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Time-Inconsistent Optimal Quantity of Debt*

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Abstract

A key property of the Aiyagari-type heterogeneous-agent models is that the equilibrium interest rate of public debt lies below the time discount rate. This fundamental property, however, implies that the Ramsey planner's fiscal policy may be time-inconsistent because the forward-looking planner would have a dominant incentive to issue plenty of debt such that all households are fully self-insured against idiosyncratic risk. But such a full self-insurance allocation may be paradoxical because, to achieve it, the optimal labor tax rate may approach 100% and aggregate consumption may approach zero. This is puzzling from an intuitive perspective because near the point of full self-insurance the marginal gains of increasing debt should be less than the marginal costs of financing the debt under distortionary taxes. We show that this puzzling behavior originates from the assumption that the planner must commit to future plans at time zero. Under such a full commitment, the Ramsey planner opts to exploit the low interest cost of borrowing to front-load consumption by sacrificing future consumption in the long run because future utilities are heavily discounted compared to the inverse of the interest rate on government bonds. We demonstrate our points analytically using a tractable heterogeneous-agents model featuring non-linear preferences and a well-defined distribution of household wealth.

JEL Classification: E13; E62; H21; H30

Key Words: Time Inconsistency, Optimal Debt, Ramsey Problem, Incomplete Markets.

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1 Introduction

A *signature* property of standard heterogeneous-agents models is that the market risk-free interest rate is lower than the time discount rate (regardless of capital).[1] This property emerges because precautionary saving motives under imperfect risk sharing entice agents to hold an excess amount of liquid assets even if their rate of return is dominated by the time discount rate, suggesting that a liquidity premium is paid by holding such assets under borrowing constraints and non-diversifiable risk.

This property, however, has important implications for optimal government policies but has not been thoroughly investigated or well understood in the existing literature. One particular implication among them is the existence of an "arbitrage opportunity" for the Ramsey planner—who can borrow cheaply by issuing debt at the lower interest rate and roll over the debt indefinitely as long as that interest rate stays below the time discount rate. Arbitrate will continue until the model converges to an interior Ramsey steady state (if it exists) where households are fully self-insured by holding a sufficient amount of public debt. In such an interior Ramsey steady state, agents are no longer borrowing constrained and the interest rate equals the time discount rate.[2] In other words, the Ramsey planner opts to issue plenty of debt such that asset holders (savers) are satiated with savings against idiosyncratic risks.

But this Ramsey outcome, or intention to issue a sufficiently large stock of government bonds to enable borrowing-constrained agents to achieve a full self-insurance (FSI) allocation may be puzzling: This is because intuition tells us that the marginal benefit of reducing consumption risk must decline with an increasing stock of public debt, while the cost of financing the debt with distortionary taxes must rise. Therefore, an FSI allocation does not seem to be optimal.

Yet, consistent with the finding of Chien and Wen (2019), this paper illustrates that an FSI allocation is indeed a Ramsey outcome even if (i) the implied steady-state labor tax rate is near 100% in order to support the FSI allocation and (ii) the model's preferences are

[1] By "standard" models we mean the class of models similar to that studied by Aiyagari (1994) in which agents are ex ante identical but ex post heterogenous. Therefore, our paper focuses on this class of models and does not consider other popular types of models with ex ante heterogenous agents such as the two-class model of Judd (1985) or the OLG models.

[2] There may exist two types of Ramsey steady states in general: an interior one and a non-interior one. If all quantity variables converge to finitely positive values, it is called an interior steady state. It is called a non-interior steady state if one or more quantity variables (such as aggregate consumption) converge to zero.

standard (i.e., non-linear) and the distribution of wealth is not degenerated (in contrast to the model of Chien and Wen (2019)).

The insight obtained from our analysis is that the FSI Ramsey allocation, driven by the gap between the interest rate and the time discount rate, is a consequence of the assumption of full commitment by the Ramsey planner at time zero. If the Ramsey planner has opportunities to readjust policies, or if the objective is to maximize the steady-state welfare of the competitive equilibrium instead of the dynamic path of welfare at time zero, then an FSI allocation can be shown to be suboptimal.

This inconsistency in the optimal quantity of public debt between maximizing the steady-state welfare and the dynamic welfare arises from the fact that the Ramsey planner does not discount the future when using policies to pick the best competitive steady state of the model, in contrast to policies that pick the best dynamic path of the competitive equilibrium. In the former case, the difference between the interest rate and the time discount rate does not matter, since nothing is discounted in the steady state; but in the latter case this difference provides an "arbitrage" opportunity for the Ramsey planner to exploit—since the interest cost of borrowing funds to finance debt is lower than the time discount rate. It is thus always beneficial to issue plenty of public debt to improve households' time-zero welfare by front-loading consumption when the future is discounted at a rate higher than the inverse of the interest rate, even if this implies that the optimal steady-state tax rate is close to 100%.

In other words, if the Ramsey planner does not abide by their original plans made at time zero, or if they have the opportunity to readjust their original plans, the planner will choose to deviate away from the steady-state FSI allocation. This suggests that the optimal long-run quantity of public debt committed at time zero is not time consistent.

Our contributions in this paper are thus four-fold: First, we prove that when an interior Ramsey steady state exists and is unique, then it must feature FSI. Second, we show that the optimal quantity of public debt committed at time zero to support such an FSI allocation in the long run is time inconsistent, and we conjecture that this feature may be a general property of standard heterogeneous-agents models.[3] Third, we also show that a non-interior Ramsey steady state—featuring zero aggregate consumption—is possible in certain parameter spaces, such as when the idiosyncratic income shock is highly persistent. Fourth, we provide an analytically tractable model to demonstrate our points.

[3]In representative-agent models without capital, such as the model of Lucas and Stokey (1983), the issue of time inconsistency does not arise, because in the absence of aggregate uncertainty the equilibrium interest rate always equals the time discount rate. But the time-inconsistency issue arises in heterogeneous-agents models because the long-term interest rate is no higher than the time discount rate.

Model tractability is crucial. It is well known that analyzing the Ramsey problem in heterogeneous-agents models is often technically very challenging because such models are generally non-tractable due to the curse of the history-dependent distribution of individuals' wealth. Even when the competitive equilibrium has closed-form solutions, the Ramsey problem may still be intractable due to possible non-convexities, let alone when the competitive equilibrium is itself intractable. Without tractability, it is hard to know whether a Ramsey steady state exists or not. Consequently, an alternative approach in the literature is to appeal to numerical solutions under the critical assumption that an interior Ramsey steady state *does* exist. But as mentioned above, if the Ramsey planner has a dominant incentive to achieve an FSI allocation, then an interior Ramsey steady state may not exist if such an allocation can only be supported by an extremely high debt-to-GDP (debt to domestic gross product) ratio or by a tax rate arbitrarily close to 100%. So the optimal fiscal policies drawn from the analysis under the assumed (instead of proven) Ramsey steady state may be misleading, since they hinge critically on the validity of such an assumption.

To overcome the daunting challenge of model tractability, our paper follows a strand of the literature that assumes an imperfect risk-sharing technology to render the distribution of wealth degenerate or partially degenerate. More specifically, in the spirit of Heathcote and Perri (2018) and Bilbiie and Ragot (2020), we adopt the multiple-member household metaphor (Lucas (1990)) to allow family members to partially pool their wealth at the beginning of each period according to their current employment status, regardless of their employment history. Consequently, every family member belongs to either a high-wealth group or a low-wealth group in each period. Because idiosyncratic risk for each member's employment status is only partially shared under this imperfect risk-sharing arrangement, precautionary saving motives are well preserved; the equilibrium distribution of wealth is not fully degenerate; and the competitive-equilibrium interest rate lies below the time discount rate. In such a model, we provide the necessary and sufficient conditions for an interior Ramsey steady state to exist and prove that such an interior Ramsey steady state must feature FSI.

More importantly, we show that the Ramsey planner's long-run policies to sustain the FSI allocation are time inconsistent: If the planner has the chance to readjust their policies or can ignore the transitional dynamics of the competitive equilibrium by focusing only on the steady-state welfare of the competitive equilibrium, then the FSI allocation is no longer optimal. In other words, the Ramsey planner has incentives to deviate from the FSI allocation either when they have the chance to readjust their plans or when they target only the steady-state welfare instead of maximizing the entire dynamic path of social welfare at time zero.

1.1 A Brief Literature Review

In the Aiyagari-type heterogeneous-agents framework, several previous works have studied optimal fiscal policies by maximizing the steady-state welfare of competitive equilibrium, such as Aiyagari and McGrattan (1998), Flodén (2001), and Azzimonti and Yared (2019). These studies rely on numerical methods to search for the optimal policy mix. Their results show that public debt can improve steady-state welfare but that the Ramsey allocation does not feature FSI. Our study complements this literature in the sense that we explain why an FSI allocation does not appear to be optimal in these studies by pointing out the time-inconsistency problem of the Ramsey plan in heterogeneous-agents models. In particular, by taking into account the entire dynamic path of expected future welfare at time zero, we show that the Ramsey planner has a dominant incentive to achieve FSI at "all costs" because of the wedge between the market interest rate and the time discount rate under precautionary saving motives—which is a hallmark feature of the Aiyagari-type heterogeneous-agents models. Yet, such a dynamic consideration of the Ramsey planner is missing in steady-state welfare maximization.

It is in fact well documented in the existing literature that optimal policies may look dramatically different between steady-state welfare analysis and time-zero dynamic welfare analysis because the former ignores the transition path of the Ramsey allocation. For example, Domeij and Heathcote (2004) show that a capital tax could decrease steady-state welfare but improve welfare during the transition. Rohrs and Winter (2017) find that taking transitional dynamics into consideration has important implications for government debt policy. A significant short-run macroeconomic effect of an income tax cut is also demonstrated by Heathcote (2005). In this regard, our paper contributes to clarifying the issues involved between steady-state welfare maximization and time-zero dynamic welfare maximization by pointing out that the culprit is the time-inconsistency problem of the Ramsey plan.

In terms of solution methodology, it is well known that assuming a simple risk-sharing arrangement (technology) can dramatically simplify heterogeneous-agent models and produce fruitful research outcomes. A good reference is the work of Lucas (1990), which assumes that household members belong to a "big representative family" and can share information or resources to get around some technical difficulties in solving heterogeneous-agents models. Following this spirit, Heathcote and Perri (2018) solve their heterogeneous-agents model analytically by assuming that agents can reshuffle their asset holdings in the end of each period so that the wealth distribution is completely degenerate. Similarly, Bilbiie and Ragot (2020), Challe and Ragot (2011), Challe, Matheron, Ragot, and Rubio-Ramirez (2017), Sterk and Tenreyro (2018), and Bilbiie (2019), among others, further extend the idea of Lucas (1990) by assuming a partial risk-sharing technology that allows heterogenous agents to share their asset holdings according to their types. More specifically, Bilbiie (2019) designs a tractable heterogeneous-agent New-Keynesian (HANK) model that captures analytically key micro-heterogeneity channels of quantitative-HANK models and uses it for a full-fledged New-Keynesian macro analysis regarding interest-rate rules, a forward-guidance puzzle, amplification-multipliers, liquidity traps, and optimal policy. This paper makes a particular assumption regarding risk-sharing technology to make the model tractable, which is closely related to our approach. Bilbiie and Ragot (2020) build a tractable general equilibrium New Keynesian (NK) model with heterogeneous households where aggregate demand depends on liquidity, defined as monetary assets used for self-insurance as in the seminal Bewley model and Wen (2015). They show that price stability is no longer optimal with household heterogeneity, in particular when households self-insure against idiosyncratic risk using scarcely available liquid assets. They identify a liquidity-insurance channel for optimal policy and show that this changes the standard stabilization objectives (of inflation and real activity) by introducing a linear term in the otherwise standard quadratic objective function of the central bank, capturing the lack of consumption insurance. To make their model tractable, the authors also assume a specific risk-sharing technology in the spirit of Lucas (1990) to render their model tractable. In addition, Le Grand and Ragot (2019) extend this approach further by assuming that households can form risk-sharing pools according to their truncated histories of idiosyncratic shocks, and they show that the equilibrium property of their model approaches the original Aiyagari model in the limit if the truncated history is long enough.[4]

Our work is also related to the recent work by Bassetto and Cui (2020), who study optimal fiscal policies in a heterogeneous-agents environment with endogenous financial constraints on capital investment. They show that the Ramsey planner opts to supply enough government debt to provide public liquidity up to a point where financial constraints are all slack.

[4]Assuming quasi-liner utility is an alternative approach to simplify the heterogenous-agents models. For example, see Wen (2009), Challe and Ragot (2011) and Wen (2015). In addition, Lagos and Wright (2005) is also a well-known example in the money-search literature.

Similar to the result found by Chien and Wen (2020), these authors show that the optimal capital tax is zero in the Ramsey steady state, but they do not study the time-inconsistency issue of such government policies as we do here.

The work by Bhandari, Evans, Golosov, and Sargent (2017) studies the optimal quantity of public debt and optimal lump-sum transfers together with distortionary taxes in an environment with enforceable private debt contracts. But the intractability of their model makes it hard to answer our question in hand regarding the distinctive roles of public debt in mitigating consumption inequality due to uninsurable income risk.

Our model framework is in the class of models pioneered by Aiyagari (1994) in which infinitely lived agents are ex-ante identical but ex-post heterogenous. But future research can also study the time-inconsistency problem in ex-ante heterogenous-agent models, such as the two-class model of Judd (1985) or the overlapping generation models. For example, in a two-period ex-ante heterogeneous-agent model, Azzimonti and Yared (2017) show that the Ramsey planner optimally limits the supply of bond such that not all households are slack in their borrowing constraints—because the Ramsey planner can borrow more cheaply and hence can relax its budget constraints by keeping some of the agents constrained. Given that the setup of such models is dramatically different from ours, it is worthwhile to investigate whether time inconsistency arises in such frameworks in the future.

The rest of the paper is organized as follows: Section 2 sets up the model and defines the competitive equilibrium. Section 3 solves the Ramsey problem using the primal approach, proves the existence and uniqueness of the Ramsey steady state, and provides interpretations on the FSI Ramsey steady-state allocation. Section 4 examines the time-inconsistency problem of the long-run Ramsey policies. Section 5 concludes.

2 The Model

Firm. Time is discrete and indexed by $t = 0, 1, 2, ..., \infty$. A representative firm produces output according to a linear production technology, $Y_t = N_t$, where N_t is the aggregate labor input. The firm hires labor from households by paying a competitive real wage rate w_t. Perfect competition implies

$$w_t = \frac{\partial Y_t}{\partial N_t} = 1.$$

Government. In each period t, the government issues risk-free bonds B_{t+1} and receives labor-income tax revenues under a flat tax rate τ_t. Denote the price of bonds in period t by

Q_{t+1}, which pays one unit of consumption goods in period $t+1$; then the risk-free interest rate is given by $r_{t+1} = Q_{t+1}^{-1}$. The flow government budget constraint is then given by

$$\tau_t w_t N_t + Q_{t+1} B_{t+1} \geq B_t \tag{1}$$

for all $t \geq 0$, where the initial level of government bonds B_0 is exogenously given. For simplicity, the government spending is assumed to be zero.

Households. There is a unit measure of individual households who are subject to an idiosyncratic employment-status shock in each period, denoted by $\theta_t \in \{e, u\}$. The shock is identically and independently distributed (iid) across individuals and follows a Markov process. If $\theta_t = e$, then individuals can work and receive their labor income; otherwise, if $\theta_t = u$, individuals cannot work and have no labor income. Let $\pi(e)$ and $\pi(u)$ denote the unconditional probability of employment and unemployment shocks, respectively. By the law of large numbers, these probability measures also represent the share of employed or unemployed individuals in total population, respectively. For simplicity, we assume that $\pi(e)$ and $\pi(u)$ also represent the initial period's share of employed and unemployed individual households.

