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The Ramsey Steady-State Conundrum in Heterogeneous-Agent Economies*

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Abstract

This paper makes two fundamental contributions: (i) We prove that the interior Ramsey steady state commonly assumed in the literature may not exist in a standard Aiyagari model-in particular, a steady state with the modified golden rule and a positive capital tax is shown to be feasible but not optimal. (ii) We design a modified, analytically tractable version of the standard Aiyagari model to reveal the necessary and/or sufficient conditions for the existence of a Ramsey steady state. We characterize the basic properties of both interior and non-interior Ramsey steady states and show that researchers may draw fundamentally misleading conclusions about optimal fiscal policy (such as the optimal capital tax rate) from their analysis if an interior Ramsey steady state is incorrectly assumed.

JEL Classification: E13; E62; H21; H30

Keywords: Ramsey steady state, optimal capital tax, optimal government debt, intertemporal distortions, consumption front-loading, heterogeneous agents.

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1 Introduction

When solving the Ramsey taxation problem in a standard Aiyagari model, the existence of a Ramsey steady state is often assumed rather than proved (see, e.g., Aiyagari (1995) and the literature that follows), because proving the existence of the Ramsey steady state in such models is a daunting challenge due to their intractability. For example, Aiyagari (1995) openly admits that "it seems quite difficult to guarantee that a solution to the optimal tax problem converges to a steady state." (See Aiyagari (1995) Footnote 14.) However, without such an "existence assumption," the Ramsey allocation is difficult to analyze and not numerically solvable. But, optimal tax policies derived from the analyses depend critically on the validity of such an "existence assumption." In this paper we refer to this difficult situation as the Ramsey steady-state conundrum.[1]

We tackle the Ramsey steady-state conundrum by first proving that an interior Ramsey steady state with the properties proposed by Aiyagari (1995) or generally assumed in the literature cannot exist in a standard Aiyagari model.[2] For this reason, the assumption of the existence of an interior Ramsey steady state may have led researchers to draw fundamentally misleading conclusions about optimal fiscal policy, such as the conclusion that the optimal capital tax is positive, when in fact it could be zero or negative.

The intractability of the standard Aiyagari model is at the root of the Ramsey steady-state conundrum and stems from the infinite history dependence of individual wealth on past idiosyncratic shocks. This property implies that the wealth distribution in the Aiyagari model can become an infinite-dimensional object, making it difficult to study the full set of the Ramsey first-order conditions (FOCs). However, this daunting challenge does not prevent us from shedding light on this puzzle. Using the power utility function over consumption, $u(c) = c^{1-\sigma}/(1-\sigma)$, we prove that the interior Ramsey steady state does not exist in a standard Aiyagari model if $\sigma \geq 1$.

This paper also goes beyond such a negative result for the conundrum. To uncover the mechanism behind our alarming result, we design a modified, analytically tractable version of the standard Aiyagari model by introducing a partial risk-sharing technology to reduce the state space of the model so as to explain why the commonly assumed interior Ramsey steady state may or may not exist. A nice feature of our modified model is that it allows for an interior Ramsey steady state when $\sigma \geq 1$, but it converges to the standard Aiyagari model in the limit as the risk-sharing technology in our model is gradually eliminated. We use our modified model as a laboratory to study the necessary and/or sufficient conditions for the existence of an interior Ramsey steady state and find that the conditions are quite

[1] This problem was first raised by Chen, Chien, and Yang (2019).

[2] In general, there can be two types of Ramsey steady states: interior and non-interior. If all quantity variables converge to finitely positive values, it is called an interior steady state. Otherwise, it is called a non-interior steady state if one or more quantity variables (such as total consumption) converge to zero.

stringent and sensitive to the Ramsey planner's fiscal capacity to achieve an unconstrained allocation, which depends critically on the persistence of idiosyncratic shocks, the degree of risk aversion, and the degree of historical dependence of individual wealth on idiosyncratic shocks.[3]

In particular, we show analytically that, under the normal parameter condition of $\sigma \geq 1$, the following results hold: (i) If unconstrained allocation is feasible for the Ramsey planner, then there exists a unique interior Ramsey steady state where the MGR holds and the optimal capital tax is zero. (ii) If unconstrained allocation is not feasible, then there is no interior Ramsey steady state. In this case, an erroneous "existence assumption" of the Ramsey steady state leads to a positive capital tax in order to be consistent with the MGR; but, the only possible Ramsey steady state in this case is a non-interior one in which aggregate consumption approaches zero, the optimal labor tax goes to 100%, and the capital tax is indeterminate. (iii) An unconstrained interior Ramsey steady state can be ensured to exist in our modified Aiyagari model for a properly chosen degree of effectiveness of the risk-sharing technology; however, the interior Ramsey steady state rapidly converges to the non-interior Ramsey steady state as our modified model approaches the standard Aiyagari model by gradually eliminating the effectiveness of the risk-sharing technology in our model.

On the other hand, if the IES parameter $\sigma < 1$ such that the degree of risk aversion is low, we show that the only possible Ramsey steady state is one in which the Ramsey Lagrangian multiplier associated with the aggregate resource constraint diverges, the MGR does not hold, and the interest rate is below the time discount rate.

By studying the modified Aiyagari model, we clearly learn that the underlying driving force of the non-existence result in the standard Aiyagari model is its characteristic feature that the market interest rate r is always lower than the time discount rate β^{-1} in any competitive equilibrium. Given this intertemporal wedge, the government has a dominant incentive to front-load consumption by borrowing more cheaply in the short run as a trade-off for low consumption in the long run, because the future utility cost of debt financing is heavily discounted by a time discount factor β that is lower than the market discount rate $1/r$. This incentive to arbitrage and front-load consumption never disappears unless $r = 1/\beta$, which is infeasible in the standard Aiyagari model, leading to a non-interior Ramsey steady state with an immiseration outcome where consumption approaches zero. Such a dynamic implication for the Ramsey allocation is consistent with the finding of Albanesi and Armenter (2012), who argue that front-loading intertemporal distortions induces a first-order welfare gain in a broad class of second-best economies.

However, when $\sigma < 1$, we show that an interior Ramsey steady state can exist, but it is characterized

[3]Here, an unconstrained allocation is defined as a competitive equilibrium allocation in which no individual's ad hoc borrowing constraint is strictly binding. This unconstrained allocation is also referred to in this paper as the full self-insurance (FSI) allocation.

by the failure of MGR. The intuition is as follows: The general principle of front-loading consumption still holds in the case of $\sigma < 1$, but the government's incentive to issue debt cannot be satisfied in the market due to the weak precautionary saving motive (or weak household demand for assets), leading to a divergent multiplier and the failure of the MGR.

In addition, we show that solving optimal fiscal policy in Aiyagari-type models by assuming that the Ramsey planner maximizes only the steady-state welfare of a competitive equilibrium can trivially ensure the existence of an interior Ramsey steady state, but the result distorts the picture of Ramsey allocation in a dynamic setting that maximizes expected welfare at time zero. This distortion occurs because the steady-state welfare approach ignores the transitional dynamics of the Ramsey problem. Although it is well known in the literature that optimal policies can look dramatically different between steady-state welfare analysis and dynamic welfare analysis at time zero (see, e.g., Domeij and Heathcote (2004), Heathcote (2005), and, Rohrs and Winter (2017)), our analytical approach makes a further contribution to the literature by showing the underlying mechanism that drives the sharp differences between these two approaches. The culprit behind the different results for optimal fiscal policy between steady-state and dynamic welfare analysis is the arbitrage opportunity arising from the gap between the market interest rate and the time discount rate; this gap does not matter when maximizing steady-state welfare, but it matters a lot when maximizing expected welfare at time zero. This is because, in a dynamic setting, the Ramsey planner chooses to take advantage of the cheap interest rate on debt by front-loading consumption; however, such a front-loading incentive disappears in static welfare analysis.

2 Literature Review

Our work is motivated by Straub and Werning (2020), who challenged the classical zero-capital-taxation result of Judd (1985) by considering the possibility of a non-interior Ramsey steady state where aggregate consumption approaches zero. This paper instead challenges the positive-capital-taxation result of Aiyagari (1995) by showing that an interior Ramsey steady state with the properties assumed by Aiyagari does not exist in a standard Aiyagari model with constant relative risk aversion (CRRA) preferences. The underlying mechanism of our non-interior Ramsey steady state result is also completely different from that of Straub and Werning (2020).

Our paper is related to the work of Bassetto and Cui (2020), who argue that when the government's fiscal capacity is insufficient to support an unconstrained allocation, the optimal Ramsey allocation can converge to a constrained interior steady state where the Lagrangian multiplier diverges. They use a numerical method to find such an interior Ramsey steady state under $\sigma < 1$. Consistent with their result, we are able to prove with certainty that such a constrained interior Ramsey steady state can arise only when $\sigma < 1$ and the multiplier diverges.

Similarly, our paper makes contact with Angeletos, Collard, and Dellas (2020), who show that when risk-free government bonds contribute to the supply of liquidity to alleviate private agents' borrowing constraints, issuing more debt increases welfare by improving the allocation of resources. Similar to our results, theirs show that the heterogeneous-agent structure justifies a higher optimal level of public debt and introduces an interesting transition path to a Ramsey steady state along which a departure from tax smoothing becomes desirable.[4]

Our modified framework in Section 5 builds on the truncation method of Le Grand and Ragot (2022) to reduce the state space of our model and to make it analytically tractable. Their work is part of a large literature investigating the optimal responses of fiscal policy to aggregate shocks, stemming from the work of Barro (1979) and Lucas and Stokey (1983) in the representative-agent framework. There is a strong tradition and renewed interest in extending this literature to a heterogeneous-agent framework, such as Bassetto (2014) and Bhandari, Evans, Golosov, and Sargent (2021). However, the existence of an interior stationary Ramsey allocation in heterogeneous-agent models with both aggregate and idiosyncratic uncertainty is often assumed rather than proved. We believe that our modified Aiyagari model can be extended to include aggregate risks and thus complement this literature by providing a more-transparent analysis of the dynamics around the Ramsey steady state (which can be proved to exist).

For the same reason above, our theoretical analysis provides a cautionary note to the growing literature that relies on numerical methods to solve Ramsey taxation problems in Aiyagari-type economies. To the best of our knowledge, most of the numerical approaches rely on the existence assumption of an interior Ramsey steady state to proceed. For example, Acikgoz, Hagedorn, Holter, and Wang (2018) consider the same utility function as ours while directly assuming the existence of a unique interior Ramsey steady state (see their assumption 1) without proof. Our theoretical analysis proves that such an interior Ramsey steady state does not exist when the risk aversion parameter $\sigma \geq 1$. However, in a follow-up paper, Acikgoz, Hagedorn, Holter, and Wang (2023) change the household preference from the power utility function to the Greenwood-Hercowitz-Huffman (GHH) form. The work of Dyrda and Pedroni (2023) adopts the KPR utility function (King, Plosser, and Rebelo (1988)) to solve the Ramsey problem numerically under the existence assumption of an interior Ramsey steady state. Since the GHH and KPR preferences are different from ours, it is not clear to us whether their existence assumption of an interior Ramsey steady state is correct. Nevertheless, in their Appendix M, Dyrda and Pedroni (2023) compare

[4] In contrast to our model, their model does not necessarily feature unconstrained allocation as an optimal goal, and their optimal debt level preserves financial frictions to lower the interest rate on public debt. The main reason for this difference from ours may be that their model does not have enough policy instruments to match the number of goods (i.e., their model has an incomplete tax system at the macroeconomic level).

their numerical result with that of Acikgoz, Hagedorn, Holter, and Wang (2018) by using the same power utility function as that in our analysis, and they claim to find an interior Ramsey steady state under a risk aversion parameter $\sigma = 2$, which we prove to be incorrect. Thus, our analysis serves as a warning to a large literature that relies on numerical analysis to study optimal fiscal policy.

Finally, our work includes as a special case our previous work in Chien and Wen (2021a), which uses a tractable heterogeneous-agent model with quasi-linear preferences to show that the optimal capital tax must be zero in an unconstrained Ramsey steady state, which can be shown to exist. The intuition of Chien and Wen (2021a) shows that the Ramsey planner's desire to front-load consumption by issuing increasing amounts of debt never disappears unless the market discount rate equals the time discount rate, which can only be achieved in an unconstrained allocation. We show in this paper that this mechanism carries over to our more general model, which includes the standard Aiyagari model as a special limiting case, and we use it to prove that the infeasibility of the Ramsey planner's never-ending pursuit of an unconstrained allocation leads to a non-interior Ramsey steady state.

The rest of the paper is organized as follows: Section 3 sets up the standard Aiyagari model and defines the competitive equilibrium. Section 4 shows that the commonly assumed interior Ramsey steady state may not exist in a standard Aiyagari model. Section 5 builds a modified Aiyagari model to reveal the mechanism behind the Ramsey steady-state conundrum and to provide necessary and/or sufficient conditions for the existence of various types of Ramsey steady states. Section 6 considers a steady-state welfare-maximizing analysis to further explore the underlying mechanism of our results. Finally, Section 7 concludes.

3 A Standard Aiyagari Model

Firms. A representative firm produces output according to the Cobb-Douglas technology with constant returns to scale, $Y_t = F(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha}$, where Y, K, and N denote total output, capital, and labor, respectively. The firm rents capital and hires labor by paying a competitive rental rate and real wage, denoted by q_t and w_t, respectively. The firm's optimal conditions for profit maximization at time t satisfy

$$w_t = \frac{\partial F(K_t, N_t)}{\partial N_t} \equiv MPN_t, \tag{1}$$

$$q_t = \frac{\partial F(K_t, N_t)}{\partial K_t} \equiv MPK_t. \tag{2}$$

Government. In each period t, the government can issue bonds, B_{t+1}, and levy both a linear labor tax on labor $\tau_{n,t}$ and a linear capital tax on capital $\tau_{k,t}$. Let Q_{t+1} be the price of the risk-free government bonds in period t, which pay for one unit of consumption goods in period $t+1$; then, the risk-free interest rate is given by $r_{t+1} \equiv Q_{t+1}^{-1}$. The flow government budget constraint in period t is

$$\tau_{n,t} w_t N_t + \tau_{k,t} q_t K_t + Q_{t+1} B_{t+1} \geq B_t, \tag{3}$$

where the initial level of government bonds B_0 is exogenously given. For simplicity, government spending is assumed to be zero.

Individuals. There is a unit measure of ex ante identical individuals with initial wealth $a_0 > 0$. Ex post, each individual is subject to an idiosyncratic labor productivity shock in each period. The shock process follows a first-order finite state Markov process $\theta_t \in Z$. We denote $\theta^t \equiv \{\theta_0, \theta_1, ..., \theta_t\}$ as the shock history of an individual up to period t; $\pi(\theta^t)$ as the unconditional probability of the realization of state θ^t; and $\pi(\theta^{t+1}|\theta^t)$ as the transition probability from event θ^t to θ^{t+1}, which is equal to $\pi(\theta_{t+1}|\theta_t)$ since the shock process is first-order Markov.

Let $\widehat{w}_t \equiv (1 - \tau_{n,t})\, w_t$ be the after-tax wage rate. In period t, given the shock history θ^t, let $a_{t+1}(\theta^t)$, $n_t(\theta^t)$, $c_t(\theta^t)$, and $z_t(\theta^t)$ be an individual's wealth, labor supply, consumption, and labor productivity levels, respectively. The budget constraint for an individual with history θ^t is given by

$$a_t(\theta^{t-1}) + \widehat{w}_t z_t(\theta^t) n_t(\theta^t) - c_t(\theta^t) - Q_{t+1} a_{t+1}(\theta^t) \geq 0, \text{ for all } t \geq 0, \tag{4}$$

where a_0 is the exogenously given initial wealth. All individuals are subject to the following ad hoc borrowing constraints for all $t \geq 0$ and θ^t:

$$a_{t+1}(\theta^t) \geq 0. \tag{5}$$

The individual's welfare criterion is given by

$$U = \sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \left[u(c_t(\theta^t)) - v\left(n_t(\theta^t)\right) \right] \pi(\theta^t), \tag{6}$$

where $\beta \in (0, 1)$ is the time-discounting factor and the utility function takes the standard power form:

$$u(c) = \frac{1}{1-\sigma} c^{1-\sigma} \text{ and } v(n) = \frac{1}{1+\gamma} n^{1+\gamma},$$

where the IES parameter $\sigma \in (0, \infty)$ and the Frisch elasticity parameter $\gamma > 0$.

Given the market prices $\{Q_{t+1}, \widehat{w}_t\}_{t=0}^{\infty}$, the government policies $\{\tau_{n,t}, \tau_{k,t}, B_{t+1}\}_{t=0}^{\infty}$, and the initial wealth a_0, each individual chooses a plan $\{c_t(\theta^t), n_t(\theta^t), a_{t+1}(\theta^t)\}_{t=0}^{\infty}$ to maximize (6) subject to (4) and

(5). The FOCs with respect to $c_t(\theta^t)$, $n_t(\theta^t)$, and $a_{t+1}(\theta^t)$ imply the following optimal conditions

$$n_t(\theta^t)^\gamma = c_t(\theta^t)^{-\sigma} \widehat{w}_t z_t(\theta^t), \tag{7}$$

and

$$Q_{t+1} c_t(\theta^t)^{-\sigma} \geq \beta \sum_{\theta^{t+1}} c_{t+1}(\theta^{t+1})^{-\sigma} \pi(\theta^{t+1}|\theta^t), \tag{8}$$

where equality holds if the household borrowing constraint is not strictly binding in the state θ^t of period t.

There is no aggregate uncertainty. Government bonds and capital are perfect substitutes as stores of value for individuals. As a result, the gross after-tax return on capital must equal the gross risk-free rate (no arbitrage condition):

$$(1 - \tau_{k,t}) q_t + 1 - \delta = r_t \equiv \frac{1}{Q_t}, \text{ for all } t \geq 0. \tag{9}$$

3.1 Competitive Equilibrium

Definition 1. (Competitive Equilibrium) Let A_{t+1} and C_t be the aggregate wealth and aggregate consumption, respectively, at period t. Given the initial asset holdings a_0, the initial supply of government bonds B_0, the initial capital stock K_0, and the sequence of policies $\{\tau_{n,t}, \tau_{k,t}, B_{t+1}\}_{t=0}^\infty$, a competitive equilibrium is defined as the sequence of prices $\{w_t, Q_{t+1}\}_{t=0}^\infty$, aggregate allocation $\{C_t, N_t, K_{t+1}, A_{t+1}\}_{t=0}^\infty$, and individual allocation $\{c_t(\theta^t), n_t(\theta^t), a_{t+1}(\theta^t)\}_{t=0}^\infty$, such that

1. given $\{Q_{t+1}, w_t, \tau_{n,t}\}_{t=0}^\infty$, the allocation $\{c_t(\theta^t), n_t(\theta^t), a_{t+1}(\theta^t)\}_{t=0}^\infty$ solves the individual problem;
2. the no-arbitrage condition holds: $1/Q_t = 1 + (1 - \tau_{k,t})\, q_t - \delta$ for all $t \geq 0$;
3. given $\{w_t, q_t\}_{t=0}^\infty$, the path of total quantities $\{N_t, K_t\}_{t=0}^\infty$ solves the representative firm problem;
4. all markets clear for $t \geq 0$:

$$\begin{aligned}
F(K_t, N_t) + (1 - \delta) K_t &= C_t + K_{t+1}, \\
N_t &= \sum_{\theta^t} n_t(\theta^t) z_t(\theta^t) \pi(\theta^t), \\
C_t &= \sum_{\theta^t} c_t(\theta^t) \pi(\theta^t), \\
B_{t+1} + \frac{K_{t+1}}{Q_{t+1}} &= \sum_{\theta^t} a_{t+1}(\theta^t) \pi(\theta^t) \equiv A_{t+1},
\end{aligned} \tag{10}$$

and the government flow budget constraint holds for $t \geq 0$:

$$\tau_{n,t} w_t N_t + \tau_{k,t} q_t K_t + Q_{t+1} B_{t+1} \geq B_t.$$

Define an unconstrained competitive equilibrium allocation as the allocation in which all individual borrowing constraints are slack, regardless of their shock history.[5] It is easy to show that the unconstrained allocation must imply $Q = \beta$ in the steady state. However, it is well known that in any steady state of a competitive equilibrium of the standard Aiyagari model, it must be the case that $Q > \beta$. Otherwise, the individual asset demand goes to infinity, which cannot constitute a competitive equilibrium. In other words, unconstrained allocation is impossible to achieve in the standard Aiyagari model (see Aiyagari (1994) and Ljungqvist and Sargent (2012)).

Therefore, in the standard Aiyagari model, a positive measure of individual borrowing constraints must be strictly binding, resulting in $Q > \beta$, and there must be aggregate allocative inefficiency due to overaccumulation of capital in a laissez-faire competitive equilibrium. For this reason, Aiyagari (1995) argues that the best outcome the Ramsey planner can achieve is an allocation where the MGR is satisfied by taxing the capital stock in the steady state so that the aggregate allocative efficiency can be restored in the long run.

