Are Government Bonds Net Wealth or a Liability? ---Optimal Debt and Taxes in an OLG Model with Uninsurable Income Risk

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Are Government Bonds Net Wealth or a Liability? —Optimal Debt and Taxes in an OLG Model with Uninsurable Income Risk

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Abstract

The rapidly growing national debt in the U.S. since the 1970s has alarmed and intrigued the academic world. Consequently, the concept of dynamic (in)efficiency in an overlapping-generations (OLG) world and the importance of the heterogeneous-agents and incomplete-markets (HAIM) hypothesis to justify a high debt-to-GDP ratio have been extensively studied. Two important consensus emerge from this literature: (i) The optimal quantity of public debt is positive—due to insufficient private liquidity to support private saving and investment (see, e.g., Barro (1974), Woodford (1990), and Aiyagari and McGrattan (1998)); (ii) the optimal capital tax is positive—because of precautionary saving and the consequent failure of the modified golden rule (see, e.g., Aiyagari (1995)). But these two consensus views are seldom derived jointly in the same model, so the dynamic relationship between optimal debt and optimal taxation remains unclear in HAIM models, especially considering that the optimal quantity of debt must be judged by the golden-rule saving rate and any debt must be financed by future taxes. We use a primal Ramsey approach to analytically characterize optimal debt and tax policy in an OLG-HAIM model. We show that since precautionary saving and oversaving are not necessarily the same thing, they have different policy implications—the Ramsey planner opts to issue bonds to crowd out private savings \textit{if and only if} a competitive equilibrium is dynamically inefficient regardless of precautionary savings. In other words, optimal debt can be negative even if households cannot insure themselves against idiosyncratic risk under borrowing constraints. The sign and magnitude of the optimal quantity of debt in turn dictate the sign and magnitude of optimal taxes as well as the priority order of tax tools such as a labor tax vs. a capital tax.

JEL Classification: E13; E62; H21; H30

Key Words: Role of Public Debt, Optimal Fiscal Policy, Ramsey Problem, Overlapping Generation, Incomplete Markets.

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1 Introduction

The optimal quantity of debt and optimal tax rates are intrinsically intertwined. This is so not only because these fiscal policies provide alternative means to financing public spending, but also because public debt itself needs to be financed by future taxes. The intrinsic interactions between the two types of fiscal policies has been best illustrated under aggregate uncertainty by the tax-smoothing literature pioneered by Barro (1979) and Lucas and Stokey (1983). However, even in the absence of aggregate uncertainty, these two types of fiscal policies are also closely intertwined whenever the Ricardian equivalence principle fails to hold.

In a seminal paper, Barro (1974) studies the conditions under which government bonds are perceived by the private sector as net wealth—i.e., the value of government bonds exceeds the costs of financing them in terms of tax burden on the private sector. Similarly, Woodford (1990) and Aiyagari and McGrattan (1998) also show that the optimal quantity of public debt is strictly positive under incomplete financial markets with idiosyncratic risk and borrowing constraints—because of the shortage of private liquidity. However, this literature does not study optimal tax policy and often assumes that the interest on public debt is financed by lump-sum taxes instead of distortionary taxes on capital and labor.

On the other hand, since Aiyagari’s (1994) seminal work, a large body of literature has been devoted to studying optimal tax policy under heterogeneous agents and incomplete markets (HAIM). But this literature often assumes away the role of government bonds or does not simultaneously study the issue of the optimal quantity of debt in conjunction with optimal taxation.

