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Abstract

We design an infinite-horizon heterogeneous-agents and incomplete-markets model to demonstrate analytically that in the absence of any redistributional effects of government policies, optimal capital tax is zero despite capital overaccumulation under precautionary savings and borrowing constraints. Our result indicates that public debt is a better tool than capital taxation to restore aggregate productive efficiency.

JEL Classification: C61; E22; E62; H21; H30

Key Words: Capital Taxation; Government Bonds; Heterogeneous Agents; Incomplete Markets; Modified Golden Rule; Ramsey Problem; Wealth Distribution.

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1 Introduction

In a representative-agent framework, the Ramsey taxation literature has shown that the best way for the government to finance its expenditures in the long run is to tax labor but not capital (see Judd (1985) and Chamley (1986)). This "zero capital tax" result is surprising and has attracted many studies to examine its robustness.[1] A significant progress comes from the seminal work of Aiyagari (1995), which shows that it is optimal to tax capital in a heterogenous-agent incomplete-markets (HAIM) framework. However, despite several important revisits, such as Chamley (2001), Conesa, Kitao, and Krueger (2009), Dávila, Hong, Krusell, and Ríos-Rull (2012), and many others, some key issues regarding optimal capital taxation in a HAIM economy (á la Aiyagari (1994)) remain unsettled, as recently discussed by Chien and Wen (2020).

For example, in an infinite-horizon HAIM economy, is a positive tax levied on capital by the Ramsey planner motivated chiefly by correcting the failure of the modified golden rule (MGR) in light of capital overaccumulation, by wealth redistribution in light of income inequality under borrowing constraints, or both?

These questions are intertwined not only because both government bonds and a capital tax can change individuals' incentives for saving, but also because they both tend to have a redistributional effect on household wealth.

This paper designs a tractable HAIM model to address these questions by shutting down the redistributional channel. Our analytical approach is built upon the work of Heathcote and Perri (2018). We show analytically that (i) the Ramsey planner will never tax capital in a Ramsey steady state despite capital overaccumulation and (ii) the MGR holds provided that the government can amass a sufficiently large stock of bonds to enable households to achieve full self-insurance.

These results are consistent with the recent of Chien and Wen (2020), which proves in a HAIM model with quasi-linear preferences that the optimal capital tax is always zero. However, Chien and Wen (2020) relied critically on quasi-linear preferences to derive their results. The drawback of quasi-linear preferences is that the marginal utility cost of a labor tax is constant and this property may be driving their "zero capital tax" result.

In this paper, we relax the assumption of quasi-linear preferences and assume instead that households can reshuffle asset holdings at the end of each period such that the distribution

[1] For example, in the introduction of Straub and Werning (2020): "One may even say that the result (zero capital tax) remains downright puzzling, as witnessed by the fact that economists have continued to take turns putting forth various intuitions to interpret it, none definitive nor universally accepted."

of households' end-of-period wealth is degenerated, as in Heathcote and Perri (2018). This feature shuts down the wealth redistribution effects of government policies and at the same time allows our HAIM model to be analytically tractable despite nonlinear preferences, idiosyncratic risk, and precautionary saving motives.

2 The Model

2.1 Model Setup

Time is discrete and indexed by $t = 0, 1, 2, ..., \infty$. A representative firm's production technology is $Y_t = F(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha}$, where Y, K, and N denote aggregate output, capital, and labor, respectively. The firm rents capital and hires labor from households by paying a competitive rental rate q_t and real wage w_t. Profit maximization gives

$$w_t = \frac{\partial F(K_t, N_t)}{\partial N_t}, \tag{1}$$

$$q_t = \frac{\partial F(K_t, N_t)}{\partial K_t}. \tag{2}$$

The government can issue one-period bonds and levy flat-rate time-varying taxes on labor and capital, denoted by $\tau_{n,t}$ and $\tau_{k,t}$, respectively. The flow government budget constraint is given by

$$\tau_{n,t} w_t N_t + \tau_{k,t}(q_t - \delta)K_t + \frac{B_{t+1}}{R_{t+1}} = B_t, \tag{3}$$

where R_{t+1} is the risk-free gross interest rate between t and $t+1$, $\delta \in (0, 1)$ is the depreciation rate of capital, and B_{t+1} is the amount of government bonds issued at time t. The government is assumed to fully commit to a sequence of future taxes and debts issued, given the initial bond supply B_0 at time 0.