In the spirit of Bilbiie (2019) and Bilbiie and Ragot (2020), we adopt the family metaphor of Lucas (1990) to simplify our analysis. Each individual household belongs to a big representative family. In each representative family, there is a family head who maximizes the intertemporal welfare of all family members using a utilitarian welfare criterion (all family members are equally weighted) but who faces some limits to the amount of risk sharing that it can do. Specifically, in each period after the realization of the idiosyncratic shock to employment status, the head of family can reshuffle asset holdings among individual households who have the same current status of employment. However, the family head cannot reshuffle resources across households with different employment status.

Given that there are only two employment statuses, consequently there will be two groups of households in every time period: a employed group and an unemployed group. Namely, each household's initial wealth in period t takes only two possible values that depend on the current state of idiosyncratic shock but that are independent of the household's past history of employment status. As a result, the consumption and saving choices of individual households are identical within each group.

Denote the level of consumption, labor supply, and asset holdings (savings) for the employed households in period t as c_t^e, n_t^e, and $a_{t+1}^e/\pi(e)$, respectively; and similarly let c_t^u

and $a^u_{t+1}/\pi(u)$ denote consumption and savings for the unemployed households (who do not have a job in this period). In such a setup, under the partial risk-sharing technology the employed households' initial asset holdings in the beginning of period t are given by $\frac{a^e_t \pi(e|e)+a^u_t \pi(e|u)}{\pi(e)}$, where $\pi(e|e)$ and $\pi(e|u)$ denote the transition probability from employed and unemployed states to an employed state, respectively. Similarly, the initial asset holdings for unemployed households are given by $\frac{a^e_t \pi(u|e)+a^u_t \pi(u|u)}{\pi(u)}$, where $\pi(u|e)$ and $\pi(u|u)$ denote the transition probability from employed and unemployed states to an unemployed state, respectively. Note that the unconditional probabilities $\pi(e) = \frac{\pi(e|u)}{\pi(e|u)+\pi(u|e)}$ and $\pi(u) = \frac{\pi(u|e)}{\pi(e|u)+\pi(u|e)}$ satisfy $\pi(e) + \pi(u) = 1$. This specified wealth-pooling arrangement simplifies the model dramatically and renders it analytically tractable.

Hence, for $t \geq 1$, the household budget constraints for the employed and unemployed households are given, respectively, by

$$\frac{a^e_t \pi(e|e) + a^u_t \pi(e|u)}{\pi(e)} + \widehat{w}_t n^e_t - c^e_t - Q_{t+1}\frac{a^e_{t+1}}{\pi(e)} \geq 0, \tag{2}$$

$$\frac{a^e_t \pi(u|e) + a^u_t \pi(u|u)}{\pi(u)} - c^u_t - Q_{t+1}\frac{a^u_{t+1}}{\pi(u)} \geq 0, \tag{3}$$

where $\widehat{w}_t \equiv (1-\tau_t)\, w_t$ is the after-tax wage rate. For the initial period ($t = 0$), the household budget constraints for the employed and unemployed households are given, respectively, by

$$\frac{a^e_0}{\pi(e)} + \widehat{w}_0 n^e_0 - c^e_0 - Q_1\frac{a^e_1}{\pi(e)} \geq 0, \tag{4}$$

$$\frac{a^u_0}{\pi(u)} - c^u_0 - Q_1\frac{a^u_1}{\pi(u)} \geq 0, \tag{5}$$

where $\frac{a^e_0}{\pi(e)}$ and $\frac{a^u_0}{\pi(u)}$ are the period-0 initial asset holdings for the employed and unemployed households, respectively. Note that $a^e_0 + a^u_0 = B_0$. In addition, households are subject to borrowing constraints for all $t \geq 0$:

$$a^e_{t+1} \geq 0, \tag{6}$$

$$a^u_{t+1} \geq 0. \tag{7}$$

Since the labor supply of the unemployed households is zero, the utilitarian welfare criterion of the family head is given by

$$U = \sum_{t=0}^{\infty} \beta^t \left\{ \left[u(c_t^e) - v\left(n_t^e\right) \right] \pi(e) + u(c_t^u) \pi(u) \right\}, \tag{8}$$

where $\beta \in (0,1)$ is the time discounting factor, and the utility functions take the standard form:

$$u(c) = \frac{1}{1-\sigma} c^{1-\sigma} \text{ and } v(n) = \frac{1}{1+\gamma} n^{1+\gamma},$$

where $\sigma > 0$ and $\gamma > 0$.

Given market prices, $\{Q_{t+1}, \widehat{w}_t\}_{t=0}^{\infty}$, and initial asset holdings, $\{a_0^e, a_0^u\}$, the family head chooses a sequence of $\{c_t^e, c_t^u, n_t^e, a_{t+1}^e, a_{t+1}^u\}_{t=0}^{\infty}$ to maximize (8) subject to (2)-(7). Let $\beta^t \xi_t^e \pi(e)$, $\beta^t \xi_t^u \pi\left(u\right)$, $\xi_0^e \pi(e)$, $\xi_0^u \pi\left(u\right)$, $\beta^t \kappa_t^e \pi(e)$ and $\beta^t \kappa_t^u \pi(u)$ be the Lagrangian multipliers associated with constraints (2)-(7), respectively; the FOCs with respect to c_t^e, c_t^u, a_{t+1}^e, a_{t+1}^u, and n_t^e are given, respectively, by

$$u_{c,t}^e = \xi_t^e, \tag{9}$$

$$u_{c,t}^u = \xi_t^u, \tag{10}$$

$$Q_{t+1} \xi_t^e = \beta \left[\xi_{t+1}^e \pi(e|e) + \xi_{t+1}^u \pi(u|e) \right] + \kappa_t^e, \tag{11}$$

$$Q_{t+1} \xi_t^u = \beta \left[\xi_{t+1}^e \pi(e|u) + \xi_{t+1}^u \pi(u|u) \right] + \kappa_t^u, \tag{12}$$

$$v_{n,t}^e = \xi_t^e \widehat{w}_t; \tag{13}$$

where $u_{c,t}^e$ and $u_{c,t}^u$ denote the marginal utility of consumption for the employed and unemployed households in period t; similarly, $v_{n,t}^e$ denotes the marginal disutility of labor for the employed households in period t.

2.1 Competitive Equilibrium

Definition 1. *Given the initial asset holdings (a_0^e, a_0^u), the initial government bonds $B_0 = a_0^e + a_0^u > 0$, and the sequence of policies $\{\tau_t, B_{t+1}\}_{t=0}^{\infty}$, a competitive equilibrium is defined as the sequences of prices $\{w_t, Q_{t+1}\}_{t=0}^{\infty}$, aggregate allocations $\{C_t, N_t\}_{t=0}^{\infty}$, and individual allocation plans $\{c_t^e, c_t^u, n_t^e, a_{t+1}^e, a_{t+1}^u\}_{t=0}^{\infty}$, such that*

1. *$\{c_t^e, c_t^u, n_t^e, a_{t+1}^e, a_{t+1}^u\}_{t=0}^{\infty}$ solves the family head's problem;*

2. *$\{N_t\}$ solves the representative firm's problem;*

3. *the government flow budget constraint holds:*

$$\tau_t N_t + Q_{t+1} B_{t+1} \geq B_t;$$

and

4. *all markets clear for $t \geq 0$:*

$$\begin{aligned} B_{t+1} &= a^e_{t+1} + a^u_{t+1}, \\ N_t &= C_t, \\ N_t &= n^e_t \pi(e), \\ C_t &= c^e_t \pi(e) + c^u_t \pi(u). \end{aligned} \tag{14}$$

To facilitate our analysis, we make the following assumptions for the rest of this paper.

Assumption 1. *1. The autocorrelation of the idiosyncratic shock process is either zero or positive: $\pi(e|e) + \pi(u|u) \geq 1$.*

2. In period 0*, employed households have higher initial wealth than unemployed households, $\frac{a^e_0}{\pi(e)} > \frac{a^u_0}{\pi(u)}$.*

Assumption 1 ensures that under the wealth-pooling technology, the initial wealth of the employed households in each period $t \geq 1$ should be no less than that of the unemployed households. More specifically, if $\pi(e|e)+\pi(u|u) \geq 1$ and $\frac{a^e_0}{\pi(e)} > \frac{a^u_0}{\pi(u)}$, then it must be the case that $\frac{a^e_t \pi(e|e)+a^u_t \pi(e|u)}{\pi(e)} \geq \frac{a^e_t \pi(u|e)+a^u_t \pi(u|u)}{\pi(u)}$ for all $t \geq 1$ and $\frac{a^e_{t+1}}{\pi(e)} > \frac{a^u_{t+1}}{\pi(u)}$ for all $t \geq 0$. Notice that a strict equality holds if the idiosyncratic shock is iid; namely, if $\pi(e|e) + \pi(u|u) = 1$ and $\frac{a^e_0}{\pi(e)} > \frac{a^u_0}{\pi(u)}$, then it must be true that $\frac{a^e_t \pi(e|e)+a^u_t \pi(e|u)}{\pi(e)} = \frac{a^e_t \pi(u|e)+a^u_t \pi(u|u)}{\pi(u)}$ for all $t \geq 1$ and that $\frac{a^e_{t+1}}{\pi(e)} > \frac{a^u_{t+1}}{\pi(u)}$ for all $t \geq 0$. In other words, this assumption rules out the uninteresting case of a wealth-pooling arrangement such that unemployed households are wealthier than employed households from time to time; this uninteresting possibility arises if the employment shock is negatively autocorrelated (i.e., $\pi(e|e) + \pi(u|u) < 1$). Therefore, our analysis focuses only on the cases where the employment shock is non-negatively autocorrelated.

Proposition 1. *Under Assumption 1 and the exogenously given policy $B_t > 0$ for all t, the competitive equilibrium has the following properties:*

1. *For all* $t \geq 0$, *it must be true that* $c^e_t \geq c^u_t$ *and* $\frac{a^e_{t+1}}{\pi(e)} > \frac{a^u_{t+1}}{\pi(u)} \geq 0$. *That is, the borrowing constraints of the employed households are always slack (*$\frac{a^e_{t+1}}{\pi(e)} > 0$*, implying that the Lagrangian multiplier* $\kappa^e_t = 0$*); depending on the level of* B_t*, the borrowing constraints of the unemployed households may or may not be binding (*$\frac{a^u_{t+1}}{\pi(u)} \geq 0$*).*

2. *The intertemporal price* Q_{t+1} *is determined by*

$$Q_{t+1} = \beta \left[\frac{u^e_{c,t+1}}{u^e_{c,t}} \pi(e|e) + \frac{u^u_{c,t+1}}{u^e_{c,t}} \pi(u|e) \right]. \tag{15}$$

3. *In the steady state, if the asset holdings of the unemployed households are not binding such that the corresponding multiplier* $\kappa^u = 0$*, then the competitive equilibrium features full self-insurance with* $c^e = c^u$ *and* $Q = \beta$ *(or* $r = \beta^{-1}$*).*[5]

Proof. See Appendix A.1. □

Proposition 1 states that if the asset holdings a^e_{t+1} and a^u_{t+1} are both sufficiently large such that all households' borrowing constraints are slack, then in the steady state they can obtain the same level of consumption regardless of their employment status. In this case, the steady-state market interest rate equals the time discount rate. We refer to this allocation as a FSI allocation.

In our model, the FSI can be achieved even at a finite level of asset holdings because of the special risk-sharing technology among family members. Specifically, in any period t, if the savings of the employed households a^e_t in the last period are sufficiently large, then the period-t initial wealth of the unemployed agents, $\frac{a^e_t \pi(u|e) + a^u_t \pi(u|u)}{\pi(u)}$, can be high enough such that their borrowing constraints are also slack. As a result, household savings (or asset demand) are bounded away from infinity even at the point $r = 1/\beta$. This property does not hold in the original Aiyagari model—where asset demand goes to infinity when the interest rate equals the time discount rate—because the wealth distribution in that model does not converge in the absence of the risk-sharing technology assumed in our paper. However, this assumption is not what gives rise to the time inconsistency of Ramsey policies, as will become clear shortly.

Obviously, by the asset market-clearing condition (14), FSI is feasible if the supply of government bonds is sufficiently high. In other words, if FSI is achieved in the competitive

[5]Throughout this paper, a variable without subscript t is a steady-state value.

steady state, then the risk-free interest rate r ($\equiv 1/Q$) must be equal to the time discount rate β^{-1}: $r = \beta^{-1}$. Otherwise it must be true that $c^e > c^u$ and $r < \beta^{-1}$.

To make our Ramsey problem interesting, we assume in the rest of the paper that the initial bond supply B_0 and the initial distribution of household wealth $\{a_0^e, a_0^u\}$ are such that the competitive equilibrium (without further policy intervention) does not feature FSI. Namely, the competitive equilibrium features consumption inequality $c^e > e^u$ and precautionary saving behaviors such that $r < \beta^{-1}$. The central question of our analysis is whether the Ramsey planner is willing to increase the bond supply to achieve FSI at the extra costs of distortionary taxes and why.

3 Solving the Ramsey Problem

Note that the competitive equilibrium defined above is in general a function of the path of government policies $\{\tau_t, B_{t+1}\}_{t=0}^{\infty}$. Namely, each different path of government policies corresponds to a different competitive equilibrium. The Ramsey problem is to select a particular path of government policies such that the corresponding competitive equilibrium yields the maximum social welfare.

We use the primal approach to solve the Ramsey problem. Under the primal approach, we first substitute out all market prices and policy variables by using a subset of the competitive equilibrium's first-order conditions and then choose the path of allocation, $\{c_t^e, c_t^u, n_t^e, a_{t+1}^e, a_{t+1}^u\}_{t=0}^{\infty}$, to maximize social welfare subject to the rest of the equilibrium conditions. The solution under such a primal approach is called a Ramsey allocation/outcome or a Ramsey plan.

3.1 Conditions to Support a Competitive Equilibrium

To ensure that a Ramsey plan constitutes a competitive equilibrium, we must show first that all possible allocations in the choice set of the Ramsey planner, $\{c_t^e, n_t^e, c_t^u, a_{t+1}^e, a_{t+1}^u\}_{t=0}^{\infty}$ (after substituting out all market prices and policy variables but before solving the Ramsey maximization problem), constitute a competitive equilibrium. The following proposition states the conditions for any constructed Ramsey allocation to satisfy in order to constitute a competitive equilibrium.

