From Population Growth to TFP Growth

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From Population Growth to TFP Growth*

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Abstract
A slowdown in population growth causes a decline in business dynamism by increasing the share of old businesses. But how does it affect productivity growth? We answer this question by extending a standard firm dynamics model to include endogenous productivity growth. Theoretically, the growth rate of the size of surviving old businesses is a "sufficient statistic" for determining the direction and magnitude of the impact of population growth on TFP growth. Quantitatively, this effect is significant across balanced growth paths for the United States and Japan. TFP growth in the United States falls by 0.10-0.23 percentage points because of the slowing in population growth between 1900 and 2060. The same driving force produces a noticeably bigger response in Japan. Despite the significant long-run effect, we discover that changes in TFP growth are slow in reaction to population growth changes due to two short-run counterbalancing factors.

Keywords: population growth, economic growth, firms dynamics, demographics, productivity, innovation, TFP, Japan.

JEL Codes: E20, J11, O33, O41.

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1 Introduction

In developed countries, there is an increasing concern that the slowing of population growth may be leading to a decline in economic growth.\(^1\) Several recent studies (Karahan, Pugsley and Sahin, 2019; Peters and Walsh, 2021; Hopenhayn, Neira and Singhania, 2022) demonstrated that a slowing in population growth in the United States led to a decline in business dynamism by increasing the share of old businesses. But how does this shift in population and business demographics affect productivity growth? Our aim is to provide an answer to this specific question.

To achieve this goal, this paper incorporates population growth and endogenous productivity growth into a business-dynamics model. As in Hopenhayn (1992), businesses begin with low productivity and increase their productivity throughout their life cycle. However, the productivity of younger businesses is determined by the intensity of innovation within the business, which is endogenous. A business innovation improves on previous innovations, as in Romer (1990) and Aghion and Howitt (1992). This is the first driver of growth in our model. Older businesses’ productivity also increases with age. This represents the second engine of growth in our model and captures technology advancements made by mature, successful businesses. Therefore, for our model to display balanced growth, one of these two forces—innovation by young businesses or technological advancements by established and successful businesses—must be present. In addition, every period, businesses exit with an exogenous probability and new businesses enter.\(^2\)

The main theoretical result is that in comparing BGPs, the sign and magnitude of the impact of population growth on productivity growth are determined by the shape of the businesses’ life-cycle profile. In particular, we identify a “sufficient

\(^1\)Although we usually refer to it as population growth, the driving force in our study is labor force growth.

\(^2\)The exogenous exit assumption is useful for the theoretical characterization. In a subsequent section, this assumption is relaxed.
statistic”—namely the growth rate of surviving old businesses. If the growth rate of the size of surviving old businesses is negative, a fall in the population growth rate will result in a decrease in the rate of aggregate productivity growth. Two components make up the mechanism for this result. First, as population growth increases, so does the growth rate in the number of businesses; otherwise, the average firm size will diverge. As a result, an economy with a higher population growth rate will have a higher proportion of young businesses (a significant number of new businesses relative to existing businesses indicates that most businesses are young). The second element of the mechanism underlying the aforementioned result is based on the productivity growth of old businesses relative to overall productivity growth. The “sufficient statistic” is precisely a measure of these two growth rates. If the size of surviving old businesses decreases over the life-cycle, it means that their productivity is expanding at a slower rate than average productivity. Thus, putting these two factors together, as the labor force expands more slowly and the proportion of old businesses grows, if the productivity growth of old businesses is lower than the average, the total productivity growth will be lower.

The main quantitative findings are based on a comparison of BGPs with different rates of population growth in the US and Japan. We conclude that population growth has a significant impact on productivity growth. In particular, a drop in population growth, as projected for the US for 1900-2060, implies a long-run decline in productivity growth of about 0.10-0.23 percentage points. Similarly, for Japan, the predicted drop in population growth for 1960-2060 implies, in the long run, a 0.36- to 0.59-percentage-point reduction in productivity growth.

Next, we compute transitional dynamics for the economies calibrated to the US and Japan. The experiment involves giving the model time series for labor force growth, ranging from around 1900 to 2060, to provide projections for total produc-
tivity (TFP) growth. An important result of the impact of population growth on TFP growth is that it takes a long time to occur. For example, in 2060, when we assume the drop in population growth ends, only 82% and 67% of the impact on TFP growth will have occurred for the US and Japan, respectively. Most of the rest of the effect will transpire in the next 50 years. We also investigate why the response of TFP growth is sluggish and discover two major factors: a business size effect and a level-composition effect. Both effects fade in the long term, leading to our main result comparing BGPs; but in the meantime, they partially offset the drop in TFP growth.

The paper’s next section validates the mechanism proposed here. The dynamic correlation between labor force growth and productivity growth produced by the model is very similar to the correlation found in data for US states. This result, obtained using local projections, lends support to the proposed mechanism as well as its quantitative significance. Furthermore, instrument variable regressions suggest a causal effect of labor force growth on productivity growth. The paper’s final section demonstrates that a direct amplification effect is added by considering an endogenous exit decision. The slower rate of population growth implies that wage growth will also be slower. The exit rate then decreases endogenously for all ages, resulting in a higher share of old businesses and, consequently, a higher decline in TFP growth than in the case of an exogenous exit rate. However, that section also shows that endogenous exit can also mitigate the drop in TFP growth brought on by a decline in population growth. Businesses may find it most advantageous to choose to innovate more when they have an endogenous exit decision because they understand that the exit rate will be lower at every age.

3 We define $TFP \equiv \frac{Y}{(K^\alpha L^{1-\alpha})}$ in accordance with a long tradition in macroeconomics (Caselli, 2005), where $K$ represents aggregate capital, $L$ is aggregate labor, and $\alpha$ is the share of capital in production. In the context of BGP analysis, it turns out that TFP growth corresponds to average productivity growth. However, during transitional periods, they are different, as TFP growth exhibits an additional factor, namely the average size of businesses.

4 Additionally, in online appendix B.6, we show that the model generates the recent slowdown in business dynamism in the United States, which has been examined in several recent papers.
Two recent studies analyze the relationship between population growth and business dynamism using similar firm-dynamics models. According to Karahan, Pugsley and Sahin (2019) and Hopenhayn, Neira and Singhania (2022), the slowing of labor force growth has resulted in a startup deficit, which can explain what is widely known as a reduction in business dynamism. Related, Engbom (2018) focuses on the age of workers and the dynamism of businesses. Several characteristics of our framework are shared by these papers. As a validation exercise, we show that our model can also reproduce the reduction in business dynamism in the US. However, we focus on the impact of the same driving force on TFP growth rather than on business dynamism.\textsuperscript{5} Peters and Walsh (2021) also focus on the relationship between population growth and business dynamism, but it is more associated with our work because it also displays endogenous growth. While we purposely abstract away from scale effects to focus on our new mechanism, they use a semi-endogenous growth model.\textsuperscript{6} Their research supplements ours by exploring the role of product variety. They don’t look into the mechanism discussed here, because they assume a business’ own innovation is independent of the age of the business. We bring an original and quantifiable mechanism relating population growth to TFP growth that complements their work by focusing on business growth over the life-cycle.\textsuperscript{7}

There are two recent studies linking population growth to productivity growth. First, Jones (2020) investigates the extreme case of long-run negative population growth in the context of models of ideas, which include a variety of endogenous and semi-endogenous growth models. He discovers that negative population growth

\textsuperscript{5}Although it is not the focus of Engbom (2018)’s analysis, the transitional dynamics shown in that paper’s figure 10 reveal a slight decline in growth between 1970 and 2050.

\textsuperscript{6}For a survey on the importance of scale effects see Jones (2022). In the online appendix B.7, we argue that the magnitude of our results is smaller but comparable to the scale effects described there.

\textsuperscript{7}Several other papers have been written about the importance of population growth to economic prosperity. Cooley, Henriksen and Nusbaum (2019), for example, investigate the impact of this demographic change on output growth via capital accumulation and labor productivity. Vandenbroucke (2021) investigates the slowdown in output-per-worker growth in the 1960s and 1970s.
leads to stagnant living standards as the population vanishes. Recently, Kalyani (2022) finds a negative association between inventors’ creativity and age, and argues that a larger proportion of older workers in the labor force will result in lower productivity growth because inventors are, on average, less creative.

2 Model

The economy is made up of businesses and households. Households own businesses and make decisions about consumption and investment. Businesses are the most important part of the framework since they innovate, hire workers, and rent capital. In equilibrium, slower labor-force growth reduces the number of startups. This will change business demographics, which is critical for determining the relationship between population growth and productivity growth.

2.1 Household

A representative household populates the economy and solves

$$\max_{\{c_t, k_t\}} \sum_{t=1}^{\infty} \frac{\beta^{t-1} c_t^{1-\epsilon}}{1-\epsilon} \quad s.t. \quad c_t + g_{M,t+1} k_{t+1} = w_t + s_t + r_t k_t + (1-\delta)k_t,$$

where $k_t \equiv K_t / M_t$ is capital per person, $s_t \equiv \sum_i S_{i,t} / M_t$ is business surplus per person, $\delta$ is depreciation rate, $\beta$ is discount factor, and $g_{M,t+1}$ is population growth rate. Note that this representative household also owns the businesses.

