Should Capital Be Taxed?

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Abstract

We design an infinite-horizon heterogeneous-agents and incomplete-markets model to demonstrate analytically that in the absence of any redistributional effects of government policies, optimal capital tax is zero despite capital overaccumulation under precautionary savings and borrowing constraints. Our result indicates that in the long run public debt is a better tool than capital taxation to restore aggregate productive efficiency.

JEL Classification: C61; E22; E62; H21; H30
Key Words: Optimal Capital Taxation; Government Bonds; Heterogeneous Agents; Incomplete Markets; Modified Golden Rule; Ramsey Problem; Wealth Distribution.

1   Introduction

In a representative-agent framework, the Ramsey taxation literature has shown that the best way for the government to finance its expenditures in the long run is to tax labor but not capital (see Chamley (1986) and Chari, Christiano, and Kehoe (1994)). This "zero capital tax" result is surprising and has attracted many studies to examine its robustness.[1] A significant progress comes from the seminal work of Aiyagari (1995), which shows that it is optimal to tax capital in a heterogeneous-agent incomplete-markets (HAIM) framework. However, despite several important revisits, such as Chamley (2001), Conesa, Kitao, and Krueger (2009), Dávila, Hong, Krusell, and Ríos-Rull (2012), and many others, some key issues regarding optimal capital taxation in a HAIM economy (á la Aiyagari (1994)) remain unsettled, as recently discussed by Chien and Wen (2020).

For example, in an infinite-horizon HAIM economy, is a positive tax levied on capital by the Ramsey planner motivated chiefly by correcting the failure of the modified golden rule (MGR) in light of capital overaccumulation, by wealth redistribution in light of income inequality under borrowing constraints, or both?

These questions are intertwined not only because both government bonds and a capital tax can change individuals' incentives for saving, but also because they both tend to have a redistributional effect on household wealth.

This paper designs a tractable HAIM model to address these questions by shutting down the redistributional channel. Our analytical approach is built upon the work of Heathcote and Perri (2018). We show analytically that (i) the Ramsey planner will never tax capital in a Ramsey steady state despite capital overaccumulation and (ii) the MGR holds provided that the government can amass a sufficiently large stock of bonds to enable households to achieve full self-insurance.

These results are consistent with the recent study of Chien and Wen (2020), which proves in a HAIM model with quasi-linear preferences that the optimal capital tax is always zero. However, Chien and Wen (2020) relied critically on quasi-linear preferences to derive their results. The drawback of quasi-linear preferences is that the marginal utility cost of a labor tax is constant and this property may be driving their "zero capital tax" result.

In this paper, we relax the assumption of quasi-linear preferences and assume instead that households can reshuffle asset holdings at the end of each period such that the distribution

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[1]For example, in the introduction of Straub and Werning (2020): "One may even say that the result (zero capital tax) remains downright puzzling, as witnessed by the fact that economists have continued to take turns putting forth various intuitions to interpret it, none definitive nor universally accepted."

of households' end-of-period wealth is degenerated, as in Heathcote and Perri (2018). This feature shuts down the wealth redistribution effects of government policies and at the same time allows our HAIM model to be analytically tractable despite nonlinear preferences, idiosyncratic risk, and precautionary saving motives.

2 The Model

2.1 Model Setup

Time is discrete and indexed by $t = 0, 1, 2, ..., \infty$. A representative firm's production technology is $Y_t = F(K_t, N_t) = K_t^\alpha N_t^{1-\alpha}$, where Y, K, and N denote aggregate output, capital, and labor, respectively. The firm rents capital and hires labor from households by paying a competitive rental rate q_t and real wage w_t. Profit maximization gives

$$
\begin{aligned}
w_t &= \frac{\partial F(K_t, N_t)}{\partial N_t} \equiv MP_{N,t}, \\
q_t &= \frac{\partial F(K_t, N_t)}{\partial K_t} \equiv MP_{K,t}.
\end{aligned}
$$

The government can issue one-period bonds and levy flat-rate time-varying taxes on labor and capital, denoted by $\tau_{n,t}$ and $\tau_{k,t}$, respectively. The flow government budget constraint is given by

$$
\tau_{n,t} w_t N_t + \tau_{k,t}(q_t - \delta)K_t + \frac{B_{t+1}}{R_{t+1}} = B_t, \tag{1}
$$

where R_{t+1} is the risk-free gross interest rate between t and $t+1$, $\delta \in (0,1)$ is the depreciation rate of capital, and B_{t+1} is the amount of government bonds issued at time t. The government is assumed to fully commit to a sequence of future taxes and debts issued, given the initial bond supply B_0 at time 0.

There is a unit measure of *ex ante* identical agents belonging to a representative household. Individual agents face an idiosyncratic shock to their employment status, denoted by $\theta \in \{e, u\}$. The shock is iid over time and across individuals. If $\theta = e$, an individual can work and receive labor income; if $\theta = u$, an individual cannot work and has no labor income. Let $\pi(e)$ and $\pi(u)$ denote the corresponding probability of employment status.

There are two subperiods within each period. In the first subperiod, individual agents are separated from their representative households and receive the employment shock θ. In addition, their consumption and labor supply decisions (if employed) are made in the

first subperiod. In the second subperiod, individual agents are reunited back with their representative households and share their assets. Therefore, the idiosyncratic shock is not fully insurable but the wealth distribution is degenerated. Hence, individuals still have precautionary saving motives. This setup not only eliminates the wealth redistribution effects of any government policies but also renders the model analytically tractable.

