (Equation 6) and, thus, lengthens the time necessary for Blue to reach its political threshold: $\tau^B$ increases. Again, this is because additional Blue weapons imply a higher flow of Red casualties (Equation 7) and, thus, impair Red’s ability to destroy Blue weapons. The higher flow of Red casualties shortens the time necessary for Red to reach its political threshold: $\tau^R$ decreases. The effects of Red resources on $\tau^B$ and $\tau^R$ have similar explanations.

Casualties The casualties of the country suing for peace are given by its political threshold. How are its opponents’ casualties affected by resources? I show (Appendix F) that

$$\frac{\partial D^R_{\tau^B}}{\partial X^B} > 0, \quad \frac{\partial D^R_{\tau^B}}{\partial X^R} < 0, \quad \frac{\partial D^R_{\tau^B}}{\partial K^B_0} > 0, \quad \frac{\partial D^R_{\tau^B}}{\partial k^R_0} < 0,$$
and 
\[ \frac{\partial D^B}{\partial X^B} < 0, \quad \frac{\partial D^B}{\partial X^R} > 0, \quad \frac{\partial D^B}{\partial K^B_0} < 0, \quad \frac{\partial D^B}{\partial k^R_0} > 0. \]

At a political conclusion initiated by Blue, at date \( \tau^B \), Red casualties are increasing in Blue resources and decreasing with Red resources. Symmetrically, if Red initiates the political conclusion at date \( \tau^R \), Blue casualties are increasing in Red resources and decreasing with Blue resources.

Consider a conclusion initiated by Blue at \( \tau^B \). When Blue commits additional resources to the war, either via \( K^B_0 \) or via \( X^B \), two effects raise Red casualties: First, Red flow-casualties are higher at each point in time (Equation 7). Second, the war is longer because the date \( \tau^B \) is pushed into the future (Equation 17). If Red commits additional resources to the war, two effects lower Red casualties: First Red flow-casualties are lower—this is the symmetric effect of that described in Equation (6) for \( d^B_t \). Second the war is shorter (Equation 17). A similar explanation applies to the effects of resources on \( D^B_{\tau^R} \).

3 Using the model

3.1 Rich v. poor countries

In December 1941 Japan declared war on the United States even though the latter was a larger economy: The gross domestic product (GDP) difference was 5-fold and the population difference was almost 2-fold, both in favor of the United States.\(^5\) In February 2022 Russia invaded Ukraine. The latter was significantly smaller: The GDP difference was 8-fold and the population difference was more than 3-fold, both in favor of Russia.\(^6\) How are GDP differences relevant for the outcomes of war?

Let \( Y^R \) and \( Y^B \) denote the (constant) GDP of Red and Blue, respectively. I assume that, before the war, Red and Blue are at steady states where they

\(^5\)Maddison Project Database 2020.
\(^6\)2018 figures from the Maddison Project Database 2020.
allocate constant fractions of GDP to their military. Thus, at the start of war
their initial weapons stocks are proportional to their GDP:

\[ K_0^B = s^BY^B \quad \text{and} \quad K_0^R = s^RY^R. \]

I refer to \( s^R \) and \( s^B \) as prewar (military) saving rates for the sake of exposition.\(^7\)
I further assume that, in wartime, Red and Blue allocate constant fractions
(possibly different than in peacetime) of their GDP to reinforcements

\[ X^B = \sigma^BY^B \quad \text{and} \quad X^R = \sigma^RY^R, \]

where \( \sigma^B \) and \( \sigma^R \) are wartime investment rates in weapons.

The prevailing side The condition for a Blue military victory in Equation
(10) can be written \( \mathcal{B}/Y^B > 0 \), yielding

\[ s^B + \frac{\sigma^B}{\sqrt{\theta^R\theta^B}} > \frac{\sqrt{\theta^R}Y^R}{\sqrt{\theta^B}} \left( s^R + \frac{\sigma^R}{\sqrt{\theta^R\theta^B}} \right). \]  \( (19) \)

I refer to \( \sqrt{\theta^B}Y^B \) as the fighting strength of Blue GDP. That is, if Blue
converted the entirety of its GDP into weapons, the fighting strength it would
obtain is precisely \( \sqrt{\theta^B}Y^B \). I refer to \( \sqrt{\theta^R}Y^R \) similarly.

The ratio \( \mathcal{Y} \) of Red-to-Blue fighting strength of GDP, in Equation (19), indicates
the role of each country’s GDP in determining the prevailing side in a
military conclusion. Suppose, for instance, that \( s^R > s^B \) and \( \sigma^R > \sigma^B \). That
is, Red allocates a larger fraction of its GDP to the military both in peacetime
and in wartime. This does not imply a Red victory, however, since Blue’s GDP
could be larger. But how exactly does GDP matter? The answer is through
\( \mathcal{Y} \). With \( \mathcal{Y} \) small enough, Blue obtains a military victory.

\(^7\)With a constant saving rate, as in the Solow model for instance, the capital-to-output
ratio is proportional to the saving rate, albeit not equal to it.
As with condition (10), the role of military technology, via the attrition coefficients, is small relative to the role of GDP: The elasticity of $\mathcal{Y}$ with respect to relative technology $\theta^R/\theta^B$ is $1/2$, while the elasticity of $\mathcal{Y}$ with respect to relative GDP is 1. This is illustrated in Figure 7, which shows level curves of $\mathcal{Y}$ in the $(Y^R/Y^B, \theta^R/\theta^B)$ plane. As Red GDP becomes smaller relative to Blue GDP, the relative technology advantage in favor of Red, necessary to maintain $\mathcal{Y}$, increases quadratically. A poor country is thus at a disadvantage relative to a rich country not only because it has fewer resources, but also because the resource gap is increasingly difficult to offset with better technology the poorer the country is.