There is a unit measure of *ex ante* identical agents belonging to a representative household. Individual agents face an idiosyncratic shock to their employment status, denoted by $\theta \in \{e, u\}$. The shock is iid over time and across individuals. If $\theta = e$, an individual can work and receive labor income; if $\theta = u$, an individual cannot work and has no labor income. Let $\pi(e)$ and $\pi(u)$ denote the corresponding probability of employment status.

There are two subperiods within each period. In the first subperiod, individuals are separated and the shock θ is realized, and they must choose consumption and labor-supply decisions (if employed). In the second subperiod, individuals are reunited and share their assets. Therefore, the idiosyncratic shock is not fully insurable but the wealth distribution is degenerated. Hence, individuals still have precautionary saving motives. This setup not only eliminates the wealth redistribution effects of any government policies but also renders the model analytically tractable.

The lifetime utility of the representative household is given by

$$U = \sum_{t=0}^{\infty} \beta^t \left\{ \left[\log c_t^e + v\left(1 - n_t^e\right)\right] \pi(e) + \left[\log c_t^u + v\left(1\right)\right] \pi(u) \right\}, \tag{4}$$

where $\beta \in (0, 1)$ is the discount factor; c_t^e and c_t^u denote the consumption for employed and unemployed individuals at time t; and n_t^e is the labor supply of the employed individuals. Note that the labor supply is zero for unemployed individuals. Let $v(\cdot)$ satisfy $v' > 0, v'' < 0$, and $\lim_{n \to 1} v' = \infty$. The budget constraints for employed and unemployed individuals are given, respectively, by:

$$c_t^e \leq a_t + \widehat{w}_t n_t^e, \tag{5}$$

$$c_t^u \leq a_t; \tag{6}$$

and the representative household's flow budget constraint is thus given by

$$c_t^e \pi(e) + c_t^u \pi(u) + \frac{a_{t+1}}{R_{t+1}} \leq a_t + \widehat{w}_t n_t^e \pi(e), \tag{7}$$

where $\widehat{w}_t = (1 - \tau_{l,t}) w_t$ is the after-tax wage rate and a_t is the asset holdings carried over from period $t - 1$. Notice that all individuals by assumption have the same asset holdings a_t.

Finally, government bonds and capital are perfect substitutes as stores of value for individuals. As a result, the after-tax gross rate of return to capital must equal the gross risk-free rate:

$$R_{t+1} = 1 + (1 - \tau_{k,t+1})(q_{t+1} - \delta), \tag{8}$$

which constitutes a no-arbitrage condition for capital and bonds.

2.2 Competitive Equilibrium

Definition 1. *Given the initial capital K_0 and initial government bond B_0, a competitive equilibrium is defined as sequences of tax rates and government bonds $\{\tau_{n,t}, \tau_{k,t}, B_{t+1}\}_{t=0}^{\infty}$, and sequences of prices $\{w_t, q_t, R_t\}_{t=0}^{\infty}$, aggregate allocations $\{C_t, N_t, K_{t+1}\}_{t=0}^{\infty}$ and individual* *allocations* $\{c_t^e, c_t^u, n_t^e, a_{t+1}\}_{t=0}^{\infty}$, *such that*

1. *given prices, the allocation* $\{c_t^e, c_t^u, n_t^e, a_{t+1}\}$ *solves the household problem;*

2. *given prices,the allocation* $\{N_t, K_t\}$ *solves the firm problem;*

3. *the no-arbitrage condition, equation (8), holds for all* t*;*

4. *the government budget constraint, equation (3), holds for all* t*;*

5. *all markets clear for all* $t \geq 0$*:*

$$\begin{aligned} B_{t+1} + R_{t+1}K_{t+1} &= a_{t+1} && (9) \\ N_t &= n_t^e \pi(e), \\ C_t &= c_t^e \pi(e) + c_t^u \pi(u), \\ F(K_t, N_t) &= C_t + K_{t+1} - (1-\delta)K_t. && (10) \end{aligned}$$

