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Are Government Bonds Net Wealth or a Liability?

—Optimal Debt and Taxes under Uninsurable Income Risk*

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Abstract

A positive national debt is often rationalized either by the assumption of dynamic inefficiency in an overlapping-generations (OLG) model, or by the hypothesis of heterogeneous-agents and incomplete-markets (HAIM) in an infinite horizon model. Both assumptions imply insufficient private liquidity to support private saving and investment, thus calling for a positive level of public debt to improve social welfare. However, since public debt is financed often by distortionary future taxes, optimal debt and tax policies ought to be studied jointly in a single framework. In this paper we use a primal Ramsey approach to analytically characterize optimal debt and tax policy in an OLG-HAIM model. We show that (i) public debt can be a liability instead of net wealth, despite insufficient private liquidity to support private saving and investment, and (ii) such a debt policy can dramatically change the government's optimal tax scheme since the sign and magnitude of the optimal quantity of debt dictate the sign and magnitude of optimal taxes as well as the priority order of tax tools (such as a labor tax vs. a capital tax) in financing the public debt.

JEL Classification: E13; E62; H21; H30

Key Words: Role of Public Debt, Optimal Fiscal Policy, Ramsey Problem, Overlapping Generation, Incomplete Markets.

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1 Introduction

The optimal quantity of debt and optimal rates of taxation are intrinsically intertwined. This is so not only because these fiscal policies provide alternative means to financing public spending, but also because public debt itself needs to be financed by future taxes. Such intrinsic interactions have been best illustrated under aggregate uncertainty by the tax-smoothing literature pioneered by Barro (1979) and Lucas and Stokey (1983). However, even in the absence of aggregate uncertainty, these two types of fiscal policies are also closely intertwined whenever the Ricardian equivalence principle fails to hold.

In a seminal paper, Barro (1974) studies the conditions under which government bonds are perceived by the private sector as net wealth—i.e., the social value of government bonds exceeds the costs of financing them in terms of tax burden on the private sector. Similarly, Woodford (1990) and Aiyagari and McGrattan (1998) also show that the optimal quantity of public debt is strictly positive under incomplete financial markets with idiosyncratic risk and borrowing constraints—because of the shortage of private liquidity. However, this literature does not study optimal tax policy and often assumes that the interest on public debt is financed by lump-sum taxes instead of distortionary taxes on capital and labor.

On the other hand, since Aiyagari's (1994) seminal work, a large body of literature has been devoted to studying optimal tax policy under heterogeneous agents and incomplete markets (HAIM). But this literature often assumes away the role of government bonds or does not simultaneously study the issue of the optimal quantity of debt in conjunction with optimal taxation.[1]

In this paper, we determine optimal debt and tax policies jointly in a single framework by modifying the overlapping-generations (OLG) model to a setting with imperfect/incomplete financial markets and uninsurable income risk (in the spirit of Aiyagari (1994)).

Specifically, households in our model have a two-period life span but are overlapped by another generation at any point in time; they save when young, supply labor in both periods, and earn capital returns when old. They face uninsurable income risk when old and cannot leave positive/negative bequests. The introduction of idiosyncratic risk into the old generation's labor income implies that precautionary saving by households when young is never adequate for providing self-insurance in old age in the absence of two-sided altruism. There is no aggregate uncertainty or exogenous government expenditures. The government has the following policy tools to improve welfare: public debt, a lump-sum transfer to the old, a capital tax, and a generation-specific labor tax on the young and the old households. These fiscal variables are allowed to be negative, and we

[1] The work of Aiyagari and McGrattan (1998) does not focus on optimal taxation problem. To finance public debt, the authors simply assume that the rates of capital tax and labor tax are proportional to each other.

will also study optimal policies when public debt and/or lump-sum taxes/transfers are exogenously constrained by ad hoc limits.

We use the primal Ramsey approach to solve optimal policies in this OLG-HAIM model. Under our approach, we define the first-best allocation as the solution to a social planner's problem where the government has enough tools to eliminate all market frictions. We show that the first-best allocation implies two efficiency conditions in the steady state: (i) modified golden rule (MGR) holds and (ii) the marginal utility of consumption is the same for old and young households in all possible states.[2] These two efficiency conditions often fail to hold under incomplete markets in a *laissez faire* competitive equilibrium. But we investigate whether a Ramsey planner can restore these conditions using fiscal policies in various scenarios and study the implied signs and magnitudes of fiscal variables.

Clearly, the first condition (MGR) ensures aggregate allocative efficiency (AAE) of capital, and the second condition ensures individual allocative efficiency (IAE) of household consumption and saving. These two efficiency conditions are not necessarily the same thing and do not necessarily imply each other—recall that only in infinite-horizon representative-agent models does IAE automatically holds in the steady state. In our OLG-HAIM model, without idiosyncratic uncertainty, equality between the interest rate and the time discount rate implies IAE; but with uninsurable idiosyncratic uncertainty, equality between the interest rate and time discount rate implies only expected IAE (EIAE)—namely, the marginal utility of consumption of the young equals the expected marginal utility of consumption of the old.

Within such a conceptual framework, the net-wealth effect of government bonds is measured by the extent to which they improve social welfare: Namely, the net wealth from government bonds is zero if they play no role in improving social welfare, positive if they can help improve the efficiency of resource allocations along either of the two margins outlined above, and negative if the optimal quantity of public debt is negative—such that government bonds are net liabilities (instead of wealth) for the private sector.[3]

For convenience and with some abuse of terminology, we define our model economy as "dynamically efficient/inefficient" if the steady-state capital stock in the laissez-faire competitive equilibrium is below/above the capital stock implied by the MGR.[4]

[2]See Diamond (1965). These two efficiency conditions are thus the necessary conditions for the first-best allocation in the long run.

[3]When public debt is negative, it simply means that the government opts to hold privately issued debt and receive interest payments. For simplicity we assume there are no government expenditures other than interest payments if public debt is positive.

[4]Our definition is analogous to the notion of "dynamic efficiency" in the OLG literature. Under our definition, social welfare can still be Pareto improved even if the economy is "dynamically efficient"—if the marginal product of capital is sufficiently high in a competitive equilibrium, then the social welfare cannot be Pareto improved by issuing a positive amount of government bonds, but can be Pareto improved by issuing a negative amount of government

In the meantime, since government revenues are needed to finance the interest payments on public debt (if positive), our Ramsey approach also simultaneously determines the optimal paths of future taxes (or subsidies if the debt is negative) in both the transition and the steady states.

We obtain the following results: (i) In the absence of idiosyncratic income uncertainty (i.e., with only cross-generational heterogeneity), the *laissez faire* competitive equilibrium cannot achieve the first-best allocation, and government debt can improve social welfare by achieving the first-best outcome. However, the optimal quantity of debt is positive if and only if the economy is "dynamically inefficient" (according to our definition)—otherwise the optimal quantity of public debt is negative. In particular, under a positive debt level the Ramsey planner opts to rely only on lump-sum taxes on the old to finance the interest payments on government debt so that the optimal tax rate is zero for both labor and capital. Nonetheless, if lump-sum taxes are not available, then the optimal tax rate remains zero for capital but becomes positive for both the young and old households' labor income. In either cases, the sign of optimal debt depends only on the condition for "dynamic efficiency". In addition, if the capacity for the government to issue debt is restricted from above by an ad hoc debt limit and the debt-limit constraint binds, then neither AAE nor IAE can be achieved in a Ramsey equilibrium, further suggesting that government bonds are net wealth of the private economy so long as the economy is "dynamically inefficient". In contrast, if the economy is "dynamically efficient" (the market interest rate exceeds the time discount rate), then the optimal policies simply flip signs—i.e., the optimal quantity of public debt is negative (implying positive private debt held by the government) and taxes become subsidies, or transfers.[5]

(ii) With uninsurable idiosyncratic income uncertainty, the Ramsey steady state is not a first-best allocation—it achieves the MGR but not IAE (not even in the sense of the expected marginal utilities of old household across states), and the tax rates on capital and old households' labor income are non-zero despite lump-sum taxes/transfers to old households. Furthermore, the labor tax/subsidy for young households is zero unless lump-sum taxes/transfers to old households are not feasible. We also provide a sufficient condition for "dynamically efficiency" that dictates the sign of the optimal quantity of public debt. In addition, under a binding constraint on the government's capacity to issue debt (e.g., $B = 0$), neither the MGR nor IAE (even in the sense of expected value) can be achieved in the Ramsey steady state regardless of the availability of lump-sum taxes/transfers to old households.

Hence, our theoretical analysis not only relates the optimal quantity of public debt to a notion of "dynamic efficiency," but also connects and contributes to the large literature on optimal taxation.

bonds.

[5]Notice that the lack of private credit markets by no means implies that the government cannot lend to the private sector or purchase privately issued debt.

Pivotal work by Judd (1985) and Chamley (1986) has shown that the best way for the government to finance its expenditures in the long run is to tax labor but not capital when the option of a lump-sum tax is not available. This zero-capital taxation result is surprising and has attracted many studies to examine its robustness.[6] Erosa and Gervais (2002) and Garriga (2019) have extended this result to OLG models and show that the classical result of a zero capital tax still holds in life-cycle economies, provided that labor tax rates can be varied according to age. However, this issue is relatively less explored in the HAIM framework with uninsurable income risk, especially in conjunction with the determination of optimal quantity of debt. Consequently, despite several important attempts such as Aiyagari (1995), Conesa, Kitao, and Krueger (2009), and Dávila, Hong, Krusell, and Ríos-Rull (2012), some issues still remain unsettled.

In our view, there are two major issues in studying optimal tax polices in the HAIM framework that cannot be adequately addressed in a representative-agent infinite-horizon model: (i) What is the role of precautionary saving in determining the optimal quantity of public debt in a HAIM economy? (ii) How does the sign of the optimal quantity of public debt influence/determine the sign and path of optimal taxes?

In a representative-agent framework, since the Ricardian equivalent principle holds and government bonds are neither net wealth nor net liabilities of the private sector (in the language of Barro (1974)), the choice between debt or taxes financing government expenditure is irrelevant. However, in an OLG framework and especially in OLG-HAIM models, government debt is no longer neutral for the economy and the key consideration of the Ramsey planner is thus fundamentally different, especially when precautionary saving is involved and when precautionary saving is not necessarily equivalent to oversaving (in the sense of dynamic inefficiency). Therefore, government bonds are not necessarily net wealth of the private sector despite incomplete credit markets and uninsurable idiosyncratic risk under borrowing constraints. An important criterion for the determination of public debt becomes dynamic (in)efficiency. Consequently, optimal tax policies are also changed in a fundamental way, depending both on the sign and on the magnitude of the optimal quantity of public debt. This interdependence of public debt and tax policies is of great importance because on the one hand the bulk of theoretical literature justifies a positive quantity of public debt in the U.S. economy by relying on the assumption of incomplete financial markets (e.g., Barro (1974), Woodford (1990), and, Aiyagari and McGrattan (1998)); on the other hand the bulk of the empirical literature claims that the U.S. economy and the economies of other major OECD nations are dynamically efficient (e.g., Abel, Mankiw, Summers, and Zeckhauser (1989)).

