**Time-Inconsistent Optimal Quantity of Debt**

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Time-Inconsistent Optimal Quantity of Debt*

YiLi Chien Yi Wen

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Abstract

A key feature of the infinite-horizon heterogeneous-agents incomplete-markets (Inf-HAIM) framework is that the equilibrium interest rate of public debt lies below the time discount rate (regardless of capital). This happens because of a positive liquidity premium on asset returns due to imperfect risk sharing. This fundamental property of standard Inf-HAIM models, however, implies that the Ramsey planner’s fiscal policy may be time-inconsistent—because the planner has a dominate incentive to issue plenty of debt such that all households are fully self-insured against idiosyncratic risk whenever the interest rate of government borrowing is lower than the household time discount rate. But such a full self-insurance allocation may be infeasible—because to achieve it the optimal quantity of debt may approach infinity or the optimal labor tax rate may approach 100%. This is puzzling from an intuitive perspective because near the point of full self-insurance the marginal gains of increasing debt should be less than the marginal costs of financing the debt under distortionary taxes. We show that this puzzling behavior originates from the assumption that the planner must commit to future plans at time zero. Under such a full commitment, the Ramsey planner opts to exploit the low interest cost of borrowing to front load consumption by sacrificing future consumption in the long run—because future utilities are heavily discounted compared to the inverse of the interest rate on government bonds. We demonstrate our points analytically using a tractable Inf-HAIM model featuring non-linear preferences and a well-defined distribution of household wealth.

JEL Classification: E13; E62; H21; H30
Key Words: Time Inconsistency, Optimal Debt, Ramsey Problem, Incomplete Markets.

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1 Introduction

A signature property of standard infinite-horizon heterogeneous-agents incomplete-markets (Inf-HAIM) models is that the market interest rate is lower than the time discount rate (regardless of capital). This property emerges because precautionary saving motives under imperfect risk sharing entice agents to hold an excess amount of liquid assets even if their rate of return is dominated by the time discount rate, suggesting that a liquidity premium be paid by holding such assets under borrowing constraint and non-diversifiable risk.

This property, however, has important implications for optimal government policies but has not been thoroughly investigated or well understood in the existing literature. One particular implication among them is the existence of an “arbitrage opportunity” for the Ramsey planner—who can borrow cheaply by issuing debt at the low interest rate and rollover the debt indefinitely as long as the interest rate lies below the time discount rate—until the model converges to a Ramsey steady state (if it exists). Such a Ramsey steady state features full self-insurance (FSI) where all agents are no longer borrowing constrained and the interest rate equals the time discount rate. In other words, under borrowing constraints and imperfect risk sharing, a Ramsey planner opts to issue plenty of debt such that asset holders (savers) are satiated with savings and fully self-insured against idiosyncratic risks.

But this Ramsey outcome or intention to amass a sufficiently large stock of government bonds to enable borrowing-constrained agents to achieve an FSI allocation may be not only infeasible but also puzzling. It may be infeasible because the required amount of government debt to achieve an FSI allocation could be infinite, or it may require a labor tax rate close to 100% to finance the colossal amount of public debt. It is also puzzling because intuition tells us that the marginal benefit of reducing consumption risk must decline with an increasing stock of public debt while the cost of financing the debt by distortionary taxes must rise. Therefore, an FSI allocation cannot always be optimal/feasible.

Yet, consistent with the finding of Chien and Wen (2019), this paper illustrates that an FSI allocation is indeed a Ramsey outcome even if (i) the implied steady-state labor tax rate is near 100% in order to support the FSI allocation and (ii) the model’s preferences are standard (i.e., non-linear) and the distribution of wealth is not degenerated (in contrast to

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1By “standard” Inf-HAIM models we mean the class of models similar to that studied by Aiyagari (1994) in which agents are ex ante identical but ex post heterogenous. Therefore, our paper focuses on this class of models and does not consider other popular types of models with ex ante heterogenous agents such as the two-class model of Judd (1985) or the OLG models. However, since our model permits certain degree of ex ante heterogeneity, our analysis is not restricted completely to models with ex ante identical agents.
the model of Chien and Wen (2019)).

The insight obtained from our analysis is that the FSI Ramsey allocation, driven by the gap between the interest rate and the time discount rate, is a consequence of the assumption of full commitment by the Ramsey planner at time zero. If the Ramsey planner has opportunities to readjust its policies, or if its objective is to maximize the steady-state welfare of the competitive equilibrium instead of the dynamic path of welfare at time zero, then an FSI allocation can be shown to be sub-optimal.

This inconsistency in the optimal quantity of public debt between maximizing dynamic welfare and steady-state welfare arises from the fact that the Ramsey planner does not discount the future when using policies to pick the best competitive steady state of the model, in contrast to policies that pick the best dynamic path of the competitive equilibrium. In the former case, the difference between the interest rate and the time discount rate does not matter since nothing is discounted in the steady state, but in the latter case this difference provides an “arbitrage” opportunity for the Ramsey planner to exploit—since the interest cost of borrowing funds to finance debt is lower than the time discount rate, it is thus always beneficial to issue plenty of public debt to improve households’ time-zero welfare by front-loading consumption when the future is discounted at a rate higher than the inverse of the interest rate, even if this implies that the optimal steady-state tax rate is close to 100%.

In other words, if the Ramsey planner does not abide to its original plans made at time zero, or if it has the opportunity to readjust its original plans, it will choose to deviate away from the steady-state FSI allocation. This suggests that the optimal long-run quantity of public debt committed at time zero is not time consistent.

Our contribution in this paper is thus three-fold: First, we prove that when a Ramsey steady state exists and is unique, then it must feature FSI. Second, we show that the optimal quantity of public debt committed at time-zero to support such an FSI allocation in the long run is time-inconsistent, and we conjecture that this feature may be a general property of standard Inf-HAIM models. Third, we provide an analytically tractable model to demonstrate our points.²

Model tractability is crucial. It is well known that analyzing the Ramsey problem in Inf-HAIM models is often technically very challenging because such models are generally non-tractable due to the curse of history-dependent distribution of individuals’ wealth. Even

²In representative-agent models without capital, such as the model of Lucas and Stokey (1983), the issue of time inconsistency does not arise, because in the absence of aggregate uncertainty the equilibrium interest rate always equals the time discount rate. But the time-inconsistency issue arises in Inf-HAIM models because the interest rate lies below the time discount rate.
when the competitive equilibrium has closed-form solutions, the Ramsey problem may still be intractable, let alone when the competitive equilibrium is itself intractable. Without tractability, it is hard to know whether a Ramsey steady state exists or not. Consequently, a general approach in the literature is to appeal to numerical solution techniques under the critical assumption that a Ramsey steady state does exist. But as mentioned above, if the Ramsey planner has a dominate incentive to achieve an FSI allocation, then a Ramsey steady state may not exist if such an allocation can only be supported by an infinite amount of government debt or by a tax rate arbitrarily close to 100%. So the optimal fiscal policies drawn from the analysis under the assumed (instead of proved) Ramsey steady state becomes misleading, since they hinge critically on the validity of such an assumption (see the analyses and examples provided in Chen, Chien, and Yang (2020) and Chien and Wen (2020)).

To overcome the daunting challenge in model tractability, in this paper we design a model in which households face employment-status shocks but the distribution of household wealth is partially degenerated. In such a model parameter restrictions for the existence and uniqueness of a Ramsey steady state can be analytically proven, thanks to the model’s tractability. However, unlike the model of Chien and Wen (2019), we do not rely on the assumption of quasi-linear preferences and the implied degenerated wealth distribution to obtain tractability. Under quasi-linear preferences and idiosyncratic preference shocks, Chien and Wen (2019) show that their Inf-HAIM model has closed-form solutions for the competitive equilibrium, and the Ramsey problem is analytically tractable—thanks to the completely degenerated wealth distribution in their model. They show that under such circumstances a Ramsey steady state not only exists but is also unique and features FSI.

However, since quasi-linear preferences imply that households can elastically supply labor to target any optimal level of household wealth for expected self-insurance, it is not clear whether the FSI Ramsey allocation in the model of Chien and Wen (2019) is a consequence of the quasi-linear preference, the completely degenerated distribution of household wealth under iid idiosyncratic shocks, or something else.

This paper designs an Inf-HAIM model that is more general than the model of Chien and Wen (2019). This model has standard non-linear preferences, a well-defined wealth distribution, and highly persistent idiosyncratic shocks. What makes this model tractable, however, is the assumption of an exogenous risk-sharing technology that allows households to partially pool their wealth at the beginning of each period regardless of their employment history such that there exist only two levels of wealth at any point in time. Consequently, every household belongs to either a high-wealth group or a low-wealth group. Because
idiosyncratic risk is only partially (not completely) shared under this risk-sharing technology, precautionary saving motives are well preserved, the wealth distribution is time-varying, and the competitive-equilibrium interest rate lies below the time discount rate. We show analytically that under certain parameter conditions a Ramsey steady state exists and is unique, and such a Ramsey steady state necessarily features FSI.

Moreover, we also show that the Ramsey planner’s long-run FSI policies are time inconsistent—if the planner has the chance to readjust its policies or can ignore the transitional dynamics of the competitive equilibrium by focusing only on the steady-state welfare of the competitive equilibrium, then an FSI allocation is no longer optimal. In other words, the Ramsey planner has incentives to deviate from the FSI allocation either when it has the chance to readjust its plans or when it targets only the steady-state welfare over the space of competitive equilibria instead of maximizing the entire dynamic path of social welfare at time zero over the space of competitive equilibria.