The lifetime utility of the representative household is given by

$$U = \sum_{t=0}^{\infty} \beta^t \left\{ [\log c_t^e + \log (1 - n_t^e)] \pi(e) + \log c_t^u \pi(u) \right\}, \tag{2}$$

where $\beta \in (0, 1)$ is the discount factor; c_t^e and c_t^u denote the consumption for employed and unemployed individuals at time t; and n_t^e is the labor supply of the employed individuals. Note that the labor supply is zero for unemployed individuals. The budget constraints for employed and unemployed individuals are given, respectively, by:

$$c_t^e \leq a_t + \widehat{w}_t n_t^e, \tag{3}$$

$$c_t^u \leq a_t; \tag{4}$$

and the representative household's flow budget constraint is thus given by

$$c_t^e \pi(e) + c_t^u \pi(u) + \frac{a_{t+1}}{R_{t+1}} \leq a_t + \widehat{w}_t n_t^e \pi(e), \tag{5}$$

where $\widehat{w}_t = (1 - \tau_{n,t}) w_t$ is the after-tax wage rate and a_t is the asset holdings carried over from period $t - 1$.

Finally, government bonds and capital are perfect substitutes as stores of value for individuals. As a result, the after-tax gross rate of return to capital must equal the gross risk-free rate:

$$R_{t+1} = 1 + (1 - \tau_{k,t+1})(q_{t+1} - \delta), \tag{6}$$

which constitutes a constitutes a no-arbitrage condition for capital and bonds.

2.2 Competitive Equilibrium

Definition 1. *Given the initial capital K_0 and initial government bond B_0, a competitive equilibrium is defined as sequences of tax rates and government bonds $\{\tau_{n,t}, \tau_{k,t}, B_{t+1}\}_{t=0}^{\infty}$, and sequences of prices $\{w_t, q_t, R_t\}_{t=0}^{\infty}$, aggregate allocations $\{C_t, N_t, K_{t+1}\}_{t=0}^{\infty}$ and individual*

allocations $\{c_t^e, c_t^u, n_t^e, a_{t+1}\}_{t=0}^{\infty}$, *such that*

1. *given prices,* $\{c_t^e, c_t^u, n_t^e, a_{t+1}\}_{t=0}^{\infty}$ *solves the household problem;*

2. *given prices,* $\{N_t, K_t\}_{t=0}^{\infty}$ *solves the firm problem;*

3. *the no-arbitrage condition (6) holds for all t;*

4. *the government budget constraint, equation (1), holds for all t;*

5. *all markets clear for all* $t \geq 0$:

$$B_{t+1} + R_{t+1}K_{t+1} = a_{t+1} \tag{7}$$
$$N_t = n_t^e \pi(e),$$
$$C_t = c_t^e \pi(e) + c_t^u \pi(u),$$
$$F(K_t, N_t) = C_t + K_{t+1} - (1-\delta)K_t. \tag{8}$$

Proposition 1. *The competitive equilibrium has the following two properties:*

1. c_t^u *can be expressed as a function of* c_t^e *and* a_t :

$$c_t^u = \begin{cases} c_t^e & \text{if } c_t^e \leq a_t, \\ a_t & \text{if } c_t^e > a_t. \end{cases} \tag{9}$$

2. *The risk-free interest rate and the time discount rate are related to each other by*

$$\frac{1}{R_{t+1}} = \beta \frac{\left(1/c_{t+1}^e\right)\pi(e) + \left(1/c_{t+1}^u\right)\pi(u)}{1/c_t^e}. \tag{10}$$

Proof. Please refer to Appendix A.1. ☐

Proposition 1 implies that if the asset holdings (or self-insurance position) a_t are suffi-ciently large such that $c_t^e \leq a_t$, then individuals can obtain the same level of consumption ($c_t^u = c_t^e$) regardless of their employment status. We call this situation a full self-insurance allocation. Obviously, by the asset market clearing condition (7), this could be achieved if the capital stock or the supply of government bonds is sufficiently high. In addition, if a full self-insurance allocation is achieved in the steady state, then the steady-state risk-free rate R is equal to the time discount rate $1/\beta$. Otherwise, $R < 1/\beta$.

In order to make our analysis interesting and the Ramsey problem meaningful, the model parameters have to be restricted in a way such that the full self-insurance allocation is not achieved in the steady state of laissez-faire competitive equilibria. Otherwise, there is no issue of capital overaccumulation since $R = 1/\beta$ and the MGR holds in the steady state. The following proposition provides a necessary and sufficient condition for such a parameter restriction.

Proposition 2. *The steady state of laissez-faire competitive equilibria does not feature full self-insurance if and only if the model parameters satisfy the following condition:*

$$\beta < \frac{1 - \alpha}{\alpha + (1 - \alpha)(1 - \delta)}. \tag{11}$$

Proof. Please refer to Appendix A.2. $\square$

Under condition (11) and without any government intervention, precautionary-saving motives reduce the equilibrium interest rate below $1/\beta$ and there is overaccumulation of capital in the sense that the steady-state capital stock exceeds the level implied by the MGR. We then study the Ramsey outcome assuming that condition (11) holds.