2.2 Businesses

There are $N_t$ businesses in the economy. They have decreasing returns to scale and solve

$$S_{i,t}(x_{i,t}, w_t, r_t) = \max_{x_{i,t}, k_{i,t}, l_{i,t}} \left\{ x_{i,t}^\alpha k_{i,t}^{\beta} l_{i,t}^{1-\alpha-\beta} - w_t l_{i,t} - r_t k_{i,t} \right\},$$
taking as given wages $w_t$ and capital rental rate $r_t$. The solutions for $l_{i,t}$, $k_{i,t}$, and $y_{i,t}$ are linear in productivity,

$$l_i = x_i \left[ \left( \frac{\alpha}{r} \right)^a \left( \frac{1 - \alpha - \zeta}{w} \right)^{1-a} \right]^{\frac{1}{\zeta}}$$

$$k_i = x_i \left[ \left( \frac{\alpha}{r} \right)^{a+\zeta} \left( \frac{1 - \alpha - \zeta}{w} \right)^{1-a-\zeta} \right]^\frac{1}{\zeta}$$

$$y_i = x_i \left[ \left( \frac{\alpha}{r} \right)^a \left( \frac{1 - \alpha - \zeta}{w} \right)^{1-a-\zeta} \right]^\frac{1}{\zeta}.$$

Also, the average productivity in the economy is $X \equiv \frac{1}{N} \sum_i x_i$. When average productivity is combined with the expressions above, we get useful expressions for aggregate variables that will be used later to define the economy’s equilibrium. Therefore, output, labor, and capital can be written as

$$L = \left[ \left( \frac{\alpha}{r} \right)^a \left( \frac{1 - \alpha - \zeta}{w} \right)^{1-a} \right]^\frac{1}{\zeta} \text{NX}, \quad K = \left[ \left( \frac{\alpha}{r} \right)^{a+\zeta} \left( \frac{1 - \alpha - \zeta}{w} \right)^{1-a-\zeta} \right]^\frac{1}{\zeta} \text{NX},$$

$$Y = \left[ \left( \frac{\alpha}{r} \right)^a \left( \frac{1 - \alpha - \zeta}{w} \right)^{1-a-\zeta} \right]^\frac{1}{\zeta} \text{NX}.$$

### 2.3 Innovation

Innovators use the ideas of successful businesses to generate their own new ideas at any given time. Let $\chi$ be the average productivity of successful businesses. An innovator will then choose an innovation step size $g$, which measures the difference between the innovator’s potential productivity, $\hat{x}$, and the reference productivity, $\chi$. Thus, the cost of research for generating $\hat{x}$ is proportional to how far ahead of the pack the project is, $R(\hat{x}/\chi) = \frac{1}{z_R} \left( \frac{\hat{x}}{\chi} \right)^i$, with $i > 2$.

After the research stage, innovators develop ideas to start their businesses. The probability of entering the market ($\sigma$) hinges on the amount of money spent on developing the project, $D(\sigma) = \sigma^2/(2z_D)$.

The value of a project started with potential productivity $\hat{x}$ is

$$I(\hat{x}; \{w_t\}; \{r_t\}) = \sum_{i=1}^{\infty} \beta_i \mathbb{E}[S(x_i, w_i, r_i) | \hat{x}],$$
where $\hat{\beta}_t$ is the market discount factor. At the time of innovation, an innovator chooses $\sigma$ and $\hat{x}$ to maximize its payoff,

$$V(\{w_t\}, \{r_t\}, \chi_t) = \max_{\sigma_t, \hat{x}_t} \sigma_t I(\hat{x}_t; \{w_t\}, \{r_t\}) - w_t R(\hat{x}_t/\chi_t) - w_t D(\sigma_t).$$

In partial equilibrium, solving this problem yields the innovator’s chosen step size of innovation, $g^*$, the probability of starting the project, $\sigma^*$, and the maximized payoff, $V^*$. The value $V^*$ is important because the household is willing to start a business if the initial fixed cost is covered by this value. As a result, in an equilibrium with entry, the following free-entry condition must be met:

$$V_t \leq w_t c_E.$$  (4)

The assumption that the entry cost increases one-to-one with wages makes the model tractable and is common in growth models (e.g. Klette and Kortum, 2004). The assumption is also supported by the data presented in Klenow and Li (2022).

### 2.4 Life-cycle profile of productivity

Consider a project with potential productivity of $g\chi$. Figure 1 depicts this business’s life-cycle profile up to age 4. At age=1, the project may be successful, with productivity equal to its potential, $g\chi$, or unsuccessful, with productivity equal to $\theta g\chi$. Each period, a fraction $\lambda_a$ of age-$a$ unsuccessful projects succeed, and their productivity increases from $\theta g\chi$ to $g\chi$. While unsuccessful, the project’s productivity remains constant. However, the productivity of successful projects grows at a constant rate $g_S$. We assume that $\lambda_a$ is decreasing in age $a$, which is consistent with Greenwood, Han and Sanchez (2022)’s finding that the odds of success by venture capital funding round decreases with the age of the project.

The survival rates $s_{S,a}$ and $s_{U,a}$ for successful and unsuccessful businesses are different. In particular, we assume that unsuccessful businesses exit at a higher rate than successful businesses: $s_{S,a} \geq s_{U,a}$. In the calibration section, although we

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8Specifically, $\hat{\beta}_t = \prod_{j=1}^{t-1} \frac{1}{(1+r_j-\sigma_j)}$. 

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do not impose this condition, we find that successful businesses are more likely to survive than unsuccessful businesses to capture the growth over the life-cycle of average business size and size of surviving businesses.

Figure 1: A business’s life-cycle (example up to age 4)

The reason for this simplified structure for productivity is that it allows us to construct some useful expressions for a business life-cycle. For unsuccessful businesses, the “expected productivity relative to its potential” at age \( a \) is

\[
\Lambda_U, a \equiv \theta \left( \prod_{k=0}^{a-1} (1 - \lambda_k) \right) \left( \prod_{k=1}^{a-1} s_{U,k} \right).
\]

One way of interpreting this expression is that relative productivity is \( \theta \) if the business remains unsuccessful and 0 if it either exits or becomes successful.

Similarly, the expected productivity relative to its potential for a successful business is

\[
\Lambda_S, a (g_S) \equiv \sum_{j=1}^{a} g_S^{j-1} \left( \prod_{k=0}^{a-j-1} (1 - \lambda_k) \right) \left( \prod_{k=1}^{a-j} s_{U,k} \right) \left( \prod_{k=a-j+1}^{a-1} s_{S,k} \lambda_{a-j} \right).
\]

With this notation at hand, the probability that a business will survive as successful and unsuccessful are \( \Lambda_S, a (1) \) and \( \Lambda_U, a / \theta \), respectively. We can also use it to com-
pute the expected productivity based on their potential productivity. If a project starts with its potential productivity $\hat{x}$, the expected productivity at age $a$ is

$$\mathbb{E}[x_a|\hat{x}] = (\Lambda_{S,a}(S) + \Lambda_{U,a}) \hat{x}. \quad (5)$$

Similarly, the survival probability up to age $a$ is $\Lambda_{S,a}(1) + \Lambda_{U,a}/\theta$.

2.5 Law of motion for the number of projects

We can write the law of motion for the number of projects (by type and total) given the number of entrants $n_t$ at a given time $t$ as

$$N_{U,t} = \sum_a n_{t-a+1} \Lambda_{U,a}/\theta, \quad (6)$$

$$N_{S,t} = \sum_a n_{t-a+1} \Lambda_{S,a}(1). \quad (7)$$

$$N_t = N_{U,t} + N_{S,t}. \quad (8)$$

2.6 Market-clearing conditions

To close the model, three market-clearing conditions must be met. The labor-market-clearing condition $M = L$ is simply

$$M = \left[ \left( \frac{\alpha}{r} \right)^a \left( \frac{1 - \alpha - \zeta}{w} \right)^{1-a} \right]^{\frac{1}{\delta}} N \times X. \quad (9)$$

Likewise, the capital-market-clearing condition is

$$K_t = \left[ \left( \frac{\alpha}{r} \right)^{a+\zeta} \left( \frac{1 - \alpha - \zeta}{w} \right)^{1-a-\zeta} \right]^{\frac{1}{\delta}} N \times X \quad (10)$$

where $K_t = k_t M_t$. Finally, the goods-market-clearing condition is

$$Y_t = C_t + I_t, \quad (11)$$

where $C_t = c_t M_t$ and $I_t = K_{t+1} - (1 - \delta)K_t$.

2.7 Equilibrium

We now define the notion of equilibrium in the economy.
**Definition 1.** Given a sequence for labor supply \( \{M_t\} \), an equilibrium is a sequence of prices \( \{w_t, r_t\} \), business choices \( \{l_{i,t}, k_{i,t}, g_t, \sigma_t\} \), household choices \( \{c_t, k_t\} \), a measure of entrants \( \{n_t\} \), and the number of successful projects, \( \{N_{t,S}, N_{t,U}, N_t\} \), such that: (a) \( c_t \) and \( k_t \) solve the optimization problem of a household (1), (b) \( l_{i,t} \) and \( k_{i,t} \) solve the business’s static problem (2), (c) \( \sigma_t \) and \( g_t \) are the innovation choices that results from problem (3), (d) The free entry condition (4) is satisfied, (e) \( N_{t,S}, N_{t,U}, \) and \( N_t \) are in accordance with the laws of motion (6), (7), and (8), (f) The clearing conditions for the labor market (9), the capital market (10) and the goods market (11) are satisfied.

### 2.8 The equilibrium step size of innovation

The optimal step size of innovation is obtained by solving (3). When we incorporate the labor market clearing condition (9) and the free entry condition (4) into the solution for the step size of innovation, we find that the step size of innovation is constant in equilibrium. The next lemma presents this result. *All proofs are in appendix A.1.*

**Lemma 1 (Step size of innovation).** The equilibrium step size of innovation is constant,

\[
g^* = \left( \frac{2cEZ}{i - 2} \right)^{\frac{1}{i}}.
\]

This lemma implies that the step size of innovation is determined by only three parameters: the slope of the cost of innovation, the entry cost, and research efficiency. The free entry condition is crucial to this result. Many important aspects of the economy influence income levels, but not the size of innovation, as in Atkeson and Burstein (2010).\(^9\) This result simplifies the analysis because it implies that the productivity growth rate of young businesses, determined by \( g^* \), and the productivity growth rate of old businesses, determined by \( g_S \), will be constant.

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\(^9\) Technically, having a constant step size requires that expected profit be a monomial function of potential productivity \( \hat{x} \).
3 Balanced growth path

In this section, we characterize a balanced growth path for this economy and investigate how it is impacted by changes in the constant population growth rate \( g_M \). The following lemma characterizes the economy’s BGP equilibrium.