**The duration of war** Assume that the condition for a Blue military victory is satisfied (Equation 19) and that $K^R_{\tau} = 0$. The role of GDP for the duration
of war can be gauged by deriving, from Equations (B.1)–(B.4) that

\[
\frac{ds^B}{ds^R}\bigg|_{\tau^R=0} = \mathcal{Y} \left( \frac{-\cosh(\tau^B \lambda_1)}{\sinh(\tau^B \lambda_1)} \right) \quad \text{and} \quad \frac{d\sigma^B}{d\sigma^R}\bigg|_{\tau^R=0} = \mathcal{Y} \left( \frac{\sinh(\tau^B \lambda_1)}{1 - \cosh(\tau^B \lambda_1)} \right).
\]

These equations describe the level curves of \(\tau\) in the pre- and wartime saving rates planes. Consider increases in Red’s pre- and/or wartime investment, \(s^R\) and/or \(\sigma^R\). Since Blue is poised to obtain a military victory, these increases lengthen the war (Equation 11). Blue can offset these effects by raising \(s^B\) and/or \(\sigma^B\). The lower \(\mathcal{Y}\), the cheaper it is for Blue to do so: That is, the lower \(\mathcal{Y}\), the smaller the increase in \(s^B\) and/or \(\sigma^B\) is needed for Blue to offset the increase in Red’s pre- and/or wartime investment.

Blue may reach its political threshold before it obtains the military victory, however. Furthermore, Red can hasten such political conclusion by raising \(s^R\) and/or \(\sigma^R\) and thus committing additional resources to the war (Equation 17). Equations (E.1)-(E.4) imply

\[
\frac{ds^B}{ds^R}\bigg|_{\tau^B=0} = \mathcal{Y} \sinh\left(\tau^B \lambda_1\right) \frac{1}{1 - \cosh(\tau^B \lambda_1)} \quad \text{and} \quad \frac{d\sigma^B}{d\sigma^R}\bigg|_{\tau^B=0} = \mathcal{Y} \left( \frac{1 - \cosh(\tau^B \lambda_1)}{\sinh(\tau^B \lambda_1) - \tau^B \lambda_1} \right).
\]

Thus, Blue can offset the effects of \(s^R\) and/or \(\sigma^R\) on \(\tau^B\) by raising \(s^B\) and/or \(\sigma^B\) in turn. As above, the lower \(\mathcal{Y}\), the cheaper it is for Blue to do so.

**The Pacific war**  To emphasize the usefulness of the results in this section, I submit that they suggest an interpretation of the outcome of the Pacific war opposing the U.S. to Japan which, to the best of my knowledge, has not been emphasized in the historical literature.

Historical evidence (e.g., Nolan, 2017, Toll, 2012) indicate that the Japanese military (i) wanted a short war because they knew that Japanese GDP could not sustain a protracted competition against U.S. GDP, and (ii) expected to
obtain a rapid political victory because they assumed the U.S. would quickly sue for peace. The results above emphasize a weaknesses in this reasoning. The weakness I refer to, however, was not to misjudge the U.S. willingness to take casualties. Instead it was the failure to note that the larger U.S. GDP, which was conducive to a Japanese military defeat in the long-run, was not conducive to a Japanese political victory in the short run. In other words, the same reason that made a “long” war unwinnable for Japan also implied that the war would be “long” before the U.S. reached its political threshold as the Japanese leaders expected.

In the language of the model (where Blue is the U.S. and Red is Japan), Equation (19) held at the start of the War because \( Y \) was low enough, mostly due to a low \( Y^R/Y^B \). Thus, the U.S. was poised to obtain a military victory. The Japanese military recognized that, but assumed that the U.S. political threshold, \( \bar{D}^B \), was so low that \( \tau^B < \tau \) and that, therefore, the U.S. would sue for peace before obtaining a military victory. Furthermore, the Japanese military could hasten a favorable (to them) political conclusion by inflicting casualties on the U.S., i.e., by exploiting Equation (17) and lowering \( \tau^B \). Yet, Equations (20) indicate that the U.S. could offset the effects of Japanese military spending on \( \tau^B \) relatively easily because \( Y \) was low enough. Thus, the large U.S. GDP, relative to that of Japan, played two roles in the war. First it ensured that Equation (19) held and that the U.S. could eventually obtain a military victory—a point that has been emphasized by numerous authors (e.g., footnote 2). Second, it ensured that the date at which the U.S. would sue for peace, \( \tau^B \), could be pushed into the future—a point not emphasized, to the best of my knowledge, in the existing historical literature. Note that this second point remains true regardless of Blue’s political threshold \( D^B \): No matter how high or low \( D^B \) is, the higher Blue’s GDP the farther into the future will \( \bar{D}^B \) be reached. Formally, this discussion can be summarized as

\[
\frac{\partial B}{\partial Y^B} > 0, \quad \text{and} \quad \frac{\partial \tau^B}{\partial Y^B} > 0,
\]

\(^8\)Maybe the Japanese did, indeed, misjudge the U.S. willingness to take casualties. My work, however, cannot shed any insights on this debate.
where the first inequality (which derives from Equation 19) represents the first effect above: a larger GDP is conducive to a military victory. The second inequality (which derives from Equation 17) represents the second effect above: a larger GDP is conducive to a late unfavorable political conclusion.\(^9\)

### 3.2 Foreign support

In the Russia-Ukraine war started in February 2022, Ukraine is assessed by the international community to be unable to withstand the Russian military on its own. This assessment prompted the United States and other countries to provide military support to Ukraine. How does a third party support matter for the outcomes of war?

Consider a conflict where Red is poised for a military victory, i.e., \(B < 0\). Assume that a third party, e.g., a coalition of foreign countries, supports Blue with a one-time transfer of weapons, \(S_K\), at date 0 and/or a commitment to a flow of reinforcements \(S_X\) at each point in time. Define \(K_{0, \text{new}}^B = K_0^B + S_K\) and \(X_{\text{new}}^B = X^B + S_X\) and define \(K_{\text{new}}^R\) and \(B_{\text{new}}\) accordingly.