3 The Ramsey Outcome

3.1 Ramsey Problem

As shown in the Appendix A.3, the Ramsey problem can be represented as maximizing (2) by choosing $\{c_t^e, c_t^u, n_t^e, a_{t+1}, K_{t+1}\}_{t=0}^{\infty}$, and subject to the resource constraint

$$F(K_t, n_t^e \pi(e)) + (1 - \delta)K_t - c_t^e \pi(e) - c_t^u \pi(u) - K_{t+1} \geq 0, \tag{12}$$

and the implementability condition

$$\frac{c_t^e}{1 - n_t^e} n_t^e \pi(e) + a_t - c_t^e \pi(e) - c_t^u \pi(u) - a_{t+1} \beta \frac{\left(1/c_{t+1}^e\right) \pi(e) + \left(1/c_{t+1}^u\right) \pi(u)}{1/c_t^e} \leq 0, \tag{13}$$

for $t \geq 0$, where

$$c_t^u = \begin{cases} c_t^e & \text{if } c_t^e \leq a_t, \\ a_t & \text{if } c_t^e > a_t. \end{cases}$$

5

In addition, $a_0 = (1 + (1 - \tau_{k,0})(MP_{K,0} - \delta))K_0 + B_0$, where $\tau_{k,0}$, K_0 and B_0 are exogenously given.

3.2 Ramsey Steady State

Our theoretical findings focus on the long-run Ramsey outcome, but we also study the transitional dynamics of the Ramsey allocation. In a recent paper, Straub and Werning (2020) revisited the classical Chamley-Judd result, showing that the common assumption that the endogenous multipliers associated with the Ramsey problem will always converge in the limit is not necessarily correct. However, in our model we can prove that not only the Ramsey allocation but also the Lagrangian multipliers associated with conditions (12) and (13) all converge to finite positive values (see the proof of Proposition 3). In subsection 3.3, we numerically solve for the entire transitional path of the Ramsey problem, showing that the Ramsey allocation converges to the Ramsey steady state. Hence, Straub and Werning (2020)'s criticism regarding the models of Chamley (1986) and Judd (1985) does not apply to our model.

Proposition 3. *Under condition (11), a Ramsey steady state exists and it features the following properties:[2]*

1. *The full self-insurance allocation is achieved, namely, $a = c^e = c^u$, and $R = 1/\beta$.*

2. *The MGR holds without taxing capital, $\tau_k = 0$.*

3. *The optimal level of government bonds is strictly positive.*

4. *The bond interest payments are financed solely by revenues from a labor income tax, $\tau_n > 0$.*

Finally, there cannot exist a Ramsey steady state featuring $c^e > c^u$ and $R < 1/\beta$.

Proof. See Appendix A.4. □

There are several key points here. First, instead of taxing capital in the long run, the Ramsey planner always opts to supply enough government bonds to solve the problem of production inefficiency (i.e., the failure of the MGR) caused by the households' precautionary savings. In other words, a long-run capital tax is never optimal in the Ramsey steady state.

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[2]We denote variables without the subscript t as their steady state values.

Hence, the government bond is a better tool than the capital tax to restore the production efficiency.

This result is fully consistent with Chien and Wen (2020), who derive a similar result using a different type of HAIM model. The intuition behind the result is that government bonds can play a role in crowding out overaccumulated capital and improve the self-insurance position of the households. In contrast, although imposing a steady-state capital tax can alleviate the problem of capital overaccumulation, it does so at the expense of worsening the problem of insufficient self-insurance caused by incomplete markets and borrowing constraints.

Second, the optimal level of debt is positive only if there is a lack of full self-insurance in the laissez-faire competitive equilibrium. This result highlights again that the purpose of supplying public debt is to satisfy the need for precautionary savings. In other words, with a sufficient amount of public debt, the Ramsey planner can achieve full self-insurance as well as production efficiency.

Finally, Proposition 3 shows that the Ramsey steady state is unique in the sense that the Ramsey planner never settles in a steady state without full self-insurance and with $R < 1/\beta$. This result is consistent with the finding in the study by Chen, Chien, and Yang (2019), which shows that in the absence of government debt limits a Ramsey steady state featuring the property $R < 1/\beta$ does not exist.
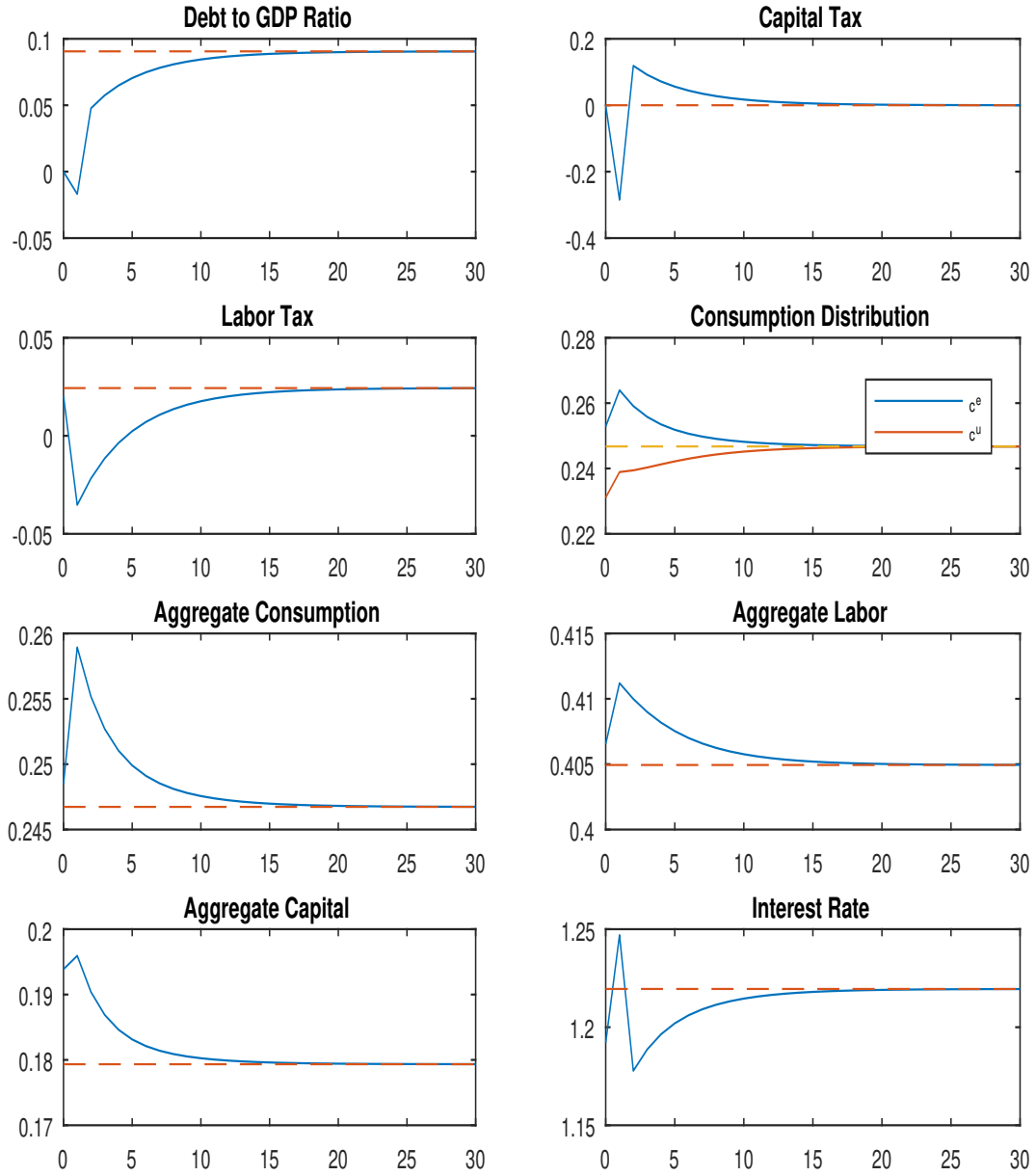
3.3   Ramsey Transition Path