**Lemma 2** (Characterization of the balanced growth path). *Given a constant growth rate of the labor supply greater than the old businesses’ survival rate, \( g_M > s_{S,\infty} \), there is a unique BGP equilibrium in which the following occurs: (a) Aggregate variables \( Y, K, \) and \( C \) grow at constant rates, (b) Wages grow at the same rate, \( g_w = (g_X)^{(1-\alpha)/\xi} \), (c) The interest rate is fixed at \( r = \frac{(g_w)^{\epsilon}}{\beta} - (1 - \delta) \), (d) The step size \( g \) and the probability of starting business \( \sigma \) are constant, (e) Business size is constant because the number of businesses grows at the same rate as the population, \( g_N = g_M \), and (f) Average productivity and successful business productivity grow at the same rate, \( g_X = g_X \).*

In the BGP described above, the average productivity of all projects, \( X \), is a function of the potential productivity of new projects today, \( \hat{x}_1 \), and other parameters:

\[
X = \frac{\hat{x}_1 \sum_{a=1}^{\infty} \left( \frac{1}{g_N} \right)^{a-1} \left( (\Lambda_{S,a}(g_S) + \Lambda_{U,a}) \right)}{\sum_{a=1}^{\infty} \left( \frac{1}{g_N} \right)^{a-1} \left( \Lambda_{S,a}(1) + \Lambda_{U,a} / \theta \right)},
\]

where the growth rate of the number of businesses, \( g_N \), is used to account for the increase in the number of businesses over time. Similarly, \( \chi \), which is today’s reference productivity for innovators, is

\[
\chi = \frac{\hat{x}_1 \sum_{a=2}^{\infty} \left( \frac{1}{g_X} \right)^{a-1} \left( \frac{1}{g_N} \right)^{a-1} \Lambda_{S,a}(g_S)}{\sum_{a=2}^{\infty} \left( \frac{1}{g_N} \right)^{a-1} \Lambda_{S,a}(1)}.
\]

Because potential productivity today is equal to the step size of innovation multiplied by the average productivity of successful projects; i.e., \( \hat{x}_1 = g \chi \), we can derive an equation that defines the relationship between the step size \( g \), productivity growth rate \( g_X \), and the number of businesses growth rate \( g_N \). This equation
implies that $g_X$ solves

$$g = \frac{\sum_{a=2}^{\infty} \left( \frac{1}{g_M} \right)^{a-1} \Lambda_{S,a}(1)}{\sum_{a=2}^{\infty} \left( \frac{1}{g_Xg_M} \right)^{a-1} \Lambda_{S,a}(g_S)}.$$  \hspace{1cm} (15)

We can immediately see in equation (15) the two sources of growth determining $g_X$: $g_S$ and $g$. We can also see the potential role of population growth $g_M$, which will be the focus of the next subsection. To gain more intuition on the workings of the model, consider for a moment a case in which all new businesses become successful at age 1 ($\lambda_0 = 1$) and with a constant survival rate over the life-cycle ($s_{S,a} = s$). The cost of this simplification is that in this simpler case, we can distinguish only between entrants and incumbents (not between young and old businesses), and all incumbents’ productivity growth and exit rates will be the same.

In this case, however, we can find a closed-form solution for $g_X$ as a function of $g$, $g_S$, and the share of incumbent businesses. In particular, we find that

$$\text{share of incumbent} = \frac{(s/g_M) \times n + (s/g_M)^2 \times n + \ldots}{n + (s/g_M) \times n + (s/g_M)^2 \times n + \ldots} = \frac{s}{g_M},$$

and total productivity growth is simply

$$g_X = g_S \times (s/g_M) + g \times (1 - s/g_M). \hspace{1cm} (16)$$

This equation makes it immediately clear that there will be positive total productivity growth ($g_X > 1$) even if only successful businesses have positive productivity growth ($g_S > 1, g = 1$) or if only new businesses have positive innovation ($g > 1, g_S = 1$). In addition, note that this result is regardless of the value of population growth (as long as $g_M > s$).

3.1 The “sufficient statistic”

Before studying the impact of population growth on productivity growth, we show that the employment-size growth rate of surviving old businesses converges to the simple ratio of productivity growth rates, $g_S/g_X$.

**Lemma 3** (Growth rate of the size of surviving old businesses). *In a balanced growth...*
equilibrium, the employment growth rate of surviving businesses converges monotonically to $g_S/g_X$ as the age $\rightarrow \infty$. 

As we will show below, this variable will be a “sufficient statistic” for characterizing the influence of population growth on productivity growth.

3.2 The impact of population growth on TFP growth

Substituting in the definition of TFP the equations for $Y$, $K$, $L$ and using equation (9) to substitute for the wage, we obtain

$$TFP = \left( \frac{NX}{M} \right) \zeta.$$ (17)

As a consequence, growth in TFP in this economy can be written as

$$\log(g_{TFP}) = \zeta \left( \log(g_X) - (\log(g_M) - \log(g_N)) \right).$$

Given the equality between labor force growth and the growth in the number of businesses in the BGP ($g_M = g_N$), the key question for understanding the impact of population growth on TFP is how population growth affects average productivity growth, or $\frac{dg_X}{dg_M}$. The following lemma, which is the paper’s main theoretical result, employs equation (15) to characterize the impact of $g_M$ on $g_X$.

**Lemma 4** (The sign of the impact of population growth on productivity growth).

*In a balanced growth equilibrium, if the growth rate of the size of surviving old businesses is negative, then an increase in the labor force growth rate $g_M$ raises average productivity $g_X$; i.e., if $g_S/g_X < 1 \Rightarrow \frac{dg_X}{dg_M} > 0$.***

We explain this result after we present the next result, which describes how the same “sufficient statistic” determines the size of the impact of population growth on productivity growth.

**Lemma 5** (Magnitude of the impact of population growth on productivity growth).

*Suppose the growth rate of the size of surviving old businesses is negative. Suppose there are two economies with the same TFP growth and the same labor force growth, but the*
growth rate in the size of old businesses decreases faster in one than in the other. Then the impact of population growth on productivity growth, $dX/dM$, is larger in the economy in which the growth rate in the size of old businesses decreases faster.

These findings are better understood by considering that there are two mechanisms. First, recall that the growth in the number of businesses must be equal to the growth rate in the number of new businesses. Therefore, an increase in $g_M$ (and consequently in $g_N$) reduces the share of old businesses. Second, total productivity growth equals the weighted average of productivity growth of businesses of various ages. As a result, if the productivity growth of old businesses is lower than the average, a decline in $g_M$ (and an increase in the share of old businesses) will have a negative effect on total productivity growth. Because the ratio of the average productivity growth rate to the productivity growth rate of old businesses equals the growth rate of the size of surviving old businesses, we referred to this variable as a "sufficient statistic," as it is the only information required to identify the sign of the impact of population growth on productivity growth.

To see this logic more clearly in an equation, recall the case in which all new businesses become successful at age 1 ($\lambda_0 = 1$) and with a constant survival rate over the life-cycle ($s_{S,a} = s$). There, total productivity growth is given by equation (16). Clearly, productivity growth ($g_X$) is the weighted average of incumbent businesses’ productivity growth ($g_S$) and the step size of innovation by new businesses ($g$). Decreasing population growth increases the weight on incumbents ($s/g_M$) and it has a negative impact on total productivity growth ($g_X$) as long as the size of incumbent businesses declines over their life-cycle (what happens if $g_S/g_X$).

These findings also imply that the calibration of the productivity life-cycle profile is crucial for the quantitative results presented in the next sections of this paper.
3.3 Two new features for the quantitative model

Before proceeding to the quantitative analysis of the model, we add two realistic features: congestion of entering firms and spillovers to older businesses. These extensions will depend on two key parameters: \( \phi \) and \( \gamma \). By setting \( \phi = \gamma = 0 \), these two extensions can be removed. This exercise will be used to assess the importance of these features in terms of our quantitative results.

3.3.1 Congestion among entering firms

We modify the free entry condition (4) to account for potential “congestion.” According to Hopenhayn (1992), the working assumption in the model described above is that as long as the free entry condition is satisfied, the number of entrants is perfectly elastic, so \( n_t \) can be chosen to scale up or down the number of businesses and clear the labor market. We modify the free entry as in Karahan, Pugsley and Sahin (2019) to add a more realistic response of the number of businesses to economic conditions. Now, this condition is 

\[
V_t \leq w_t c_E (n_t/M_t) \phi
\]

which means that in order to increase the number of entrants into the economy, the value of entry must also increase. The key parameter is \( \phi \), which we will calibrate based on previous estimates.

The main implication of this feature is that the step-size of innovation is no longer independent of the growth rate of the population. Because of congestion, equation (12) is replaced by 

\[
g^* = \left( \frac{2 c_E (n/M) \phi z_R}{1-2} \right)^{\frac{1}{2}}.
\]

Now, the share of entry in the population, \( n/M \), affects the step-size of innovation. Therefore, the economy’s productivity growth \( g_X \) is also influenced by \( g_M \).\(^{10}\)

3.3.2 Productivity spillover to successful businesses

We consider that the productivity growth of successful projects, \( g_S \), may be a function of the last period productivity growth of successful businesses. This equation captures the idea that successful businesses may benefit (with some de-

\(^{10}\)The full expression for \( n/M \) is included in online appendix (B.2).
lay) from younger businesses’ innovation. Thus, the productivity growth rate of already-successful businesses is

$$g_{S,t} = \bar{g}_S + \gamma (g_{\chi,t-1} - \bar{g}_\chi),$$

(18)

where $\bar{g}_S$ is a constant representing the productivity growth rate of successful businesses in normal times and $\bar{g}_\chi$ is a constant growth rate for successful businesses’ productivity in normal times (i.e., our reference period 1980-1999). The key parameter is $\gamma$, which we will estimate using data on the relationship between employment growth by mature businesses and overall productivity growth (more on this in the calibration section).

Since the productivity growth of successful projects $g_S$ depends on $g_\chi$, which is equal to $g_\chi$ along a balanced growth path, expected productivity for successful projects $\Lambda_{S,a}(g_S)$ will depend on $g_\chi$; i.e., $\Lambda_{S,a}(g_S(g_\chi))$. Thus, equation (15), which determines the growth rate of productivity in the economy, is replaced by

$$\left(\frac{2c_{EZ}}{i-2}\right)^\frac{1}{i} = \frac{\sum_{a=2}^{\infty} \left(\frac{1}{\bar{g}_N}\right)^{a-1} \Lambda_{S,a}(1)}{\sum_{a=2}^{\infty} \left(\frac{1}{\bar{g}_X\bar{g}_N}\right)^{a-1} \Lambda_{S,a}(g_S(g_\chi))}.$$

If spillover is positive, this extension will amplify the effect of $g_M$ on $g_\chi$.

4 Calibration

We calibrate the model to aggregate statistics and business dynamics data for the US and Japan in this section. This section demonstrates how, with a relatively small number of parameters, the model can reproduce key stylized facts for these economies.