The rest of the paper is organized as follows. Section 2 sets up the model and defines the competitive equilibrium. Section 3 solves the Ramsey problem using the primal approach, proves the existence and uniqueness of the Ramsey steady state, and provides interpretations on the FSI Ramsey steady-state allocation. Section 4 examines the time-inconsistency problem of the long-run Ramsey policies. Section 5 provides a brief literature review and Section 6 concludes.

2 The Model

Government. Time is discrete and indexed by \( t = 0, 1, 2, \ldots, \infty \). In each period \( t \), the government issues risk-free bonds \( B_{t+1} \) and receives labor-income tax revenues under a flat tax rate \( \tau_t \). Denote the price of bonds in period \( t \) by \( Q_{t+1} \), which pays one unit of consumption goods in period \( t+1 \), then the risk-free interest rate is given by \( r_{t+1} = Q_{t+1}^{-1} \). The flow government budget constraint is then given by

\[
\tau_t N_t + Q_{t+1} B_{t+1} \geq B_t
\]

for all \( t \geq 0 \), where the initial level of government bonds \( B_0 \) is exogenously given. For simplicity, the government spending is assumed to be zero.

Firm. A representative firm produces output according to a linear production technology, \( Y_t = N_t \), where \( N_t \) is the aggregate labor input. The firm hires labor from households by
paying a competitive real wage rate \( w_t \). Perfect competition implies

\[
w_t = \frac{\partial Y_t}{\partial N_t} = 1.
\]

**Households.** There is a unit measure of *ex ante* identical households, who are subject to idiosyncratic employment-status shocks in each period, denoted by \( \theta_t \in \{e, u\} \). The shocks are identically and independently distributed (iid) across individuals and follows a Markov process. If \( \theta_t = e \), then individuals can work and receive their labor income; otherwise, if \( \theta_t = u \), individuals cannot work and have no labor income.

At time zero, all households agree to participate in a risk-sharing contract, which specifies that in the beginning of every period \( t \) after receiving an employment shock, households who have the same (un)employment status should pool their asset holdings and divide the aggregate wealth equally among them. As a result, in every period \( t \) each household’s initial wealth takes only two possible values that depends only on the current state of idiosyncratic shock and are independent of the household’s past history of employment status.\(^3\)

Let \( \pi(e) \) and \( \pi(u) \) denote the unconditional probability of employment and unemployment shocks, respectively. By the law of large numbers, these probability measures also represent the share of employed or unemployed households in total population, respectively. For simplicity, we assume that \( \pi(e) \) and \( \pi(u) \) also represent the initial period’s share of employed and unemployed households. Denote the level of consumption, labor supply, and savings for the employed individual households in period \( t \) as \( c_e^t, n_e^t, \) and \( a_{e+1}^t/\pi(e) \), respectively; and similarly, let \( c_u^t \) and \( a_{u+1}^t/\pi(u) \) denote consumption and savings for the unemployed individual households.

In such a setup, the employed households’ initial asset holdings in the beginning of period \( t \geq 1 \) can be written as

\[
a_e^t = a_{e+1}^t/\pi(e) + a_u^t/\pi(e|u),
\]

where \( \pi(e|e) \) and \( \pi(e|u) \) denote the transition probability from employed and unemployed states to an employed state, respectively. Similarly, the initial asset holdings for unemployed households are

\[
a_u^t = a_{u+1}^t/\pi(u) + a_u^t/\pi(u|u),
\]

where \( \pi(u|e) \) and \( \pi(u|u) \) denote the transition probability from employed and unemployed states to an

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\(^3\)This exogenously specified risk-sharing technology simplifies the model dramatically and renders it analytically tractable. This tactic for analytical tractability is an extension of that adopted by Heathcote and Perri (2018). Notice that if our exogenous risk-sharing technology is perfect such that all households have identical wealth in the beginning of each period, then the barebone of our model reduces to that of Heathcote and Perri (2018). On the other hand, if we assume that the wealth-sharing technology is such that households with the same history of idiosyncratic shocks up to a finite \( k \) periods from today can pool their asset holdings perfectly, then the model should behave more and more like a standard Inf-HAIM model (as in Aiyagari (1994)) as \( k \) goes to infinity (see Grand and Ragot (2019)).
unemployed state, respectively. Note that the unconditional probabilities 
\( \pi(e) = \frac{\pi(e|u)}{\pi(e|u) + \pi(u|e)} \)
and 
\( \pi(u) = \frac{\pi(u|e)}{\pi(e|u) + \pi(u|e)} \)
satisfy \( \pi(e) + \pi(u) = 1 \).

Hence, for \( t \geq 1 \), the household budget constraints for the employed and unemployed households are given, respectively, by

\[
\frac{a^e_t \pi(e|e) + a^u_t \pi(e|u)}{\pi(e)} + \hat{\omega}_t n^e_t - c^e_t - Q_{t+1} \frac{a^e_{t+1}}{\pi(e)} \geq 0,
\]

\[
\frac{a^u_t \pi(u|e) + a^u_t \pi(u|u)}{\pi(u)} - c^u_t - Q_{t+1} \frac{a^u_{t+1}}{\pi(u)} \geq 0,
\]

where \( \hat{\omega}_t \equiv (1 - \tau_t) w_t \) is the after-tax wage rate. For the initial period \( (t = 0) \), the household budget constraints for the employed and unemployed households are given, respectively, by

\[
\frac{a^e_0 \pi(e)}{\pi(e)} + \hat{\omega}_0 n^e_0 - c^e_0 - Q_1 \frac{a^e_1}{\pi(e)} \geq 0,
\]

\[
\frac{a^u_0 \pi(u)}{\pi(u)} - c^u_0 - Q_1 \frac{a^u_1}{\pi(u)} \geq 0,
\]

where \( \frac{a^e_0}{\pi(e)} \) and \( \frac{a^u_0}{\pi(u)} \) are the initial asset holdings for the employed and unemployed households, respectively. Note that \( a^e_0 + a^u_0 = B_0 \). In addition, households are subject to borrowing constraints:

\[
a^e_{t+1} \geq 0,
\]

\[
a^u_{t+1} \geq 0.
\]

The lifetime utility of households is given by

\[
U = \sum_{t=0}^{\infty} \beta^t \{ [u(c^e_t) - v(n^e_t)] \pi(e) + u(c^u_t)\pi(u) \},
\]

where \( \beta \in (0, 1) \) is the time discounting factor. Note that the labor supply for unemployed households is zero.

**Household Problem.** Given market prices, \( \{Q_{t+1}, \hat{\omega}_t\}_{t=0}^{\infty} \), and initial asset holdings, \( \{a^e_0, a^u_0\} \), households choose a sequence of \( \{c^e_t, c^u_t, n^e_t, a^e_{t+1}, a^u_{t+1}\}_{t=0}^{\infty} \) to maximize (8) subject to (2)-(7). Let \( \beta^t \xi^e_t \pi(e), \beta^t \xi^u_t \pi(u), \xi^e_0 \pi(e), \xi^u_0 \pi(u), \beta^t \kappa^e_t \pi(e) \) and \( \beta^t \kappa^u_t \pi(u) \) be the Lagrangian multipliers associated with constraints (2)-(7), respectively; the FOCs with respect to \( c^e_t, c^u_t, \)
\( a_{t+1}^e, a_{t+1}^u, \) and \( n_t^e \) are given, respectively, by

\[
u_{c,t}^e = \xi_t^e, \tag{9}\]
\[
u_{c,t}^u = \xi_t^u, \tag{10}\]

\[
Q_{t+1}\xi_t^e = \beta \left[ \xi_{t+1}^e \pi(e|e) + \xi_{t+1}^u \pi(u|e) \right] + \kappa_t^e, \tag{11}\]
\[
Q_{t+1}\xi_t^u = \beta \left[ \xi_{t+1}^e \pi(e|u) + \xi_{t+1}^u \pi(u|u) \right] + \kappa_t^u, \tag{12}\]

\[
v_{n,t}^e = \xi_t^e \tilde{w}_t; \tag{13}\]

where \( \nu_{c,t}^e \) and \( \nu_{c,t}^u \) denote the marginal utility of consumption for the employed and unemployed households in period \( t \); similarly, \( v_{n,t}^e \) denotes the marginal disutility of labor for the employed households in period \( t \).

### 2.1 Competitive Equilibrium

**Definition 1.** Given the initial asset holdings \( (a_0^e, a_0^u) \), initial government bonds \( B_0 = a_0^e + a_0^u \), \( a_0^u > 0 \), and the sequence of policies \( \{\tau_t, B_{t+1}\}_{t=0}^\infty \), a competitive equilibrium is defined as the sequences of prices \( \{w_t, Q_{t+1}\}_{t=0}^\infty \), aggregate allocations \( \{C_t, N_t\}_{t=0}^\infty \), and individual allocation plans \( \{c_t^e, c_t^u, n_t^e, a_{t+1}^e, a_{t+1}^u\}_{t=0}^\infty \), such that

1. \( \{c_t^e, c_t^u, n_t^e, a_{t+1}^e, a_{t+1}^u\}_{t=0}^\infty \) solves the household problem;

2. \( \{N_t\} \) solves the representative firm’s problem;

3. the government flow budget constraint holds:

\[
\tau_t N_t + Q_{t+1} B_{t+1} \geq B_t; \tag{14}\]

and

4. all markets clear for \( t \geq 0 \):

\[
B_{t+1} = a_{t+1}^e + a_{t+1}^u, \tag{15}\]
\[
N_t = C_t, \]
\[
N_t = n_t^e \pi(e), \]
\[
C_t = c_t^e \pi(e) + c_t^u \pi(u). \]
Proposition 1. The competitive equilibrium has the following properties:

1. For all \( t \geq 0 \), the asset holdings of the employed households satisfy \( a_{t+1}^e > 0 \) (implying that the Lagrangian multiplier \( \kappa_t^e = 0 \)), provided that the aggregate supply of government bonds \( B_t > 0 \).