The prevailing side  The condition for a Blue military victory with foreign support is \(B_{\text{new}}/Y^B > 0\), that is,

\[
s^B + S_K/Y^B + \frac{\sigma^B + S_X/Y^B}{\sqrt{\theta^R \theta^B}} > Y \left( s^R + \frac{\sigma^R}{\sqrt{\theta^R \theta^B}} \right).
\]

(21)

The effect of a time-0 transfer of weapons, that is, \(S_K > 0\) and \(S_X = 0\), is represented in Panel A of Figure 8. The initial condition is below the stable branch and thus, Blue is on a trajectory to a military defeat (black). If \(S_K\) is large enough, the initial condition is above the stable branch and Blue obtains a military victory (orange).

The effect of a commitment to reinforcements at each point in time, that is

\(^9\)I am not arguing that the U.S. did not suffer high casualties during the Pacific war.
$S_K = 0$ and $S_X > 0$, is represented on Panel B. Additional reinforcements raise the stalemate value of Red’s weapons stock to $\tilde{K}^R_{\text{new}} = X^B_{\text{new}}/\theta^R > \tilde{K}^R$, making it harder for Red to attrit the Blue force. Graphically, the stable and unstable branches translate to the right while the initial condition $(K^R_0, K^B_0)$ does not change. Instead of being below the stable branch, i.e., $\mathcal{B} < 0$, the initial condition is above the new stable branch, i.e., $\mathcal{B}_{\text{new}} > 0$. The without-support (black) arrow represents the dynamics of war absent foreign support: Blue’s weapons stock goes to zero. The with-support (orange) arrow represents the dynamics with foreign support: Red’s weapons stock goes to zero.

**Figure 8: The effect of foreign support to Blue**

The cost of support  What is the cost-minimizing support (from the point of view of the third party) that ensures a Blue military victory? If the support is a one-time transfer of weapons, then the answer to this question is trivial: It is $S_{K, \text{min}}$ such that Equation (21) is satisfied at equality with $S_K = S_{K, \text{min}}$ and $S_X = 0$. With such support the system is on the stable branch and the
war in a stalemate: Its duration is infinity. Any support marginally above $S_{K,\min}$ would permit a Blue military victory. Thus, the cost-minimizing one-time transfer maximizes the duration of war before a military conclusion and increases, therefore, the likelihood of a political conclusion initiated by either belligerent.

If support is a commitment to reinforcements at each point in time, $S_X > 0$, the cost of support at a military conclusion for the foreign coalition is $\tau S_X$. Recall from Equation (11) that raising Blue reinforcements lowers the duration of war, $\tau$. Thus, higher support has two opposite effects on the cost to the foreign coalition: a direct effect increasing $\tau S_X$ via $S_X$, and an indirect effect reducing $\tau S_X$ via $\tau$. Figure 9 illustrates the trade off. A low support (orange) that would move the stable branch such that the initial condition is just above the stable branch implies a long war. A high support (purple) implies a shorter war. The cost-minimizing support for a military victory must
satisfy \((\partial \tau / \tau) / (\partial X^B / X^B) = -1\).

4 Empirical application: Iwo Jima

In this section, I present an empirical application of the model to the case of a battle for lack of data about an entire war. My goal is to replicate the analysis of the battle of Iwo Jima, first presented by Engel (1954), and to argue that the Lanchester model provides an empirically relevant description of the process of military attrition.

Iwo Jima is a small volcanic island in the Western Pacific Ocean, almost halfway between Tokyo and Guam in the Mariana archipelago. During World War II, Iwo Jima was deemed a strategic objective by the United States and assaulted on February 19, 1945 (D-day). Organized resistance by Japanese troops ceased on March 26.

Let Blue represent the United States and Red represent Japan. Since the existing data pertain to the number of troops, let \(K_t^B\) and \(K_t^R\) indicate the stock of U.S. and Japanese troops fighting on day \(t\), respectively. Reinforcements, that is, \(X_t^B\) and \(X_t^R\), are then the flow of troops landing on day \(t\).

The U.S. data consist of the number of casualties (killed, wounded, or missing) per day (Morehouse, 1946) and number of newly landed troops per day. One set of estimates for newly landed troops is from Engel (1954) who reports a total of 73,000 troops coming ashore: 54,000 on D-day, 6,000 on D-day+2, and 13,000 on D-day+5. Samz (1972) argues that Engel’s pattern of troop landing is too high and proposes alternative figures with a total of 71,245 troops landing between D-day and D-day+10. In what follows, I consider both Engel’s and Samz’s estimates and construct \(K_t^B\), the stock of fighting U.S. troops on Iwo Jima, via

\[ K_{t+1}^B = K_t^B - \text{Casualities}_t + X_t^B, \]
Figure 10: Two estimates of U.S. troops \((K_t^B)\) on Iwo Jima, Feb. 19 to Mar. 26, 1945

with the initial condition \(K_0^B = 0\) (no U.S. troops on the island at the start of D-day) and \(X_t^B\) given by either the Engel or the Samz patterns of reinforcements. Figure 10 shows \(K_t^B\) under each scenario.

There are no available data for the stock of Japanese fighting troops per day or Japanese casualties per day. The existing record indicates, however, that there was a stock \(K_0^R = 21,500\) of Japanese troops on Iwo Jima on the eve of the U.S. invasion. Both Engel and Samz use this figure. The Japanese garrison received no reinforcements throughout the battle, i.e., \(X_t^R = 0\). Both Engel and Samz estimate that there remained no Japanese troops fighting at the end of the battle, that is, \(K_{35}^R = 0.10\).

A discrete-time version of Equations (1) and (2) with time-varying reinforce-

\(^{10}\)Recently, Toll (2020, p. 516) reports a garrison of “about 22,000” Japanese on the island at the start of the battle and that, except for a few hundred taken prisoner, the entire garrison was killed.
ments is
\[ K_{i+1}^i - K_i^i = -\theta_i K_i^{i-i} + X_i^i, \]  
(22)
where \((i, -i)\) stands for either \((B, R)\) or \((R, B)\). Engel (1954) proposed a technique for estimating attrition coefficients using the existing data. In the discrete version of the model presented here, Engel’s technique can be described as follows: Summing over \(t\), Equation (22) for \((i, -i) = (R, B)\) implies
\[ K_{35}^R - K_0^R = -\theta B \sum_{t=0}^{35} K_t^B + \sum_{t=0}^{35} X_t^R. \]  
(23)
The left-hand side is observed Japanese casualties, 21,500; the right-hand side is the sum of Japanese reinforcements, which is zero; and, finally, \(\sum_{t=0}^{35} K_t^B\) can be deduced from the record of U.S. casualties and reinforcement. Under Engel’s scenario for U.S. reinforcements, this figure is 1,971,820. It follows that an estimate of the U.S. attrition coefficient is \(\hat{\theta}^B = 0.011\) with Engel’s data. A similar calculation with Samz’s data yields \(\hat{\theta}^B = 0.012\).