Proposition 2. *Given the initial asset holdings (a_0^e, a_0^u) and initial government bonds $B_0 =$*

$a_0^e + a_0^u > 0$, the sequence $\{c_t^e, c_t^u, n_t^e, a_{t+1}^e, a_{t+1}^u\}_{t=0}^{\infty}$ can be supported as a competitive equilibrium if and only if it satisfies the following conditions:

1. *Resource constraints:*

$$n_t^e \pi(e) - c_t^e \pi(e) - c_t^u \pi(u) \geq 0, \ \forall t \geq 0. \tag{16}$$

2. *Implementability conditions: for $t = 0$,*

$$u_{c,0}^e c_0^e \pi(e) - v_{n,0}^e n_0^e \pi(e) + Q_1 u_{c,0}^e a_1^e - u_{c,0}^e a_0^e = 0, \tag{17}$$

$$u_{c,0}^e c_0^u \pi(u) + Q_1 u_{c,0}^e a_1^u - u_{c,0}^e a_t^u = 0, \tag{18}$$

and, for $t \geq 1$,

$$u_{c,t}^e c_t^e \pi(e) - v_{n,t}^e n_t^e \pi(e) + Q_{t+1} u_{c,t}^e a_{t+1}^e - u_{c,t}^e \left[a_t^e \pi(e|e) + a_t^u \pi(e|u)\right] = 0, \tag{19}$$

$$u_{c,t}^e c_t^u \pi(u) + Q_{t+1} u_{c,t}^e a_{t+1}^u - u_{c,t}^e \left[a_t^e \pi(u|e) + a_t^u \pi(u|u)\right] = 0, \tag{20}$$

where

$$Q_{t+1} u_{c,t}^e = \beta \left[u_{c,t+1}^e \pi(e|e) + u_{c,t+1}^u \pi(u|e)\right].$$

3. *Borrowing constraints and complementary slackness conditions: $\forall t \geq 0$,*

$$a_{t+1}^u \geq 0, \tag{21}$$

$$g_t(c_t^e, c_t^u, c_{t+1}^e, c_{t+1}^u) \geq 0, \tag{22}$$

and

$$g_t(c_t^e, c_t^u, c_{t+1}^e, c_{t+1}^u) a_{t+1}^u = 0, \tag{23}$$

where the function g_t is defined as

$$g_t(c_t^e, c_t^u, c_{t+1}^e, c_{t+1}^u) \equiv \frac{u_{c,t}^u}{u_{c,t+1}^e \pi(e|u) + u_{c,t+1}^u \pi(u|u)} - \frac{u_{c,t}^e}{u_{c,t+1}^e \pi(e|e) + u_{c,t+1}^u \pi(u|e)}.$$

Proof. See Appendix A.2. ∎

3.2 Ramsey Allocation

The Ramsey problem under the primal approach can then be represented by the following maximization problem:

$$\max_{\{c_t^e, c_t^u, n_t^e, a_{t+1}^e, a_{t+1}^u\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \left\{ \left[u(c_t^e) - v\left(n_t^e\right) \right] \pi(e) + \left[u(c_t^u) \right] \pi(u) \right\} \tag{24}$$

subject to constraints (16)-(23) listed in Proposition 2.

Before presenting the Ramsey outcome, we first define the Ramsey steady state in our economy:

Definition 2. *Given $\{B_0, a_0^e, a_0^u\}$, a Ramsey steady state is a long-run Ramsey allocation where the variables $\{N_t, C_t, B_{t+1}, c_t^e, c_t^u, n_t^e\}$ all converge to finite non-negative values. In addition, a Ramsey steady state is called "interior" if none of the variables converges to zero; otherwise, the Ramsey steady state is called "non-interior" if one or more of these variables (such as consumption variables C_t, c_t^e, c_t^u) converge to zero.*

Proposition 3.

1. *Under the parameter condition $(1-\beta)\frac{\pi(u)}{\pi(u|e)} < 1$, an interior Ramsey steady state exists and it has the following properties:*

   (a) The allocation features FSI with $0 = a^u < a^e$, $c^e = c^u$, and $r = \beta^{-1}$.

   (b) The optimal tax rate is given by $\tau = (1-\beta)\frac{\pi(u)}{\pi(u|e)} \in (0,1)$.

   (c) In addition, if $\sigma \geq 1$, then the only possible Ramsey steady state is an interior steady state.

2. *Under the parameter condition $(1-\beta)\frac{\pi(u)}{\pi(u|e)} \geq 1$, either the Ramsey allocation does not converge or it converges to a non-interior steady state. Moreover, if $\sigma \geq 1$, then the only possible Ramsey steady state is non-interior where aggregate consumption C_t converges to zero and the optimal tax rate τ_t converges to 1.*

Proof. See Appendix A.3. □

3.3 Interpretations

The most surprising result in Proposition 3 is that under proper parameter conditions the Ramsey planner opts to achieve FSI even at the cost of a possibly very high long-run tax rate:

Notice that the optimal tax rate τ is bounded above from 1 if and only if β is sufficiently large for any given probability distribution.[6] So $\tau = (1-\beta)\frac{\pi(u)}{\pi(u|e)}$ can become very close to 100% if β is small enough. Also, τ is increasing in the probability of unemployment and in the persistence of the idiosyncratic shock, suggesting that a higher ratio of government debt to gross domestic product (GDP) is needed to provide FSI when the risk of unemployment is high or the shock is more persistent.

In other words, to achieve FSI by equalizing the individual consumption levels across the two types of households, the optimal long-run tax rate may have to approach 100% in the Ramsey steady state. A high steady-state tax rate implies not only a high debt-to-GDP ratio but also that steady-state consumption is very low because household labor income is almost completely taxed away when τ is near 100%.[7]

Figure 1 shows the FSI steady-state policies and other endogenous variables in the Ramsey allocation as we reduce the time discount factor β. It shows that as the value of β decreases (from left to right), the implied optimal tax rate τ (top-left panel) rises from 0% to nearly 100%; consequently, the aggregate consumption (bottom-left panel) declines toward zero. This suggests that when households are very impatient, and for a given probability distribution of the employment shock, in order to achieve FSI the Ramsey planner opts to impose a very high tax rate in the long run such that the level of steady-state consumption and hours worked are extremely low. A high tax rate and a low labor supply also imply that total tax revenues will eventually be zero as well (a maximum is reached around $\beta = 0.925$, see the bottom-right panel); consequently, the optimal quantity of debt to support FSI will also decrease to zero when β becomes sufficiently small (top-right panel).

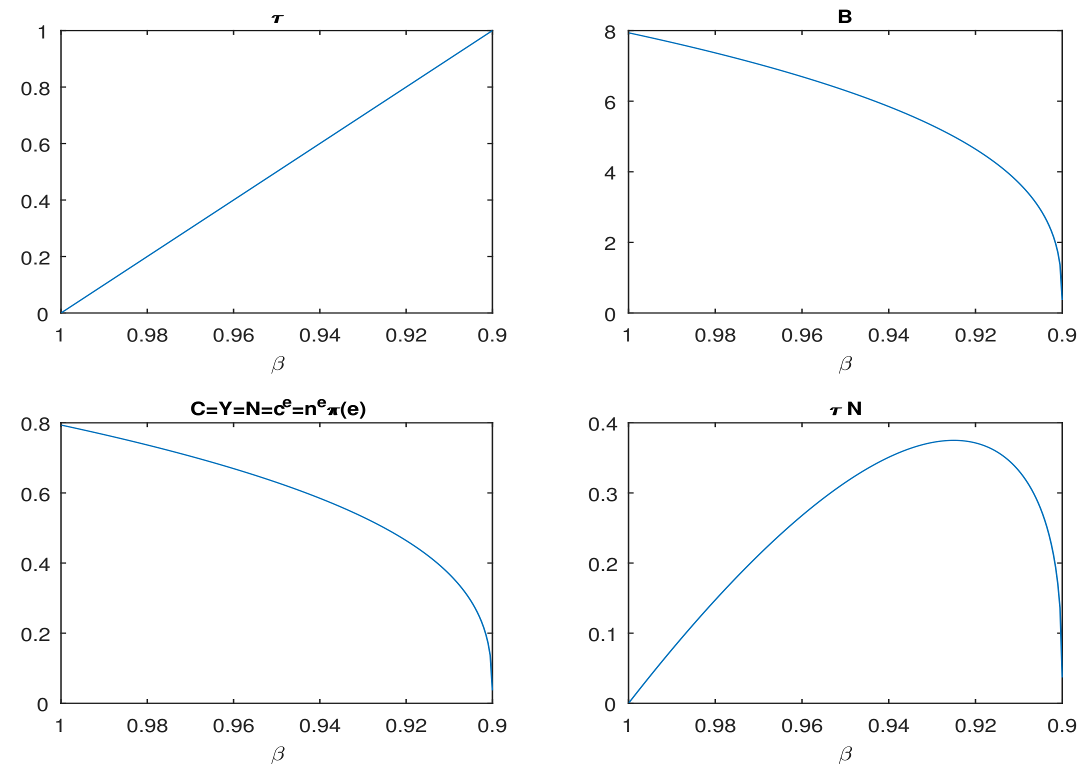
How can such steady-state FSI allocations be optimal? Common sense seems to tell us that the marginal benefit of reducing consumption risk to achieve FSI must decline with an increasing stock of public debt (or debt-to-GDP ratio), while the cost of financing the debt under distortionary taxes must also increase. Therefore, an FSI Ramsey allocation at all costs seems puzzling.

The fundamental reason for such a counterintuitive result is as follows. Since β is less

[6]Notice that if the idiosyncratic shock is not autocorrelated, then $\frac{\pi(u)}{\pi(u|e)} = 1$ and hence the parameter condition, $(1-\beta)\frac{\pi(u)}{\pi(u|e)} = (1-\beta) < 1$, is automatically satisfied. Consequently, the Ramsey steady state must be interior. The intuition is that under shocks with zero autocorrelation it is easy to achieve FSI and never optimal to impose a 100% tax rate on employed households.

[7]The fact that the optimal tax rate depends entirely on the discounting factor and employment probability but not on the curvature of the utility function could be a consequence of the ad hoc wealth-redistribution technology.

Figure 1: FSI Steady-State Relationship



Notes: Parameter values are $\gamma = \sigma = 2$, $\pi(e|e) = \pi(u|u) = 0.95$, $\pi(u|e) = \pi(e|u) = 0.05$ *and* $\pi(e) = \pi(u) = 0.5$.

than 1 and the market interest rate is lower than the time discount rate ($r < \beta^{-1}$), the Ramsey planner opts to increase (front-load) household consumption in the short run as a trade-off for low consumption in the long run. Because the future is heavily discounted by the high discounting factor compared to the interest rate, in the short run the labor tax rate can even be negative and the level of public debt can be higher than its steady-state value so as to stimulate work efforts and enhance risk sharing across households with different employment states. Consequently, the Ramsey planner has to finance the "sky-rocketing" debt level (relative to income) incurred during the transition period by increasing the steady-state tax rate even if this implies low consumption in the long run.

On the other hand, if the parameter condition $(1-\beta)\frac{\pi(u)}{\pi(u|e)} \geq 1$ holds, then there is no interior Ramsey steady state. Instead, the Ramsey steady state (if it exists) must be non-interior. Furthermore, if $\sigma \geq 1$, then the only possible Ramsey steady state is non-interior and it features zero consumption $\lim_{t\to\infty} C_t = 0$ and a 100% percent tax rate $\lim_{t\to\infty} \tau_t = 1$ in the limit.[8]

In what follows, we confirm our theoretical results with numerical simulations, which not only confirm the convergence property of the interior Ramsey allocation but also illustrate the pattern of optimal transition paths of the Ramsey allocation. Such numerical analyses are valid because the interior Ramsey steady state has been proved to exist and is unique under our parameter specifications. Notice that our numerical algorithm does not assume convergence.

Parameter Values. We set the preference parameters to $\gamma = \sigma = 2$, the time discounting factor to $\beta = 0.96$, and the transition probability matrix of (un)employment shocks to

$$\pi = \begin{bmatrix} \pi(u|u) & \pi(e|u) \\ \pi(u|e) & \pi(e|e) \end{bmatrix} = \begin{bmatrix} 0.95 & 0.05 \\ 0.05 & 0.95 \end{bmatrix}. \tag{25}$$

The transition probability matrix implies that the unconditional probability of employment and unemployment is $\pi(e) = \pi(u) = 0.5$ and that the shock is highly persistent.[9]

Ramsey Transition Paths. In Figure 2, we set the initial debt level B_0 to 50% of its Ramsey steady-state value. The initial values of the individual household wealth are then given by $a_0^e = B_0\pi(e|e)$ and $a_0^u = B_0\pi(u|e)$. These initial conditions are consistent with the requirement that the competitive equilibrium does not feature FSI.

The path of government debt in the figure (panel [2,1]—i.e., 2nd row, 1st column) shows a jump in the second period and over-shooting of its long-run steady state and then a gradual decrease toward the interior Ramsey steady state. However, when translating the debt level into the debt-to-GDP ratio (panel [3,2]), it implies a very high initial level of debt-to-GDP ratio of 692%; then this ratio jumps even higher, up to 852% in period 1, and then keeps increasing over time toward its steady-state level of 1000%. Such an enormously high debt-to-GDP ratio indicates strong incentives for the Ramsey planner to provide self insurance to the households when the shocks are highly persistent.

[8]It can be shown easily that the same kind of non-interior long-run outcome exists in a representative-agent model if the interest rate is exogenously given and lies below the time discount rate, which encourages the household to borrow heavily against future income in order to front-load consumption in the short run. The key difference here is that the interest rate is endogenous in our general-equilibrium model with heterogeneous agents and incomplete markets.

[9]Since our goal is not to simulate a realistic real-world economy, we do not intend to calibrate the model parameters to match real-world data. Instead, these numerical analyses are meant to demonstrate the mechanisms behind Proposition 3 and reveal how the economy transits to an FSI Ramsey steady state with a potentially very high long-run optimal tax rate.

Meanwhile, the optimal tax (panel [2,2]) jumps up in the first period to a very high rate of $\tau_0 = 61\%$, but drops immediately down to a large negative value of $\tau_1 = -60.5\%$ in the second period and then gradually increases toward a long-run rate of $\tau = 40\%$. Meanwhile, the gross interest rate $r = Q^{-1}$ (panel [3,1]) starts very high at 237%, jumps down to about 93% (implying a negative net interest rate of -7.24%) and then gradually increases over time to approach the time discount rate $\beta^{-1} = 1.04$ in the long run.

The path of aggregate consumption (panel [1,1]) shows a clear pattern of front-loading by jumping up sharply in the second period (period 1) and then gradually decreasing back to its long-run steady state. The reason that aggregate consumption does not jump up immediately in the first period to its maximum level is because the level of the initial bond supply is fixed and only the newly issued bonds can exhibit a sharp jump in the next period. Consequently, for the government budget to balance, the tax rate cannot decline too much in the first period, so we see a sharp increase in aggregate consumption only in the second period, and such a high level of consumption above its long-run value is supported by a negative tax rate in the same period.

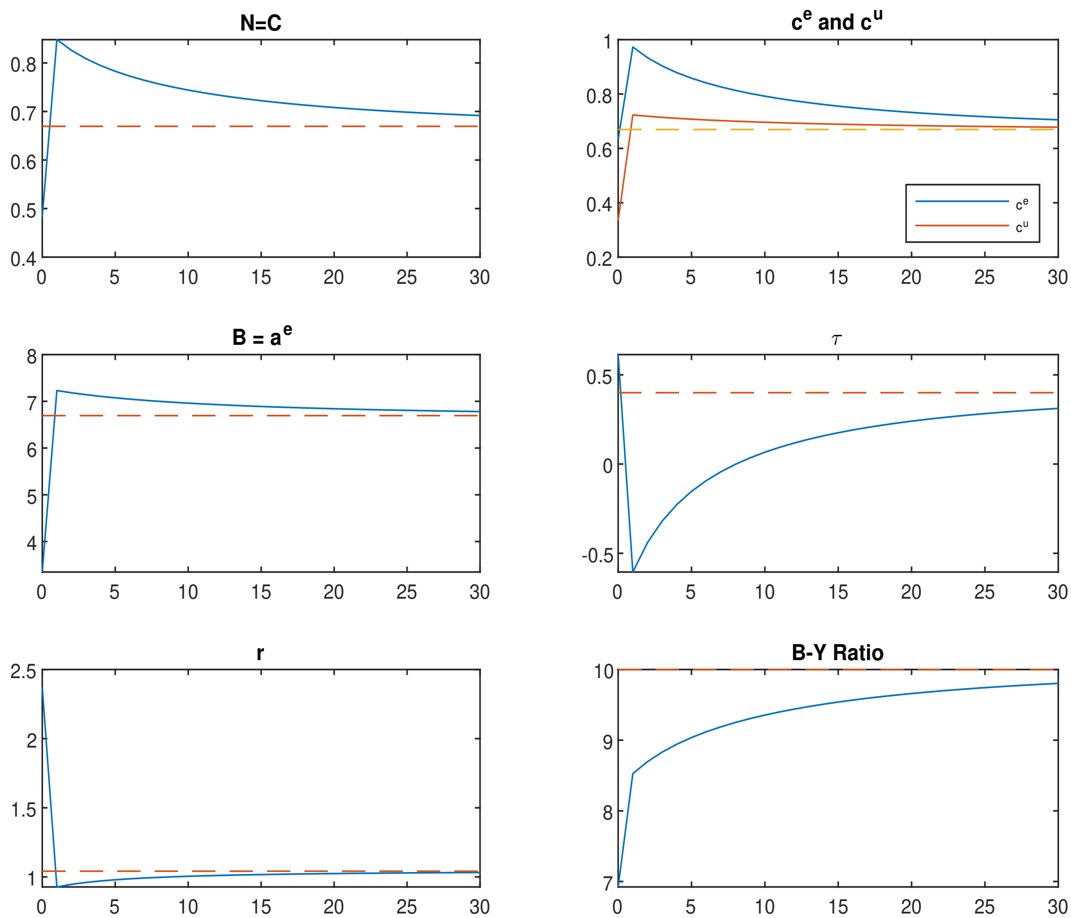
The individual consumption levels for employed and unemployed households also show big front-loading behavior (panel [1,2]). However, the panel shows a tremendous degree of consumption inequality during the initial phase of the front-loading period, with employed households consuming 47% and 26% more in the first two periods than unemployed households, respectively, despite the existence of a wealth-redistribution technology for risk sharing. However, the consumption inequality shrinks rapidly along the transition and eventually disappears completely in the long run when FSI is achieved.