We calibrate the model to reproduce the average from 1980 to 1999. The calibrated parameters based on previous papers or obtained directly from data are shown in the top panel of Table 1. We assign values to the remaining parameters, which are shown in the bottom panel of Table 1, in order to reproduce key stylized facts such as establishment size, life-cycle profiles, and exit rates.11

11We use establishment-level data to capture product or project-level activity, which is more in
Table 1: Parameters’ values and targets of calibration

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
<th>Basis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Entry cost, $c_E$</td>
<td>1</td>
<td>Normalization</td>
</tr>
<tr>
<td>Decreasing returns, $\zeta$</td>
<td>0.2</td>
<td>Standard</td>
</tr>
<tr>
<td>Capital share, $\alpha$</td>
<td>0.28</td>
<td>Standard</td>
</tr>
<tr>
<td>Depreciation rate, $\delta$</td>
<td>0.07</td>
<td>Standard</td>
</tr>
<tr>
<td>Risk aversion, $\epsilon$</td>
<td>2</td>
<td>Standard</td>
</tr>
<tr>
<td>Discount factor, $\beta$</td>
<td>0.994</td>
<td>Interest rate 4%</td>
</tr>
<tr>
<td>Labor force growth rate, $g_M$</td>
<td>$(1.0143, 1.0103)$</td>
<td>Average $g_M$ 1980-1999</td>
</tr>
<tr>
<td>Research cost exponent, $\iota$</td>
<td>2.56</td>
<td>GHS</td>
</tr>
<tr>
<td>Convexity of aggregate entry cost, $\phi$</td>
<td>0.55</td>
<td>KPS</td>
</tr>
<tr>
<td>Elasticity of $g_S$ to $g_M$, $\gamma$</td>
<td>0.342</td>
<td>See appendix A.2.</td>
</tr>
<tr>
<td>Research cost slope, $z_R$</td>
<td>$(0.634, 1.990)$</td>
<td>Average prod. growth</td>
</tr>
<tr>
<td>Jump of prod. at success, $1/\theta$</td>
<td>$(65.14, 43.70)$</td>
<td>Average size by age</td>
</tr>
<tr>
<td>Success probability, $\lambda_a$</td>
<td>See Figure 8</td>
<td>Growth of estab.</td>
</tr>
<tr>
<td>Productivity growth of successful estab., $g_S$</td>
<td>$(1.057, 1.043)$</td>
<td>Growth of old estab.</td>
</tr>
<tr>
<td>Survival of successful estab., $s_S$</td>
<td>$(0.969, 0.969)$</td>
<td>Exit rate of old estab.</td>
</tr>
<tr>
<td>Survival of unsuccessful estab., $s_{UL,a}$</td>
<td>See Figure 8</td>
<td>Life-cycle profile of exit rate</td>
</tr>
</tbody>
</table>

Note: The parameters that have different values for the United States and Japan are shown in parenthesis, with the United States representing the first number and Japan representing the second. GHS is an abbreviation for Greenwood, Han and Sanchez (2022) and KPS is an abbreviation for Karahan, Pugsley and Sahin (2019).

Thus, the value of $g_M$ in the US, 1.0143, and in Japan, 1.0103, are the averages for the years 1980-1999. The exponent of the research cost function, $\iota$, is set to the same value as in Greenwood, Han and Sanchez (2022). They estimated it to be equivalent to the impact of innovation expenditures on a firm stock market value. Similarly, the parameter $\phi$ that determines the degree of congestion is set as in Karahan, Pugsley and Sahin (2019). The parameter $\gamma$, which influences the diffusion of innovation from new to old businesses, is calibrated using the value estimate with the process of innovation that we model. Establishment-level data are frequently used as a proxy of project-level analysis, assuming that one establishment produces one product (e.g., Klenow and Li (2020) and Garcia-Macia, Hsieh and Klenow (2019)).
imated in appendix A.2. There, we compare various specifications for regressing current old-establishment productivity growth on the economy’s past productivity growth.

We now discuss the rationale behind the selection of those parameters that match specific targets. To start, it should be mentioned that since all parameters’ values are determined during the process of matching moments, there isn’t a one-to-one link between parameters and targets.

First, because the value of productivity for research, \( z_R \), affects the growth rate of average productivity in the economy, we calibrate the BGP to replicate the average productivity growth in the United States and Japan from 1980 to 1999. The model then reproduces the life-cycle profiles of establishment size and employment growth rate of surviving establishments by using the values of the productivity jump at success, \( 1/\theta \). Of course, in order to reproduce these profiles, the success probability’s life-cycle profile must also be calibrated. Because the increase in productivity at success is so large, successful projects will be significantly larger than unsuccessful projects on average. Figure 8 in the appendix A.3 depicts the resulting life-cycle profile of success probabilities.\(^{12}\) Both the United States and Japan have extremely low success probabilities. As a result, successful businesses are uncommon. Also, old businesses’ employment growth is \( g_S/g_X \) as discussed in Lemma 3. As a result, we calibrate \( g_S \) so that the model accurately reproduces the employment growth for old establishments in the data.

Finally, we calibrate the set of parameters that determine the survival probability life-cycle profile. We assume that the probability of successful businesses surviving is constant. The resulting survival probability, 0.97, is very similar for the United States and Japan and reflects the fact that the exit rate of old establishments is quite low in both countries’ data. Because successful businesses are rare

\(^{12}\) We calibrate the success probability \( \lambda_a \) for ages \( a \geq 2 \) by assuming it exponentially decays with age. Hence, only the initial probability and decay constant are needed. \( \lambda_0 \) and \( \lambda_1 \) are selected separately, as they result in a better fit.
in the model, the resulting life-cycle profile of survival probabilities for unsuccessful businesses is very closely related to the survival probability for establishments in the data.\textsuperscript{13} The resulting profiles for $s_S$ and $s_U$ for the United States and Japan are also shown in Figure 8 in appendix A.3.

Figure 2 depicts the fit of calibration targets.\textsuperscript{14} Figure 2’s left panel shows how well the model matches the exit rate life-cycle patterns for the US and Japan. Exit rates in both countries fall with age, and exit rates in the US are greater than in Japan, especially for newer establishments. The model accurately predicts these trends, which is critical for determining the relative relevance of young and old businesses.

The middle panel of Figure 2 depicts the fit of the life-cycle profile of establishment size as measured in employment. We set the employment of age-one establishments to one because the model allows us to normalize the level of employment. Two facts should be emphasized. First, the model accurately reproduces the profiles for the US and Japan. Second, Japan’s profile is much lower than that of the US. While establishments 27 years or older in the US are approximately 3.5 times larger than those one year old, the same ratio for establishments 29 years or older in Japan is only 1.5.

Finally, Figure 2’s right panel shows the growth of surviving businesses. Contrary to the average establishment size by age, these profiles are unaffected by business survival selection. Thus, the differences between the middle panel and the right panel help identify the differences in survival rates of successful and un-

\textsuperscript{13}In the same manner as the choice of success probability, we calibrate the survival probability $s_U, s$ using an exponential decay function, while allowing it to converge to a non-zero value. Hence, the initial probability, long-run probability, and decay constant are calibrated.

\textsuperscript{14}The source of data for the United States is the Business Dynamics Statistics (BDS), which is produced by the United States Census Bureau. In Figure 2, the average of 1980-2019 is taken for all three measures. The source of data for Japan is the Economics Census and Establishment and Enterprise Census conducted by the Statistics Bureau. The exit rate and growth of surviving establishments by age are based on data in 2004, as these measures are only available in 2004, by comparing them with the data in 2001. For the employment size, we have extracted the year-of-birth fixed effect as the life-cycle profile for Japan, unlike the one for the US, is influenced by the year of birth. More on data sources in the online appendix B.1.
successful businesses.\textsuperscript{15} As our theoretical analysis showed, it is very important to reproduce the growth of surviving businesses. Critically, there is a diminishing growth profile with respect to age, and old businesses experience a decline in size on average. Recall this was our “sufficient statistic” described in Lemma 3. The patterns in the data indicate that a drop in population growth will lead to a fall in aggregate productivity growth along the BGP, as stated by our Lemma 4. Also, relatively old businesses grow at a lower rate in Japan than in the US. This suggests, according to our Lemma 5, that we are likely to find a larger impact of labor force growth on productivity growth in Japan than in the US. In terms of the calibration, the growth rate of the size of surviving old businesses is the prime target for calibrating $g_S$. The data implies $g_S = 1.057$ for the US and $g_S = 1.043$ for Japan.

\textsuperscript{15}In particular, if the growth rate of the average employment size would be equal to the growth rate of surviving businesses, then the exit rate of successful and unsuccessful businesses should be equal.
5 Quantitative impact of population growth on the BGP

This section shows how variations in the population growth rate impact the productivity growth rate along the BGP. We first offer the benchmark findings before showing how spillovers and congestion affect the results. The exercise is straightforward. We take the model’s BGP calibrations for the US and Japan from 1980 to 1999, modify $g_M$, and find the new value of $g_X$, which is proportional to $g_{TFP}$ in the BGP. This exercise is repeated for various values of $g_M$. Figure 3 shows the results for the US (left panel) and Japan (right panel).

Figure 3: Impact of population growth on TFP (comparison across BGPs)

The calibrated point is indicated by the stars in Figure 3, and the lines depict how $g_{TFP}$ changes as $g_M$ increases. Additionally, there are vertical lines representing historical times of greater labor force growth and others indicating the labor force growth projections for the years 2050 to 2060.\footnote{The labor force growth projections are taken from the Bureau of Labor Statistics (BLS) for the US and Cabinet Office (CAO) for Japan.} Using these time frames as examples, we illustrate how drops in labor force growth imply major shifts in the pace of productivity growth of the economy. Our model predicts a 0.23-percentage-point drop in productivity growth for the US as a result. For Japan, population growth declined by more than 3 percentage points between the indi-
cated 100 years (1950-1960 to 2050-2060), which implies a 0.59 percentage point reduction in productivity growth.

What role do the key characteristics of the model play in the outcomes shown in Figure 3? To address this, we compare the results for our benchmark model (column A) with those for three alternative models in Table 2. The findings of a model with no congestion ($\phi = 0$), a model with no spillovers ($\gamma = 0$), and the simplest model with neither congestion nor spillovers ($\phi = \gamma = 0$) are shown in columns B, C, and D, respectively.