[6]For example, in the introduction of Straub and Werning (2020): "One may even say that the result (zero capital tax) remains downright puzzling, as witnessed by the fact that economists have continued to take turns putting forth various intuitions to interpret it, none definitive nor universally accepted."

Our modeling framework is close to that of Krueger and Ludwig (2018), which also features uninsurable idiosyncratic uncertainty in an OLG environment. However, these authors' focus is exclusively on optimal capital taxation (in the absence of labor taxes and government debt). Our paper studies optimal quantity of debt in light of the net-wealth effect of government bonds in conjunction with optimal tax policies. In doing so, our analysis also sheds light on the role of government bonds in determining and shaping optimal tax policies.

The rest of the paper is organized as follows. Section 2 presents the model and characterizes the competitive equilibrium. Section 3 solves the Ramsey problem. The Ramsey equilibrium is characterized in two scenarios: one without idiosyncratic shocks and one with idiosyncratic shocks. Section 4 provides a more detailed literature review of closely related works. Section 5 concludes the paper.

2 The Model

2.1 Firms

Time is discrete and indexed by $t = 0, 1, 2, 3...$ A representative firm produces output according to the constant-returns-to-scale Cobb-Douglas technology, $Y_t = F(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha}$, where Y, K, and N denote aggregate output, capital, and labor, respectively. The firm rents capital and hires labor from households by paying a competitive rental rate q_t and real wage w_t. The firm's optimal conditions for profit maximization at time t thus satisfy

$$w_t = \frac{\partial F(K_t, N_t)}{\partial N_t} \equiv MP_{N,t}, \quad (1)$$

$$q_t = \frac{\partial F(K_t, N_t)}{\partial K_t} \equiv MP_{K,t}. \quad (2)$$

2.2 Households

In each period t, a new generation of households is born and lives for two periods: t and $t+1$. An old generation also exists in the very beginning of time $t = 0$. Thus, at any point in time $t \geq 0$, there is a young and an old generation of households. The population size of each generation remains constant over time and is normalized to unity. The labor productivity of the young and the old are different, which are denoted by κ^y and κ^o, respectively. Given the competitive wage rate w_t, while all households have the same constant labor productivity κ^y (or wage income $\kappa^y w_t$ per hour) when young, there is an idiosyncratic shock to labor productivity when old, denoted by $\theta \sim \Psi(\theta)$, such that the wage income per hour for the old is given by $\theta_t \kappa^o w_t$. The mean value of the idiosyncratic shocks is normalized to unity:

$$\int \theta d\Psi = 1.$$

Each new-born generation t of households has the lifetime expected utility:

$$V_t = u(c_t^y) - v(n_t^y) + \beta \int (u(c_{t+1}^o) - v(n_{t+1}^o))d\Psi,$$

where $\beta \in (0, 1)$ is the discount factor, c_t^y and n_t^y denote consumption and raw labor supply when young, respectively; similarly, c_{t+1}^o and n_{t+1}^o denote consumption and raw labor supply, respectively, when old in period $t + 1$. Note that the optimal decisions of old households are affected by the idiosyncratic shock θ. The budget constraints of generation-t households when young and old are given, respectively, by

$$c_t^y + a_{t+1} = (1 - \tau_{n,t}^y)w_t\kappa^y n_t^y, \tag{3}$$

$$c_{t+1}^o = a_{t+1}r_{t+1} + (1 - \tau_{n,t+1}^o)w_{t+1}\theta_{t+1}\kappa^o n_{t+1}^o + T_{t+1}, \tag{4}$$

where a_{t+1} is the savings of the young, and $\tau_{n,t}^y$ and $\tau_{n,t}^o$ denote the labor tax rates of young and old households, respectively. In addition, r_t is the gross risk-free rate of the return to savings and T_{t+1} denotes a lump-sum transfer (or tax if negative) received from the government when old.[7]

There is a tremendous consumption risk that the idiosyncratic shock θ imposes on households: No matter how much they opt to save when young, their savings a_{t+1} will never be adequate to fully smooth their consumption when old—because given that $t+1$ is the end period of life, consumption of the old must fluctuate together with total income. Therefore, we allow for a lump-sum transfer T_{t+1} to old households to buffer the income risk and show how the idiosyncratic risk θ_{t+1} and the lump-sum transfer T_{t+1} affect the optimal level of public debt in the model economy.

Hence, the utility-maximization problem of generation t is given by

$$\max_{\{c_t^y, n_t^y, c_{t+1}^o, n_{t+1}^o, a_{t+1}\}} u(c_t^y) - v(n_t^y) + \beta \int (u(c_{t+1}^o) - v(n_{t+1}^o))d\Psi$$

subject to (3) and (4). Notice that the aggregate effective labor units in the entire economy in each period t is given by

$$N_t = \kappa^y n_t^y + \kappa^o \int \theta_t n_t^o d\Psi,$$

which will be called (with slight abuse of terminology) the labor market clearing condition in the

[7]In the laissez-faire competitive equilibrium, the asset market clearing condition implies that $a_{t+1} = K_{t+1} > 0$, which suggests that the no borrowing constraint of young households, $a_{t+1} > 0$, never binds. Hence, we simply ignore this no borrowing constraint in the setup.

model.

The first-order conditions (FOCs) of generation-t households for $\left\{c_t^y, n_t^y, c_{t+1}^o, n_{t+1}^o, a_{t+1}\right\}$ can be summarized by three equations:

$$\begin{aligned} v_{n,t}^y &= u_{c,t}^y (1-\tau_{n,t}^y)\kappa^y w_t, \\ v_{n,t+1}^o &= u_{c,t+1}^o (1-\tau_{n,t+1}^o)\kappa^o \theta_{t+1} w_{t+1}, \\ u_{c,t}^y &= \beta r_{t+1} \int u_{c,t+1}^o d\Psi; \end{aligned}$$

where $u_{c,t}^i$ and $v_{n,t}^i$ denote, respectively, the marginal utility of consumption and marginal disutility of labor for the young ($i = y$) cohort and old ($i = o$) cohort in period t. The first two equations indicate that the marginal rate of substitution along the indifference curve for consumption and leisure equals the marginal rate of transformation along the budget constraint for the young and the old, respectively. The last equation indicates the households' intertemporal optimality condition for saving—namely, the marginal-utility cost of forgone consumption when young equals the expected future marginal-utility gains under the stochastic discounting rate βr_{t+1}. The Ramsey planner will respect the households' FOCs when designing policies for public debt and taxes to improve the market allocations.

2.3 Government

The government in each period t can issue bonds, B_{t+1}, collect lump-sum taxes or provide transfers to the currently old households in the amount T_t, or levy a time-varying capital tax $\tau_{k,t}$ and generation-specific labor taxes $\left\{\tau_{n,t}^y, \tau_{n,t}^o\right\}$ to improve welfare.[8] The budget constraint for the government in period t is given by

$$\tau_{k,t} q_t K_t + \tau_{n,t}^y w_t \kappa^y n_t^y + \tau_{n,t}^o w_t \kappa^o \int \theta_t n_t^o d\Psi + B_{t+1} \geq T_t + r_t B_t, \text{ for all } t \geq 0,$$

where the left-hand side is the source of government revenue and the right-hand side is the outlay of government expenditures.

There is no aggregate uncertainty in the economy, and government bonds and productive capital are perfect substitutes as stores of value for households. As a result, the after-tax gross rate of return to capital must equal the gross risk-free rate:

$$r_t = 1 + (1-\tau_{k,t}) q_t - \delta,$$

[8]For simplicity and without loss of generality, we assume that government spending $G_t = 0$ for all $t \geq 0$.

which constitutes a no-arbitrage condition for capital and bonds.

Note that the asset holdings of the initially old households in period 0 is exogenously given by $a_0 = K_0 + B_0$. In addition, for simplicity we assume that the generation-0 old households are not subject to any lump-sum taxes or transfers ($T_0 = 0$) and do not supply labor ($n_0^o = 0$). Therefore, their consumption is pinned down by their budget constraints: $c_0^o = a_0 r_0$.

2.4 Competitive Equilibrium

Definition 1. *Given the initial capital K_0, the initial government bonds B_0, and the sequence of government policies $\{\tau_{k,t}, \tau_{n,t}^y, \tau_{n,t+1}^o, T_{t+1}, B_{t+1}\}_{t=0}^{\infty}$, a competitive equilibrium is defined as the sequences of market prices $\{w_t, r_t, q_t\}_{t=0}^{\infty}$, aggregate allocations $\{C_t, N_t, K_{t+1}\}_{t=0}^{\infty}$, and individual allocation plans $\{c_t^y, c_t^o, n_t^y, n_t^o, a_{t+1}\}_{t=0}^{\infty}$, such that the following hold:*

1. *Given the sequences of government policies $\{\tau_{k,t}, \tau_{n,t}^y, \tau_{n,t+1}^o, T_{t+1}, B_{t+1}\}_{t=0}^{\infty}$ and market prices $\{w_t, r_t\}_{t=0}^{\infty}$, the sequence $\{c_t^y, c_{t+1}^o, n_t^y, n_{t+1}^o, a_{t+1}\}_{t=0}^{\infty}$ solves the household problem for all generations, given the initial conditions $n_0^o = 0$ and $c_0^o = (K_0 + B_0)r_0$.*

2. *Given the sequences of government policies $\{\tau_{k,t}, \tau_{n,t}^y, \tau_{n,t+1}^o, T_{t+1}, B_{t+1}\}_{t=0}^{\infty}$ and market prices $\{w_t, q_t\}_{t=0}^{\infty}$, the sequence of aggregate labor and capital $\{N_t, K_t\}$ solves the representative firm's problem for all period t.*

3. *The households' no-arbitrage condition holds: $r_t = (1 - \tau_{k,t})q_t + 1 - \delta$ for all $t \geq 0$.*

4. *The government budget constraint holds for all $t \geq 0$:*

$$\tau_{k,t} q_t K_t + \tau_{n,t}^y w_t \kappa^y n_t^y + \tau_{n,t}^o w_t \kappa^o \int \theta n_t^o d\Psi + B_{t+1} \geq T_t + r_t B_t,$$

5. *All markets clear for all $t \geq 0$:*

$$\begin{aligned}
a_{t+1} &= B_{t+1} + K_{t+1} \\
N_t &= \kappa^y n_t^y + \kappa^o \int \theta n_t^o d\Psi \\
C_t &= c_t^y + \int c_t^o d\Psi \\
F(K_t, N_t) &= C_t + K_{t+1} - (1-\delta)K_t,
\end{aligned}$$

   where the first equation is the asset market clearing condition, the second is the effective labor's market clearing condition, the third the identify of aggregate consumption, and the last the aggregate goods market clearing condition.