4 Time Inconsistency

The key reason for the peculiar behavior of the Ramsey steady-state allocation illustrated in Figure 1—namely, the extremely high tax rate and low consumption level when β is relatively small—is that the Ramsey policies are time inconsistent. In this section we address the issue of time inconsistency in two ways: First, we numerically simulate the transitional Ramsey dynamics by setting the initial allocation (the total bond supply and asset distribution) to their respective Ramsey steady-state values and examine whether the Ramsey planner has incentives to deviate from this initial condition.[10] Second, we resolve the Ramsey problem

[10]Our simulation method is consistent with the Ramsey problem of the theoretical section in the sense that the planner takes the steady-state asset allocation as given and then resolves a dynamic date-0 optimization

Figure 2: Ramsey Transition Paths of Benchmark Parameter Values



Notes: Transition paths in the Ramsey economy (solid lines) and their corresponding steady states (dashed lines).

by maximizing the steady-state welfare of the competitive equilibrium instead of the time-zero dynamic welfare of the competitive equilibrium and check if it gives the same Ramsey allocation as that in the previous analysis.

problem. The Ramsey transitional dynamics would be in general quite different from that of the "timeless" reoptimization method by King and Wolman (1999) who additionally set the lagged Lagrangian multipliers (under the recursive approach) to their steady-state values.

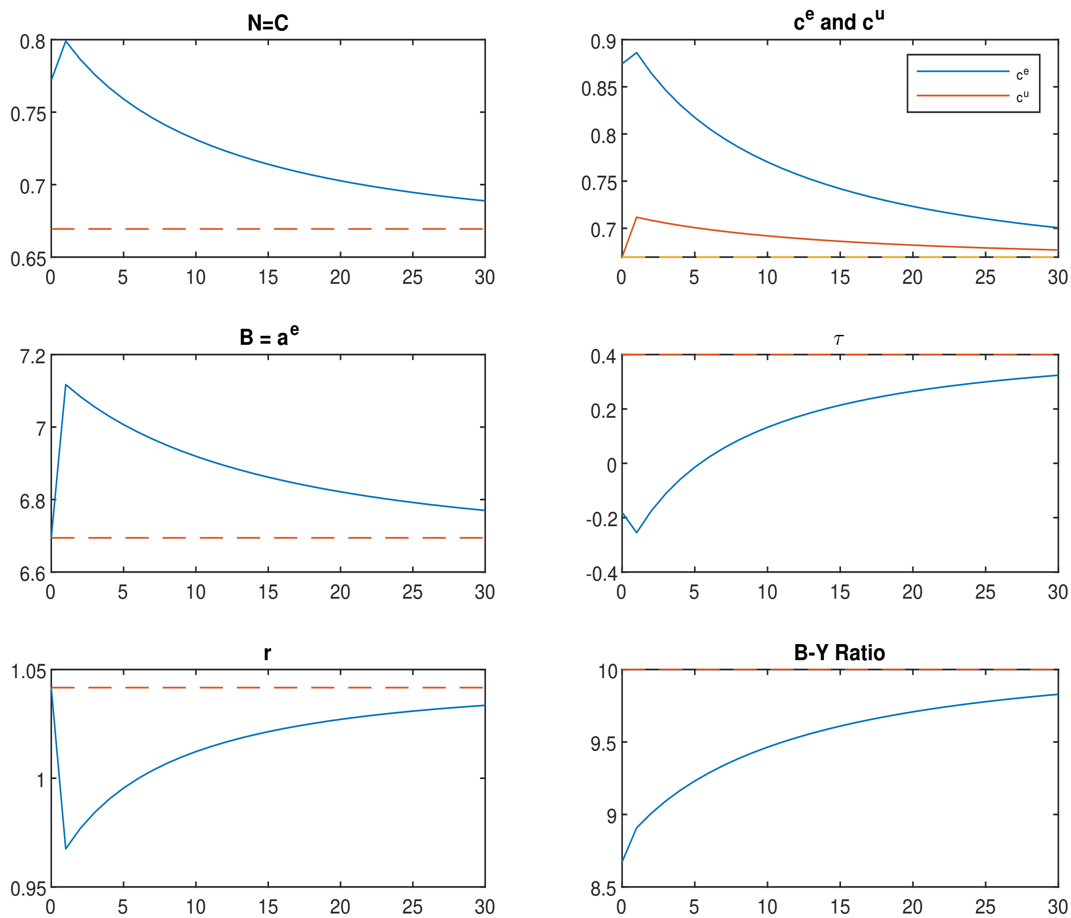
4.1 Numerical Evidence

Scenario 1. Since we know analytically where the Ramsey steady state is, in our numerical simulation we can set the initial bond supply and asset distribution to be equal to their respective Ramsey steady-state values. The choice of such initial values is deliberate and not innocuous: If the Ramsey policy is time consistent, then given that the initial allocation is already in the Ramsey steady state, the policy should be to let the initial allocation remain where it is forever without any policy changes. Conversely, if the Ramsey planner opts to re-adjust its policies temporarily and converge back to the same steady state in the long run, it is then a clear sign of time inconsistency in the Ramsey plan.

Figure 3 plots the transitional Ramsey path under the benchmark parameter values and the initial conditions identical to the interior Ramsey steady state. It is interesting to notice in the figure that the Ramsey planner "dislikes" the initial Ramsey steady-state allocation and opts to change its policies in the following manner: (i) In the first two periods the optimal tax rate (panel [2,2]) drops sharply from its long-run value to -18% and -26%, respectively; it then rises gradually over time back to its original steady-state value of 40%. (ii) In the meantime, the level of public debt (panel [2,1]) increases from its initial value of 6.7 to a level of 7.1 and then gradually declines back to the original value of 6.7 in the long run. Also, because labor and total output have increased under the negative tax rate (panel [1,1]), the public debt-to-GDP ratio (panel [3,2]) drops significantly in the beginning by 13% to 867%, and then converges back gradually to the original long-run value of 1000% over time. The higher-than-steady-state values of aggregate consumption and individual consumption in the top row of the figure indicate once again the policy maker's front-loading behavior because the Ramsey planner seizes the opportunity of policy change to boost consumption by reducing the tax rate. The long period of excessively low interest rate below the time discount rate (panel [3,1]) also entices the Ramsey planner to further boost bond supply above its steady-state value. However, because total output increases relatively more than the stock of public debt, the debt-to-GDP ratio falls below its steady-state value during the entire transition period (panel [3,3]).

Such a significant but transitory deviation from the initial Ramsey steady state via readjustment of policies clearly indicates that the Ramsey policy is time inconsistent—if there is any chance to readjust its steady-state policies and break away from its time-zero commitment, the Ramsey planner will do so in such a way that the original long-run policies appear to be "incorrect" or too aggressive, which discourages household work efforts and

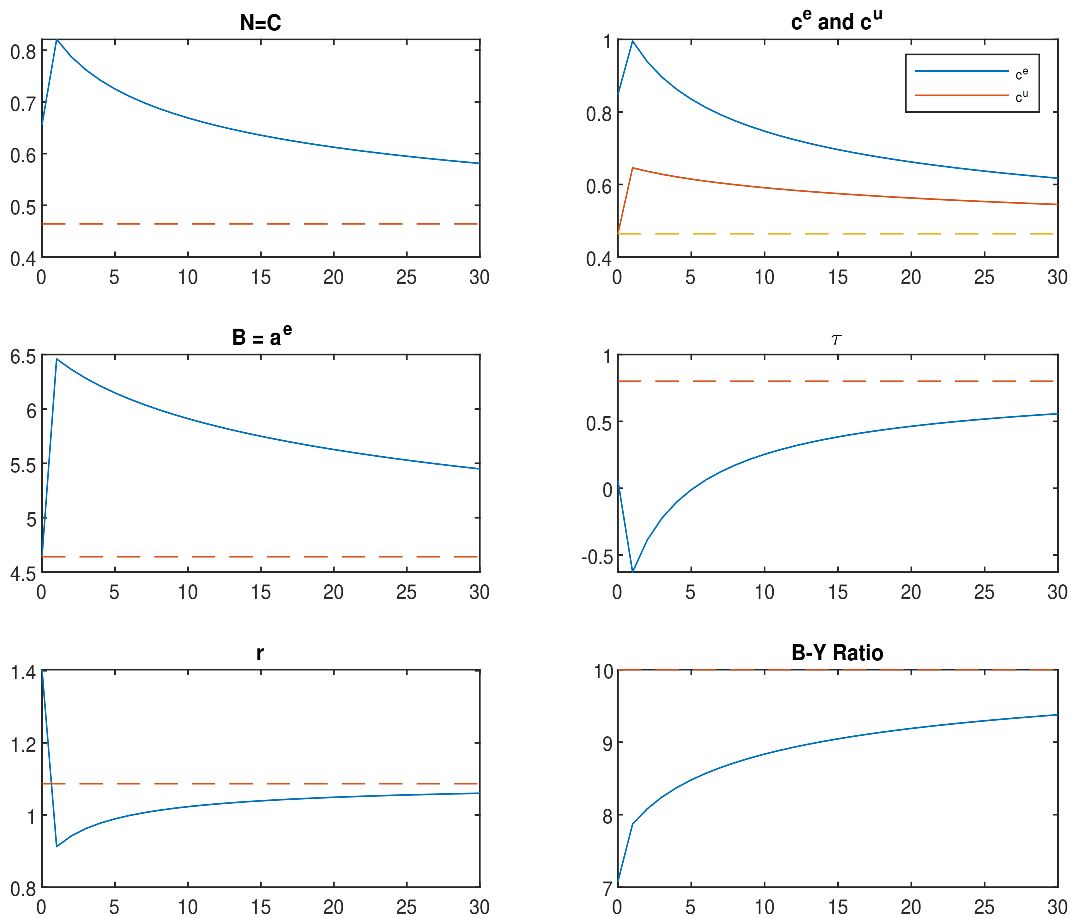
Figure 3: Transition Path Starting with Steady-State Wealth Distribution



Notes: Transition paths in the Ramsey economy (solid lines) and their corresponding steady states (dashed lines).

consumption. Therefore, the Ramsey planner opts to relax such policies sharply in the beginning (except for B_0) so as to stimulate work effort and raise consumption by lowering the tax rate. Since consumption for the employed and unemployed households does not rise in the same proportion, the original FSI allocation is broken so that more public debt is needed to improve self insurance. But since such a change or readjustment in policies is only a once-and-forever action and the planner knows that it must commit to the new policies forever, the Ramsey allocation will eventually converge back to the same initial Ramsey steady state as if no changes have taken place in the long run. In other words, the readjustment of policies is not a permanent deviation.

Figure 4: Transition Path with Low β



Notes: Transition paths in the Ramsey economy (solid lines) and their corresponding steady states (dashed lines).

Scenario 2. Our next experiment is to see what happens to the Ramsey steady state and the time-inconsistency policy if we reduce the parameter value of β from 0.96 to 0.92, such that households become less patient. As before, we assume that the initial bond supply and distribution of household wealth are already in the Ramsey steady state.

Figure 4 shows that the transition is different from that in Figure 3. First, as in Figure 3, the Ramsey planner clearly dislikes its initial allocation and opts to readjust its policies by sharply reducing its optimal tax rate downward to -0.6 and increasing its new bond supply to a level of 6.5 (about 812% of GDP).

Second, the steady-state level of public debt is much lower than before (4.64 v.s 6.69) and the steady-state tax rate is much higher than before ($\tau = 80\%$ v.s 40%). Consequently, the new steady-state levels of work effort and consumption are about 30% lower than before, thanks to the higher steady-state tax rate to discourage work and consumption.

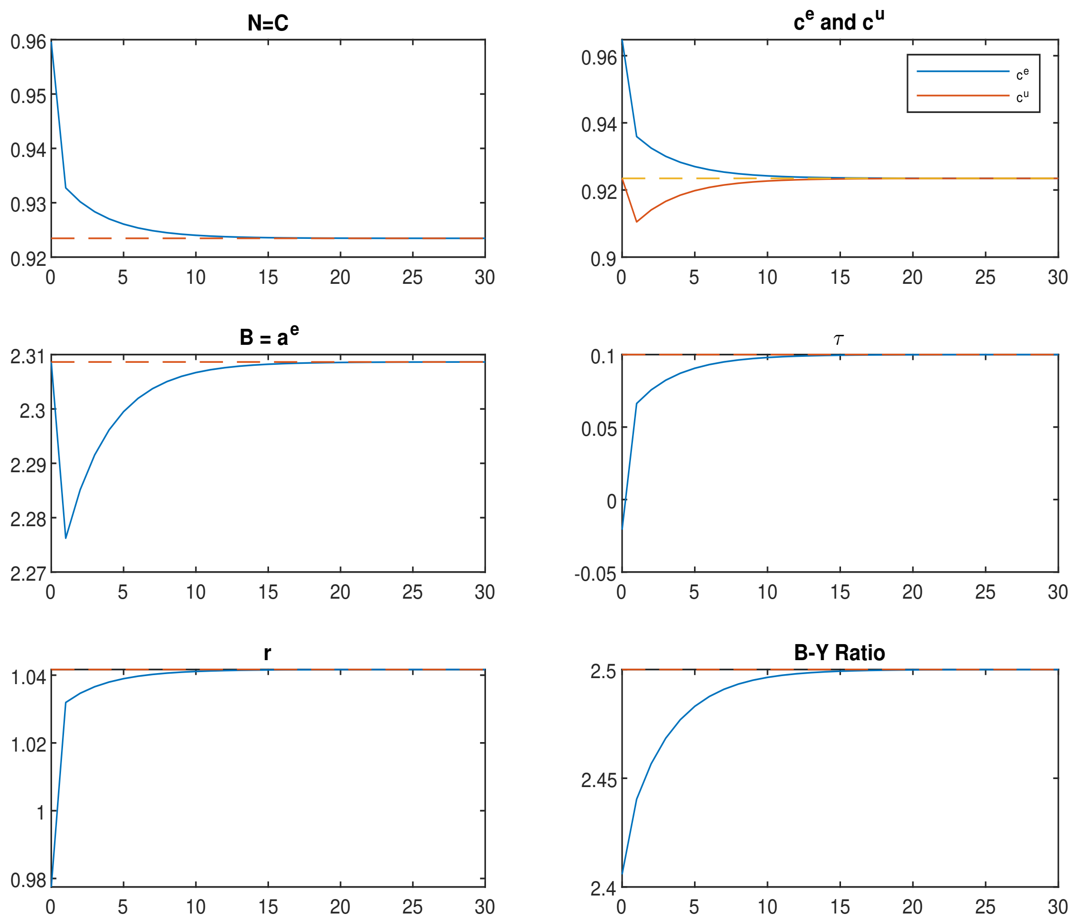
The reason is that the Ramsey planner opts to be more aggressive in front-loading consumption in the short run, knowing that the value of β is smaller and the interest rate tends to be much lower than the time discount rate. Hence, to pay for the short-run subsidies and high debt costs, the new steady-state policy appears too tight and the Ramsey planner will "regret" it and opt to relax it when it gets the chance, so the planner deviates from its steady-state policies more sharply than in Figure 3. Consequently, in period 1 consumption jumps up by more than 120% above the steady-state value for the employed households and more than 42% for the unemployed households. This enormously high level of consumption increase in the early phase of the transition is supported by a high level of work effort of the employed households (89% higher than its steady-state value), thanks to the very negative labor-income tax rate of $\tau_t = -111\%$ in the short run. In the meantime, the debt level rises by 42% above its steady state to support risk-sharing between the employed and unemployed household groups. All these short-run stimulating policies to front-load consumption are paid for by the very high tax rate ($\tau = 80\%$) in the long run after FSI is achieved.