Given \(\hat{\theta}^B\) and \(K_t^B\), Equation (22) yields an estimate of the stock of Japanese troops per day: \(\hat{K}_t^R\). The equivalent of Equation (23) for U.S. casualties is then
\[ K_{35}^B - K_0^B = -\theta R \sum_{t=0}^{35} \hat{K}_t^R + \sum_{t=0}^{35} X_t^B. \]  
The left-hand side and the sum of U.S. reinforcements on the right-hand side are obtained from Morehouse (1946): 52,150 and 73,000, respectively. The simulated \(\hat{K}_t^R\) yields \(\sum_{t=0}^{35} \hat{K}_t^R = 39,436\). It follows that an estimate for the Japanese attrition coefficient is \(\hat{\theta}^R = 0.052\). A similar calculation with Samz’s data yields \(\hat{\theta}^R = 0.050\).

Panel A of Figure 11 shows the stock of fighting U.S. troops in both the model and the data. Recall that \((\hat{\theta}^B, \hat{\theta}^R)\) is not a minimum-distance estimator, i.e., the estimates were not constructed to fit the time series \(K_t^B\) in Panel A of Figure 11. Yet, the estimated parameters yield a close fit between the observed and predicted stock of fighting U.S. troops throughout the 36 days of battle.
Figure 11: The Lanchester model’s implications for the battle of Iwo Jima

Note: In panel A, the stock of active U.S. troops is under either Engel’s assumption for U.S. reinforcement or Samz’s.

Source: Author’s calculations, Morehouse (1946), Engel (1954), and Samz (1972).

This observation, first made by Engel (1954), indicates that the Lanchester model is a quantitatively relevant model to study attrition patterns during battles and/or wars.

Panel B of Figure 11 shows the model’s implications for U.S. casualties and the unobserved Japanese troops and casualties. The model fits the daily accumulation of U.S. casualties closely, as it does for the stock of remaining U.S. troops on the island. Again, this is not the result of fitting the observed time series but is an indication of the model’s empirical relevance.

Observe that the estimated stock of Japanese fighting troops is decaying monotonically because the Japanese did not receive reinforcements during the battle. Observe also that the accumulated U.S. casualties were higher than that of the Japanese and ended at nearly 21,000 in both the model and Morehouse’s data. More recent work (e.g. O’Brien, 2015, p. 452) reports higher figures (more than 26,000) for U.S. casualties, making Iwo Jima one of the few battles of World War II where U.S. casualties exceeded those of the opposing force.
For consistency’s sake, however, I could not use the more recent estimate of total U.S. casualties without a description of the daily flow, as in Morehouse (1946).

Iwo Jima illustrates the notion of attrition well: Despite their higher attrition coefficient and thus higher ability to inflict casualties, the Japanese could not withstand the mass of U.S. military resources they faced on Iwo Jima.

5 Conclusion

Despite the large literature on war finances, there is little or no work studying how war-related expenditures affect the outcomes of war which, I argue, are economically relevant. My goal in this paper is to suggest how to fill this gap.

Historians have argued that wars are often decided by attrition instead of “decisive” battles and genius-like generalship. Attrition, in turn, emphasizes the importance of resources in determining the outcomes of war. Hence, I use a model of resource attrition derived from combat models à la Lanchester (1916) to represent war.

I consider military conclusions, where one side cannot fight anymore for lack of resources, and political conclusions, where one side does not fight anymore for lack of political will. Under each scenario, I describe how resources determine the duration of the war, the destruction and casualties, and the prevailing side.

The military technology I assume, labelled “aimed fire” in the Operations Research literature, has important implications for the role of resources in the outcomes of war. The technology assumes that a belligerent’s casualties are independent of the mass of resources it commits to war. Some of the implied results are as follows: (i) a country obtaining a military victory can shorten the war by allocating more resources to it and, thus, reduce casualties for both sides; (ii) higher GDP makes the condition for a military victory more favorable and the condition for suing for peace on political grounds less favorable; and
(iii) there is a well-defined cost-minimizing level of support from a third-party to a small country fighting a war against a larger country.

Resources are exogenous in the model I presented. Endogenizing production (e.g. à la Solow) and the destruction of productive capacities seems a natural extension of this work, which I leave to future research. The modeling of decisions, such as the allocation of resources toward the production of consumption goods versus the production of military equipment also seems a natural extension. Finally, the collection of data and the testing of the model on an actual war instead of a battle (as in Section 4) is yet another avenue for future work.

References


A Solution of Lanchester model

Dynamics of weapons stocks  Let $\mathcal{P}$ represent the matrix of eigenvectors of $\mathcal{M}$ in Equation (3). Standard methods imply that the solution of (3) can be expressed as

$$
\begin{pmatrix}
\tilde{K}_t^R \\
\tilde{K}_t^B
\end{pmatrix} = \mathcal{P} \begin{pmatrix}
e^{\lambda_1 t} & 0 \\
0 & e^{\lambda_2 t}
\end{pmatrix} \mathcal{P}^{-1} \begin{pmatrix}
\tilde{K}_0^R \\
\tilde{K}_0^B
\end{pmatrix},
$$

where

$$
= \begin{pmatrix}
e^{\lambda_1 t} & e^{\lambda_2 t} \\
v_1 e^{\lambda_1 t} & v_2 e^{\lambda_2 t}
\end{pmatrix} \begin{pmatrix}(v_2 \tilde{K}_0^R - \tilde{K}_0^B)/(v_2 - v_1) \\
(\tilde{K}_0^B - v_1 \tilde{K}_0^R)/(v_2 - v_1)
\end{pmatrix}.
$$

Recall that $v_2 = -v_1$. The system can then be written as

$$
\tilde{K}_i^R = \frac{1}{2} \left[ e^{\lambda_1} A - e^{\lambda_2 B} \right] \frac{1}{v_1}, \quad (A.1)
$$

$$
\tilde{K}_i^B = \frac{1}{2} \left[ e^{\lambda_1} A + e^{\lambda_2 B} \right], \quad (A.2)
$$

where $A = \tilde{K}_0^B + v_1 \tilde{K}_0^R$ and $B = \tilde{K}_0^B - v_1 \tilde{K}_0^R$.