Table 2: Role of key model’s features on the impact of population growth on TFP

<table>
<thead>
<tr>
<th>Periods</th>
<th>Data’s growth in labor force, %</th>
<th>Model’s growth in TFP, %</th>
<th>(A) Benchmark</th>
<th>(B) No congestion</th>
<th>(C) No spillover</th>
<th>(D) Simplest</th>
</tr>
</thead>
<tbody>
<tr>
<td>United States</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1900-1910</td>
<td>2.57</td>
<td>1.37</td>
<td>1.33</td>
<td>1.34</td>
<td>1.31</td>
<td></td>
</tr>
<tr>
<td>1980-1999</td>
<td>1.43</td>
<td>1.26</td>
<td>1.26</td>
<td>1.26</td>
<td>1.26</td>
<td></td>
</tr>
<tr>
<td>2050-2060</td>
<td>0.25</td>
<td>1.15</td>
<td>1.19</td>
<td>1.18</td>
<td>1.21</td>
<td></td>
</tr>
<tr>
<td>Difference in pp</td>
<td>-2.32</td>
<td>-0.23</td>
<td>-0.14</td>
<td>-0.16</td>
<td>-0.10</td>
<td></td>
</tr>
<tr>
<td>Japan</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1950-1960</td>
<td>1.94</td>
<td>1.31</td>
<td>1.28</td>
<td>1.27</td>
<td>1.25</td>
<td></td>
</tr>
<tr>
<td>1980-1999</td>
<td>1.03</td>
<td>1.15</td>
<td>1.15</td>
<td>1.15</td>
<td>1.15</td>
<td></td>
</tr>
<tr>
<td>2050-2060</td>
<td>-1.34</td>
<td>0.72</td>
<td>0.78</td>
<td>0.84</td>
<td>0.89</td>
<td></td>
</tr>
<tr>
<td>Difference in pp</td>
<td>-3.27</td>
<td>-0.59</td>
<td>-0.50</td>
<td>-0.43</td>
<td>-0.36</td>
<td></td>
</tr>
</tbody>
</table>

The results in Table 2 show that congestion and spillovers are important, although around half of the impact would still be seen absent them. The overall impact for the US decreases from 0.23 percentage points to 0.10 percentage points when both factors are removed. Japan’s decrease falls from 0.59 to 0.36 percentage points. As a consequence, we conclude that the impact is a reduction of about 0.04-0.10 percentage points of productivity growth for every percentage point reduction in the US population growth, and between 0.11 and 0.18 percentage points
of productivity growth for every percentage point reduction in Japan’s population growth.

In addition, Table 6 in online appendix B.3 presents the results of a sensitivity analysis for the size of the impact of $g_M$ on $g_{TFP}$. We find that the most important parameters affecting the size of the impact are the survival of successful businesses, $s_S$, and their productivity growth, $g_S$. We find that an increase in $s_S$ or a decrease in $g_S$ would increase the size of the impact of $g_M$ on $g_{TFP}$ significantly. Note that the last result is in line with Lemma 5.

6 Transitional Dynamics

The concern that declining population growth will have an influence on TFP growth is clearly long-term. However, there are a number of interesting questions that can be addressed by computing transitional dynamics. In this section, we compute transitional dynamics and use the results to answer three questions. The first question is: How important has population growth been for the slowdown in TFP in the US and Japan in the last 50 years? We show that the share accounting for population growth is significant but relatively small. This result suggests that the transition is slow and most of the impact of the decline in population growth on TFP growth will be realized in the future. Thus, a natural question is: How slowly is the transition between the two different BGPs (for example, as the ones specified in Table 2)? There, we demonstrate that transitions are fairly sluggish. Finally, we ask: What are the causes of the TFP growth’s sluggish reaction to population growth changes? We show that the interplay of composition and business-size effects is partially responsible for this.

6.1 Impact on TFP growth in the last 50 years

We use transitions between BGPs to compare the drop caused by the decline in population growth with data for the United States and Japan. We select as the starting BGPs those that correspond to labor force growth in the first year for which we
have data (1900 for the US and 1930 for Japan). We utilize the BGPs corresponding to the value of \( g_M \) in 2060 for the final BGP in both countries because we have projections for \( g_M \) up to that year.\(^{17}\) The entire transitions for the main variables are shown in Figure 9 in the online appendix.

Table 3 describes the changes in the last 50 years focusing on two sub-periods.\(^{18}\) The numbers for the “slowdown” in TFP growth are the difference in TFP growth (in percentage points) between the average growth in 1970-99 and 2000-19. For instance, TFP growth in the US was 1.105% in 1970-99 and 0.960 in 2000-19, so the difference is 0.145 pp. The decline in TFP growth was more severe in Japan: on average, it was 1.244% in 1970-99 and 0.572% in 2000-19.

Table 3: TFP growth slowdown since 1970-99 to 2000-19, data-model comparison

<table>
<thead>
<tr>
<th></th>
<th>United States</th>
<th>Japan</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Change in ( g_{TFP} )</td>
<td>Share accounted for</td>
</tr>
<tr>
<td>1970-99 - 2000-19</td>
<td>0.145</td>
<td>13.63%</td>
</tr>
<tr>
<td>Data</td>
<td>0.020</td>
<td>8.69%</td>
</tr>
<tr>
<td>Benchmark</td>
<td>0.013</td>
<td>6.43%</td>
</tr>
<tr>
<td>No congestion</td>
<td>0.009</td>
<td>6.43%</td>
</tr>
<tr>
<td>Simplest</td>
<td>0.006</td>
<td>4.31%</td>
</tr>
</tbody>
</table>

How much of the drop in TFP growth can be attributed to the slowdown in population growth? We find that the drop in population growth accounts for 14% of the decline in TFP growth in the US using the benchmark model. If we ab-

---

\(^{17}\)The Christiano and Fitzgerald (2003) filter is used to extract the slow-moving trend of labor force growth. We chose this filter because it allows us to choose the parameters to capture the long-run movement in the labor force. We set the parameters at 2 and 100 and also consider the changes using other values.

\(^{18}\)We begin the analysis in 1970 since other circumstances, such as World War II, would likely be influential in previous decades. Recall that in the previous section we calibrated the model to have a balanced growth rate of TFP equal to the average growth of TFP between 1980 and 1999. To facilitate the comparison with data, for the transitions, we recalibrated \( z_R \) for the US and Japan to make sure that the average growth in the period 1970-1999 is equal in the model in the data. Only very small changes were necessary; the new values are 0.585 and 1.920 for the US and Japan, respectively.
stract away from congestion, this share is reduced by one-third, while removing spillovers decreases it by nearly half. In the simplest scenario, the drop in population growth explains just 4.3% of the decline in TFP growth in the US over that time period. The analysis is similar in Japan, although congestion and spillovers appear to have a lower influence. The benchmark model accounts for roughly 10% of the drop in TFP growth, whereas the simplest model (without congestion and spillovers) accounts for around 6%.

Combining the results in the previous section analyzing BGPs with the findings in Table 3 suggests that the transition must be slow. The next two subsections investigate this issue.

### 6.2 Speed of the transition

We assess the speed of the transition by displaying the share of the overall change in $g_M$ and $g_{TFP}$ between the initial and final BGPs as described in the previous subsection. Focus on the left panel of Figure 4. The blue line represents “$g_M$ share” equal to $(g_{M,t} - g_{M,1900})/(g_{M,2060} - g_{M,1900})$. Since $g_{M,1900}$ is the largest population growth rate for the United States during the sample period, the numerator and denominator are always negative. The non-monotonicity is a result of the period known as the “Baby Boom.” We used vertical lines to identify three years. First, 1900 marks the beginning of the period in which $g_M$ varies. Second, 2020 corresponds to the most recent data available. Finally, 2060 is the last year for which we have a population growth rate forecast. The blue square on the blue line indicates that 71% of the overall change in $g_M$ between the two BGPs has occurred by 2020.

The green line on the same plot is more interesting because it describes the response of TFP growth in our model. In particular, the green line represents “$g_{TFP}$ share” equal to $(g_{TFP,t} - g_{TFP,1900})/(g_{TFP,2200} - g_{TFP,1900})$. Here we select the year 2200 rather than 2060 as the final point to capture how long it takes the economy to
achieve the BGP corresponding to $g_{M,2060}$.\footnote{Note that this share rises even before 1900. This occurs because in the transition, agents are aware of the approaching change in $g_M$.} Two main observations from this plot lead us to conclude that in the model there is a slow response of $g_{TFP}$ to changes in $g_M$. By 2020, while 71% of the overall change in $g_M$ has occurred, the green square on the green line shows that only 48% of the overall change in $g_{TFP}$ has occurred. A similar point is shown by the green circle on the green line. Only 82% of the overall response of $g_{TFP}$ has occurred by the year 2060, when all of the change in $g_M$ has occurred.

The comparable analysis for Japan is presented in the right panel of Figure 4. The only difference in the definition of shares is that the starting point is 1930 rather than 1900 due to data availability. The results for Japan are more extreme, as only 58% of the drop in population growth has occurred by 2020, and just one-fourth of the whole decline in TFP growth has occurred. The model forecasts that 67% of the drop in TFP will have happened by 2060.

6.3 Why is the response so slow? Two counterbalancing factors

Why is there a slow reaction of $g_{TFP}$ to $g_M$ as mentioned in the preceding subsection? To answer this question, we evaluate the economy’s response to a one-period unanticipated change in population growth.\footnote{The transition presented in the previous section was anticipated by agents, and this had an impact in the years preceding the first change in $g_M$. We examine an unanticipated shock here to...} We carry out this experi-
ment in the model calibrated for the US, starting with the BGP corresponding to population growth of 2%.

Figure 5 depicts the economy’s response to the change in population growth rates that we feed into the model. The shock is a permanent fall in population growth from 2% to 1% in the year designated as zero. Prior to the shock, population (panel A), wages (panel B), and the number of businesses (panel D) were all increasing at a steady rate. Also, like in any BGP, \( g_M = g_N \); therefore the average size of businesses has zero growth (panel C). Also as a result of \( g_M = g_N \), prior to the shock, TFP (panel F) grows at the same pace as average productivity (panel E), \( g_X = g_{TFP} \).\(^{21}\)

![Figure 5: Response to an unexpected decline in population growth](image)

Following the realization of the shock, there is a sharp rise in wage growth (panel B), which then declines and gradually converges to a lower growth rate (consistent with lower growth in the new BGP). This pattern is triggered by the free entry condition. Because wage growth will be slower in the future, wages must be higher now to ensure that the free entry condition is met; otherwise, the benefits of starting a business would exceed the costs.

The rise in wages during the shock period causes the decrease in the business

\(^{21}\)Here, we normalize the growth in average productivity, \( g_X \), and the growth in size, \( g_M/g_N \), by taking their power of \( \zeta \).
size indicated in panel C. This is consistent with the reduction in the number of businesses shown in panel D, which is lower than the decline in population growth due to the smaller size of businesses. This mechanism, referred to as the business size effect, is the first major element in explaining the slow response of TFP growth to population growth since reductions in business size drive up TFP growth. In the period of the shock, the contribution of these forces is slightly less than 0.08 percentage points, which is close to 63% of the total increase in TFP growth in that period. The impact of this force on TFP growth eventually vanishes as business size growth (panel C) converges to zero.