However, Aiyagari's argument is based on only one of the many Ramsey FOCs and, more importantly, on the implicit assumption that there is an interior Ramsey steady state with convergent Ramsey Lagrangian multiplier(s). In this paper we will go beyond Aiyagari's approach by deriving more than one Ramsey FOC and showing that the assumption of the existence of an interior Ramsey steady state leads to contradictions or is inconsistent with additional Ramsey FOCs.

4 Ramsey Outcome in a Standard Aiyagari Model

4.1 Conditions to Support a Competitive Equilibrium

To ensure that a Ramsey outcome constitutes a competitive equilibrium, we must show first that all possible allocations in the choice set of the Ramsey planner constitute a competitive equilibrium. The following proposition states the conditions that any constructed Ramsey allocation must satisfy to constitute a competitive equilibrium[6]:

[5] Throughout this paper, we follow the literature and consider only ad hoc household borrowing constraints, which are assumed to be tighter than the natural borrowing limits.

[6] As is standard in the literature, we prevent the planner from choosing $\tau_{k,0}$ in the Ramsey problem.

Proposition 1. ***(Conditions to Support a Competitive Equilibrium)*** *Given the initial asset holdings a_0, the initial capital tax $\tau_{k,0}$, the initial government bond B_0, and the initial capital stock K_0, the allocation $\{c_t(\theta^t), n_t(\theta^t), a_{t+1}(\theta^t), K_{t+1}\}_{t=0}^{\infty}$ and the price sequence $\{\widehat{w}_t, Q_{t+1}\}_{t=0}^{\infty}$ can be supported as a competitive equilibrium if and only if they satisfy the following conditions:*

1. *the aggregate resource constraint:*

$$F\left(\sum_{\theta^t} n_t(\theta^t) z_t(\theta^t)\pi(\theta^t), K_t\right) + (1-\delta)K_t - \sum_{\theta^t} c_t(\theta^t)\pi(\theta^t) - K_{t+1} \geq 0,\ \forall t \geq 0; \tag{11}$$

2. *the implementability condition:*

$$c_t(\theta^t)^{1-\sigma} - n_t(\theta^t)^{1+\gamma} + Q_{t+1}c_t(\theta^t)^{-\sigma}a_{t+1}(\theta^t) - c_t(\theta^t)^{-\sigma}a_t(\theta^{t-1}) = 0 \tag{12}$$

   for all $t \geq 0$ and θ^t;

3. *the initial-period asset market-clearing condition:*

$$\frac{K_0}{Q_0} + B_0 = a_0, \tag{13}$$

   in which the initial bond price satisfies

$$\frac{1}{Q_0} = 1 + (1-\tau_{k,0})MPK_0 - \delta;$$

4. *the individual marginal substitution conditions:*

$$\widehat{w}_t z_t(\theta^t) c_t(\theta^t)^{-\sigma} - n_t(\theta^t)^{\gamma} = 0 \tag{14}$$

   for all $t \geq 0$ and θ^t;

5. *the borrowing constraints and their associated complementary slackness conditions:*

$$a_{t+1}(\theta^t) \geq 0, g_t(\theta^t) \geq 0,\ g_t(\theta^t)a_{t+1}(\theta^t) = 0 \tag{15}$$

   for all $t \geq 0$ and all θ^t, where the function $g_t(\theta^t)$ is defined as

$$g_t(\theta^t) \equiv Q_{t+1}c_t(\theta^t)^{-\sigma} - \beta \sum_{\theta_{t+1}} c_{t+1}(\theta_{t+1})^{-\sigma}\pi(\theta_{t+1}|\theta^t).$$

Proof. See Appendix A.1. □

4.2 Ramsey Outcome

Armed with Proposition 1, the Ramsey problem can be written as

$$\max_{\{c_t(\theta^t),n_t(\theta^t),a_{t+1}(\theta^t),K_{t+1},\widehat{w}_t,Q_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \sum_{\theta^t} \left(\frac{c_t(\theta^t)^{1-\sigma}}{1-\sigma} - \frac{n_t(\theta^t)^{1+\gamma}}{1+\gamma} \right) \pi(\theta^t)$$

subject to constraints (11) to (15). But before solving the Ramsey problem, we first define the Ramsey steady state in our economy:

Definition 2. (Ramsey Steady State) Given $\{K_0, B_0, a_0\}$, a Ramsey steady state is a long-run Ramsey allocation in which the aggregate variables $\{N_t, C_t, K_{t+1}, A_{t+1}\}$ all converge to constant values and the individual variables $\{c_t(\theta^t), n_t(\theta^t), a_{t+1}(\theta^t)\}$ converge to stationary distributions. In addition, a Ramsey steady state is called "interior" if all of the aggregate variables are strictly positive; otherwise, if one or more of these aggregate variables (such as consumption C_t) converges to zero, the Ramsey steady state is called "non-interior."[7]

Denote $\beta^t \mu_t$ and $\beta^t \lambda_t(\theta^t)\pi(\theta^t)$ as the Ramsey Lagrangian multipliers for constraints (11) and (12), respectively. The Ramsey FOC with respect to K_{t+1} is given by

$$\mu_t = \beta \mu_{t+1} \left(MPK_{t+1} + 1 - \delta \right), \tag{16}$$

which is identical to that in Aiyagari (1995).

Relying on only equation (16), Aiyagari (1995) obtains his famous result of $\tau_k > 0$ based on the following critical assumptions: (i) there is an interior Ramsey steady state, and (ii) the Ramsey Lagrangian multiplier μ_t associated with the aggregate resource constraint (11) converges to a positive constant. More specifically, under the assumption that μ_t converges, equation (16) implies that the MGR holds in the assumed steady state: $1 = \beta \left(MPK + 1 - \delta \right)$. The optimal steady-state capital tax τ_k is chosen so that the no-arbitrage condition, $1 = Q \left((1-\tau_k) MPK + 1 - \delta \right)$, is consistent with the MGR; hence,

$$\tau_k = 1 - \frac{\frac{1}{Q} - (1-\delta)}{\frac{1}{\beta} - (1-\delta)} > 0,$$

which is strictly positive because $Q > \beta$ in any competitive equilibrium. This implies that the Ramsey planner "chooses" to impose a (permanent) capital tax in order to select a long-run competitive

[7]For example, using the two-class model of Judd (1985), Straub and Werning (2020) show that the Ramsey outcome could converge to a non-interior Ramsey steady state.

equilibrium consistent with the MGR.[8]

However, we can derive three more optimal conditions of the Ramsey problem; namely, the Ramsey FOCs with respect to individual wealth $a_{t+1}(\theta^t)$, individual labor supply $n_t\left(\theta^t\right)$, and individual consumption $c_t\left(\theta^t\right)$. As shown in the proof of Proposition 2, the assumption of an interior Ramsey steady state with a convergent multiplier μ_t is incompatible with these optimal conditions and leads to contradictions. The results are formally stated in the following proposition:

Proposition 2. ***(Ramsey Allocation in the Aiyagari Model)***

1. *Under the parameter condition $\sigma \geq 1$, there is no interior Ramsey steady state; and the only possible Ramsey steady state (if it exists) must be non-interior with $C_t \to 0$.*

2. *If $\sigma < 1$, an interior Ramsey steady state may exist; however, if it exists, it must have a divergent Ramsey multiplier μ_t and the failure of the MGR.*

Proof. See Appendix A.2. □

Proposition 2 shows that the interior Ramsey steady state commonly assumed in the literature (such as Aiyagari (1995)) does not exist in a standard Aiyagari model with CRRA preference. If a Ramsey steady state exists at all, it must be non-interior with zero aggregate consumption. The crux of the problem is the hallmark feature of the Aiyagari model: $\frac{Q}{\beta} > 1$. Thus, the common practice in the literature of simply assuming the existence of an interior Ramsey steady state without proof is dangerous.

In addition, this proposition shows that the result is also sensitive to the utility curvature parameter σ, which determines consumers' saving behavior.[9] In particular, if $\sigma < 1$ such that the IES is sufficiently high, then an interior Ramsey steady state can exist, but only if the Lagrangian multiplier μ_t diverges such that the MGR fails. (The intuition shall become clear in Section 5.)

5 A Modified Aiyagari Model

It may be surprising and even puzzling that an interior Ramsey steady state, commonly assumed in the literature, may not exist in a standard Aiyagari model. Note that a steady state with MGR and a

[8] It is well acknowledged that Aiyagari (1995) also introduces an endogenous government spending in household utilities. However, the introduction of endogenous government spending does not contribute to the different results found in this paper. The critical assumptions to uphold the result of Aiyagari (1995) are the interior Ramsey steady state and the convergence of Ramsey Lagrangian multiplier. It is straightforward to show that even with endogenous government spending, our results remain unchanged.

[9] The parameter σ determines both the degree of risk aversion and the inverse degree of IES. Higher risk aversion implies a stronger incentive for precautionary saving to avoid consumption fluctuations, and at the same time a lower IES implies a lower substitutability between current and future consumption (or a stronger income effect than substitution effect). As these two aspects are captured by the same parameter, we use the two terms interchangeably in this paper.

positive capital tax as described by Aiyagari (1995) is certainly feasible, but not optimal for the Ramsey planner, as shown in Proposition 2. The question is, why does the Ramsey allocation not converge to such a feasible steady state to maintain the MGR by taxing capital? Also, what would be the optimal level of public debt and the optimal tax policy in the Aiyagari model?

To answer these questions, in this section we develop a modified and tractable version of the standard Aiyagari model to show clearly the conditions under which an interior Ramsey steady state may or may not exist. This modified Aiyagari model can also help us understand the properties of both the interior and the non-interior Ramsey steady states whenever they exist, as well as determine the optimal level of public debt and the optimal tax policy in each corresponding Ramsey steady state.

Specifically, we introduce an ad hoc wealth-pooling technology into an otherwise standard Aiyagari model to allow partial risk sharing among individuals, thereby reducing the state space of the wealth distribution in the model. As a result, the model becomes analytically tractable. Importantly, the modified model converges to the standard Aiyagari model in the limit as the effectiveness of the risk-sharing technology is gradually reduced.

Our wealth-pooling technology follows the spirit of Lucas (1990), Heathcote and Perri (2018), and Bilbiie and Ragot (2021). In particular, our truncation method builds on Le Grand and Ragot (2022). This wealth-pooling technology allows individuals with identical idiosyncratic-shock histories in the last κ-periods ($\infty > \kappa \geq 0$) to share risk by pooling their wealth together in the beginning of each period after the idiosyncratic shock is realized. As a result, individuals with the same truncated κ-period shock history make the same consumption and saving decisions, leading to a partially degenerate wealth distribution and to less precautionary saving motives. Moreover, our modified model can become arbitrarily close to the standard Aiyagari model as κ increases to infinity so that the effect of the wealth-pooling technology becomes ineffective or non-existent—in which case the probability of any two individuals having identical histories goes to zero as $\kappa \to \infty$. On the other extreme where $\kappa = 0$, individuals can pool their wealth in the beginning of every period t as long as their current idiosyncratic-shock status is the same, leading to an almost complete degenerate wealth distribution.[10] The technical details of the risk-sharing technology is described below.

Individuals and Families. We introduce a unit measure of representative families and assume that within each representative family there is a unit measure of heterogeneous individuals. Within each representative family, there is a family head who is equipped with a wealth-pooling technology that allows partial risk sharing among family members according to each individual's truncated history $\kappa \geq 0$. Denote the κ-period truncated history of shocks in period t as $h_t^\kappa = \{\theta_{t-\kappa}, \theta_{t-\kappa+1}, .., \theta_t\} \in Z^\kappa$. In

[10] Our model thus includes both the standard Aiyagari model and the model of Chien and Wen (2021b) as special limiting cases without the need to appeal to log-linear preferences (as in Chien and Wen (2021a)) to gain model tractability.

particular, the head of a family can redistribute wealth among individual family members with the same truncated history h^{κ}. However, the family head cannot redistribute resources among individual members with different h^{κ}. Thus, the family head can provide a limited amount of risk sharing among certain family members but cannot completely eliminate the idiosyncratic risk faced by individuals.

As time passes from one period to the next, the transition probability from individuals of type h^{κ} to individuals of type $h^{\kappa'}$ is denoted by $\pi(h^{\kappa'}|h^{\kappa})$, which is determined by the transition probability of the first-order Markov process of θ. Furthermore, the invariant probability of each group h^{κ} is denoted by $\pi(h^{\kappa})$. We also denote the group of individuals who experience the highest shock and the lowest shock in each period during the entire truncated history as $\boldsymbol{h}_h^{\kappa}$ and $\boldsymbol{h}_l^{\kappa}$, respectively.

For simplicity and without loss of generality, we assume that $\pi(h^{\kappa})$ also represents the initial period's share of individuals at time 0. The utilitarian welfare criterion of a family head is then given by

$$U = \sum_{t=0}^{\infty} \beta^t \sum_{h^{\kappa}} \left[\frac{1}{1-\sigma} c_t(h^{\kappa})^{1-\sigma} - \frac{1}{1+\gamma} n_t(h^{\kappa})^{1+\gamma} \right] \pi(h^{\kappa}). \tag{17}$$

Let $z_t(h^k)$ be the period-t (current period) labor productivity shock for group h^{κ}. The budget constraints for type-h^{κ} individuals in period 0 are given by

$$a_0(h^{\kappa}) + \widehat{w}_0 z_0(h^{\kappa}) n_0(h^{\kappa}) - c_0(h^{\kappa}) - Q_1 a_1(h^{\kappa}) \geq 0. \tag{18}$$

Under the wealth-pooling technology, the total assets available for type-h^{κ} individuals in the beginning of period $t \geq 1$ is given by $\sum_{h_{-1}^{\kappa}} a_t(h_{-1}^{\kappa})\pi(h_{-1}^{\kappa})\pi(h^{\kappa}|h_{-1}^{\kappa})$. Therefore, for all $t \geq 1$, the budget constraint for type-h^{κ} individuals can be written as

$$\sum_{h_{-1}^{\kappa}} \frac{a_t(h_{-1}^{\kappa})\pi(h_{-1}^{\kappa})\pi(h^{\kappa}|h_{-1}^{\kappa})}{\pi(h^{\kappa})} + \widehat{w}_t z_t(h^{\kappa}) n_t(h^{\kappa}) - c_t(h^{\kappa}) - Q_{t+1} a_{t+1}(h^{\kappa}) \geq 0. \tag{19}$$

The individual borrowing constraint is still given by

$$a_{t+1}(h^{\kappa}) \geq 0 \text{ for all } t \geq 0 \text{ and } h^{\kappa}. \tag{20}$$

Finally, each family head chooses a plan of $\{c_t(h^{\kappa}), n_t(h^{\kappa}), a_{t+1}(h^{\kappa})\}_{t=0}^{\infty}$ to maximize (17) subject to (18), (19), and (20).

To make the ad hoc wealth-pooling technology meaningful in analytically addressing our problems at hand, we assume that the idiosyncratic shock process is non-negatively autocorrelated such that in each period $t \geq 1$ the initial wealth of type-$\boldsymbol{h}_h^{\kappa}$ individuals is no less than that of the other individuals in the population. This assumption rules out the uninteresting case where any individual may become wealthier than type-$\boldsymbol{h}_h^\kappa$ individuals from time to time. More specifically, if $a_t(\boldsymbol{h}_h^\kappa) > a_t(h^\kappa)$ for all h^κ, then under the assumption that the idiosyncratic shock process is non-negatively autocorrelated, the following inequality must hold:

$$\sum_{h_{-1}^\kappa} \frac{a_t(h_{-1}^\kappa)\pi(h_{-1}^\kappa)\pi(\boldsymbol{h}_h^\kappa|h_{-1}^\kappa)}{\pi(\boldsymbol{h}_h^\kappa)} \geq \sum_{h_{-1}^\kappa} \frac{a_t(h_{-1}^\kappa)\pi(h_{-1}^\kappa)\pi(h^\kappa|h_{-1}^\kappa)}{\pi(h^\kappa)} \text{ for all } h^\kappa \neq \boldsymbol{h}_h^\kappa,$$

which says that the period-t initial asset holdings of $\boldsymbol{h}_h^\kappa$ individuals are no less than that of any other type of individuals.

As we show in the following proposition, the competitive equilibrium in the modified Aiyagari model could feature an allocation where the ad hoc borrowing constraints are not binding for all individuals if their asset holdings are sufficiently large. We define such an equilibrium as unconstrained allocation.

Proposition 3. ***(Competitive Equilibrium in the Modified Model)*** *For simplicity, assume that in period 0 the initial asset holdings satisfy $a_0(\boldsymbol{h}_h^\kappa) > a_0(h^\kappa) \geq 0$ for all $h^\kappa \neq \boldsymbol{h}_h^\kappa$. The competitive equilibrium of the modified Aiyagari model must have the following properties:*

1. *For all $t \geq 0$, it must be true that $c_t(\boldsymbol{h}_h^\kappa) \geq c_t(h^\kappa)$ and $a_{t+1}(\boldsymbol{h}_h^\kappa) > a_{t+1}(h^\kappa) \geq 0$ for all $h^\kappa \neq \boldsymbol{h}_h^\kappa$. That is, the ad hoc borrowing constraints of type-$\boldsymbol{h}_h^\kappa$ individuals are always slack: $a_{t+1}(\boldsymbol{h}_h^\kappa) > 0$, which implies that the Lagrangian multiplier associated with constraint (20) is $\psi_t(\boldsymbol{h}_h^\kappa) = 0$. Also, depending on the level of B_t, the borrowing constraints of the currently unemployed individuals may or may not be binding: $a_{t+1}(h^\kappa) \geq 0$.*

2. *Because $\psi_t(\boldsymbol{h}_h^\kappa) = 0$, the intertemporal price Q_{t+1} can be expressed as*

$$Q_{t+1} = \beta \sum_{h^{\kappa\prime}} \frac{u_{c,t+1}(h^{\kappa\prime})}{u_{c,t}(\boldsymbol{h}_h^\kappa)} \pi(h^{\kappa\prime}|\boldsymbol{h}_h^\kappa) \geq \beta \sum_{h^{\kappa\prime}} \frac{u_{c,t+1}(h^{\kappa\prime})}{u_{c,t}(h^\kappa)} \pi(h^{\kappa\prime}|h^\kappa) \quad \textit{for all } t \textit{ and } h^\kappa \neq \boldsymbol{h}_h^\kappa. \tag{21}$$

3. *In the steady state, if the multiplier $\psi_t(h^\kappa) = 0$ for all individuals regardless of their truncated history h^κ, then the competitive equilibrium features unconstrained allocation with two properties: (i) consumption equality $c(\boldsymbol{h}_h^\kappa) = c(h^\kappa)$ for all h^κ and (ii) a zero liquidity premium with $Q = \beta$ (or $r = \beta^{-1}$). Otherwise, in the case of only partial self-insurance (constrained allocation) it must be true that $Q > \beta$ (or $r < \beta^{-1}$).*

Proof. See Appendix A.3. ∎

Proposition 3 states that if the asset holdings $a_{t+1}(h^\kappa)$ are sufficiently large for all individuals, regardless of their truncated history h^κ, such that everyone's ad hoc borrowing constraint is slack, then they can achieve the same level of steady-state consumption, regardless of their truncated idiosyncratic history. In this unconstrained competitive equilibrium, the steady-state market interest rate is equal to the time discount rate: $Q = \beta$ (or $r = \beta^{-1}$).

It is well known that an unconstrained competitive equilibrium (where $r = \beta^{-1}$ and each individual is fully self-insured with sufficient precautionary savings so that his borrowing constraint never binds) is not possible in the standard Aiyagari model. However, in our modified Aiyagari model, the unconstrained steady state can be achieved with only a finite level of asset holdings because the level of asset demand to achieve full self-insurance (FSI) does not go to infinity even at $Q = \beta$, thanks to the wealth-pooling technology that allows partial risk sharing among individuals.

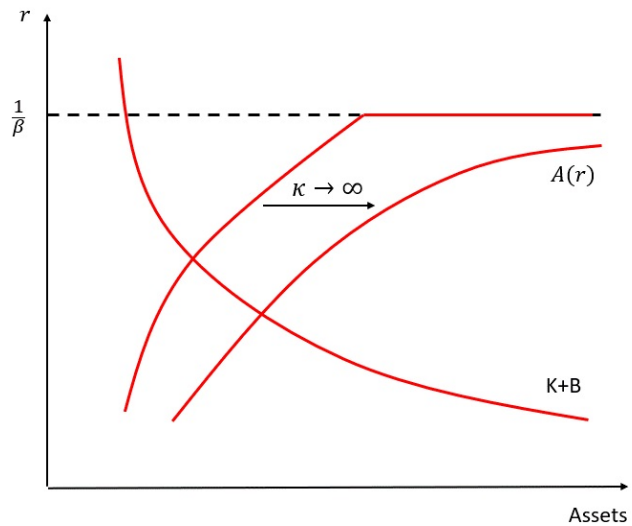
Figure 1 shows the main difference in the asset demand functions between the standard Aiyagari model and the modified model. In our modified model, household asset demand remains finite even as the interest rate r approaches $1/\beta$, whereas in the standard Aiyagari model asset demand goes to infinity when $r = 1/\beta$. However, as κ increases, the asset demand curve in our modified model shifts to the right and approaches that of the Aiyagari model.