We then define the notion of "dynamic efficiency" in our model as follows:

Definition 2. *Our model economy is called "dynamically efficient/inefficient" if and only if the steady-state capital stock in the laissez-faire competitive equilibrium is below/above the capital stock implied by the MGR.*

The above definition also implies that our model economy is "dynamically efficient/inefficient" if and only if the laissez-faire market interest rate lies above/below the time discount rate. To connect our notion of "dynamic efficiency" to the sign of the optimal quantity of public debt, we express the young households' steady-state excess savings, $\widetilde{S}(r)$, as follows:

$$\widetilde{S} \equiv \frac{w\kappa^y n^y}{Y} - \frac{\tau_n^y w\kappa^y n^y}{Y} - \frac{c^y}{Y} - \frac{K}{Y} - \frac{B}{Y}.$$

In what follows, we show that the equilibrium interest rate r could be pinned down by setting $\widetilde{S} = 0$. To facilitate our analysis, we parameterize the utility function as follows:

Assumption 3. *Assume power utility functions for consumption and labor:*

$$\begin{aligned} u(c) &= \frac{1}{1-\sigma} c^{1-\sigma}, \\ v(n) &= \frac{1}{\gamma+1} n^{\gamma+1}, \end{aligned}$$

where $\sigma > 0$ and $\gamma > 0$. Note that the utility function exhibits constant relative risk aversion:

$$\begin{aligned} \frac{u_{cc}c}{u_c} &= -\sigma < 0, \\ \frac{v_{nn}n}{v_n} &= \gamma > 0. \end{aligned}$$

Proposition 4. *In a competitive equilibrium, the excess-saving is given by*

$$\widetilde{S} = \frac{(1-\alpha)}{1+\widetilde{\omega}} - \frac{1}{(1+(\beta r)^{\frac{1}{\sigma}})} - \left(1 - \delta \frac{1}{(1+(\beta r)^{\frac{1}{\sigma}})}\right) \frac{\alpha}{r-1+\delta} - \frac{B}{Y}, \tag{5}$$

where

$$\widetilde{\omega} = \left(\beta r \frac{(1-\tau_n^y)}{(1-\tau_n^o)}\right)^{-\frac{1}{\gamma}} \left(\frac{\kappa^y}{\kappa^o}\right)^{-\frac{1}{\gamma}-1}. \tag{6}$$

Consequently, in a laissez-faire competitive equilibrium (where $B = T = \tau_k = \tau_n^y = \tau_n^o = 0$), the laissez-faire steady-state interest rate r^* is the solution to

$$S(r^*) \equiv \widetilde{S}\Big|_{B=\tau_n^y=\tau_n^o=0} = 0. \tag{7}$$

Furthermore, the model economy is "dynamically efficient" if $r^* \geq 1/\beta$ and "dynamically inefficient" if $r^* < 1/\beta$.

Proof. See Appendix A.1. $\square$

3 Ramsey Problem

Note that the competitive equilibrium defined above is in general a function of the path of government policies $\left\{\tau_{k,t}, \tau^y_{n,t}, \tau^o_{n,t+1}, T_{t+1}, B_{t+1}\right\}_{t=0}^{\infty}$. Namely, each different path of government policies corresponds to a different competitive equilibrium. The Ramsey problem is to select a particular path of government policies such that the corresponding competitive equilibrium yields the maximum social welfare.

We use the primal approach to solve the Ramsey problem. Under the primal approach, we first substitute out all market prices and policy variables by using a subset of the competitive equilibrium's FOC conditions, and then choose the path of resource allocation $\{c^y_t, c^o_{t+1}, n^y_t, n^o_{t+1}, C_t, N_t, K_{t+1}\}_{t=0}^{\infty}$ to maximize social welfare subject to the rest of the equilibrium conditions. This implies that the more policy tools the Ramsey planner has, the smaller the number of constraints the planner is subject to in solving the Ramsey problem. The solution under such a primal approach is called a Ramsey allocation (outcome) or a Ramsey plan.[9]

3.1 Conditions to Support a Competitive Equilibrium

To ensure that a Ramsey plan constitutes a competitive equilibrium, we must show first that all possible allocations in the choice set of the Ramsey planner, $\{c^y_t, c^o_{t+1}, n^y_t, n^o_{t+1}, C_t, N_t, K_{t+1}\}_{t=0}^{\infty}$ (after substituting out all market prices and policy variables but before solving the Ramsey maximization problem), constitute a competitive equilibrium. The following proposition states the conditions that the constructed allocations $\{c^y_t, c^o_{t+1}, n^y_t, n^o_{t+1}, C_t, N_t, K_{t+1}\}_{t=0}^{\infty}$ must satisfy in order to constitute a competitive equilibrium.

Proposition 5. *Define $N^o_t \equiv \int \theta n^o_t d\Psi$, $C^o_t \equiv \int c^o_t d\Psi$, and $U^o_{C,t} \equiv \int u^o_{c,t} d\Psi$. Given the initial capital K_0, government bonds B_0, and capital tax $\tau_{k,0}$, the sequence of allocations $\{c^y_t, c^o_{t+1}, n^y_t, n^o_{t+1}, C_t, N_t, K_{t+1}\}_{t=0}^{\infty}$ can be supported as a competitive equilibrium if and only if the following conditions hold:*[10]

[9] For example, we have five policy variables and a total of eleven equilibrium conditions to solve for a competitive equilibrium. After using up eight equilibrium conditions to substitute out the five policy variables and three market prices, we have three conditions left (plus one government budget constraint) to serve as constraints for the Ramsey planner in determining the Ramsey plan.

[10] The initial capital tax rate, $\tau_{k,0}$, should be a choice variable for the Ramsey planner. For the sake of simplicity, we restrict the planner's ability to choose $\tau_{k,0}$ in the Ramsey problem.

1. *The aggregate resource constraint holds ($t \geq 0$):*

$$F(K_t, \kappa^y n_t^y + N_t^o \kappa^o) - c_t^y - C_t^o - K_{t+1} + (1-\delta)K_t \geq 0. \tag{8}$$

2. *The budget constraint for each old household is satisfied ($t > 0$):*

$$\frac{v_{n,t+1}^o}{u_{c,t+1}^o} n_{t+1}^o - \int \frac{v_{n,t+1}^o}{u_{c,t+1}^o} d\Psi \times N_{t+1}^o \geq c_{t+1}^o - C_{t+1}^o. \tag{9}$$

3. *The old households' FOCs are satisfied ($t > 0$):*

$$v_{n,t+1}^o = u_{c,t+1}^o \theta_{t+1} \int \frac{v_{n,t+1}^o}{u_{c,t+1}^o} d\Psi. \tag{10}$$

4. *The aggregate consumption C_t and labor N_t can be chosen to satisfy the equilibrium conditions: $N_t = \kappa^y n_t^y + N_t^o \kappa^o$ and $C_t = c_t^y + C_t^o$.*

5. *The implementability condition holds ($t \geq 0$):*

$$u_{c,t}^y c_t^y - v_{n,t}^y n_t^y + \beta U_{C,t+1}^o \left[C_{t+1}^o - \int \frac{v_{n,t+1}^o}{u_{c,t+1}^o} d\Psi \times N_{t+1}^o \right] \geq T_{t+1} \beta U_{C,t+1}^o. \tag{11}$$

6. *Finally, the equilibrium conditions also imply that the choice of government bonds must satisfy the asset market clearing condition ($t \geq 0$):*

$$B_{t+1} = \frac{v_{n,t}^y}{u_{c,t}^y} n_t^y - c_t^y - K_{t+1}. \tag{12}$$

Proof. See Appendix A.2. □

Note that the last two conditions (11) and (12) are redundant because under the primal approach we have already used these conditions to substitute out the policy variables T_{t+1} and B_{t+1}. This implies that, in the Ramsey problem (studied in the next subsection), the Lagrangian multipliers associated with these two conditions will be exclusively zero. As we will show later, however, adding these two conditions into the set of constraints in the Ramsey problem has an advantage when we study the net wealth of government bonds in special cases where lump-sum taxes and/or transfer are not available, or when the government's capacity to issue debt or to lend is limited. In such special cases ($T_{t+1} = 0$ or $T_{t+1} \geq 0$, and $B_{t+1} = 0$ or $B_{t+1} \leq \overline{B}$), the conditions (11) and (12) must be imposed on (respected by) the Ramsey planner and hence their Lagrangian multipliers are no longer zero. So for convenience, we replace the condition (12) by the following two-sided constraints:

$$\overline{B} \geq B_{t+1} = \frac{v^y_{n,t}}{u^y_{c,t}} n^y_t - c^y_t - K_{t+1} \geq \underline{B}, \tag{13}$$

where $\overline{B} \geq 0$ and $\underline{B} \leq 0$. Notice that when government bonds are negative ($B_{t+1} < 0$), it implies that the government is lending credit to the private sector (or holding privately issued debt).

Moreover, if there is no idiosyncratic risk, conditions (9) and (10) are trivially satisfied (they become nil identity $0 = 0$) and do not impose any restrictions on the Ramsey planner. In short, the setup in Proposition 5 is general enough and offers the convenience of investigating the Ramsey outcomes under several special scenarios.

3.2 Optimal Ramsey FOCs

Since $C_t = c^y_t + \int c^o_t d\Psi$ and $N_t = \kappa^y n^y_t + \kappa^o \int \theta n^o_t d\Psi$, the choice set of the Ramsey planner can be simplified from a sequence of seven variables to a sequence five variables: $\{K_{t+1}, c^y_t, n^y_t, c^o_{t+1}, n^o_{t+1}\}^{\infty}_{t=0}$. Then by Proposition 5, the Ramsey problem under the primal approach is given by

$$\max_{\{K_{t+1}, c^y_t, n^y_t, c^o_{t+1}, n^o_{t+1}\}^{\infty}_{t=0}} \sum_{t=0}^{\infty} \omega_t \left(u(c^y_t) - v(n^y_t) + \beta \int \left(u(c^o_{t+1}) - v(n^o_{t+1}) \right) d\Psi \right) \tag{14}$$

subject to the four constraints (8)-(11) listed in Proposition 5 together with the debt-limit constraints (13). In addition, ω_t in the planner's social-welfare function denotes the weight of generation t.[11] The solution to this maximization problem is called a Ramsey plan or a Ramsey allocation (outcome).