Panel E depicts the second pattern that is relevant in explaining the slow response of TFP growth to population growth. During the shock period and for a few years afterward, average productivity growth is larger than in the original BGP. This is referred to as the level-composition effect. Remember that in the long term (across BGP), TFP growth declines as population growth decreases. This is due to a rise in the proportion of older businesses in the economy, which have lower productivity growth than the average. However, on impact, a higher proportion of older businesses has a beneficial influence on TFP since younger businesses are less productive than older businesses. Thus, in the short run, the lower level of productivity of younger businesses outweighs their greater productivity growth, and average productivity rises. This force alone increases TFP growth by 0.045 percentage points, which is close to 37% of the total increase in TFP growth during that period. The business size effect and the level-composition effect operate together to drive TFP growth in the short run in the opposite direction that it moves in the long run as population growth changes. As a result, these forces partially counterbalance the short-run effect on TFP growth and create its sluggish response to changes in population growth.

We also study the sensitivity of the speed of convergence to the model’s parameters in Table 6 in the online appendix B.3. The most important parameters are
$s_s$, $g_s$, and $\beta$. We find that declines in $s_s$, $g_s$, and $\beta$ would increase the speed of convergence significantly.

## 7 Validation using US state-level data

This section analyzes some of the model’s predictions using US data. The purpose is to validate the proposed mechanism. First, we study the impact of population growth on productivity growth throughout US states using local projections. The focus is on studying if the dynamics after a change in population growth described in the previous section can be found in the data. Second, we consider the possibility of endogeneity in the regressions using an instrumental variable approach. The results in the instrumental variable approach resemble the results in the natural experiment studied by Peters (2022).

For the analysis in this section, we would ideally need lengthy time series of state-level TFPs, which are not available in the US. As a result, we use real GDP per worker, which is referred to as labor productivity. Our data on real GDP per worker and labor force range from 1977 to 2019. To keep the analysis comparable, we also use labor productivity from the model, which is calculated using the following expression: \[ \log(g_{\text{prod}, t}) = \frac{1}{(1 - \alpha)} \times \log(g_{\text{TFP}, t}) - \frac{\alpha}{(1 - \alpha)} \times \log(g_{r, t}). \]

To generate simulated time series from the model, we run an anticipated transition from 1900 to 2060 for the 10 largest US states.\(^{22}\)

### 7.1 Local projections

A difficulty with the analysis of the relationship between changes in productivity growth and labor force growth is that, as shown in the previous section, it is not monotonic. Productivity growth rises at first and then falls in reaction to a slowdown in labor force growth, as shown in Figure 5. This pattern suggests that we should estimate a dynamic model. As a result, we follow Jordà (2005) and employ

\(^{22}\)Not surprisingly, if we just use simulated aggregate US data or the simulation for only two states, the estimates are very comparable.
local projections.

Our left-hand-side variable is the change in labor productivity growth rate between $i$ years after the shock and the year before the shock, for each state $s$,

$$\Delta(g_{prod})_{t+i,t-1}^s = g_{prod,t+i}^s - g_{prod,t-1}^s.$$  

We regress this variable on the change in the growth rate of the labor force,

$$\Delta(g_{prod})_{t+i,t-1}^s = \beta_0^i + \beta_1^i \times \Delta(g_{M})_{t,t-1}^s + \text{controls},$$

for $i = 0, 1, \ldots, 10$. Note that these are eleven different regressions, one for each value of $i$. The controls included are four lags of $\Delta(g_{M})_{t,t-1}^s$, four lags of $\Delta(g_{prod})_{t,t-1}^s$, and a quadratic polynomial in year.

**Figure 6:** Change in labor productivity growth after a 1pp *decline* in labor force growth

![Graph showing percentage points over years](chart.png)

Note: The shaded areas represent one (darker) and two (lighter) standard error bands.

The green dashed line in Figure 6 depicts the outcome of estimating local projections in data generated with the model for US-state transitions. The results show that after a one-point decline in labor force growth, the early response of labor productivity growth is positive and large (0.505), similar to the impulse responses shown in Figure 5. After the shock, labor productivity growth slows and becomes negative for several years (-0.141, -0.135, -0.106, and -0.108 in years 2, 3, 4, and 5, respectively) until it is indistinguishable from zero.

The blue line in Figure 6 depicts the estimated effect on data for US states. Surprisingly, its shape closely resembles the model. During the shock period, a reduction in labor force growth has a positive (0.186) and significant (at 1%) effect
on labor productivity growth. Then it decreases and becomes negative (-0.104) and significant (at 5%) three years after the shock, and it continues to be negative (-0.171) and significant (at 1%) four years later, and five years later (-0.152 and significant at 1%).

7.2 Cross-sectional IV regressions

Although in the analysis in the previous subsection the data and model show similar correlations between labor force growth and labor productivity growth, this analysis does not allow us to determine the causal impact of labor force growth on labor productivity growth. One possible explanation for our result, for example, is that workers relocate to states with greater expected labor productivity growth. Nonetheless, our mechanism for the effect of labor force growth on labor productivity growth is quite specific, as it is based on a decrease in the number of new businesses. Karahan, Pugsley and Sahin (2019) validates this mechanism by demonstrating a causal relationship between labor force growth and the number of startups or the startup rate. In particular, they identify that a 1-percentage-point decrease in the working-age population growth rate roughly translates to a nearly 1-percentage-point decrease in the startup rate. Because our model was constructed using a firm-dynamics model similar to Karahan, Pugsley and Sahin (2019)’s framework, it is not surprising that we attain the same kind of relationship, as shown in Figure 5 panels (A) and (D).

Karahan, Pugsley and Sahin (2019), inspired by Shimer (2001), used a past birth rate as an instrumental variable for labor force growth. This variable is a powerful instrument because, as previous research has shown, there is a close connection between current labor force growth and the birth rate some 20 years ago. Furthermore, in our scenario, the birth rate many years ago is unlikely to have a direct impact on current labor productivity growth. Unfortunately, we find that this instrument is too weak to be used for the yearly dynamics across states examined above using local projections. Lagged state-level birth rate, on the other hand, is a
significant predictor of differences in labor force growth after averaging state data over a 30-year span. This fact enables us to use cross-sectional regressions in an attempt to identify the causal effect of labor force growth on labor productivity growth.

For the 30 years from 1990 to 2019, we average labor productivity growth, $g_{prod}$, and labor force growth, $g_M$. We use the birth rate pushed back 15 years as an instrument for $g_M$, so the average is from 1975 to 2004.\footnote{We also perform the analysis using the average between 1975 and 1989 to avoid period overlap for the averages and the results are very similar.} We control by two potentially important variables. First, we control by the state’s initial level of income per capita (average from 1986 to 1989), because state convergence would suggest a negative link between the initial level of development and future growth. Second, we include the state’s population (average from 1990 to 2019), as many growth theories indicate that scale effects may exist.

The findings of six specifications are presented in Table 4. The first three columns show the results of OLS regressions, while the next three columns show the results of Instrumental Variable (IV) regressions. The three alternative specifications for each approach differ in how state-level observations are weighted. In the first instance, all observations are weighted equally. The weight in the second case is the logarithm of the state’s population, and the weight in the third case is the state’s population.

Table 4, regardless of specification, demonstrates that labor force growth has an effect on labor productivity of around 0.2 percentage point change in labor productivity growth for every 1-percentage-point change in labor force growth. It should be noted that these estimates are comparable to the effect estimated in the local projections for the years following the shock. In general, the estimates in the OLS regressions are more significant than in the IV regressions, though all of the coefficients for the effect of $g_M$ on $g_{prod}$ are significant at a 5% level. Furthermore, except when states are weighted by population, the F statistics of the first-stage regres-
Table 4: Impact of labor force growth on labor productivity growth, cross-sectional regressions for US states

<table>
<thead>
<tr>
<th>Dependent variable</th>
<th>OLS</th>
<th>IV, lagged birth rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average $g_{prod}$</td>
<td>0.182</td>
<td>0.190</td>
</tr>
<tr>
<td></td>
<td>(0.000)</td>
<td>(0.000)</td>
</tr>
<tr>
<td>log(Initial income pc)</td>
<td>0.007</td>
<td>0.007</td>
</tr>
<tr>
<td></td>
<td>(0.026)</td>
<td>(0.026)</td>
</tr>
<tr>
<td>log(Population)</td>
<td>-0.001</td>
<td>-0.000</td>
</tr>
<tr>
<td></td>
<td>(0.239)</td>
<td>(0.276)</td>
</tr>
<tr>
<td>R-squared</td>
<td>0.310</td>
<td>0.312</td>
</tr>
<tr>
<td>First-stage reg F stat</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>Observations</td>
<td>49</td>
<td>49</td>
</tr>
</tbody>
</table>

Note: There is also a constant in each regression and the values in parenthesis are the p-values corresponding to robust standard errors. The states include all the US states, except Alaska and Hawaii, plus the District of Columbia.

Evaluations show that the lagged birth rate is a very powerful instrument. Note also that the magnitudes are comparable, although slightly smaller, than those estimated by Peters (2022) using forced population expulsions in post-war Germany.

8 Endogenous exit

In this section, we consider an extension of the model that incorporates endogenous exit. To give businesses a reason to exit, we incorporate a fixed cost shock $c_{a\omega\varepsilon}$ to the profits of unsuccessful businesses that depend on the age of the business. If the pre-fixed costs expected discounted profits $I_{a}$ is larger than the fixed cost, they pay the fixed costs and continue their businesses. If not, they exit the market.

We also keep an exogenous probability of exit so the survival probability for unsuccessful businesses is given by $s_{l,a} = s_{l,\infty} \Pr(c_{a\omega\varepsilon} \leq I_{a}) = s_{l,\infty} \times F(I_{a}/(wca))$, where $F$ is the distribution of $\varepsilon$. For tractability, we assume that $F$ is Type III extreme value (or Weibull) distribution; i.e., $\varepsilon \sim Weibull(1, \theta)$. With this assumption,

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24Successful businesses could also have to pay a fixed cost, but we assume it is negligible compared to their revenue so they would not exit for this reason (they are still subject to the exogenous probability of survival).
the survival probability for unsuccessful businesses is given by

$$s_{i,a} = s_{i,\infty} \left[ 1 - \exp \left( - \left( \frac{I_a}{\omega c_a} \right)^\theta \right) \right],$$

where $\theta$ is a parameter of the distribution. In addition, we incorporate in expected profits the expected fixed cost given survival. We calibrated $c_a$ to get the same age profile of the exit rate as the exogenous exit case. Also, we chose $\theta = 1.25$ such that the effect of population growth on the economy’s exit rate across BGPs coincides with the effect in Hopenhayn, Neira and Singhania (2022).