Figure 1 and Proposition 3 also imply that even if the laissez-faire competitive equilibrium does not feature FSI due to an insufficient initial supply of assets, the Ramsey planner can potentially achieve the unconstrained allocation by issuing sufficient public debt (i.e., shifting the supply curve to the right) if desired. Therefore, to make our Ramsey problem interesting in the modified Aiyagari model, we assume that the initial capital K_0 and bond supply B_0, as well as the initial distribution of household wealth $a_0(h^\kappa)$, are such that the laissez-faire competitive equilibrium (without further policy intervention) *does not* achieve the unconstrained allocation. Namely, in the absence of further government intervention ($B_t = B_0$ for all $t > 0$), the competitive equilibrium features consumption inequality $c(\boldsymbol{h}_h^\kappa) > c(h^\kappa)$ and precautionary saving behavior with a positive liquidity premium: $r < \beta^{-1}$.

5.1 Ramsey Outcome in the Modified Aiyagari Model

To facilitate the analysis below, we define A as the minimum level of aggregate wealth required to achieve an unconstrained allocation in the Ramsey steady state, and $\phi \equiv \frac{A}{C}$ as the ratio of A to aggregate consumption C in an unconstrained allocation. Note that the equilibrium value of ϕ depends on the persistence of idiosyncratic shocks and the effectiveness of the risk-sharing technology (parameter κ); and ϕ essentially captures the feasibility of the policy space to achieve an unconstrained interior Ramsey steady state in our modified Aiyagari model.

Figure 1: Asset Market in the Modified Aiyagari Model



Proposition 4. ***(Ramsey Allocation in the Modified Model)***

1. *If the feasibility condition $\phi(1-\beta) < 1$ is satisfied, then under $\sigma \geq 1$ there exists a unique interior Ramsey steady state with the following properties:*

   (a) *The allocation is unconstrained, with all individuals having FSI and the same steady-state consumption, and the Lagrangian multiplier associated with the aggregate resource constraint, μ_t, converges to finite positive value.*

   (b) *The risk-free rate satisfies $r = 1/\beta$, the MGR holds, and the steady-state capital tax is zero: $\tau_k = 0$.*

   (c) *The optimal labor tax rate τ_n and the optimal debt-to-output ratio B/Y depend on ϕ and are given by*

$$\tau_n = (1-\beta)\frac{\phi(1-\beta+\delta\beta(1-\alpha))-\alpha}{(1-\alpha)(1-\beta+\delta\beta)} \in (0,1),$$

$$\frac{B}{Y} = \frac{(1-\beta+\delta(1-\alpha)\beta)\,\phi-\alpha}{(1-\beta+\delta\beta)};$$

   where $\tau_n < 1$ if and only if the feasibility condition $\phi(1-\beta) < 1$ holds.

2. *On the other hand, if $\phi(1-\beta) \geq 1$ such that the feasibility condition does not hold, then the only possible Ramsey steady state is non-interior with $C \to 0$ and $\tau_n \to 1$.*

3. *Moreover, under the parameter conditions $\sigma < 1$, the only possible Ramsey steady state is an interior allocation with partial self-insurance (i.e., consumption inequality) and a divergent Lagrangian multiplier μ_t.*

Proof. See Appendix A.4. □

One of the main insights of Proposition 4 is that, under the parameter restriction $\sigma \geq 1$ and the feasibility condition $\phi(1-\beta) < 1$, the only possible Ramsey steady state is an unconstrained interior Ramsey steady state that necessarily exists; and, more importantly, the Ramsey planner chooses to achieve it even at the cost of a possibly very high steady-state labor tax rate while still setting the optimal long-run capital tax rate to zero—reminiscent of the classical result in representative-agent models.[11]

In such an interior Ramsey steady state the optimal labor tax rate τ_n is bounded above by 1—only because of the feasibility constraint $(1-\beta)\phi < 1$—but τ_n could be arbitrarily close to 100%, depending on the parameter values that influence the value of ϕ. In particular, the value of ϕ depends critically on the length of the truncated history κ for effective risk sharing. A labor tax rate close to 100% then implies steady-state output and consumption close to zero. Indeed, the Ramsey planner chooses to achieve consumption equality at "any cost."

Moreover, in this interior steady state, the optimal capital tax is zero, suggesting that the Ramsey planner will never levy a steady-state capital tax to achieve MGR even if the labor tax is close to 100%; MGR is instead achieved by having a sufficiently high public debt-to-GDP ratio such that the borrowing constraints of all individuals are slack, even though it is feasible to use a capital tax to achieve MGR. In other words, the Ramsey planner never uses capital taxation to reduce the burden of labor taxation.

The basic reason for such a counterintuitive result is as follows: Given that the market discount rate (interest rate) is lower than the time discount rate ($r < \beta^{-1}$ or $Q > \beta$)—a characteristic feature of Aiyagari-type models—the Ramsey planner chooses to front-load consumption by borrowing more cheaply in the short run as a trade-off for low consumption in the long run because the future utility cost of debt financing is heavily discounted by a lower time discount rate β than by the market discount rate $1/r$. Moreover, the front-loading of consumption must be supported by a higher labor supply to increase output, which requires a lower or even negative labor tax in the transition period to incentivize hard

[11] A similar zero capital tax result holds in representative-agent models even when a high labor tax is required to finance sufficiently high exogenous government spending. For a survey of the literature on optimal capital taxation in representative-agent models, see Atkeson, Chari, and Kehoe (1999) or more recent work by Chari, Nicolini, and Teles (2020)

work. However, a rapidly growing debt and a low labor tax rate in the short run must imply a high tax burden in the long run to finance the skyrocketing public debt (relative to total output).

Therefore, when $\sigma \geq 1$, consumers' low IES is consistent with the Ramsey planner's intention to increase the debt supply and bond growth to support consumption front-loading because the expected high future labor tax to finance the government debt burden leads to higher current saving when the income effect dominates the substitution effect.[12] Namely, individuals are willing to hold more government debt in the short run in anticipation of a high labor tax rate in the long run when $\sigma \geq 1$. This facilitates the government's strategy of front-loading consumption since the Ramsey planner can then issue debt or increase the supply of bonds more rapidly. This incentive to front-load consumption never disappears unless the equilibrium interest rate becomes equal to the time discount rate, which can only be achieved with an excessive supply of government bonds (relative to output) in the unconstrained allocation. Consequently, the Ramsey planner must continue to increase and finance the "skyrocketing" debt by raising the future labor tax rate, even if this implies low (or near zero) consumption (but better equality) in the steady state. Therefore, if the labor tax rate required to support unconstrained allocation exceeds 100%, i.e., the feasibility condition $(1-\beta)\phi < 1$ is violated, then the Ramsey steady state becomes non-interior with $C \rightarrow 0$ and $\tau_n \rightarrow 1$.[13]

The above discussion of the case of $\sigma \geq 1$ also shows that when $\sigma < 1$, because the substitution effect dominates the income effect, the expected future increase in labor taxes will lead to a reduction rather than an increase in individuals' current saving. As a result, despite the planner's intention to front-load consumption, the weaker saving motive significantly limits the government's ability to increase debt, leading to a divergent multiplier and creating a counterforce to the planner's pursuit of an unconstrained allocation. In other words, the government is unable to issue as much debt as necessary to support front-loading consumption in the short run, resulting in a lower future tax burden in the steady state. Therefore, the non-interior Ramsey steady state with zero consumption is deterred or diverted to an interior steady state where the interest rate is lower than the time discount rate ($Q > \beta$) and the MGR does not hold because of the divergent multiplier.[14]

A nice property of this modified Aiyagari model is that, as κ increases (or as the effectiveness of the wealth-pooling technology deteriorates), the model converges to the standard Aiyagari model, in which

[12]Interestingly, the result found by Straub and Werning (2020) also depends on the IES parameter. The intuition also depends on the response of households' current saving to the change in the future tax rate, which in their model is called the "anticipatory savings effects."

[13]Such a Ramsey policy has to do with the planner's commitment at time zero and may be time-inconsistent (see the analysis of Chien and Wen (2021b))

[14]Note that the Ramsey planner's incentive to front-load consumption is always present regardless of the IES parameter $\sigma \in (0, \infty)$. However, σ governs the saving behavior of individuals in response to a permanent future tax increase. Note also that the feasibility condition $\phi(1-\beta) < 1$ (which is equivalent to the condition of $\tau_n < 1$) only matters in the case of $\sigma \geq 1$. This feasibility condition becomes irrelevant in the parameter space $\sigma < 1$.

case the demand for assets (or the wealth-to-consumption ratio ϕ) increases with κ to reflect the strong and increasing demand for self-insurance under $\sigma \geq 1$. As a result, the feasibility condition $\phi(1-\beta) < 1$ becomes increasingly difficult to satisfy. Eventually, when the value of κ is sufficiently large, the condition $\phi(1-\beta) < 1$ is violated and, consequently, the interior Ramsey steady state with MGR disappears and turns into the non-interior Ramsey steady state.

5.2 Numerical Illustration

In the following, we use numerical simulations to illustrate how the unconstrained interior Ramsey steady state converges to the non-interior Ramsey steady state as κ increases by taking different values. In particular, we numerically solve the unconstrained Ramsey steady state of the modified Aiyagari model by ensuring that all Ramsey FOCs are satisfied under proper parameter values and a given set of values for κ. We set the preference parameters to $\sigma = \gamma = 2$, the capital depreciation rate to $\delta = 0.1$, and the capital share to $\alpha = 0.35$, which are standard in the macroeconomic literature. For simplicity, we consider a two-state Markov process where $Z = \{e, u\}$, $z(e) = 1$, and $z(u) = 0$. In other words, an individual can work and receive labor income if $\theta = e$; otherwise, if $\theta = u$, the individual cannot work and has no labor income. In addition, to make the mechanism sharper, we deliberately set the time discount factor to a low value of $\beta = 0.65$, which leaves less room to increase the value of κ. (Otherwise, the changes in the Ramsey allocation in our numerical simulations would be less visible before κ becomes extremely large.) Thus, when $\kappa = 0$, the steady-state labor tax rate is very low because of the low interest cost burden in the unconstrained Ramsey allocation; but as κ increases from 0 to 10, the optimal steady-state labor tax rate rises rapidly to close to 100%. Finally, the transition probability matrix of the θ shock is given by

$$\pi = \begin{bmatrix} \pi(u|u) & \pi(e|u) \\ \pi(u|e) & \pi(e|e) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}. \tag{22}$$

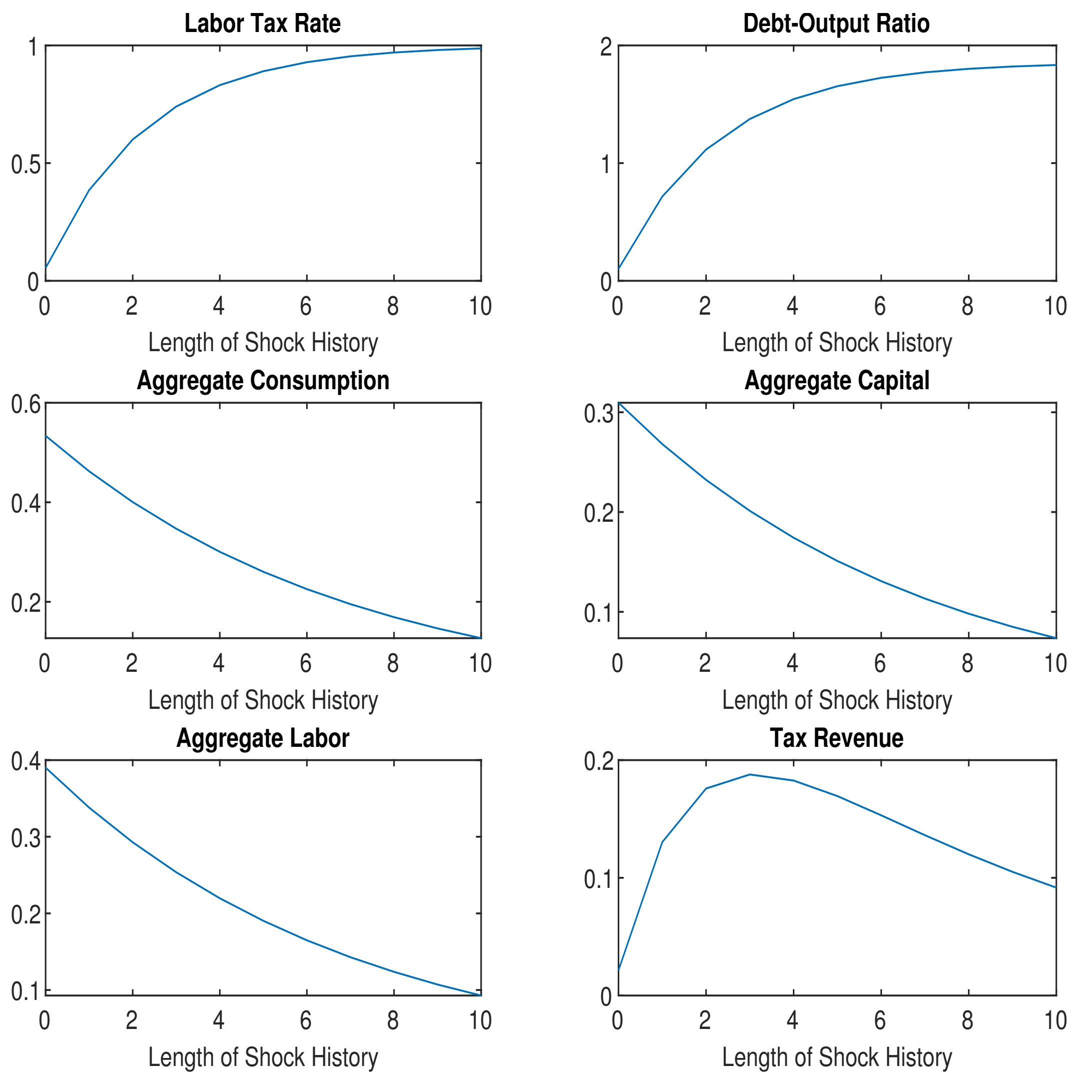
Such a numerical analysis is valid because the interior Ramsey steady state has been proved to exist in Proposition 4 and can therefore be found numerically by solving all of the Ramsey FOCs. We find that as κ increases from 0 to 10, the implied optimal debt-to-output ratio required for an unconstrained interior Ramsey steady state grows rapidly, the steady-state labor tax τ_n approaches 100%, and the steady-state aggregate consumption approaches zero. Namely, the interior Ramsey steady state approaches a non-interior Ramsey steady state as κ increases. Since we have proved theoretically that the unconstrained allocation is the only possible interior Ramsey steady state when $\sigma \geq 1$, the numerical result suggests that the interior Ramsey steady state with the MGR will eventually disappear even within a finite value of κ, leaving the non-interior Ramsey steady state as the only possible Ramsey allocation in the long run for a sufficiently large κ such that $\phi(1-\beta) \geq 1$, consistent with Proposition 4.

Figure 2 shows the Ramsey allocation and policy as we extend κ from 0 to 10. It shows that, as κ increases by taking different values, that is, as the risk-sharing capacity introduced by wealth-pooling technology becomes less effective, the implied optimal steady-state labor tax rate τ_n (top-left panel or panel[1,1]) increases from 5.5% to nearly 100%. This is because the optimal debt-to-output ratio required to support an unconstrained Ramsey allocation (top-right panel or panel [1,2]) increases rapidly as the degree of risk embodied in wealth increases. As a result, the levels of aggregate consumption, aggregate capital, and aggregate labor (panel [2,2], panel [2,2] and panel [3,1], respectively) fall towards zero. That is, as κ increases, the interior Ramsey steady state moves toward the non-interior Ramsey steady state. During this process of prolonging the shock history κ, a gradually increasing labor tax rate and decreasing labor supply also imply that the total tax revenue will initially increase (e.g., for $\kappa < 4$) but eventually decline toward zero, as shown in the bottom-right panel.

Since our model converges to the standard Aiyagari model (which has no risk-sharing technology) as κ approaches infinity, our numerical result is thus consistent with our theoretical result that under parameter value $\sigma \geq 1$ there is no interior Ramsey steady state in the standard Aiyagari model. The intuition is as follows: Because the Ramsey planner chooses to pursue an unconstrained allocation to completely eliminate the household borrowing constraint—driven by the arbitrage opportunity when the market interest rate is below the time discount rate—the demand for assets in the standard Aiyagari model will approach infinity as the interest rate approaches the time discount rate (i.e., as $Q \to \beta$). Thus, as our modified Aiyagari model approaches the standard Aiyagari model by reducing the risk-sharing effectiveness of wealth-pooling technology (implied by increasing κ), the optimal level of debt required to maintain an unconstrained allocation will correspondingly increase to infinity. This infinite bond demand makes the unconstrained interior Ramsey steady state infeasible as a competitive equilibrium. As a result, the unconstrained interior steady state in our modified model pushes the labor tax rate toward 100% and thus eventually disappears, giving way to a non-interior Ramsey steady state as κ increases.

Therefore, the numerical results are consistent with the theoretical proof that, if the IES parameter satisfies $\sigma \geq 1$, the only possible Ramsey steady state in the standard Aiyagari model is a non-interior one in which the aggregate consumption, aggregate capital stock, aggregate labor, and aggregate output are all zero and the optimal labor tax rate is 100%.

Figure 2: The Unconstrained Ramsey Steady-State with Extended Truncated History



5.3 Characterization of a Ramsey Steady State with $\kappa = 0$

In this subsection, we consider a further simplified version of our model by setting $\kappa = 0$ to shed light on how the persistence of idiosyncratic shocks affects the Ramsey allocation in our modified Aiyagari model. We then further utilize this simplified model to prove the existence of different Ramsey steady states and to characterize their corresponding parameter spaces.

5.3.1 The Role of Persistent Shocks

Continuing to assume a two-state Markov process for the idiosyncratic shock process, as in Section 5.2, this means that there are only two types of individuals in each period, denoted by types e and u, so that the wealth distribution in the model is sufficiently degenerate. In this further simplified case, we can go one step further by analytically expressing the role of shock persistence in determining the feasibility condition parameter ϕ and proving the existence of the Ramsey steady state, whether it be interior or non-interior.

When $\kappa = 0$, the implementability condition in the unconstrained Ramsey steady state can be simplified to

$$\frac{a^e \pi(e)\pi(u|e)}{\pi(u)} = c^u = c^e,$$

and the asset-to-consumption ratio ϕ in the feasibility condition can be simplified to $\phi \equiv \frac{A}{C} = \frac{a^e \pi(e)}{c^e} = \frac{\pi(u)}{\pi(u|e)}$. According to Proposition 4, under the parameter conditions $(1-\beta)\frac{\pi(u)}{\pi(u|e)} < 1$ and $\sigma \geq 1$, an unconstrained interior Ramsey steady state necessarily and uniquely exists where the optimal labor tax rate and debt-to-output ratio are given by

$$\tau_n = \frac{(1-\beta+\delta(1-\alpha)\beta)\,\frac{\pi(u)}{\pi(u|e)}(1-\beta) - \alpha(1-\beta)}{(1-\alpha)\,(1-\beta+\delta\beta)} \in (0,1),$$

and

$$\frac{B}{Y} = \frac{(1-\beta+\delta(1-\alpha)\beta)\,\frac{\pi(u)}{\pi(u|e)} - \alpha}{(1-\beta+\delta\beta)}.$$

Clearly, the optimal B/Y ratio required to support unconstrained allocation is proportional to $\frac{\pi(u)}{\pi(u|e)}$, which can be rewritten as $\frac{\pi(u)}{\pi(u|e)} = \frac{1}{2-(\pi(u|u)+\pi(e|e))}$. In a simple two-state Markov process, $\pi(u|u)+\pi(e|e)$ represents the persistence of idiosyncratic shocks.

For example, if the idiosyncratic shock becomes permanent, then $\pi(u|u) + \pi(e|e)$ becomes 2 and the value of $\frac{\pi(u)}{\pi(u|e)}$ goes to infinity. In other words, the required B/Y ratio to support unconstrained allocation can be arbitrarily close to infinity even in the case of a fairly effective risk-sharing technology under $\kappa = 0$. This result suggests that once the idiosyncratic shock process becomes highly persistent or permanent, an unconstrained Ramsey steady state cannot exist, because the condition $(1-\beta)\frac{\pi(u)}{\pi(u|e)} < 1$ is violated and the steady-state labor tax rate exceeds 1, precluding the interior Ramsey steady state.

5.3.2 The Existence of a Non-interior Ramsey Steady State

In the previous analysis, we proved that there is no interior Ramsey steady state when the feasibility condition is violated and that the only possible Ramsey steady state (if it exists) is non-interior, so we only conjectured that there may be a non-interior Ramsey steady state when $\sigma \geq 1$ and an interior one when $\sigma < 1$. In this simplified model (with two shock states and $\kappa = 0$), we can further prove the existence of these two Ramsey steady states when the feasibility condition is violated. Note that the strategy of this proof is different from that of the previous analysis.

