On the Economic Mechanics of Warfare

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On the Economic Mechanics of Warfare*

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Abstract

The literature on war deals with finances, causes, or consequences. But, how do war-related expenditures affect economically-relevant outcomes at a war’s conclusion (e.g., prevailing side, duration, and casualties)? I present a model of attrition and characterize the effects of GDP at a military conclusion (one side cannot fight anymore) and a political conclusion (one side quits). The estimated model fits the data for the battle of Iwo Jima well. Analyzing data for the current Russo-Ukrainian war through the lenses of the model suggests that additional support to Ukraine could yield a shorter, cheaper war with less destruction on both sides.

JEL: E6, H56, N4
Keywords: War; Attrition; Military spending.

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1 Introduction

War is a frequent and costly activity of mankind. Naturally, there is a large economic literature concerned with war. Most studies ask one of three questions: How to finance war? What causes it? What are its consequences? There is little theory, however, to understand how the costs borne by belligerents affect economically-relevant outcomes such as duration, destruction, or the prevailing side. This paper offers such a theory.

Duration is economically relevant because it affects the final cost of war. Duration also has other economic, demographic, and political consequences since it magnifies the effects of the disruptions caused by war. These disruptions are, for example, to marriage patterns (e.g., due to a deficit of men) and to fertility (e.g., due to mobilization and the increased risk of men dying) during and after the war. These disruptions also affect labor markets and gender inequality (e.g., due to the increased labor force participation of women).

Destruction, either material or human, and the resources remaining at the end of war, are economically relevant as well. First, the material destruction affects well-being (e.g., the destruction of housing), productivity (e.g., the destruction of productive capital), and sometimes domestic and international politics (e.g., the Marshall Plan). Casualties also matter for well-being and the size and composition of the postwar labor force. Second, at the end of war, a country’s remaining military equipment can be sold or retain value for peacekeeping and/or deterrence.

Which side prevails is economically relevant because the prevailing side often imposes payments in land (e.g., any war of conquest), treasure (e.g., the treaty of Versailles), or people (i.e., slaves). Also relevant is the reason why one side ceases to fight: for lack of resources (e.g., Germany and Japan at the end of World War 2) or for lack of political will (e.g., the U.S. in Vietnam).

To analyze the role of expenditures in war, I present a model of how wars unfold and conclude. That is what I mean by “warfare.” It is not a “model of
war” explaining why countries go to war, however interesting this question is. Instead, I assume a state of war to exist between two countries. The countries are endowed with initial “weapons” stocks and military technologies that use weapons to destroy their opponent’s weapons. They also receive per-period exogenous flows of weapons (“reinforcements”).

I interpret the initial weapons stocks as resulting from prewar investments, and reinforcements as resulting from wartime expenditures. These are the resources the effects of which I analyze.

The destruction experienced by countries during the war is mitigated by their reinforcements and the destruction they inflict on their opponents. The model’s outcome is thus the joint dynamics of weapons accumulation and of weapons destruction and casualties. I discuss two possible conclusions: In the first, which I label “military,” the war concludes when a country’s weapons stock reaches an exogenously-determined low threshold and the country cannot fight anymore. In the second, which I label “political,” the war concludes when a country’s casualties reach an exogenously-determined high threshold and political forces request that the fighting ends.

Observations about the theory

Battles v. Attrition Wars are sometimes viewed as idiosyncratic events with outcomes largely determined by genius-like generalship and “decisive” battles. I do not take this view. My approach is inspired, instead, by observations from military historians on the one hand, and by Operations Research models on the other hand.

Historians such as O’Brien (2015) and Nolan (2017) argue against a battle-centric view of wars. O’Brien’s first sentence is “There were no decisive battles in World War II” (p. 1). Nolan insists that attrition, more than battles, is key to understanding the outcomes of conflict such as the Punic wars, the Hundred Years war, the Napoleonic wars, and the two World Wars of the 20th century.
In the same vein, Parshall and Tully (2005) argue that, although important, the battle of Midway was not decisive.\footnote{They write “(...) win or loose at Midway, the vast industrial resources of the United States gave its navy an absolutely irrevocable writ of strategic dominance in the Pacific War.” (p. 424) and “(...) Midway stands as the most important battle of the Pacific War, not because it was decisive in an absolute sense, and not because it won the war in a day, but because of its (...) effects on American military options in the Pacific.” (p. 430).}

The notion of a decisive battle is not useful because “decisive” is not well-defined. What is the decided outcome? Moreover, regardless of the definition, assessing whether a battle is decisive requires a counterfactual which cannot be evaluated without a theory of decision making in war. (I discuss decisions below.) For these reasons, the model I present does not have “battles.” Instead, it represents war as a continuous process of attrition.

Combat models developed in Operations Research since the advent of the so-called Lanchester model (Lanchester, 1916) are systems of differential equations describing the attrition of opposing forces during battles. For the reasons I described above, such emphasis on attrition in Lanchester-type models makes them appealing to think about war as a whole. That is therefore my approach: I use a Lanchester-type model and interpret it as a representation of war.

\textbf{Literature} \hspace{1em} The literature on the “economics of war” is too vast to review here. There exists excellent surveys, some of which I mention below. For my purpose, which is to indicate this paper’s contribution, it is useful to think of the literature as divided into three non-mutually-exclusive categories.

The first category is the literature dealing with war finance. Keynes (1940) poses the problem as a trade-off between civilian consumption and the requirements of war. Ohanian (1997) uses the neoclassical growth model to compare the financing scheme used by the U.S. during World War 2 (mostly debt) with that used during the Korean War (mostly taxes). Hall and Sargent (2014, 2021) study the financing of U.S. wars from independence to 1975. In general, the literature on war finance is a subset of the optimal taxation literature, e.g.,
Barro (1979) and Lucas and Stokey (1983).

The second category is the literature discussing the causes of war and/or the decisions to go to war. A fraction of this literature is rooted in game theory and the analysis of conflicts as illustrated in Kimbrough et al. (2020)’s survey. Recently, for instance, Caselli and Coleman (2013) discuss the causes of ethnic conflicts; and Ransom (2018) and Wolford (2019) analyze the decisions that led to World War 1. (See also the review by Myerson, 2023). There is also an empirical literature that seeks to identify correlates of war. Collier and Hoeffler (2007) and Blattman and Miguel (2010), for instance, review the empirical determinant of civil wars; Lagerlöf (2010) ties the occurrence of war to the development process in the context of a growth model; Acemoglu et al. (2022) discuss the rise of fascism during the inter-war period in Italy.

The third category is the literature investigating the consequences of war. McGrattan and Ohanian (2010) assess the impact of World War 2 on the U.S. economy using neoclassical theory. The depressing effect of wars on fertility is document by Caldwell (2004), and Vandenbroucke (2014) offers an explanation of this effect in the case of World War 1. Bethmann and Kvasnicka (2013) discuss the effects of World War 2 on out-of-wedlock births. Doepke et al. (2015) argue that World War 2 caused a postwar baby boom because it disrupted the labor market participation of women. Acemoglu et al. (2004) also argue that World War 2 disrupted the labor market, with consequences for the earnings of both men and women. A literature also considers the effects of war on marriage patterns via its effect on the sex ratio, e.g., Abramitzky et al. (2011), Brainerd (2017), and Knowles and Vandenbroucke (2019).

This paper does not fall into any of these categories. Its contribution to the “economics of war” literature follows. Importantly, this paper does not contribute to the Operations Research literature. The basic insight from the combat model I use is tactical (the principle of concentration) and well known to Operations Research scholars (Taylor, 1980, ch. 2). My goal is not to generate tactical insights. Instead, it is to use a well-established and compelling
representation of warfare and ask of it questions relevant to economists.\footnote{The existing literature is not silent about these questions. In Tullock-like models, for instance, the probability of success in a contest depends on expenditures. Such models do not have a notion of destruction/casualties, though.}

**Decisions v. Mechanics of warfare** I do not model decisions. The model represents the mechanics of attrition in a manner similar to the Solow model’s representation of the mechanics of capital accumulation. Adding decisions would be a natural extension, just like the Ramsey-Cass-Koopmans model is a natural extension of the Solow model. Yet, absent a theory of why there is a war, it is difficult to assign objective functions to the belligerents.

My contention is that, before understanding whether a decision is optimal with respect to an objective, it is worth understanding how the decision affects, mechanically, the variables of the model. In the model, for example, when a country commits additional resources to the war given its opponent’s resources, casualties decrease for both under some conditions. This may be optimal, or not, since a country’s objective may be to minimize its casualties or to maximize its opponent’s. This mechanism operates regardless of a country’s objective. Understanding how is the point of the exercise I propose.

**Organization of the paper**

I develop the model in Section 2. I describe the setup in 2.1 and the dynamics of weapons accumulation and casualties in 2.2. In 2.3, I describe the model’s characterization of a military conclusion and, in 2.4, that of a political conclusion. In Section 3, I show that the model is a compelling description of attrition by estimating it with data from a particular battle: Iwo Jima. In Section 4 I discuss the Russo-Ukrainian war started in 2022 through the lenses of the model. I conclude in Section 5.
Some results

In Section 2.3, I show that, at a military conclusion, casualties on both sides are decreasing with the resources committed by the prevailing country, and increasing with the resources committed by its opponent. That is because the country obtaining the military victory can shorten the war by allocating more resources to it, thereby reducing destruction and casualties for both sides. I explain how this finding relates to an assumption about the military technology used by the belligerents to destroy their opponents’ weapons.

In Sections 2.3 and 2.4, I describe the role of the relative Gross Domestic Product (GDP) of belligerents. All else equal, a country’s high GDP makes the condition for a military victory more favorable and the need to sue for peace on political grounds less pressing. I argue that this mechanism yields an interpretation of the U.S. victory in the Pacific war not emphasized in the literature: The Japanese failure to note that the larger U.S. GDP, which was conducive to a Japanese military defeat in the long-run, was not conducive to a Japanese political victory in the short run.