Scenario 3. Notice that in the previous episodes, the probability matrix in equation (25) features a high persistence of shocks such that risk-sharing across employed and unemployed households is quite difficult—since the wealth-redistribution technology is quite imperfect when the shock is very persistent. This explains why the optimal debt-to-GDP ratio needs to be very high to support FSI allocation in the previous examples.

Now we consider another experiment where the unemployment shock becomes much less persistent with $\pi(u|u) = 0.65$, which implies $\pi(e|u) = 0.35$ and $\pi(u) = 0.125$.

The transition dynamics are shown in Figure 5. Notice the dramatic difference from the previous cases: First, the transition period is extremely short: Starting from the initial Ramsey steady-state allocation, the transition (deviation) under policy-readjustment lasts only about 15 periods in this case in converging back to the same Ramsey steady state, as opposed to several hundred periods in the previous scenarios. Second, the steady-state level of tax rate τ is now very small—only about 10% as opposed to an earlier 80%—and the steady-state bond supply is also much lower than before—about 50% lower than before—since it requires a smaller amount of bonds to support FSI when the shocks are less persistent and the consumption inequality is less severe. Consequently, the steady-state work effort is much higher than before ($N = 0.9235$ v.s 0.6694), which implies that the FSI steady-state

Figure 5: Transition Path with Lower Unemployed Households



Notes: Transition paths in the Ramsey economy (solid lines) and their corresponding steady states (dashed lines).

consumption for both types of households is now also much higher than before.

Again, once getting a chance of readjustment the Ramsey planner "regrets" its steady-state policies and opts to readjust the tax rate downward initially. Nonetheless, despite the time-inconsistency problem, the magnitude of the readjustment is very small compared with the previous cases. The reason is that under zero-correlated or weakly positive-correlated shocks, the wealth redistribution technology provides a very good channel of risk sharing and self insurance, such that the need (or room) for the Ramsey planner to use policies to achieve FSI allocation is much smaller than the cases with highly persistent shocks. Therefore, the need for readjustment is also smaller.

Scenario 4. Our last experiment sets $\beta = 0.999$, which is very close to one. As anticipated, Figure 6 shows that the incentives for front-loading consumption are much reduced now when households are extremely patient, such that the Ramsey planner opts to keep the original steady-state allocation virtually intact with only small changes in policies. For example, the maximum readjustment of the bond position is only about 0.06% relative to its steady state, and the readjustment in labor tax is only about 15 basis points. This high parameter value of β also implies that the gap between the interest rate and the time discount rate is quite small for the Ramsey planner to take advantage of (the difference is only 0.2 basis points), resulting in a much lower steady-state level of public debt (the steady-state level of debt is now about 2.2) and a much lower steady-state tax rate ($\tau = 1\%$), and such long-run policies are good enough to support FSI allocation in the long run.

4.2 Steady-State Welfare Analysis

The above analysis shows that the Ramsey planner has a dominating incentive to pursue FSI allocation even at the costs of extremely high distortionary tax rates. We asked ourselves why the Ramsey planner must pursue an FSI allocation when the marginal benefit of increasing debt is diminishing while the marginal cost of distortionary future taxes is rising.

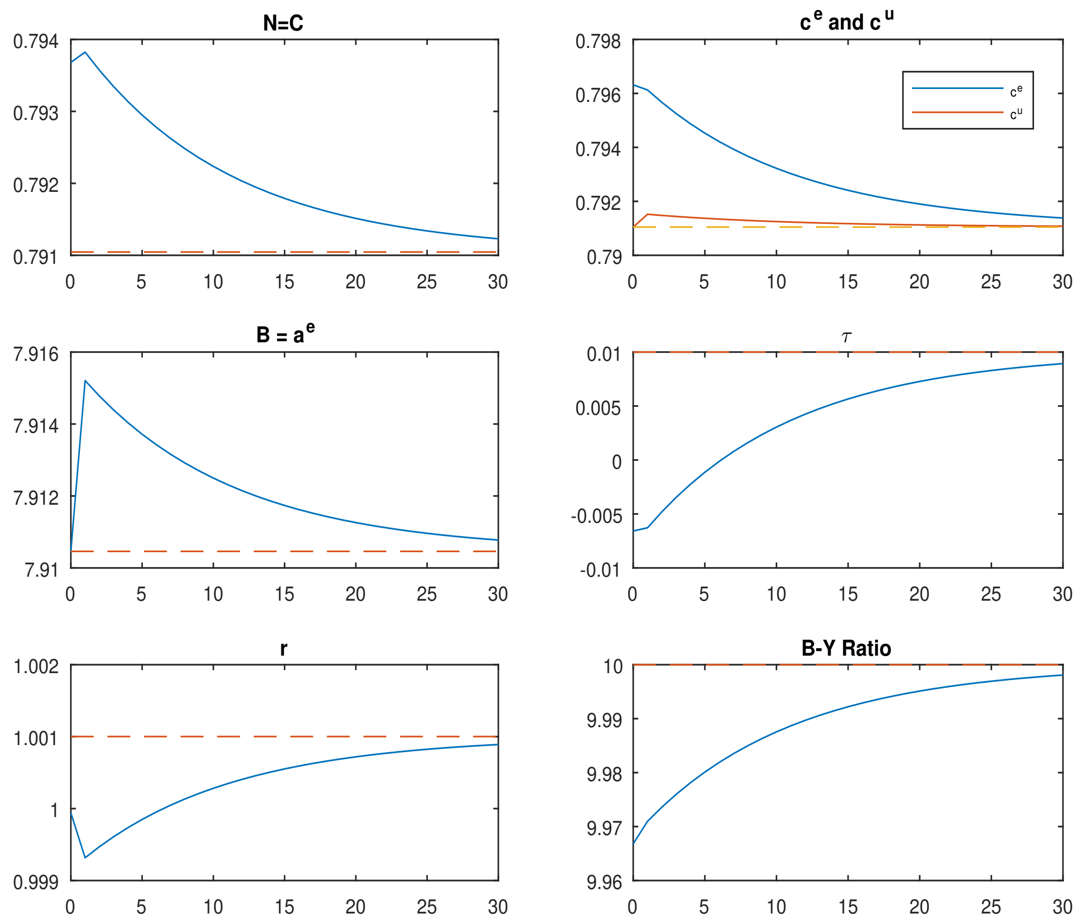
Our answer provided above was that as long as $\beta < 1$ and $r < \beta^{-1}$, the marginal social cost of borrowing is strictly less than the marginal social benefit of providing self-insurance. Therefore, the Ramsey planner has incentives to increase bond supply to pursue FSI allocation and front-load consumption even if this implies a close to 100% tax rate in the long run to finance the sky-rocketing public debt-to-GDP ratio.

We also pointed out in the above analysis that such a policy is time-inconsistent—that is, it is optimal only ex ante, but not ex post. Our numerical analysis confirmed this conjecture by showing that whenever the planner gets the chance to readjust its steady-state policies, it will do so even if the economy is initially already in the Ramsey steady state; although, such a deviation is only temporary because of new commitment.

To further support our argument that the Ramsey policies are time-inconsistent, here we conduct a different kind of analysis by supposing that the Ramsey planner maximizes only the steady-state welfare of the competitive equilibrium instead of the time-zero present value of the dynamic path of social welfare. We will see how the Ramsey planner designs its optimal debt and tax policies in this situation.

To maximize the steady-state welfare of the economy, the Ramsey problem under the

Figure 6: Transition Path with β close to 1



Notes: Transition paths in the Ramsey economy (solid lines) and their corresponding steady states (dashed lines).

primal approach becomes:

$$\max_{\{c^e,c^u,n^e,a^e,a^u\}} \left[u(c^e) - v\left(n^e\right)\right]\pi(e) + u(c^u)\pi(u)$$

subject to

$$n^e\pi(e) - c^e\pi(e) - c^u\pi(u) \geq 0, \tag{26}$$

$$u_c^e c^e\pi(e) - v_n^e n^e\pi(e) + a^e\beta(u_c^e\pi(e|e) + u_c^u\pi(u|e)) - u_c^e(a^e\pi(e|e) + a^u\pi(e|u)) = 0, \tag{27}$$

$$u_c^e c^u\pi(u) + a^u\beta(u_c^e\pi(e|e) + u_c^u\pi(u|e)) - u_c^e(a^e\pi(u|e) + a^u\pi(u|u)) = 0, \tag{28}$$

$$a^u \geq 0, \tag{29}$$

$$g(c^e, c^u) \equiv \pi(e|e) + \frac{u_c^u}{u_c^e}\pi(u|e) - \frac{u_c^e}{u_c^u}\pi(e|u) - \pi(u|u) \geq 0, \tag{30}$$

$$a^u g(c^e, c^u) = 0. \tag{31}$$

Note that this Ramsey problem features no dynamics and that there is no dynamic consideration for the Ramsey planner. In other words, the "future" is no longer "discounted" compared to the "present." We prove that the FSI allocation is *not optimal*, as shown in the following Proposition:

Proposition 4. *If the Ramsey planner considers only the steady-state welfare of the competitive equilibrium, then an FSI allocation is not optimal. Namely, it is optimal to set* $c^e > c^u$ *and to have the borrowing constraint of the unemployed households strictly binding:* $a^u = 0$.

Proof. See Appendix A.4. □

The result in Proposition 4 is intuitive. By maximizing only the steady-state welfare of the competitive economy, the Ramsey planner no longer has the incentive to exploit the difference between the interest rate and the time discount rate since the time discount rate is no longer relevant in maximizing the steady-state welfare. Consequently, the issue of front-loading consumption also becomes irrelevant. In this case, the Ramsey planner does not choose FSI allocation by equalizing consumption across the employed and unemployed households—because the cost of doing so by issuing too much debt is too high at the margin when there is no time discounting. In other words, from the competitive equilibrium's steady-state welfare point of view, the marginal benefit of achieving FSI by increasing public debt is at some point dominated by the marginal cost of distortionary taxation, such that the Ramsey planner will issue bonds only to a certain level but not all the way to achieve FSI.

5 Conclusion

A *signature* property of standard infinite-horizon heterogenous-agents models is that the market interest rate is lower than the time discount rate (regardless of capital). This property emerges because precautionary saving motives under imperfect risk sharing entice agents to hold an excess amount of liquid assets even if their rate of return is dominated by the time discount rate, suggesting a liquidity premium to be paid by holding such assets under borrowing constraint and non-diversifiable risk.

This fundamental property of standard heterogeneous-agents models, however, implies that the Ramsey planner's fiscal policy may be time inconsistent. This is because the planner has a dominant incentive to front load consumption until all households are fully self-insured whenever the interest cost of borrowing is lower than the time discount rate (i.e., $r < \beta^{-1}$). But such an intended full self-insurance allocation may be either infeasible or time inconsistent because, to achieve it, the optimal labor tax rate may approach 100% such that the Ramsey planner is willing to change its long-run policies ex post when it gets a chance to do so.

This is puzzling from an intuitive perspective because near the point of full self-insurance, the marginal gains of increasing debt should be less than the marginal costs to finance the debt under distortionary taxes. We argue in this paper that this seemingly puzzling behavior of the Ramsey planner achieving full self-insurance in the long run at "all costs" originates from the assumption that the planner must commit to any future plans at time zero. Under such a full commitment, the Ramsey planner opts to exploit the low interest cost of borrowing to front load consumption by sacrificing future consumption in the long run because future utilities are heavily discounted compared to the inverse of the interest rate on government bonds.

We demonstrate our points both analytically and numerically using a tractable infinite-horizon heterogeneous-agents model featuring non-linear preferences and a well-defined distribution of household wealth. We also use both numerical simulations and steady-state welfare analysis to show that the Ramsey polices are time-inconsistent, such that the planner opts to deviate from its long-run policies whenever it gets a chance or if it does not need to discount the future according to the household time discounting factor.

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A Appendix

A.1 Proof of Proposition 1

A.1.1 Proof of $a^e_{t+1} > 0$

First, it is impossible for both $a^e_t = 0$ and $a^u_t = 0$ if $B_t > 0$, since it violates the asset market-clearing condition.

Second, if $\pi(e|e) + \pi(u|u) \geq 1$ and $\frac{a^e_t}{\pi(e)} > \frac{a^u_t}{\pi(u)}$, then it must be true that $\frac{a^e_t \pi(e|e) + a^u_t \pi(e|u)}{\pi(e)} \geq \frac{a^e_t \pi(u|e) + a^u_t \pi(u|u)}{\pi(u)}$ and $\frac{a^e_{t+1}}{\pi(e)} > \frac{a^u_{t+1}}{\pi(u)}$ for all $t \geq 0$; namely, the period-t initial wealth of employed households is no less than that of unemployed households, and the newly accumulated wealth of the employed households is greater than that of the unemployed households. This can be seen by the following steps:

(i) Given $\pi(e|e) + \pi(u|u) \geq 1$, then $\frac{a^e_t}{\pi(e)} > \frac{a^u_t}{\pi(u)}$ can be rewritten as

$$\frac{a^e_t}{\pi(e)} \left[\pi(e|e) + \pi(u|u) - 1\right] \geq \frac{a^u_t}{\pi(u)} \left[\pi(u|u) - 1 + \pi(e|e)\right],$$

which together with $\pi(e) = \frac{\pi(e|u)}{\pi(e|u)+\pi(u|e)}$ and $\pi(u) = \frac{\pi(u|e)}{\pi(e|u)+\pi(u|e)}$ implies

$$\frac{a^e_t}{\pi(e)} \left[\pi(e|e) - \frac{\pi(e)}{\pi(u)}\pi(u|e)\right] \geq \frac{a^u_t}{\pi(u)} \left[\pi(u|u) - \frac{\pi(u)}{\pi(e)}\pi(e|u)\right],$$

where the equality holds if $\pi(e|e) + \pi(u|u) = 1$. By rearranging terms in the equation above, we obtain

$$\frac{a^e_t \pi(e|e) + a^u_t \pi(e|u)}{\pi(e)} \geq \frac{a^e_t \pi(u|e) + a^u_t \pi(u|u)}{\pi(u)}.$$

which means that in every period $t \geq 0$, if $\frac{a^e_t}{\pi(e)} > \frac{a^u_t}{\pi(u)}$, then it must be true that $\frac{a^e_t \pi(e|e) + a^u_t \pi(e|u)}{\pi(e)} \geq \frac{a^e_t \pi(u|e) + a^u_t \pi(u|u)}{\pi(u)}$.

(ii) Suppose that the employed households are no poorer than the unemployed households in the beginning of each period $t \geq 0$ (namely, $\frac{a^e_t \pi(e|e) + a^u_t \pi(e|u)}{\pi(e)} \geq \frac{a^e_t \pi(u|e) + a^u_t \pi(u|u)}{\pi(u)}$), which together with their higher labor income suggests that the employed agents must have higher consumption, $c^e_t \geq c^u_t$. Moreover, in order to smooth consumption, the employed households have precautionary saving motives to self-insure against the positive possibility of switching to an unemployed state in the future; as a result, the employed households also have higher savings: $\frac{a^e_{t+1}}{\pi(e)} > \frac{a^u_{t+1}}{\pi(u)}$.