Let $\beta^t \mu_t$, $\beta^{t+1} \lambda^1_{t+1}$, $\beta^{t+1} \lambda^2_{t+1}$, and $\beta^t \phi_t$ denote, respectively, the Lagrangian multipliers for the first four conditions in Proposition 5—which represent (i) the resource constraint, (ii) old households' budget constraint, (iii) old households' FOCs, and (iv) the implementability condition. Denote $\beta^t \eta^H_t$ and $\beta^t \eta^L_t$ as the Lagrangian multipliers for the upper and lower debt-limit constraints in condition (13). Hence, a Ramsey allocation is characterized by these constraints together with the following five Ramsey FOCs with respect to $\{K_{t+1}, c^y_t, n^y_t, c^o_{t+1}, n^o_{t+1}\}$:

$$\mu_t + \eta^L_t - \eta^H_t = \beta \mu_{t+1} (MP_{K,t+1} + 1 - \delta), \tag{15}$$

$$\omega_t u^y_{c,t} + \beta^t \phi_t (u^y_{cc,t} c^y_t + u^y_{c,t}) + \beta^t (\eta^H_t - \eta^L_t)(1 + \frac{v^y_{n,t} n^y_t}{(u^y_{c,t})^2} u^y_{cc,t}) = \beta^t \mu_t, \tag{16}$$

[11] By our simplifying assumption, the period-0 old households supply no labor and their consumption is exogenously given to the Ramsey planner by $c^o_0 = (K_0 + B_0)[(1 - \tau_{k,0}) q_0 + 1 - \delta]$.

$$\omega_t v^y_{n,t} + \beta^t \phi_t (v^y_{nn,t} n^y_t + v^y_{n,t}) + \beta^t \frac{(\eta^H_t - \eta^L_t)}{u^y_{c,t}} (v^y_{nn,t} n^y_t + v^y_{n,t}) = \beta^t \mu_t \kappa^y MP_{N,t}, \tag{17}$$

$$\begin{aligned} & \omega_t \beta u^o_{c,t+1} + \beta^{t+1} \phi_t u^o_{cc,t+1} \left[C^o_{t+1} - \int \frac{v^o_{n,t+1}}{u^o_{c,t+1}} d\Psi \times N^o_{t+1} \right] \\ & + \beta^{t+1} \phi_t U^o_{C,t+1} \left[1 + u^o_{cc,t+1} \frac{v^o_{n,t+1}}{(u^o_{c,t+1})^2} \times N^o_{t+1} \right] \\ = \;& \beta^{t+1} \mu_{t+1} + \beta^{t+1} \frac{u^o_{cc,t+1}}{u^o_{c,t+1}} v^o_{n,t+1} \zeta_{t+1}, \end{aligned} \tag{18}$$

and

$$\begin{aligned} & \omega_t \beta v^o_{n,t+1} + \beta^{t+1} \phi_t U^o_{C,t+1} \left(\frac{v^o_{nn,t+1}}{u^o_{c,t+1}} N^o_{t+1} + \int \frac{v^o_{n,t+1}}{u^o_{c,t+1}} d\Psi \times \theta_{t+1} \right) \\ = \;& \beta^{t+1} \mu_{t+1} MP_{N,t+1} \theta_{t+1} + \beta^{t+1} v^o_{nn,t+1} \zeta_{t+1}, \end{aligned} \tag{19}$$

where ζ_{t+1} is defined as

$$\zeta_{t+1} \equiv \left[\lambda^1_{t+1} \left(n^o_{t+1} - \frac{1}{\theta_{t+1}} \int \theta n^o_{t+1} d\Psi \right) + \lambda^2_{t+1} (1 - \theta_{t+1}) \right].$$

Recall that the multipliers $\{\lambda^1, \lambda^2, \phi\}$ are possibly zero in the different scenarios considered below.

3.3 Characterization of the Ramsey Allocation

We define aggregate allocative efficiency (AAE) as a competitive-equilibrium allocation in which the MGR holds, and individual allocative efficiency (IAE) as a competitive-equilibrium allocation in which a young household's marginal utility of consumption equals his/her marginal utility of consumption when old—which also implies that the market interest rate r equals the time discount rate $1/\beta$ if there is no idiosyncratic uncertainty. Similarly, we define the expected IAE (EIAE) as a competitive-equilibrium allocation in which the marginal utility of consumption when young equals the expected marginal utility of consumption when old across idiosyncratic states.

In addition, we define the first-best allocation as the optimal solution by a social planner that maximize the social-welfare function (14) subject only to the aggregate resource constraint (8). Notice that the social planner's solution may not constitute a competitive equilibrium because it ignores the other constraints in Proposition 5. We can, however, show easily that the first-best allocation implies AAE and IAE (as well as EIAE).

Assumption 6. *Assume the social welfare weight of generation t is $\omega_t = \beta^t$.*

With these conceptual definitions in mind, we can define a Ramsey steady state as follows:

Definition 7. *A Ramsey steady state is a long-run Ramsey allocation where all variables, $\{K, N, C, c^y, c^o, n^y, n^o\}$, converge to finite positive constants and the optimal level of government debt B converges to a finite constant.*

In what follows, we derive the Ramsey allocation under several special scenarios in order to gain insights and intuitions behind the net-wealth effects of government bonds and implied taxes. We consider two scenarios: (i) with no idiosyncratic uncertainty and (ii) with idiosyncratic uncertainty. In each scenario, we investigate how the net wealth of government bonds and optimal tax scheme would change according to the condition of dynamic (in)efficiency and the availability of different government policy tools.

3.3.1 Scenario 1: Ramsey Allocation without Idiosyncratic Uncertainty

Scenario 1 is also the benchmark model studied by Barro (1974) without bequest motives and imperfect capital markets. In this scenario the optimal Ramsey FOCs are dramatically simplified since the conditions (9) and (10) are trivially satisfied and do not impose any restrictions on the Ramsey planner; consequently, the multipliers λ^1_{t+1} and $\lambda^2_{t+1}$are zero. Also, as discussed previously, the multipliers associated with the conditions (11) and (13), $\left\{\phi_t, \eta^H_t, \eta^L_t\right\}$, are zero if there are no restrictions on T_{t+1} and B_{t+1} and non-zero if there are restrictions.

Proposition 8. *The Ramsey steady state in the absence of idiosyncratic uncertainty features the following properties:*

1. *If the debt constraints (13) do not bind (i.e., the absolute values of $\left|\overline{B}\right|$ and $|\underline{B}|$ are sufficiently large) and a lump-sum tax/transfer to old households is feasible and unconstrained, then the Ramsey steady state achieves the first-best allocation, which exhibits both AAE and IAE. All distortionary taxes (capital and labor taxes for the young and old) are zero. In addition, if the excess saving function $S(r)$ (defined in equation (7)) is monotonically increasing in r, then the optimal supply of bonds is negative/positive if and only if the economy is "dynamically efficient/inefficient". Finally, the sign of the optimal quantity of public debt (or the net wealth of government bonds) determines the sign of the lump-sum transfer.*

2. *If the debt constraints (13) do not bind and $T = 0$ (lump-sum taxes/transfers are not feasible), the Ramsey steady state still achieves both AAE and IAE. Moreover, if the excess saving function $S(r)$ (defined in equation (7)) is monotonically increasing in r, then the*

optimal supply of bonds is negative/positive if and only if the economy is "dynamically efficient/inefficient". The optimal labor taxes are identical across age cohorts ($\tau_n^y = \tau_n^0$) and their sign follows the sign of optimal debt. Finally, the capital tax remains zero ($\tau_k = 0$).

3. *If the government debt limit binds on either end of the debt-limit constraints (13), then in the Ramsey steady state neither the MGR nor IAE holds regardless of the availability of lump-sum taxes/transfers T. The steady-state capital tax is not zero. Notice that even though it is feasible for the Ramsey planner to use a non-zero capital tax (subsidy) to achieve the MGR, it is not optimal to do so.*

4. *A sufficient condition for the excess-saving function $S(r)$ to be monotonically increasing is $r > \alpha + 1 - \delta$, namely,*
$$\left.\frac{dS(r)}{dr}\right|_{r>\alpha+1-\delta} > 0.$$

Proof. See Appendix A.3. □

Result (1) in Proposition 8 should not be confused with Ricardian equivalence. Although the Ramsey planner can achieve the first-best allocation featuring MGR and IAE, it is because government bonds have played the role of lump-sum taxes/transfers to young households in addition to lump-sum taxes/transfers (T) to old households. In other words, the demand for (supply of) bonds by young households acts as a lump-sum tax (or transfer) that shifts resources between the young and the old age without distortions. In the Ramsey steady state, the interest cost of public debt is financed also by a lump-sum tax on the old ($T < 0$). If the public debt is negative, then the government receives interest payments from young households and provides lump-sum transfers to the old ($T > 0$).

Precisely because of the need to finance public debt (or lend credit to young households if $B < 0$), the second result in Proposition 8 shows that the labor tax is positive/negative for the old if a lump-sum tax/transfer to the old is not feasible. This result is reminiscent of the results found in the literature that it is not optimal to tax capital when age-specific labor taxes are available (see, e.g., Erosa and Gervais (2002) and Garriga (2019)). Given that there is no exogenous government spending, the Ramsey planner uses the interest payments received from the private sector to subsidize labor if the optimal quantity of public debt is negative. In addition, there is a if-and-only-if relationship between the sign of public debt and the "dynamically efficient (inefficient)" economy. In other word, for both result (1) and (2), the positive (negative) supply of public debt is welfare improving in the view of Ramsey planner if and only if the economy is "dynamically efficient (inefficient)".

The third result in Proposition 8 clearly demonstrates the importance of government bonds (or credit) in improving social welfare. It highlights that public debt must be either net wealth or a liability to the private sector depending on whether the economy is dynamically inefficient or not. It also shows that if the Ramsey planner cannot utilize government bonds as a tool to achieve IAE because of a binding constraint on public debt (credit), the MGR must fail to hold, although it is still feasible to achieve the MGR by taxing or subsiding capital. Hence, our result is consistent to the study by Chien and Wen (2019), where they finds that government bonds are a better tool than capital tax to achieve the MGR in an infinite-live agent model.