In Section 4, I use data to analyze the current Russo-Ukrainian conflict through the lenses of the model. I find that if no political solution is sought after and if Ukraine’s allies (i) maintain their support, then Ukraine is on a trajectory to a favorable military conclusion, (ii) increase their support, then the war will be shorter, cheaper, and cause fewer casualties for both belligerents.

2 The model

2.1 Setup

The model is a version of Lanchester (1916)’s model. Time is continuous and there is no uncertainty. There are two countries, Red and Blue, with weapons stocks denoted \( K_t^R \) and \( K_t^B \), respectively. A country’s weapons stock is an
input into its military “production function,” the output of which is a flow of destruction inflicted on the opposite side’s weapons stock. Let $\theta^R > 0$ denote Red’s attrition coefficient, which is the flow of Blue weapons destroyed per Red weapon at each point in time. Blue’s attrition coefficient, $\theta^B > 0$, has a similar interpretation. Let $X^R \geq 0$ and $X^B \geq 0$ denote constant reinforcement flows at each point in time. The laws of motion for $K^R_t$ and $K^B_t$ are

$$
\frac{dK^R_t}{dt} = -\theta^B K^B_t + X^R, \tag{1}
$$
$$
\frac{dK^B_t}{dt} = -\theta^R K^R_t + X^B, \tag{2}
$$

where the initial conditions $K^R_0$ and $K^B_0$ are given. I use the term “resources” to refer to initial weapons stocks and/or reinforcements.

I interpret a weapon as a combination of human and physical capital, e.g., soldier & rifle or aircraft & crew. I assume complementarity between human and physical capital and interpret “destruction” as both casualties and material destruction. (I use these terms interchangeably). I do not distinguish between lethal and non-lethal casualties. It is conceivable that one aircraft & crew effects as much destruction as a large number of soldiers & rifles, suggesting substitutability between different weapon types. I consider one aggregate weapon type for simplicity.

It is worth noting an important assumption. The flow of Blue casualties, $\theta^R K^R_t$, increases with the stock of Red weapons but is independent of the stock of Blue weapons (and vice versa). Thus, additional Blue weapons are not destroyed by Red when the Red stock is constant. This assumption, referred to as “aimed fire” in the Operations Research literature (e.g., Taylor, 1980), can be opposed to an “area fire” assumption, where the flow of Blue casualties increases with the Blue stock. I use the aimed fire assumption because of its simplicity. I show the model’s ability to match data in Section 3.

I assume no destruction other than that of weapons and, thus, abstract from some form of destruction found in war: First, there is no destruction of the productive capacities of a country. Second, there are no civilians casualties.
I present, in Appendix F, a version of the model with civilian casualties and show that the dynamics are isomorphic to that of the model without civilians.

The steady state \((\bar{K}^R, \bar{K}^B)\) of system (1)-(2) is a stalemate where each country’s reinforcements are destroyed by its opponent: \(\bar{K}^R = X^B/\theta^R\) and \(\bar{K}^B = X^R/\theta^B\). Define \(\tilde{K}_t^R = K_t^R - \bar{K}^R\) and \(\tilde{K}_t^B = K_t^B - \bar{K}^B\). Then, (1)-(2) become

\[
\begin{pmatrix}
\frac{d\tilde{K}_t^R}{dt} \\
\frac{d\tilde{K}_t^B}{dt}
\end{pmatrix} =
\begin{pmatrix}
0 & -\theta^B \\
-\theta^R & 0
\end{pmatrix}
\begin{pmatrix}
\tilde{K}_t^R \\
\tilde{K}_t^B
\end{pmatrix}.
\]

(3)

Let \(\lambda_1\) and \(\lambda_2\) be the eigenvalues of \(\mathcal{M}\) with corresponding eigenvectors \([1, v_1]'\) and \([1, v_2]'\), respectively: \(\lambda_1 = -\lambda_2 = -\sqrt{\theta^R\theta^B}\) and \(v_1 = -v_2 = \sqrt{\theta^R/\theta^B}\).

2.2 Dynamics

**Weapons stocks**  Standard methods (Appendices A) yield the solution

\[
\begin{align*}
\tilde{K}_t^R &= \frac{1}{2} \left[ e^{t\lambda_1} A - e^{t\lambda_2} B \right] \frac{1}{v_1}, \\
\tilde{K}_t^B &= \frac{1}{2} \left[ e^{t\lambda_1} A + e^{t\lambda_2} B \right],
\end{align*}
\]

(4)

where the constant \(A\) and \(B\) depend on initial conditions:

\[A = \tilde{K}_0^B + v_1 \tilde{K}_0^R\quad\text{and}\quad B = \tilde{K}_0^B - v_1 \tilde{K}_0^R.\]

Note that \(\lambda_1 < 0\) and \(\lambda_2 > 0\). Thus, the stalemate is a saddle-point. Note also, from Equation (3), that \(d\tilde{K}_t^R/dt > 0\) whenever \(K_t^R < \bar{K}^R\) because Red does not offset Blue reinforcements and, thus, \(K_t^B\) increases. Conversely, if \(K_t^R\) is above its stalemate, Blue reinforcements do not offset the destruction caused by Red and, thus, \(K_t^B\) decreases. The same logic applies to the Red weapons stock. Figure 1 summarizes these observations in a phase diagram.

Weapons stocks’ trajectories need not be monotonic. The blue arrow starting
off in the light-shaded area of Figure 1 represents a case where both weapons stocks are initially below their stalemate: The Blue stock increases monotonically; the Red stock increases until $K^B_t = \bar{K}^B$, when $t = \ln(-B/A)/(2\lambda_1)$, and then decreases. It is the reverse with the blue arrow starting off in the dark-shaded area: The Red stock decreases monotonically; the Blue stock decreases until $K^R_t = \bar{K}^R$, when $t = \ln(B/A)/(2\lambda_1)$, and then increases. With initial conditions in the northwest quadrant, stocks evolve monotonically.

**Casualties**  The flows of casualties are $d^B_t = \theta^R K^R_t$ for Blue and $d^R_t = \theta^B K^B_t$ for Red. Their dynamics mimic the weapons stocks dynamics, so they need not be monotonic. I show (Equations A.7 and A.9) that, at any time $t$, the Red stock is decreasing with the initial Blue stock and with Blue reinforcements. This follows from the “aimed fire” assumption (Section 2.1): Given the Red stock, additional Blue weapons are not exposed to destruction, but
they destroy Red weapons. This, in turn, implies fewer Blue destruction. It follows that $d^B_t$ is decreasing in $K^B_0$ and $X^B$:

$$\frac{\partial K^R_t}{\partial X^B} < 0 \Rightarrow \frac{\partial d^R_t}{\partial X^B} < 0 \quad \text{and} \quad \frac{\partial K^R_t}{\partial K^B_0} < 0 \Rightarrow \frac{\partial d^B_t}{\partial K^B_0} < 0.$$ (6)

I also show (Equations A.3 and A.5) that the Blue stock is increasing with $K^B_0$ and $X^B$, implying that $d^R_t$ is increasing as well:

$$\frac{\partial K^R_t}{\partial X^B} > 0 \Rightarrow \frac{\partial d^R_t}{\partial X^B} > 0 \quad \text{and} \quad \frac{\partial K^B_t}{\partial K^B_0} > 0 \Rightarrow \frac{\partial d^R_t}{\partial K^B_0} > 0.$$ (7)

Symmetric results hold for the effect of Red resources on $d^B_t$ and $d^R_t$. Thus a country’s flow of casualties is decreasing in the resources the country commits to the war and increasing in the resources committed by its opponent.

Let $D^R_t$ and $D^B_t$ denote total casualties at $t$ for Red and Blue, respectively:

$$D^R_t = \int_0^t d^R_u \, du \quad \text{and} \quad D^B_t = \int_0^t d^B_u \, du.$$ (8)

In the remainder of the paper I refer to $d^B_t$ and $d^R_t$ as “flow-casualties” and to $D^B_t$ and $D^R_t$ as “casualties.” I show (Equations A.11 and A.12) that

$$D^R_t = tX^R + K^R_0 - K^R_t,$$ (8)
$$D^B_t = tX^B + K^B_0 - K^B_t.$$ (9)

A country’s casualties at $t$ are the initial stock and reinforcements committed until $t$, net of the remaining stock. Casualties at the war’s conclusion depend on how and when the war concludes.

### 2.3 Military conclusion

I adopt the following definition: A “military” conclusion is when the war ends because a belligerent’s weapons stock reaches a critically low, exogenously
determined, threshold. I assume the threshold is zero. Not all wars end with a military conclusion, but World War 2 is an example. The fighting ability of both Germany and Japan was close to nil by the end of the war.

**The prevailing side** Figure 1 illustrates the condition under which a military conclusion is a victory for Blue or for Red, or is a stalemate. Initial conditions above the stable branch imply that the Red stock eventually reaches 0, while the Blue stock reaches a positive value. This can be seen from Equations (4) and (5) since $e^{t\lambda_1}$ converges to 0 while $e^{t\lambda_2}$ diverges. It follows that, when $B > 0$, $\tilde{K}_t^B$ eventually increases while $\tilde{K}_t^R$ eventually decreases. Thus,

$$B > 0 \Rightarrow \text{Blue victory},$$

$$B < 0 \Rightarrow \text{Red victory},$$

$$B = 0 \Rightarrow \text{stalemate}.$$

The condition for a Blue military victory can be expressed as

$$\sqrt{\theta^B}K_0^B + X^B/\sqrt{\theta^R} > \sqrt{\theta^R}K_0^R + X^R/\sqrt{\theta^B}. \quad (10)$$

I label the units in Equation (10) “efficiency units” or “fighting strength” (as in the Operations Research literature). I use these terms interchangeably. The general form for efficiency units is

$$\sqrt{\text{attrition coefficient}} \times \text{quantity of weapon}.$$  

The Blue fighting strength (left-hand side of 10), is the sum of that arising from the initial weapons stock, $\sqrt{\theta^B}K_0^B$, and that arising from reinforcements, $X^B/\sqrt{\theta^R}$. Note that the latter is also $\sqrt{\theta^R}\tilde{K}_t^B$, so it is indeed in the same units, and that a higher $\tilde{K}_t^R$ raises the Blue fighting strength. That is because a higher stalemate makes it harder for a country to attrit its opponent. The Red fighting strength is defined in the same manner.