Proposition 5. ***(Ramsey Allocation when*** $\kappa = 0$***)***

1. *If* $(1-\beta)\frac{\pi(u)}{\pi(u|e)} \geq 1$ *and* $\sigma \geq 1$*, there is uniquely a non-interior steady state in which:*

   (a) *Aggregate consumption* C_t*, aggregate capital* K_{t+1}*, and aggregate labor* N_t*, all converge to zero.*

   (b) *The optimal labor tax rate* $\tau_{n,t}$ *converges to 100%.*

   (c) *The optimal capital tax is indeterminate.*

   (d) *The multiplier* μ_t *diverges to infinity.*

2. *In addition, under* $\sigma < 1$*, there exists an interior Ramsey steady state with a divergent multiplier* μ_t *and partial self-insurance (constrained allocation) with* $Q > \beta$*.*

Proof. See Appendix A.5. □

6 Maximizing Steady-State Welfare

In Section 5, we mention that when $Q > \beta$, the Ramsey planner has incentives to increase the supply of bonds in order to front-load consumption and pursue the unconstrained allocation, even if this implies a nearly 100% labor tax rate in the long run to finance the skyrocketing public debt-to-GDP ratio. To further support our argument and the intuition behind it, here we perform a different kind of analysis by assuming that the Ramsey planner maximizes only the steady-state welfare of the competitive equilibrium (as in the work of Aiyagari and McGrattan (1998) and Floden (2001)) instead of the time-zero present value of the dynamic path of social welfare. We will see that the Ramsey planner's design of debt and tax policy in this situation is fundamentally different from that discussed above, when the incentive to front-load consumption is no longer present in a static optimization problem.

To maximize the steady-state welfare of the economy, the Ramsey problem becomes

$$\max_{\{c(h^\kappa),n(h^\kappa),a(h^\kappa),\widehat{w},Q,K\}_{t=0}^{\infty}} \sum_{h^\kappa} \left[\frac{1}{1-\sigma}c(h^\kappa)^{1-\sigma} - \frac{1}{1+\gamma}n(h^\kappa)^{1+\gamma}\right]\pi(h^\kappa)$$

subject to

$$F(\sum_{h^\kappa} n(h^\kappa)z(h^\kappa)\pi(h^\kappa), K) - \delta K - \sum_{h^\kappa} c(h^\kappa)\pi(h^\kappa) \geq 0, \tag{23}$$

$$c(h^\kappa) - \widehat{w}z(h^\kappa)n(h^\kappa) + Qa(h^\kappa) - \sum_{h_{-1}^\kappa} \frac{a(h_{-1}^\kappa)\pi(h_{-1}^\kappa)\pi(h^\kappa|h_{-1}^\kappa)}{\pi(h^\kappa)} = 0, \forall h_\kappa, \tag{24}$$

$$\chi(h^\kappa)\pi(h_\kappa) : \widehat{w}c(h^\kappa)^{-\sigma}z(h^\kappa) - n(h^\kappa)^\gamma = 0, \forall h^\kappa$$

$$a(h^\kappa) \geq 0, \forall h^\kappa \tag{25}$$

$$g(h^\kappa) \geq 0, \ \forall h^\kappa$$

and

$$g(h^\kappa)a(h^\kappa) = 0, \ \forall h^\kappa \tag{26}$$

where $g(h^\kappa)$ is defined as

$$g(h^\kappa) \equiv Qc(h^\kappa)^{-\sigma} - \sum_{h^{\kappa'}} c(h^{\kappa'})^{-\sigma}\pi(h^{\kappa'}|h^\kappa) \geq 0.$$

Note that this Ramsey problem has no dynamics because there is no dynamic consideration for the Ramsey planner. In other words, the "future" is no longer "discounted" relative to the "present." We prove that unconstrained allocation is *no longer optimal*, as shown in the following proposition:

Proposition 6. ***(Ramsey Allocation in Steady-State Welfare Maximization)*** *If the Ramsey planner is concerned only with the steady-state welfare of the competitive equilibrium, then unconstrained allocation is not optimal regardless of IES, even if unconstrained allocation is feasible. Instead, it is optimal to set $c(\boldsymbol{h}_h^\kappa) > c(h^\kappa) > c(\boldsymbol{h}_l^\kappa)$ and $Q > \beta$ and to make the borrowing constraints of the low-income individuals strictly binding.*

Proof. See Appendix A.6. □

It is easy to show that the result in Proposition 6 also holds for a standard Aiyagari model where $\kappa = \infty$. This result is quite intuitive. By maximizing only the steady-state welfare of the competitive economy, the Ramsey planner no longer has an incentive to exploit the difference between the interest rate and the time discount rate, since the time discount rate is no longer relevant in maximizing the steady-state welfare. Consequently, without transitional dynamics, the issue of front-loading consumption becomes irrelevant. In such a case, the Ramsey planner chooses not to pursue an unconstrained allocation by equalizing consumption across employed and unemployed individuals, because the cost of doing so in terms of levying distortionary taxes and issuing too much debt is too high at the margin, where there is no time discounting. In other words, from the point of view of the steady-state welfare of the competitive equilibrium, the marginal benefit of achieving unconstrained allocation by increasing public debt is at some point dominated by the marginal cost of distortionary taxation, so the Ramsey planner will stop issuing bonds at some level before the unconstrained allocation is achieved.

7 Conclusion

Throughout history, capital taxation has been a major source of government revenue and is often seen as an important means of reducing income/wealth inequality. However, macroeconomic theory was unable to rationalize this popular practice using representative-agent models until the seminal work of Aiyagari (1995) broke the ice. Aiyagari (1995) argued that in models with heterogeneous agents and incomplete markets, it is optimal for the government to tax capital—because capital is overaccumulated due to borrowing constraints and associated precautionary saving motives. However, Aiyagari's analysis relies on the critical assumption, without proof, that there is an interior Ramsey steady state.

In this paper, we analyze the Ramsey steady-state conundrum in a standard Aiyagari model with CRRA preference. We prove that in this model the assumption of the existence of an interior Ramsey steady state with a positive capital tax and convergent Lagrangian multiplier(s) (commonly made in both the theoretical and numerical literature) is incorrect. Instead, we show that if a Ramsey steady state exists at all in a standard Aiyagari model, it must be non-interior if $\sigma \geq 1$; alternatively, if $\sigma < 1$, an interior Ramsey steady state (if it exists) must have a divergent Ramsey Lagrangian multiplier and the MGR does not hold.

We then develop a modified and tractable version of the Aiyagari model to uncover the mechanisms behind our alarming result and study the conditions for the existence of an interior Ramsey steady state. We find that the conditions are quite demanding and sensitive to the values of structural parameters related to the economy's ability to sustain public debt and mitigate idiosyncratic risk. In particular, we prove that an interior Ramsey steady state can exist under certain feasibility conditions, but the steady state is characterized by either a zero capital tax (under $\sigma \geq 1$) or the failure of MGR (under $\sigma < 1$); both results are in sharp contrast to the claims of Aiyagari (1995) and the subsequent literature, which use heterogeneous-agent and incomplete-market models to justify positive capital taxes in the real world. We also prove that when the feasibility condition is violated (as it will be in the standard Aiyagari model), the only Ramsey steady state is non-interior under normal parameter values for risk aversion (i.e., $\sigma \geq 1$).

The main reasons for our unconventional results are as follows: Because of the arbitrage opportunity created by the gap between the interest rate and the time discount rate in Aiyagari-type models, the Ramsey planner has a dominant incentive to issue a sufficiently large amount of debt to front-load consumption and crowd out capital during the transition in order to raise the interest rate and achieve an unconstrained allocation in the long run, even at the cost of an extremely high steady-state labor tax to finance public debt. If the labor tax rate is feasible and the individual saving incentive is sufficiently strong ($\sigma \geq 1$) to accommodate a large amount of public debt (or debt-to-output ratio), the Ramsey allocation has an interior steady state where no one's borrowing is constrained and the optimal capital tax is zero. The optimal capital tax is zero in the unconstrained steady state because the only source of allocative inefficiency (due to the borrowing constraint) is fully addressed by a sufficient supply of government debt–which can be fully financed by a labor tax. This result is in sharp contrast to the Ramsey steady state considered by Aiyagari (1995).

On the other hand, if the Ramsey planner's dominant motive to pursue unconstrained allocation leads to unsustainable levels of government debt, then an interior Ramsey steady state cannot exist. Thus, the only possible Ramsey steady state in a standard Aiyagari model with normal IES parameter values ($\sigma \geq 1$) is non-interior with zero aggregate consumption and a 100% labor tax rate in the limit.[15]

However, if individual saving motives are too weak ($\sigma < 1$), the Ramsey planner's intention to front-load consumption still exists but cannot be supported by a rising debt level due to individuals' weak asset demand for government bonds, leading to an interior Ramsey steady state with a divergent Lagrangian multiplier; hence the MGR must fail—still in sharp contrast to the interior Ramsey steady state imagined by Aiyagari.

Therefore, our analysis suggests not only that Aiyagari-type models in their current form may not yet rationalize positive capital taxation in the real world where household preferences are often characterized by $\sigma \geq 1$, but also that any result obtained in the heterogeneous-agent literature (both theoretical and numerical) under the common practice of assuming (without proof) the existence of an interior Ramsey steady state may be dubious and must be interpreted with caution.

Nevertheless, our analysis points to several avenues for future research. One avenue is to explicitly consider the government's natural borrowing limit in the Aiyagari model when solving the Ramsey problem. Our analysis suggests that because of the dominant incentive to pursue unconstrained allocation by the Ramsey planner, the government's natural borrowing limit may become strictly binding, leading to an interior steady state in which the optimal capital tax may be positive. Another way to justify a positive capital tax in the real world may be to relax the neoclassical assumption that factors of production are always paid at their marginal product. There is ample historical evidence that workers have never been paid at their marginal product (see, for example, Karl Marx's critique of capitalism). In such a case, it may be socially optimal to tax capital.

[15]Note that the traditional Laffer curve argument doesn't work here, because the government's dominant concern is to achieve consumption equality by relaxing everyone's borrowing constraint.

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A Appendix

A.1 Proof of Proposition 1(Conditions to Support a Competitive Equilibrium)

A.1.1 The "If" Part

Given the initial values of $(B_0, K_0, a_0, \tau_{k,0})$, the allocation $\{c_t(\theta^t), n_t(\theta^t), a_{t+1}(\theta^t), K_{t+1}\}_{t=0}^{\infty}$, and the price sequences $\{Q_{t+1}, \widehat{w}_t\}_{t=0}^{\infty}$, a competitive equilibrium can be constructed by using the conditions in Proposition 1 and by following the steps below to uniquely back up the sequences of the other aggregate and policy variables:

1. N_t and C_t are determined by $N_t = \sum_{\theta^t} n_t(\theta^t) z_t(\theta^t)\pi(\theta^t)$ and $C_t = \sum_{\theta^t} c_t(\theta^t)\pi(\theta^t)$, respectively.

2. w_t and q_t are determined by $w_t = MPN_t$ and $q_t = MPK_t$, respectively.

3. $\tau_{n,t}$ is determined by
$$\widehat{w}_t = (1 - \tau_{n,t})MPN_t. \tag{27}$$

4. $\tau_{k,t+1}$ is determined by $\frac{1}{Q_{t+1}} = 1 + (1 - \tau_{k,t+1})MPK_{t+1} - \delta$.

5. B_{t+1} is determined by the asset market-clearing condition
$$B_{t+1} + \frac{K_{t+1}}{Q_{t+1}} = A_{t+1}, \text{ for all } t \geq 0.$$

6. The following constraints are satisfied:

   (a) The resource constraint is given in equation (11) after replacing N_t and C_t by individual consumption and labor.

   (b) The individual optimal condition (8) is given in equation (14).

   (c) The implementability condition, given in equation (12), is the household budget constraint multiplied by $c_t(\theta^t)^{-\sigma}$ after replacing $\widehat{w}_t$ by using equation (14).

   (d) The asset constraint in period 0 is given by equation (13).

   (e) The individual optimal condition (7) and the borrowing constraint are given in equation (15) as constraints in Proposition 1.

7. Finally, it is straightforward to verify that the sum of all implementability conditions together with the aggregate resource constraint imply the government budget constraint.

A.1.2 The "Only If" Part

The constraints given in Proposition 1 are trivially satisfied because they are part of the competitive equilibrium conditions.

A.2 Proof of Proposition 2 (Ramsey Allocation in the Aiyagari Model)

A.2.1 Ramsey Problem

According to Proposition A.1, the Ramsey problem can be written as the following (while noting that the Ramsey Lagrangian multiplier is listed before each constraint)

$$\max_{\{c_t(\theta^t),n_t(\theta^t),a_{t+1}(\theta^t),\widehat{w}_t,Q_{t+1},K_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{1-\sigma}\sum_{\theta^t} c_t(\theta^t)^{1-\sigma} - \frac{1}{1+\gamma}\sum_{\theta^t} n_t(\theta^t)^{1+\gamma}\right]\pi(\theta^t)$$

subject to

$$\beta^t\mu_t : F\left(\sum_{\theta^t} n_t(\theta^t)z_t(\theta^t)\pi(\theta^t), K_t\right) + (1-\delta)K_t - \sum_{\theta^t} c_t(\theta^t)\pi(\theta^t) - K_{t+1} \geq 0,\ \forall t \geq 0,$$

$$\lambda_0(\theta^0)\pi(\theta^0) : c_0(\theta^0)^{1-\sigma} - n_0(\theta^0)^{1+\gamma} + Q_1 c_0(\theta^0)^{-\sigma}a_1(\theta^0) - c_0(\theta^0)^{-\sigma}a_0 = 0,$$

$$\beta^t\lambda_t(\theta^t)\pi(\theta^t) : c_t(\theta^t)^{1-\sigma} - n_t(\theta^t)^{1+\gamma} + Q_{t+1}c_t(\theta^t)^{-\sigma}a_{t+1}(\theta^t) - c_t(\theta^t)^{-\sigma}a_t(\theta^{t-1}) = 0, \forall t \geq 1,$$

$$\beta^t\chi_t(\theta^t)\pi(\theta^t) : \widehat{w}_t z_t(\theta^t)c_t(\theta^t)^{-\sigma} - n_t(\theta^t)^{\gamma} = 0,$$

$$\zeta^0 : (1 + (1-\tau_{k,0})MPK_0 - \delta)K_0 + B_0 - a_0 = 0,$$

$$\beta^{t+1}\zeta_t^1(\theta^t)\pi(\theta^t) : a_{t+1}(\theta^t) \geq 0,\ \forall t \geq 0,$$

$$\beta^{t+1}\zeta_t^2(\theta^t)\pi(\theta^t) : g_t(\theta^t) \geq 0,\ \forall t \geq 0,$$

$$\beta^{t+1}\zeta_t^3(\theta^t)\pi(\theta^t) : g_t(\theta^t)a_{t+1}(\theta^t) = 0,\ \forall t \geq 0;$$

where the function $g_t(\theta^t)$ in the last constraint is defined as

$$g_t(\theta^t) \equiv Q_{t+1}c_t(\theta^t)^{-\sigma} - \beta\sum_{\theta_{t+1}} c_{t+1}(\theta_{t+1})^{-\sigma}\pi(\theta_{t+1}|\theta^t) \geq 0.$$

A.2.2 Ramsey FOCs

The Ramsey FOCs with respect to $a_{t+1}(\theta^t)$, $n_t(\theta^t)$, and $c_t(\theta^t)$ are given, respectively, by

$$\begin{aligned} &\lambda_t(\theta^t)Q_{t+1}c_t(\theta^t)^{-\sigma} - \beta \sum_{\theta_{t+1}} \lambda_{t+1}(\theta^{t+1})c_{t+1}(\theta^{t+1})^{-\sigma}\pi(\theta^{t+1}|\theta^t) \\ &+\beta^{t+1}\zeta_t^1(\theta^t) + \beta^{t+1}\zeta_t^3(\theta^t)g_t(\theta^t) = 0, \end{aligned} \tag{28}$$

$$n_t(\theta^t)^\gamma + (1+\gamma)\lambda_t(\theta^t)n_t(\theta^t)^\gamma + \gamma\chi_t(\theta^t)n_t(\theta^t)^{\gamma-1} = \mu_t MPN_t z_t(\theta^t), \tag{29}$$

$$\begin{aligned} &c_t(\theta^t)^{-\sigma} + \lambda_t(\theta^t)(1-\sigma)c_t(\theta^t)^{-\sigma} - \sigma\lambda_t(\theta^t)c_t(\theta^t)^{\sigma-1}(Q_{t+1}a_{t+1}(\theta^t) - a_t(\theta^{t-1})) \\ &-\sigma\widehat{w}_t\chi_t(\theta^t)z_t(\theta^t)c_t(\theta^t)^{-\sigma-1} + \beta\left[\zeta_t^2(\theta^t) + \zeta_t^3(\theta^t)a_{t+1}(\theta^t)\right]\frac{\partial g_t(\theta^t)}{\partial c_t(\theta^t)} \\ &+\left[\zeta_{t-1}^2(\theta^{t-1}) + \zeta_{t-1}^3(\theta^{t-1})a_t(\theta^{t-1})\right]\frac{\partial g_{t-1}(\theta^{t-1})}{\partial c_t(\theta^t)}\frac{\pi(\theta^{t-1})}{\pi(\theta^t)} = \mu_t; \end{aligned} \tag{30}$$

where

$$\begin{aligned} \frac{\partial g_t(\theta^t)}{\partial c_t(\theta^t)} &= -\sigma c_t(\theta^t)^{-\sigma-1}Q_{t+1}, \\ \frac{\partial g_{t-1}(\theta^{t-1})}{\partial c_t(\theta^t)} &= \beta\sigma c_t(\theta^t)^{-\sigma-1}\pi(\theta_t|\theta^{t-1}). \end{aligned}$$

In addition, the Ramsey FOCs with respect to $\widehat{w}_t$ and Q_{t+1} are given, respectively, by

$$\sum_{\theta^t} \chi_t(\theta^t)c_t(\theta^t)^{-\sigma}z_t(\theta^t)\pi(\theta^t) = 0, \tag{31}$$

$$\sum_{\theta_t} \lambda_t(\theta^t)c_t(\theta^t)^{-\sigma}a_{t+1}(\theta^t)\pi(\theta^t) + \beta\sum_{\theta_t}\left(\zeta_t^2(\theta^t) + \zeta_t^3(\theta^t)a_{t+1}(\theta^t)\right)c_t(\theta^t)^{-\sigma}\pi(\theta^t) = 0. \tag{32}$$

Finally, the Ramsey FOC with respect to K_{t+1} is given by

$$\mu_t = \beta\mu_{t+1}\left(MPK_{t+1} + 1 - \delta\right). \tag{33}$$

A.2.3 Three Key Equations

To facilitate our analysis and proof, it is useful to obtain the following three key equations derived from the above system of Ramsey FOCs.

1. Consider an *ad hoc* borrowing limit $\underline{a}_t(\theta^t)$ (currently $\underline{a}_t(\theta^t) = 0$ in the model) and a marginal increase of the *ad hoc* borrowing limit $\underline{a}_t(\theta^t)$. The change in social welfare due to the increase in the

borrowing limit must be non-increasing. From the envelope condition with respect to the borrowing constraint under this *ad hoc* borrowing limit, we obtain

$$0 \geq \frac{\partial \mathcal{V}}{\partial \underline{a}_t(\theta^t)} = \frac{\partial \mathcal{L}^*}{\partial \underline{a}_t(\theta^t)} = -\beta^{t+1}\left[\zeta_t^1(\theta^t) + \zeta_t^3(\theta^t) g_t(\theta^t)\right],$$

where $\mathcal{V}$ and $\mathcal{L}^*$ denote the social welfare and the associated Lagrangian, respectively. So it must be true that $\left[\zeta_t^1(\theta^t) + \zeta_t^3(\theta^t) g_t(\theta^t)\right] \geq 0$ (which implies $\zeta_t^3 \geq 0$). Then the FOC with respect to $a_{t+1}(\theta^t)$ can be rewritten as

$$\left[\lambda_t(\theta^t) c_t(\theta^t)^{-\sigma}\right] \frac{Q_{t+1}}{\beta} \leq \sum_{\theta_{t+1}} \left[\lambda_{t+1}(\theta^{t+1}) c_{t+1}(\theta^{t+1})^{-\sigma}\right] \pi(\theta^{t+1}|\theta^t). \tag{34}$$

This is the first key equation we need.