Proposition 1. *The competitive equilibrium has the following two properties:*

1. c_t^u *can be expressed as a function of* c_t^e *and* a_t *:*

$$c_t^u = \begin{cases} c_t^e \text{ if } c_t^e \leq a_t, \\ a_t \text{ if } c_t^e > a_t. \end{cases} . \tag{11}$$

2. *The risk-free interest rate and the time discount rate are related to each other by*

$$\frac{1}{R_{t+1}} = \beta \frac{\left(1/c_{t+1}^e\right)\pi(e) + \left(1/c_{t+1}^u\right)\pi(u)}{1/c_t^e}. \tag{12}$$

Proof. Please refer to Appendix A.1. □

Proposition 1 implies that if the asset holdings (or self-insurance position) a_t are sufficiently large such that $c_t^e \leq a_t$, then individuals can obtain the same level of consumption ($c_t^u = c_t^e$) regardless of their employment status. We call this situation a full self-insurance allocation. Obviously, by the asset market clearing condition (9), this could be achieved only if the supply of government bonds is sufficiently high. In addition, if a full self-insurance allocation is achieved in the steady state, then the steady-state risk-free rate R is equal to the time discount rate $1/\beta$. Otherwise, $R < 1/\beta$.

Proposition 2. *In a laissez-faire competitive equilibrium, the steady-state risk-free rate R is lower than the time discount rate $1/\beta$ if and only if the model parameters satisfy the following condition:*

$$\beta < \frac{1-\alpha}{\alpha + (1-\alpha)(1-\delta)}. \tag{13}$$

Proof. Please refer to Appendix A.2. $\square$

Proposition 2 provides a necessary and sufficient condition for the interest rate to be lower than the time discount rate in a laissez-faire competitive equilibrium. In this case, precautionary-saving motives reduce the equilibrium interest rate and there is overaccumulation of capital in the sense that the steady-state capital stock exceeds the level implied by the MGR. This outcome is clearly inefficient from a social point of view because individuals can consume more and save less capital without lowering aggregate output. So, to make our analysis interesting and the Ramsey problem meaningful, we assume that condition (13) holds in this paper.

3 The Ramsey Plan

3.1 Ramsey Problem

As shown in the Appendix A.3, the Ramsey problem can be represented as maximizing (4) by choosing $\{c_t^e, c_t^u, n_t^e, a_{t+1}, K_{t+1}\}_{t=0}^{\infty}$, and subject to the resource constraint

$$F(K_t, n_t^e \pi(e)) + (1-\delta)K_t - c_t^e \pi(e) - c_t^u \pi(u) - K_{t+1} \geq 0, \tag{14}$$

and the implementability condition

$$\frac{v'(1-n_t^e)}{1/c_t^e} n_t^e \pi(e) + a_t - c_t^e \pi(e) - c_t^u \pi(u) - a_{t+1}\beta \frac{\left(1/c_{t+1}^e\right)\pi(e) + \left(1/c_{t+1}^u\right)\pi(u)}{1/c_t^e} \leq 0, \tag{15}$$

for $t \geq 0$, K_0, B_0, and $\tau_{k,0}$ given; where

$$c_t^u = \begin{cases} c_t^e \text{ if } c_t^e \leq a_t, \\ a_t \text{ if } c_t^e > a_t. \end{cases}.$$

3.2 Ramsey Outcome

Proposition 3. *Under condition (13), a Ramsey steady state exists and it features:*

1. *a full self-insurance allocation with equalized consumption:* $c^u = c^e \leq a$*;*

2. *that the MGR holds with* $R = 1/\beta$*;*

3. *that the optimal capital tax rate* $\tau_k = 0$*;*

4. *that the optimal level of government bonds is strictly positive and the bond interest payments are financed solely by a labor income tax.*

Proof. See Appendix A.4. □

The intuition behind the "zero capital tax" result is straightforward: In the absence of any redistributional effects, taxing capital in the steady state permanently hinders individuals' self-insurance positions and thus the Ramsey planner opts to issue debt rather than impose a steady-state capital tax to correct the capital-overaccumulation problem. The interest payment of the debt is financed by a labor tax. This is exactly the same message delivered by Chien and Wen (2020), despite the difference between the two models.