Equation (10) is a modified version of the so-called Lanchester Square Law and
deserves some comments. Consider the case where \( X^B = X^R = 0 \). The Blue fighting strength is then \( \sqrt{\theta^B B^B_0} \), which increases faster with the weapons stocks than with the attrition coefficient. That is because additional Blue weapons destroy Red weapons and dilute Red’s ability to attrit Blue. A higher Blue attrition coefficient serves the first purpose but not the second. (The same logic applies for Red). This argument is an implication of “aimed fire” technology (Equations 1 and 2): Since the destruction experienced by one side is independent of its weapons stock, fielding more weapons has no adverse effect. The property that the size of the weapons stocks matters more than the quality (measured by the attrition coefficient) is the main tactical insight of the Lanchester model and has often been viewed as a rationalization of the practice of concentrating forces.

In the general case, when \( X^R, X^B \geq 0 \), there is an additional benefit from a higher attrition coefficient: It reduces the opposing force’s fighting strength arising from reinforcements. A higher Blue attrition coefficient, for instance, reduces the contribution of Red reinforcements to Red fighting strength. That is because Blue destroys Red reinforcements with a lower stalemate stock, and, thus, it is easier for Blue to exceed its stalemate and attrit Red.

I assume, for the remainder of this section, that \( B > 0 \), so that Blue is poised to obtain a military victory at \( \tau \), that is

\[
K^R_\tau = 0.
\]

**Duration of war** I show (Equations B.1–B.4) that

\[
\frac{\partial \tau}{\partial X^B} < 0, \quad \frac{\partial \tau}{\partial X^R} > 0, \quad \frac{\partial \tau}{\partial K^B_0} < 0, \quad \frac{\partial \tau}{\partial K^R_0} > 0.
\]  

(11)

The duration of war before a military conclusion, \( \tau \), is decreasing in the resources of the country obtaining the military victory, and increasing in the resources of its opponent.
When Blue allocates more resources to the war, via $K^B_0$ and/or $X^B$, Blue’s ability to attrit Red is heightened and the Red stock depletes faster. Hence, the war is shorter. Recall that the additional Blue resources are not causing more Blue casualties because of the aimed fire technology. If Red allocates more resources to the war while $B$ remains positive, Blue suffers additional flow-casualties. This slows down Blue’s ability to attrit Red, and, therefore, the war takes longer before Blue prevails militarily.

The results in (11) imply that the level curves of $\tau$ are straight lines,

$$
\left. \frac{dK^B_0}{dK^R_0} \right|_{\tau=0} = v_1 \left( \frac{-\cosh(\tau \lambda_1)}{\sinh(\tau \lambda_1)} \right) \quad \text{and} \quad \left. \frac{dX^B}{dX^R} \right|_{\tau=0} = v_1 \left( \frac{\sinh(\tau \lambda_1)}{1 - \cosh(\tau \lambda_1)} \right),
$$

where $\cosh$ and $\sinh$ are the hyperbolic cosine and sine functions, respectively. Consider the $(K^R_0, K^B_0)$ plane. The slopes are (i) positive, (ii) steeper than the $B = 0$-locus, and (iii) decreasing with $\tau$. Point (i) indicates that Blue needs a larger initial stock to offset an increase in Red’s initial stock and maintain the duration of war constant. Point (ii) indicates that such increase in $K^B_0$ cannot “just” maintain the condition for a military victory, i.e., $dB = 0$ as this would imply extra time at war to destroy the additional Red weapons. It must be more. Point (iii) indicates that, as the war gets shorter, each additional Red weapon must be destroyed faster for the duration of war to be constant. The level curves in the $(X^R, X^B)$ plane have similar properties.

**Casualties** Casualties at $\tau$ follow from Equations (8) and (9):

$$
D^R_\tau = \tau X^R + K^R_0 \quad \text{and} \quad D^B_\tau = \tau X^B + K^B_0 - K^B_\tau.
$$

Red loses all the resources it commits to the war. Blue casualties are mitigated by the end-of-war Blue weapons stock. I show (Appendix C) that

$$
\frac{\partial D^R_\tau}{\partial X^B} < 0, \quad \frac{\partial D^R_\tau}{\partial X^R} > 0, \quad \frac{\partial D^R_\tau}{\partial K^B_0} < 0, \quad \frac{\partial D^R_\tau}{\partial K^R_0} > 0,
$$

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Figure 2: The effect of \( K_0^B \) on casualties when \( B > 0 \)

**Note:** In panel A, the vertical axis measures Blue flow-casualties, \( d_B^t \). Thus, the area under a line and up to time \( t \) represents Blue casualties, \( D_B^t = \int_0^t d_B^u \, du \). The left-pointing arrow under the horizontal axis indicates the shorter duration of war. Panel B reads similarly.

\[
\frac{\partial D_R^t}{\partial X^B} < 0, \quad \frac{\partial D_R^t}{\partial X^R} > 0, \quad \frac{\partial D_R^t}{\partial K_B^0} < 0, \quad \frac{\partial D_R^t}{\partial K_R^0} > 0. \tag{15}
\]

That is, in a Blue military victory, Blue casualties are reduced by Blue resources committed to the war and increased by Red resources. Red casualties behave in the same (not symmetric) manner. That is, Red casualties are also reduced by Blue resources and increased by Red resources.

To understand these results, consider the effects of an increase in the initial stock of Blue weapons, from \( K_0^B \) to \( K_{0,\text{new}}^B \), illustrated in Figure 2. Panel A represents Blue flow-casualties (the solid and dashed lines) and Blue casualties (the areas under the lines). Blue flow-casualties converge to zero because the stock of Red weapons converges to zero in a Blue military victory. Recall that Blue flow-casualties are lower at each point in time when \( K_0^B \) is higher.
Figure 3: The effect of $X^R$ on casualties when $B > 0$

Note: In panel A the vertical axis measures Red flow-casualties, $d^R_t$. Thus, the area under a line and up to time $t$ represents Red casualties, $D^R_t = \int_0^t d^R_u \, du$. The right-pointing arrow under the horizontal axis indicates the longer duration of war. Panel B reads similarly.

(Equation 6), implying that the dashed line is below the solid line. Thus, the light-shaded area in Panel A indicates a reduction in Blue casualties. This reduction combines two effects acting in the same direction: the lower flow-casualties at each point in time and the shorter war (Equation 11).

Panel B represents Red flow-casualties and Red casualties. Two effects operate in opposite directions. First, Red flow-casualties are higher at each point in time (Equation 7). Second, the war is shorter. The dark-shaded area represents the increase in Red casualties due to the first effect. The light-shaded area represents the decrease due to the second effect. In the end, the second effect dominates, as indicated in Equation (14). Thus, in a Blue military victory, Red casualties are reduced when the initial stock of Blue weapons is larger. The effect of an increase in Blue reinforcements, $X^B$ can be understood in a similar way.
Figure 3 shows the effect of an increase in Red resources, namely $X^R$. In Panel A, Blue flow-casualties are higher at each point in time because $X^R$ is higher—this is the symmetric effect of that described in Equation (7) for $d^R_t$. Thus, the dark shaded area indicates an increase in Blue casualties resulting from two effects acting in the same direction: the higher flow-casualties and the longer war. Panel B shows Red flow-casualties, which are lower at each point in time—this is the symmetric effect of that described in Equation (6) for $d^B_t$. The effect on Red casualties combines two effects acting in opposite directions: The light-shaded area represents a reduction of Red casualties due to fewer flow-casualties. The dark-shaded area represents an increase due to the longer war. In the end, as indicated in Equation (14), the second effect dominates and Red casualties increase as a result of increased Red reinforcements. The effect of the initial Red weapons stock can be understood similarly.

The results in Equations (14) and (15) can be stated more generally as: At a military conclusion, both sides’ casualties decrease with the resources of the victorious belligerent and increase with the resources of the defeated.

The role of GDP Let $Y^R$ and $Y^B$ denote the (constant) GDP of Red and Blue, respectively. I assume that, before the war, Red and Blue are at steady states where they allocate constant fractions of GDP to their military. Thus, at the start of war their initial weapons stocks are proportional to their GDP:

$$K^B_0 = s^B Y^B \quad \text{and} \quad K^R_0 = s^R Y^R.$$

I refer to $s^R$ and $s^B$ as prewar (military) saving rates for the sake of exposition. I further assume that, in wartime, Red and Blue allocate constant fractions (possibly different than in peacetime) of their GDP to reinforcements:

$$X^B = \sigma^B Y^B \quad \text{and} \quad X^R = \sigma^R Y^R,$$

3With a constant saving rate, as in the Solow model for instance, the capital-to-output ratio is proportional to the saving rate, albeit not equal to it.
where $\sigma^B$ and $\sigma^R$ are wartime investment rates in weapons.

The condition for a Blue military victory (Equation 10) becomes

$$s^B + \frac{\sigma^B}{\sqrt{\theta^B \theta^R}} > \frac{\theta^R \gamma^R}{\sqrt{\theta^B \gamma^B}} \left( s^R + \frac{\sigma^R}{\sqrt{\theta^R \theta^B}} \right).$$  \hspace{1cm} (16)

The ratio $\gamma$ indicates the role of each country’s GDP in determining the prevailing side in a military conclusion. Red may allocate larger fractions of its GDP to the military in peacetime and wartime ($s^R > s^B$ and $\sigma^R > \sigma^B$), but that is not enough for a military victory if Blue’s GDP is high. This, per se, is not a “finding,” however, since the preceding paragraphs have already established that larger resources are conducive to a military victory (e.g., Equation 10). The lesson to draw from Equation (16) is how exactly does GDP matter? And how much? The answer is through $\gamma$.