Armed with these insights from Proposition 8, we are now ready to study the Ramsey plan under idiosyncratic uncertainty and incomplete financial markets.

3.3.2 Scenario 2: Ramsey Allocation with Idiosyncratic Uncertainty

Proposition 9. *Under uninsurable idiosyncratic risk for labor productivity in old age, a Ramsey steady state has the following properties:*

1. *If the debt constraints (13) do not bind and a lump-sum tax/transfer to old households is feasible and unconstrained, the Ramsey steady state achieves the MGR but not IAE (nor EIAE), and the tax/subsidize rates on capital and old-household labor income are non-zero despite lump-sum taxes/transfers to old households. Furthermore, the labor tax/subsidy on young households is zero. In addition, if the excess-saving function* $S(r)$ *is monotonically increasing and the parameters satisfy the condition:* $1-\alpha < \frac{\alpha}{1/\beta-1+\delta}$*, then the optimal supply of bonds is negative and the economy is "dynamically efficient".*

2. *If the debt constraints (13) do not bind and if a lump-sum tax/transfer to old households is not feasible* ($T = 0$)*, the Ramsey steady state achieves the MGR but not IAE (nor EIAE), and the taxes on capital and labor are all non-zero regardless of age. In addition, if the excess-saving function* $S(r)$ *is monotonically increasing and the parameters satisfy the condition:* $1-\alpha < \frac{\alpha}{1/\beta-1+\delta}$*, then the optimal supply of bonds is negative and the economy is "dynamically efficient".*

3. *Under a binding constraint on the government's capacity to issue debt (i.e.,* $B = 0$*), neither the MGR nor IAE or EIAE can be achieved in the Ramsey steady state regardless of the availability of lump-sum taxes/transfers to old households.*

The result (1) and (2) of Proposition 9 show that precautionary saving motives do not necessarily lead to overaccumulation of capital from the view point of Ramsey planner. Under certain parameter restrictions, the optimal quantity of public debt is negative despite shortages of private liquidity under imperfect credit markets and uninsurable income risk. In other words, government bonds are not necessarily net wealth of the private sector (but are instead a net liability) even if financial markets are incomplete and idiosyncratic income risk cannot be effectively diversified by private credit markets. Namely, borrowing constraints and market incompleteness per se do not imply that government bonds are necessarily net wealth.

In addition, despite the existence of idiosyncratic risks, the Ramsey planner still prefers to archive the MGR through public debt/credit instead of capital tax/subsidy. This can be seen from the third result of our proposition that the MGR is feasible (through a capital tax) but not optimal.

Out result, nonetheless, is fully consistent with the conventional wisdom (see Aiyagari and McGrattan (1998)) that the optimal level of government debt can be positive and sufficiently high in an infinite-horizon HAIM economy. The reason is that in such an economy the market interest rate lies strictly below the time discount rate and the *laissez faire* equilibrium is "dynamically inefficient."

4 A Brief Literature Review

In a seminal paper, Barro (1974) studied the conditions under which government bonds are perceived by the private sector as net wealth—i.e., situations where the value of government bonds exceeds the cost of financing them in terms of the future tax burden on the private sector. In particular, Barro (1974) studied three scenarios in which government bonds are net wealth to the private sector because they can improve welfare: (i) the consumer life span is finite (as in an OLG economy) and voluntary intergenerational transfers of resources are ruled out or hindered; (ii) consumers within the same age-cohort are heterogeneous in their time discounting factor and the credit market is imperfect for competitive lending and borrowing; and (iii) government bonds can provide better liquidity services than private debt. Our paper complement to his by studying the relationships between the sign of optimal quantity of debt and "dynamically inefficient".

In a similar spirit, Woodford (1990) shows in an infinite-horizon model with heterogeneous agents and uninsurable individual income risk that public debt can improve welfare if the market interest rate lies below the time discount rate because of liquidity constraints. But similar to Barro (1974), Woodford (1990) does not study optimal tax policy in conjunction with optimal debt policy.

The work of Aiyagari (1995) is the first attempt at investigating optimal Ramsey taxation in HAIM economies. Under the assumption of the existence of an interior Ramsey state steady, Aiyagari (1995) shows that the Ramsey planner opts to restore the MGR by taxing capital in the steady state even though a labor tax is also available. However, Aiyagari (1995) did not focus on the issue of optimal quantity of debt and its interactions with optimal taxes.

Aiyagari and McGrattan (1998) build on the Aiyagari (1994) model to determine the optimal quantity of debt by studying the trade-offs in benefits and costs of varying the quantity of debt. Their analysis mainly focus on steady-state welfare through numerical methods under a critical assumption that the proportional tax rates on labor and capital income are levied equally across households to finance public debt. This assumption rules out the possibility that the optimal capital tax may be zero. Our results provide an explanation of their finding because in their model the interest rate lies below the time discount rate and their economy is always "dynamically inefficient."

Angeletos, Collard, and Dellas (2016) study the Ramsey policy problem in the Lagos and Wright (2005) framework with HAIM properties. They show that when risk-free government bonds contribute to the supply of liquidity to alleviate private agents' borrowing constraints, issuing more debt raises welfare by improving the allocation of resources. The key reason is that in their model the market interest rate lies below the time discount rate and hence the economy is "dynamically inefficient".

Bassetto and Kocherlakota (2004) show that the paths of government debt can be irrelevant under distortionary taxes. In particular, they show that if the government collects taxes in a given period based only on incomes earned in previous periods, then the government can adjust its tax policy so as to attain any debt path without affecting equilibrium allocations or prices.

Compared with the aforementioned literature, our model framework is more closely related to Krueger and Ludwig (2018). But that paper also focuses exclusively on optimal capital taxation (in the absence of labor taxes and government debt), instead of on the interactions between optimal tax and optimal debt policy.

5 Conclusion

The rapidly growing national debt in the U.S. since the 1980s has alarmed and intrigued the academic world. Consequently, the concept of dynamic (in)efficiency in an OLG world and the importance of the HAIM hypothesis to justify a high debt-to-GDP ratio have been extensively studied. Two important consensus views have emerged from this literature: (i) The optimal quantity of public debt is positive if private liquidity is insufficient to support private saving and investment demand (due to borrowing constraints and incomplete markets), and (ii) the optimal capital tax is positive—because of precautionary saving and the consequent failure of the MGR. But these two views are seldom derived jointly in the same model; consequently, the dynamic relationship between optimal debt and optimal tax remains unclear, especially considering that the optimal quantity of debt is judged by the notion of golden-rule saving rate and any debt must be financed by future taxes.

In this paper we use a primal Ramsey approach to analytically determine the net-wealth effect of government bonds in an OLG-HAIM model with capital. We show that even if private liquidity is insufficient for providing an adequate buffer to smooth consumption and precautionary holdings of public debt do improve self-insurance against idiosyncratic income risk, government bonds are still not necessarily net wealth to households. The reason is that precautionary saving and oversaving are not necessarily the same thing, and they can have very different policy implications—i.e., the Ramsey planner may opt to issue a negative amount of government bonds to improve social welfare in spite of market incompleteness, unless the economy is "dynamically inefficient". In other words, optimal debt can be negative even if households cannot insure themselves against idiosyncratic risk due to borrowing constraints and would like to hold government bonds in a competitive equilibrium.

More importantly, since the sign and magnitude of the optimal quantity of debt dictate the sign and magnitude of optimal taxes as well as the priority order of tax tools such as a labor tax vs. a capital tax, we argue that optimal tax policy and optimal debt policy must be studied jointly. Yet the existing literature often studies these two types of fiscal policy separately.

A formidable challenge to studying optimal debt policy and optimal tax policy jointly in a HAIM framework is the intractability of Ramsey problem. We overcome this difficulty by using a two-period OLG-HAIM model (a la Krueger and Ludwig (2018)) that allows us to solve the Ramsey problem analytically.

Since the criterion for the determination of the sign of public debt is "dynamic (in)efficiency", optimal tax policies must also be studied accordingly, as they depend both on the sign and on the magnitude of the optimal quantity of public debt. Calling attention to this interdependence of public debt and tax policies is of great importance: On the one hand the bulk of the theoretical literature justifies a positive quantity of public debt by relying on the assumption of incomplete financial markets and the lack of perfect risk sharing (e.g., Barro (1974), Woodford (1990), Aiyagari and McGrattan (1998)); but on the other hand the bulk of the empirical literature claims that the U.S. economy and the economies of other major OECD nations are dynamically efficient (e.g., Abel, Mankiw, Summers, and Zeckhauser (1989)), which may imply that the optimal quantity of public debt should be negative.

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A Appendix

A.1 Proof of Proposition 4

The steady state version of household FOCs with respect to consumption give

$$(c^y)^{-\sigma} = \beta r (c^o)^{-\sigma},$$

which together with $c^y + c^o = C$ suggest that the consumption share of young can be expressed as a function of r :

$$\frac{c^y}{C} = \frac{1}{(1 + (\beta r)^{\frac{1}{\sigma}})}. \tag{20}$$

In steady states, the young and old household FOCs with respect to labor are

$$\begin{aligned}(n^y)^\gamma (c^y)^\sigma &= (1 - \tau_n^y)\kappa^y w n^y, \\ (n^o)^\gamma (c^o)^\sigma &= (1 - \tau_n^o)\kappa^o w n^o,\end{aligned}$$

which implies labor share of the young is $\frac{1}{1+\widetilde{\omega}}$, where

$$\widetilde{\omega} \equiv \left(\beta r \frac{(1 - \tau_n^y)}{(1 - \tau_n^o)}\right)^{-\frac{1}{\gamma}} \left(\frac{\kappa^y}{\kappa^o}\right)^{-\frac{1}{\gamma}-1}.$$

Hence, the labor income to output ratio of young is

$$\frac{w\kappa^y n^y}{Y} = \frac{1-\alpha}{1+\widetilde{\omega}}, \tag{21}$$

where the total labor income's share is $1-\alpha$.