Given the discussions above, it must be true that $c^e_t \geq c^u_t$ and $\frac{a^e_{t+1}}{\pi(e)} > \frac{a^u_{t+1}}{\pi(u)} \geq 0$ if (a)

$\pi(e|e)+\pi(u|u)\geq 1$, (b) $\frac{a_0^e}{\pi(e)} > \frac{a_0^u}{\pi(u)}$ and (c) $B_t > 0$ for all $t \geq 0$. In other words, $\kappa_t^e = 0$ and $\kappa_t^u \geq 0$ for all t.

A.1.2 Proof of Equation (15)

Equation (11) together with $\kappa_t^e = 0$ gives equation (15).

A.1.3 Proof of Full Self-Insurance

Suppose $\kappa^e = \kappa^u = 0$ in the steady state, then equations (11) and (12) imply

$$\frac{\xi^e}{\xi^u} = \frac{\pi(e|e) + \frac{\xi^u}{\xi^e}\pi(u|e)}{\pi(e|u) + \frac{\xi^u}{\xi^e}\pi(u|u)},$$

which can be rewritten as

$$\left(\frac{\xi^e}{\xi^u}\right)^2 \pi(e|u) + \frac{\xi^e}{\xi^u}\left(\pi(u|u) - \pi(e|e)\right) - \pi(u|e) = 0.$$

The equation above has one positive root and one negative root. The positive root implies $\xi^u = \xi^e$, which suggests $c^e = c^u$. The negative root violates the requirement that both $c^u > 0$ and $c^e > 0$. Given that $c^e = c^u$, equation (15) implies $Q = \beta$ in the steady state.

A.2 Proof of Proposition 2

A.2.1 The "If" Part:

Given the initial value of B_0 as well as the allocation $\{c_t^e, n_t^e, c_t^u, a_{t+1}^e, a_{t+1}^u\}_{t=0}^{\infty}$, a competitive equilibrium can be constructed by using the two conditions in Proposition 2 and by following the steps below that uniquely back up the sequences of the other variables.

1. Aggregate C_t and N_t are chosen to satisfy

$$\begin{aligned} N_t &= n_t^e \pi(e), \\ C_t &= c_t^e \pi(e) + c_t^u \pi(u). \end{aligned}$$

2. w_t is set to 1.

3. τ_t is chosen to satisfy

$$\frac{v^e_{n,t}}{u^e_{c,t}} = \widehat{w}_t = (1-\tau_t). \tag{32}$$

4. Q_{t+1} is chosen to satisfy the Euler equation

$$Q_{t+1}u^e_{c,t} = \beta(u^e_{c,t+1}\pi(e|e) + u^u_{c,t+1}\pi(u|e)). \tag{33}$$

5. B_{t+1} is pinned down by the asset market-clearing condition

$$B_{t+1} = a^e_{t+1} + a^u_{t+1}.$$

6. The following constraints are satisfied:

   (a) By plugging equations (32) and (33) into household budget constraints we can obtain the implementability conditions displayed in equations (17), (18), (19), and (20).

   (b) The resource constraint can be rewritten as equation (16).

   (c) To satisfy the households' FOCs (12) and borrowing constraints for unemployed households, we have listed equations (21), (22) and (23) as constraints in Proposition 2.

7. Finally, it is straightforward to verify that the implementability condition together with the resource constraint implies the government budget constraint.

A.2.2 The "Only If" Part:

The constraints listed in Proposition 2 are trivially satisfied because they are part of the competitive-equilibrium conditions.

A.3 Proof of Proposition 3

A.3.1 Ramsey FOCs

We first state the Ramsey FOCs. Denote $\beta^t\mu_t$, λ^e_0, λ^u_0, $\beta^t\lambda^e_t$, $\beta^t\lambda^u_t$, $\beta^{t+1}v^1_t$, $\beta^{t+1}v^2_t$, and $\beta^{t+1}v^3_t$ as the Lagrangian multipliers for conditions (16)-(23), respectively. For all $t \geq 0$, the FOCs of the Ramsey problem with respect to n_t^e, a_{t+1}^e, and a_{t+1}^u are given, respectively, by

$$v_{n,t}^e + \lambda_t^e(v_{n,t}^e + v_{nn,t}^e n_t^e) = \mu_t, \text{ for } t \geq 0 \tag{34}$$

$$\lambda_t^e(u_{c,t+1}^e \pi(e|e) + u_{c,t+1}^u \pi(u|e)) = \lambda_{t+1}^e u_{c,t+1}^e \pi(e|e) + \lambda_{t+1}^u u_{c,t+1}^e \pi(u|e), \text{ for } t \geq 0 \tag{35}$$

and

$$\begin{aligned} \lambda_t^u(u_{c,t+1}^e \pi(e|e) + u_{c,t+1}^u \pi(u|e)) &= \lambda_{t+1}^u u_{c,t+1}^e \pi(u|u) + \lambda_{t+1}^e u_{c,t+1}^e \pi(e|u) \\ &\quad + v_t^1 + v_t^3 g(c_t^e, c_t^u, c_{t+1}^e, c_{t+1}^u), \end{aligned} \tag{36}$$

respectively.

For all $t \geq 1$, the FOCs of the Ramsey problem with respect to c_t^e and c_t^u are given, respectively, by

$$\begin{aligned} & (u_{c,t}^e - \mu_t)\pi(e) + \lambda_t^e(u_{c,t}^e + u_{cc,t}^e c_t^e)\pi(e) - \lambda_t^e u_{cc,t}^e (a_t^e \pi(e|e) + a_t^u \pi(e|u)) \\ & + \lambda_{t-1}^e a_t^e u_{cc,t}^e \pi(e|e) + \lambda_t^u u_{cc,t}^e c_t^u \pi(u) - \lambda_t^u u_{cc,t}^e (a_t^e \pi(u|e) + a_t^u \pi(u|u)) \\ & + \lambda_{t-1}^u a_t^u u_{cc,t}^e \pi(e|e) + \beta v_t^2 \frac{\partial g_t}{\partial c_t^e} + v_{t-1}^2 \frac{\partial g_{t-1}}{\partial c_t^e} + \beta a_{t+1}^u v_t^3 \frac{\partial g_t}{\partial c_t^e} + a_t^u v_{t-1}^3 \frac{\partial g_{t-1}}{\partial c_t^e} \\ = {} & 0, \end{aligned} \tag{37}$$

and

$$\begin{aligned} & (u_{c,t}^u - \mu_t)\pi(u) + \lambda_t^u u_{c,t}^e \pi(u) + \lambda_{t-1}^u a_t^u u_{cc,t}^u \pi(u|e) \\ & + \lambda_{t-1}^e a_t^e u_{cc,t}^u \pi(u|e) + \beta v_t^2 \frac{\partial g_t}{\partial c_t^u} + v_{t-1}^2 \frac{\partial g_{t-1}}{\partial c_t^u} + \beta a_{t+1}^u v_t^3 \frac{\partial g_t}{\partial c_t^u} + a_t^u v_{t-1}^3 \frac{\partial g_{t-1}}{\partial c_t^u} \\ = {} & 0. \end{aligned} \tag{38}$$

For $t = 0$, the FOCs of the Ramsey problem with respect to c_0^e and c_0^u are given, respectively, by

$$\begin{aligned} & (u_{c,0}^e - \mu_0)\pi(e) + \lambda_0^e(u_{c,0}^e + u_{cc,0}^e c_0^e)\pi(e) - \lambda_0^e u_{cc,0}^e a_0^e \\ & + \lambda_0^u u_{cc,0}^e c_0^u \pi(u) - \lambda_0^u u_{cc,0}^e a_0^u + \beta v_0^2 \frac{\partial g_0}{\partial c_0^e} + \beta a_1^u v_0^3 \frac{\partial g_0}{\partial c_0^e} \\ = {} & 0, \end{aligned}$$

and

$$(u^u_{c,0} - \mu_0)\pi(u) + \lambda^u_0 u^e_{c,0}\pi(u) + \beta v^2_0 \frac{\partial g_0}{\partial c^u_0} + \beta a^u_1 v^3_0 \frac{\partial g_0}{\partial c^u_0} = 0.$$

Note that

$$\begin{aligned}
\frac{\partial g_t}{\partial c^u_t} &= \frac{u^u_{cc,t}}{u^e_{c,t+1}\pi(e|u) + u^u_{c,t+1}\pi(u|u)}, \\
\frac{\partial g_t}{\partial c^e_t} &= -\frac{u^e_{cc,t}}{u^e_{c,t+1}\pi(e|e) + u^u_{c,t+1}\pi(u|e)},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial g_{t-1}}{\partial c^u_t} &= -\frac{u^u_{c,t-1}u^u_{cc,t}\pi(u|u)}{\left(u^e_{c,t}\pi(e|u) + u^u_{c,t}\pi(u|u)\right)^2} + \frac{u^e_{c,t-1}u^u_{cc,t}\pi(u|e)}{\left(u^e_{c,t}\pi(e|e) + u^u_{c,t}\pi(u|e)\right)^2}, \\
\frac{\partial g_{t-1}}{\partial c^e_t} &= -\frac{u^u_{c,t-1}u^e_{cc,t}\pi(e|u)}{\left(u^e_{c,t}\pi(e|u) + u^u_{c,t}\pi(u|u)\right)^2} + \frac{u^e_{c,t-1}u^e_{cc,t}\pi(e|e)}{\left(u^e_{c,t}\pi(e|e) + u^u_{c,t}\pi(u|e)\right)^2}.
\end{aligned}$$

A.3.2 Existence of FSI Interior Ramsey Steady State

By the following steps, we conjecture and verify that under the parameter condition $(1 - \beta)\frac{\pi(u)}{\pi(u|e)} < 1$, there exists an FSI interior Ramsey steady state featuring (i) $c^e = c^u > 0$, (ii) $a^e > a^u = 0$, (iii) $v^1_t = 0$, and (iv) $Q = \beta$.

1. When $c^e = c^u$ and $v^1_t = 0$, the steady-state version of the Ramsey FOCs with respect to a^e and a^u can be rewritten as

$$\lambda^e_t = \lambda^e_{t+1}\pi(e|e) + \lambda^u_{t+1}\pi(u|e),$$

$$\lambda^u_t = \lambda^u_{t+1}\pi(u|u) + \lambda^e_{t+1}\pi(e|u),$$

where the second equation utilizes the fact that $g(c^e, c^u, c^e, c^u) = 0$ if $c^e = c^u$. These two equations imply $\lambda^e = \lambda^u$.

2. When $c^e = c^u$, $v^1 = 0$, and $a^u = 0$, the FOC with respect to c^u_t is given by

$$\begin{aligned}
&(u^u_c - \mu)\pi(u) + \lambda^u u^e_c\pi(u) + \lambda^u a^e u^u_{cc,t}\pi(u|e) \\
&+\beta v^2 \frac{\partial g}{\partial c^u} + v^2 \frac{\partial g_{-1}}{\partial c^u} = 0,
\end{aligned}$$

and the FOC with respect to c_t^e is given by

$$
\begin{aligned}
&(u_c^e-\mu)\pi(e)+\lambda^e(u_c^e+u_{cc}^e c^e)\pi(e)-\lambda^e u_{cc}^e a^e \\
&+\lambda^e a^e u_{cc}^e \pi(e|e)+\lambda^u u_{cc}^e c^u \pi(u)+\beta v^2\frac{\partial g}{\partial c^e}+v^2\frac{\partial g_{-1}}{\partial c^e}=0.
\end{aligned}
$$

Note that under $c^e=c^u$, we have

$$
\frac{\partial g}{\partial c^u}=\frac{u_{cc}^e}{u_c^e}=-\frac{\partial g}{\partial c^e},
$$

$$
\begin{aligned}
\frac{\partial g_{-1}}{\partial c^u} &= -\frac{u_{cc}^e\pi(u|u)}{u_c^e}+\frac{u_{cc,t}^e\pi(u|e)}{u_c^e}, \\
\frac{\partial g_{-1}}{\partial c^e} &= -\frac{u_{cc}^e\pi(e|u)}{u_c^e}+\frac{u_{cc}^e\pi(e|e)}{u_c^e}.
\end{aligned}
$$

In the steady state, combining the above two FOCs together with $c^e=c^u$ leads to

$$
u_c^e+\lambda^e u_c^e(1-\sigma)=\mu,
$$

which, together with the FOC with respect to v_n^e:

$$
v_n^e+\lambda^e v_n^e(1+\gamma)=\mu,
$$

solves for λ^e and μ in the Ramsey steady state. The value of v^2 can then be solved by using the FOC with respect to c_t^e.

3. Finally, the optimal long-run tax rate τ can be solved by the following steps:

   (a) $Q=\beta$ by equation (15).

   (b) The condition (20) is simplified to $a^e\pi(u|e)-c^u\pi(u)=0$, which implies

   $$
   a^e=c^e\frac{\pi(u)}{\pi(u|e)}.
   $$

   (c) The resource condition (16) gives $n^e=c^e/\pi(e)$.

   (d) Given $Q=\beta$ and $B=a^e$, the government budget constraint becomes

   $$
   \tau n^e\pi(e)=(1-\beta)a^e,
   $$

which together with $n^e\pi(e) = c^e$ and $a^e = c^e\frac{\pi(u)}{\pi(u|e)}$ implies

$$\tau = \frac{(1-\beta)}{n^e\pi(e)}a^e = \frac{(1-\beta)}{c^e}a^e = (1-\beta)\frac{\pi(u)}{\pi(u|e)}.$$

(e) Given equation (32), τ must be less than 1 since $v_n^e > 0$ and $u_c^e > 0$. Hence, for the Ramsey steady state to be interior, it requires a restriction on the parameter values satisfying

$$(1-\beta)\frac{\pi(u)}{\pi(u|e)} < 1.$$

4. We can verify that this steady-state allocation satisfies all Ramsey FOCs since the rest of the constraints, (21), (22) and (23), are trivially satisfied.

A.3.3 Uniqueness of FSI Interior Ramsey Steady State

We now prove that this FSI interior Ramsey steady state is unique—i.e., there cannot exist an interior Ramsey steady state featuring consumption inequality or partial self-insurance—provided that the additional parameter restriction for the elasticity of intertemporal substitution $\sigma \geq 1$ is satisfied. We will also prove in the next subsection that a non-interior Ramsey steady state is impossible if $(1-\beta)\frac{\pi(u)}{\pi(u|e)} < 1$ and $\sigma \geq 1$.

We prove this by contradiction. Suppose this is not true such that $c^e > c^u$; then given that the borrowing constraint for the unemployed households must be strictly binding with $a^u = 0$ and $v_t^1 > 0$, we can consider the following arguments:

1. From the Ramsey FOC with respect to n^e, we know that for a steady state to exist, the growth rate of λ_t^e and μ_t have to be the same. Denote their steady state growth rate by g_λ^e and g_μ, respectively.

2. The FOCs with respect to a^e and a^u give

$$1 < \frac{Q}{\beta} = \pi(e|e) + \frac{u_c^u}{u_c^e}\pi(u|e) = g_\lambda^e\pi(e|e) + g_\lambda^u\frac{\lambda_t^u}{\lambda_t^e}\pi(u|e) < g_\lambda^u\pi(u|u) + g_\lambda^e\frac{\lambda_t^e}{\lambda_t^u}\pi(e|u), \quad (39)$$

where g_λ^e and g_λ^u denote the growth rate of λ_t^e and λ_t^u, respectively. We can show that $g_\lambda^e = g_\lambda^\mu$ by considering each of the following cases:

(a) Suppose $g_\lambda^e < g_\lambda^u$, then $\frac{\lambda_t^u}{\lambda_t^e} \to \infty$. Equation (39) becomes

$$\begin{aligned} \pi(e|e) + \frac{u_c^u}{u_c^e}\pi(u|e) &= g_\lambda^e \pi(e|e) + g_\lambda^u \frac{\lambda_t^u}{\lambda_t^e}\pi(u|e) \\ &\Longrightarrow \infty < g_\lambda^u \pi(u|u), \end{aligned}$$

which is impossible.