2. The Ramsey FOC with respect to $\widehat{w}_t$ is $\sum_{\theta^t} \chi_t(\theta^t) c_t(\theta^t)^{-\sigma} z_t(\theta^t) \pi(\theta^t) = 0$, which, together with the Ramsey constraint $n_t(\theta^t)^\gamma = \widehat{w}_t z_t(\theta^t) c_t(\theta^t)^{-\sigma}$, gives

$$0 = \sum_{\theta^t} \chi_t(\theta^t) c_t(\theta^t)^{-\sigma} z_t(\theta^t) \pi(\theta^t) = \frac{1}{\widehat{w}_t} \sum_{\theta^t} \chi_t(\theta^t) n_t(\theta^t)^\gamma \pi(\theta^t). \tag{35}$$

Multiplying equation (29) (the FOC with respect to $n_t(\theta^t)$) by $n_t(\theta^t)$ and then summing it over θ^t gives

$$\begin{aligned}
& \sum_{\theta^t} n_t(\theta^t)^{\gamma+1} \pi(\theta^t) + (1+\gamma) \sum_{\theta^t} \lambda_t(\theta^t) n_t(\theta^t)^{\gamma+1} \pi(\theta^t) \\
& + \gamma \sum_{\theta^t} \chi_t(\theta^t) n_t(\theta^t)^\gamma \pi(\theta^t) \\
= \; & \mu_t MPN_t \sum_{\theta^t} n_t(\theta^t) z_t(\theta^t) \pi(\theta^t).
\end{aligned}$$

Using equation (35), the above equation can be rewritten as

$$\begin{aligned}
& \sum_{\theta^t} \frac{n_t(\theta^t)^{\gamma+1}}{N_t} \pi(\theta^t) + (1+\gamma) \widehat{w}_t \sum_{\theta^t} \lambda_t(\theta^t) c_t(\theta^t)^{-\sigma} \frac{z_t(\theta^t) n_t(\theta^t)}{N_t} \pi(\theta^t) \\
= \; & \mu_t MPN_t.
\end{aligned} \tag{36}$$

This is the second key equation we need.

3. Multiplying the Ramsey FOC with respect to $c_t(\theta^t)$ (equation 30) by $c_t(\theta^t)\pi(\theta^t)$ gives

$$\begin{aligned}
& c_t(\theta^t)^{1-\sigma}\pi(\theta^t) + \lambda_t(\theta^t)(1-\sigma)c_t(\theta^t)^{1-\sigma}\pi(\theta^t) \\
& -\sigma\lambda_t(\theta^t)c_t(\theta^t)^{-\sigma}(Q_{t+1}a_{t+1}(\theta^t) - a_t(\theta^{t-1}))\pi(\theta^t) \\
& -\sigma\widehat{w}_t\chi_t(\theta^t)\pi(\theta^t)z_t(\theta^t)c_t(\theta^t)^{-\sigma} \\
& -\sigma Q_{t+1}\beta\left[\zeta_t^2(\theta^t) + \zeta_t^3(\theta^t)a_{t+1}(\theta^t)\right]c_t(\theta^t)^{-\sigma}\pi(\theta^t) \\
& +\sigma\beta\left[\zeta_{t-1}^2(\theta^{t-1}) + \zeta_{t-1}^3(\theta^{t-1})a_t(\theta^{t-1})\right]c_t(\theta^t)^{-\sigma}\pi(\theta_t|\theta^{t-1})\pi(\theta^{t-1}) \\
= \quad & \mu_t c_t(\theta^t)\pi(\theta^t).
\end{aligned}$$

Summing the above equation over θ^t gives

$$\begin{aligned}
& \sum_{\theta^t} c_t(\theta^t)^{1-\sigma}\pi(\theta^t) + \sum_{\theta^t}\lambda_t(\theta^t)(1-\sigma)c_t(\theta^t)^{1-\sigma}\pi(\theta^t) \\
& -\sigma\sum_{\theta^t}\lambda_t(\theta^t)c_t(\theta^t)^{-\sigma}(Q_{t+1}a_{t+1}(\theta^t) - a_t(\theta^{t-1}))\pi(\theta^t) \\
& -\sigma\widehat{w}_t\sum_{\theta^t}\chi_t(\theta^t)\pi(\theta^t)z_t(\theta^t)c_t(\theta^t)^{-\sigma} \\
& -\sigma Q_{t+1}\beta\sum_{\theta^t}\left[\zeta_t^2(\theta^t) + \zeta_t^3(\theta^t)a_{t+1}(\theta^t)\right]c_t(\theta^t)^{-\sigma} \\
& +\sigma\beta\sum_{\theta^t}\left[\zeta_{t-1}^2(\theta^{t-1}) + \zeta_{t-1}^3(\theta^{t-1})a_t(\theta^{t-1})\right]c_t(\theta^t)^{-\sigma}\pi(\theta_t|\theta^{t-1})\pi(\theta^{t-1}) \\
= \quad & \mu_t\sum_{\theta^t}c_t(\theta^t)\pi(\theta^t).
\end{aligned}$$

Applying $\sum_{\theta^t} \chi_t(\theta^t) z_t(\theta^t) c_t(\theta^t)^{-\sigma}\pi(\theta^t) = 0$ to the equation above and rearranging terms gives

$$
\begin{aligned}
& \sum_{\theta^t} c_t(\theta^t)^{1-\sigma}\pi(\theta^t) + (1-\sigma)\sum_{\theta^t} \lambda_t(\theta^t) c_t(\theta^t)^{1-\sigma}\pi(\theta^t) \\
& \underbrace{-\sigma Q_{t+1} \sum_{\theta^t} \lambda_t(\theta^t) c_t(\theta^t)^{-\sigma} a_{t+1}(\theta^t)\pi(\theta^t)}_{\text{Part 1}} \\
& \underbrace{-\sigma Q_{t+1} \sum_{\theta^t} \beta \left[\zeta_t^2(\theta^t) + \zeta_t^3(\theta^t) a_{t+1}(\theta^t)\right] c_t(\theta^t)^{-\sigma}\pi(\theta^t)}_{\text{Part 2}} \\
& \underbrace{+\sigma \sum_{\theta^{t-1}} \beta \left[\zeta_{t-1}^2(\theta^{t-1}) + \zeta_{t-1}^3(\theta^{t-1}) a_t(\theta^{t-1})\right] \sum_{\theta_t} c_t(\theta^t)^{-\sigma}\pi(\theta_t|\theta^{t-1})\pi(\theta^{t-1})}_{\text{Part 3}} \\
& \underbrace{+\sigma \sum_{\theta^t} \lambda_t(\theta^t) c_t(\theta^t)^{-\sigma} a_t(\theta^{t-1})\pi(\theta^t)}_{\text{Part 4}} \\
= \; & \sum_{\theta^t} \mu_t \pi(\theta^t) c_t(\theta^t).
\end{aligned} \tag{37}
$$

We can show that (i) Part 1 + Part 2 = 0 and (ii) Part 3 + Part 4 = 0 in the above equation by the following steps:

(a) To show that Part 1 + Part 2 = 0, we use the Ramsey FOC with respect to Q_{t+1}:

$$\sum_{\theta_t} \lambda_t(\theta^t) c_t(\theta^t)^{-\sigma} a_{t+1}(\theta^t)\pi(\theta^t) + \beta \sum_{\theta_t} \left(\zeta_t^2(\theta^t) + \zeta_t^3(\theta^t) a_{t+1}(\theta^t)\right) c_t(\theta^t)^{-\sigma}\pi(\theta^t) = 0.$$

Multiplying the above equation by $-\sigma Q_{t+1}$ leads to

$$-\sigma Q_{t+1} \sum_{\theta_t} \lambda_t(\theta^t) c_t(\theta^t)^{-\sigma} a_{t+1}(\theta^t)\pi(\theta^t) - \sigma Q_{t+1}\beta \sum_{\theta_t} \left(\zeta_t^2(\theta^t) + \zeta_t^3(\theta^t) a_{t+1}(\theta^t)\right) c_t(\theta^t)^{-\sigma}\pi(\theta^t) = 0,$$

which shows exactly that Part 1 + Part 2 = 0 in equation (37).

(b) To show that Part 3 + Part 4 = 0, we follow the following 2 steps:

i. We know that $\zeta_{t-1}^2(\theta^{t-1}) g_{t-1}(\theta^{t-1}) = 0$ and $\zeta_{t-1}^3(\theta^{t-1}) g_{t-1}(\theta^{t-1}) a_t(\theta^{t-1}) = 0$ for all θ^{t-1}, which together implies

$$\left[\zeta_{t-1}^2(\theta^{t-1}) + \zeta_{t-1}^3(\theta^{t-1}) a_t(\theta^{t-1})\right] g_{t-1}(\theta^{t-1}) = 0.$$

Plugging the definition $g_{t-1}(\theta^{t-1}) \equiv Q_t c_{t-1}(\theta^{t-1})^{-\sigma} - \beta \sum_{\theta_t} c_t(\theta_t)^{-\sigma}\pi(\theta_t|\theta^{t-1})$ into the above equation gives

$$\begin{aligned} & \left[\zeta_{t-1}^{2}(\theta^{t-1})+\zeta_{t-1}^{3}(\theta^{t-1})a_t(\theta^{t-1})\right]Q_t \\ = & \left[\zeta_{t-1}^{2}(\theta^{t-1})+\zeta_{t-1}^{3}(\theta^{t-1})a_t(\theta^{t-1})\right]\beta\sum_{\theta_t}\frac{c_t(\theta_t)^{-\sigma}}{c_{t-1}(\theta^{t-1})^{-\sigma}}\pi(\theta_t|\theta^{t-1}). \end{aligned} \tag{38}$$

ii. We then consider Part 3:

$$\begin{aligned} \text{Part 3} &= \sigma\beta\sum_{\theta^{t-1}}\left[\zeta_{t-1}^{2}(\theta^{t-1})+\zeta_{t-1}^{3}(\theta^{t-1})a_t(\theta^{t-1})\right]\sum_{\theta_t}c_t(\theta^t)^{-\sigma}\pi(\theta_t|\theta^{t-1})\pi(\theta^{t-1}) \\ &= \sigma\sum_{\theta^{t-1}}\left[\zeta_{t-1}^{2}(\theta^{t-1})+\zeta_{t-1}^{3}(\theta^{t-1})a_t(\theta^{t-1})\right]c_{t-1}(\theta^{t-1})^{-\sigma}\times \\ &\quad \sum_{\theta_t}\beta\frac{c_t(\theta^t)^{-\sigma}}{c_{t-1}(\theta^{t-1})^{-\sigma}}\pi(\theta_t|\theta^{t-1})\pi(\theta^{t-1}) \\ &= \sigma\sum_{\theta^{t-1}}\left[\zeta_{t-1}^{2}(\theta^{t-1})+\zeta_{t-1}^{3}(\theta^{t-1})a_t(\theta^{t-1})\right]c_{t-1}(\theta^{t-1})^{-\sigma}Q_t\pi(\theta^{t-1}), \end{aligned}$$

where the second equality utilizes equation (38). The equation above together with the Ramsey FOC with respect to Q_t implies

$$\begin{aligned} & \sum_{\theta^{t-1}}\lambda_{t-1}(\theta^{t-1})c_{t-1}(\theta^{t-1})^{-\sigma}a_t(\theta^{t-1})\pi(\theta^{t-1}) \\ = & -\beta\sum_{\theta^{t-1}}\left(\zeta_{t-1}^{2}(\theta^{t-1})+\zeta_{t-1}^{3}(\theta^{t-1})a_t(\theta^{t-1})\right)c_{t-1}(\theta^{t-1})^{-\sigma}\pi(\theta^t), \end{aligned}$$

which leads to

$$\begin{aligned} \text{Part 3} &= -\sigma\frac{Q_t}{\beta}\sum_{\theta^{t-1}}\lambda_{t-1}(\theta^{t-1})c_{t-1}(\theta^{t-1})^{-\sigma}a_t(\theta^{t-1})\pi(\theta^{t-1}) \\ &= -\sigma\sum_{\theta^t}\lambda_t(\theta^t)c_t(\theta^t)^{-\sigma}a_t(\theta^{t-1})\pi(\theta^t), \end{aligned}$$

where the second equality utilizes equation (28). Therefore, Part 3 and Part 4 in equation (37) exactly cancel each other out, so Part 3 + Part 4 = 0.

(c) Equation (37) is then simplified to

$$\sum_{\theta^t}\frac{c_t(\theta^t)^{1-\sigma}}{C_t}\pi(\theta^t)+(1-\sigma)\sum_{\theta^t}\lambda_t(\theta^t)\frac{c_t(\theta^t)^{1-\sigma}}{C_t}\pi(\theta^t)=\mu_t, \tag{39}$$

which is the third key equation we need.

A.2.4 Non-Existence of Interior Ramsey Steady State When $\sigma \geq 1$

Here we prove that under the parameter condition $\sigma \geq 1$ there is no interior Ramsey steady state with $1 > Q > \beta$. The proof is done by contradiction in the following steps:

1. The first key equation (34) can be expressed as

$$\sum_{\theta_{t+1}} x_{t+1}(\theta^{t+1})\pi(\theta^{t+1}|\theta^t) \geq \frac{Q_{t+1}}{\beta} x_t(\theta^t), \tag{40}$$

   where $x_t \equiv \lambda_t(\theta^t)c_t(\theta^t)^{-\sigma}$. The above equation can be rewritten (after taking unconditional expectations on both sides) as

$$\mathbb{E}x_{t+1} = \rho_t \mathbb{E}x_t, \tag{41}$$

   where $\rho_t \geq \frac{Q_{t+1}}{\beta} > 1$, suggesting that $\mathbb{E}x_t$ is a stochastically growing process with a positive growth rate $\rho > 1$. Since $\mathbb{E}x_t$ is equivalent to the multiplier for aggregate household budget constraints (which together with resource constraint gives government budget constraint), it must be positive: $\mathbb{E}x_t > 0$. Hence, equation (41) implies that $\mathbb{E}x_t$ must diverge to positive infinity: $\lim_{t\to\infty} \mathbb{E}x_t = \infty$. Note that even if equation (40) only holds with equality, which is true for those households whose borrowing constraints are not binding, it remains true that $\lim_{t\to\infty} \mathbb{E}x_t = \infty$ for this set of households, because $\frac{Q_{t+1}}{\beta} > 1$ for all t.

2. The second key equation (36) can be rewritten as

$$\frac{1}{(1+\gamma)\widehat{w}_t}\mathbb{E}\left(\frac{n_t^{\gamma+1}}{N_t}\right) + \mathbb{E}\left(x_t \frac{z_t n_t}{N_t}\right) = \mu_t \left[\frac{MPN_t}{(1+\gamma)\widehat{w}_t}\right]. \tag{42}$$

   If an interior Ramsey steady state exists, then $\{n_t, N_t, \hat{w}_t, MPN_t\}$ and their ratios must all converge to a positive value or distribution. Since the labor share $\frac{z_t n_t}{N_t} > 0$ with $\mathbb{E}\frac{z_t n_t}{N_t} = 1$, we have

$$\mathbb{E}\left(x_t \frac{z_t n_t}{N_t}\right) = \mathbb{E}x_t + cov\left(x_t, \frac{z_t n_t}{N_t}\right). \tag{43}$$

   Given that both consumption c_t and labor income $z_t n_t$ respond positively to the labor productivity shock z_t,[16] we have

$$\text{sign of } cov\left(x_t, z_t n_t\right) = \text{sign of } cov\left(x_t, c_t\right).$$

   Since x_t is the shadow price of implementing the household's budget constraint in the Ramsey

[16]Note that equation (43) can be written as $\mathbb{E}\left(x_t \frac{z_t n_t}{N_t}\right) = \frac{1}{W_t N_t}\{\mathbb{E}x_t + cov\left(x_t, z_t n_t W_t\right)\}$, where $z_t n_t W_t$ co-move with consumption because it is is precisely the labor income of type-θ^t households.

problem, which must be positively correlated with household wealth position; namely, all else equal, a higher household wealth position a_t implies a higher consumption level c_t and a tighter constraint (or higher shadow price x_t) in the Ramsey problem. Then it must be the case that $cov\,(x_t, c_t) > 0$. Hence, $\mathbb{E}x_t + cov\left(x_t, \frac{z_t n_t}{N_t}\right) \to \infty$. Given this, equation (42) implies that μ_t on the right hand side of the equation must also diverge to positive infinity. Thus, Aiyagari's stationarity assumption for μ_t is inconsistent with the Ramsey FOCs with respect to a_{t+1} and n_t. However, this does not necessarily rule out an interior Ramsey steady state; it only means that the multiplier μ_t cannot be stationary if an interior Ramsey steady state does indeed exist.

3. The third key equation (39) can be expressed as

$$\mathbb{E}\left(\frac{c_t(\theta^t)^{1-\sigma}}{C_t}\right) + (1-\sigma)\mathbb{E}\left(x_t \frac{c_t}{C_t}\right) = \mu_t, \tag{44}$$

where $\frac{c_t}{C_t} > 0$ is consumption share with $\mathbb{E}\left(\frac{c_t}{C_t}\right) = 1$ and

$$\mathbb{E}\left(x_t \frac{c_t}{C_t}\right) = \mathbb{E}x_t + \frac{1}{C_t} cov\,(x_t, c_t)\,.$$

Since $cov\,(x_t, c_t) > 0$ from step (2), the term $\mathbb{E}\left(x_t \frac{c_t}{C_t}\right)$ must diverge to positive infinity. Since $\mu_t \to \infty$, equation (44) implies that the second term $(1-\sigma)\mathbb{E}\left(x_t \frac{c_t}{C_t}\right)$ on the left-hand side must also diverge to positive infinity in order to balance the equation. However, this is impossible because this term can only go to negative infinity if $\sigma > 1$ or be zero if $\sigma = 1$. Therefore, the combination of the three key equations implies that the assumption of an interior Ramsey steady state (with or without a divergent multiplier μ_t) leads to a contradiction when $\sigma \geq 1$.

4. The above proof also shows that if $\sigma < 1$, then it is possible to have an interior Ramsey steady state, provided that the multiplier μ_t diverges to positive infinity. However, without studying all of the Ramsey FOCs, we cannot prove that such an interior Ramsey steady state with a divergent multiplier does indeed (necessarily) exist under $\sigma < 1$.

5. Finally, the above proof also shows that under $\sigma \geq 1$, the only possible Ramsey steady state (if it exists) must be non-interior with $C_t \to 0$. Again, without studying all of the Ramsey FOCs, we cannot prove that such a non-interior Ramsey steady state necessarily exists.

A.3 Proof of Proposition 3 (Competitive Equilibrium in the Modified Model)

Given that $a_0(\boldsymbol{h}_h^\kappa) > a_0(h^\kappa)$ and the assumption that the autocorrelation of the shock process is non-negative, then it must be the case that

$$a_{t+1}(\boldsymbol{h}_h^\kappa) > a_{t+1}(h^\kappa) \geq 0 \text{ for all } t \geq 0 \text{ and } h^\kappa \neq \boldsymbol{h}_h^\kappa,$$

which implies that the associated multipliers on the borrowing constraints satisfy $\psi_t(\boldsymbol{h}_h^\kappa) = 0$ and $\psi_t(h^\kappa) \geq 0$. This result, together with the household FOCs with respect to $a_{t+1}(h^\kappa)$ and $c_t(h^\kappa)$, leads to equation (21).

Given such a wealth-pooling technology, the individual steady-state asset demand may remain finite even when $Q = \beta$. This fact opens up the possibility that an unconstrained steady state may exist in such an economy. In such a steady state where $\psi(h^\kappa) = 0$ for all h^κ, the steady-state version of equation (21) must hold with equality for all h^κ, which implies that (i) $Q = \beta$ and (ii) $c(\boldsymbol{h}_h^\kappa) = c(h^\kappa)$ for all h^κ.

A.4 Proof of Proposition 4 (Ramsey Allocation in the Modified Model)

The first part of the proof is analogous to that in Appendix A.2.