As with condition (10), the role of military technology, via the attrition coefficients, is small relative to the role of GDP: The elasticity of $\gamma$ with respect to relative attrition is $1/2$, while its elasticity with respect to relative GDP is 1. As Red GDP becomes smaller relative to Blue’s, the relative attrition in favor of Red, necessary to maintain $\gamma$ constant, increases quadratically. A poor country is thus at a disadvantage relative to a rich country not only because it has fewer resources, but also because the resource gap is increasingly difficult to offset with better technology the poorer the country is.

I equated “rich” with large GDP, but made no reference to GDP per capita. In the model, GDP per capita does not play a role. It is conceivable that countries with high GDP per capita innovate more and have access to better military technologies, i.e., more favorable relative attrition, than countries with low GDP per capita. This effect is trumped by the size effect I described.

The role of GDP for the duration of war can be gauged by rewriting the level
curves of $\tau$ from Equation (12),

$$\left. \frac{ds^B}{ds^R} \right|_{d\tau=0} = Y \left( -\frac{\cosh(\tau\lambda_1)}{\sinh(\tau\lambda_1)} \right) \quad \text{and} \quad \left. \frac{d\sigma^B}{d\sigma^R} \right|_{d\tau=0} = Y \left( \frac{\sinh(\tau\lambda_1)}{1 - \cosh(\tau\lambda_1)} \right).$$

Consider increases in Red’s prewar and/or wartime investment, $s^R$ and/or $\sigma^R$. Since Blue is poised to obtain a military victory, these increases lengthen the war (Equation 11). Blue can offset the increased duration of war by raising $s^B$ and/or $\sigma^B$. The lower $Y$, the cheaper it is for Blue to do so: That is, the lower $Y$, the smaller the increase in $s^B$ and/or $\sigma^B$ is needed for Blue to offset the increase in Red’s prewar and/or wartime investment.

### 2.4 Political conclusion

I assume there are exogenously-determined casualties thresholds at which countries sue for peace. Let $\bar{D}^R$ and $\bar{D}^B$ denote these thresholds for Red and Blue, respectively. I label such conclusions “political” because the fighting strength of the country suing for peace need not be lower than its opponent’s. For example, Blue can sue for peace when $B > 0$.

The Vietnam war is an example of a conflict without a military conclusion. No belligerent was incapable of fighting at the conclusion. Instead, political forces compelled decision makers to reduce the U.S. involvement in the war. The peak of U.S. troops in Vietnam was in April 1969 (Anderson, 2002, p. 187), and evidence of political discontent with the war is numerous, e.g., the anti-war movement or the repeal of the Tonkin Gulf Resolution in 1970.

World War 1 is another example, although more controversial. Allied forces were not in Germany when the war ended, and Douglas Haig, the commander of the British Expeditionary Force, said of the November 1918 armistice: “Germany is not broken in a military sense” (Liddell Hart, 2012, ch. 13).
Duration of war  Blue sues for peace at $\tau^B < \tau$ if casualties reach their threshold value and Red has not yet sued for peace, i.e., if $D^B_{\tau^B} = \bar{D}^B$ or, using Equation (9),

$$K^B_{\tau^B} = \tau^B X^B + K^B_0 - \bar{D}^B.$$  

Note that if $\tau^B > 0$ exists, it is unique because the slope of the left-hand side (with respect to time) is less than that of the right-hand side. Similarly, Red sues for peace at $\tau^R < \tau$ if Blue has not done so already and

$$K^R_{\tau^R} = \tau^R X^R + K^R_0 - \bar{D}^R.$$  

Recall that casualties are cumulative so that $D^R_t$ and $D^B_t$ are monotonically increasing over time. It follows that, if $B > 0$,

the conclusion is $\begin{cases} \text{military at } \tau & \text{if } \bar{D}^R \geq D^R_{\tau^R} \text{ and } \bar{D}^B \geq D^B_{\tau^B}, \\ \text{political at } \min \{ \tau^B, \tau^R \} & \text{otherwise,} \end{cases}$

where $D^R_{\tau^R}$ and $D^B_{\tau^B}$ are given by Equation (13).\footnote{If $B < 0$ the same rule applies, but the date $\tau$ of a military conclusion is different from that discussed in Section 2.3 and end-of-war casualties, $D^B_{\tau^B}$ and $D^R_{\tau^R}$, are different from those given in Equation (13).}

How do resources allocated to war by either belligerent affect the date at which one of them initiates a political conclusion? I show (Equations D.1-D.8) that

$$\frac{\partial \tau^B}{\partial X^B} > 0, \quad \frac{\partial \tau^B}{\partial X^R} < 0, \quad \frac{\partial \tau^B}{\partial K^B_0} > 0, \quad \frac{\partial \tau^B}{\partial K^R_0} < 0,$$  \hspace{1cm} (17)

and

$$\frac{\partial \tau^R}{\partial X^B} < 0, \quad \frac{\partial \tau^R}{\partial X^R} > 0, \quad \frac{\partial \tau^R}{\partial K^B_0} < 0, \quad \frac{\partial \tau^R}{\partial K^R_0} > 0.$$  \hspace{1cm} (18)

The date at which a country reaches its political threshold is increasing with the resources committed by the country and decreasing with the resources committed by its opponent. These results are independent of the sign of $B$. They hold regardless of which country is poised to obtain a military victory.
Suppose Blue commits additional resources to the war, either via $K^B_0$ or via $X^B$. This reduces Blue flow-casualties at each point in time (Equation 6) and, thus, lengthens the time necessary for Blue to reach its political threshold: $\tau^B$ increases. Again, this is because additional Blue weapons imply higher Red flow-casualties (Equation 7) and, thus, impair Red’s ability to destroy Blue weapons. Higher Red flow-casualties shorten the time necessary for Red to reach its political threshold: $\tau^R$ decreases. The effects of Red resources on $\tau^B$ and $\tau^R$ have similar explanations.

Casualties The casualties of the country suing for peace are given by its political threshold. How are its opponent’s casualties affected by resources? I show (Appendix E) that

$$\frac{\partial D^R_{\tau^B}}{\partial X^B} > 0, \quad \frac{\partial D^R_{\tau^R}}{\partial X^R} < 0, \quad \frac{\partial D^R_{\tau^B}}{\partial K^B_0} > 0, \quad \frac{\partial D^R_{\tau^R}}{\partial K^R_0} < 0,$$

and

$$\frac{\partial D^B_{\tau^R}}{\partial X^B} < 0, \quad \frac{\partial D^B_{\tau^R}}{\partial X^R} > 0, \quad \frac{\partial D^B_{\tau^B}}{\partial K^B_0} < 0, \quad \frac{\partial D^B_{\tau^R}}{\partial K^R_0} > 0.$$

Consider a political conclusion initiated by Red at $\tau^R$, i.e., a Blue political victory. Blue casualties increase with Red resources and decrease with Blue resources. Red resources have two effects: First more Red resources imply higher Blue flow-casualties—this is the symmetric effect of that described in Equation (7) for $d^R_t$. Second the war is longer (Equation 18). Symmetrically, more Blue resources lower Blue flow-casualties and lead to a shorter war.

The role of GDP and the Pacific war’s outcome I assume a country’s resource are proportional to its GDP, as I did in Section 2.3. It then follows from the discussion above that (i) the date at which a country reaches its political threshold is increasing with its GDP and decreasing with that of its opponent, (ii) a rich country suing for peace inflicts more casualties on its opponent than a poor country suing for peace.
It is useful to analyze the level curves of $\tau^B$. Equations (D.1)-(D.4) imply

$$
\frac{ds^B}{ds^R} \bigg|_{d\tau^B=0} = \mathcal{Y} \frac{\sinh (\tau^B \lambda_1)}{1 - \cosh (\tau^B \lambda_1)} \quad \text{and} \quad \frac{d\sigma^B}{d\sigma^R} \bigg|_{d\tau^B=0} = \mathcal{Y} \frac{1 - \cosh (\tau^B \lambda_1)}{\sinh (\tau^B \lambda_1) - \tau^B \lambda_1}.
$$

When Red spends a larger fraction of its GDP on its military, i.e, $s^B$ and/or $\sigma^B$ increase, the date at which Blue sues for peace decreases. That is, Blue sues for peace earlier. Blue can offset the effects of $s^R$ and/or $\sigma^R$ on $\tau^B$ by raising $s^B$ and/or $\sigma^B$. The lower $\mathcal{Y}$, the cheaper it is for Blue to do so.

To illustrate the implications of these results, I submit that they suggest an interpretation of the outcome of the Pacific war which, to the best of my knowledge, has not been emphasized in the historical literature.

When Japan started a war with the United States in December 1941, the latter was a larger economy: The GDP difference was 5-fold and the population difference was almost 2-fold, both in favor of the U.S.\(^5\) Historical evidence (e.g., Nolan, 2017, Toll, 2012) indicate that the Japanese military leaders (i) wanted a short war because they knew that Japanese GDP could not sustain a protracted competition against U.S. GDP, and (ii) expected to obtain a rapid political victory because they assumed the U.S. would quickly sue for peace.

The results above emphasize a weaknesses in the Japanese military leaders’ reasoning. The weakness I refer to was not to misjudge the U.S. willingness to take casualties, however.\(^6\) Instead, it was the failure to note that the larger U.S. GDP, which was conducive to a Japanese military defeat in the long-run, was not conducive to a Japanese political victory in the short run. In other words, the same reason that made a “long” war unwinnable for Japan also implied that the war would be “long” before the U.S. reached its political threshold as the Japanese leaders expected.