4 Conclusion

We have demonstrated analytically that (i) optimal capital tax is zero despite capital over-accumulation under precautionary savings and incomplete markets and (ii) the MGR can be achieved through supplying enough government bonds. Hence, consistent with Chien and Wen (2020), our result indicates that public debt is a better tool than capital tax to restore production efficiency.

Of course, in the original Aiyagari (1994) model, since a full self-insurance allocation is not feasible unless the optimal quantity of government debt goes to infinity; therefore, it is not clear whether a Ramsey steady state exists in his model without debt limits. Our current study and the previous work of Chien and Wen (2020) suggest that in the absence of redistributional effects and any binding debt-limit constraints on government bonds, the Ramsey planner always opts to issue enough bonds to equalize the interest rate and the time discount rate so long as a Ramsey steady state can be proven to exist, regardless of the preference structure of the households.

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A Online Appendix

A.1 Proof of Proposition 1

The household chooses a sequence of $\{c_t^e, c_t^u, n_t^e, a_{t+1}\}_{t=0}^{\infty}$ to maximize (4) subject to (7), (5) and (6). Let $\beta^t\lambda_t^e\pi(e)$, $\beta^t\lambda_t^u\pi(u)$ and $\beta^t\mu_t$ denote Lagrangian multipliers attached to (5), (6) and (7) respectively. The FOCs with respect to c_t^e, c_t^u, n_t^e and a_{t+1}, are given, respectively, by

$$\frac{1}{c_t^e} = \lambda_t^e + \mu_t, \tag{16}$$

$$\frac{1}{c_t^u} = \lambda_t^u + \mu_t, \tag{17}$$

$$v'(1-n_t^e) = \widehat{w}_t(\lambda_t^e + \mu_t), \tag{18}$$

$$\beta(\lambda_{t+1}^e\pi(e) + \lambda_{t+1}^u\pi(e)) + \beta\mu_{t+1} = \frac{\mu_t}{R_{t+1}}. \tag{19}$$

By equations (5) and (6), we know that $c_t^u \le c_t^e$ so that $\lambda_t^u \ge \lambda_t^e$. Now suppose $\lambda_t^e > 0$, it follows that $\lambda_t^u > 0$. Given $\lambda_t^e > 0$ and $\lambda_t^u > 0$, both (5) and (6) must be binding. Inserting $c_t^e = a_t + \widehat{w}_t n_t^e$ and $c_t^u = a_t$ back to (7), we obtain $a_{t+1} = 0$, which together with (6) imply that $c_{t+1}^u = 0$. This leads to a contradiction since $u'(0) = \infty$. As a consequence, λ_t^e must be zero. Next, consider two possibilities for λ_t^u. One is that $\lambda_t^u > 0$ so that $c_t^u = a_t$ and $c_t^u < c_t^e$ (recall that $\lambda_t^e = 0$). The other is that $\lambda_t^u = 0$, along with $\lambda_t^u = \lambda_t^e = 0$, which leads to $c_t^e = c_t^u \le a_t$. Hence, we can express c_t^u as a function of c_t^e and a_t as shown by (11). In addition, using $\lambda_t^e = 0$, we can rewrite (19) as (12).

A.2 Proof of Proposition 2

Under the laissez-faire competitive equilibrium, $\tau_{l,t} = \tau_{k,t} = B_t = 0$ and $a_t = R_t K_t$. Provided that $c^e > c^u = RK$ is true, we can conclude that in the steady state $\beta R < 1$. Given this, we first prove that if $c^e > c^u = RK$, then $\beta < \frac{1-\alpha}{\alpha+(1-\alpha)(1-\delta)}$ must hold.

Based on $R = 1 + \alpha K^{\alpha-1}N^{1-\alpha} - \delta$, we express $\left(\frac{K}{N}\right)^{\alpha-1}$ as $\frac{R+\delta-1}{\alpha}$. In the steady state, we have Euler equation (12) and the resource constraint imply

$$1 = \beta R\left(\pi(e) + \frac{c^e}{c^u}\pi(u)\right), \tag{20}$$

$$
\begin{aligned}
c^e\pi(e) + c^u\pi(u) &= K^\alpha N^{1-\alpha} - \delta K \\
&= \left(\alpha K^{\alpha-1}N^{1-\alpha} - \delta\right) K + (1-\alpha) K^{\alpha-1}N^{1-\alpha}K \\
&= (R-1) K + (1-\alpha)\left(\frac{R+\delta-1}{\alpha}\right) K,
\end{aligned} \tag{21}
$$

where we replace $K^{\alpha-1}N^{1-\alpha}$ with $\frac{R+\delta-1}{\alpha}$ in (21). Dividing each term by RK in (21) gives

$$
\frac{c^e}{RK}\pi(e) + \frac{c^u}{RK}\pi(u) = \frac{R-1}{R} + \frac{1-\alpha}{\alpha}\frac{R+\delta-1}{R}. \tag{22}
$$