Figure 7 displays the amplification of the effect of the population growth rate across BGPs. For Japan’s calibration and for an increase in the population growth rate relative to the calibrated BGP, the amplification is the largest. For example, if the growth rate of the population would be 2.5% instead of 1% (1.5pp larger), the amplification would be 18%; i.e., the growth rate of TFP would increase by 0.19pp instead of 0.16pp. However, for declines in population growth, the amplification is significantly smaller than that for both calibrations.

Figure 7: Amplification effect of endogenous exit

Why is the amplification so small? Because there are two counterbalancing forces. First, a decline in population growth produces an increase in survival rate for each age because wages grow at a slower pace. This effect on the survival rate for every age increases the share of old businesses, which magnifies the effect on

In particular, we match that when population growth declines from 2.66% to 0.78%, the exit rate declines by 0.88 percentage points. As a reference, with exogenous exit, the composition effect due to firms getting older would imply a decline of 0.72 percentage points.
productivity growth due to the growth composition effect. Second, there is also a change in the step size of innovation, \( g \). Given that the survival rates increase endogenously for each age, businesses may find it beneficial to innovate more.

9 Conclusions

At least since Solow (1957), the persistent improvement in living standards around the globe has been largely attributed to TFP growth. The trend growth in TFP, however, has recently slowed in industrialized economies (Cette, Fernald and Mojon, 2016; Fernald et al., 2017). On the other hand, population growth has declined in most developed economies and is projected to continue this trend in the next decades. In fact, the latest United Nations projections suggest that the whole world’s population could reach zero growth during the 2080s (UN, 2022). Therefore, figuring out the possible impact of population growth on TFP growth is crucial. We offer a theory that ties these two trends together. According to our theory, TFP growth will likely continue to fall in the coming decades as a result of the slowing population growth.

References


A Appendix

A.1 Proofs

A.1.1 Proof of Lemma 1

First, the value of a project with potential productivity $\hat{x}_t$ is

$$I_t(\hat{x}_t; \{w_t\}, \{r_t\}) = \sum_{j=t}^{\infty} \hat{\beta}_j \mathbb{E}[S(x_j; w_j, r_j) | \hat{x}_t].$$

Note that using equation (2), we have that

$$S_t = \zeta x_t \left[ \left( \frac{\alpha}{r_t} \right)^{1 - \alpha - \zeta} \left( \frac{1 - \alpha - \zeta}{w_t} \right)^{1 - \alpha - \zeta} \right]^{\frac{1}{\zeta}},$$

so we can rewrite $I_t$ as

$$I_t(\hat{x}_t; \{w_t\}, \{r_t\}) = \sum_{j=t}^{\infty} \hat{\beta}_j \zeta \left[ \left( \frac{\alpha}{r_j} \right)^{1 - \alpha - \zeta} \left( \frac{1 - \alpha - \zeta}{w_j} \right)^{1 - \alpha - \zeta} \right]^{\frac{1}{\zeta}} \mathbb{E}[x_j | \hat{x}_t] = \Gamma(w_t, r_t) \hat{x}_t,$$

where

$$\Gamma(w_t, r_t) = \zeta \alpha \zeta \left( 1 - \alpha - \zeta \right) \sum_{j=t}^{\infty} \hat{\beta}_j (\Lambda_{S,j-t+1}(gS) + \Lambda_{U,j-t+1}(r_j))^{-\frac{1}{\zeta}} (w_j)^{-\frac{1}{\zeta}}.$$
using equation (5) to substitute for \( E[x_t | \hat{x}_t] \).

Then, we can solve equation (3). This equation is altered to

\[
V(\{w_t\}, \{r_t\}, \chi_t) = \max_{\sigma_t, g_t} \sigma_t \Gamma(w_t, r_t) g_t \chi_t - \frac{1}{z_R}(g_t)^t w_t - \frac{\sigma_t^2}{2z_D} w_t, \tag{19}
\]

where \( g_t \equiv \hat{x}_t / \chi_t \) is the step size of innovation. Note that the FOCs with respect to \( \sigma_t \) and \( g_t \) are

\[
\frac{\partial V_t}{\partial \sigma_t} = \Gamma(w_t, r_t) g_t \chi_t - \frac{w_t}{z_D} \sigma_t = 0, \quad \frac{\partial V_t}{\partial g_t} = \Gamma(w_t, r_t) \sigma_t \chi_t - \frac{w_t}{z_R} g_t^{t-1} = 0.
\]

The solutions are

\[
g_t^* = \left( z_R z_D \frac{\Gamma(w_t, r_t)}{t} \left( \frac{\chi_t}{w_t} \right)^2 \right)^{-\frac{1}{2}}, \tag{20}
\]

\[
\sigma_t^* = \frac{z_D \Gamma(w_t, r_t)}{w_t} \chi_t g_t^*. \tag{21}
\]

Substituting equations (20) and (21) into (19), we obtain

\[
V_t = \frac{t - 2}{2z_R} \left( z_R z_D \frac{\Gamma(w_t, r_t)}{t} \left( \frac{\chi_t}{w_t} \right)^2 \right)^{-\frac{1}{2}} w_t = \frac{t - 2}{2z_R} (g_t^*)^t w_t. \tag{22}
\]

Finally, we replace equation (22) into the free entry condition (equation (4)),

\[
V_t = \frac{t - 2}{2z_R} (g_t^*)^t w_t = c_E w_t,
\]

which determines the step size \( g^* \),

\[
g^* = \left( \frac{2c_E z_R}{t - 2} \right)^{\frac{1}{t}}.
\]

Since this solution is equal to equation (12), this concludes the proof of Lemma 1.

**A.1.2 Proof of Lemma 2**

Let \( \cdot' \) denote values in the next period. Suppose that \( g_M > s_{S, \infty} \) and \( \{\sigma, g, c, k\} \) solves the old problem. The existence of a balanced growth path will be shown when \( \{\sigma, g, g_w c, g_w k\} \) solves the new one for \( M' = g_M M \).

First, the Euler equation derived from equation (1), \( (g_w)^e = \beta(1 + r - \delta) \), shows \( r \) is constant, and so \( \hat{\beta}' = \hat{\beta}_t \).

\[
\hat{\beta}' = \hat{\beta}_t.
\]
Second, when we observe \( M' = g_M M \), we will get a certain value of \( g_X \) from equation (15). Note that the right-hand side of equation (15) is a monotonic (increasing) function of \( g_X \) and \( \lim_{g_X \to 0} RHS \to 0 \) and \( \lim_{g_X \to \infty} RHS \to \infty \) for any \( g_M \in (0, \infty) \). Given the LHS is positive, we have a unique \( g_X \) for every \( g_M \).

In addition, equations (13) and (14) give \( \chi' = g_X \chi \) and \( \hat{x}' = g_X \hat{x} \). equation (20) gives \( w' = (g_X)^{\frac{\tau}{1-\kappa}} w \), and it is immediate from equation (9) that \( N' = g_M N \). Goods and capital and clearing market conditions give \( K' = g_w g_M K \), \( Y' = g_w g_M Y \), \( C' = g_w g_M C \).

To see the firm side, from equation (2), \( S(\hat{x}', w', r') = g_w S(\hat{x}, w, r) \), and so \( I(\hat{x}', w', r') = g_w I(\hat{x}, w, r) \). Therefore, the objective function for the value of projects in equation (3) inflates by the factor \( g_w \). Note that \( R(\hat{x}' / \chi') = R(\hat{x} / \chi) \) and \( D(\sigma') = D(\sigma) \). Also, both sides of equation (4) inflates by \( g_w \), so the new solution still satisfies the free entry condition.

Lastly, consider the budget constraint in equation (1). At the conjectured solution, both sides inflate by the factor \( g_w \). As such, the budget constraint also holds with the new solution.

Thus, by canceling out this factor of proportionality, the new problem reverts back to the old one and we have shown that it is a BGP.

Finally, we show that population growth must be higher than the old businesses’ survival rate, \( g_M > s_{S, \infty} \) to have a BGP. For this, note that the lower bound for the growth in the number of business is \( s_{S, \infty} \). This is the case because (i) in that lower bound the number of entrants is at its lower bound (zero) and (ii) we assumed that successful firms live longer than unsuccessful ones, \( s_{S, \infty} \geq s_{U, \infty} \). Hence, if \( g_M < s_{S, \infty} \), then the population growth rate is lower than the growth rate of the number of businesses, \( g_M < g_N \), which contradicts the condition \( g_M = g_N \) that should hold in any balanced growth path.
A.1.3 Proof of Lemma 3

This proof consists of two parts: (i) Prove that $g_X > g_S$ if and only if the employment growth rate of surviving successful businesses is negative, and (ii) prove that the employment growth rate of surviving old businesses is asymptotically equivalent to that of surviving successful businesses.

First, the employment of a business $i$ at time $t$ is

$$l_{i,t} = x_{i,t} \left[ \left( \frac{\alpha}{r_t} \right)^{a} \left( \frac{1 - \alpha - \zeta}{w_t} \right)^{1-a} \right]^{\frac{1}{\zeta}}.$$ 

Given that the business is successful at time $t$ and surviving at time $t+1$, the business’s employment at time $t+1$ is written as

$$l_{i,t+1} = g_S x_{i,t} \left[ \left( \frac{\alpha}{r_{t+1}} \right)^{a} \left( \frac{1 - \alpha - \zeta}{w_{t+1}} \right)^{1-a} \right]^{\frac{1}{\zeta}}.$$ 

Since $r_{t+1} = r_t$ and $w_{t+1} = (g_X)^{\frac{\zeta}{1-a}} w_t$ in a BGP, the employment growth surviving of successful businesses $l_{i,t+1}/l_{i,t} = g_S/g_X$.

Next, to show the asymptotic equivalence between the employment growth rate of surviving old businesses and that of surviving successful businesses, recall that we assume that old successful businesses are more likely to survive than old unsuccessful businesses, $s_{S,\infty} \geq s_{U,\infty}$. It implies that the share of unsuccessful businesses converges to zero, so the employment growth rate of old businesses becomes equivalent to that of surviving businesses. This concludes the proof of Lemma 3.