We now turn to the transitional dynamics of the Ramsey allocation through numerical simulations of the model, which not only substantiate our theoretical results but also illustrate the pattern of optimal transition paths of the Ramsey allocation. Such numerical analyses are valid because the Ramsey steady state has been proven to exist under our parameter specifications.

Parameter Values. Each model period is assumed to be five years and hence β and δ are set to be 0.82 and 0.35, respectively. The capital share α is 0.33. Note that condition (11) is met under this parameter setting. The population share of employment and unemployment agents are $\pi(e) = 0.8$ and $\pi(u) = 0.2$, respectively. The initial capital is set to be its steady state value in the laissez-faire competitive equilibrium. Finally, we set both the initial bond supply and capital tax to zero, namely, $B_0 = \tau_{k,0} = 0$.

Ramsey Transition Paths. Figure 1 plots the transition path of the Ramsey outcome. First, the initial conditions in period 0 cause some temporary adjustment in period 1, but

Figure 1: Ramsey Transition Paths

Notes: Transition paths in the Ramsey economy (solid lines) and their corresponding steady states (dashed lines).

starting from the second period, the optimal debt-to-GDP ratio (panel [1,1]) increases gradually over time (after a temporary drop in period 1) so as to boost household savings, thus improving households' self-insurance position and crowding out aggregate capital. Hence, as the household self-insurance position improves, the consumption inequality (panel [2,2]) shrinks and the overaccumulated aggregate capital stock (panel [4,1]) declines. The path of aggregate consumption (panel [3,1]) shows a clear pattern of front-loading: it jumps up sharply in the second period and then gradually decreases back to its long-run steady state. The aggregate labor supply (panel [3,2]) shows a similar pattern because more work efforts are required to support consumption.

The increasing amount of public debt clearly requires financing from tax revenues. The Ramsey planner opts to put the pressure of revenue collection on capital taxes in the short run (after a temporary drop in period 1) and shift the burden to labor taxes in the longer run—such that short-run capital tax reaches its highest level in period 2 to about 12% and gradually reduces to 0% in the long run; meanwhile, the labor tax rate is negative (panel [2,1]) in the short run so as to incentivize hard working behavior, and gradually rises to a steady-state value of 2.43% in the long run. This suggests that the source of government revenues to finance public debt after period 1 comes mainly from a positive capital tax in the short run but exclusively from a labor tax in the long run. This result differs clearly from that obtained form a representative agent model with CRRA utility, where the optimal capital tax becomes zero after the first period.

Also notice that the optimal capital tax (panel [1,2]) is momentarily negative in period 1 (before turning to positive in the following periods) because the optimal bond position is negative in period 1, which may have to do with our assumption that the initial bond supply and capital tax are both zero in period 0. Hence, the only way to boost consumption in period 1 is to lend to the households by issuing a negative amount of government bonds and use the interest payments from consumers to subsidize capital in period 1. This fact also explains why the market interest rate (panel [4,2]) jumps up momentarily in period 1 before falling below the time discount rate $(1/\beta)$ in period 2 and beyond. In the long run, however, the interest rate approaches the time discount rate from below.