Substituting $c^u = RK$ into (20) and (22) yields

$$
\frac{1}{R} - \beta\pi(u)\frac{c^e}{RK} - \beta\pi(e) = 0. \tag{23}
$$

$$
\frac{\alpha + (1-\alpha)(1-\delta)}{\alpha}\frac{1}{R} + \frac{c^e}{RK}\pi(e) + \pi(u) - \frac{1}{\alpha} = 0, \tag{24}
$$

Combining (23) and (24) leads to

$$
f\left(\frac{c^e}{RK}\right) = 0, \tag{25}
$$

where $f : \frac{c^e}{RK} \to \frac{\alpha+(1-\alpha)(1-\delta)}{\alpha}\beta\left(\pi(u)\frac{c^e}{RK} + \pi(e)\right) + \frac{c^e}{RK}\pi(e) + \pi(u) - \frac{1}{\alpha}$. Since $f' > 0$ and $f(1) = \frac{\beta(\alpha+(1-\alpha)(1-\delta))+\alpha-1}{\alpha} < 0$ by specification, we know $\frac{c^e}{RK} > 1$ solves (25).

Second, we prove that the condition $\beta < \frac{1-\alpha}{\alpha+(1-\alpha)(1-\delta)}$ guarantees that the steady-state allocation satisfies $c^e > c^u = RK$ in a laissez-faire competitive equilibrium. Combining (21) and (20) gives

$$
\beta\left(\pi(e) + \frac{c^e}{c^u}\pi(u)\right) = \frac{1}{R} = \frac{\alpha}{\alpha + (1-\alpha)(1-\delta)}\left(\frac{1}{\alpha} - \frac{C}{RK}\right) \tag{26}
$$

Given (11), we know $c^e = c^u$ if $C \leq RK$ and $c^e > c^u$ if $C > RK$. Consider the case $C \leq RK$. Then from (26), we have

$$
\beta = \frac{\alpha}{\alpha + (1-\alpha)(1-\delta)}\left(\frac{1}{\alpha} - \frac{C}{RK}\right) \geq \frac{1-\alpha}{\alpha + (1-\alpha)(1-\delta)}, \tag{27}
$$

where we know $c^e = c^u$ since $C \leq RK$ and the inequality is due to $\frac{C}{RK} \leq 1$.

It follows that if (27) does not hold, then $C \leq RK$ cannot be true. If $C > RK$ is true, then we have $c^e > c^u$. Therefore, if $\beta < \frac{1-\alpha}{\alpha+(1-\alpha)(1-\delta)}$, then $C > RK$ and thus $c^e > c^u$ holds.

A.3 Construction of Ramsey Problem

We first show that, given the initial B_0 and K_0 as well as the allocation $\{c_t^e, n_t^e, a_{t+1}, K_{t+1}\}_{t=0}^{\infty}$, a competitive equilibrium can be constructed by using the two conditions (14) and (15) as well as following the steps below that uniquely back up the sequences of the other variables:

1. Given c_t^e, n_t^e and a_{t+1}, c_t^u is chosen such that

$$c_t^u = \begin{cases} c_t^e \text{ if } c_t^e \leq a_t \\ a_t \text{ if } c_t^e > a_t \end{cases}$$

   Note that $a_t - c_t^u \geq 0$ is satisfied.

2. Aggregate C_t and N_t are chosen such that

$$\begin{aligned} N_t &= n_t^e \pi_e, \\ C_t &= c_t^e \pi^e + c_t^e \pi^e. \end{aligned}$$

3. w_t and q_t are chosen such that $w_t = F_L(K_t, N_t)$ and $q_t = F_K(K_t, N_t)$. As a result, the firm's problem is solved.