2. The intertemporal price \( Q_{t+1} \) is determined by

\[
Q_{t+1} = \beta \left[ \frac{u_{c,t+1}^e}{u_{c,t}^e} \pi(c|e) + \frac{u_{u,t+1}^e}{u_{c,t}^e} \pi(u|e) \right].
\] (16)

3. In the steady state, if the asset holdings of the unemployed households are such that \( \kappa_u = 0 \), then the competitive equilibrium features \( c^e = c^u \) and \( Q = \beta \) (or \( r = \beta^{-1} \)).

Proof. See Appendix A.1. \( \square \)

Proposition 1 suggests that if the asset holdings \( a_t^e \) and \( a_t^u \) are both sufficiently large such that all households’ borrowing constraints are slack, then they can obtain the same level of consumption regardless of their employment status. We refer to this allocation as a full self-insurance (FSI) allocation. Obviously, by the asset market-clearing condition (15), FSI is feasible if the supply of government bonds is sufficiently high. In addition, if FSI is achieved in the steady state, then the risk-free interest rate \( r \) (\( \equiv 1/Q \)) is equal to the time discount rate \( \beta^{-1} \). Otherwise it must be true that \( r < \beta^{-1} \).

To make our Ramsey problem interesting, we assume in the rest of the paper that the initial bond supply \( B_0 \) and the initial distribution of household wealth \( \{a_0^e, a_0^u\} \) are such that the competitive equilibrium does not feature FSI. Namely, without future changes on the initial bond position \( B_0 \), the competitive equilibrium features \( c^e > c^u \) and \( r < \beta^{-1} \). The central question of our analysis is to ask whether the Ramsey planner is willing to increase bond supply to achieve FSI and why.

3 Solving the Ramsey Problem

Note that the competitive equilibrium defined above is in general a function of the path of government policies \( \{\tau_t, B_{t+1}\}_{t=0}^{\infty} \). Namely, each different path of government policies corresponds to a different competitive equilibrium. The Ramsey problem is to select a particular path of government policies such that the corresponding competitive equilibrium yields the maximum social welfare.
We use the primal approach to solve the Ramsey problem. Under the primal approach, we first substitute out all market prices and policy variables by using a subset of the competitive equilibrium’s FOC conditions, and then choose the path of allocation, \( \{c_t^e, c_t^u, n_t, a_{t+1}^e, a_{t+1}^u\}_{t=0}^\infty \), to maximize social welfare subject to the rest of the equilibrium conditions. The solution under such a primal approach is called a Ramsey allocation (outcome) or a Ramsey plan.

### 3.1 Conditions to Support a Competitive Equilibrium

To ensure that a Ramsey plan constitutes a competitive equilibrium, we must show first that all possible allocations in the choice set of the Ramsey planner, \( \{c_t^e, c_t^u, n_t, a_{t+1}^e, a_{t+1}^u\}_{t=0}^\infty \) (after substituting out all market prices and policy variables but before solving the Ramsey maximization problem), constitute a competitive equilibrium. The following proposition states the conditions that any constructed Ramsey allocation must satisfy in order to constitute a competitive equilibrium.

**Proposition 2.** Given the initial asset holdings \((a_0^e, a_0^u)\) and initial government bonds \(B_0 = a_0^e + a_0^u > 0\), the sequence of allocations \(\{c_t^e, c_t^u, n_t, a_{t+1}^e, a_{t+1}^u\}_{t=0}^\infty\) can be supported as a competitive equilibrium if and only if they satisfy the following conditions:

1. **Resource constraints:**
   \[
   n_t^e \pi(e) - c_t^e \pi(e) - c_t^u \pi(u) \geq 0, \quad \forall t \geq 0. \tag{17}
   \]

2. **Implementability conditions:** for \(t = 0\),
   \[
   u_{c,0}^e c_0^e \pi(e) - v_{n,0}^e n_0 \pi(e) + Q_1 u_{c,0}^e a_1^e - u_{c,0}^e a_0^e = 0, \tag{18}
   \]
   \[
   u_{c,0}^e c_0^u \pi(u) + Q_1 u_{c,0}^u a_1^u - u_{c,0}^e a_0^u = 0, \tag{19}
   \]
   and, for \(t \geq 1\),
   \[
   u_{c,t}^e c_t^e \pi(e) - v_{n,t}^e n_t \pi(e) + Q_{t+1} u_{c,t}^e a_{t+1}^e - u_{c,t}^e [a_t^e \pi(e|e) + a_t^u \pi(e|u)] = 0, \tag{20}
   \]
   \[
   u_{c,t}^e c_t^u \pi(u) + Q_{t+1} u_{c,t}^u a_{t+1}^u - u_{c,t}^e [a_t^e \pi(u|e) + a_t^u \pi(u|u)] = 0, \tag{21}
   \]

where
   \[
   Q_{t+1} u_{c,t+1}^e = \beta [u_{c,t+1}^e \pi(e|e) + u_{c,t+1}^u \pi(u|e)].
   \]
3. Borrowing constraints and complementary slackness conditions: $\forall t \geq 0,$

$$g_t(c^e_t, c^u_t, c^e_{t+1}, c^u_{t+1})a^u_{t+1} = 0,$$

$$a^u_{t+1} \geq 0,$$

$$g_t(c^e_t, c^u_t, c^e_{t+1}, c^u_{t+1}) \geq 0,$$

where the function $g_t$ is defined as

$$g_t(c^e_t, c^u_t, c^e_{t+1}, c^u_{t+1}) \equiv \frac{u^u_{c,t}}{u^e_{c,t+1} \pi(e|u) + u^u_{c,t+1} \pi(u|u)} - \frac{u^e_{c,t}}{u^e_{c,t+1} \pi(e|e) + u^u_{c,t+1} \pi(u|e)}.$$

**Proof.** See Appendix A.2. \qed

### 3.2 Ramsey Allocation

The Ramsey problem under the primal approach can then be represented by the following maximization problem:

$$\max_{\{c^e_t, c^u_t, c^e_{t+1}, c^u_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left\{ [u(c^e_t) - v(n^e_t)] \pi(e) + [u(c^u_t)] \pi(u) \right\}$$

subject to constraints (17)-(22) listed in Proposition 2. To facilitate our analysis, we make the following assumptions on the utility function’s relative risk aversion parameters $\{\gamma_c, \gamma_n\}$:

$$\gamma_c \equiv \frac{u_{cc}}{u_c} < 0,$$

$$\gamma_n \equiv \frac{v_{nn}}{v_n} > 0,$$

so the utility function exhibits constant relative risk aversion (CRRA).

Before presenting the Ramsey outcome, we first define the Ramsey steady state in our economy:

**Definition 2.** Given $\{B_0, a^e_0, a^u_0\}$, a Ramsey steady state is a long-run Ramsey allocation where aggregate and individual variables $\{N_t, C_t, B_{t+1}, c^e_t, c^u_t, n^e_t\}$ all converge to finitely positive non-zero constants.

**Proposition 3.** Given $\pi(u|e) > 0$ and a sufficiently large $\beta$, there exists a Ramsey steady state with the following properties:
1. The allocation features FSI with $0 = a^u < a^e$, $c^e = c^u$, and $r = \beta^{-1}$.

2. The optimal tax rate is given by $\tau = (1 - \beta) \frac{\pi(u)}{\pi(u|e)} \in (0, 1)$.

3. The Ramsey steady state is unique if $(1 + \gamma_c) \leq 0$; or alternatively, there does not exist any Ramsey steady state with inadequate self insurance such that $c^e > c^u$ and $r < \beta^{-1}$, if the CRRA parameter satisfies $(1 + \gamma_c) \leq 0$.

**Proof.** See Appendix A.3.

### 3.3 Interpretations

The most surprising result in Proposition 3 is that the Ramsey planner opts to achieve FSI even at the cost of a very high long-run tax rate—notice that the optimal tax rate $\tau$ is bounded above from 1 if and only if $\beta$ is sufficiently large for any given probability distribution. So $\tau = (1 - \beta) \frac{\pi(u)}{\pi(u|e)}$ can become very close to 100% if $\beta$ is small. Also, $\tau$ is increasing in the probability of unemployment and in the persistence of the shocks, suggesting that more government debt is needed to provide FSI when the risk of unemployment is high or the shock is more persistent.

In other words, to achieve FSI by equalizing the consumption level across the two types of households, the optimal long-run tax rate may have to approach 100% in the Ramsey steady state. A high steady-state tax rate implies that the steady-state consumption is very low—because household labor income is almost completely taxed away when $\tau$ is near 100%.4

Figure 1 shows the FSI steady-state policies and other endogenous variables in the Ramsey allocation as we reduce the time discount factor $\beta$. It shows that as the value of $\beta$ decreases (from left to right), the implied optimal tax rate $\tau$ (top left panel) rises from 0% toward nearly 100%; consequently, the aggregate consumption (bottom left panel) declines toward zero. This suggests that when households are very impatient, and for given probability distribution of the shocks, in order to achieve FSI the Ramsey planner opts to impose a very high tax rate in the Ramsey steady state such that the level of long-run consumption and hours worked are extremely low. A high level of tax rate and a low level of labor supply also implies that total tax revenues will eventually be zero as well (a maximum is reached around $\beta = 0.925$, see the bottom right panel); consequently, the optimal quantity of debt to support FSI will also decrease to zero when $\beta$ becomes sufficiently small (top right panel).