The effect of resources on weapons stocks dynamics  The derivatives of $K_i^B$ are

$$
\frac{\partial K_i^B}{\partial X^B} = \frac{1}{2} \left( e^{\lambda_1} \left( -\frac{v_1}{\theta^R} \right) + e^{-t \lambda_1} \left( \frac{v_1}{\theta^B} \right) \right) = \frac{\sinh(t \lambda_1)}{\lambda_1} > 0, \quad (A.3)
$$

$$
\frac{\partial K_i^B}{\partial X^R} = \frac{1}{\theta^B} + \frac{1}{2} \left( e^{\lambda_1} \left( -\frac{1}{\theta^B} \right) + e^{-t \lambda_1} \left( -\frac{1}{\theta^B} \right) \right) = \frac{1 - \cosh(t \lambda_1)}{\theta^B} < 0, \quad (A.4)
$$

$$
\frac{\partial K_i^B}{\partial K_0^B} = \frac{1}{2} \left( e^{\lambda_1} + e^{-t \lambda_1} \right) = \cosh(t \lambda_1) > 0, \quad (A.5)
$$

$$
\frac{\partial K_i^B}{\partial K_0^R} = \frac{1}{2} \left( e^{t \lambda_1} (v_1) + e^{-t \lambda_1} (-v_1) \right) = v_1 \sinh(t \lambda_1) < 0, \quad (A.6)
$$

where $\cosh$ and $\sinh$ are the hyperbolic cosine and sine functions, respectively. The derivatives of $K_i^R$ are

$$
\frac{\partial K_i^R}{\partial X^B} = \frac{1}{\theta^R} + \frac{1}{2v_1} \left( e^{t \lambda_1} \left( -\frac{v_1}{\theta^R} \right) - e^{-t \lambda_1} \left( \frac{v_1}{\theta^R} \right) \right) = \frac{1 - \cosh(t \lambda_1)}{v_1 \theta^R} < 0, \quad (A.7)
$$

$$
\frac{\partial K_i^R}{\partial X^R} = \frac{1}{2v_1} \left( e^{\lambda_1} \left( -\frac{1}{\theta^B} \right) - e^{-t \lambda_1} \left( -\frac{1}{\theta^B} \right) \right) = \frac{\sinh(t \lambda_1)}{\lambda_1} > 0, \quad (A.8)
$$

$$
\frac{\partial K_i^R}{\partial K_0^B} = \frac{1}{2v_1} \left( e^{t \lambda_1} - e^{-t \lambda_1} \right) = \frac{\sinh(t \lambda_1)}{v_1} < 0, \quad (A.9)
$$

$$
\frac{\partial K_i^R}{\partial K_0^R} = \frac{1}{2v_1} \left( e^{t \lambda_1} (v_1) - e^{-t \lambda_1} (-v_1) \right) = \cosh(t \lambda_1) > 0. \quad (A.10)
$$
The following relationships are useful:

\[
AB = \left( \tilde{K}_B^R \right)^2 - \left( v_1 \tilde{K}_B^R \right)^2, \quad (A.11)
\]
\[
A - B = 2v_1 \tilde{K}_B^R, \quad (A.12)
\]
\[
A + B = 2 \tilde{K}_B^R. \quad (A.13)
\]

**Casualties**  
Red casualties are \( D_R^t = \int_0^t \theta_B K_u^R du \), that is

\[
D_R^t = \theta_B \int_0^t \left[ \tilde{K}_u^B + \tilde{K}_u^R \right] du = \theta_B \frac{1}{2} \int_0^t \left[ e^{v_1 \lambda_1} A + e^{v_1 \lambda_2} B \right] du + tX^R,
\]

\[
= \frac{\theta_B}{2 \lambda_1} \left[ A (e^{t \lambda_1} - 1) + B (1 - e^{-t \lambda_1}) \right] + tX^R.
\]

Using \( \theta_B / \lambda_1 = -1 / v_1 \), Equation (A.1), and Equation (A.12) yields

\[
D_R^t = tX^R + K_t^R - K_0^R. \quad (A.14)
\]

and Blue casualties are \( D_B^t = \int_0^t \theta_B K_u^R du \):

\[
D_B^t = \theta_B \int_0^t \left[ \tilde{K}_u^B + \tilde{K}_u^R \right] du = \theta_B \frac{1}{2v_1} \left[ A \int_0^t e^{v_1 \lambda_1} du - B \int_0^t e^{v_1 \lambda_2} du \right] + tX^B,
\]

\[
= \frac{\theta_B}{2v_1 \lambda_1} \left[ A (e^{t \lambda_1} - 1) - B (1 - e^{-t \lambda_1}) \right] + tX^B.
\]

Using \( v_1 \lambda_1 = -\theta_B \), Equation (A.2), and Equation (A.13) yields

\[
D_B^t = tX^B + K_t^B - K_0^B. \quad (A.15)
\]