4. $\tau_{n,t}$ is chosen such that $\widehat{w}_t = (1 - \tau_{n,t})w_t$ satisfies $v'(1 - n_t^e) = \frac{\widehat{w}_t}{c_t^e}$.

5. R_{t+1} is chosen by the household Euler equation

$$\frac{1}{R_{t+1}} = \beta \frac{\left(1/c_{t+1}^e\right)\pi(e) + \left(1/c_{t+1}^u\right)\pi(u)}{1/c_t^e},$$

   and $\tau_{k,t+1}$ is determined by

$$R_{t+1} = 1 + (1 - \tau_{k,t+1})(q_{t+1} - \delta).$$

6. Given K_{t+1} and a_{t+1}, B_{t+1} is pinned down by the asset market-clearing condition:

$$B_{t+1} + R_{t+1}K_{t+1} = a_{t+1}.$$

7. There are only two conditions left: the resource constraint and the representative household's budget constraint (or the implementability condition), which can be expressed as equations (14) and (15), respectively.

Second, note that the constraints (14) and (15) are trivially satisfied because they are part of the competitive-equilibrium conditions.

A.4 Proof of Proposition 3

Denote $\beta^t \chi_t$ and $\beta^t \zeta_t$ as the Lagrangian multipliers for conditions (14) and (15), respectively. The FOC of the Ramsey problem with respect to K_{t+1} is given by

$$\chi_t = \beta \chi_{t+1} (MP_{K,t+1} + 1 - \delta), \tag{28}$$

where $MP_{K,t+1}$ denotes $\frac{\partial F(K_{t+1}, N_{t+1})}{\partial K_{t+1}}$. The FOC (29) is given by

$$-v'\left(1-n_t^e\right) + \chi_t F_{N,t} - \zeta_t \left[\frac{v'(1-n_t^e)}{1/c_t^e} - \frac{v''(1-n_t^e)}{1/c_t^e} n_t^e \right] = 0, \tag{29}$$

which implies that the growth rates of χ_t and λ_t^p in the steady state are the same. To see this, rewrite (29) as

$$\chi_t \frac{F_{N,t}}{v'\left(1-n_t^e\right)} - \zeta_t \frac{1+\eta_t}{1/c_t^e} = 1,$$

where η_t is defined as the Frisch labor supply elasticity, $\eta_t \equiv \frac{\hat{w}}{n^e} \frac{dn^e}{d\hat{w}}\Big|_{u_c} = -\frac{v'(1-n_t^e)}{v''(1-n_t^e) n_t^e}$. Hence, it must be the case where χ_t and ζ_t grow at the same rate in the Ramsey steady state.

In what follows, we show that there exists a Ramsey steady state exhibiting full self-insurance and there is no Ramsey steady state with $R < 1/\beta$. We discuss three cases respectively below: (1) $a_t > c_t^e = c_t^u$, (2) $a_t = c_t^e = c_t^u$, and (3) $c_t^e > c_t^u = a_t$.

1. Consider the case that $a_t > c_t^e$ implying $c_t^e = c_t^u = c_t$. The FOCs with respect to a_{t+1} and c_t satisfy

$$a_{t+1} : \zeta_{t+1} = \zeta_t \frac{c_t^e}{c_{t+1}^e}, \tag{30}$$

$$c : \frac{1}{c_t} - \chi_t + \zeta_t - a_t \zeta_{t-1} \frac{c_{t-1}^e}{\left(c_t^e\right)^2} + a_{t+1} \beta \zeta_t \left(\frac{\pi(e)}{c_{t+1}^e} + \frac{\pi(u)}{c_{t+1}^u} \right) - \zeta_t v'(1-n_t^e) n_t^e \pi(e) = 0. \tag{31}$$

In Ramsey steady state, we have $\zeta_{t+1} = \zeta_t = \zeta$ and $\chi_{t+1} = \chi_t = \chi$ according to (29) and (30). Given this, (31) becomes

$$\frac{1}{c^e} - \chi + \zeta \left(1 - \frac{a}{c^e} (1-\beta) - \frac{\hat{w} n^e}{c^e} n^e \pi(e) \right) = 0, \tag{32}$$

where $\frac{\hat{w}}{c^e} = v'(1-n^e)$.