(b) Suppose $g_\lambda^e > g_\lambda^u$, so the FOC with respect to c^u (under $v^2 = 0$, $a^u = 0$, and $a^e\pi(u|e) = c^u\pi(u)$) becomes

$$u_c^u + g_\lambda^u \lambda_{t-1}^u u_c^e + \lambda_{t-1}^e u_{cc}^u c^u = g\mu_{t-1}\mu_t,$$

which implies

$$\frac{u_c^u}{\lambda_{t-1}^e} + g_\lambda^u \frac{\lambda_{t-1}^u}{\lambda_{t-1}^e} u_c^e + u_{cc}^u c^u = g_\mu \frac{\mu_{t-1}}{\lambda_{t-1}^e}.$$

As $t \to \infty$, the left-hand side is negative and the right-hand side is positive, a contradiction.

(c) Then, it must be true that $g_\lambda^e = g_\lambda^u$.

3. Equation (39) can be rewritten as

$$\left(\frac{\lambda_t^u}{\lambda_t^e} - 1\right)\pi(u|e) < \left(\frac{\lambda_t^e}{\lambda_t^u} - 1\right)\pi(e|u),$$

which implies $\lambda_t^e > \lambda_t^u$. In addition, equation (39) implies that

$$\left(\frac{u_c^u}{u_c^e} - g_\lambda^u \frac{\lambda_t^u}{\lambda_t^e}\right)\pi(u|e) = (g_\lambda^e - 1)\pi(e|e). \tag{40}$$

Under $v^2 = 0$, $a^u = 0$, and $a^e\pi(u|e) = c^u\pi(u)$, the FOC with respect to c^e can be rewritten as

$$u_c^e + \lambda_t^e u_c^e(1-\sigma) = \mu_t + u_{cc}^e c^u \frac{\pi(u)}{\pi(u|e)}\frac{\pi(e|e)}{\pi(e)}(g_\lambda^e - 1)\lambda_{t-1}^e, \tag{41}$$

and the FOC with respect to c^u can be rewritten as

$$u_c^u + \lambda_{t-1}^e u_c^u(1-\sigma) = \mu_t - \lambda_t^u u_c^e + \lambda_{t-1}^e u_c^u. \tag{42}$$

Now, with the above three equations, consider the following cases:

(a) The growth rates satisfy $g_\lambda^e = g_\lambda^u = 1$. Without growth, λ^e must converge. Equation (40) then implies $\lambda^u u_c^e = \lambda^e u_c^u$. The difference between equation (41) and equation (42) gives

$$(u_c^e - u_c^u) + \lambda^e(1-\sigma)(u_c^e - u_c^u) = 0,$$

which implies $u_c^e = u_c^u$ and this leads to a contradiction. Notice that this result holds for all possible values of σ.

(b) The growth rates satisfy $g_\lambda^e = g_\lambda^u > 1$. The sum of FOCs with respect to c^e and c^u can be written as

$$\begin{aligned} & \frac{u_c^e\pi(e) + u_c^u\pi(u)}{\lambda_{t-1}^e} + \frac{\lambda_t^e}{\lambda_{t-1}^e} u_c^e(1-\sigma)\pi(e) + u_c^u(1-\sigma)\pi(u) \\ = \; & \frac{\mu_t}{\lambda_{t-1}^e} - \frac{\lambda_t^u}{\lambda_{t-1}^e} u_c^e\pi(u) + u_c^u\pi(u) + u_{cc}^e c^u \frac{\pi(u)}{\pi(u|e)}\pi(e|e)(g_\lambda^e - 1). \end{aligned}$$

Since under positive growth $\lambda_{t-1}^e \to \infty$, then the above equation becomes

$$\begin{aligned} & g_\lambda^e u_c^e(1-\sigma)\pi(e) + u_c^u(1-\sigma)\pi(u) \\ = \; & g_\mu \frac{\mu_{t-1}}{\lambda_{t-1}^e} - g_\lambda^u \frac{\lambda_{t-1}^u}{\lambda_{t-1}^e} u_c^e\pi(u) + u_c^u\pi(u) + u_{cc}^e c^u \frac{\pi(u)}{\pi(u|e)}\pi(e|e)(g_\lambda^e - 1), \end{aligned}$$

which together with (40) implies

$$\begin{aligned} & g_\lambda^e u_c^e(1-\sigma)\pi(e) + u_c^u(1-\sigma)\pi(u) \\ = \; & g_\mu \frac{\mu_{t-1}}{\lambda_{t-1}^e} + \pi(u)u_c^e\left(\frac{u_c^u}{u_c^e} - g_\lambda^u\frac{\lambda_{t-1}^u}{\lambda_{t-1}^e}\right) + u_{cc}^e c^u\pi(u)\left(\frac{u_c^u}{u_c^e} - g_\lambda^u\frac{\lambda_{t-1}^u}{\lambda_{t-1}^e}\right) \\ = \; & g_\mu \frac{\mu_{t-1}}{\lambda_{t-1}^e} + \pi(u)\left(\frac{u_c^u}{u_c^e} - g_\lambda^u\frac{\lambda_{t-1}^u}{\lambda_{t-1}^e}\right)(u_c^e + u_{cc}^e c^u) \\ > \; & \pi(u)\left(\frac{u_c^u}{u_c^e} - g_\lambda^u\frac{\lambda_{t-1}^u}{\lambda_{t-1}^e}\right)u_c^e\left(1 - \sigma\frac{c^u}{c^e}\right) \\ > \; & \pi(u)\left(u_c^u - g_\lambda^u\frac{\lambda_{t-1}^u}{\lambda_{t-1}^e}u_c^e\right)(1-\sigma), \end{aligned}$$

where the last two inequalities utilize the facts that (i) $g_\mu\frac{\mu_{t-1}}{\lambda_{t-1}^e} > 0$ and (ii) $\frac{c^u}{c^e} < 1$. Now consider the parameter value $\sigma \geq 1$; the above inequality is simplified to the following two possible relationships:

$$
\begin{aligned}
0 &< g_\lambda^e u_c^e \pi(e) < -g_\lambda^u \frac{\lambda_t^u}{\lambda_t^e} u_c^e \pi(u) < 0, \text{ if } \sigma > 1, \\
\text{or } 0 &< 0 \text{ if } \sigma > 1;
\end{aligned}
$$

and

$$
\begin{aligned}
0 &> \left(\frac{u_c^u}{u_c^e} - g_\lambda^u \frac{\lambda_{t-1}^u}{\lambda_{t-1}^e} \right) (1-\sigma) = 0, \text{ if } \sigma = 1, \\
\text{or } 0 &> 0 \text{ if } \sigma = 1;
\end{aligned}
$$

both of which are self-contradictions.

To summarize, the previous subsection A.3.2 proves that under the parameter condition $(1-\beta)\frac{\pi(u)}{\pi(u|e)} < 1$, there is an interior steady state featuring FSI. This current subsection A.3.3 further proves that if $\sigma \geq 1$, then there cannot exist any other interior Ramsey steady states that feature imperfect insurance. Hence, the FSI interior Ramsey steady state exists and is the only possible interior steady state if $(1-\beta)\frac{\pi(u)}{\pi(u|e)} < 1$ and $\sigma \geq 1$. However, there may exist Ramsey steady states that are non-interior; for this we now turn.

A.3.4 Non-Interior Ramsey Steady State

In this subsection, we further prove that (i) under the parameter restrictions $(1-\beta)\frac{\pi(u)}{\pi(u|e)} < 1$ and $\sigma \geq 1$, the only possible Ramsey steady state is interior; and (ii) if $(1-\beta)\frac{\pi(u)}{\pi(u|e)} \geq 1$ and $\sigma \geq 1$, then the only possible Ramsey steady state is non-interior. The proof proceeds by considering all possible situations of a non-interior Ramsey steady state.

Case 1 Consider a non-interior steady-state allocation where (i) $\lim_{t\to\infty} c_t^e = c^e > 0$ and (ii) $\lim_{t\to\infty} c_t^u = 0$. The resource constraint then implies that $c^e = \lim_{t\to\infty} c_t^e = \lim_{t\to\infty} n_t^e = n^e > 0$. We show by the following steps that this allocation is not optimal and must be ruled out.

1. This allocation implies the following:

   (a) Given that $\lim_{t\to\infty} c_t^e > 0$ and $\lim_{t\to\infty} c_t^u = 0$, we can assume $a_t^u = 0$ without loss of generality. By the implementability condition of unemployed agents, we have

$c_t^u = a_t^e \frac{\pi(u|e)}{\pi(u)}$, so $a_t^e \to 0$. Hence, $B_t \to 0$ according to the asset market clearing condition.

(b) The asset value $Q_{t+1} a_{t+1}^e$ has to converge in the limit, otherwise the implementability condition of employed agents is violated, making this allocation infeasible. By equation (15) and $c_t^u = a_t^e \frac{\pi(u|e)}{\pi(u)}$, the limit of $Q_{t+1} a_{t+1}^e$ is given by

$$\lim_{t\to\infty} Q_{t+1} a_{t+1}^e = \frac{\beta}{u_c^e} \pi(u) \lim_{t\to\infty} (c_{t+1}^u)^{1-\sigma} = \begin{cases} 0 & \text{if } \sigma < 1 \\ \frac{\beta}{u_c^e}\pi(u) & \text{if } \sigma = 1 \\ \infty & \text{if } \sigma > 1 \end{cases}, \tag{43}$$

which imply that case 1 is feasible only if $\sigma \leq 1$. Since the parameter restriction requires $\sigma \geq 1$, in what follows we only need to consider the case where $\sigma = 1$ and $\lim_{t\to\infty} Q_{t+1} a_{t+1}^e = \frac{\beta}{u_c^e}\pi(u)$.

(c) As $t \to \infty$, the implementability condition of employe agents then becomes

$$(1 - \frac{v_n^e}{u_c^e}) n^e \pi(e) = -\frac{\beta}{u_c^e}\pi(u),$$

which together with $\tau = (1 - \frac{v_n^e}{u_c^e})$ implies that $\tau_t \to \tau < 0$.

2. Now, we show that this allocation violates the Ramsey FOCs and hence cannot be a Ramsey allocation. There are two possibilities: μ_t either converges or diverges.

(a) Consider the case where $\mu_t \to \infty$.

i. In the limit, the Ramsey FOC with respect to n_t^e in equation (34) becomes

$$\lim_{t\to\infty} \frac{\mu_t}{\lambda_t^e} = (1+\gamma) v_n^e,$$

which implies that $\lambda_t^e \to \infty$.

ii. Given $v_t^1 > 0$, $v_t^2 = 0$, $a_t^u = 0$, $c_t^u = a_t^e \frac{\pi(u|e)}{\pi(u)}$ and $\sigma = 1$, the Ramsey FOC with respect to c_t^e can be rewritten as

$$u_{c,t}^e + \left(\lambda_{t-1}^e - \lambda_t^e\right) u_{cc,t}^e c_t^u \frac{\pi(u)}{\pi(u|e)} \frac{\pi(e|e)}{\pi(e)} = \mu_t,$$

which can be further transformed to

$$\frac{1}{\lambda_t^e} + \left(1 - \frac{\lambda_{t-1}^e}{\lambda_t^e}\right) \frac{c_t^u}{c_t^e} \frac{\pi(u)}{\pi(u|e)} \frac{\pi(e|e)}{\pi(e)} = \frac{\mu_t}{\lambda_t^e} \frac{1}{u_{c,t}^e},$$

where we have utilized the fact that $\frac{u_{cc,t}^e c_t^e}{u_{c,t}^e} = -1$. As $t \to \infty$, the equation above becomes

$$0 = \lim_{t\to\infty} \left(1 - \frac{\lambda_{t-1}^e}{\lambda_t^e}\right) \frac{\lim_{t\to\infty} c_t^u}{c^e} \frac{\pi(u)}{\pi(u|e)} \frac{\pi(e|e)}{\pi(e)} = \lim_{t\to\infty} \frac{\mu_t}{\lambda_t^e} \frac{1}{u_{c,t}^e} = (1+\gamma) \frac{v_n^e}{u_c^e} > 0,$$

which leads to a contradiction.

(b) Consider the case where $\mu_t \to \mu < \infty$.

i. The Ramsey FOC with respect to n_t^e in equation (34) implies $\lambda_t^e \to \lambda^e < \infty$ and in the limit becomes

$$v_n^e (1 + \lambda^e (1+\gamma)) = \mu. \tag{44}$$

ii. Given $v_t^1 > 0$, $v_t^2 = 0$, $a_t^u = 0$, $\sigma = 1$ and $c_t^u = a_t^e \frac{\pi(u|e)}{\pi(u)}$, the Ramsey FOC with respect to c_t^e can be rewritten as

$$u_{c,t}^e + \left(\lambda_{t-1}^e - \lambda_t^e\right) u_{cc,t}^e c_t^u \frac{\pi(u)}{\pi(u|e)} \frac{\pi(e|e)}{\pi(e)} = \mu_t,$$

which in the limit can be simplified to

$$u_c^e = \mu. \tag{45}$$

iii. Combing equations (44) and (45) gives

$$\frac{v_n^e}{u_c^e} = \frac{1}{1 + \lambda^e (1+\gamma)} < 1,$$

which implies $\tau = 1 - \frac{v_n^e}{u_c^e} > 0$, a contradiction to the property of this allocation shown in step 1 where $\tau < 0$.

Case 2 Consider the allocation where (i) $c_t^e = c_t^u > 0$ in transition for a finite period of time $t > 0$ but (ii) $\lim_{t\to\infty} c_t^e = \lim_{t\to\infty} c_t^u = 0$. Clearly, in this allocation, $\lim_{t\to\infty} n_t^e = 0$ by the resource constraint. Now, we show that this allocation violates the Ramsey FOCs and hence cannot be an optimal allocation.

1. The Ramsey FOC with respect to n_t^e is given by

$$\frac{v_{n,t}^e}{\lambda_t^e} + v_{n,t}^e(1+\gamma) = \frac{\mu_t}{\lambda_t^e},$$

   which together with $v_{n,t}^e \to 0$ implies $\frac{\mu_t}{\lambda_t^e} \to 0$ and $\lambda_t^e \to \infty$ since $\mu_t > 0$.

2. Given that $c_t^e = c_t^u > 0$ for a certain period of time $t > 0$, we know that $v_t^1 = 0$ and $g_t(c_t^e, c_t^u, c_{t+1}^e, c_{t+1}^u) = 0$ starting from time t. Beyond period t, the Ramsey FOCs with respect to a_{t+1}^u can be rewritten as

$$1 = \frac{\lambda_{t+1}^u}{\lambda_t^u}\pi(u|u) + \frac{\lambda_{t+1}^e}{\lambda_t^u}\pi(e|u),$$

   which together with $\lambda_t^e \to \infty$ implies $\lambda_t^u \to \infty$. In addition, the equation above can be rewritten as

$$\left(\frac{\lambda_{t+1}^u}{\lambda_t^u}\right)\left(\frac{\lambda_{t+1}^e}{\lambda_{t+1}^u}\right)\pi(e|u) = 1 - \frac{\lambda_{t+1}^u}{\lambda_t^u}\pi(u|u) < \pi(e|u),$$

   where the last inequality utilizes the fact that $\frac{\lambda_{t+1}^u}{\lambda_t^u} > 1$. Similarly, by using the fact that $\frac{\lambda_{t+1}^e}{\lambda_t^e} > 1$, the Ramsey FOCs with respect to a_{t+1}^e can be rewritten as

$$\left(\frac{\lambda_{t+1}^e}{\lambda_t^e}\right)\left(\frac{\lambda_{t+1}^u}{\lambda_{t+1}^e}\right)\pi(u|e) = 1 - \frac{\lambda_{t+1}^e}{\lambda_t^e}\pi(e|e) < \pi(u|e).$$

   Multiplying the above two equations gives

$$\frac{\lambda_{t+1}^u}{\lambda_t^u}\frac{\lambda_{t+1}^e}{\lambda_t^e} < 1,$$

   which leads to a contradiction since $\frac{\lambda_{t+1}^u}{\lambda_t^u} > 1$ and $\frac{\lambda_{t+1}^e}{\lambda_t^e} > 1$. Hence, case 2 cannot satisfy all of the Ramsey FOCs.