---

4The fact that the optimal tax rate depends entirely on the discounting factor and employment probability
How can such steady-state FSI allocations be optimal? Common sense seems to tell us that the marginal benefit of reducing consumption risk to achieve FSI must decline with an increasing stock of public debt, while the cost of financing the debt under distortionary taxes must be rising. Therefore, an FSI Ramsey allocation seems puzzling.

The fundamental reason for such a “counter-intuitive” result is as follows. Since $\beta$ is less than one and the market interest rate is lower than the time discount rate ($r < \beta^{-1}$), the Ramsey planner opts to increase (front load) household consumption in the short run and trade off with a low consumption in the long run. Because the future is heavily discounted compared to the interest rate, in the short run the labor tax rate can be even negative and the level of public debt can be even higher than its steady-state value so as to stimulate

Notes: Utility functions are assumed to be $u(c) = \frac{1}{1-\sigma}c^{1-\sigma}$ and $v(n) = \frac{1}{\gamma}n^{\gamma}$, where $\gamma = \sigma = 2$. Parameter values are $\pi(e|e) = \pi(u|u) = 0.95$, $\pi(u|e) = \pi(e|u) = 0.05$ and $\pi(e) = \pi(u) = 0.5$. 

but not on the curvature of the utility function could be a consequence of the ad hoc wealth-redistribution technology.
work efforts and enhance risk sharing across employment states. Consequently, the Ramsey planner has to finance the “sky-rocketing” debt level incurred during the transition period by increasing the steady-state tax rate even if this implies low consumption in the long run.

In what follows, we confirm our intuition by numerical simulations of the model, which not only substantiate our theoretical results but also illustrate the pattern of optimal transition paths of the Ramsey allocation. Such numerical analyses are valid because the Ramsey steady state is proved to exist and is unique under our parameter specifications.

**Parameter Values.** First, we assume a power utility function with \( u(c) = \frac{1}{1-\sigma} c^{1-\sigma} \) and \( v(n) = \frac{1}{\gamma} n^\gamma \), where \( \gamma = \sigma = 2 \). Second, we set \( \beta = 0.96 \) and the transition probability matrix of (un)employment shocks as

\[
\pi = \begin{bmatrix}
\pi(u|u) & \pi(e|u) \\
\pi(u|e) & \pi(e|e)
\end{bmatrix} = \begin{bmatrix}
0.95 & 0.05 \\
0.05 & 0.95
\end{bmatrix},
\]

which implies that the unconditional probability of employment and unemployment is \( \pi(e) = \pi(u) = 0.5 \). Notice that the probability matrix implies that the shocks are highly persistent.\(^5\)

**Ramsey Transition Paths.** In Figure 2, we set the initial debt level \( B_0 \) to 50% of its Ramsey steady-state value. The initial values of the individual household wealth are then given by \( a_0^e = B_0 \pi(e|e) \) and \( a_0^u = B_0 \pi(u|e) \). These initial conditions are consistent with the requirement that the competitive equilibrium does not feature FSI.

The path of government debt in the figure (panel [2,1]) shows a jump in the second period and over-shoots its long-run steady state, and then it gradually decreases toward the Ramsey steady state. However, when translating the debt level into the debt-to-output ratio (panel [3,2]), it implies a very high initial level of debt-to-output ratio of 692%; then this ratio jumps even higher, up to 852% in period 1, and then keeps increasing over time toward its steady-state level of 1000%. Such an enormously high debt-to-GDP (gross domestic product) ratio indicates strong incentives of the Ramsey planner for providing self insurance to the households when the shocks are highly persistent.

Meanwhile, the optimal tax rate (panel [2,2]) jumps up in the first period to a very high value of \( \tau_0 = 61\% \), but drops immediately down to a very negative value of \( \tau_1 = -60.5\% \) in the second period and then gradually increases toward a long-run rate of \( \tau = 40\% \).

\(^5\)Since our goal is not to simulate a realistic real-world economy, we do not intend to calibrate the model parameters to match real-world data. Instead, these numerical analyses are meant to demonstrate the mechanisms behind Proposition 3 and reveal how the economy transits to an FSI Ramsey steady state with a potentially very high tax rate.
Meanwhile, the gross interest rate \( r = Q^{-1} \) (panel [3,1]) starts very high at 237\%, jumps down to about 93\% (implying a negative net interest rate of \(-7.24\%\)) and then gradually increases over time to approaches the time discount rate \( \beta^{-1} = 1.04 \) in the long run.

The path of aggregate consumption (panel [1,1]) shows a clear pattern of front loading by jumping up sharply in the second period (period 1) and then gradually decreasing back to its long-run steady state. The reason that aggregate consumption does not jump up immediately in the first period to its maximum level is because that the level of initial bond supply is fixed and only the newly issued bonds can exhibit a sharp jump in the next period; consequently, to balance the government budget the tax rate is unable to decline too much in the first period, so we see a sharp increase in aggregate consumption only in the second period, and such a high level of consumption above its long-run value is supported by a negative tax rate in the second period.

The individual consumption levels for the employed and unemployed households also show big front-loading behavior (panel [1,2]). However, the panel shows a tremendous degree of consumption inequality during the initial phase of the front-loading period, with employed households consuming 47\% and 26\% more in the first two periods than the unemployed households, respectively, despite the existence of a wealth redistribution technology for risk sharing. However, the consumption inequality shrinks rapidly along the transition and eventually disappears completely in the long run when FSI is achieved.

\[
4 \quad \text{Time Inconsistency}
\]

The key reason for the peculiar behavior of the Ramsey steady-state allocation illustrated in Figure 1—namely, the extremely high tax rate and low consumption level when \( \beta \) is relatively small—is that the Ramsey policies are time inconsistent. In this section we address the issue of time inconsistency in two ways: First, we numerically simulate the transitional Ramsey dynamics by setting the initial allocation (the total bond supply and asset distribution) to their respective Ramsey steady-state values. Second, we re-solve the Ramsey problem by maximizing the steady-state welfare of the competitive equilibrium instead of the time-zero dynamic welfare of the competitive equilibrium.
4.1 Numerical Evidence

**Scenario 1.** Since we know where the Ramsey steady state is, in our numerical simulation we can set the initial bond supply and asset distribution to be equal to their respective Ramsey steady-state values. The choice of such initial values is deliberate and not innocuous: If the Ramsey policy is time consistent, then given that the initial allocation is already in the Ramsey steady state, the policy should be to let the initial allocation remain where it is forever without any policy changes. Conversely, if the Ramsey planner opts to re-adjust its policies temporarily and converge back to the same steady state in the long run, it is then a clear sign of time inconsistency in the Ramsey plan.
Figure 3 plots the transitional Ramsey path under the benchmark parameter values and the initial conditions featuring a Ramsey steady state allocation. Notice in the figure that the Ramsey planner “dislikes” the initial Ramsey steady state allocation and opts to change its policies in the following manner: (i) In the first two periods the optimal tax rate (panel [2,2]) drops down from its long-run value sharply to $-18\%$ and $-26\%$, respectively; it then rises gradually over time back to its original steady-state value of $50\%$. (ii) In the meantime, the level of public debt (panel [2,1]) increases from its initial value of 6.7 to a level of 7.1 and then gradually declines back to the original value of 6.7 in the long run. Also, because labor and total output have increased under the negative tax rate (panel [1,1]), the public debt-to-GDP ratio (panel [3,2]) drops in the beginning by about 13% to 867%, and then converges back gradually to the original long-run value of 1000% over time. The higher-than-steady-state values of aggregate consumption and individual consumption in the top row of the figure indicate once again the policy maker’s front-loading behavior because the Ramsey planner seizes the opportunity of policy change to boost consumption by reducing the tax rate. The long period of excessively low interest rates below the time discount rate (panel [3,1]) also entices the Ramsey planner to further boost bond supply above its steady-state value. However, because total output increases relatively more than the stock of public debt, the debt-to-GDP ratio falls below its steady-state value during the entire transition period (panel [3,3]).

Such a significant but transitory deviation from the initial Ramsey steady state via readjustment of policies clearly indicates that the Ramsey policy is time inconsistent—if there is any chance to readjust its steady-state policies and break away from its time-zero commitment, the Ramsey planner will do so in such a way that the original long-run policies appear to be “incorrect” or too aggressive, which discourages household work efforts and consumption. Therefore, the Ramsey planner opts to relax such policies sharply in the beginning (except for $B_0$) so as to stimulate work effort and raise consumption by lowering the tax rate. Since consumption for the employed and unemployed households does not rise in the same proportion, the original FSI allocation is broken so that more public debt is needed to improve self insurance. But since such a change or readjustment in policies is only a once-and-forever action and the planner knows that it must commit to the new policies forever, the Ramsey allocation will eventually converge back to the same initial Ramsey steady state as if no changes have taken place in the long run. In other words, the readjustment of policies is not a permanent deviation.

Scenario 2. Our next experiment is to see what happens to the Ramsey steady state
and the time-inconsistency policy if we reduce the parameter value of $\beta$ from 0.96 to 0.92, such that households become less patient. As before, we assume that the initial bond supply and distribution of household wealth are already in the Ramsey steady state.