4   Conclusion

In a HAIM environment with degenerated wealth distribution, we prove the existence of a unique type of Ramsey steady featuring the following properties: (i) optimal long-run capital

tax is zero despite capital overaccumulation under precautionary savings and incomplete markets and (ii) the MGR can be achieved through supplying enough government bonds such that households are fully self-insured against idiosyncratic risk. These theoretical results are also confirmed by our numerical analysis. Along the transitional path of the Ramsey allocation, optimal capital tax is positive in order to finance the increasing amount of public debt, while optimal labor tax is negative in order to stimulate work efforts and consumption when the interest rate is below the time discount rate. Hence, our result indicates that public debt is a better tool than capital tax to restore long-run production efficiency hindered by capital overaccumulation under precautionary savings.

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A Online Appendix

A.1 Proof of Proposition 1

The household chooses a sequence of $\{c_t^e, c_t^u, n_t^e, a_{t+1}\}_{t=0}^{\infty}$ to maximize (2) subject to (5), (3) and (4). Let $\beta^t \lambda_t^e \pi(e)$, $\beta^t \lambda_t^u \pi(u)$ and $\beta^t \mu_t$ denote Lagrangian multipliers attached to (3), (4) and (5) respectively. The FOCs with respect to c_t^e, c_t^u, n_t^e and a_{t+1}, are given, respectively, by

$$\frac{1}{c_t^e} = \lambda_t^e + \mu_t, \tag{14}$$

$$\frac{1}{c_t^u} = \lambda_t^u + \mu_t, \tag{15}$$

$$\frac{1}{1 - n_t^e} = \widehat{w}_t (\lambda_t^e + \mu_t), \tag{16}$$

$$\beta(\lambda_{t+1}^e \pi(e) + \lambda_{t+1}^u \pi(e)) + \beta \mu_{t+1} = \frac{\mu_t}{R_{t+1}}. \tag{17}$$

By equations (3) and (4), we know that $c_t^u \leq c_t^e$ so that $\lambda_t^u \geq \lambda_t^e$. Now suppose $\lambda_t^e > 0$, it follows that $\lambda_t^u > 0$. Given $\lambda_t^e > 0$ and $\lambda_t^u > 0$, both (3) and (4) must be binding. Inserting $c_t^e = a_t + \widehat{w}_t n_t^e$ and $c_t^u = a_t$ back to (5), we obtain $a_{t+1} = 0$, which together with (4) imply that $c_{t+1}^u = 0$. This leads to a contradiction since $u'(0) = \infty$. As a consequence, λ_t^e must be zero. Next, consider two possibilities for λ_t^u. One is that $\lambda_t^u > 0$ so that $c_t^u = a_t$ and $c_t^u < c_t^e$ (recall that $\lambda_t^e = 0$). The other is that $\lambda_t^u = 0$, along with $\lambda_t^u = \lambda_t^e = 0$, which leads to $c_t^e = c_t^u \leq a_t$. Hence, we can express c_t^u as a function of c_t^e and a_t as shown by (9). In addition, using $\lambda_t^e = 0$, we can rewrite (17) as (10).

A.2 Proof of Proposition 2

Under the laissez-faire competitive equilibrium, $\tau_{l,t} = \tau_{k,t} = B_t = 0$ and $a_t = R_t K_t$. Provided that $c^e > c^u = RK$ is true, we can conclude that in the steady state $\beta R < 1$. Given this, we first prove that if $c^e > c^u = RK$, then $\beta < \frac{1-\alpha}{\alpha + (1-\alpha)(1-\delta)}$ must hold.

Based on $R = 1 + \alpha K^{\alpha-1} N^{1-\alpha} - \delta$, we express $\left(\frac{K}{N}\right)^{\alpha-1}$ as $\frac{R+\delta-1}{\alpha}$. In the steady state, we have Euler equation (10) and the resource constraint imply

$$1 = \beta R \left(\pi(e) + \frac{c^e}{c^u} \pi(u) \right), \tag{18}$$

12

$$
\begin{aligned}
c^e \pi(e) + c^u \pi(u) &= K^\alpha N^{1-\alpha} - \delta K \\
&= \left(\alpha K^{\alpha-1} N^{1-\alpha} - \delta \right) K + (1-\alpha) K^{\alpha-1} N^{1-\alpha} K \\
&= (R-1) K + (1-\alpha) \left(\frac{R+\delta-1}{\alpha} \right) K, \quad (19)
\end{aligned}
$$

where we replace $K^{\alpha-1} N^{1-\alpha}$ with $\frac{R+\delta-1}{\alpha}$ in (19). Dividing each term by RK in (19) gives

$$
\frac{c^e}{RK} \pi(e) + \frac{c^u}{RK} \pi(u) = \frac{R-1}{R} + \frac{1-\alpha}{\alpha} \frac{R+\delta-1}{R}. \quad (20)
$$

Substituting $c^u = RK$ into (18) and (20) yields

$$
\frac{1}{R} - \beta \pi(u) \frac{c^e}{RK} - \beta \pi(e) = 0. \quad (21)
$$

$$
\frac{\alpha + (1-\alpha)(1-\delta)}{\alpha} \frac{1}{R} + \frac{c^e}{RK} \pi(e) + \pi(u) - \frac{1}{\alpha} = 0, \quad (22)
$$

Combining (21) and (22) leads to

$$
f \left(\frac{c^e}{RK} \right) = 0, \quad (23)
$$

where $f : \frac{c^e}{RK} \rightarrow \frac{\alpha + (1-\alpha)(1-\delta)}{\alpha} \beta \left(\pi(u) \frac{c^e}{RK} + \pi(e) \right) + \frac{c^e}{RK} \pi(e) + \pi(u) - \frac{1}{\alpha}$. Since $f' > 0$ and $f(1) = \frac{\beta(\alpha+(1-\alpha)(1-\delta))+\alpha-1}{\alpha} < 0$ by specification, we know $\frac{c^e}{RK} > 1$ solves (23).