Case 3 Consider the allocation where (i) $c_t^e > c_t^u > 0$ for all $t < \infty$ and (ii) $\lim_{t\to\infty} c_t^e = \lim_{t\to\infty} c_t^u = 0$. We show through the following steps that this allocation is both feasible and optimal if and only if $(1-\beta)\frac{\pi(u)}{\pi(u|e)} \geq 1$ and $\sigma \geq 1$. In other words, under the parameter restrictions $(1-\beta)\frac{\pi(u)}{\pi(u|e)} \geq 1$ and $\sigma \geq 1$, this non-interior steady state could exist and is the only possible Ramsey steady state.

1. This allocation implies the following:

   (a) $\lim_{t\to\infty} n_t^e = 0$ by the resource constraint.

   (b) In order to induce $\lim_{t\to\infty} c_t^e = \lim_{t\to\infty} n_t^e = 0$, the tax rate τ_t must converge to 1. Otherwise, it violates the households' FOCs.

   (c) Given that $c_t^e > c_t^u > 0$ for $t < \infty$ and $a_t^u = 0$, by the implementability condition of the unemployed agents, we have $c_t^u = a_t^e \frac{\pi(u|e)}{\pi(u)}$; so $\lim_{t\to\infty} c_t^u = 0$ implies $a_t^e \to 0$. Hence, $B_t \to 0$ according to the asset market-clearing condition.

   (d) The implementability condition of the employed agents is reduced to

   $$c_t^e \pi(e) - \frac{v_{n,t}^e}{u_{c,t}^e} n_t^e \pi(e) + Q_{t+1} a_{t+1}^e - a_t^e \pi(e|e) = 0,$$

   which is satisfied in the limit given that $Q_{t+1} B_{t+1} = Q_{t+1} a_{t+1}^e \to 0$ and $\lim_{t\to\infty} c_t^e = \lim_{t\to\infty} \frac{v_{n,t}^e}{u_{c,t}^e} n_t^e = 0$.

   (e) The borrowing constraints and complementary slackness conditions of the Ramsey problem are trivially satisfied.

   So far we have shown that this non-interior steady-state allocation can satisfy all constraints of the Ramsey planner problem.

2. Now, we further show that this allocation satisfies all of the Ramsey FOCs by properly choosing convergent properties of the Ramsey multipliers:

   (a) Given that $c_t^e > c_t^u > 0$ for $t < \infty$ and that $a_t^u = 0$, it must be true that $v_t^1 > 0$ and $v_t^2 = 0$ for all $t < \infty$.

   (b) Let $\mu_t \to \infty$, $\lambda_t^e \to \infty$, and $\frac{\mu_t}{\lambda_t^e} \to 0$; the Ramsey FOC with respect to n_t^e in equation (34) is satisfied in the limit as $t \to \infty$:

   $$\lim_{t\to\infty} \frac{\mu_t}{\lambda_t^e} = (1+\gamma) \lim_{t\to\infty} v_{n,t}^e = 0.$$

   (c) Given $v_t^1 > 0$, $v_t^2 = 0$, $a_t^u = 0$, and $c_t^u = a_t^e \frac{\pi(u|e)}{\pi(u)}$, the Ramsey FOC with respect to c_t^e can be rewritten as

   $$u_{c,t}^e \left(1 + \lambda_t^e (1-\sigma)\right) + \left(\lambda_{t-1}^e - \lambda_t^e\right) u_{cc,t}^e c_t^u \frac{\pi(u)}{\pi(u|e)} \frac{\pi(e|e)}{\pi(e)} = \mu_t,$$

which can be further transformed to

$$\frac{1}{\sigma\lambda_t^e}+\left(1-\frac{\lambda_{t-1}^e}{\lambda_t^e}\right)\frac{c_t^u}{c_t^e}\frac{\pi(e|e)}{1-\pi(u|u)}=\frac{\mu_t}{\lambda_t^e}\frac{1}{u_{c,t}^e}\frac{1}{\sigma}+1-\frac{1}{\sigma},$$

where we have utilized the fact that $\pi(e)=\frac{\pi(e|u)}{\pi(e|u)+\pi(u|e)}$ and $\pi(u)=\frac{\pi(u|e)}{\pi(e|u)+\pi(u|e)}$. As $t\to\infty$, the equation above becomes

$$\left(1-\lim_{t\to\infty}\frac{\lambda_{t-1}^e}{\lambda_t^e}\right)\lim_{t\to\infty}\frac{c_t^u}{c_t^e}=\frac{1-\pi(u|u)}{\pi(e|e)}(1-\frac{1}{\sigma})\leq(1-\frac{1}{\sigma}),\tag{46}$$

where the inequality on the right-hand side holds because $\pi(e|e)+\pi(u|u)\geq 1$. Hence, the above equation can be satisfied only if $\sigma\geq 1$. We then have two subcases to consider:

i. $\sigma>1$. Equation (46) can be satisfied if $\frac{\lambda_{t-1}^e}{\lambda_t^e}$ and $\frac{c_t^u}{c_t^e}$ both converge to finite positive (less than 1) constants.

ii. $\sigma=1$. Equation (46) can be satisfied if $\lim_{t\to\infty}\frac{\lambda_{t-1}^e}{\lambda_t^e}=1$ and $\frac{c_t^u}{c_t^e}$ converges to a finite positive value.

Hence, both subcases above are possible and do not lead to contradictions.

(d) Under the conditions that $v_t^1>0$, $v_t^2=0$, $a_t^u=0$, and $c_t^u=a_t^e\frac{\pi(u|e)}{\pi(u)}$, the Ramsey FOC with respect to c_t^u is simplified to

$$u_{c,t}^u+\lambda_t^u u_{c,t}^e+\lambda_{t-1}^e c_t^u u_{cc,t}^u=\mu_t,$$

which can be rewritten as

$$\frac{1}{\lambda_{t-1}^e}+\frac{\lambda_t^u}{\lambda_{t-1}^e}\frac{u_{c,t}^e}{u_{c,t}^u}-\sigma=\frac{\mu_t}{\lambda_t^e}\frac{\lambda_t^e}{\lambda_{t-1}^e}\frac{1}{u_{c,t}^u}.$$

Since $\lim_{t\to\infty}\frac{\mu_t}{\lambda_t^e}\frac{\lambda_t^e}{\lambda_{t-1}^e}\frac{1}{u_{c,t}^u}=0$, in the limit the equation above becomes

$$\lim_{t\to\infty}\frac{\lambda_t^u}{\lambda_{t-1}^e}\frac{u_{c,t}^e}{u_{c,t}^u}=\sigma.\tag{47}$$

We then have two subcases to consider:

i. $\sigma>1$. Equation (47) can be satisfied if $\frac{\lambda_t^u}{\lambda_{t-1}^e}$ converges to a finite positive constant. From 2(c), we know that $\frac{u_{c,t}^e}{u_{c,t}^u}$ also converges to a finite positive constant given the convergence of $\frac{c_t^u}{c_t^e}$.

ii. $\sigma = 1$. Equation (47) can be satisfied if both $\frac{\lambda_t^u}{\lambda_{t-1}^e}$ and $\frac{c_t^u}{c_t^e}$ converge to finite positive values.

Hence, both subcases above are possible and do not lead to contradictions.

(e) The FOCs of a_{t+1}^e and a_{t+1}^u in equations (35) and (36) can be rewritten, respectively, as

$$\pi(e|e) + \frac{u_{c,t+1}^u}{u_{c,t+1}^e}\pi(u|e) = \frac{\lambda_{t+1}^e}{\lambda_t^e}\pi(e|e) + \frac{\lambda_{t+1}^u}{\lambda_t^e}\pi(u|e), \tag{48}$$

$$\pi(e|e)+\frac{u_{c,t+1}^u}{u_{c,t+1}^e}\pi(u|e) = \frac{\lambda_{t+1}^u}{\lambda_t^u}\pi(u|u)+\frac{\lambda_{t+1}^e}{\lambda_t^u}\pi(e|u)+\frac{v_t^1 + v_t^3 g(c_t^e, c_t^u, c_{t+1}^e, c_{t+1}^u)}{\lambda_t^u u_{c,t+1}^e}. \tag{49}$$

Given $\sigma \geq 1$ and from 2(c) and 2(d), we know that $\frac{u_{c,t}^u}{u_{c,t}^e}$, $\frac{\lambda_{t+1}^e}{\lambda_t^e}$, $\frac{\lambda_{t+1}^u}{\lambda_t^u}$, $\frac{\lambda_{t+1}^e}{\lambda_t^u}$, and $\frac{\lambda_{t+1}^u}{\lambda_t^e}$ all converge to finite positive constants. Hence, equation (48) is satisfied. In addition, $g_t(c_t^e, c_t^u, c_{t+1}^e, c_{t+1}^u)$ also converges to a finite positive constant according to its definition, so equation (49) can be satisfied if $\lim_{t\to\infty} \frac{v_t^1+v_t^3 g(c_t^e,c_t^u,c_{t+1}^e,c_{t+1}^u)}{\lambda_t^u u_{c,t+1}^e}$ is chosen to be a finite constant; which is possible and does not lead to contradictions.

3. Finally, under the parameter restriction $\sigma \geq 1$, we prove that case 3 is possible only if $(1-\beta)\frac{\pi(u)}{\pi(u|e)} \geq 1$.

(a) From the proof of case 3 above, recall that for $t < \infty$, $c_t^e > c_t^u > 0$, $a_t^u = 0$, $c_t^u = a_t^e \frac{\pi(u|e)}{\pi(u)}$, and $a_t^e = B_t$. In addition, as $t \to \infty$ we have $a_t^e \to 0$, $B_t \to 0, \tau_t \to 1$, $0 < \lim_{t\to\infty} \frac{c_t^u}{c_t^e} \leq 1$, and $\lim_{t\to\infty} c_t^e = \lim_{t\to\infty} n_t^e = \lim_{t\to\infty} c_t^u = 0$. Also, notice that since $\frac{c_t^u}{c_t^e}$ approaches a positive constant in the limit, the growth rates of consumption must be equal in the limit: $\lim_{t\to\infty} \frac{c_{t+1}^e}{c_t^e} = \lim_{t\to\infty} \frac{c_{t+1}^u}{c_t^u}$.

(b) For $t < \infty$, equation (15) implies

$$Q_{t+1} = \beta\left[\frac{u_{c,t+1}^e}{u_{c,t}^e}\pi(e|e) + \frac{u_{c,t+1}^u}{u_{c,t}^e}\pi(u|e)\right] > \beta\frac{u_{c,t+1}^e}{u_{c,t}^e}, \tag{50}$$

where the inequality comes from the facts that $u_{c,t+1}^e < u_{c,t+1}^u$ and $\pi(e|e)+\pi(u|e) = 1$.

(c) By the government budget constraint and equation (50), we have the following inequality:

$$\tau_t n_t^e \pi(e) = B_t - Q_{t+1}B_{t+1} < B_t - \beta\frac{u_{c,t+1}^e}{u_{c,t}^e}B_{t+1},$$

which together with $B_t = a_t^e = c_t^u \frac{\pi(u)}{\pi(u|e)}$ implies the following inequalities:

$$\tau_t < \frac{c_t^u}{n_t^e \pi(e)} \left(1 - \beta \left(\frac{c_{t+1}^e}{c_t^e}\right)^{-\sigma} \frac{c_{t+1}^u}{c_t^u}\right) \frac{\pi(u)}{\pi(u|e)}, \text{ for } t < \infty;$$

$$\lim_{t\to\infty} \tau_t \leq \lim_{t\to\infty} \frac{c_t^u}{n_t^e \pi(e)} \left(1 - \beta \lim_{t\to\infty} \left(\frac{c_{t+1}^e}{c_t^e}\right)^{-\sigma} \frac{c_{t+1}^u}{c_t^u}\right) \frac{\pi(u)}{\pi(u|e)}, \text{ for } t \to \infty.$$

Given that $\lim_{t\to\infty} \frac{c_t^u}{n_t^e \pi(e)} \leq 1$, $\lim_{t\to\infty} \tau_t = 1$, and $\lim_{t\to\infty} \frac{c_{t+1}^e}{c_t^e} = \lim_{t\to\infty} \frac{c_{t+1}^u}{c_t^u}$, the equation above then becomes

$$1 \leq \left(1 - \beta \lim_{t\to\infty} \left(\frac{c_{t+1}^e}{c_t^e}\right)^{1-\sigma}\right) \frac{\pi(u)}{\pi(u|e)}.$$

Since consumption declines overtime, we have $\frac{c_t^e}{c_{t+1}^e} > 1$ and consequently $\left(\frac{c_{t+1}^e}{c_t^e}\right)^{1-\sigma} \geq 1$ if $\sigma \geq 1$. Hence, the equation above leads to the inequality $1 \leq (1-\beta) \frac{\pi(u)}{\pi(u|e)}$. Namely, case 3 is possible only if $1 \leq (1-\beta) \frac{\pi(u)}{\pi(u|e)}$, provided that $\sigma \geq 1$.

In short, we can conclude that (i) if the parameter requirements $(1-\beta)\frac{\pi(u)}{\pi(u|e)} < 1$ and $\sigma \geq 1$ are satisfied, then the interior FSI steady state is the only possible Ramsey steady state, and (ii) if $(1-\beta)\frac{\pi(u)}{\pi(u|e)} \geq 1$ and $\sigma \geq 1$, then the non-interior steady state in case 3 is not only feasible but also the only possible Ramsey steady state.

A.4 Proof of Proposition 4

We first state the Ramsey FOCs. Denote μ, λ^e, λ^u, v^1, v^2, and v^3 as the Lagrangian multipliers for conditions (26)-(31), respectively. The Ramsey FOCs with respect to n_t^e, a_{t+1}^e, and a_{t+1}^u become

$$\lambda^e \beta(u_c^e \pi(e|e) + u_c^u \pi(u|e)) = \lambda^e u_c^e \pi(e|e) + \lambda^u u_c^e \pi(u|e), \tag{51}$$

$$\lambda^u \beta(u_c^e \pi(e|e) + u_c^u \pi(u|e)) = \lambda^u u_c^e \pi(u|u) + \lambda^e u_c^e \pi(e|u) - v^1 - v^3 g(c^e, c^u). \tag{52}$$

Consider a Ramsey steady state featuring full self-insurance; that is, $a^u \geq 0$ or $v^1 = 0$, $c^e = c^u$ and $g(c^e, c^u) = 0$. Then, the FOC with respect to a^e is simplified as

$$\lambda^e(\beta - \pi(e|e)) = \lambda^u \pi(u|e),$$

which implies

$$\frac{\lambda^u}{\lambda^e} = \frac{\beta - \pi(e|e)}{1 - \pi(e|e)} < 1. \tag{53}$$

However, the Ramsey FOC with respect to a^u becomes

$$\lambda^u(\beta - \pi(u|u)) = \lambda^e \pi(e|u),$$

which implies

$$\frac{\lambda^u}{\lambda^e} = \frac{1 - \pi(u|u)}{\beta - \pi(u|u)} > 1,$$

thus contradicting with equation (53). Hence, $c^e = c^u$ cannot be the outcome of this Ramsey problem, which maximizes the steady-state welfare of the competitive equilibrium.