\(^5\)Maddison Project Database 2020.
\(^6\)Maybe the Japanese did, indeed, misjudge the U.S. willingness to take casualties. I have no insights on this debate.
Using the model (U.S. is Blue and Japan is Red), Equation (16) held at the start of the War because $Y$ was low enough, mostly due to a low $Y^R/Y^B$. Thus, the U.S. was poised to obtain a military victory. The Japanese military recognized that, but assumed that the U.S. political threshold, $\bar{D}^B$, was so low that $\tau^B < \tau$ and that, therefore, the U.S. would sue for peace before it could obtain a military victory. Furthermore, the Japanese military could hasten a favorable (to them) political conclusion by inflicting casualties on the U.S., i.e., by exploiting Equation (17) and lowering $\tau^B$. Yet, Equation (19) indicates that the U.S. could offset the effects of Japanese military spending on $\tau^B$ relatively easily because $Y$ was low enough. Thus, the large U.S. GDP, relative to that of Japan, played two roles. First, it ensured that Equation (16) held and that the U.S. could eventually obtain a military victory—a point emphasized by many authors (e.g., footnote 1). Second, it ensured that the date at which the U.S. would sue for peace, $\tau^B$, could be pushed into the future—a point not emphasized, to the best of my knowledge, in the historical literature. Note that this second point remains true regardless of the U.S. political threshold $\bar{D}^B$: No matter how high or low $\bar{D}^B$ is, the higher the U.S. GDP the farther into the future will $\bar{D}^B$ be reached. Formally,

$$\frac{\partial B}{\partial Y^B} > 0, \quad \text{and} \quad \frac{\partial \tau^B}{\partial Y^B} > 0.$$  

The first inequality (derived from 16) is the first effect: a large U.S. GDP is conducive to a U.S. military victory. The second inequality (derived from 17) is the second effect: a large U.S. GDP is not conducive to an early Japanase political victory.\(^7\)

3 IWO JIMA

In this section, I present an empirical application of the model to the case of a battle for lack of data about an entire war. My goal is to replicate the

\(^7\)I am not arguing that the U.S. did not suffer high casualties during the Pacific war.
analysis of the battle of Iwo Jima, first presented by Engel (1954), and to argue that the Lanchester model provides an empirically relevant description of the process of military attrition.

Iwo Jima is a small volcanic island in the Western Pacific Ocean, almost halfway between Tokyo and Guam in the Mariana archipelago. During World War 2, Iwo Jima was deemed a strategic objective by the United States and assaulted on February 19, 1945 (D-day). Organized resistance by Japanese troops ceased on March 26.

Let Blue represent the U.S. and Red represent Japan. Let time be measured in days. Since the data pertain to the number of troops, let $K^B_t$ and $K^R_t$ indicate the stock of U.S. and Japanese troops fighting on day $t$, respectively. Reinforcements, that is, $X^B_t$ and $X^R_t$, are the flow of troops landing on day $t$.

The U.S. data consist of the number of casualties (killed, wounded, or missing) per day (Morehouse, 1946) and number of newly landed troops per day. One set of estimates for newly landed troops is from Engel (1954) who reports a total of 73,000 troops coming ashore: 54,000 on D-day, 6,000 on D-day+2, and 13,000 on D-day+5. Samz (1972) argues that Engel’s pattern of troop landing is too high and proposes alternative figures with a total of 71,245 troops landing between D-day and D-day+10. In what follows, I consider both Engel’s and Samz’s estimates and construct $K^B_t$, the stock of fighting U.S. troops on Iwo Jima, via

$$K^B_{t+1} = K^B_t - \text{Casualties}_t + X^B_t,$$

with the initial condition $K^B_0 = 0$ (no U.S. troops on the island at the start of D-day) and $X^B_t$ given by either the Engel or the Samz figures for reinforcements. Figure 4 shows $K^B_t$ under each scenario.

There are no data for the stock of Japanese troops or casualties per day. The existing record indicates, however, that there was a stock $K^R_0 = 21,500$ of Japanese troops on Iwo Jima on the eve of the U.S. invasion. Both Engel
and Samz use this figure. The Japanese garrison received no reinforcements throughout the battle, i.e., $X_t^R = 0$. Both Engel and Samz estimate that there remained no Japanese troops at the end of the battle, that is, $K_{35}^R = 0$.  

A discrete-time version of Equations (1) and (2) with time-varying reinforcements is

$$K_{t+1}^i - K_t^i = -\theta_i K_{t}^{-i} + X_t^i. \quad (20)$$

where $(i, -i)$ stands for either $(B, R)$ or $(R, B)$. Engel (1954) proposed a technique for estimating attrition coefficients using the existing data. In the discrete version of the model presented here, Engel’s technique can be described as follows: Summing over $t$, Equation (20) for $(i, -i) = (R, B)$ implies

$$K_{35}^R - K_0^R = -\theta^B \sum_{t=0}^{35} K_t^B + \sum_{t=0}^{35} X_t^R. \quad (21)$$

Recently, Toll (2020, p. 516) reports a garrison of “about 22,000” Japanese on the island at the start of the battle and that, except for a few hundred taken prisoner, the entire garrison was killed.

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Figure 4: Estimates of U.S. troops on Iwo Jima, Feb. 19 to Mar. 26, 1945

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8Recently, Toll (2020, p. 516) reports a garrison of “about 22,000” Japanese on the island at the start of the battle and that, except for a few hundred taken prisoner, the entire garrison was killed.
The left-hand side is observed Japanese casualties, 21,500; the right-hand side is the sum of Japanese reinforcements, which is zero; and, finally, \[ \sum_{t=0}^{35} K_t^B \] can be deduced from the record of U.S. casualties and reinforcement. Under Engel’s scenario for U.S. reinforcements, this figure is 1,971,820. It follows that an estimate of the U.S. attrition coefficient is \( \hat{\theta}^B = 0.011 \) with Engel’s data. A similar calculation with Samz’s data yields \( \hat{\theta}^B = 0.012 \).

Given \( \hat{\theta}^B \) and \( K_0^B \), Equation (20) yields an estimate of the stock of Japanese troops per day: \( \hat{K}_t^R \). The equivalent of Equation (21) for the U.S. is then

\[
K_{35}^B - K_0^B = -\theta^R \sum_{t=0}^{35} \hat{K}_t^R + \sum_{t=0}^{35} X_t^B.
\]

The left-hand side and the sum of U.S. reinforcements on the right-hand side are obtained from Morehouse (1946): 52,150 and 73,000, respectively. The simulated \( \hat{K}_t^R \) yields \( \sum_{t=0}^{35} \hat{K}_t^R = 39,436 \). It follows that an estimate for the Japanese attrition coefficient is \( \hat{\theta}^R = 0.052 \). A similar calculation with Samz’s data yields \( \hat{\theta}^R = 0.050 \).

Panel A of Figure 5 shows the stock of U.S. troops in both the model and the data. Recall that \( (\hat{\theta}^B, \hat{\theta}^R) \) is not a minimum-distance estimator, i.e., the estimates were not constructed to fit the time series \( K_t^B \) in Panel A of Figure 5. Yet, the estimated parameters yield a close fit between the observed and predicted stock of fighting U.S. troops throughout the 36 days of battle.

Panel B of Figure 5 shows the model’s implications for U.S. casualties and the unobserved Japanese troops and casualties. The model fits the daily accumulation of U.S. casualties closely. Again, this is not from fitting the observed time series. It is, instead, an indication of the model’s empirical relevance under the assumption of constant attrition coefficients and aimed fire.

There are lessons to draw from this exercise. First, the model fits the data well. This observation, first made by Engel (1954), indicates that the Lanchester model is quantitatively relevant to study attrition. Second, Iwo Jima provides a good illustration of attrition: Despite their higher ability to inflict casualties,
Figure 5: The Lanchester model’s implications for the battle of Iwo Jima

Note: In panel A, the stock of active U.S. troops is under either Engel’s assumption for U.S. reinforcement or Samz’s.

Source: Author’s calculations, Morehouse (1946), Engel (1954), and Samz (1972).

the Japanese could not withstand the mass of U.S. military resources they faced on Iwo Jima. Finally, the exercise points to the type of data one needs in order to estimate a similar model for a whole war. This would not be an easy task, however, and it is an interesting direction for future research.

4 Russia and Ukraine

In February 2022 Russia (Red) invaded Ukraine (Blue). The latter is significantly smaller: The GDP difference is 8-fold and the population difference more than 3-fold, both in favor of Russia. The international community, therefore, assessed that Ukraine was unable to withstand the Russian military on its own. This prompted the United States and other countries to provide military support to Ukraine.

In the first part of this section, I discuss the mechanics of foreign support in

\footnote{2018 figures from the Maddison Project Database 2020.}
the model. In the second, part I use the model and existing data (arguably of questionable quality) to assess some possible outcomes of the conflict.

**Foreign support** Assume Red is poised for a military victory, i.e., $B < 0$, and that a third party, e.g., a coalition of foreign countries, supports Blue with a one-time transfer of weapons, $S_K$, at date 0 and/or a commitment to a flow of reinforcements $S_X$ at each point in time. Define $K^B_{0,\text{new}} = K^B_0 + S_K$ and $X^B_{\text{new}} = X^B + S_X$ and define $\tilde{K}^R_{\text{new}}$ and $B_{\text{new}}$ accordingly.

The condition for a Blue military victory (the equivalent of Equation 16) is

$$s^B + S_K/Y^B + \frac{\sigma^B + S_X/Y^B}{\sqrt{\theta^R\theta^B}} > \mathcal{Y}\left(s^R + \frac{\sigma^R}{\sqrt{\theta^R\theta^B}}\right).$$

(22)

The effect of a time-0 transfer of weapons, that is, $S_K > 0$ and $S_X = 0$, is represented in Panel A of Figure 6. The initial condition is below the stable branch and Red prevails (black). If $S_K$ is large enough, the initial condition is above the stable branch and Blue prevails (orange).