Second, we prove that the condition $\beta < \frac{1-\alpha}{\alpha+(1-\alpha)(1-\delta)}$ guarantees that the steady-state allocation satisfies $c^e > c^u = RK$ in a laissez-faire competitive equilibrium. Combining (19) and (18) gives

$$
\beta \left(\pi(e) + \frac{c^e}{c^u} \pi(u) \right) = \frac{1}{R} = \frac{\alpha}{\alpha + (1-\alpha)(1-\delta)} \left(\frac{1}{\alpha} - \frac{C}{RK} \right) \quad (24)
$$

Given (9), we know $c^e = c^u$ if $C \leq RK$ and $c^e > c^u$ if $C > RK$. Consider the case $C \leq RK$. Then from (24), we have

$$
\beta = \frac{\alpha}{\alpha + (1-\alpha)(1-\delta)} \left(\frac{1}{\alpha} - \frac{C}{RK} \right) \geq \frac{1-\alpha}{\alpha + (1-\alpha)(1-\delta)}, \quad (25)
$$

where we know $c^e = c^u$ since $C \leq RK$ and the inequality is due to $\frac{C}{RK} \leq 1$.

It follows that if (25) does not hold, then $C \leq RK$ cannot be true. If $C > RK$ is true, then we have $c^e > c^u$. Therefore, if $\beta < \frac{1-\alpha}{\alpha+(1-\alpha)(1-\delta)}$, then $C > RK$ and thus $c^e > c^u$ holds.

A.3 Construction of Ramsey Problem

We first show that, given the initial B_0 and K_0 as well as the allocation $\{c_t^e, n_t^e, a_{t+1}, K_{t+1}\}_{t=0}^\infty$, a competitive equilibrium can be constructed by using the two conditions (12) and (13) as well as following the steps below that uniquely back up the sequences of the other variables:

1. Given c_t^e, n_t^e and a_{t+1}, c_t^u is chosen such that

$$c_t^u = \begin{cases} c_t^e \text{ if } c_t^e \leq a_t \\ a_t \text{ if } c_t^e > a_t \end{cases}$$

Note that $a_t - c_t^u \geq 0$ is satisfied.

2. Aggregate C_t and N_t are chosen such that

$$N_t = n_t^e \pi(e),$$
$$C_t = c_t^e \pi(e) + c_t^u \pi(u).$$

3. w_t and q_t are chosen such that $w_t = MP_{N,t}$ and $q_t = MP_{K,t}$. As a result, the firm's problem is solved.

4. $\tau_{n,t}$ is chosen such that $\widehat{w}_t = (1 - \tau_{n,t})w_t$ satisfies $\frac{1}{1-n_t^e} = \frac{\widehat{w}_t}{c_t^e}$.

5. R_{t+1} is chosen by the household Euler equation

$$\frac{1}{R_{t+1}} = \beta \frac{\left(1/c_{t+1}^e\right)\pi(e) + \left(1/c_{t+1}^u\right)\pi(u)}{1/c_t^e},$$

and $\tau_{k,t+1}$ is determined by

$$R_{t+1} = 1 + (1 - \tau_{k,t+1})(q_{t+1} - \delta).$$

6. Given K_{t+1} and a_{t+1}, B_{t+1} is pinned down by the asset market-clearing condition:

$$B_{t+1} + R_{t+1}K_{t+1} = a_{t+1}.$$

7. There are only two conditions left: the resource constraint and the representative household's budget constraint (or the implementability condition), which can be expressed as equations (12) and (13), respectively.

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Second, note that the constraints (12) and (13) are trivially satisfied because they are part of the competitive-equilibrium conditions.

A.4 Proof of Proposition 3

Denote $\beta^t \chi_t$ and $\beta^t \zeta_t$ as the Lagrangian multipliers for conditions (12) and (13), respectively. The FOC of the Ramsey problem with respect to K_{t+1} is given by

$$\chi_t = \beta \chi_{t+1}(MP_{K,t+1} + 1 - \delta). \tag{26}$$

The FOC (27) is given by

$$\chi_t MP_{N,t} = \frac{1}{(1 - n_t^e)} + \zeta_t \frac{c_t^e}{(1 - n_t^e)^2}, \tag{27}$$

which implies that the growth rates of χ_t and ζ_t in the steady state are the same. Hence, it must be the case where χ_t and ζ_t grow at the same rate in the Ramsey steady state.

In what follows, we show that there exists a Ramsey steady state exhibiting full self-insurance and there is no Ramsey steady state with $R < 1/\beta$. We discuss three cases respectively below: (1) $a_t > c_t^e = c_t^u$, (2) $a_t = c_t^e = c_t^u$, and (3) $c_t^e > c_t^u = a_t$.

1. Consider the case that $a_t > c_t^e$ implying $c_t^e = c_t^u = c_t$. The FOCs with respect to a_{t+1} and c_t are

$$\zeta_{t+1} = \zeta_t \frac{c_t^e}{c_{t+1}^e}, \tag{28}$$

and

$$\frac{1}{c_t^e} - \chi_t + \zeta_t - a_t \zeta_{t-1} \frac{c_{t-1}^e}{(c_t^e)^2} + a_{t+1} \beta \zeta_t \left(\frac{\pi(e)}{c_{t+1}^e} + \frac{\pi(u)}{c_{t+1}^u} \right) - \zeta_t \frac{n_t^e}{1 - n_t^e} \pi(e) = 0, \tag{29}$$

respectively. In Ramsey steady state, we have $\zeta_{t+1} = \zeta_t = \zeta$ and $\chi_{t+1} = \chi_t = \chi$ according to (27) and (28). Given this, (29) in steady state becomes

$$\frac{1}{c^e} - \chi + \zeta \left(1 - \frac{a}{c^e}(1 - \beta) - \frac{\hat{w} n^e}{c^e} \pi(e) \right) = 0, \tag{30}$$

where $\frac{\hat{w}}{c^e} = \frac{1}{1 - n^e}$.