**B  Time to military conclusion**

Assume that \( B > 0 \) such that, eventually, \( K_t^R = 0 \). Implicit differentiation yields

\[
\frac{\partial \tau}{\partial X^R} = -\frac{\partial K_t^R / \partial X^B}{\partial K_t^R / \partial \tau} = \frac{(1 - \cosh(\tau \lambda_1))) / \theta_B K_t^R}{\theta_B K_t^R} < 0, \quad (B.1)
\]
\[
\frac{\partial \tau}{\partial X^R} = -\frac{\partial K_t^R / \partial X^R}{\partial K_t^R / \partial \tau} = \frac{\sinh(\tau \lambda_1) / \lambda_1}{\theta_B K_t^R} > 0, \quad (B.2)
\]
\[
\frac{\partial \tau}{\partial K_0^R} = -\frac{\partial K_t^R / \partial K_0^R}{\partial K_t^R / \partial \tau} = \frac{\sinh(\tau \lambda_1) / v_1}{\theta_B K_t^R} < 0, \quad (B.3)
\]
\[
\frac{\partial \tau}{\partial K_0^R} = -\frac{\partial K_t^R / \partial K_0^R}{\partial K_t^R / \partial \tau} = \frac{\cosh(\tau \lambda_1)}{\theta_B K_t^R} > 0. \quad (B.4)
\]
### End-of-war Capital

Equations (A.1) and (A.2) imply

\[
4 \left( v_1 \tilde{K}_t^R \right)^2 = e^{2t\lambda_1} \Delta^2 + e^{2t\lambda_2} \mathcal{B}^2 - 2AB,
\]

\[
4 \left( \tilde{K}_t^B \right)^2 = e^{2t\lambda_1} \Delta^2 + e^{2t\lambda_2} \mathcal{B}^2 + 2AB.
\]

Adding up yields

\[
\left( \tilde{K}_t^B \right)^2 - \left( v_1 \tilde{K}_t^R \right)^2 = AB. \tag{C.1}
\]

Note this result is holds at date 0 (Equation A.11), and must also hold at date \( \tau \) when \( \tilde{K}_\tau^R = 0 \), implying

\[
\tilde{K}_\tau^B = \tilde{K}^B + \sqrt{\left( v_1 \tilde{K}_R \right)^2 + AB}. \]

Thus, \( \tilde{K}_\tau^B > \tilde{K}^B \) and it is immediate that \( \tilde{K}_\tau^B > \tilde{K}_0^B \) whenever \( \tilde{K}^B > \tilde{K}_0^B \). Suppose now that \( \tilde{K}_0^B > \tilde{K}^B \), then \( \tilde{K}_\tau^B > \tilde{K}_0^B \) whenever

\[
\sqrt{\left( v_1 \tilde{K}_R \right)^2 + AB} > K_0^B - \tilde{K}^B,
\]

\[
2 \tilde{K}^R > K_0^R,
\]

where the second line follows from Equation (A.11). Then

\[
\frac{\partial K_\tau^B}{\partial K_0^B} = \frac{\partial}{\partial K_0^B} \left[ \left( v_1 \tilde{K}_R \right)^2 + AB \right]^{1/2} = \frac{K_0^B - \tilde{K}^B}{K_\tau^B - \tilde{K}^B}. \tag{C.2}
\]

\[
\frac{\partial K_\tau^B}{\partial X^B} \frac{\partial}{\partial X^B} \left[ \left( v_1 \tilde{K}_R \right)^2 + AB \right]^{1/2} = \frac{1}{\theta^B} \frac{K_0^B}{K_\tau^B - \tilde{K}^B}. \tag{C.3}
\]

\[
\frac{\partial K_\tau^B}{\partial K_0^R} = \frac{1}{2} \left( \left( v_1 \tilde{K}_R \right)^2 + AB \right)^{1/2-1} \frac{\partial}{\partial K_0^R} \left( \left( v_1 \tilde{K}_R \right)^2 + AB \right) = -v_1 \frac{K_0^R - \tilde{K}^R}{K_\tau^B - \tilde{K}^B}. \tag{C.4}
\]

and

\[
\frac{\partial K_\tau^B}{\partial X^R} = \frac{1}{\theta^B} + \frac{1}{2} \left( \left( v_1 \tilde{K}_R \right)^2 + AB \right)^{1/2-1} \frac{\partial}{\partial X^R} \left( \left( v_1 \tilde{K}_R \right)^2 + AB \right) = \frac{1}{\theta^B} \frac{K_\tau^B - K_0^R}{K_\tau^B - \tilde{K}^B}. \tag{C.5}
\]
D Casualties at Military Conclusion

Assume that \( B > 0 \) such that, eventually, \( K^R_\tau = 0 \). The derivatives of \( D^B_\tau \) are

\[
\frac{\partial D^B_\tau}{\partial X^B} = \frac{\partial}{\partial X^B} \int_0^\tau \frac{\partial K^B_u}{\partial X^B} du = \frac{\partial}{\partial X^B} K^B_\tau \frac{\partial \tau}{\partial X^B} + \frac{\partial}{\partial X^B} \int_0^\tau \frac{\partial K^B_u}{\partial X^B} du
\]

\[
= \frac{K^B_\tau}{K^B_{\tau}} \left( 1 - \cosh (\tau \lambda_1) \right) + \frac{\cosh (\tau \lambda_1) - 1}{\theta^R} = \frac{1 - \cosh (\tau \lambda_1)}{\theta^R} \left( \frac{K^B_\tau}{K^B_{\tau}} \right) < 0, \quad (D.1)
\]

\[
\frac{\partial D^R_\tau}{\partial X^R} = \frac{\partial}{\partial X^R} \int_0^\tau \frac{\partial K^B_u}{\partial X^R} du = \frac{\partial}{\partial X^R} K^B_\tau \frac{\partial \tau}{\partial X^R} + \frac{\partial}{\partial X^R} \int_0^\tau \frac{\partial K^B_u}{\partial X^R} du,
\]

\[
= \frac{K^B_\tau}{K^B_{\tau}} \sinh (\tau \lambda_1) + \tau - \frac{\sinh (\tau \lambda_1)}{\lambda_1} = \tau + \frac{\sinh (\tau \lambda_1)}{\lambda_1} \frac{K^B_\tau}{K^B_{\tau}} > 0, \quad (D.2)
\]

\[
\frac{\partial D^B_\tau}{\partial K^B_0} = \frac{\partial}{\partial K^B_0} \int_0^\tau \frac{\partial K^B_u}{\partial K^B_0} du = \frac{\partial}{\partial K^B_0} K^B_\tau \frac{\partial \tau}{\partial K^B_0} + \frac{\partial}{\partial K^B_0} \int_0^\tau \frac{\partial K^B_u}{\partial K^B_0} du,
\]