The effect of a commitment to reinforcements at each point in time, that is $S_K = 0$ and $S_X > 0$, is represented on Panel B. Additional reinforcements raise the stalemate value of Red’s weapons stock to $\tilde{K}^R_{\text{new}} = X^B_{\text{new}}/\theta^R > \tilde{K}^R$, making it harder for Red to attrit the Blue force. The stable and unstable branches translate to the right while the initial condition $(K^R_0, K^B_0)$ does not change. Instead of being below the stable branch, i.e., $B < 0$, the initial condition is above the new stable branch, i.e., $B_{\text{new}} > 0$. The without-support (black) arrow represents the dynamics of war absent foreign support: Blue’s weapons stock goes to zero. The with-support (orange) arrow represents the dynamics with foreign support: Red’s weapons stock goes to zero.

What is the cost-minimizing support (from the point of view of the third party) to ensure a Blue military victory? If the support is a one-time transfer of weapons, the answer is trivial: Let $S_{K,\text{min}}$ solve Equation (22) at equality with $S_X = 0$. With such support the system is on the stable branch and the war
in a stalemate: Its duration is infinity. Any support marginally above \( S_{K_{\text{min}}} \) yields a Blue military victory. Thus, the cost-minimizing one-time transfer maximizes the duration of war before a military conclusion and increases, therefore, the likelihood of a political conclusion initiated by either belligerent.

If support is a commitment to reinforcements at each point in time, \( S_X > 0 \), the cost of support at a military conclusion for the foreign coalition is \( \tau S_X \). Recall from Equation (11) that raising Blue reinforcements lowers the duration of war, \( \tau \). Thus, higher support has two opposite effects on the cost to the foreign coalition: a direct effect increasing \( \tau S_X \) via \( S_X \), and an indirect effect reducing \( \tau S_X \) via \( \tau \). The cost-minimizing support for a military victory must satisfy

\[
\frac{\partial \tau}{\tau} \frac{X_{\text{new}}}{X_{\text{new}}} = -1.
\]

**Numbers** How to bring data to bear on the war in Ukraine? Since the conflict is ongoing, the empirical strategy of Section 3 is not useful. One approach is as follows.
Let time be measured in days. Consider mid-2023, that is 18 months (approximately 550 days) after the start of the war. So, let $t = 550$. Write Equations (8) and (9) as

$$X^R = \frac{D_t^R + K_t^R - K_0^R}{t} \quad \text{and} \quad X^B = \frac{D_t^B + K_t^B - K_0^B}{t}.$$ 

With data on initial weapons stocks, weapons stock at time $t$, and casualties at time $t$, these equations yield an estimate for constant reinforcements.

The CIA’s world factbook indicates that, on the eve of the Russian invasion, in February 2022, there were 200,000 active Armed Forces troops in Ukraine.\textsuperscript{10} Let $K_0^B = 200$. Following the invasion, Ukraine announced a general mobilization. Applying to Ukraine’s population (43 million in 2023) the rate of mobilization of the United States in 1942 (i.e., the first full year of World War 2 for the U.S.), that is 2.8%, yield a potential size for Ukraine’s armed forces of 1.2 millions. Let $K_0^B = 1,200$. For Russia, the CIA’s world factbook estimates that Russia invaded Ukraine with 150,000 troops in February 2022. Let $K_0^R = 150$. To assess the time $t$ level of Russian troops in Ukraine, applying a 2.8% mobilization rate to the Russian population of 142 millions in 2023 would lead to a figure of nearly 4 million. However, the CIA’s world factbook indicates that, in December 2022, the Russian government announced a target level of 1.15 million troops (at an unknown future date), and a further increase to 1.5 million troops by 2026.\textsuperscript{11} Thus, a figure of 4 million troops appears too high; at least too high relative to the announcement of the Russian government itself. Yet another data point is from the Institute for the Study of War (ISW), which reports information from European diplomats citing 300,000 Russian troops in Ukraine in the middle of 2023.\textsuperscript{12} In what follows I will then consider two values for $K_{550}^R$: 300 and 1,150.

For casualties, I use figures reported by the New York Times in August 2023.\textsuperscript{13}

\textsuperscript{10}From CIA world factbook (here)—retrieved on 10/04/2023.
\textsuperscript{11}From CIA world factbook (here)—retrieved on 10/04/2023.
\textsuperscript{12}From ISW (here)—retrieved on 10/04/2023.
\textsuperscript{13}See the New York Times (here)—retrieved on 10/04/2023.
Russia’s casualties appear to include 120,000 killed and 175,000 wounded, for a total of 295,000. Let $D^R_{550} = 295$. Ukraine’s casualties appear to be 70,000 killed and 110,000 wounded. Let $D^B_{550} = 180$. In both cases I assume that wounded troops do not return to the fighting.

Using these figures, the model indicates

$$X^R = \frac{295 + \left\{ \begin{array}{c} 300 \\ 1,150 - 150 \\ 550 \end{array} \right\}}{0.81 \quad 2.35} \quad \text{and} \quad X^B = \frac{180 + \left\{ \begin{array}{c} 1,200 - 200 \\ 550 \end{array} \right\}}{2.14}.$$

A few remarks are in order. First, depending upon one’s assessment of the stock of Russian troops in Ukraine in mid-2023, the estimated reinforcement flow for Russian troops varies significantly. Second, in the first scenario $X^B/X^R = 2.14/0.81 = 2.6$ and in the second scenario $X^B/X^R = 2.14/2.35 = 0.9$. Both numbers are high in comparison with the size of Ukraine relative to that of Russia: In 2023, the Ukraine-to-Russia population ratio was 0.3; and in 2021 the Ukraine-to-Russia GDP ratio was 0.1. Thus, in either scenario, the size of Ukrainian reinforcements suggested by the data is likely to reflect foreign support. It is clear, however, that foreign support has not been provided to Ukraine in the form of troops, but rather in the form of military equipment. In the context of the model equipment and troops are complementary, though. So, the estimate of large Ukrainian troop reinforcements reflects the fact that Ukrainian troops had been fighting with foreign-provided materiel.

To estimate the attrition coefficients, I simulate the model using $(K_0^B, X^B)$ and $(K_0^R, X^R)$, and iterate on $(\theta^R, \theta^B)$ until the model’s stocks of Ukrainian and Russian weapons at $t = 550$ is close (in a least-square sense) to the targeted values of $K^B_{550} = 1,200$ for Ukraine, and $K^R_{550} = 300$ (Estimation 1) or $K^R_{550} = 1,150$ (Estimation 2) for Russia. Table 1 indicates the results. The main difference between the two estimations is the smaller $\theta^R$ in Estimation 2. This is because, to inflict the same casualties on Ukraine, at $t = 550$, with reinforcements almost three times larger in Estimation 2 than in Estimation 1 (i.e., $2.35/0.81 \simeq 3$), the attrition coefficient of Russia must be smaller. The
attrition coefficient of Ukraine changes marginally, allowing the Russian stock of weapons to be almost four times higher in Estimation 2 than in Estimation 1 at date $t = 550$, thanks to larger Russian reinforcements.

It is interesting to plot level curves of $B$ in the $(\theta^R, \theta^B)$ plane around the estimates of these coefficients. These curves are presented in Figure 7. A few points are worth making. First, in each estimation, the attrition coefficients lie in the area of the plane where $B$ is positive, indicating that Ukraine is poised to obtain a military victory.

Second, in Estimation 1, the war lasts over 3 years, while in Estimation 2 it lasts 12 years. These results hold under the assumption of constant flows of reinforcement for each belligerent. These results are also contingent upon no political conclusion being sought by either belligerent. In Estimation 1 the Ukrainian and Russian casualties, at the military conclusion, are 332,000 and 1,079,000, respectively (without distinguishing between killed and wounded). In Estimation 2, these figures rise to over 3 million for Ukraine and over 10 million for Russia. In Section 2.4 I discussed the mechanics of political conclusion for given casualties thresholds, but the model is silent about the determinants of these thresholds.

Assuming the war proceeds to a military conclusion, Figure 8 shows the total

<table>
<thead>
<tr>
<th>Estimation</th>
<th>$\theta^R$</th>
<th>$\theta^B$</th>
<th>$K^R_{550}$</th>
<th>$K^B_{550}$</th>
<th>$D^R_{550}$</th>
<th>$D^B_{550}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimation 1</td>
<td>0.0013</td>
<td>0.0008</td>
<td>302</td>
<td>1,194</td>
<td>294</td>
<td>184</td>
</tr>
<tr>
<td>Data</td>
<td></td>
<td></td>
<td>300</td>
<td>1,200</td>
<td>295</td>
<td>180</td>
</tr>
<tr>
<td>Estimation 2</td>
<td>0.0005</td>
<td>0.0007</td>
<td>1,152</td>
<td>1,191</td>
<td>291</td>
<td>187</td>
</tr>
<tr>
<td>Data</td>
<td></td>
<td></td>
<td>1,150</td>
<td>1,200</td>
<td>295</td>
<td>180</td>
</tr>
</tbody>
</table>

*Source:* See text; author’s calculations.
cost of wartime spending for Ukraine and its allies, $\tau X^B$, for values of $X^B$ around its estimate, 2.14. It is important to discuss an interpretation of $\tau X^B$: Recall that $\tau$ is measured in days and $X^B$ in thousands of troops. In Estimation 1, for instance, with the war lasting slightly over 3 years, there would be a total of over $\tau X^B = 3 \times 365 \times 2.14$ thousand Ukrainian troops engaged in the war. Assuming a constant ratio of troop-to-materiel, the behavior of $\tau X^B$ is an indication of the financial cost of war for Ukraine and its allies. That is the interpretation I adopt here.\(^{14}\)

Each panel of Figure 8 indicates that the cost of wartime spending for Ukraine and its allies is non monotonic in $X^B$. I explained this in the previous paragraph: Since the duration of war before a military conclusion, $\tau$, is decreasing in $X^B$ (Equation 11), the product $\tau X^B$ has a well defined minimum. On each panel, the Red dot represents the estimated cost of the war in the calibrated model. The green dot indicates the minimum cost consistent with Ukraine

\(^{14}\)This interpretation abstracts from civilian casualties and the destruction of non-military capital, e.g., housing. See Appendix F for a discussion of civilian casualties in the context of the model.
and its allies prevailing militarily.