Given that $c^e = c^u$, we know $\beta R = 1$ in the steady state. Then we can rewrite the

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budget constraint as

$$\widehat{w}n^e\pi(e) + a\,(1 - \beta) - c = 0. \tag{31}$$

Combining (29) and (31) leads to

$$\frac{1}{c^e} = \frac{1}{c^u} = \chi > 0.$$

Since in Ramsey steady state $\chi_{t+1} = \chi_t = \chi > 0$, we know that MGR holds; that is, $\beta\,(MP_K + 1 - \delta) = 1$. Thus we obtain $\tau_k = 0$ in the Ramsey steady state.

2. We then consider the case where $a_t = c_t^e = c_t^u$ and that the FOC with respect to c_t^e satisfies

$$\begin{aligned}
0 &= \beta^t \frac{1}{a_t} - \beta^t \chi_t + \beta^{t-1}\zeta_{t-1}\beta\left[\left(c_{t-1}^e/c_t^e\right)\pi(e) + \left(c_{t-1}^e/c_t^u\right)\pi(u)\right] - \beta^t\zeta_t \\
&\quad + \beta^t\zeta_t a_{t+1}\beta\left[\frac{\pi\,(e)}{c_{t+1}^e} + \frac{\pi(u)}{c_{t+1}^u}\right] + \beta^t\zeta_t \\
&\quad - \beta^t\zeta_t \frac{n_t^e}{1 - n_t^e}\pi(e) - \beta^{t-1}\zeta_{t-1}a_t\beta\left[\frac{c_{t-1}^e\pi(e)}{c_t^e c_t^e} + \frac{c_{t-1}^e\pi(u)}{c_t^u c_t^u}\right].
\end{aligned} \tag{32}$$

The steady-state version of the equation above and the implementability condition evaluated at $a = c^e = c^u$ are

$$\chi_t = \frac{1}{a} + \zeta_t\left(\beta - \frac{n_t^e}{1 - n_t^e}\pi(e)\right),$$

$$\frac{n_t^e}{1 - n_t^e}\pi(e) = \beta,$$

respectively. It follows that

$$\chi = \frac{1}{a} = \frac{1}{c^e} > 0.$$

As a result, χ_t converges to a finite positive value, which implies MGR holds and $\tau_k = 0$.

3. Finally, we show that there is no Ramsey steady state featuring $R < 1/\beta$. In this case,

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$c_t^e > c_t^u = a_t$. The FOCs with respect to c_t^e and c_t^u satisfy

$$c_t^e \quad : \quad \frac{\pi(e)}{c_t^e} - \chi_t \pi(e) - \zeta_t \frac{n_t^e}{1 - n_t^e} \pi(e) + \zeta_t \pi(e)$$

$$+ \zeta_t a_{t+1} \beta \left[(1/c_{t+1}^e) \pi(e) + (1/c_{t+1}^u) \pi(u) \right] - \zeta_{t-1} a_t \frac{c_{t-1}^e}{c_t^e c_t^e} \pi(e)$$

$$= \quad 0, \tag{33}$$

and

$$c_t^u \quad : \quad \frac{\pi(u)}{c_t^u} - \chi_t \pi(u) + \zeta_t \pi(u) - \zeta_t$$

$$+ \zeta_{t-1} \left[\left(c_{t-1}^e / c_t^e \right) \pi(e) + \left(c_{t-1}^e / c_t^u \right) \pi(u) \right] - \zeta_{t-1} a_t \frac{c_{t-1}^e}{c_t^u c_t^u} \pi(u)$$

$$= \quad 0. \tag{34}$$

By implementability condition, we rewrite (33) as

$$\frac{\pi(e)}{c_t^e} - \chi_t \pi(e) + \zeta_t \frac{c_t^u}{c_t^e} \pi(e) - \zeta_{t-1} c_t^u \frac{c_{t-1}^e}{c_t^e c_t^e} \pi(e) = 0. \tag{35}$$

Rearranging (34) and (35) and evaluating them at steady state, we obtain

$$\frac{\pi(u)}{c^u} - \chi_t \pi(u) = (\zeta_t - \zeta_{t-1}) \pi(e), \tag{36}$$

$$\frac{\pi(e)}{c^e} - \chi_t \pi(e) = (\zeta_{t-1} - \zeta_t) \frac{c^u}{c^e} \pi(e). \tag{37}$$

Let g denote the growth rate of χ_t as well as ζ_t (recall that χ_t and ζ_t must grow at the same rate in steady state). Given $\frac{1}{c^u} > \frac{1}{c^e}$, from (36) and (37) we obtain $(\zeta_t - \zeta_{t-1}) \frac{\pi(e)}{\pi(u)} > (\zeta_{t-1} - \zeta_t) \frac{c^u}{c^e}$, which can be expressed as

$$(\zeta_t - \zeta_{t-1}) \frac{\pi(e)}{\pi(u)} - (\zeta_{t-1} - \zeta_t) \frac{c^u}{c^e} = (\zeta_t - \zeta_{t-1}) \left(\frac{\pi(e)}{\pi(u)} + \frac{c^u}{c^e} \right) > 0,$$

which implies $g \equiv \frac{\zeta_t}{\zeta_{t-1}} > 1$. Manipulating (36) by $\frac{\pi(u)}{c_t^u} = \frac{\pi(u)}{c_{t-1}^u}$ gives

$$\chi_t \pi(u) + (\zeta_t - \zeta_{t-1}) \pi(e) = \chi_{t-1} \pi(u) + (\zeta_{t-1} - \zeta_{t-2}) \pi(e).$$