A.4.1 Ramsey Problem

It is straightforward to verify that the Ramsey planner's problem can be written as

$$\max_{\{c_t(h^\kappa), n_t(h^\kappa), a_{t+1}(h^\kappa), \widehat{w}_t, Q_{t+1}, K_{t+1}\}_{t=0}^\infty} \sum_{t=0}^{\infty} \beta^t \sum_{h^\kappa} \left[\frac{1}{1-\sigma} c_t(h^\kappa)^{1-\sigma} - \frac{1}{1+\gamma} n_t(h^\kappa)^{1+\gamma} \right] \pi(h^\kappa)$$

subject to the following (noting that the Lagrangian multiplier for the associated constraint is listed before each constraint)

$$\beta^t \mu_t : F(\sum_{h^\kappa} n_t(h^\kappa) z_t(h^\kappa) \pi(h^\kappa), K_t) + (1-\delta) K_t - \sum_{h^\kappa} c_t(h^\kappa) \pi(h^\kappa) - K_{t+1} \geq 0 \ \forall t \geq 0,$$

$$\lambda_0(h^0)\pi(h^0) : c_0(h^\kappa)^{1-\sigma} - n_0(h^\kappa)^{1+\gamma} + Q_1 c_0(h^\kappa)^{-\sigma} a_1(h^\kappa) - c_0(h^\kappa)^{-\sigma} a_0(h^\kappa) = 0 \ \forall h^\kappa,$$

$$\begin{aligned} \beta^t \lambda_t(h^\kappa)\pi(h^\kappa) \quad &: \quad c_t(h^\kappa)^{1-\sigma} - n_t(h^\kappa)^{1+\gamma} + Q_{t+1} c_t(h^\kappa)^{-\sigma} a_{t+1}(h^\kappa) - c_t(h^\kappa)^{-\sigma} \sum_{h_{-1}^\kappa} \frac{a_t(h_{-1}^\kappa)\pi(h_{-1}^\kappa)\pi(h^\kappa|h_{-1}^\kappa)}{\pi(h^\kappa)} \\ &= \quad 0 \ \forall h_\kappa \text{ and } t \geq 1, \end{aligned}$$

$$\beta^t \chi_t(h^\kappa)\pi(h_\kappa) : \widehat{w}_t c_t(h^\kappa)^{-\sigma} z_t(h^\kappa) - n_t(h^\kappa)^\gamma = 0 \ \forall h^\kappa \text{ and } t \geq 0,$$

$$\zeta^0 : (1 + (1 - \tau_{k,0})MPK_0 - \delta)K_0 + B_0 - \sum_{h_\kappa} a_0(h_\kappa) = 0,$$

and

$$\beta^{t+1}\zeta_t^1(h^\kappa) : a_{t+1}(h^\kappa) \geq 0,$$

$$\beta^{t+1}\zeta_t^2(h^\kappa) : g_t(h^\kappa) \geq 0,$$

$$\beta^{t+1}\zeta_t^3(h^\kappa) : g_t(h^\kappa)a_{t+1}(h^\kappa) = 0,$$

where $g_t(h^\kappa)$ in the last constraint is defined as

$$g_t(h^\kappa) \equiv Q_{t+1}c_t(h^\kappa)^{-\sigma} - \beta \sum_{h^{\kappa'}} c_{t+1}(h^{\kappa'})\pi(h^{\kappa'}|h^\kappa).$$

A.4.2 Ramsey FOCs

For all $t \geq 0$, the FOCs of the Ramsey problem with respect to $K_{t+1}, \widehat{w}_t, Q_{t+1}$, and $a_{t+1}(h^\kappa)$ are given, respectively, by

$$\mu_t = \beta\mu_{t+1}\left(MP_{K,t+1} + 1 - \delta\right),$$

$$\sum_{h_\kappa} \chi_t(h^\kappa)c_t(h^\kappa)^{-\sigma} z_t(h^\kappa)\pi(h_\kappa) = 0, \tag{45}$$

$$\sum_{h^\kappa} \lambda_t(h^\kappa)c_t(h^\kappa)^{-\sigma} a_{t+1}(h^\kappa)\pi(h^\kappa) + \beta \sum_{h^\kappa} \left(\zeta_t^2(h^\kappa) + \zeta_t^3(h^\kappa)a_{t+1}(h^\kappa)\right) c_t(h^\kappa)^{-\sigma}\pi(h^\kappa) = 0, \tag{46}$$

and

$$\lambda_t(h_\kappa)c_t(h^\kappa)^{-\sigma}Q_{t+1} = \beta \sum_{h^{\kappa'}} \lambda_{t+1}(h^{\kappa'})c_{t+1}(h^{\kappa'})^{-\sigma}\pi(h^{\kappa'}|h^\kappa) + \zeta_t^1(h^\kappa) + \zeta_t^3(h^\kappa)g_t(h^\kappa). \tag{47}$$

For all $t \geq 1$, the FOCs of the Ramsey problem with respect to $n_t(h^\kappa)$ and $c_t(h^\kappa)$ are given, respectively, by

$$n_t(h^\kappa)^\gamma + (1+\gamma)\lambda_t(h^\kappa)n_t(h^\kappa)^\gamma + \gamma\chi_t(h^\kappa)n_t(h^\kappa)^{\gamma-1} = \mu_t MPN_t z_t(h^\kappa) \tag{48}$$

and

$$
\begin{aligned}
&c_t(h^\kappa)^{-\sigma} + (1-\sigma)\lambda_t(h^\kappa)c_t(h^\kappa)^{-\sigma} \\
&-\sigma\lambda_t(h^\kappa)c_t(h^\kappa)^{-\sigma-1}\left(Q_{t+1}a_{t+1}(h^\kappa) - \sum_{h^\kappa_{-1}} \frac{a_t(h^\kappa_{-1})\pi(h^\kappa_{-1})\pi(h^\kappa|h^\kappa_{-1})}{\pi(h^\kappa)}\right) \\
&+\sigma\widehat{w}_t\chi_t(h^\kappa)z_t(h^\kappa)c_t(h^\kappa)^{-\sigma-1} \\
&-\sigma\beta Q_{t+1}\left[\zeta^2_t(h^\kappa) + \zeta^3_t(h^\kappa)a_{t+1}(h^\kappa)\right]c_t(h^\kappa)^{-\sigma-1} \\
&+\beta\sigma\left[\zeta^2_{t-1}(h^\kappa_{-1}) + \zeta^3_{t-1}(h^\kappa_{-1})a_t(h^\kappa_{-1})\right]c_t(h^\kappa)^{-\sigma-1}\pi(h^\kappa|h^\kappa_{-1})\frac{\pi(h^\kappa_{-1})}{\pi(h^\kappa)} = \mu_t.
\end{aligned}
\tag{49}
$$

A.4.3 Three Key Equations

Similar to the proof above in Appendix A.2, it is useful to obtain the following three key equations based on the Ramsey FOCs.

1. Consider a marginal change in an ad hoc borrowing constraint for household type h^κ in period t. By the envelope theorem, we obtain that $\zeta^1_t(h^\kappa) + \zeta^3_t(h^\kappa)g_t(h^\kappa) \geq 0$. As a result, the FOC with respect to $a_{t+1}(h^\kappa)$ can be rewritten as

$$
\lambda_t(h_\kappa)c_t(h^\kappa)^{-\sigma}Q_{t+1} \leq \beta\sum_{h^{\kappa'}}\lambda_{t+1}(h^{\kappa'})c_{t+1}(h^{\kappa'})^{-\sigma}\pi(h^{\kappa'}|h^\kappa). \tag{50}
$$

This is the first key equation.

2. The FOC with respect to $\widehat{w}_t$ is given by $\sum_{h^\kappa}\chi_t(h^\kappa)c_t(h^\kappa)^{-\sigma}z_t(h^\kappa)\pi(h^\kappa) = 0$, which together with the Ramsey constraint $n_t(h^\kappa)^\gamma = \widehat{w}_t z_t(h^\kappa)c_t(h^\kappa)^{-\sigma}$, gives

$$
0 = \sum_{h^\kappa}\chi_t(h^\kappa)c_t(h^\kappa)^{-\sigma}z_t(h^\kappa)\pi(h^\kappa) = \frac{1}{\widehat{w}_t}\sum_{h^\kappa}\chi_t(h^\kappa)n_t(h^\kappa)^\gamma\pi(h^\kappa). \tag{51}
$$

Multiplying the FOC with respect to n_t by $n_t(h^\kappa)\pi(h^\kappa)$ and summing it over h^κ, together with equation (51), gives

$$
\begin{aligned}
&\sum_{h^\kappa}\frac{n_t(h^\kappa)^{\gamma+1}}{N_t}\pi(h^\kappa) + (1+\gamma)\frac{\widehat{w}_t}{N_t}\sum_{h^\kappa}\lambda_t(h^\kappa)c_t(h^\kappa)^{-\sigma}z_t(h^\kappa)n_t(h^\kappa)\pi(h^\kappa) \\
=\;& \mu_t MPN_t.
\end{aligned}
\tag{52}
$$

This is the second key equation.

3. Multiplying the FOC with respect to $c_t(h^\kappa)$ in equation (49) by $c_t(h^\kappa)\pi(h^\kappa)$ gives

$$
\begin{aligned}
& c_t(h^\kappa)^{1-\sigma}\pi(h^\kappa) + (1-\sigma)\lambda_t(h^\kappa)c_t(h^\kappa)^{1-\sigma}\pi(h^\kappa) \\
& -\sigma\lambda_t(h^\kappa)c_t(h^\kappa)^{-\sigma}\left(Q_{t+1}a_{t+1}(h^\kappa) - \sum_{h^\kappa_{-1}}\frac{a_t(h^\kappa_{-1})\pi(h^\kappa_{-1})\pi(h^\kappa|h^\kappa_{-1})}{\pi(h^\kappa)}\right)\pi(h^\kappa) \\
& +\sigma\widehat{w}_t\chi_t(h^\kappa)z_t(h^\kappa)c_t(h^\kappa)^{-\sigma}\pi(h^\kappa) \\
& -\sigma\beta Q_{t+1}\left[\zeta^2_t(h^\kappa) + \zeta^3_t(h^\kappa)a_{t+1}(h^\kappa)\right]c_t(h^\kappa)^{-\sigma}\pi(h^\kappa) \\
& +\beta\sigma\left[\zeta^2_{t-1}(h^\kappa_{-1}) + \zeta^3_{t-1}(h^\kappa_{-1})a_t(h^\kappa_{-1})\right]c_t(h^\kappa)^{-\sigma}\pi(h^\kappa|h^\kappa_{-1})\pi(h^\kappa_{-1}) = \mu_t c_t(h^\kappa)\pi(h^\kappa).
\end{aligned}
$$

Summing the above equation over h^κ gives

$$
\begin{aligned}
& \sum_{h^\kappa} c_t(h^\kappa)^{1-\sigma}\pi(h^\kappa) + (1-\sigma)\sum_{h^\kappa}\lambda_t(h^\kappa)c_t(h^\kappa)^{1-\sigma}\pi(h^\kappa) \\
& -\sigma\sum_{h^\kappa}\lambda_t(h^\kappa)c_t(h^\kappa)^{-\sigma}\left(Q_{t+1}a_{t+1}(h^\kappa) - \sum_{h^\kappa_{-1}}\frac{a_t(h^\kappa_{-1})\pi(h^\kappa_{-1})\pi(h^\kappa|h^\kappa_{-1})}{\pi(h^\kappa)}\right)\pi(h^\kappa) \\
& +\sigma\widehat{w}_t\sum_{h^\kappa}\chi_t(h^\kappa)z_t(h^\kappa)c_t(h^\kappa)^{-\sigma}\pi(h^\kappa) \\
& -\sigma\beta Q_{t+1}\sum_{h^\kappa}\left[\zeta^2_t(h^\kappa) + \zeta^3_t(h^\kappa)a_{t+1}(h^\kappa)\right]c_t(h^\kappa)^{-\sigma}\pi(h^\kappa) \\
& +\beta\sigma\sum_{h^\kappa_{-1}}\left[\zeta^2_{t-1}(h^\kappa_{-1}) + \zeta^3_{t-1}(h^\kappa_{-1})a_t(h^\kappa_{-1})\right]\sum_{h^\kappa}c_t(h^\kappa)^{-\sigma}\pi(h^\kappa|h^\kappa_{-1})\pi(h^\kappa_{-1}) = \sum_{h^\kappa}\mu_t c_t(h^\kappa)\pi(h^\kappa).
\end{aligned}
$$

Applying the FOC with respect to $\widehat{w}_t$ to the equation above and rearranging terms gives

$$\begin{aligned}
&\sum_{h^\kappa} c_t(h^\kappa)^{1-\sigma}\pi(h^\kappa) + (1-\sigma)\sum_{h^\kappa}\lambda_t(h^\kappa)c_t(h^\kappa)^{1-\sigma}\pi(h^\kappa) \\
&\underbrace{-\sigma Q_{t+1}\sum_{h^\kappa}\lambda_t(h^\kappa)c_t(h^\kappa)^{-\sigma}a_{t+1}(h^\kappa)\pi(h^\kappa)}_{\text{Part 1}} \\
&\underbrace{-\sigma\beta Q_{t+1}\sum_{h^\kappa}\left[\zeta_t^2(h^\kappa)+\zeta_t^3(h^\kappa)a_{t+1}(h^\kappa)\right]c_t(h^\kappa)^{-\sigma}\pi(h^\kappa)}_{\text{Part 2}} \\
&\underbrace{+\beta\sigma\sum_{h_{-1}^\kappa}\left[\zeta_{t-1}^2(h_{-1}^\kappa)+\zeta_{t-1}^3(h_{-1}^\kappa)a_t(h_{-1}^\kappa)\right]\sum_{h^\kappa}c_t(h^\kappa)^{-\sigma}\pi(h^\kappa|h_{-1}^\kappa)\pi(h_{-1}^\kappa)}_{\text{Part 3}} \\
&\underbrace{+\sigma\sum_{h_{-1}^\kappa}\sum_{h^\kappa}\lambda_t(h^\kappa)c_t(h^\kappa)^{-\sigma}\pi(h^\kappa|h_{-1}^\kappa)a_t(h_{-1}^\kappa)\pi(h_{-1}^\kappa)}_{\text{Part 4}} \\
= \;&\sum_{\theta^t}\mu_t\pi(\theta^t)c_t(\theta^t).
\end{aligned} \tag{53}$$

We can simplify the above equation by showing that (a) Part 1 + Part 2 = 0 and (b) Part 3 + Part 4 = 0.

(a) To show that Part 1 + Part 2 = 0, we first multiply the FOC with respect to Q_{t+1} by $-\sigma Q_{t+1}$, and this gives

$$\begin{aligned}
&-\sigma Q_{t+1}\sum_{h^\kappa}\lambda_t(h^\kappa)c_t(h^\kappa)^{-\sigma}a_{t+1}(h^\kappa)\pi(h^\kappa) \\
= \;&\sigma Q_{t+1}\beta\sum_{h^\kappa}\left(\zeta_t^2(h^\kappa)+\zeta_t^3(h^\kappa)a_{t+1}(h^\kappa)\right)c_t(h^\kappa)^{-\sigma}\pi(h^\kappa),
\end{aligned}$$

which shows that the sum of Part 1 and Part 2 in equation (37) is zero.

(b) To show that Part 3 + Part 4 = 0, we consider the following steps:

i. We know that

$$\left(\zeta_{t-1}^2(h_{-1}^\kappa)+\zeta_{t-1}^3(h_{-1}^\kappa)a_t(h_{-1}^\kappa)\right)g_{t-1}(h_{-1}^\kappa)=0.$$

Plugging the definition of $g_{t-1}(h_{-1}^\kappa) \equiv Q_t c_{t-1}(h_{-1}^\kappa)^{-\sigma} - \beta\sum_{h^\kappa} c_t(h^\kappa)^{-\sigma}\pi(h^\kappa|h_{-1}^\kappa)$ into the above equation gives

$$\begin{aligned}
& \left[\zeta_{t-1}^{2}(h_{-1}^{\kappa})+\zeta_{t-1}^{3}(h_{-1}^{\kappa})a_t(h_{-1}^{\kappa})\right]Q_t \\
= & \left[\zeta_{t-1}^{2}(h_{-1}^{\kappa})+\zeta_{t-1}^{3}(h_{-1}^{\kappa})a_t(h_{-1}^{\kappa})\right]\beta\sum_{h^{\kappa}}\frac{c_t(h^{\kappa})^{-\sigma}}{c_{t-1}(h_{-1}^{\kappa})^{-\sigma}}\pi(h^{\kappa}|h_{-1}^{\kappa}).
\end{aligned} \tag{54}$$

ii. Part 3 can be written as

$$\begin{aligned}
& \sigma\sum_{h_{-1}^{\kappa}}\left[\zeta_{t-1}^{2}(h_{-1}^{\kappa})+\zeta_{t-1}^{3}(h_{-1}^{\kappa})a_t(h_{-1}^{\kappa})\right]c_{t-1}(h_{-1}^{\kappa})^{-\sigma}\times \\
& \beta\sum_{h^{\kappa}}\frac{c_t(h^{\kappa})^{-\sigma}}{c_{t-1}(h_{-1}^{\kappa})^{-\sigma}}\pi(h^{\kappa}|h_{-1}^{\kappa})\pi(h_{-1}^{\kappa}) \\
= & \sigma Q_t\sum_{h_{-1}^{\kappa}}\left[\zeta_{t-1}^{2}(h_{-1}^{\kappa})+\zeta_{t-1}^{3}(h_{-1}^{\kappa})a_t(h_{-1}^{\kappa})\right]c_{t-1}(h_{-1}^{\kappa})^{-\sigma}\pi(h_{-1}^{\kappa}),
\end{aligned}$$

where the last equality utilizes equation (54). The equation above together with the FOC with respect to Q_t leads to

$$\begin{aligned}
\text{Part 3} &= -\sigma\frac{Q_t}{\beta}\sum_{h_{-1}^{\kappa}}\lambda_{t-1}(h_{-1}^{\kappa})c_{t-1}(h_{-1}^{\kappa})^{-\sigma}a_t(h_{-1}^{\kappa})\pi(h_{-1}^{\kappa}) \\
&= -\sigma\sum_{h_{-1}^{\kappa}}\sum_{h^{\kappa}}\lambda_t(h^{\kappa})c_t(h^{\kappa})^{-\sigma}\pi(h^{\kappa}|h_{-1}^{\kappa})a_t(h_{-1}^{\kappa})\pi(h_{-1}^{\kappa}),
\end{aligned}$$

where the second equality utilizes equation (47). Hence, Part 3 and Part 4 in equation (37) exactly cancel each other out.

(c) Equation (37) is then simplified to

$$\sum_{h^{\kappa}}c_t(h^{\kappa})^{1-\sigma}\pi(h^{\kappa})+(1-\sigma)\sum_{h^{\kappa}}\lambda_t(h^{\kappa})c_t(h^{\kappa})^{1-\sigma}\pi(h^{\kappa})=C_t\mu_t. \tag{55}$$

This is the third key equation.

A.4.4 Existence of an Unconstrained Interior Ramsey Steady State

By the following steps, we conjecture and verify that there exists an unconstrained interior Ramsey steady state with (i) $c(h^{\kappa})=c>0$ for all h^{κ}, (ii) $\zeta^{1}(h^{\kappa})=0$ for all h^{κ}, (iii) $Q=\beta$, and (iv) $0<\mu<\infty$. In the following, we show that such an interior Ramsey steady state satisfies all Ramsey FOCs and the constraints in the Ramsey problem.

1. Consider the steady-state Ramsey allocation with $a(\boldsymbol{h}_l^\kappa) = 0$. The steady-state allocation, which includes $(c(h^\kappa) = c, n(h^\kappa), \{a(h^\kappa)\}_{h^\kappa \neq \boldsymbol{h}_l^\kappa}, K, \widehat{w}))$, can be solved by the following steady-state equations. Denote the number of states of shock θ_t as ω. Note that the number of unknowns is equal to the number of equations, which is $2\omega^\kappa + 2$.

   (a) The FOC with respect to K_{t+1} in the steady state is (1 equation)

   $$1 = \beta \left(MPK + 1 - \delta \right).$$

   (b) The steady state resource constraint is (1 equation)

   $$F(\sum_{h^\kappa} n(h^\kappa) z(h^\kappa) \pi(h^\kappa), K) - \delta K = c.$$

   (c) Given $c = c(h^\kappa)$, the intratemporal marginal conditions are given by (ω^κ equations)

   $$\widehat{w}_t c^{-\sigma} z_t(h^\kappa) = n_t(h^\kappa)^\gamma.$$

   (d) The implementability conditions of each type-$h_\kappa$agent are given by (ω^κ equations)

   $$c - \widehat{w}_t z(h^\kappa) n(h^\kappa) + \beta a(h_\kappa) = \sum_{h_{-1}^\kappa} \frac{a(h_{-1}^\kappa) \pi(h_{-1}^\kappa) \pi(h^\kappa | h_{-1}^\kappa)}{\pi(h^\kappa)}.$$

2. Note that this Ramsey steady-state allocation satisfies all other constraints of the Ramsey problem trivially. We then show that all Lagrangian multipliers of the Ramsey problem can be correctly solved such that all Ramsey FOCs are satisfied:

   (a) Set $\zeta^1(h^\kappa) = 0$ for all h^κ.

   (b) Given $Q = \beta$, $\zeta^1(h^\kappa) = 0$, $c = c(h^\kappa)$, and $g(h^\kappa) = 0$, the steady-state version of the Ramsey FOCs with respect to $a_{t+1}(h_\kappa)$ in equation (47) can be rewritten as

   $$\lambda(h^\kappa) = \sum_{h^{\kappa\prime}} \lambda(h^{\kappa\prime}) \pi(h^{\kappa\prime} | h^\kappa),$$

   which can be satisfied only if $\lambda(h^\kappa) = \lambda$ for all h^κ.

   (c) The multipliers μ and λ can be solved by using the two equations (52) and (55) in the steady state, together with $c(h^\kappa)=c$, $\lambda(h^\kappa)=\lambda$, and $Q=\beta$, giving

$$\sum_{h^\kappa}\frac{n(h^\kappa)^{\gamma+1}}{N_t}\pi(h^\kappa)+(1+\gamma)\frac{\widehat{w}}{N}\lambda c^{-\sigma}\sum_{h^\kappa}z_t(h^\kappa)n_t(h^\kappa)\pi(h^\kappa)=\mu MPN,$$

and

$$c^{-\sigma}+(1-\sigma)\lambda c^{-\sigma}=\mu.$$

(d) $\chi_t(h^\kappa)$ is chosen such that the steady-state Ramsey FOC with respect to $n_t(h^\kappa)$ in equation (48) is satisfied.

(e) The steady-state FOC with respect to $\widehat{w}_t$ is satisfied because it is implied by combining the aggregate of equation (48) with equation (52).

(f) The steady-state FOC with respect to Q_{t+1} is satisfied since it is implied by combining the aggregate of equation (49) with equation (55).

(g) Finally, by properly choosing $\zeta^2(h^\kappa)$ and $\zeta^3(h^\kappa)$, the FOC with respect to $c_t(h^\kappa)$ can be satisfied.