We then can express the steady-state excess saving as a function of r. Using young household budget constraint, the excess savings of young households, denoted by, $\widetilde{S}$, is

$$\widetilde{S} = \frac{w\kappa^y n^y}{Y} - \frac{C}{Y}\frac{c^y}{C} - \frac{K}{Y} - \frac{B}{Y},$$

which together with steady state resource constraint $\frac{C}{Y} = 1 - \delta\frac{K}{Y}$, we get

$$\widetilde{S} = \frac{w\kappa^y n^y}{Y} - \left(1 - \delta\frac{K}{Y}\right)\frac{c^y}{C} - \frac{K}{Y} - \frac{B}{Y}.$$

By using (20) and (21) and the no-arbitrage condition, the excess saving of young households can be rewritten as

$$\widetilde{S}(r) \equiv \frac{(1-\alpha)}{1+\widetilde{\omega}} - \frac{1}{(1+(\beta r)^{\frac{1}{\sigma}})} - \left(1 - \delta\frac{1}{(1+(\beta r)^{\frac{1}{\sigma}})}\right)\frac{\alpha}{r-1+\delta} - \frac{B}{Y}.$$

A.2 Proof of Proposition 5

The "If" Part: Given the initial B_0 and K_0 as well as the allocation $\{c_t^y, c_t^o, n_t^o, n_t^y, K_{t+1}\}_{t=0}^{\infty}$, a competitive equilibrium can be constructed by using the 5 conditions in Proposition 5 and following the steps below that uniquely back up the sequences of the 11 variables, $\{N_t, C_t, a_{t+1}\}_{t=0}^{\infty}$, $\{w_t, q_t, r_t\}_{t=0}^{\infty}$, and $\left\{\tau_{k,t}, \tau_{n,t}^y, \tau_{n,t}^o, T_{t+1}, B_{t+1}\right\}_{t=0}^{\infty}$, respectively:

1. Given $n_0^o = 0$, the aggregate labor N_t is determined by the labor market clearing condition:

$$N_t = \kappa^y n_t^y + \kappa^o \int \theta n_t^o d\Psi.$$

2. The aggregate consumption C_t is determined by the consumption market clearing condition:

$$C_t = c_t^y + \int c_t^o d\Psi.$$

3. Given $\{N_t, K_t\}_{t=0}^{\infty}$, the prices w_t and q_t are determined by the firm's FOCs for all $t \geq 0$:

$$\begin{aligned} w_t &= MP_{N,t} \\ q_t &= MP_{K,t}. \end{aligned}$$

4. Given $\left\{c_t^y, c_{t+1}^o\right\}$, the two variables $\{r_{t+1}, \tau_{k,t+1}\}$ are determined by

$$\begin{aligned} r_{t+1} &= \frac{u_c(c_t^y)}{\beta \int u_c(c_{t+1}^o) d\Psi} \\ \tau_{k,t+1} &= 1 - \frac{r_{t+1} - 1 + \delta}{q_{t+1}}. \end{aligned}$$

   In addition, r_0 and c_0^o are determined by $r_0 = (1 - \tau_{k,0})q_0 + 1 - \delta$ and $c_0^o = (K_0 + B_0)r_0$, respectively.

5. The labor tax rates are determined by

$$
\begin{aligned}
(1-\tau^y_{n,t}) &= \frac{v_n(n^y_t)}{u_c(c^y_t)w_t\kappa^y}, \\
(1-\tau^o_{n,t+1}) &= \frac{1}{w_{t+1}\kappa^o}\int \frac{v_n(n^o_{t+1})}{u_c(c^o_{t+1})}d\Psi.
\end{aligned}
$$

6. Household savings a_{t+1} are determined by the young households' budget constraint:

$$
a_{t+1} = \frac{v_n(n^y_t)}{u_c(c^y_t)}n^y_t - c^y_t.
$$

7. Bond holdings B_{t+1} are determined by the asset market clearing condition: $B_{t+1} = a_{t+1} - K_{t+1} = \frac{v_n(n^y_t)}{u_c(c^y_t)}n^y_t - c^y_t - K_{t+1}$.

8. The lump-sum tax/transfer to old households T_{t+1} is determined by the implementability condition. We derive the implementability condition by plugging the aggregate resource constraint into the government's budget constraint:

$$
\begin{aligned}
&-(1-\tau_{k,t})q_tK_t - (1-\tau^y_{n,t})w_t\kappa^y n^y_t - (1-\tau^o_{n,t})w_t\kappa^o\int \theta_t n^o_t d\Psi + B_{t+1} \\
&-r_tB_t + c^y_t + \int c^o_t(\theta)d\Psi + K_{t+1} - (1-\delta)K_t \geq T_t.
\end{aligned}
$$

We then substitute out taxes and prices in the above equation to get

$$
\begin{aligned}
& -(1-\tau_{k,t})q_tK_t - \frac{v^y_{n,t}}{u^y_{c,t}}n^y_t - \int \frac{v_n(n^o_t)}{u_c(c^o_t)}d\Psi\int \theta_t n^o_t d\Psi + \frac{v^y_{n,t}}{u^y_{c,t}}n^y_t - c^y_t \\
& -K_{t+1} - \frac{u^y_{c,t-1}}{\beta\int u^o_{c,t}d\Psi}\left(\frac{v^y_{n,t-1}}{u^y_{c,t-1}}n^y_{t-1} - c^y_{t-1} - K_t\right) + c^y_t + \int c^o_t(\theta)d\Psi + K_{t+1} - (1-\delta)K_t \\
\geq\ & T_t,
\end{aligned}
$$

which can be simplified to

$$
\frac{u^y_{c,t-1}}{\beta\int u^o_{c,t}d\Psi}c^y_{t-1} - \frac{v^y_{n,t-1}}{\beta\int u^o_{c,t}d\Psi}n^y_{t-1} + \int c^o_t d\Psi - \int \frac{v_n(n^o_t)}{u_c(c^o_t)}d\Psi\int \theta_t n^o_t d\Psi \geq T_t.
$$

Rearranging terms, the above equation becomes

$$
u^y_{c,t-1}c^y_{t-1} - v^y_{n,t-1}n^y_{t-1} + \beta\int u^o_{c,t}d\Psi\int c^o_t d\Psi - \beta\int u^o_{c,t}d\Psi\int \frac{v^o_{n,t}}{u^o_{c,t}}d\Psi\int \theta_t n^o_t d\Psi \geq T_t\beta\int u^o_{c,t}d\Psi,
$$

which gives equation (11) by updating to period t.

9. The above steps leave the following three constraints to the Ramsey planner in determining the optimal policy mix:

   i. The aggregate resource constraint can be rewritten as equation (8) by using the labor and consumption market clearing conditions.

   ii. The old households' FOC is expressed as equation (10) after substituting out the labor tax for old households.

   iii. By substituting out the lump-sum transfer T_{t+1} using the government budget constraint, the old households' budget constraint is rewritten as

$$\begin{aligned} & \tau_{k,t+1} q_{t+1} K_{t+1} + \tau^y_{n,t+1} w_{t+1} \kappa^y n^y_{t+1} + \tau^o_{n,t+1} w_{t+1} \kappa^o \int \theta_{t+1} n^o_{t+1} d\Psi + B_{t+2} - r_{t+1} B_{t+1} \\ \geq \; & c^o_{t+1} - a_{t+1} r_{t+1} - \frac{v_n(n^o_{t+1})}{u_c(c^o_{t+1})} n^o_{t+1}, \end{aligned}$$

   which can be further rewritten as

$$\begin{aligned} & q_{t+1} K_{t+1} + (1-\delta) K_{t+1} + w_{t+1} N_{t+1} - (1 - \tau^y_{n,t+1}) \kappa^o w_{t+1} n^y_{t+1} \\ & -(1 - \tau^o_{n,t+1}) w_{t+1} \int n^o_{t+1} d\Psi + B_{t+2} \\ \geq \; & c^o_{t+1} - \frac{v_n(n^o_{t+1})}{u_c(c^o_{t+1})} n^o_{t+1}. \end{aligned}$$

   By substituting out labor taxes and using the aggregate resource constraint in the above expression, we arrive at the old households' budget constraint:

$$\begin{aligned} & c^y_{t+1} + \int c^o_{t+1}(\theta) d\Psi + K_{t+2} - \frac{v_n(n^y_{t+1})}{u_c(c^y_{t+1})} n^y_{t+1} \\ & - \int \frac{v_n(n^o_{t+1})}{u_c(c^o_{t+1})} d\Psi \int \eta n^o_{t+1} d\Psi + B_{t+2} \\ \geq \; & c^o_{t+1} - \frac{v_n(n^o_{t+1})}{u_c(c^o_{t+1})} n^o_{t+1}. \end{aligned}$$

   Finally, the above equation becomes (9) by using the young households' budget constraint.

Notice that without idiosyncratic uncertainty, the old households' budget constraint and the old households' FOCs are trivially satisfied, leaving the aggregate resource constraint as the only constraint on the Ramsey problem. Since this situation is equivalent to a social planner's problem, the resulting Ramsey plan must be the first-best allocation.

The "Only If" Part: The constraints listed in Proposition 5 are trivially satisfied because they are part of the competitive-equilibrium conditions.

A.3 Proof of Proposition 8

A.3.1 With Unconstrained B and T

With both B_{t+1} and T_t unconstrained and without idiosyncratic uncertainty, the Ramsey problem becomes the social planner's problem since the only constraint left in Proposition 5 is the aggregate resource constraint. In this case, it is straight forward to verify that IAE is achieved in the steady state according to the simplified FOCs with respect to c_t^y and c_{t+1}^o after setting the Lagrangian multipliers $\left\{\lambda^1, \lambda^2, \phi, \eta^L, \eta^H\right\}$ to zero:

$$u_{c,t}^y = \mu_t$$

$$u_{c,t+1}^o = \mu_{t+1}.$$

Since $\mu_t = \mu_{t+1}$ in the steady state, we must have $u_c^y = u_c^o$. In addition, the MGR holds according to the simplified Ramsey FOC with respect to K_{t+1}:

$$\mu_t = \beta\mu_{t+1}(MP_{K,t+1} + 1 - \delta),$$

which implies that the capital tax is zero not only in the steady state but also in the entire transition period. Also, the simplified Ramsey FOCs with respect to n_t^y and n_{t+1}^o are given by

$$v_{n,t}^y = \mu_t MP_{N,t}\kappa^y$$

and

$$v_{n,t+1}^o = \mu_{t+1}MP_{N,t+1}\kappa^o,$$

which suggest that labor taxes for both young and old are zero:

$$\begin{aligned}
1 - \tau_{n,t}^y &= \frac{v_n(n_t^y)}{u_c(c_t^y)\kappa^y w_t} = \frac{\mu_t}{\mu_t} = 1 \\
1 - \tau_{n,t+1}^o &= \frac{v_{n,t+1}^o}{u_{c,t+1}^o w_{t+1}\kappa^o} = \frac{\mu_{t+1}}{\mu_{t+1}} = 1.
\end{aligned}$$

Hence, the interest payments on government bonds must be financed by lump-sum taxes on the old: $T_{t+1} < 0$.