Figure 4 shows that the transition is different from that in Figure 3. First, as in Figure 3, the Ramsey planner clearly dislikes its initial allocation and opts to readjust its policies by sharply reducing its optimal tax rate downward to $-0.6$ and increasing its new bond supply to a level of 6.5 (about 812% of GDP).

Second, the steady-state level of public debt is much lower than before (4.64 v.s 6.69) and the steady-state tax rate is much higher than before ($\tau = 80\%$ v.s 40%). Consequently,
Figure 4: Transition Path with Low $\beta$

Notes: Transition paths in the Ramsey economy (solid lines) and their corresponding steady states (dashed lines).

The new steady-state levels of work effort and consumption are about 30% lower than before, thanks to the higher steady-state tax rate to discourage work and consumption.

The reason is that the Ramsey planner opts to be more aggressive in front-loading consumption in the short run, knowing that the value of $\beta$ is smaller and the interest rate tends to be much lower than the time discount rate. Hence, to pay for the short-run subsidies and high debt costs, the new steady-state policy appears too tight and the Ramsey planner will “regret” it and opt to relax it when it gets the chance, so the planner deviates from its steady-state policies more sharply than in Figure 3. Consequently, in period 1 consumption jumps up by more than 120% above the steady-state value for the employed households and
more than 42% for the unemployed households. This enormously high level of consumption increase in the early phase of the transition is supported by a high level of work effort of the employed households (89% higher than its steady state value), thanks to the very negative labor-income tax rate of $\tau_t = -111\%$ in the short run. In the meantime, the debt level rises by 42% above its steady state to support risk-sharing between the employed and unemployed household groups. All these short-run stimulating policies to front-load consumption are paid for by the very high tax rate ($\tau = 80\%$) in the long run after FSI is achieved.

**Scenario 3.** Notice that in the previous episodes, the probability matrix in equation (26) features a high persistence of shocks such that risk-sharing across employed and unemployed households is quite difficult—since the wealth redistribution technology is very imperfect when shocks are very persistent. This explains why the optimal debt-to-GDP ratio needs to be very high to support FSI allocation in the previous examples.

Now we consider another experiment where the unemployment shock becomes much less persistent with $\pi(u|u) = 0.65$, which implies $\pi(e|u) = 0.35$ and $\pi(u) = 0.125$.

The transition dynamics are shown in Figure 5. Notice the dramatic difference from the previous cases: First, the transition period is extremely short, starting from the initial Ramsey steady-state allocation, the transition (deviation) under policy-readjustment lasts only about 15 periods in this case in converging back to the same Ramsey steady state, as opposed to several hundred periods in the previous scenarios. Second, the steady-state level of tax rate $\tau$ is now very small—only about 10% as opposed to an earlier 80%, and the steady-state bond supply is also much lower than before—about 50% lower than before—since it requires less amount of bonds to support FSI when the shocks are less persistent and the consumption inequality is less severe. Consequently, the steady-state work effort is much higher than before ($N = 0.9235$ v.s 0.6694), which implies that the FSI steady-state consumption for both types of households is now also much higher than before.

Again, once getting a chance the Ramsey planner “regrets” its steady-state policies and opts to readjust the tax rate downward initially. Nonetheless, despite the time-inconsistency problem, the magnitude of the readjustment is very small compared with the previous cases. The reason is that under iid shocks or shocks close to the iid process, the wealth redistribution technology provides a very good channel of risk sharing and self insurance, such that the need (or room) for the Ramsey planner to use policies to achieve FSI allocation is much smaller than the cases with highly persistent shocks. Therefore, the need of readjustment is also smaller. It is thus expected that when the shocks are truly iid, the Ramsey planner will
Figure 5: Transition Path with Lower Unemployed Households

Notes: Transition paths in the Ramsey economy (solid lines) and their corresponding steady states (dashed lines).

not readjust its policies if the initial condition is already in the Ramsey steady state.\(^6\)

**Scenario 4.** Our last experiment sets \(\beta = 0.999\), which is very close to one. As anticipated, Figure 6 shows that the incentives for front-loading consumption are much reduced now when households are extremely patient, such that the Ramsey planner opts to keep the original steady-state allocation virtually intact with only small changes in policies. For example, the maximum readjustment of bond position is only about 0.06% relative to its steady state, and the readjustment in labor tax is only about 15 basis points. This high pa-

\(^6\)When the shocks are iid, we have \(\pi(e|u) = \pi(u|e) = 0.5\), so the distribution of household wealth is completely degenerated. Consequently, starting from the Ramsey steady state, the planner will not readjust its policies at all.

20
rameter value of $\beta$ also implies that the gap between the interest rate and the time discount rate is quite small for the Ramsey planner to take advantage of (the difference is only 0.2 basis points), resulting in a much lower steady-state level of public debt (the steady-state level of debt is now about 2.2) and a much lower steady-state tax rate ($\tau = 1\%$), and such long-run policies are good enough to support FSI allocation in the long run.

Figure 6: Transition Path with $\beta$ close to 1

Notes: Transition paths in the Ramsey economy (solid lines) and their corresponding steady states (dashed lines).

4.2 Steady-State Welfare Analysis

The above analysis shows that the Ramsey planner has dominate incentives to pursue FSI allocation even at the cost of levying extremely high distortionary tax rates. We asked
ourselves why the Ramsey planner must pursue an FSI allocation when the marginal benefit of increasing debt is diminishing while the marginal cost of distortionary future taxes is rising.

Our answer provided above was that as long as $\beta < 1$ and $r < \beta^{-1}$, the marginal social cost of borrowing is strictly less than the marginal social benefit of providing self-insurance. Therefore, the Ramsey planner has incentives to increase bond supply to pursue FSI allocation and front-load consumption even if this implies a close to 100% tax rate in the long run to finance the sky-rocketing public debt-to-GDP ratio.

We also pointed out in the above analysis that such a policy is time-inconsistent—that is, it is optimal only ex ante, but not ex post. Our numerical analysis confirmed this conjecture by showing that whenever the planner has the chance to readjust its steady-state policies even if the economy is already in the Ramsey steady state, it will do so, albeit only temporarily.

To further support our argument that the Ramsey policies are time-inconsistent, here we conduct a different kind of analysis by supposing that the Ramsey planner maximizes only the steady-state welfare of the competitive equilibrium instead of the time-zero present value of the dynamic path of social welfare. We will see how the Ramsey planner designs its optimal debt and tax policies in this situation.

To maximize the steady-state welfare of the economy, the Ramsey problem under the primal approach becomes:

$$\max_{\{c^e, c^u, n^e, a^e, a^u\}} \left[ u(c^e) - v(n^e) \right] \pi(e) + u(c^u)\pi(u)$$

subject to

$$n^e\pi(e) - c^e\pi(e) - c^u\pi(u) \geq 0, \quad (27)$$

$$u^e_c c^e\pi(e) - v^e_n n^e\pi(e) + a^e\beta(u^e_c\pi(e|e) + u^u_d\pi(u|e)) - u^e_c(a^e\pi(e|e) + a^u\pi(e|u)) = 0, \quad (28)$$

$$u^e_u c^u\pi(u) + a^u\beta(u^e_u\pi(e|e) + u^u_c\pi(u|e)) - u^e_c(a^e\pi(u|e) + a^u\pi(u|u)) = 0, \quad (29)$$

$$a^u \geq 0, \quad (30)$$

$$g(c^e, c^u) \equiv \pi(e|e) + \frac{u^u_c}{u^e_c} \pi(u|e) - \frac{u^e_c}{u^u_c} \pi(e|u) - \pi(u|u) \geq 0, \quad (31)$$

$$a^u g(c^e, c^u) = 0. \quad (32)$$

Note that this Ramsey problem features no dynamics and that there is no dynamic consideration for the Ramsey planner. In other words, the “future” is no longer “discounted”
compared to the “present.” We prove that the FSI allocation is not optimal, as shown in the following Proposition:

**Proposition 4.** If the Ramsey planner considers only the steady-state welfare of the competitive equilibrium, then an FSI allocation is not optimal. Namely, it is optimal to set $c^e > c^u$ and to have the borrowing constraint of the unemployed households to be strictly binding: $a^u = 0$.

*Proof.* See Appendix A.4.

The result in Proposition 4 is intuitive. By maximizing only the steady-state welfare of the competitive economy, the Ramsey planner no longer has the incentive to exploit the difference between the interest rate and the time discount rate since the time discount rate is no longer relevant in maximizing the steady-state welfare. Consequently, the issue of front-loading consumption also becomes irrelevant. In this case, the Ramsey planner does not choose FSI allocation by equalizing consumption across the employed and unemployed households—because the cost of doing so by issuing too much debt is too high at the margin when there is no time discounting. In other words, from the competitive equilibrium’s steady-state welfare point of view, the marginal benefit of achieving FSI by increasing public debt is at some point dominated by the marginal cost of distortionary taxation, such that the Ramsey planner will issue bonds only to a certain level but not all the way to achieve FSI.

5 A Brief Literature Review

Aiyagari and McGrattan (1998) show that the optimal quantity of public debt is strictly positive under incomplete financial markets with idiosyncratic risk and borrowing constraints because of the shortage of private liquidity. Their analysis is based on maximizing only the steady-state welfare of the competitive equilibrium, and hence they do not study the transitional dynamics of the Ramsey allocation.

Woodford (1990) shows in an Inf-HAIM model that public debt can improve welfare if the market interest rate lies below the time discount rate because of liquidity constraints. However, Woodford (1990) does not study optimal tax and debt policies.