3. The optimal long-run policies, $\{B,\tau_n,\tau_k\}$, are determined by the following steps:

(a) $Q=\beta$ by equation (8). Given $Q=\beta$, the MGR implies a steady-state capital tax of zero:

$$\tau_k=1-\frac{\frac{1}{Q}-(1-\delta)}{\frac{1}{\beta}-(1-\delta)}=0.$$

(b) The government debt can be solved by using the asset market-clearing condition: $B=\sum_{h^\kappa}a(h^\kappa)\pi(h^\kappa)-\frac{K}{Q}=A-\frac{K}{Q}$.

(c) By plugging (i) $\sum_{h^\kappa}a(h^\kappa)\pi(h^\kappa)=\phi c$, (ii) the asset market-clearing condition, $B=\sum_{h^\kappa}a(h^\kappa)\pi(h^\kappa)-\frac{K}{Q}$, (iii) the steady-state resource constraint $\frac{c^e}{K}=\frac{MP_K}{\alpha}-\delta=\frac{1-\beta+\delta\beta}{\alpha\beta}-\delta$, (iv) $\phi=\frac{A}{C}=\frac{\sum_{h^\kappa}a(h^\kappa)\pi(h^\kappa)}{c}$, and (v) the MGR implied by the FOC with respect to K into the steady-state government budget constraint, the optimal long-run labor tax rate is determined by

$$\begin{aligned}\tau_n &= (1-\beta)\frac{B}{MPN\times N}=(1-\beta)\frac{A-\frac{K}{\beta}}{MPN\times N}\\ &= (1-\beta)\frac{(\phi\frac{C}{K}-\frac{1}{\beta})}{\frac{MPN\times N}{Y}}\frac{K}{Y}=(1-\beta)\frac{\phi(\frac{1-\beta+\delta\beta}{\alpha\beta}-\delta)-\frac{1}{\beta}}{(1-\alpha)}\frac{K}{Y}\\ &= \frac{(1-\beta)}{(1-\alpha)}\frac{\phi(1-\beta+\delta\beta(1-\alpha))-\alpha}{(1-\beta+\delta\beta)}.\end{aligned}$$

In addition, the debt-to-GDP ratio can be expressed as

$$(1-\beta)\frac{B}{Y} = \tau_n \frac{MPN \times N}{Y} = \tau_n(1-\alpha),$$

and hence

$$\frac{B}{Y} = \frac{\phi\left(1-\beta+\delta\beta(1-\alpha)\right)-\alpha}{(1-\beta+\delta\beta)},$$

which is an increasing function of ϕ.

4. Note that this interior steady state is only feasible if $\tau_n < 1$; otherwise, it violates the FOCs of the employed individuals. In the following two steps we verify that $\tau_n < 1$ if and only if $\phi(1-\beta) < 1$:

   (a) If $\phi(1-\beta) < 1$, then

   $$\tau_n = \frac{(1-\beta+\delta(1-\alpha)\beta)\,\phi(1-\beta)-\alpha(1-\beta)}{(1-\alpha)\,(1-\beta+\delta\beta)} < \frac{(1-\beta+\delta(1-\alpha)\beta)-\alpha(1-\beta)}{(1-\alpha)\,(1-\beta+\delta\beta)} = 1.$$

   (b) If $\tau_n < 1$, then

   $$(1-\beta+\delta(1-\alpha)\beta)\,\phi(1-\beta)-\alpha(1-\beta) < (1-\alpha)\,(1-\beta+\delta\beta)\,,$$

   which can be simplified as

   $$(1-\beta+\delta(1-\alpha)\beta)\,\phi(1-\beta) < (1-\alpha)\delta\beta+(1-\beta)\,,$$

   which can be further simplified as $\phi(1-\beta) < 1$. Note that ϕ is a function of κ and $\lim_{\kappa\to\infty}\phi = \infty$.

A.4.5 Uniqueness of the Unconstrained Interior Ramsey Steady State under $\sigma \geq 1$

We show that under the parameter condition $\sigma \geq 1$ there cannot be an interior Ramsey steady state with $Q > \beta$. The proof is done by contradiction and is similar to the argument in Appendix A.2. Suppose there is an interior Ramsey steady state with $1 > Q > \beta$:

1. The first key equation in equation (50) implies that

$$\sum_{h^{\kappa'}} x_{t+1}(h^{\kappa'})\pi(h^{\kappa'}|h^{\kappa}) \geq \frac{Q_{t+1}}{\beta}x_t(h^{\kappa}),$$

where $x_t\left(h_\kappa\right) \equiv \lambda_t(h_\kappa)c_t(h^\kappa)^{-\sigma}$. Taking unconditional expectations on both sides gives

$$\mathbb{E}x_{t+1} \geq \frac{Q_{t+1}}{\beta}\mathbb{E}x_t.$$

Given $\lim_{t\to\infty}\frac{Q_{t+1}}{\beta} = \frac{Q}{\beta} > 1$, the above inequality suggests that $\mathbb{E}x_t \to \infty$.

2. The second key equation (52),

$$\begin{aligned} & \sum_{h^\kappa}\frac{n_t(h^\kappa)^{\gamma+1}}{N_t}\pi(h^\kappa) + (1+\gamma)\widehat{w}_t\sum_{h^\kappa}x_t(h^\kappa)\frac{z_t(h^\kappa)n_t(h^\kappa)}{N_t}\pi(h^\kappa) \\ = \; & \mu_t MPN_t, \end{aligned}$$

is analogous to equation (42), where $\frac{z_t(h^\kappa)n_t(h^\kappa)}{N_t} > 0$ is labor share with $\sum_{h^\kappa}\frac{z_t(h^\kappa)n_t(h^\kappa)}{N_t}\pi(h^\kappa) = 1$. The above equation can be rewritten as

$$\mathbb{E}\left(\frac{n_t^{\gamma+1}}{N_t}\right) + (1+\gamma)\,\widehat{w}_t\mathbb{E}\left(x_t\frac{z_tn_t}{N_t}\right) = \mu_t MPN_t, \tag{56}$$

where

$$\mathbb{E}\left(x_t\frac{z_tn_t}{N_t}\right) = \mathbb{E}x_t + \frac{1}{N_t}cov\left(x_t, z_tn_t\right). \tag{57}$$

Given that both consumption c_t and labor income z_tn_t respond positively to the labor productivity shock z_t, we have

$$\text{sign of } cov\left(x_t, z_tn_t\right) = \text{sign of } cov\left(x_t, c_t\right).$$

Since x_t is the shadow price of implementing the household's budget constraint in the Ramsey problem, which must be positively correlated with household wealth position; namely, all else equal, a higher household wealth position a_t implies a higher consumption level c_t and a tighter constraint (or higher shadow price x_t) in the Ramsey problem. Then it must be the case that $cov\left(x_t, c_t\right) > 0$. Hence, $\mathbb{E}\left(x_t\frac{z_tn_t}{N_t}\right) = \mathbb{E}x_t + \frac{1}{N_t}cov\left(x_t, z_tn_t\right) \to \infty$. Given this, μ_t on the right hand side of the equation must also diverge to positive infinity (the argument is the same as that for equation (42) in Appendix A.2).

3. The third key equation (55) can be written as

$$\sum_{h^\kappa}\frac{c_t(h^\kappa)^{1-\sigma}}{C_t}\pi(h^\kappa) + (1-\sigma)\sum_{h^\kappa}x_t(h^\kappa)\frac{c_t(h^\kappa)}{C_t}\pi(h^\kappa) = \mu_t,$$

or simply

$$\mathbb{E}\left(\frac{{c_t}^{1-\sigma}}{C_t}\right)+(1-\sigma)\mathbb{E}\left(x_t\frac{c_t}{C_t}\right)=\mu_t, \tag{58}$$

where $\frac{c_t}{C_t}>0$ is consumption share with $\mathbb{E}\left(\frac{c_t}{C_t}\right)=1$ and $\mathbb{E}\left(x_t\frac{c_t}{C_t}\right)=\mathbb{E}x_t+\frac{1}{C_t}cov\left(x_t,c_t\right)$. Since $cov\left(x_t,c_t\right)>0$ from step (2), the term $\mathbb{E}\left(x_t\frac{c_t}{C_t}\right)$ must diverge to positive infinity. Since μ_t on the right hand side diverges to positive infinity, the above equation leads to a contradiction if $\sigma\geq 1$ because the left hand side of the above equation goes either to negative infinity (if $\sigma>1$) or a finite positive value (if $\sigma=1$).

To recapitulate, the above argument shows that under $\phi(1-\beta)<1$, we have the following results: (i) If $\sigma\geq 1$, the unconstrained interior steady state (with $Q=\beta$) exists and is unique; thus, it is impossible to have an interior Ramsey steady state with $Q>\beta$. (ii) If $\sigma<1$, it is possible to have an interior Ramsey steady state with $Q>\beta$, provided that the multiplier μ_t diverges. Furthermore, under $\phi(1-\beta)\geq 1$ and $\sigma\geq 1$, it is impossible for any form of interior Ramsey steady state to exist, and the only possible Ramsey steady state is non-interior with $C_t\to 0$.

A.5 The Existence of a Ramsey Steady State When $\kappa=0$ and $Z=\{e,u\}$

A.5.1 Ramsey Problem

With $\kappa=0$ and $Z=\{e,u\}$, there are only two groups of individuals, denoted by the e and u groups. Any variable denoted by a superscript e or u then represents its value for the e or u group, respectively. The Ramsey planner's problem can be written as

$$\max_{\{c_t^e,c_t^u,n_t^e,a_{t+1}^e,a_{t+1}^u,K_{t+1}\}_{t=0}^{\infty}}\sum_{t=0}^{\infty}\beta^t\left\{\left[u(c_t^e)-v\left(n_t^e\right)\right]\pi(e)+u(c_t^u)\pi(u)\right\}$$

subject to

$$\beta^t\mu_t: F(n_t^e\pi(e),K_t)+(1-\delta)K_t-c_t^e\pi(e)-c_t^u\pi(u)-K_{t+1}\geq 0\ \forall t\geq 0,$$

$$\lambda_0^e: u_{c,0}^e c_0^e\pi(e)-v_{n,0}^e n_0^e\pi(e)+Q_1u_{c,0}^e a_1^e-u_{c,0}^e a_0^e=0,$$

$$\lambda_0^u: u_{c,0}^e c_0^u\pi(u)+Q_1u_{c,0}^e a_1^u-u_{c,0}^e a_t^u=0,$$

$$\beta^t\lambda_t^e\pi(e): u_{c,t}^e c_t^e-v_{n,t}^e n_t^e+Q_{t+1}u_{c,t}^e a_{t+1}^e-u_{c,t}^e\left[\frac{a_t^e\pi(e)\pi(e|e)+a_t^u\pi(u)\pi(e|u)}{\pi(e)}\right]=0,$$

$$\beta^t\lambda_t^u\pi(u): u_{c,t}^e c_t^u+Q_{t+1}u_{c,t}^e a_{t+1}^u-u_{c,t}^e\left[\frac{a_t^e\pi(e)\pi(u|e)+a_t^u\pi(u)\pi(u|u)}{\pi(u)}\right]=0,$$

$$\zeta^0: (1+(1-\tau_{k,0})MP_{K,0}-\delta)K_0+B_0-a_0^e-a_{0^u}=0,$$

$$\beta^{t+1}\zeta_t^1 : a_{t+1}^u \geq 0,$$

$$\beta^{t+1}\zeta_t^2 : g_t^u \geq 0,$$

$$\beta^{t+1}\zeta_t^3 : g_t^u a_{t+1}^u = 0,$$

where $Q_{t+1}u_{c,t}^e$ is given by

$$Q_{t+1}u_{c,t}^e = \beta\left[u_{c,t+1}^e\pi(e|e) + u_{c,t+1}^u\pi(u|e)\right],$$

and the function g_t^u is defined as

$$g_t^u \equiv \frac{u_{c,t}^u}{u_{c,t+1}^e\pi(e|u) + u_{c,t+1}^u\pi(u|u)} - \frac{u_{c,t}^e}{u_{c,t+1}^e\pi(e|e) + u_{c,t+1}^u\pi(u|e)}.$$

A.5.2 Ramsey FOCs

For all $t \geq 0$, the FOCs of the Ramsey problem with respect to K_{t+1}, a_{t+1}^e, and a_{t+1}^u are given, respectively, by

$$\mu_t = \beta\mu_{t+1}\left(MPK_{t+1} + 1 - \delta\right), \tag{59}$$

$$\lambda_t^e Q_{t+1}u_{c,t}^e = \beta u_{c,t+1}^e\left(\lambda_{t+1}^e\pi(e|e) + \lambda_{t+1}^u\pi(u|e)\right) \text{ for } t \geq 0, \tag{60}$$

$$\begin{aligned}\lambda_t^u Q_{t+1}u_{c,t}^e \quad = \quad & \beta u_{c,t+1}^e(\lambda_{t+1}^u\pi(u|u) + \lambda_{t+1}^e\pi(e|u)) \\ & +\zeta_t^1 + \zeta_t^3 g(c_t^e, c_t^u, c_{t+1}^e, c_{t+1}^u).\end{aligned} \tag{61}$$

For all $t \geq 1$, the FOCs of the Ramsey problem with respect to n_t^e, c_t^e, and c_t^u are given, respectively, by

$$v_{n,t}^e + \lambda_t^e(v_{n,t}^e + v_{nn,t}^e n_t^e) = \mu_t MPN_t, \tag{62}$$

$$\begin{aligned}& (u_{c,t}^e - \mu_t)\pi(e) + \lambda_t^e(u_{c,t}^e + u_{cc,t}^e c_t^e)\pi(e) - \lambda_t^e u_{cc,t}^e(a_t^e\pi(e)\pi(e|e) + a_t^u\pi(u)\pi(e|u)) \\ & +\lambda_{t-1}^e a_t^e\pi(e)u_{cc,t}^e\pi(e|e) + \lambda_t^u u_{cc,t}^e c_t^u\pi(u) - \lambda_t^u u_{cc,t}^e(a_t^e\pi(e)\pi(u|e) + a_t^u\pi(u)\pi(u|u)) \\ & +\lambda_{t-1}^u a_t^u\pi(u)u_{cc,t}^e\pi(e|e) + \beta\zeta_t^2\frac{\partial g_t}{\partial c_t^e} + \zeta_{t-1}^2\frac{\partial g_{t-1}}{\partial c_t^e} + \beta a_{t+1}^u\zeta_t^3\frac{\partial g_t}{\partial c_t^e} + a_t^u\zeta_{t-1}^3\frac{\partial g_{t-1}}{\partial c_t^e} \\ = \quad & 0,\end{aligned} \tag{63}$$

$$
\begin{aligned}
& (u^u_{c,t}-\mu_t)\pi(u)+\lambda^u_t u^e_{c,t}\pi(u)+\lambda^u_{t-1}a^u_t\pi(u)u^u_{cc,t}\pi(u|e) \\
& +\lambda^e_{t-1}a^e_t\pi(e)u^u_{cc,t}\pi(u|e)+\beta\zeta^2_t\frac{\partial g_t}{\partial c^u_t}+\zeta^2_{t-1}\frac{\partial g_{t-1}}{\partial c^u_t}+\beta a^u_{t+1}\zeta^3_t\frac{\partial g_t}{\partial c^u_t}+a^u_t\zeta^3_{t-1}\frac{\partial g_{t-1}}{\partial c^u_t} \\
= & \ 0.
\end{aligned} \tag{64}
$$

For $t=0$, the FOCs of the Ramsey problem with respect to n^e_0, c^e_0, and c^u_0 are given, respectively, by

$$
v^e_{n,0}+\lambda^e_0(v^e_{n,0}+v^e_{nn,0}n^e_0)=\mu_0 MP_{N,0}+\zeta^0(1-\tau_{k,0})MP_{KN,0}K_0,
$$

$$
\begin{aligned}
& (u^e_{c,0}-\mu_0)\pi(e)+\lambda^e_0(u^e_{c,0}+u^e_{cc,0}c^e_0)\pi(e)-\lambda^e_0u^e_{cc,0}a^e_0 \\
& +\lambda^u_0u^e_{cc,0}c^u_0\pi(u)-\lambda^u_0u^e_{cc,0}a^u_0+\beta\zeta^2_0\frac{\partial g_0}{\partial c^e_0}+\beta a^u_1\zeta^3_0\frac{\partial g_0}{\partial c^e_0} \\
= & \ 0,
\end{aligned}
$$

$$
(u^u_{c,0}-\mu_0)\pi(u)+\lambda^u_0u^e_{c,0}\pi(u)+\beta\zeta^2_0\frac{\partial g_0}{\partial c^u_0}+\beta a^u_1\zeta^3_0\frac{\partial g_0}{\partial c^u_0}=0.
$$

Note that

$$
\begin{aligned}
\frac{\partial g_t}{\partial c^u_t} &= \frac{u^u_{cc,t}}{u^e_{c,t+1}\pi(e|u)+u^u_{c,t+1}\pi(u|u)}, \\
\frac{\partial g_t}{\partial c^e_t} &= -\frac{u^e_{cc,t}}{u^e_{c,t+1}\pi(e|e)+u^u_{c,t+1}\pi(u|e)},
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial g_{t-1}}{\partial c^u_t} &= -\frac{u^u_{c,t-1}u^u_{cc,t}\pi(u|u)}{\left(u^e_{c,t}\pi(e|u)+u^u_{c,t}\pi(u|u)\right)^2}+\frac{u^e_{c,t-1}u^u_{cc,t}\pi(u|e)}{\left(u^e_{c,t}\pi(e|e)+u^u_{c,t}\pi(u|e)\right)^2}, \\
\frac{\partial g_{t-1}}{\partial c^e_t} &= -\frac{u^u_{c,t-1}u^e_{cc,t}\pi(e|u)}{\left(u^e_{c,t}\pi(e|u)+u^u_{c,t}\pi(u|u)\right)^2}+\frac{u^e_{c,t-1}u^e_{cc,t}\pi(e|e)}{\left(u^e_{c,t}\pi(e|e)+u^u_{c,t}\pi(u|e)\right)^2}.
\end{aligned}
$$

A.5.3 Existence of a Non-Interior Ramsey Steady State under $\kappa=0$ and $\sigma\geq 1$

By Proposition 4, we know that if $(1-\beta)\frac{\pi(u)}{\pi(u|e)}\geq 1$ and $\sigma\geq 1$, there is no interior Ramsey steady state. Here we further prove that under these parameter conditions there must be a non-interior Ramsey steady state.

For this non-interior Ramsey steady state to exist, it must be the case that $C_t\to 0$. Thus, the proof proceeds by considering an allocation path where (i) $c^e_t>c^u_t>0$ for all $t<\infty$ and (ii) $\lim_{t\to\infty}c^e_t=\lim_{t\to\infty}c^u_t=0$.

1. We first show that this non-interior steady-state allocation can satisfy all constraints and the Ramsey FOCs:

   (a) The resource constraint is satisfied if $K_t \to 0$ since $\lim_{t\to\infty} c_t^e = \lim_{t\to\infty} c_t^u = 0$.

   (b) Given that $c_t^e > c_t^u > 0$ for $t < \infty$ and $a_t^u = 0$, the implementability condition of the unemployed agents becomes $c_t^u = a_t^e \frac{\pi(e)\pi(u|e)}{\pi(u)}$, which can be satisfied in the limit by letting $a_t^e \to 0$. In addition, the implementability condition of the employed agents becomes

$$c_t^e - \frac{v_{n,t}^e}{u_{c,t}^e} n_t^e + Q_{t+1} a_{t+1}^e - a_t^e \pi(e|e) = 0,$$

   which is satisfied in the limit by the condition $\lim_{t\to\infty} Q_{t+1} a_{t+1}^e = \lim_{t\to\infty} \frac{v_{n,t}^e}{u_{c,t}^e} n_t^e \pi(e) \geq 0$.

   (c) The borrowing constraints and complementary slackness conditions of the Ramsey problem are trivially satisfied.

2. We then show that this allocation satisfies all Ramsey FOCs by properly choosing the convergence properties of the Ramsey multipliers:

   (a) The FOC with respect to K_{t+1} can be satisfied if the following condition holds

$$\lim_{t\to\infty} MPK_{t+1} = \frac{1}{\lim_{t\to\infty} \frac{\mu_{t+1}}{\mu_t}} \frac{1}{\beta} - (1-\delta).$$

   Note that the above equation also implies the following:

      i. $\lim_{t\to\infty} \frac{\mu_{t+1}}{\mu_t} < \infty$ since $\lim_{t\to\infty} MPK_{t+1} \geq 0$. We also know that $\lim_{t\to\infty} \frac{\mu_{t+1}}{\mu_t} \geq 1$. A convergence of $\frac{\mu_{t+1}}{\mu_t}$ then implies that the capital-to-labor ratio $\frac{K_t}{N_t}$ contained in MPK_t and MPN_t must also converge to a finite positive value despite the fact that $\lim_{t\to\infty} K_t = \lim_{t\to\infty} N_t = 0$.

      ii. $n_t^e \to 0$, since $\lim_{t\to\infty} MPK_t = \lim_{t\to\infty} \alpha(\frac{n_t^e \pi(e)}{K_t})^{1-\alpha} < \infty$ and $K_t \to 0$.