Given that $c^e = c^u$, we know $\beta R = 1$ in the steady state. Then we can rewrite the budget constraint as

$$\widehat{w} n^e \pi(e) + a(1-\beta) - c = 0. \tag{33}$$

Combining (31) and (33) leads to

$$\frac{1}{c^e} = \frac{1}{c^u} = \chi.$$

Since in Ramsey steady state $\chi_{t+1} = \chi_t = \chi$, we know that MGR holds; that is, $\beta(MP_K + 1 - \delta) = 1$. Thus we obtain $\tau_k = 0$ in the Ramsey steady state.

2. We then consider the case where $a_t = c_t^e = c_t^u$ and that the FOC with respect to c_t^e satisfies

$$\begin{aligned} 0 \;=\; & \beta^t \frac{1}{a_t} - \beta^t \chi_t + \beta^{t-1}\zeta_{t-1}\beta\left[\left(c_{t-1}^e/c_t^e\right)\pi(e) + \left(c_{t-1}^e/c_t^u\right)\pi(u)\right] - \beta^t\zeta_t \\ & + \beta^t\zeta_t a_{t+1}\beta\left[\frac{\pi(e)}{c_{t+1}^e} + \frac{\pi(u)}{c_{t+1}^u}\right] + \beta^t\zeta_t \\ & - \beta^t\zeta_t v'(1-n_t^e)n_t^e\pi(e) - \beta^{t-1}\zeta_{t-1}a_t\beta\left[\frac{c_{t-1}^e\pi(e)}{c_t^e c_t^e} + \frac{c_{t-1}^e\pi(u)}{c_t^u c_t^u}\right]. \end{aligned}$$

The steady-state version of the equation above and the implementability condition evaluated at $a = c^e = c^u$ are

$$\chi_t = \frac{1}{a} + \zeta_t\left(\beta - v'(1-n^e)n^e\pi(e)\right),$$

$$v'(1-n^e)n^e\pi(e) = \beta,$$

respectively. It follows that

$$\chi_t = \frac{1}{a}.$$

As a result, χ_t converges and this implies that MGR holds and $\tau_k = 0$.

3. Finally, we show that there is no Ramsey steady state featuring $R < 1/\beta$. In this case,

$c_t^e > c_t^u = a_t$. The FOCs with respect to c_t^e and c_t^u satisfy

$$\begin{aligned}
c_t^e \quad : \quad & \frac{\pi(e)}{c_t^e} - \chi_t \pi(e) - \zeta_t v'(1-n_t^e) n_t^e \pi(e) + \zeta_t \pi(e) \\
& + \zeta_t a_{t+1} \beta \left[\left(1/c_{t+1}^e\right) \pi(e) + \left(1/c_{t+1}^u\right) \pi(u) \right] - \zeta_{t-1} a_t \frac{c_{t-1}^e}{c_t^e c_t^e} \pi(e) \\
= \quad & 0, \qquad (34)
\end{aligned}$$

and

$$\begin{aligned}
c_t^u \quad : \quad & \frac{\pi(u)}{c_t^u} - \chi_t \pi(u) + \zeta_t \pi(u) - \zeta_t \\
& + \zeta_{t-1} \left[\left(c_{t-1}^e / c_t^e\right) \pi(e) + \left(c_{t-1}^e / c_t^u\right) \pi(u) \right] - \zeta_{t-1} a_t \frac{c_{t-1}^e}{c_t^u c_t^u} \pi(u) \\
= \quad & 0. \qquad (35)
\end{aligned}$$

By implementability condition, we rewrite (34) as

$$\frac{\pi(e)}{c_t^e} - \chi_t \pi(e) + \zeta_t \frac{c_t^u}{c_t^e} \pi(e) - \zeta_{t-1} c_t^u \frac{c_{t-1}^e}{c_t^e c_t^e} \pi(e) = 0. \qquad (36)$$

Rearranging (35) and (36) and evaluating them at steady state, we obtain

$$\frac{\pi(u)}{c^u} - \chi_t \pi(u) = (\zeta_t - \zeta_{t-1}) \pi(e), \qquad (37)$$

$$\frac{\pi(e)}{c^e} - \chi_t \pi(e) = (\zeta_{t-1} - \zeta_t) \frac{c^u}{c^e} \pi(e). \qquad (38)$$