In both estimations, the flow of reinforcements to Ukraine are below their cost-minimizing values. In Estimation 1, the flow of reinforcement would need to be 17.8% higher to reach its cost-minimizing value; in Estimation 2 it would need to be more than 3 times higher. I draw three implications from these observations. By increasing the flow of support to Ukraine (i) Ukraine’s allies will spend less, (ii) the war will be shorter (which is the reason why it will be cheaper), (iii) casualties will be fewer for both sides (Equations 14 and 15).

5 Conclusion

Despite the large literature dealing with war finances, the causes of war, or its consequences, there is little work on how war-related expenditures affect economically-relevant outcomes of war. My goal is to fill this gap.

Historians have argued that wars are often decided by attrition instead of “decisive” battles and genius-like generalship. Attrition, in turn, emphasizes
the importance of resources in determining the outcomes of war. Hence, I use a model of resource attrition à la Lanchester (1916) to represent war.

I consider military conclusions, where one side cannot fight anymore for lack of resources, and political conclusions, where one side does not fight anymore for lack of political will. Under each scenario, I describe how resources determine the duration of the war, the destruction and casualties, and the prevailing side.

The military technology assumes that a belligerent’s casualties are independent of the resources it commits to war. Instead, the casualties depend only on the opposite side’s resources. This assumption implies (i) a country poised to obtaining a military victory shortens the war by spending more, (ii) the shorter war leads to fewer casualties for both sides, (iii) there is a well-defined, cost-minimizing, level of wartime spending leading to a military victory, and (iv) higher GDP makes the condition for a military victory more favorable and the need to sue for peace on political grounds less pressing.

In Section 3, I show that the model matches observed patterns of attrition for the battle of Iwo Jima during World War 2. This exercise is useful as it points to the type of data needed to estimate the model. In Section 4, I use the model to analyze the conflict started in 2022 between Russia and Ukraine. Data can be used to asses the trajectory of the war through the lenses of the model. I conclude that if Ukraine’s allies (i) maintain their support and no political solution is sought after, then Ukraine is on a trajectory to prevail militarily, (ii) increase their support the war will be shorter, cheaper, and cause fewer casualties for both belligerents.

Resources are exogenous in the model. Endogenizing production and the destruction of productive capacities seems a natural extension, which I leave to future research. Modeling decisions, such as the allocation of resources toward the production of consumption goods versus the production of military equipment also seems a natural extension. Finally, the collection of data and the testing of the model on a whole war is yet another avenue for future work.
References


A Solution of Lanchester model

Dynamics of weapons stocks  Let $\mathcal{P}$ represent the matrix of eigenvectors of $\mathcal{M}$ in Equation (3). Standard methods imply that the solution of (3) can be expressed as

$\begin{pmatrix} \tilde{K}_t^R \\ \tilde{K}_t^B \end{pmatrix} = \mathcal{P} \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix} \mathcal{P}^{-1} \begin{pmatrix} \tilde{K}_0^R \\ \tilde{K}_0^B \end{pmatrix},$

$\begin{pmatrix} \tilde{K}_t^R \\ \tilde{K}_t^B \end{pmatrix} = \begin{pmatrix} e^{t\lambda_1} & e^{t\lambda_2} \\ v_1 e^{t\lambda_1} & v_2 e^{t\lambda_2} \end{pmatrix} \begin{pmatrix} (v_2\tilde{K}_0^R - \tilde{K}_0^B)/(v_2 - v_1) \\ (\tilde{K}_0^B - v_1\tilde{K}_0^R)/(v_2 - v_1) \end{pmatrix}.$

Recall that $v_2 = -v_1$. The system can then be written as

$\tilde{K}_t^R = \frac{1}{2} \left[ e^{t\lambda_1} A - e^{t\lambda_2} B \right] \frac{1}{v_1}, \quad (A.1)$

$\tilde{K}_t^B = \frac{1}{2} \left[ e^{t\lambda_1} A + e^{t\lambda_2} B \right], \quad (A.2)$

where $A = \tilde{K}_0^B + v_1\tilde{K}_0^R$ and $B = \tilde{K}_0^B - v_1\tilde{K}_0^R$.

The effect of resources on weapons stocks

$\frac{\partial K_t^B}{\partial X^B} = \frac{1}{2} \left( e^{t\lambda_1} \left( -\frac{v_1}{\theta^R} \right) + e^{-t\lambda_1} \left( \frac{v_1}{\theta^R} \right) \right) = \frac{\sinh(t\lambda_1)}{\lambda_1} > 0, \quad (A.3)$

$\frac{\partial K_t^B}{\partial X^R} = \frac{1}{\theta^B} + \frac{1}{2} \left( e^{t\lambda_1} \left( -\frac{1}{\theta^B} \right) + e^{-t\lambda_1} \left( -\frac{1}{\theta^B} \right) \right) = \frac{1 - \cosh(t\lambda_1)}{\theta^B} < (A.4)$

$\frac{\partial K_t^B}{\partial K_0^B} = \frac{1}{2} \left( e^{t\lambda_1} + e^{-t\lambda_1} \right) = \cosh(t\lambda_1) > 0, \quad (A.5)$

$\frac{\partial K_t^B}{\partial K_0^R} = \frac{1}{2} \left( e^{t\lambda_1} (v_1) + e^{-t\lambda_1} (-v_1) \right) = v_1 \sinh(t\lambda_1) < 0, \quad (A.6)$

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where cosh and sinh are the hyperbolic cosine and sine functions, respectively.

\[
\frac{\partial K_t^R}{\partial X^B} = \frac{1}{\theta^R} + \frac{1}{2v_1} \left( e^{t\lambda_1} \left( -\frac{v_1}{\theta^R} \right) - e^{-t\lambda_1} \left( \frac{v_1}{\theta^R} \right) \right) = \frac{1 - \cosh(t\lambda_1)}{\theta^R} < 0, \quad (A.7)
\]

\[
\frac{\partial K_t^R}{\partial X^R} = \frac{1}{2v_1} \left( e^{t\lambda_1} \left( -\frac{1}{\theta^B} \right) - e^{-t\lambda_1} \left( -\frac{1}{\theta^B} \right) \right) = \frac{\sinh(t\lambda_1)}{\lambda_1} > 0, \quad (A.8)
\]

\[
\frac{\partial K_t^R}{\partial K_0^B} = \frac{1}{2v_1} \left( e^{t\lambda_1} - e^{-t\lambda_1} \right) = \frac{\sinh(t\lambda_1)}{v_1} < 0, \quad (A.9)
\]

\[
\frac{\partial K_t^R}{\partial K_0^R} = \frac{1}{2v_1} \left( e^{t\lambda_1} (v_1) - e^{-t\lambda_1} (-v_1) \right) = \cosh(t\lambda_1) > 0. \quad (A.10)
\]

**Casualties** The following relationships are useful: \( A - B = 2v_1 \tilde{K}_0^R \) and \( A + B = 2\tilde{K}_0^B \). Red casualties are

\[
D_t^R = \int_0^t \theta^B K_u^R du = \theta^B \int_0^t [\tilde{K}_u^B + \tilde{K}_0^B] du:
\]

\[
D_t^R = \theta^B \frac{1}{2} \left[ A \int_0^t e^{u\lambda_1} du + B \int_0^t e^{u\lambda_2} du \right] + tX^R = tX^R + K_0^R - K_t^R. \quad (A.11)
\]

Blue casualties are

\[
D_t^B = \int_0^t \theta^R K_u^R du = \theta^R \int_0^t [\tilde{K}_u^R + \tilde{K}_0^R] du:
\]

\[
D_t^B = -\frac{\lambda_1}{2} \left[ A \int_0^t e^{u\lambda_1} du - B \int_0^t e^{-u\lambda_1} du \right] + tX^B = tX^B + K_0^B - K_t^R. \quad (A.12)
\]

**B Time to military conclusion**

Assume \( B > 0 \) such that, eventually, \( K_t^R = 0 \). Implicit differentiation yields

\[
\frac{\partial \tau}{\partial X^B} = -\frac{\partial K_t^R}{\partial X^B} = -\frac{(1 - \cosh(\tau\lambda_1))/\theta^R}{\theta^B \tilde{K}_0^B} < 0, \quad (B.1)
\]

\[
\frac{\partial \tau}{\partial X^R} = -\frac{\partial K_t^R}{\partial X^R} = -\frac{\sinh(\tau\lambda_1)/\lambda_1}{\theta^B \tilde{K}_0^B} > 0, \quad (B.2)
\]

\[
\frac{\partial \tau}{\partial K_0^B} = -\frac{\partial K_t^R}{\partial K_0^B} = -\frac{\sinh(\tau\lambda_1)/v_1}{\theta^B \tilde{K}_0^B} < 0, \quad (B.3)
\]

\[
\frac{\partial \tau}{\partial K_0^R} = -\frac{\partial K_t^R}{\partial K_0^R} = -\frac{\cos(\tau\lambda_1)}{\theta^B \tilde{K}_0^B} > 0. \quad (B.4)
\]
C Casualties at military conclusion

Assume that $B > 0$.