   (b) Let $\lambda_t^e \to \infty$ and $\frac{\mu_t}{\lambda_t^e} \to 0$. The Ramsey FOC with respect to n_t^e in equation (62) is satisfied in the limit:

$$MPN \lim_{t\to\infty} \frac{\mu_t}{\lambda_t^e} = (1+\gamma) \lim_{t\to\infty} v_{n,t}^e = 0.$$

   (c) Given $\zeta_t^1 > 0$, $\zeta_t^2 = 0$, $a_t^u = 0$, and $c_t^u = a_t^e \frac{\pi(e)\pi(u|e)}{\pi(u)}$, the Ramsey FOC with respect to c_t^e can be rewritten as

$$u_{c,t}^e \left(1 + \lambda_t^e (1-\sigma)\right) + \left(\lambda_{t-1}^e - \lambda_t^e\right) u_{cc,t}^e c_t^u \frac{\pi(u)}{\pi(u|e)} \pi(e|e) = \mu_t,$$

which can be further transformed to

$$\frac{1}{\sigma\lambda_t^e} + \left(1 - \frac{\lambda_{t-1}^e}{\lambda_t^e}\right)\frac{c_t^u}{c_t^e}\frac{\pi(u)}{\pi(u|e)}\pi(e|e) = \frac{\mu_t}{\lambda_t^e}\frac{1}{u_{c,t}^e}\frac{1}{\sigma} + 1 - \frac{1}{\sigma}.$$

As $t \to \infty$, the above equation becomes

$$0 \leq \left(1 - \lim_{t\to\infty}\frac{\lambda_{t-1}^e}{\lambda_t^e}\right)\lim_{t\to\infty}\frac{c_t^u}{c_t^e} = \frac{\pi(u|e)}{\pi(e|e)\pi(u)}\left(1 - \frac{1}{\sigma}\right). \tag{65}$$

Hence, given $\sigma \geq 1$ (a necessary condition), the above FOC can be satisfied and does not lead to contradictions.

(d) Under the conditions that $\zeta_t^1 > 0$, $\zeta_t^2 = 0$, $a_t^u = 0$, and $c_t^u = a_t^e\frac{\pi(e)\pi(u|e)}{\pi(u)}$, the Ramsey FOC with respect to c_t^u can be simplified to

$$u_{c,t}^u + \lambda_t^u u_{c,t}^e + \lambda_{t-1}^e c_t^u u_{cc,t}^u = \mu_t,$$

which can be rewritten as

$$\frac{1}{\lambda_{t-1}^e} + \frac{\lambda_t^u}{\lambda_{t-1}^e}\frac{u_{c,t}^e}{u_{c,t}^u} - \sigma = \frac{\mu_t}{\lambda_t^e}\frac{\lambda_t^e}{\lambda_{t-1}^e}\frac{1}{u_{c,t}^u}.$$

Since $\lim_{t\to\infty}\frac{\mu_t}{\lambda_t^e}\frac{\lambda_t^e}{\lambda_{t-1}^e}\frac{1}{u_{c,t}^u} = 0$, the equation above becomes

$$\lim_{t\to\infty}\frac{\lambda_t^u}{\lambda_{t-1}^e}\frac{u_{c,t}^e}{u_{c,t}^u} = \sigma. \tag{66}$$

We then have two subcases to consider:

i. $\sigma > 1$. Equation (66) can be satisfied if $\frac{\lambda_t^u}{\lambda_{t-1}^e}$ converges to a finite positive constant. From 2(c), we know that $\frac{u_{c,t}^e}{u_{c,t}^u}$ also converges to a finite positive constant given the convergence of $\frac{c_t^u}{c_t^e}$.

ii. $\sigma = 1$. Equation (66) can be satisfied if both $\frac{\lambda_t^u}{\lambda_{t-1}^e}$ and $\frac{c_t^u}{c_t^e}$ converge to finite positive values.

Hence, both sub-cases above are possible and do not lead to contradictions.

(e) The FOCs of a_{t+1}^e and a_{t+1}^u in equations (60) and (61) can be rewritten, respectively, as

$$\pi(e|e) + \frac{u_{c,t+1}^u}{u_{c,t+1}^e}\pi(u|e) = \frac{\lambda_{t+1}^e}{\lambda_t^e}\pi(e|e) + \frac{\lambda_{t+1}^u}{\lambda_t^e}\pi(u|e), \tag{67}$$

$$\pi(e|e) + \frac{u^u_{c,t+1}}{u^e_{c,t+1}}\pi(u|e) = \frac{\lambda^u_{t+1}}{\lambda^u_t}\pi(u|u) + \frac{\lambda^e_{t+1}}{\lambda^u_t}\pi(e|u) + \frac{\zeta^1_t + \zeta^3_t g(c^e_t, c^u_t, c^e_{t+1}, c^u_{t+1})}{\lambda^u_t u^e_{c,t+1}}. \tag{68}$$

Given $\sigma \geq 1$ and from 2(c) and 2(d), we know that $\frac{u^u_{c,t}}{u^e_{c,t}}$, $\frac{\lambda^e_{t+1}}{\lambda^e_t}$, $\frac{\lambda^u_{t+1}}{\lambda^u_t}$, $\frac{\lambda^e_{t+1}}{\lambda^u_t}$, and $\frac{\lambda^u_{t+1}}{\lambda^e_t}$ all converge to finite positive constants. Hence, equation (67) is satisfied. In addition, $g_t(c^e_t, c^u_t, c^e_{t+1}, c^u_{t+1})$ also converges to a finite positive constant according to its definition, so equation (68) can be satisfied if $\lim_{t\to\infty} \frac{\zeta^1_t + \zeta^3_t g(c^e_t, c^u_t, c^e_{t+1}, c^u_{t+1})}{\lambda^u_t u^e_{c,t+1}}$ is chosen to be a finite constant, which is possible and does not lead to contradictions.

In short, we have shown that a non-interior Ramsey steady state exists and it must feature divergent multipliers. Also note that the last equation in (c) step 2 is the only critical condition involving the value of σ:

$$0 \leq \left(1 - \lim_{t\to\infty} \frac{\lambda^e_{t-1}}{\lambda^e_t}\right) \lim_{t\to\infty} \frac{c^u_t}{c^e_t} = \frac{\pi(u|e)}{\pi(e|e)\pi(u)} \left(1 - \frac{1}{\sigma}\right), \tag{69}$$

which suggests that if $\sigma < 1$, the right-hand side of the above inequality is negative; hence, there cannot exist a non-interior Ramsey steady state in the simplified model with $\kappa = 0$ and $\sigma < 1$.

3. Policy implication

   (a) Given $c^e_t \to 0$, $n^e_t \to 0$, and $MPN_t \to MPN > 0$, it must be true that $\tau_{n,t} \to 1$ by equation (27).

   (b) The intertemporal price is

   $$Q_{t+1} = \beta \left[\frac{u^e_{c,t+1}}{u^e_{c,t}} \pi(e|e) + \frac{u^u_{c,t+1}}{u^u_{c,t}} \frac{u^u_{c,t}}{u^e_{c,t}} \pi(u|e) \right].$$

   We know that $\frac{u^e_{c,t+1}}{u^e_{c,t}}$, $\frac{u^u_{c,t+1}}{u^u_{c,t}}$, and $\frac{u^u_{c,t}}{u^e_{c,t}}$ all converge to finite values larger than or equal to 1, hence $\infty > \lim_{t\to\infty} Q_{t+1} \geq \beta$.

   (c) Capital tax is determined by

   $$\tau_{k,t+1} = 1 - \frac{\frac{\beta}{Q_{t+1}} - \beta(1-\delta)}{\frac{\mu_t}{\mu_{t+1}} - \beta(1-\delta)}.$$

   Therefore, the sign of the capital tax in the limit depends on the growth rate of μ_t relative to $\frac{Q_{t+1}}{\beta}$ in the limit. As $K_t \to 0$, the optimal capital tax is irrelevant.

   (d) By the asset market-clearing condition, we have $B_t \to 0$.

A.5.4 Existence of a Constrained Interior Ramsey Steady State under $\sigma < 1$ and $\mu_t \to \infty$

By Proposition 4, we know that if $(1-\beta)\frac{\pi(u)}{\pi(u|e)} \geq 1$, there is no unconstrained interior steady state with full self-insurance (FSI). Furthermore, we know from section A.5.3 that the existence of a non-interior Ramsey steady state requires the condition $\sigma \geq 1$. We now prove that there is a constrained interior Ramsey steady state under $\sigma < 1$.

Consider a constrained interior Ramsey steady state where (i) $c^e > c^u > 0$, (ii) the borrowing constraint for the unemployed is strictly binding with $a^u = 0$ and $\zeta^1 > 0$ and $\zeta^2 = 0$, and (iii) $Q > \beta$.

Let g_λ^u, g_λ^e, and g_μ denote the steady-state growth rate of λ_t^u, λ_t^e, and μ_t, respectively. We first show that this Ramsey steady state must have $g_\lambda^u = g_\lambda^e = g_\mu$.

1. From the Ramsey FOC with respect to n^e, we know that for an interior Ramsey steady state to exist, the growth rate of λ_t^e and μ_t must be equal: $g_\lambda^e = g_\mu$.

2. Furthermore, we can show that $g_\lambda^e = g_\lambda^\mu$ by the following steps:

   (a) The FOC with respect to a^e in the constrained steady state is given by

   $$1 < \frac{Q}{\beta} = \pi(e|e) + \frac{u_c^u}{u_c^e}\pi(u|e) = g_\lambda^e \pi(e|e) + g_\lambda^u \frac{\lambda_t^u}{\lambda_t^e}\pi(u|e). \tag{70}$$

   (b) Suppose $g_\lambda^e < g_\lambda^u$, then $\frac{\lambda_t^u}{\lambda_t^e} \to \infty$. Equation (70) becomes $\infty > \pi(e|e) + \frac{u_c^u}{u_c^e}\pi(u|e) = \infty$, which is impossible.

   (c) Suppose $g_\lambda^e > g_\lambda^u$, then $\frac{\lambda_t^u}{\lambda_t^e} \to 0$. The FOC with respect to c^u (under $\zeta_t^2 = 0$, $a^u = 0$, and $a^e\pi(e)\pi(u|e) = c^u\pi(u)$) becomes

   $$u_c^u + g_\lambda^u \lambda_{t-1}^u u_c^e + \lambda_{t-1}^e u_{cc}^u c^u = g\mu_{t-1}\mu_t,$$

   which implies

   $$\frac{u_c^u}{\lambda_{t-1}^e} + g_\lambda^u \frac{\lambda_{t-1}^u}{\lambda_{t-1}^e} u_c^e + u_{cc}^u c^u = g_\mu \frac{\mu_{t-1}}{\lambda_{t-1}^e}.$$

   As $t \to \infty$, the left-hand side is negative and the right-hand side is positive, which is a contradiction.

   (d) Therefore, it must be true that $g_\lambda^e = g_\lambda^u = g_\mu$.

We now show that such an interior steady state cannot exist under the condition $\sigma \geq 1$. Namely, this Ramsey steady state exists only if $\sigma < 1$. First, the following must hold:

1. Given $g_\lambda^e = g_\lambda^u$, equation (70) implies that

$$\left(\frac{u_c^u}{u_c^e} - g_\lambda^u \frac{\lambda_t^u}{\lambda_t^e}\right)\pi(u|e) = (g_\lambda^e - 1)\pi(e|e). \tag{71}$$

2. Under $\zeta_t^2 = 0$, $a^u = 0$, and $a^e\pi(e)\pi(u|e) = c^u\pi(u)$, the FOC with respect to c^e can be rewritten as

$$u_c^e + \lambda_t^e u_c^e(1-\sigma) = \mu_t + u_{cc}^e c^u \frac{\pi(u)}{\pi(u|e)}\frac{\pi(e|e)}{\pi(e)}(g_\lambda^e - 1)\lambda_{t-1}^e, \tag{72}$$

and the FOC with respect to c^u can be rewritten as

$$u_c^u + \lambda_{t-1}^e u_c^u(1-\sigma) = \mu_t - \lambda_t^u u_c^e + \lambda_{t-1}^e u_c^u. \tag{73}$$

With the above two equations in hand, consider the following cases:

   (a) No growth: $g_\lambda^e = g_\lambda^u = 1$. Without growth, λ^e must converge. Equation (71) then implies $\lambda^u u_c^e = \lambda^e u_c^u$. The difference between equation (72) and equation (73) gives

   $$(u_c^e - u_c^u) + \lambda^e(1-\sigma)(u_c^e - u_c^u) = 0,$$

   which implies $u_c^e = u_c^u$, and this contradicts the assumption $c^e > c^u$.

   (b) There is growth: $g_\lambda^e = g_\lambda^u > 1$. The sum of the FOCs with respect to c^e and c^u can be written as

   $$\begin{aligned} & \frac{u_c^e\pi(e) + u_c^u\pi(u)}{\lambda_{t-1}^e} + \frac{\lambda_t^e}{\lambda_{t-1}^e}u_c^e(1-\sigma)\pi(e) + u_c^u(1-\sigma)\pi(u) \\ = \; & \frac{\mu_t}{\lambda_{t-1}^e} - \frac{\lambda_t^u}{\lambda_{t-1}^e}u_c^e\pi(u) + u_c^u\pi(u) + u_{cc}^e c^u\frac{\pi(u)}{\pi(u|e)}\pi(e|e)(g_\lambda^e - 1). \end{aligned}$$

   Since under positive growth, $\lambda_{t-1}^e \to \infty$, the above equation becomes

   $$\begin{aligned} & g_\lambda^e u_c^e(1-\sigma)\pi(e) + u_c^u(1-\sigma)\pi(u) \\ = \; & g_\mu\frac{\mu_{t-1}}{\lambda_{t-1}^e} - g_\lambda^u\frac{\lambda_{t-1}^u}{\lambda_{t-1}^e}u_c^e\pi(u) + u_c^u\pi(u) + u_{cc}^e c^u\frac{\pi(u)}{\pi(u|e)}\pi(e|e)(g_\lambda^e - 1), \end{aligned}$$

which together with (71) implies

$$\begin{aligned}
& g_\lambda^e u_c^e(1-\sigma)\pi(e) + u_c^u(1-\sigma)\pi(u) \\
= \;& g_\mu \frac{\mu_{t-1}}{\lambda_{t-1}^e} + \pi(u) u_c^e \left(\frac{u_c^u}{u_c^e} - g_\lambda^u \frac{\lambda_{t-1}^u}{\lambda_{t-1}^e} \right) + u_{cc}^e c^u \pi(u) \left(\frac{u_c^u}{u_c^e} - g_\lambda^u \frac{\lambda_{t-1}^u}{\lambda_{t-1}^e} \right) \\
= \;& g_\mu \frac{\mu_{t-1}}{\lambda_{t-1}^e} + \pi(u) \left(\frac{u_c^u}{u_c^e} - g_\lambda^u \frac{\lambda_{t-1}^u}{\lambda_{t-1}^e} \right) (u_c^e + u_{cc}^e c^u) \\
> \;& \pi(u) \left(\frac{u_c^u}{u_c^e} - g_\lambda^u \frac{\lambda_{t-1}^u}{\lambda_{t-1}^e} \right) u_c^e \left(1 - \sigma \frac{c^u}{c^e} \right) \\
> \;& \pi(u) \left(u_c^u - g_\lambda^u \frac{\lambda_{t-1}^u}{\lambda_{t-1}^e} u_c^e \right) (1-\sigma),
\end{aligned}$$

where the last two inequalities utilize the fact that (i) $g_\mu \frac{\mu_{t-1}}{\lambda_{t-1}^e} > 0$ and (ii) $\frac{c^u}{c^e} < 1$. Now, considering the parameter value $\sigma \geq 1$, the above inequalities can be simplified to the following two possible relationships:

$$0 < g_\lambda^e u_c^e \pi(e) < -g_\lambda^u \frac{\lambda_t^u}{\lambda_t^e} u_c^e \pi(u) < 0, \text{ if } \sigma > 1,$$

or $0 < 0$ if $\sigma > 1$; and

$$0 > \left(\frac{u_c^u}{u_c^e} - g_\lambda^u \frac{\lambda_{t-1}^u}{\lambda_{t-1}^e} \right) (1-\sigma) = 0, \text{ if } \sigma = 1,$$

or $0 > 0$ if $\sigma = 1$; both are self-contradictory.

We now show that there is indeed a constrained interior Ramsey steady state with a divergent multiplier μ_t under $\sigma < 1$. We prove this by showing that this steady state satisfies all of the Ramsey FOCs:

1. In this constrained interior Ramsey steady state there are eight variables to solve for, including c^e, c^u, n^e, K, a^e, g_λ^e, $\lim_{t\to\infty} \frac{\mu_t}{\lambda_t^e}$, and $\lim_{t\to\infty} \frac{\mu_t}{\lambda_t^u}$. Note that $g_\mu = g_\lambda^u = g_\lambda^e$ and $\lim_{t\to\infty} \frac{\lambda_t^e}{\lambda_t^u}$ is known once we know $\lim_{t\to\infty} \frac{\mu_t}{\lambda_t^u}$ and $\lim_{t\to\infty} \frac{\mu_t}{\lambda_t^e}$.

2. There are eight Ramsey FOCs or constraints that can be used to solve these eight unknown variables in the steady state:

   (a) In the limit, the Ramsey FOCs with respect to $K_{t+1}, n_t^e, a_{t+1}^e, c_t^e$, and c_t^u are given, respectively, by

$$1 = \beta g_\lambda^e (MPK + 1 - \delta),$$

$$v_n^e(1+\gamma) = MPN \lim_{t\to\infty} \frac{\mu_t}{\lambda_t^e},$$

$$\left(\frac{u_c^u}{u_c^e} - g_\lambda^u \lim_{t\to\infty} \frac{\lambda_t^u}{\lambda_t^e}\right) \pi(u|e) = (g_\lambda^e - 1)\pi(e|e),$$

$$u_c^e(1-\sigma) = \lim_{t\to\infty} \frac{\mu_t}{\lambda_t^e} + u_{cc}^e c^u \frac{\pi(u)}{\pi(u|e)} \frac{\pi(e|e)}{\pi(e)} (g_\lambda^e - 1) \frac{1}{g_\lambda^e}, \tag{74}$$

and

$$u_c^e \lim_{t\to\infty} \frac{\lambda_t^u}{\lambda_t^e} + \frac{1}{g_\lambda^e} c^u u_{cc}^u = \lim_{t\to\infty} \frac{\mu_t}{\lambda_t^e}. \tag{75}$$

(b) The resource constraint and the implementability conditions for type-e and type-u individuals are given, respectively, by

$$F(n^e\pi(e), K) - \delta K - c^e\pi(e) - c^u\pi(u) = 0,$$

$$c_t^e - \frac{v_{n,t}^e}{u_{c,t}^e} n_t^e + \beta\left(\pi(e|e) + \frac{u_c^u}{u_c^e}\pi(u|e)\right) a^e - \left[\frac{a^e\pi(e)\pi(e|e)}{\pi(e)}\right] = 0,$$

and

$$c^u = \frac{a^e\pi(e)\pi(u|e)}{\pi(u)}.$$

Therefore, in the above steps we have proved the existence of a constrained interior Ramsey steady state with a divergent μ_t under $\sigma < 1$.

A.6 Proof of Proposition 6 (Ramsey Allocation in Steady-State Welfare Maximization)

Denote μ, $\lambda(h^\kappa)\pi(h^\kappa)$, $\zeta^1(h^\kappa)$, and $\zeta^3(h^\kappa)$ as the Lagrangian multipliers for constraints (23), (24), (25) and (26), respectively. The Ramsey FOCs with respect to $c(h^\kappa)$ and $a(h^\kappa)$ are given, respectively, by

$$c(h^\kappa)^{-\sigma} = \mu + \lambda(h^\kappa), \tag{76}$$

and

$$\lambda(h^\kappa)\pi(h^\kappa)Q = \sum_{h_{-1}^\kappa} \lambda(h_{-1}^\kappa)\pi(h^\kappa)\pi(h_{-1}^\kappa|h^\kappa) - \zeta^1(h^\kappa) - \zeta^3(h^\kappa)g(h^\kappa). \tag{77}$$

The proof that the Ramsey allocation in steady-state welfare maximization must have $Q > \beta$ is done by contradiction. Suppose the Ramsey allocation has FSI with $c(h^\kappa) = c(\boldsymbol{h}_h^\kappa) > 0$, $a(h^\kappa) \geq 0$, $\zeta^1(h^\kappa) = 0$, $Q = \beta < 1$, and $g(h^\kappa) = 0$ for all h^κ. Then, the Ramsey FOC with respect to consumption in equation

(77) implies $\lambda(h^\kappa) = \lambda$ for all h^κ. As a result, equation (77) can be simplified as

$$\pi(h^\kappa)Q = \sum_{h^\kappa_{-1}} \pi(h^\kappa)\pi(h^\kappa_{-1}|h^\kappa),$$

which together with $Q = \beta$ gives $\beta = 1$, a contradiction with $\beta < 1$. Thus, an unconstrained allocation where $c(h^\kappa) = c$ for all h^κ cannot be the solution to the static Ramsey problem that maximizes the steady-state welfare of the competitive equilibrium.