Bassetto and Kucherlakota (2004) show that the paths of government debt can be irrelevant under distortionary taxes. In particular, they show that if the government collects taxes in a given period based only on incomes earned in previous periods, then the government can
adjust its tax policy so as to attain any debt path without affecting equilibrium allocations or prices. Therefore, the focus of their paper is different from ours.

Angeletos, Collard, and Dellas (2016) study the Ramsey policy problem in the Lagos and Wright (2005) framework with HAIM properties. They show that when risk-free government bonds contribute to the supply of liquidity to alleviate private agents’ borrowing constraints, issuing more debt raises welfare by improving the allocation of resources. However, increasing debt also tightens the government budget constraint by raising the interest rate on public debt. They show that there are two possible Ramsey steady states. One of them is similar to ours in that the gap between the interest rate and the time discount rate vanishes in the Ramsey steady state with FSI. The other Ramsey steady state does not feature FSI. We show, however, that when the Ramsey steady state is unique, it must feature FSI and such a policy is time inconsistent.

Azzimonti and Yared (2019) studies the optimal quantity of public debt in an environment where private-issued debt can be partially substituted by public debt. They first show theoretically that it is never optimal to fully crowd out private-issued debt by public debt in a two-period setting without idiosyncratic risks. They then numerically solve an infinite-horizon Aiyagari-type model where the Ramsey planner is assumed to maximize the steady-state welfare of the competitive equilibrium. In contrast, the Ramsey planner maximizes dynamic welfare along the entire transitional path of competitive equilibrium in our model and we allow public debt to be a perfect substitute of private debt.

In a two-period two-agent model without idiosyncratic risk, Azzimonti and Yared (2017) demonstrates that FSI is not a Ramsey outcome. The key intuition behind their result is that the interest burden of public debt can be alleviated if some households remain borrowing constrained in the Ramsey allocation. However, given the absence of idiosyncratic risk in their model, households do not have precautionary saving motivations, which makes their model fundamentally different from ours.

Similarly, Azzimonti and Yared (2017) use a two-period two-agent model to demonstrate that FSI is not a Ramsey outcome because a non-FSI allocation can reduce the interest burden of public debt. However, their model is not in the class of Inf-HAIM models that we focus on in this paper.

Bhandari, Evans, Golosov, and Sargent (2017) study the determination of public debt and optimal taxation with an Inf-HAIM structure. Under the assumption that the government can set a lump-sum transfer and a linear tax on labor income, they show that if households are subject to ad hoc borrowing limits (including the case where they cannot borrow at
all), the Ricardian equivalence breaks down and social welfare can be further improved by decreasing the level of public debt. They interpret this improvement as a consequence of the government’s monopoly rents from issuing public debt without facing competing private borrowers—that is, a monopolistic government is able to lower the expected market interest rate below the time discount rate by restricting the bond supply. Their model is not analytically tractable and the focus of their paper is different from ours.

Bassetto and Cui (2020) study optimal fiscal policies in a heterogeneous-agent environment with endogenous financial constraints on capital investment and idiosyncratic risks. They show that the Ramsey planner opts to supply enough government debt to provide public liquidity up to a point where financial constraints are slack. Similar to the result found by Chien and Wen (2020), these authors show that the optimal capital tax is zero in the Ramsey steady state, but they do not study the time-inconsistency issue of such government policies as we do here.

In terms of solution methodology for Inf-HAIM models, it is well known that using a simplified model can produce fruitful research outcomes. One well-known example is the model of Lagos and Wright (2005) in the money search literature, where the distribution of money is designed to be completely degenerated to achieve analytical tractability. Following the same spirit, Heathcote and Perri (2018) consider an Inf-HAIM model where households can reshuffle their wealth in the end of each period so that the wealth distribution is completely degenerated. We extend this strategy by allowing households to partially reshuffle their wealth in the beginning of each period to make the model tractable yet preserve a well-defined wealth distribution at the same time. This new method enables us to rediscover the FSI Ramsey allocation in a more general setting than the model of Chien and Wen (2019) and to better address the time-inconsistency problem of the optimal quantity of public debt. Our modeling strategy is also related to the approach in Grand and Ragot (2019), which truncates the history of household wealth by assuming that households can form risk-sharing pools according to their histories of idiosyncratic shocks.

6 Conclusion

A signature property of standard Inf-HAIM models is that the market interest rate is lower than the time discount rate (regardless of capital). This property emerges because precautionary saving motives under imperfect risk sharing entice agents to hold excess amount of liquid assets even if their rate of return is dominated by the time discount rate, suggesting
a liquidity premium to be paid by holding such assets under borrowing constraint and non diversifiable risk.

This fundamental property of standard Inf-HAIM models, however, implies that the Ramsey planner’s fiscal policy may be time inconsistent—because the planner has a dominate incentive to keep increasing the quantity of debt until all households are fully self-insured whenever the interest cost of borrowing is lower than the time discount rate (i.e., \( r < \beta^{-1} \)). But such an intended full self-insurance allocation may be either infeasible or time inconsistent—because to achieve it the optimal quantity of debt may approach infinity or the optimal labor tax rate may approach 100% such that the Ramsey planner is willing to change its long-run policies when it gets a chance to do so.

This is puzzling from an intuitive perspective because near the point of full self-insurance the marginal gains of increasing debt should be less than the marginal costs to finance the debt under distortionary taxes. We argue in this paper that this seemingly puzzling behavior of the Ramsey planner to achieve full self-insurance in the long run at “all costs” originates from the assumption that the planner must commit to any future plans at time zero. Under such a full commitment, the Ramsey planner opts to exploit the low interest cost of borrowing to front load consumption by sacrificing future consumption in the long run—because future utilities are heavily discounted compared to the inverse of the interest rate on government bonds.

We demonstrate our points both analytically and numerically using a tractable Inf-HAIM model featuring non-linear preferences and a well-defined distribution of household wealth. We also use both numerical simulations and steady-state welfare analysis to show that the Ramsey polices are time-inconsistent such that the planner opts to deviate from its long-run policies whenever it gets a chance, or if it does not need to discount the future according to the household time discounting factor.
References


A Appendix

A.1 Proof of Proposition 1

First, it is straightforward to see that it is impossible for $a_t^e = 0$ and $a_t^u = 0$ if $B_t > 0$ since it violates the asset market-clearing condition. In order to smooth consumption, the employed households have precautionary saving motives to self-insure against the positive possibility of switching to unemployed state. Hence, $a_t^e > a_t^u \geq 0$ if $B_t > 0$ for all $t$. In other words, $\kappa_t^e = 0$ and $\kappa_t^u \geq 0$ for all $t$.

Second, equation (11) together with $\kappa_t^e = 0$ give equation (16).

Third, suppose in the steady state we have $\kappa^e = \kappa^u = 0$, then equations (11) and (12) imply

$$\frac{\xi^e}{\xi^u} = \frac{\pi(e|e) + \xi^u}{\pi(e|e)} = \frac{\xi^u}{\xi^e} \frac{\pi(u|e)}{\pi(u|u)},$$

which can be rewritten as

$$\left(\frac{\xi^e}{\xi^u}\right)^2 \pi(e|u) + \frac{\xi^e}{\xi^u} (\pi(u|u) - \pi(e|e)) - \pi(u|e) = 0.$$

The equation above has one positive root and one negative root. The positive root implies $\xi^u = \xi^e$, which suggests $c^e = c^u$. The negative root violates the requirement that both $c^u > 0$ and $c^e > 0$. Given that $c^e = c^u$, equation (16) implies $Q = \beta$ in the steady state.

A.2 Proof of Proposition 2

A.2.1 The “If” Part:

Given the initial value of $B_0$ as well as the allocation $\{c_t^e, n_t^e, c_t^u, a_{t+1}^e, a_{t+1}^u\}_{t=0}^\infty$, a competitive equilibrium can be constructed by using the two conditions in Proposition 2 and by following the steps below that uniquely back up the sequences of the other variables.

1. Aggregate $C_t$ and $N_t$ are chosen to satisfy

$$N_t = n_t^e \pi(e),$$

$$C_t = c_t^e \pi(e) + c_t^u \pi(u).$$

2. $w_t$ is set to 1.
3. \( \tau_t \) is chosen to satisfy
\[
\frac{v_{n,t}^e}{u_{c,t}^e} = \hat{w}_t = (1 - \tau_t).
\] (33)

4. \( Q_{t+1} \) is chosen to satisfy the Euler equation
\[
Q_{t+1} u_{c,t}^e = \beta(u_{c,t+1}^e \pi(e|e) + u_{c,t+1}^u \pi(u|e)).
\] (34)

5. \( B_{t+1} \) is pinned down by the asset market-clearing condition
\[
B_{t+1} = a_{t+1}^e + a_{t+1}^u.
\]

6. The following constraints are satisfied:

(a) By plugging equations (33) and (34) into household budget constraints, we can obtain the implementability conditions displayed in equations (18), (19), (20), and (21).

(b) The resource constraint can be rewritten as equation (17).

(c) To satisfy the households’ FOCs (12) and borrowing constraints for unemployed households, we have listed equations (23), (24) and (22) as constraints in Proposition 2.

7. Finally, it is straightforward to verify that the implementability condition together with resource constraint implies the government budget constraint.

A.2.2 The “Only If” Part:

The constraints listed in Proposition 2 are trivially satisfied because they are part of the competitive-equilibrium conditions.