\[
= \frac{K^B_\tau}{K^B_{\tau}} \sinh (\tau \lambda_1) + \frac{\theta^B v_1}{\lambda_1} \sinh (\tau \lambda_1) = \frac{\sinh (\tau \lambda_1)}{v_1} \frac{K^B_\tau}{K^B_{\tau}} < 0, \quad (D.3)
\]

and

\[
\frac{\partial D^R_\tau}{\partial K^R_{\lambda_1}} = \frac{\partial}{\partial K^R_{\lambda_1}} \int_0^\tau \frac{\partial K^B_u}{\partial K^R_{\lambda_1}} du = \frac{\partial}{\partial K^R_{\lambda_1}} K^B_\tau \frac{\partial \tau}{\partial K^R_{\lambda_1}} + \frac{\partial}{\partial K^R_{\lambda_1}} \int_0^\tau \frac{\partial K^B_u}{\partial K^R_{\lambda_1}} du,
\]

\[
= \frac{K^B_\tau}{K^B_{\tau}} \cosh (\tau \lambda_1) + \frac{\theta^B v_1}{\lambda_1} (\cosh (\tau \lambda_1) - 1) = 1 + \cosh (\tau \lambda_1) \frac{K^B_\tau}{K^B_{\tau}} > 0. \quad (D.4)
\]

The derivatives of \( D^B_\tau \) are

\[
\frac{\partial D^B_\tau}{\partial X^B} = \theta^R \int_0^\tau \frac{\partial K^B_u}{\partial X^B} du = \theta^R \int_0^\tau \frac{1 - \cosh (u \lambda_1)}{\partial^R} du = \tau - \frac{\sinh (\tau \lambda_1)}{\lambda_1} < 0, \quad (D.5)
\]

\[
\frac{\partial D^B_\tau}{\partial X^R} = \theta^R \int_0^\tau \frac{\partial K^B_u}{\partial X^R} du = \theta^R \int_0^\tau \frac{\cosh (u \lambda_1) - 1}{\lambda_1} du = \cosh (\tau \lambda_1) - 1 > 0, \quad (D.6)
\]

\[
\frac{\partial D^B_\tau}{\partial K^B_0} = \theta^R \int_0^\tau \frac{\partial K^B_u}{\partial K^B_0} du = \theta^R \int_0^\tau \frac{\sinh (u \lambda_1)}{v_1} du = 1 - \cosh (\tau \lambda_1) < 0, \quad (D.7)
\]

and

\[
\frac{\partial D^B_\tau}{\partial K^R_{\lambda_1}} = \theta^R \int_0^\tau \frac{\partial K^B_u}{\partial K^R_{\lambda_1}} du = \theta^R \int_0^\tau \cosh (u \lambda_1) du = -v_1 \sinh (\tau \lambda_1) > 0. \quad (D.8)
\]
E   POLITICAL CONCLUSION

Blue reaches the threshold $\bar{D}^B$ at date $\tau^B$ such that (using A.15) $\tau^B X^B + K^B_0 - K^B_{\tau^B_0} = \bar{D}^B$. Implicitly differentiating with respect to $\tau^B$ and $\bar{D}^B$ yields $\partial \tau^B / \partial \bar{D}^B = (\theta^R K^R_{\tau^R_0})^{-1} > 0$ and, further differentiating,

\[ \frac{\partial \tau^B}{\partial X^B} = \frac{\partial \tau^B}{\partial \bar{D}^B} \times \left( \frac{\sinh(\tau^B \lambda_1)}{\lambda_1} - \tau^B \right) > 0, \quad (E.1) \]
\[ \frac{\partial \tau^B}{\partial X^R} = \frac{\partial \tau^B}{\partial \bar{D}^B} \times \frac{1 - \cosh(\tau^B \lambda_1)}{\theta^B} < 0, \quad (E.2) \]
\[ \frac{\partial \tau^B}{\partial K^B_0} = \frac{\partial \tau^B}{\partial \bar{D}^B} \times (\cosh(\tau^B \lambda_1) - 1) > 0, \quad (E.3) \]
\[ \frac{\partial \tau^B}{\partial K^R_0} = \frac{\partial \tau^B}{\partial \bar{D}^B} \times v_1 \sinh(\tau^B \lambda_1) < 0. \quad (E.4) \]

Similarly, Red crosses its threshold $\bar{D}^R$ at date $\tau^R$ such that (using A.14) $\tau^R X^R + K^R_0 - K^R_{\tau^R_0} = \bar{D}^R$. This implies $\partial \tau^R / \partial \bar{D}^R = (\theta^B K^B_{\tau^B_0})^{-1} > 0$, and

\[ \frac{\partial \tau^R}{\partial X^B} = \frac{\partial \tau^R}{\partial \bar{D}^R} \times \frac{1 - \cosh(\tau^R \lambda_1)}{\theta^R} < 0, \quad (E.5) \]
\[ \frac{\partial \tau^R}{\partial X^R} = \frac{\partial \tau^R}{\partial \bar{D}^R} \times \left( \frac{\sinh(\tau^R \lambda_1)}{\lambda_1} - \tau^R \right) > 0, \quad (E.6) \]
\[ \frac{\partial \tau^R}{\partial K^B_0} = \frac{\partial \tau^R}{\partial \bar{D}^R} \times \frac{\sinh(\tau^R \lambda_1)}{v_1} < 0, \quad (E.7) \]
\[ \frac{\partial \tau^R}{\partial K^R_0} = \frac{\partial \tau^R}{\partial \bar{D}^R} \times (\cosh(\tau^R \lambda_1) - 1) > 0. \quad (E.8) \]

F   CASUALTIES AT POLITICAL CONCLUSION

Red casualties when Red initiates a political conclusion are given by $D^R_{\tau^R} = \bar{D}^R$ and Blue casualties when Blue initiates a political conclusion are given by $D^B_{\tau^B} = \bar{D}^B$.