We can see this step algebraically by considering the expression for the employment growth rate of surviving businesses of all ages:

$$\text{surviving growth} = \frac{g_S}{g_X} \left( 1 - \Delta_a \right) + \Delta_a \left( \frac{(1 - \lambda_d) + \lambda_d}{g_X} \right),$$

where $\Delta_a$ is the employment share of unsuccessful businesses. Note that it depends on the employment share, not the business share. Since successful businesses are
larger than unsuccessful businesses, the employment share converges faster than
the business share.

A.1.4 Proof of Lemma 4

Combining equations (12) and (15) and arranging it give an equilibrium expres-
sion for the relationship between $g_M$ and $g_X$:

$$\left( \frac{2cEz_R}{i - 2} \right)^{-\frac{1}{2}} = \sum_{a=2}^{\infty} \left( \frac{g_S}{g_Xg_M} \right)^{a-1} \frac{\Lambda_{S,a}(g_S)}{(g_S)^{a-1}\Lambda_{S,a}(1)} \Lambda_{S,a}(1) \sum_{a=2}^{\infty} \left( \frac{1}{g_M} \right)^{a-1} \Lambda_{S,a}(1).$$

The right-hand side can be interpreted as the weighted average of

$$\bar{x}_a \equiv \left( \frac{g_S}{g_X} \right)^{a-1} \frac{\Lambda_{S,a}(g_S)}{(g_S)^{a-1}\Lambda_{S,a}(1)} \sum_{a=2}^{\infty} \left( \frac{1}{g_M} \right)^{a-1} \Lambda_{S,a}(1).$$

with weights $(1/g_M)^{a-1}\Lambda_{S,a}(1)$, i.e.,

$$\left( \frac{2cEz_R}{i - 2} \right)^{-\frac{1}{2}} = \sum_{a=2}^{\infty} (\bar{x}_a) \times \frac{((1/g_M)^{a-1}\Lambda_{S,a}(1))}{\sum_{a=2}^{\infty} (1/g_M)^{a-1}\Lambda_{S,a}(1)}. \quad (23)$$

Now, $\bar{x}_a$ is decreasing in $g_X$ for all $a \geq 2$. As such, ceteris paribus, the increase in $g_X$ decreases the right-hand side of equation (23). Also, if $g_S < g_X$, $\bar{x}_a$ is decreasing in age $a$ since $\Lambda_{S,a+1}(g_S)/\Lambda_{S,a}(g_S) \leq g_S\Lambda_{S,a+1}(1)/\Lambda_{S,a}(1)$. Therefore, we need larger weights on young businesses to increase the right-hand size of equation (23) if $g_S < g_X$, which means that we need to increase $g_M$.

To put them together, if $g_S < g_X$, an increase in $g_M$ must increase $g_X$ to keep the right-hand side of equation (23) constant, which concludes the proof of Lemma 4.

A.1.5 Proof of Lemma 5

Start with equation (15). Totally differentiating it by $g_M$ and reorganizing it gives

$$\frac{1}{1 + \frac{g_M}{g_X} \frac{dg_X}{dg_M}} \sum_{a=2}^{\infty} \left( \frac{1}{g_M} \right)^{a-1} \Lambda_{S,a}(1)(a - 1) = \sum_{a=2}^{\infty} \left( \frac{1}{g_M} \right)^{a-1} \left( \frac{1}{g_X} \right)^{a-1} \Lambda_{S,a}(g_S)(a - 1) \sum_{a=2}^{\infty} \left( \frac{1}{g_X} \right)^{a-1} \Lambda_{S,a}(g_S). \quad (24)$$

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Therefore, for given \( g_M \) and \( g_X \), \( dg_X/dg_M \) is larger if the right-hand side of equation (24) is smaller. Note that only the right-hand side depends on \( g_S \). Since the growth rate in the size of old businesses decreases faster in the economy with smaller \( g_S \), we need to show

\[
\frac{d}{dg_S} \sum_{a=2}^{\infty} \left( \frac{1}{g_M} \right)^{a-1} \left( \frac{1}{g_X} \right)^{a-1} \Lambda_{S,a}(g_S)(a-1) > 0.
\]

Arranging it gives

\[
\sum_{a=2}^{\infty} \left( \frac{1}{g_M} \right)^{a-1} \left( \frac{1}{g_X} \right)^{a-1} \Lambda_{S,a}(g_S)(a-1) > \sum_{a=2}^{\infty} \left( \frac{1}{g_M} \right)^{a-1} \left( \frac{1}{g_X} \right)^{a-1} \Lambda_{S,a}(g_S) \frac{\Lambda'_{S,a}(g_S)}{\Lambda_{S,a}(g_S)}
\]

where \( \Lambda'_{S,a}(g_S) \equiv d\Lambda_{S,a}(g_S)/dg_S \). Note that both sides of equation (25) can be interpreted as weighted averages of \( \Lambda'_{S,a}(g_S)/\Lambda_{S,a}(g_S) \). Since LHS of equation (25) has more weights on older ages, a sufficient condition for this equation to hold is \( \Lambda'_{S,a}(g_S)/\Lambda_{S,a}(g_S) \) is increasing in age \( a \).

Now, using the definition of \( \Lambda_{S,a}(g_S) \), we can write \( \Lambda'_{S,a}(g_S)/\Lambda_{S,a}(g_S) \) as

\[
\frac{\Lambda'_{S,a}(g_S)}{\Lambda_{S,a}(g_S)} = \frac{\sum_{j=1}^{a} (j-1)\omega_{a,j}}{\sum_{j=1}^{a} \omega_{a,j}},
\]

where \( \omega_{a,j} \equiv g^j \sum_{k=0}^{a-j-1} (1 - \lambda_k) \left( \prod_{k=1}^{a-j} s_{U,k} \right) \left( \prod_{k=a-j+1}^{a-1} s_{S,k} \right) \lambda_{a-j} \). Notice that the subscript \( j \) represents how many periods have passed since each business becomes successful, so \( w_{a,j} \) is the aggregate productivity of successful businesses at age \( a \) that succeeded at age \( a - j + 1 \). Therefore, what we want to show gets to

\[
\frac{\sum_{j=1}^{a+1} (j-1)\omega_{a+1,j}}{\sum_{j=1}^{a+1} \omega_{a+1,j}} > \frac{\sum_{j=1}^{a} (j-1)\omega_{a,j}}{\sum_{j=1}^{a} \omega_{a,j}}
\]  

for all ages \( a \). Using a property \( \omega_{a+1,j+1} = g_S s_{S,a} \omega_{a,j} \), we can transform equation
\[
\sum_{a=1}^{\infty} \omega_{a,j} > \frac{\sum_{a=1}^{j-1} (j-1) \omega_{a,j}}{\sum_{j=1}^{a} \omega_{a,j}}.
\]

(27)

Since we assume that the survival probability for successful businesses is higher than that for unsuccessful \( s_{S,a} > s_{U,a} \) for all ages and the successful probability \( \lambda_a \) decreases in age, \( \omega_{a,j+1} > \omega_{a,j} \) because

\[
\frac{\omega_{a,j+1}}{\omega_{a,j}} = \frac{g_{S,S,a-j}\lambda_{a-j-1}}{S_{U,a-j}\lambda_{a-j}}.
\]

Under these assumptions, we can prove equation (27):

\[
\text{(LHS)} = \frac{\sum_{j=1}^{a} \omega_{a,j}}{\omega_{a+1,1}/g_{S,S,a}} > \frac{\sum_{j=1}^{a} \omega_{a,j}}{\omega_{a,1}} > \frac{\sum_{j=1}^{a-1} \omega_{a,j}}{\sum_{j=1}^{a} \omega_{a,j}/a} = a > \frac{\sum_{j=1}^{a} (j-1) \omega_{a,j}}{\sum_{j=1}^{a} \omega_{a,j}} = \text{(RHS)}.
\]

Hence, \( dgX/dgM \) is larger in the economy in which the growth rate in the size of old businesses decreases faster.

### A.2 Spillovers calibration

In this section, we estimate the value of \( \gamma \) for equation (18). To obtain a measure of \( g_S \), we use BDS data and the following procedure. Use the data to construct the share of old establishments \( (N_{old}/N) \) and the share of workers in old establishments \( (M_{old}/M) \), where we consider an establishment as old if it is 16 years old or older. Then, note that from the equation for aggregate labor \( L \), we can construct data on \( \log([X_{old}/X]_t) \) as it is equal to \( \log([M_{old}/M]_t) - \log([N_{old}/N]_t) \). Finally, we obtain \( g_S \) as \( g_S = \Delta \log([X_{old}/X]_t) + \Delta \log(X_t) = \Delta \log([X_{old}/X]_t) + g_X \).

The OLS estimation of equation (18) is in the first column of Table 5. That is the coefficient we use in our benchmark model. The second column is the same regression but adding a linear trend. It yields similar results. One may be concerned that overall productivity growth in \( t-1 \) may be affected by the productivity of the old businesses in period \( t \). Ideally, we want the variation in \( g_X \) that is independent of the productivity growth of already successful businesses. We chose as instrument of \( g_X \) venture capital (VC) investment because it should affect \( g_X \) through
the innovation by new firms. And VC investment should not affect directly the productivity growth of already successful businesses. Using this IV approach, we find slightly larger estimates.

<table>
<thead>
<tr>
<th>Regression for $g_s$</th>
<th>OLS</th>
<th>Instrumental Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_{s,t-1}$</td>
<td>0.342*</td>
<td>0.384*</td>
</tr>
<tr>
<td></td>
<td>(0.186)</td>
<td>(0.207)</td>
</tr>
<tr>
<td>Trend</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>R squared</td>
<td>0.124</td>
<td>0.108</td>
</tr>
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<td>First stage statistic F</td>
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</tr>
<tr>
<td>Hansen’s $\chi^2$, p value</td>
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<td>0.128</td>
</tr>
<tr>
<td>Instruments</td>
<td>-</td>
<td>VC</td>
</tr>
<tr>
<td>Observations</td>
<td>23</td>
<td>23</td>
</tr>
</tbody>
</table>

Note: “VC” stands for lag growth rate of (i) VC total investment, (ii) early stage investment, (iii) seed investment, and (iv) expansion stage investment. Similarly, “Entry” stands for lag growth rate in the entry rate.

A.3 Profiles of survival and success probabilities

Figure 8: Probability of exit and success over the life-cycle

Please note that the online appendix can be found in this link.