A.3 Proof of Proposition 3

A.3.1 Ramsey FOCs

We first state the Ramsey FOCs. Denote \( \beta \mu_t, \lambda_0^e, \lambda_0^u, \beta \lambda_t^e, \beta \lambda_t^u, \beta^{t+1} v_1^1, \beta^{t+1} v_1^2, \) and \( \beta^{t+1} v_1^3 \) as the Lagrangian multipliers for conditions (17)-(22), respectively. For all \( t \geq 0 \), the FOCs
of the Ramsey problem with respect to \( n^e_i, a^e_{i+1}, \) and \( a^u_{i+1} \) are given, respectively, by

\[
v^e_{n,t} + \lambda^e_t(v^e_{n,t} + v^e_{n,m,t}n^e_t) = \mu_t, \quad \text{for } t \geq 0
\]  \hspace{1cm} (35)

\[
\lambda^e_t(u^e_{c,t+1} \pi(e) + u^u_{c,t+1} \pi(u|e)) = \lambda^e_{t+1} u^e_{c,t+1} \pi(e) + \lambda^u_{t+1} u^e_{c,t+1} \pi(u|e), \quad \text{for } t \geq 0
\] \hspace{1cm} (36)

and

\[
\lambda^u_t(u^e_{c,t+1} \pi(e) + u^u_{c,t+1} \pi(u|e)) = \lambda^u_{t+1} u^e_{c,t+1} \pi(u|u) + \lambda^e_{t+1} u^e_{c,t+1} \pi(e|u)
+ v^1_t + v^3_t g(c^e_t, c^e_{t+1}, c^u_t, c^u_{t+1}),
\]  \hspace{1cm} (37)

respectively.

For all \( t \geq 1 \), the FOCs of the Ramsey problem with respect to \( c^e_t \) and \( c^u_t \) are given, respectively, by

\[
(u^e_{c,t} - \mu_t) \pi(e) + \lambda^e_t(u^e_{c,t} + u^e_{cc,t} c^e_t) \pi(e) - \lambda^e_t u^e_{cc,t} (a^e_t \pi(e) + a^u_t \pi(e|u))
+ \lambda^e_{t-1} a^e_t u^e_{cc,t} \pi(e|e) + \lambda^u_t u^e_{cc,t} c^u_t \pi(u) - \lambda^u_t u^e_{cc,t} (a^e_t \pi(u) + a^u_t \pi(u|u))
+ \lambda^u_{t-1} a^u_t u^e_{cc,t} \pi(e|e) + \beta v^2_t \partial g_t \partial c^e_t + v^1_t \partial g_{t-1} \partial c^e_t + \beta a^u_t v^3_t \partial g_t \partial c^e_t + a^u_t v^3_t \partial g_{t-1} \partial c^e_t
= 0,
\] \hspace{1cm} (38)

and

\[
(u^u_{c,t} - \mu_t) \pi(u) + \lambda^u_t(u^u_{c,t} + a^u_{cc,t} c^e_t) \pi(u) = \lambda^u_{t-1} a^u_t u^u_{cc,t} \pi(e|e)
+ \lambda^e_t a^u_t u^u_{cc,t} \pi(u|e) + \beta v^2_t \partial g_t \partial c^u_t + v^1_t \partial g_{t-1} \partial c^u_t + \beta a^u_t v^3_t \partial g_t \partial c^u_t + a^u_t v^3_t \partial g_{t-1} \partial c^u_t
= 0.
\] \hspace{1cm} (39)

For \( t = 0 \), the FOCs of the Ramsey problem with respect to \( c^e_0 \) and \( c^u_0 \) are given, respectively, by

\[
(u^e_{c,0} - \mu_0) \pi(e) + \lambda^e_0(u^e_{c,0} u^e_{cc,0} c^e_0) \pi(e) - \lambda^e_0 u^e_{cc,0} a^e_0
+ \lambda^u_0 u^u_{cc,0} c^u_0 \pi(u) - \lambda^u_0 u^e_{cc,0} a^u_0 + \beta v^2_0 \partial g_0 \partial c^e_0 + \beta a^u_0 v^3_0 \partial g_0 \partial c^e_0
= 0,
\]  \hspace{1cm} (40)
and

\[(u_{c,0} - \mu_0)\pi(u) + \lambda^u_0 u_{c,0} \pi(u) + \beta v_0^2 \frac{\partial g_0}{\partial c_0^u} + \beta a_1^u u_0^3 \frac{\partial g_0}{\partial c_0^u} = 0.\]

Note that

\[
\frac{\partial g_t}{\partial c_t^u} = -\frac{u_{c,t-1}^u u_{c,t}^e \pi(u) (u|e) + v_2 \partial g_{t-1}^u}{(u_{c,t}^e \pi(u|e) + u_{c,t}^u \pi(u|u))^2},
\]

\[
\frac{\partial g_t}{\partial c_t^e} = -\frac{u_{c,t-1}^u u_{c,t}^e \pi(u) (e|e) + v_2 \partial g_{t-1}^e}{(u_{c,t}^e \pi(e|e) + u_{c,t}^u \pi(e|u))^2}.
\]

### A.3.2 Proof of Existence

By the following steps, we conjecture and verify that there exists a Ramsey steady state featuring FSI with the property of (i) \(c^e = c^u\), (ii) \(a^e > a^u = 0\), (iii) \(v_1^t = 0\), and (iv) \(Q = \beta\).

1. When \(c^e = c^u\) and \(v_1^t = 0\), the steady-state version of the Ramsey FOCs with respect to \(a^e\) and \(a^u\) can be rewritten as

\[
\lambda^e_t = \lambda_{t+1}^e \pi(e|e) + \lambda_{t+1}^u \pi(u|e),
\]

\[
\lambda^u_t = \lambda_{t+1}^u \pi(u|e) + \lambda_{t+1}^e \pi(e|u),
\]

where the second equation utilizes the fact that \(g(c^e, c^u, c^e, c^u) = 0\) if \(c^e = c^u\). These two equations imply \(\lambda^e = \lambda^u\).

2. When \(c^e = c^u\), \(v_1^t = 0\), and \(a^u = 0\), the FOC with respect to \(c_t^u\) is given by

\[
(u_{c}^u - \mu)\pi(u) + \lambda^u u_{c}^e \pi(u) + \lambda^u a^e u_{c,t}^u \pi(u|e)
\]

\[
+ \beta v_0^2 \frac{\partial g}{\partial c^u} + v_2 \frac{\partial g_{-1}^u}{\partial c^u} = 0.
\]
and the FOC with respect to \( c^e \) is given by

\[
(u^e_c - \mu)\pi(e) + \lambda^e(u^e_c + u^{ue}c^e)\pi(e) - \lambda^e u^{ue}c^e
+ \lambda^e a^e u^{cc}e\pi(e|c) + \lambda^u u^{ue}c^e u\pi(u) + \beta v^2 \frac{\partial g}{\partial c^e} + v^2 \frac{\partial g_{-1}}{\partial c^e} = 0.
\]

Note that

\[
\frac{\partial g}{\partial c^u} = \frac{u^e_c}{u_c^e} = -\frac{\partial g}{\partial c^e}.
\]

\[
\frac{\partial g_{-1}}{\partial c^u} = \frac{-u^{ue}_{cc}\pi(u|u) + u^{ue}_{cc}\pi(u|e)}{u^e_c},
\]

\[
\frac{\partial g_{-1}}{\partial c^e} = \frac{-u^{ue}_{cc}\pi(e|u)}{u^e_c} + \frac{u^{ue}_{cc}\pi(e|e)}{u^e_c}.
\]

In the steady state, combining the above two FOCs together with \( c^e = c^u \) leads to

\[
u^e_c + \lambda^e u^e_c(1 + \frac{u^{ue}_{cc}c^e}{u^e_c}) = \mu,
\]

which together with the FOC with respect to \( v^e_n \):

\[
v^e_n + \lambda^e v^e_n(1 + \frac{v^e_{nn,t}v^e_n}{v_n}) = \mu,
\]

solve for \( \lambda^e \) and \( \mu \) in the Ramsey steady state. The value of \( v^2 \) can then be solved by using the FOC with respect to \( c^e_t \).

3. Finally, we solve for the optimal long-run tax rate \( \tau \) by the following steps:

(a) \( Q = \beta \) by equation (16).

(b) The condition (21) is simplified to \( a^e \pi(u|e) - c^u\pi(u) = 0 \), which implies

\[
\lambda^e = \frac{\pi(u)}{\pi(u|e)}.
\]

(c) The resource condition (17) gives \( n^e = c^e / \pi(e) \).

(d) Given \( Q = \beta \) and \( B = a^e \), the government budget constraint becomes

\[
\tau n^e \pi(e) = (1 - \beta)a^e,
\]
which together with $a^e = c^e \frac{\pi(u)}{\pi(u|e)}$ implies

$$\tau = \frac{(1 - \beta)}{n^e \pi(e)} a^e = \frac{(1 - \beta)}{c^e} a^e = (1 - \beta) \frac{\pi(u)}{\pi(u|e)}.$$

4. We can verify that this steady-state allocation satisfies all Ramsey FOCs since the rest of constraints, (23), (24) and (22), are trivially satisfied.

A.3.3 B. Proof of Uniqueness

We prove that the Ramsey steady state is unique if $(1 + \gamma c) \leq 0$. We show by contradiction that a steady-state allocation with $c^e > c^u$ is not possible if $(1 + \gamma c) \leq 0$.