Blue casualties when Red initiates a political conclusion

\[
\frac{\partial D^B_{\tau^R}}{\partial X^R} = \frac{\partial}{\partial X^B} \left( \theta^R \int_{X^R_0}^{\tau^R} K^B_0 \, du \right) = \frac{\theta^R K^R_{\tau^R} R_{\tau^R}^R}{\theta^B K^B_{\tau^B_0} \theta^R} \left( 1 - \cosh(\tau^R \lambda_1) \right) + \tau^R - \frac{\sinh(\tau^R \lambda_1)}{\lambda_1} < 0,
\]
using Equations (E.5) and (A.7).

\[
\frac{\partial D^R}{\partial X^R} = \frac{\partial}{\partial X^R} \left( \theta^R \int_0^{\tau^R} K_u^R \, du \right) = \frac{\theta^R K^R}{\theta^R K^R} \left( \frac{\sinh (\tau^R \lambda_1)}{\lambda_1} - \tau^R \right) + \frac{1}{\theta^R} (\cosh (\tau^R \lambda_1) - 1) > 0,
\]

using Equations (E.6) and (A.8).

\[
\frac{\partial D^B}{\partial K^B_0} = \frac{\partial}{\partial K^B_0} \left( \theta^R \int_0^{\tau^R} K^R \, du \right) = \frac{\theta^R K^R}{\theta^R K^R} \left( \sinh (\tau^R \lambda_1) \right) + 1 - \cosh (\tau^R \lambda_1) < 0,
\]

using Equations (E.7) and (A.9).

\[
\frac{\partial D^B}{\partial K^B_0} = \frac{\partial}{\partial K^B_0} \left( \theta^R \int_0^{\tau^R} K^R \, du \right) = \frac{\theta^R K^R}{\theta^R K^R} \left( \cosh (\tau^R \lambda_1) \right) - \frac{\theta^R}{\lambda_1} \sinh (\tau^R \lambda_1) > 0,
\]

using Equations (E.8) and (A.10).

Red casualties when Blue initiates a political conclusion

\[
\frac{\partial D^R}{\partial X^B} = \frac{\partial}{\partial X^B} \left( \theta^B \int_0^{\tau^B} K^B_u \, du \right) = \frac{\theta^B K^B}{\theta^B K^B} \left( \frac{\sinh (\tau^B \lambda_1)}{\lambda_1} - \tau^B \right) + \frac{\cosh (\tau^B \lambda_1) - 1}{\theta^B} > 0,
\]

using Equations (E.1) and (A.3).

\[
\frac{\partial D^B}{\partial X^B} = \frac{\partial}{\partial X^B} \left( \theta^B \int_0^{\tau^B} K^B_u \, du \right) = \frac{\theta^B K^B}{\theta^B K^B} \left( \frac{1 - \cosh (\tau^B \lambda_1)}{\theta^B} \right) + \frac{\tau^B - \sinh (\tau^B \lambda_1)}{\lambda_1} < 0,
\]

using Equations (E.2) and (A.4).

\[
\frac{\partial D^B}{\partial K^B_0} = \frac{\partial}{\partial K^B_0} \left( \theta^B \int_0^{\tau^B} K^B_u \, du \right) = \frac{\theta^B K^B}{\theta^B K^B} \left( \cosh (\tau^B \lambda_1) - 1 \right) - \frac{\sinh (\tau^B \lambda_1)}{\theta^B} > 0,
\]

using Equations (E.3) and (A.5).

\[
\frac{\partial D^B}{\partial K^B_0} = \frac{\partial}{\partial K^B_0} \left( \theta^B \int_0^{\tau^B} K^B_u \, du \right) = \frac{\theta^B K^B}{\theta^B K^B} \left( \sinh (\tau^B \lambda_1) + 1 - \cosh (\tau^B \lambda_1) \right) < 0,
\]
using Equations (E.4) and (A.6).

**G  Civilians**

Suppose Blue allocates a fraction $\alpha^B \in (0, 1)$ of its weapons stock to the destruction of Red weapons and the remainder to the destruction of Red civilian resources. I assume civilian resources are combinations of human and material resources, as are weapons. I further assume a constant rate of transformation, $\eta^R$, from civilian resources to weapons. Let $\alpha^R \in (0, 1)$ and $\eta^B$ have symmetric interpretations. Equations (1) and (2) become

$$\frac{dK^R_t}{dt} = -\theta^B \alpha^B K^B_t + X^R, \quad \text{and} \quad \frac{dK^B_t}{dt} = -\theta^R \alpha^R K^R_t + X^B.$$  

The flow of Red civilian resources destroyed by Blue weapons at $t$, expressed in Red weapons, is then $\eta^R \theta^B (1 - \alpha^B) K^B_t$. Red casualties become

$$D^R_t = \alpha^B \theta^B \int_0^t K^B_u du + \eta^R \theta^B (1 - \alpha^B) \int_0^t K^B_u du,$$

where the first element on the right-hand side is the casualties from fighting and the second element is the casualties from attacks on civilian resources. It follows that

$$D^R_t = \left(1 + \eta^R \frac{1 - \alpha^B}{\alpha^B}\right) \alpha^B \theta^B \int_0^t K^B_u du = \left(1 + \eta^R \frac{1 - \alpha^B}{\alpha^B}\right) \left(tX^R + K^R_0 - K^R_t\right),$$

where the last line follows from the same derivation as with Equation (A.14). Similarly, Blue casualties are

$$D^B_t = \left(1 + \eta^B \frac{1 - \alpha^R}{\alpha^R}\right) \left(tX^B + K^B_0 - K^B_t\right).$$

There are two differences between this model and the one I analyze in the main body of the paper. First, the laws of motion of the weapons stock are as in Equations (1) and (2) with modified attrition coefficients, that is $\theta^R \alpha^R$ instead of $\theta^R$ and $\theta^B \alpha^B$ instead of $\theta^B$. Second, casualties are scaled versions of that in Equations (8) and (9). Thus, this model is isomorphic to the model in the main body of the paper, and the analysis there remains valid with this representation of civilian casualties.