Let g denote the growth rate of χ_t as well as ζ_t (recall that χ_t and ζ_t must grow at the same rate in steady state). Given $\frac{1}{c^u} > \frac{1}{c^e}$, from (37) and (38) we obtain $(\zeta_t - \zeta_{t-1}) \frac{\pi(e)}{\pi(u)} > (\zeta_{t-1} - \zeta_t) \frac{c^u}{c^e}$, which can be expressed as

$$(\zeta_t - \zeta_{t-1}) \frac{\pi(e)}{\pi(u)} - (\zeta_{t-1} - \zeta_t) \frac{c^u}{c^e} = (\zeta_t - \zeta_{t-1}) \left(\frac{\pi(e)}{\pi(u)} + \frac{c^u}{c^e} \right) > 0,$$

which implies $g \equiv \frac{\zeta_t}{\zeta_{t-1}} > 1$. Manipulating (37) by $\frac{\pi(u)}{c_t^u} = \frac{\pi(u)}{c_{t-1}^u}$ gives

$$\chi_t \pi(u) + (\zeta_t - \zeta_{t-1}) \pi(e) = \chi_{t-1} \pi(u) + (\zeta_{t-1} - \zeta_{t-2}) \pi(e).$$

Given $\chi_t = g\chi_{t-1}$, $\zeta_t = g\zeta_{t-1}$ and $\zeta_{t-2} = \frac{1}{g}\zeta_{t-1}$, rewrite the equation above as

$$g\chi_{t-1}\pi(u) + (g\zeta_{t-1} - \zeta_{t-1})\,\pi(e) = \chi_{t-1}\pi(u) + \left(\zeta_{t-1} - \frac{1}{g}\zeta_{t-1}\right)\pi(e)\,.$$

Hence, we have

$$\chi_{t-1}\pi(u) = \pi(e)\,\zeta_{t-1}\left(\frac{1-g}{g}\right). \tag{39}$$

If $\zeta_{t-1} > 0$, equation (39) implies $\chi_{t-1} < 0$, which leads to a contradiction, given that both ζ_{t-1} and χ_{t-1} must be non-negative. If ζ_{t-1} equals zero, then $\chi_{t-1} = 0$ and $\zeta_t = 0$. We rule out the case that $\zeta_{t-1} = \chi_{t-1} = 0$ because $c^e = c^u \to \infty$, obtained by (37) and (38), is impossible.

Finally, we show that the optimal level of debt is strictly positive if condition (13) is met. Let's consider the lowest amount of government debt that achieve the full-self insurance. Namely, consider the steady state case where $a = c^e = c^u$.

By the flow government budget constraint in Ramsey steady state, the steady state labor tax rate can be written as a function of debt to GDP ratio, B/Y :

$$\tau_n = (1-\beta)\frac{B}{Y}\frac{1}{\frac{wN}{Y}} = \frac{(1-\beta)}{(1-\alpha)}\frac{B}{Y}, \tag{40}$$

where the last equality uses the labor share $\frac{wN}{Y} = 1-\alpha$. In addition, the steady state version of household budget constraint implies that the asset holding-to-GDP ratio is given by

$$\frac{a}{Y} = (1-\tau_n)\frac{wN}{Y}\frac{1}{\beta} = (1-\tau_n)\frac{1-\alpha}{\beta}. \tag{41}$$

In addition, we know that MGR holds in the Ramsey steady state, which gives

$$\frac{K}{Y} = \frac{\alpha}{\frac{1}{\beta} - 1 + \delta}. \tag{42}$$

By plugging the three equations above into the asset market-clearing condition, we can oblation

$$\frac{B}{Y} = \frac{a}{Y} - \frac{1}{\beta}\frac{K}{Y} = \left(1 - \frac{(1-\beta)}{(1-\alpha)}\frac{B}{Y}\right)\frac{1-\alpha}{\beta} - \frac{1}{\beta}\frac{K}{Y},$$

which solves for the steady-state B/Y ratio as

$$\frac{B}{Y} = 1 - \alpha - \frac{\alpha}{\frac{1}{\beta} - 1 + \delta}.$$

Hence, the optimal debt is positive if $1 - \alpha > \frac{\alpha}{\frac{1}{\beta}-1+\delta}$, which is exactly condition (13).