Suppose $c^e > c^u$, then the borrowing constraint for the unemployed households must be strictly binding with $a^u = 0$ and $v_1 > 0$. Consider the following arguments:

1. From the Ramsey FOC with respect to $n^e$, we know that for a steady state to exist, the growth rate of $\lambda^e$ and $\mu_t$ have to be the same. Denote their steady state growth rate by $g^e$ and $g^u$, respectively.

2. The FOCs with respect to $a^e$ and $a^u$ gives

$$1 < \frac{Q}{\beta} = \pi(e|e) + \frac{u^u}{u^e} \pi(u|e) = g^e \pi(e|e) + g^u \frac{\lambda^u}{\lambda^e} \pi(u|e) < g^u \pi(u|u) + g^e \frac{\lambda^e}{\lambda^u} \pi(e|u), \quad (40)$$

where $g^u$ and $g^u$ denotes the growth rate of $\lambda^e$ and $\lambda^u$, respectively. We can show that $g^e = g^u$ by the following cases:

(a) Suppose $g^e < g^u$, then $\frac{\lambda^u}{\lambda^e} \rightarrow \infty$. Equation (40) becomes

$$\pi(e|e) + \frac{u^u}{u^e} \pi(u|e) = g^e \pi(e|e) + g^u \frac{\lambda^u}{\lambda^e} \pi(u|e) \rightarrow \infty < g^u \pi(u|u),$$

which is impossible.

(b) Suppose, $g^e > g^u$, so the FOC with respect to $c^u$ (under $v^2 = 0$, $a^u = 0$ and $a^e \pi(u|e) = c^u \pi(u)$) becomes

$$u^u + g^u \lambda^u_{t-1} u^e + \lambda^e_{t-1} u^u e^u = g^u \lambda_{t-1} \mu_t,$$
which implies
\[
\frac{u^u_c}{\lambda^u_{t-1}} + g^u_{\lambda} \frac{\lambda^u_{t-1}}{\lambda^e_t} u^e_c + u^u_c c^u = g_{\mu} \frac{\mu_{t-1}}{\lambda^u_{t-1}}.
\]

As \( t \to \infty \), the left hand side is negative and the right hand side is positive, a contradiction.

(c) Then, it must be the case that \( g^u_{\lambda} = g^u_{\lambda} \).

3. Equation (40) can be written as
\[
\left( \frac{\lambda^u}{\lambda^e_t} - 1 \right) \pi(u|e) < \left( \frac{\lambda^e}{\lambda^e_t} - 1 \right) \pi(e|u),
\]
which implies \( \lambda^e_t > \lambda^u_t \). In addition, equation (40) implies that
\[
\left( \frac{u^u_c}{u^e_c} - g^u_{\lambda} \frac{\lambda^u_{t}}{\lambda^e_t} \right) \pi(u|e) = (g^e_{\lambda} - 1) \pi(e|e). \tag{41}
\]

Under \( v^2 = 0, a^u = 0, \) and \( a^e \pi(u|e) = c^u \pi(u), \) the FOC with respect to \( c^u \) can be rewritten as
\[
u^e_c + \lambda^e_t u^e_c (1 + \gamma_c) = \mu_t + u^e_c c^u \frac{\pi(e|e)}{\pi(e)} (g^e_{\lambda} - 1) \lambda^e_{t-1}
\]
\[
= \mu_t + u^e_c c^u \frac{\pi(u)}{\pi(u|e)} \frac{\pi(e|e)}{\pi(e)} (g^e_{\lambda} - 1) \lambda^e_{t-1}, \tag{42}
\]
the FOC with respect to \( c^u \) can be rewritten as
\[
u^u_c + \lambda^u_t u^e_c - \lambda^e_{t-1} u^u_c + \lambda^e_{t-1} u^u_c (1 + \lambda_c) = \mu_t
\]
\[
u^u_c + \lambda^e_{t-1} u^u_c (1 + \lambda_c) = \mu_t - \lambda^u_t u^e_c + \lambda^e_{t-1} u^u_c. \tag{43}
\]

Now, consider the following cases:

(a) \( g^e_{\lambda} = g^u_{\lambda} = 1 \). Then \( \lambda^e \) converges. The difference between equation (42) and (43) is
\[
(u^e_c - u^u_c) + \lambda^e(1 + \gamma_c)(u^e_c - u^u_c) = 0,
\]
which implies \( u^e_c = u^u_c \) and leads to a contradiction. This is true for all possible values of \( \gamma_c. \)
(b) $g_e^c = g_u^c > 1$. The sum of FOCs with respect to $e^c$ and $c^u$ can be rewritten as

$$\frac{u^c_e \pi(e) + u^u_c \pi(u)}{\lambda^e_{t-1}} + \frac{\lambda^u_c}{\lambda^e_{t-1}} u^c_e (1 + \gamma_c) \pi(e) + u^u_c (1 + \lambda_c) \pi(u)$$

$$= \frac{\mu_t}{\lambda^e_{t-1}} - \frac{\lambda^u_c}{\lambda^e_{t-1}} u^c_e \pi(u) + u^u_c \pi(u) + u^u_c c^u \frac{\pi(u)}{\pi(u|e)} \pi(e|e)(g^e_\lambda - 1).$$

Since $\lambda^e_{t-1} \to \infty$, then the above equation becomes

$$g^e_\lambda u^e_c (1 + \gamma_c) \pi(e) + u^u_c (1 + \gamma_c) \pi(u)$$

$$= g^e_\mu \mu_{t-1} - g^u_\lambda \lambda^u_{t-1} u^c_e \pi(u) + u^u_c \pi(u) + u^u_c c^u \frac{\pi(u)}{\pi(u|e)} \pi(e|e)(g^e_\lambda - 1),$$

which together with (41) implies

$$g^e_\lambda u^e_c (1 + \gamma_c) \pi(e) + u^u_c (1 + \gamma_c) \pi(u)$$

$$= g^e_\mu \mu_{t-1} + \pi(u) u^c_e \left( \frac{u^u_c}{u^e_c} - g^u_\lambda \lambda^u_{t-1} \lambda^e_{t-1} \right) + u^u_c c^u \frac{\pi(u)}{\pi(u|e)} \left( u^u_c - g^u_\lambda \lambda^u_{t-1} \lambda^e_{t-1} \right)$$

$$= g^e_\mu \mu_{t-1} + \pi(u) \left( \frac{u^u_c}{u^e_c} - g^u_\lambda \lambda^u_{t-1} \lambda^e_{t-1} \right) (u^e_c + u^u_c c^u)$$

$$> \pi(u) \left( \frac{u^u_c}{u^e_c} - g^u_\lambda \lambda^u_{t-1} \lambda^e_{t-1} \right) u^e_c \left( 1 + \gamma_c \frac{c^u}{e^u} \right)$$

$$> \pi(u) \left( \frac{u^u_c}{u^e_c} - g^u_\lambda \lambda^u_{t-1} \lambda^e_{t-1} \right) (1 + \gamma_c),$$

where the last two inequalities utilize the fact (i) $g^u_\mu \mu_{t-1} > 0$ and the fact (ii) $\frac{c^u}{e^u} < 1$. Now consider $1 + \gamma_c \leq 0$, the above inequality is simplified to

$$0 < g^e_\lambda u^e_c \pi(e) <- g^u_\lambda \lambda^u_{t-1} u^c_e \pi(u) < 0, \text{ if } 1 + \gamma_c < 0,$$

or

$$0 > \left( \frac{u^u_c}{u^e_c} - g^u_\lambda \lambda^u_{t-1} \lambda^e_{t-1} \right) (1 + \gamma_c) = 0, \text{ if } 1 + \gamma_c = 0,$$

both of which lead to a contradiction.

### A.4 Proof of Proposition 4

We first state the Ramsey FOCs. Denote $\mu, \lambda^e, \lambda^u, v^1, v^2,$ and $v^3$ as the Lagrangian multipliers for conditions (27)-(32), respectively. The Ramsey FOCs with respect to $n^e_t, a^e_{t+1},$ and
\(a_{t+1}^u\) become
\[
\lambda^u \beta (u_c^e \pi(e|e) + u_e^u \pi(u|e)) = \lambda^e u_c^e \pi(e|e) + \lambda^u u_e^u \pi(u|e),
\] (44)
\[
\lambda^u \beta (u_e^e \pi(e|e)) + u_e^u \pi(u|e)) = \lambda^u u_e^e \pi(u|e) + \lambda^e u_c^e \pi(e|e) - v^1 - v^3 g(c^e, c^u).
\] (45)

Consider a Ramsey steady state featuring full self-insurance, that is, \(a^u \geq 0\) or \(v^1 = 0\), \(c^e = c^u\) and \(g(c^e, c^u) = 0\). Then, the FOC with respect to \(a^e\) is simplified as
\[
\lambda^e (\beta - \pi(e|e)) = \lambda^u \pi(u|e),
\]
which implies
\[
\frac{\lambda^u}{\lambda^e} = \frac{\beta - \pi(e|e)}{1 - \pi(e|e)} < 1.
\] (46)

However, the Ramsey FOC with respect to \(a^u\) becomes
\[
\lambda^u (\beta - \pi(u|u)) = \lambda^e \pi(e|u),
\]
which implies
\[
\frac{\lambda^u}{\lambda^e} = \frac{1 - \pi(u|u)}{\beta - \pi(u|u)} > 1,
\]
thus contracting with equation (46). Hence, \(c^e = c^u\) cannot be the outcome of this Ramsey problem that maximizes the steady-state welfare of the competitive equilibrium.