From the discussion above, we know that the Ramsey steady state features the following properties: (1) $c^y = c^o = C_t/2$, (2) $\tau_n^y = \tau_n^o = 0$, (3) $\beta r = 1$, and (4) $MP_K = \frac{1}{\beta} - 1 + \delta$, which is the MGR. Hence, we can solve for $\frac{K}{Y}$ with the MGR:

$$\frac{K}{Y} = \frac{\alpha}{1/\beta - 1 + \delta}.$$

Hence, using equation (5), the steady-state B/Y ratio can be expressed as

$$\frac{B}{Y} = \frac{(1-\alpha)}{1 + (\beta r)^{-\frac{1}{\gamma}} \left(\frac{\kappa^y}{\kappa^o}\right)^{-\frac{1}{\gamma}-1}} - \frac{1}{(1 + (\beta r)^{\frac{1}{\sigma}})} - \left(1 - \delta \frac{1}{(1 + (\beta r)^{\frac{1}{\sigma}})}\right) \frac{\alpha}{r - 1 + \delta}\Bigg|_{r=\frac{1}{\beta}}$$

where right hand side is exactly equal to the excess-saving function defined in equation (7) under laissez-faire competitive equilibrium, as described in Proposition (4). Namely, the optimal Ramsey debt-to-GDP ratio is

$$\frac{B}{Y} = S(1/\beta).$$

Hence, if $S(r)$ is a monotonically increasing function in r, then the optimal supply of bonds is negative (positive) if and only if the economy is "dynamically efficient (inefficient)".

A.3.2 With B Unconstrained and $T = 0$

With unconstrained government debt limits but the constrained $T = 0$, the multipliers η_t^H and η_t^L are zero but $\phi_t \neq 0$. Under Assumption 6 and Assumption 3, the FOCs with respect to c_t^y and c_{t+1}^o are reduced to

$$(1 + \phi_t(1-\sigma))u_{c,t}^y = \mu_t$$

$$(1 + \phi_t(1-\sigma))u_{c,t+1}^o = \mu_{t+1}$$

and the Euler equation of the Ramsey planner is given by

$$1 = \beta \frac{\mu_{t+1}}{\mu_t}(MP_{K,t+1} + 1 - \delta).$$

Therefore, it must be true that (i) the capital tax is zero for all period t:

$$\tau_{k,t+1} = 1 - \frac{\frac{u_c(c_t^y)}{\beta u_c(c_{t+1}^o)} - 1 + \delta}{q_{t+1}} = 0,$$

and (ii) in the Ramsey steady state (where $\mu_t = \mu_{t+1}$) both the MGR and IAE ($u_c^y = u_c^o$) hold.

We then discuss the tax rates for labor. The young and old households' FOCs with respect to labor can be rewritten as

$$(1+\phi_t(1+\gamma))v_{n,t}^y = \mu_t MP_{N,t}\kappa^y,$$

and

$$(1+\phi_t(1+\gamma))v_{n,t+1}^o = \mu_{t+1} MP_{N,t+1}\kappa^o,$$

which together with the FOCs for young and old households' consumption give

$$\tau_{n,t}^y = 1 - \frac{v_n(n_t^y)}{u_c(c_t^y)\kappa^y w_t} = 1 - \frac{(1+\phi_t(1-\sigma))}{(1+\phi_t(1+\gamma))}$$

and

$$\tau_{n,t+1}^o = 1 - \frac{v_{n,t+1}^o}{u_{c,t+1}^o w_{t+1}\kappa^o} = 1 - \frac{(1+\phi_t(1-\sigma))}{(1+\phi_t(1+\gamma))}.$$

So the labor tax rates for young and old households are identical. The sign of the labor tax depends on the value of ϕ_t, which relates to the need for government revenue or the sign of public debt. The steady-state government budget constraint is given by

$$\tau_n^y w\kappa^y n^y + \tau_n^o w\kappa^o n^o = (\frac{1}{\beta} - 1)B.$$

Hence, the sign of τ_n is decide by the sign of B. More specifically, the steady state τ_n

$$\tau_n^y = \tau_n^o = \frac{1}{(1-\alpha)}(\frac{1}{\beta} - 1)\frac{B}{Y}$$

Next, we show that the relationship between the level of government bonds and dynamic efficiency. First, the MGR holds and can be used to solve for $\frac{K}{Y}$:

$$\frac{K}{Y} = \left.\frac{\alpha}{r - 1 + \delta}\right|_{r=\frac{1}{\beta}}.$$

In addition, by equation (5), the bond-to-output ratio is

$$\frac{B}{Y} = \frac{w\kappa^y n^y}{Y} - \frac{\tau_n^y w\kappa^y n^y}{Y} - \frac{C}{Y}\frac{c^y}{C} - \frac{K}{Y},$$

which implies

$$\frac{B}{Y}=\left.\frac{\frac{(1-\alpha)}{1+(\beta r)^{-\frac{1}{\gamma}}\left(\frac{\kappa y}{\kappa^o}\right)^{-\frac{1}{\gamma}-1}}-\frac{1}{(1+(\beta r)^{\frac{1}{\sigma}})}-\left(1-\delta\frac{1}{(1+(\beta r)^{\frac{1}{\sigma}})}\right)\frac{\alpha}{r-1+\delta}}{\left(1+\frac{1}{1+(\beta r)^{-\frac{1}{\gamma}}\left(\frac{\kappa y}{\kappa^o}\right)^{-\frac{1}{\gamma}-1}}(\frac{1}{\beta}-1)\right)}\right|_{r=\frac{1}{\beta}}.$$

Note that the nominator on the right hand side of the equation is exactly identical to $S(r)$ defined in Proposition 4 and the denominator is strictly positive. Hence, the optimal supply of bonds is negative (positive) if and only if the economy is dynamically efficient (inefficient) if S is a monotonically increasing function in r.

A.3.3 With Binding Debt Limits on B_{t+1}

In this case, either $\eta_t^H>0$ or $\eta_t^L>0$. Under Assumption 6, the FOCs with respect to c_t^y and c_{t+1}^o are given, respectively, by

$$(1+\phi_t(1-\sigma))u_{c,t}^y-\sigma\eta_t^H\frac{v_{n,t}^y n_t^y}{u_{c,t}^y c_t^y}=\mu_t-\eta_t^H+\eta_t^L$$

and

$$(1+\phi_t(1-\sigma))u_{c,t+1}^o=\mu_{t+1}.$$

The above two FOCs lead to

$$\frac{\mu_{t+1}}{\mu_t-\eta_t^H+\eta_t^L}=\frac{(1+\phi_t(1-\sigma))u_{c,t+1}^o}{(1+\phi_t(1-\sigma))u_{c,t}^y-\sigma(\eta_t^H-\eta_t^L)\frac{v_{n,t}^y n_t^y}{u_{c,t}^y c_t^y}}\begin{array}{l} >\frac{u_{c,t}^y}{u_{c,t+1}^o}=\beta r_{t+1}\text{ if }\eta_t^H>0\\ <\frac{u_{c,t}^y}{u_{c,t+1}^o}=\beta r_{t+1}\text{if }\eta_t^L>0\end{array},$$

which suggests that IAE fails to hold in the Ramsey steady state. In addition, the Ramsey FOC with respect to K_{t+1} is

$$1=\beta\frac{\mu_{t+1}}{\mu_t-\eta_t^H+\eta_t^L}(MP_{K,t+1}+1-\delta),\tag{22}$$

hence the MGR must fail in the Ramsey steady state as well since either $\eta_t^H>0$ or $\eta_t^L>0$. In addition, by comparing the equation above with

$$1=\beta\frac{u_{c,t+1}^o}{u_{c,t}^y}((1-\tau_{k,t+1})MP_{K,t+1}+1-\delta),$$

we can see that the sign of the capital tax is negative if $\eta_t^H>0$ and positive if $\eta_t^L>0$. This means that a capital tax (subsidy) is optimal if the level of government bonds hits its lower (upper) limit.

The implication for the old households' labor tax is identical to that in subsection A.3.2 since the Ramsey FOCs with respect to n^o_{t+1} and c^o_{t+1} remain unchanged. Hence, the old households' labor tax is given by

$$\tau^o_{n,t+1} = 1 - \frac{v^o_{n,t+1}}{u^o_{c,t+1} w_{t+1} \kappa^o} = 1 - \frac{(1 + \phi_t(1-\sigma))}{(1 + \phi_t(1+\gamma))},$$

which equals zero if and only if $\phi_t = 0$, namely if and only if the lump-sum policy tool T_{t+1} is available and unconstrained.

In addition, suppose we assume a lump-sum tax is not available (namely, $T_{t+1} \geq 0$), which implies $\phi_t \geq 0$. We can then show that $\tau^y_{n,t} > 0$ if $\eta^H_t > 0$. We prove this by contradiction: Suppose this is not true, then $v_n(n^y_t) \geq \kappa^y w_t u_c(c^y_t)$ by the young households' FOC. The Ramsey FOC with respect to n^y_t can be written as

$$\begin{aligned} \mu_t w_t \kappa^y &= (1 + \phi_t(1+\gamma)) v^y_{n,t} + \eta^H_t \frac{v^y_{n,t}}{u^y_{c,t}} (1+\gamma) \\ &\geq (1 + \phi_t(1+\gamma)) \kappa^y w_t u^y_{c,t} + \eta^H_t \kappa^y w_t (1+\gamma), \end{aligned}$$

which implies

$$\begin{aligned} \mu_t &\geq (1 + \phi_t(1+\gamma)) u^y_{c,t} + \eta^H_t (1+\gamma) \\ &> (1 + \phi_t(1-\sigma)) u^y_{c,t} + \eta^H_t \left(1 - \sigma \frac{v^y_{n,t} n^y_t}{u^y_{c,t} c^y_t}\right), \end{aligned}$$

where the last inequality utilizes the fact that $\gamma > 0 > -\sigma$. However, the last inequality leads to a contradiction since it violates the Ramsey FOC with respect to c^y_t.