$$\frac{\partial D^R_\tau}{\partial X^B} = \theta^B K^B_\tau \frac{\partial \tau}{\partial X^B} + \theta^B \int_0^\tau \frac{\partial K^B_u}{\partial X^B} du = 1 - \cosh(\tau \lambda_1) \left( \frac{\bar{K}^B_\tau}{\bar{K}^B_\eta} \right) < 0 \quad (C.1)$$

$$\frac{\partial D^R_\tau}{\partial X^R} = \theta^B K^B_\tau \frac{\partial \tau}{\partial X^R} + \theta^B \int_0^\tau \frac{\partial K^B_u}{\partial X^R} du = \tau + \frac{\sinh(\tau \lambda_1) \bar{K}^B_\tau}{\lambda_1} \frac{\bar{K}^B_\tau}{\bar{K}^B_\eta} > 0, \quad (C.2)$$

$$\frac{\partial D^R_\tau}{\partial K^B_0} = \theta^B K^B_\tau \frac{\partial \tau}{\partial K^B_0} + \theta^B \int_0^\tau \frac{\partial K^B_u}{\partial K^B_0} du = \frac{\sinh(\tau \lambda_1) \bar{K}^B_\tau}{v_1} \frac{\bar{K}^B_\tau}{\bar{K}^B_\eta} < 0, \quad (C.3)$$

$$\frac{\partial D^R_\tau}{\partial K^R_0} = \theta^B K^B_\tau \frac{\partial \tau}{\partial K^R_0} + \theta^B \int_0^\tau \frac{\partial K^B_u}{\partial K^R_0} du = 1 + \cosh(\tau \lambda_1) \frac{\bar{K}^B_\tau}{\bar{K}^B_\eta} > 0. \quad (C.4)$$

$$\frac{\partial D^B_\tau}{\partial X^B} = \theta^B \int_0^\tau \frac{\partial K^B_u}{\partial X^B} du = \theta^B \int_0^\tau \frac{1 - \cosh(u \lambda_1)}{\theta^B} \frac{du}{\lambda_1} = \tau - \frac{\sinh(\tau \lambda_1)}{\lambda_1} \neq 0 \quad (C.5)$$

$$\frac{\partial D^B_\tau}{\partial X^R} = \theta^B \int_0^\tau \frac{\partial K^B_u}{\partial X^R} du = \theta^B \int_0^\tau \frac{\sinh(u \lambda_1)}{\lambda_1} du = \frac{\cosh(\tau \lambda_1) - 1}{\theta^B} > 0, \quad (C.6)$$

$$\frac{\partial D^B_\tau}{\partial K^B_0} = \theta^B \int_0^\tau \frac{\partial K^B_u}{\partial K^B_0} du = \theta^B \int_0^\tau \frac{\sinh(u \lambda_1)}{v_1} \frac{du}{v_1} = 1 - \cosh(\tau \lambda_1) < 0, \quad (C.7)$$

$$\frac{\partial D^B_\tau}{\partial K^R_0} = \theta^B \int_0^\tau \frac{\partial K^B_u}{\partial K^R_0} du = \theta^B \int_0^\tau \cosh(u \lambda_1) \frac{du}{v_1} = -v_1 \sinh(\tau \lambda_1) > 0. \quad (C.8)$$

D Political conclusion

Blue reaches $\bar{D}^B$ at $\tau^B$ such that $\tau^B X^B + K^B_0 - K^B_{\tau^B} = \bar{D}^B$. Implicitly differentiating yields $\partial \tau^B / \partial \bar{D}^B = (\theta^B \bar{K}^R_{\tau^B})^{-1} > 0$ and

$$\frac{\partial \tau^B}{\partial X^B} = \frac{\partial \tau^B}{\partial \bar{D}^B} \times \left( \frac{\sinh(\tau^B \lambda_1)}{\lambda_1} \right) - \tau^B > 0, \quad (D.1)$$

$$\frac{\partial \tau^B}{\partial X^R} = \frac{\partial \tau^B}{\partial \bar{D}^B} \times \left( \frac{1 - \cosh(\tau^B \lambda_1)}{\theta^B} \right) < 0, \quad (D.2)$$

$$\frac{\partial \tau^B}{\partial K^B_0} = \frac{\partial \tau^B}{\partial \bar{D}^B} \times \left( \cosh(\tau^B \lambda_1) - 1 \right) > 0, \quad (D.3)$$

$$\frac{\partial \tau^B}{\partial K^R_0} = \frac{\partial \tau^B}{\partial \bar{D}^B} \times v_1 \sinh(\tau^B \lambda_1) < 0. \quad (D.4)$$
Red reaches $\bar{D}^R$ at $\tau^R$ such that $\tau^R X^R + K_0^R - K_\tau^R = \bar{D}^R$. This implies $\partial \tau^R / \partial \bar{D}^R = (\theta^B K_{\tau_R}^B)^{-1} > 0$, and

\[
\begin{align*}
\frac{\partial \tau^R}{\partial X^B} &= \frac{\partial \tau^R}{\partial \bar{D}^R} \times \frac{1 - \cosh (\tau^R \lambda_1)}{\theta^R} < 0, \quad (D.5) \\
\frac{\partial \tau^R}{\partial X^R} &= \frac{\partial \tau^R}{\partial \bar{D}^R} \times \left( \frac{\sinh (\tau^R \lambda_1)}{\lambda_1} - \tau^R \right) > 0, \quad (D.6) \\
\frac{\partial \tau^R}{\partial K_0^B} &= \frac{\partial \tau^R}{\partial \bar{D}^R} \times \sinh (\tau^R \lambda_1) < 0, \quad (D.7) \\
\frac{\partial \tau^R}{\partial K_0^R} &= \frac{\partial \tau^R}{\partial \bar{D}^R} \times (\cosh (\tau^R \lambda_1) - 1) > 0. \quad (D.8)
\end{align*}
\]

E Casualties at Political Conclusion

Red casualties at $\tau^R$ are $D^R = \bar{D}^R$ and Blue casualties at $\tau^B$ are $D^B = \bar{D}^B$. Blue casualties at $\tau^R$ satisfy

\[
\begin{align*}
\frac{\partial D^B}{\partial X^B} &= \frac{\partial}{\partial X^B} \left( \theta^R \int_0^{\tau^R} K_u^R du \right) = \frac{\theta^R K_{\tau^R}^R}{\theta^B K_{\tau^R}^B} \frac{1 - \cosh (\tau^R \lambda_1)}{\theta^R} + \tau^R \left( \frac{\sinh (\tau^R \lambda_1)}{\lambda_1} \right) < 0,
\end{align*}
\]

using Equations (D.5) and (A.7).

\[
\begin{align*}
\frac{\partial D^B}{\partial X^R} &= \frac{\partial}{\partial X^R} \left( \theta^R \int_0^{\tau^R} K_u^R du \right) = \frac{\theta^R K_{\tau^R}^R}{\theta^B K_{\tau^R}^B} \left( \frac{\sinh (\tau^R \lambda_1)}{\lambda_1} - \tau^R \right) + \frac{1}{\theta^B} (\cosh (\tau^R \lambda_1) - 1) > 0,
\end{align*}
\]

using Equations (D.6) and (A.8).

\[
\begin{align*}
\frac{\partial D^B}{\partial K_0^B} &= \frac{\partial}{\partial K_0^B} \left( \theta^R \int_0^{\tau^R} K_u^R du \right) = \frac{\theta^R K_{\tau^R}^R}{\theta^B K_{\tau^R}^B} \frac{\sinh (\tau^R \lambda_1)}{v_1} + \frac{1}{\theta^B} \left( \frac{\cosh (\tau^R \lambda_1)}{\lambda_1} \right) < 0,
\end{align*}
\]

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using Equations (D.7) and (A.9).

\[
\frac{\partial D_R^B}{\partial K_0^R} = \frac{\partial}{\partial K_0^R} \left( \theta_R \int_0^{\tau_R} K_u^R du \right) = \frac{\theta_R^R K_{\tau_R}^R}{\theta_B^R K_{\tau_R}^R} \left( \cosh (\tau_R \lambda_1) - 1 \right) + \frac{\theta_R^R}{\lambda_1} \sinh (\tau_R \lambda_1) > 0,
\]

using Equations (D.8) and (A.10). Red casualties at \(\tau^B\) are derived similarly.

\section{Civilians}

Let Blue allocates a fraction \(\alpha^B \in (0, 1)\) of its weapons stock to destroying Red weapons and the rest to destroying Red civilian resources. I interpret civilian resources as combinations of human and material capital, and I assume a rate of transformation, \(\eta^R\), from civilian resources to weapons. Let \(\alpha^R \in (0, 1)\) and \(\eta^B\) have symmetric interpretations. Equations (1) and (2) become

\[
dK_t^R/dt = -\theta_R^B \alpha^B K_t^B + X^R, \quad \text{and} \quad dK_t^B/dt = -\theta_R^R \alpha^R K_t^R + X^B.
\]

The flow of Red civilian resources destroyed by Blue weapons at \(t\), expressed in Red weapons, is then \(\eta^R \theta^B (1 - \alpha^B) K_t^B\). Red casualties become

\[
D_t^R = \alpha^B \theta^B \int_0^t K_u^B du + \eta^R \theta^B (1 - \alpha^B) \int_0^t K_u^B du,
\]

\[
= \left( 1 + \eta^R \frac{1 - \alpha^B}{\alpha^B} \right) \left( tX^R + K_0^R - K_t^R \right).
\]

A symmetric result holds for Blue casualties. There are two differences between this model and that in the main body of the paper. First, the laws of motion of weapons stock are like in Equations (1) and (2) with modified attrition coefficients, that is \(\theta_R^R \alpha^R\) instead of \(\theta_R^R\) and \(\theta_B^R \alpha^B\) instead of \(\theta_B^R\). Second, casualties are scaled versions of that in Equations (8) and (9). Thus, this model is isomorphic to that in the main body of the paper, and the analysis remains valid with this representation of civilian casualties.