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Given $\chi_t = g\chi_{t-1}$, $\zeta_t = g\zeta_{t-1}$ and $\zeta_{t-2} = \frac{1}{g}\zeta_{t-1}$, rewrite the equation above as

$$g\chi_{t-1}\pi(u) + (g\zeta_{t-1} - \zeta_{t-1})\pi(e) = \chi_{t-1}\pi(u) + \left(\zeta_{t-1} - \frac{1}{g}\zeta_{t-1}\right)\pi(e).$$

Hence, we have

$$\chi_{t-1}\pi(u) = \pi(e)\zeta_{t-1}\left(\frac{1-g}{g}\right). \tag{38}$$

If $\zeta_{t-1} > 0$, equation (38) implies $\chi_{t-1} < 0$, which leads to a contradiction, given that both ζ_{t-1} and χ_{t-1} must be non-negative. If ζ_{t-1} equals zero, then $\chi_{t-1} = 0$ and $\zeta_t = 0$. We rule out the case that $\zeta_{t-1} = \chi_{t-1} = 0$ because $c^e = c^u \to \infty$, obtained by (36) and (37), is impossible.

Finally, we show that all allocations and multipliers take positive values and are uniquely pinned down in the Ramsey steady state by the following steps.

1. Since issuing debt is costly to Ramsey planner, the Ramsey steady state has to feature the lowest amount of government debt that achieve the full-self insurance. Namely, the steady state must feature $a = c^e = c^u$, which together with the steady-state implementability condition implies $\frac{n^e}{1-n^e}\pi(e) = \beta$ and hence n^e can be solved analytically as $n^e = \frac{\beta}{\pi(e)+\beta} > 0$. Moreover, $N = n^e\pi(e) > 0$.

2. The MGR implies the steady state can be written as

$$\alpha\left(\frac{K}{N}\right)^{1-\alpha} = MP_K = 1/\beta - 1 + \delta > 0,$$

   which implies $K > 0$ given $N > 0$.

3. The steady state resource constraint together with $c^e = c^u$ gives

$$c^e = K\left(\left(\frac{K}{N}\right)^{\alpha-1} - \delta\right) = K\left(\frac{\alpha}{\frac{1}{\beta} - 1 + \delta} - \delta\right),$$

   which implies c^e is also positive if $\alpha > \delta(\frac{1}{\beta} - 1 + \delta)$. Hence, χ, c^u and a are all positive since $\frac{1}{\chi} = a = c^u = c^e$.

4. We then show that the optimal level of debt is strictly positive if condition (11) is met. By the flow government budget constraint in Ramsey steady state, the steady state

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labor tax rate can be written as a function of debt to GDP ratio, B/Y :

$$\tau_n = (1 - \beta)\frac{B}{Y}\frac{1}{\frac{wN}{Y}} = \frac{(1 - \beta)}{(1 - \alpha)}\frac{B}{Y}, \tag{39}$$

where the last equality uses the labor share $\frac{wN}{Y} = 1 - \alpha$. In addition, the steady state version of household budget constraint implies that the asset holding-to-GDP ratio is given by

$$\frac{a}{Y} = (1 - \tau_n)\frac{wN}{Y}\frac{1}{\beta} = (1 - \tau_n)\frac{1 - \alpha}{\beta}. \tag{40}$$

In addition, we know that MGR holds in the Ramsey steady state, which gives

$$\frac{K}{Y} = \frac{\alpha}{\frac{1}{\beta} - 1 + \delta}. \tag{41}$$

By plugging the three equations above into the asset market-clearing condition, we can oblation

$$\frac{B}{Y} = \frac{a}{Y} - \frac{1}{\beta}\frac{K}{Y} = \left(1 - \frac{(1 - \beta)}{(1 - \alpha)}\frac{B}{Y}\right)\frac{1 - \alpha}{\beta} - \frac{1}{\beta}\frac{K}{Y},$$

which solves for the steady-state B/Y ratio as

$$\frac{B}{Y} = 1 - \alpha - \frac{\alpha}{\frac{1}{\beta} - 1 + \delta}.$$

Hence, the optimal debt is positive if $1 - \alpha > \frac{\alpha}{\frac{1}{\beta} - 1 + \delta}$, which is exactly condition (11). Moreover, $\tau_n > 0$ by equation (39),

5. To show $\zeta > 0$, first notice that the household first-order conditions in steady state imply that

$$\frac{MP_N}{c^e} - \frac{1}{1 - n^e} = \frac{MP_N}{c^e}\tau_n > 0, \tag{42}$$

$$\frac{MP_N}{c^e} - \frac{1}{(1 - n_t^e)} = \frac{1 - n_t^e - c^e}{c^e(1 - n_t^e)}$$

where the last inequality is due to $\tau_n > 0$. In addition, the steady state version of equation (27) together with $\chi = \frac{1}{c^e}$ gives

$$\zeta\frac{c^e}{(1 - n^e)^2} = \frac{MP_N}{c^e} - \frac{1}{(1 - n^e)},$$

which together with equation (42) implies $\zeta > 0$ in the Ramsey steady state.