Similarly, it can be shown that $\tau^y_{n,t} < 0$ if $\eta^L_t > 0$. Suppose this is not true, then $v_n(n^y_t) \leq \kappa^y w_t u_c(c^y_t)$ by the young households' FOC. The Ramsey FOC with respect to n^y_t can be written as

$$\begin{aligned} \mu_t w_t \kappa^y &= (1 + \phi_t(1+\gamma)) v^y_{n,t} - \eta^L_t \frac{v^y_{n,t}}{u^y_{c,t}} (1+\gamma) \\ &\leq (1 + \phi_t(1+\gamma)) \kappa^y w_t u^y_{c,t} - \eta^L_t \kappa^y w_t (1+\gamma), \end{aligned}$$

which implies

$$\begin{aligned} \mu_t &\leq (1 + \phi_t(1+\gamma)) u^y_{c,t} - \eta^L_t (1+\gamma) \\ &< (1 + \phi_t(1-\sigma)) u^y_{c,t} - \eta^L_t \left(1 - \sigma \frac{v^y_{n,t} n^y_t}{u^y_{c,t} c^y_t}\right), \end{aligned}$$

where the last inequality utilizes the fact that $\gamma > 0 > -\sigma$. However, the last inequality leads to a contradiction since it violates the Ramsey FOC with respect to c_t^y.

A.3.4 A Sufficient Condition for Monotonicity of $S(r)$

By equation (7), $S(r)$ is given by

$$S(r) \equiv \frac{(1-\alpha)}{1+(\beta r)^{-\frac{1}{\gamma}}\left(\frac{\kappa^y}{\kappa^o}\right)^{-\frac{1}{\gamma}-1}} - \frac{1}{(1+(\beta r)^{\frac{1}{\sigma}})} - \left(1-\delta\frac{1}{(1+(\beta r)^{\frac{1}{\sigma}})}\right)\frac{\alpha}{r-1+\delta},$$

which implies

$$\frac{dS(r)}{dr} = \vartheta_1 + \vartheta_2,$$

where

$$\begin{aligned}
\vartheta_1 &= \frac{d\left(\frac{(1-\alpha)}{1+\omega}\right)}{dr} = \frac{1}{\gamma}\frac{(1-\alpha)}{(1+\omega)^2}\left(\frac{(1-\tau_n^o)}{(1-\tau_n^y)}\right)^{\frac{1}{\gamma}}\left(\frac{\kappa^o}{\kappa^y}\right)^{\frac{1}{\gamma}+1}\beta^{-\frac{1}{\gamma}}r^{-\frac{1}{\gamma}-1}, \\
\vartheta_2 &= -\frac{\partial\left((1+(\beta r)^{\frac{1}{\sigma}})^{-1}\right)}{\partial r} + \frac{\partial\left(-\frac{\alpha}{r-1+\delta}\right)}{\partial r} + \frac{\partial\left(\frac{\delta}{(1+(\beta r)^{\frac{1}{\sigma}})}\frac{\alpha}{r-1+\delta}\right)}{\partial r} \\
&= \frac{1}{(1+(\beta r)^{\frac{1}{\sigma}})^2}\frac{1}{\sigma}\beta^{\frac{1}{\sigma}}r^{\frac{1}{\sigma}-1}\left(1-\frac{\alpha}{r-1+\delta}\right) \\
&\quad + \frac{\alpha}{(r-1+\delta)^2}\left(1-\frac{\delta}{(1+(\beta r)^{\frac{1}{\sigma}})}\right).
\end{aligned}$$

It is straightforward to see that (i) $\vartheta_1 > 0$ and that (ii) $\vartheta_2 > 0$ if $r > \alpha + 1 - \delta > 0$. Hence, $S(r)$ is an strictly increasing function in the domain $r > \alpha + 1 - \delta$.

A.4 Proof of Proposition 9

Obviously, IAE fails in an environment with uninsurable idiosyncratic risks because IAE requires infinite tools to be achieved if the idiosyncratic shock has a continuum of realizations. However, it is possible to achieve EIAE. The implications with respect to the MGR and EIAE under different policy mix are discussed in the following cases.

A.4.1 With Both B and T Unconstrained

Under Assumption 6 as well as the conditions $\phi_t = \eta_t = 0$, the MGR is implied by combining the steady-state version of Ramsey FOC with respect to K_{t+1}.

Summing the Ramsey FOC with respect to c^o_{t+1} across old households gives

$$\int u^o_{c,t+1} d\Psi = \mu_{t+1} - \sigma \int \frac{v^o_{n,t+1}}{c^o_{t+1}} \zeta_{t+1} d\Psi,$$

which together with

$$\tau_{k,t+1} = 1 - \frac{\frac{u_c(c^y_t)}{\beta \int u^o_{c,t+1} d\Psi} - 1 + \delta}{q_{t+1}} \tag{23}$$

implies that the capital tax is non-zero unless $\int \frac{v^o_{n,t+1}}{c^o_{t+1}} \zeta_{t+1} d\Psi = 0$.

By the household FOCs, the young households' labor tax rate is

$$1 - \tau^y_{n,t} = \frac{v_n(n^y_t)}{u_c(c^y_t)\kappa^y w_t},$$

which together with the Ramsey FOCs with respect to n^y_t and c^y_t implies a zero labor tax for the young.

The old households' labor tax rate is non-zero. We can show this by contradiction. Suppose $\tau^o_{n,t+1} = 0$, then the old households' FOC with respect to n^o_{t+1} becomes

$$v^o_{n,t+1} = u^o_{c,t+1} \kappa^o \theta_{t+1} w_{t+1}. \tag{24}$$

In addition, the Ramsey FOCs with respect to c^o_{t+1} and n^o_{t+1} can be rewritten as

$$u^o_{c,t+1} = \mu_{t+1} + \frac{u^o_{cc,t+1}}{u^o_{c,t+1}} v^o_{n,t+1} \zeta_{t+1}$$

and

$$v^o_{n,t+1} = \mu_{t+1} MP_{N,t+1} \theta_{t+1} + v^o_{nn,t+1} \zeta_{t+1}.$$

Hence, for equation (24) to be valid, it must be the case that (by using the two FOCs above)

$$-MP_{N,t+1}\theta_{t+1} \frac{\sigma}{c^o_{t+1}} \frac{n^o_{t+1}}{\gamma} \zeta_{t+1} = \zeta_{t+1},$$

which cannot be true since $-MP_{N,t+1}\theta_{t+1} \frac{\sigma}{c^o_{t+1}} \frac{n^o_{t+1}}{\gamma} < 0$ and $\zeta_{t+1} \neq 0$ for all θ_{t+1}.

We now turn to the optimal level and sign of government bonds in the Ramsey steady state. Given that the MGR still holds and $\tau^y_n = 0$, which is given by

$$\frac{K}{Y} = \frac{\alpha}{1/\beta - 1 + \delta}.$$

In addition, the resource constraint implies that $\frac{C}{Y} = 1 - \delta\frac{K}{Y}$. The steady-state B/Y ratio can be solved by using the budget constraint of young households:

$$\frac{B}{Y} = \frac{w\kappa^y n^y}{Y} - \frac{c_y}{Y} - \frac{\alpha}{1/\beta - 1 + \delta} < 1 - \alpha - \frac{\alpha}{1/\beta - 1 + \delta},$$

where the last inequality utilizes the facts that $\frac{w\kappa^y n^y}{Y} < 1 - \alpha$ and $c_y > 0$. As a result, a sufficient condition for $\frac{B}{Y} < 0$ is

$$1 - \alpha < \frac{\alpha}{1/\beta - 1 + \delta}. \tag{25}$$

Finally, we show that condition (25) also ensures that the economy is "dynamically efficient" if $S(r)$ is monotonically increasing. First, Condition (25) implies that

$$\begin{aligned} 0 &> (1-\alpha)(1-\delta) - (1-\delta)\frac{\alpha}{1/\beta - 1 + \delta} \\ &> \frac{(1-\alpha)}{1 + \left(\frac{\kappa^y}{\kappa^o}\right)^{-\frac{1}{\gamma}-1}} - 1 - (1-\delta)\frac{\alpha}{1/\beta - 1 + \delta} \\ &= S(r = \frac{1}{\beta}), \end{aligned}$$

where the second inequality utilizes the facts that (i) $(1-\alpha)\delta < 1$ and (ii) $\left(\frac{\kappa^y}{\kappa^o}\right)^{-\frac{1}{\gamma}-1} > 0$. Moreover, the last equality holds by the definition of $S(r)$. As a result, if S is monotonically increasing in r, then the laissez-faire equilibrium interest rate r^* has to be greater than $1/\beta$. Namely, the economy is "dynamically efficiency".

A.4.2 With B Unconstrained and $T = 0$

Under Assumption 6 and the condition $\eta_t = 0$, the MGR is implied by the steady-state version of Ramsey FOC with respect to K_{t+1}.

The Ramsey FOCs with respect to n_t^y and c_t^y are given by

$$(1 + \phi_t(1-\sigma))u_{c,t}^y = \mu_t$$

and

$$(1 + \phi_t(1+\gamma))v_{n,t}^y = \mu_t MP_{N,t}\kappa^y,$$

which imply a non-zero labor tax for the young:

$$\tau^y_{n,t} = 1 - \frac{v_n(n^y_t)}{u_c(c^y_t)\kappa^y w_t} = 1 - \frac{(1+\phi_t(1-\sigma))}{(1+\phi_t(1+\gamma))} \neq 0,$$

where the last inequality utilizes the property that $\phi_t \neq 0$ that is implied from the strictly binding constraint $T = 0$.

Integrating the Ramsey FOC with respect to c^o_{t+1} across old households gives

$$\begin{aligned}
& \int u^o_{c,t+1} d\Psi + \int u^o_{cc,t+1} d\Psi \left[C^o_{t+1} - \int \frac{v^o_{n,t+1}}{u^o_{c,t+1}} d\Psi \times N^o_{t+1} \right] \\
& + \phi_t U^o_{C,t+1} \left[1 - \sigma \int \frac{1}{c^o_{t+1}} \frac{v^o_{n,t+1}}{u^o_{c,t+1}} d\Psi \times N^o_{t+1} \right] \\
= \; & \mu_{t+1} - \sigma \int \frac{v^o_{n,t+1}}{c^o_{t+1}} \zeta_{t+1} d\Psi,
\end{aligned}$$

which together with the capital tax formula implies that the capital tax is non-zero in general. Similarly, without the lump-sum tax/transfer (i.e., $\phi_t \neq 0$), the labor tax for the old is also non-zero.

We now turn to the optimal level of bonds in the Ramsey steady state. Given that the MGR still holds in the Ramsey steady state, the B/Y ratio faces the same upper bound as in subsection A.4.1. Hence, equation (25) is a sufficient condition to ensure negative optimal debt as well as "dynamically efficient" economy.

A.4.3 With Binding Debt Limits on B_{t+1}

Under Assumption 6, the Ramsey FOC with respect to K_{t+1} shows that the MGR fails in the Ramsey steady state.