In this paper, we determine optimal debt and tax policies jointly in a single framework by modifying the overlapping-generations (OLG) model of Barro (1974) to a setting with imperfect/incomplete financial markets and uninsurable income risk (as in Aiyagari (1994) and Krueger and Ludwig (2018)). Specifically, households in our model have a two-period life span but are overlapped by another generation at any point in time; they save when young, supply labor in both periods, and earn capital returns when old. They cannot leave positive/negative bequests and face uninsurable income risk when old. The introduction of idiosyncratic risk into the old generation’s labor income implies that precautionary saving by households when young is never adequate for providing self-insurance in old age in the absence of two-sided altruism. There is no aggregate uncertainty or exogenous government expenditures. The government has the following policy tools to improve welfare: public debt, a lump-sum transfer to the old, a capital tax, and a generation-specific labor tax on the young and the old households. These fiscal variables are allowed to be negative, and we will also study optimal policies when public debt and/or lump-sum taxes/transfers are exogenously constrained by ad hoc limits.
We use the primal Ramsey approach to solve optimal policies in this OLG-HAIM model. Under our approach, we define the first-best allocation as the solution to a social planner’s problem where the government has enough tools to eliminate all market frictions. We show that the first-best allocation implies two efficiency conditions in the steady state: (i) modified golden rule (MGR) holds and (ii) the marginal utility of consumption is the same for old and young households. Under incomplete markets these two efficiency conditions often fail to hold in a *laissez faire* competitive equilibrium. But we investigate whether a Ramsey planner can restore these conditions using fiscal policies in various scenarios and study the implied signs and magnitudes of fiscal variables.\(^1\)

Clearly, the first condition (MGR) ensures aggregate allocative efficiency (AAE) of capital, and the second condition ensures individual allocative efficiency (IAE) of household consumption and saving. These two efficiency conditions are not necessarily the same thing and do not necessarily imply each other—recall that only in infinite-horizon representative-agent models is IAE automatically satisfied in the steady state. In our OLG-HAIM model, without idiosyncratic uncertainty, IAE implies equality between the interest rate and the time discount rate; but with uninsurable idiosyncratic uncertainty, equality between the interest rate and time discount rate implies only expected IAE (EIAE)—namely, the marginal utility of consumption of the young equals the expected marginal utility of consumption of the old.

Within such a conceptual framework, the net-wealth effect of government bonds is measured by the extent to which they improve social welfare: Namely, the net wealth from government bonds is zero if they play no role in improving social welfare, positive if they can help improve the efficiency of resource allocations along either of the two margins outlined above, and negative if the optimal quantity of public debt is negative—such that government bonds are net liabilities (instead of wealth) for the private sector.\(^2\) In the meantime, since government revenues are needed to finance the interest payments on public debt (if positive), our Ramsey approach also simultaneously determines the optimal paths of future taxes (or subsidies if the debt is negative) in both the transition and the steady states.

We obtain the following results: (i) In the absence of idiosyncratic income uncertainty (i.e., with only cross-generational heterogeneity), the *laissez faire* competitive equilibrium cannot achieve the first-best allocation and government debt can significantly improve social welfare by achieving the first-best outcome; however, the optimal quantity of debt is positive if and only if the competitive equilibrium is dynamically inefficient—otherwise the optimal quantity of public debt is negative.\(^3\)

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\(^1\)It should be clear that the MGR and IAE are only the necessary conditions for the first-best allocation in the long run.

\(^2\)When public debt is negative, it simply means that the government opts to hold privately issued debt and receive interest payments. For simplicity we assume there are no government expenditures other than interest payments if public debt is positive.

\(^3\)A competitive equilibrium is dynamically efficient (inefficient) if and only if the steady-state capital stock is
In particular, under dynamic inefficiency the Ramsey planner opts to rely only on lump-sum taxes on the old to finance the interest payments on government debt so that the optimal tax rate is zero for both labor and capital. However, if lump-sum taxes are not available, then the optimal tax rate remains zero for capital but becomes positive for both the young and old households’ labor income. In addition, if the capacity for the government to issue debt is restricted from above by an ad hoc debt limit and the debt-limit constraint binds, then neither AAE nor IAE can be achieved in a Ramsey equilibrium, further suggesting that government bonds are net wealth of the private economy so long as the competitive equilibrium is dynamically inefficient. In contrast, if the competitive equilibrium is dynamically efficient (the market interest rate exceeds the time discount rate), then the optimal policies simply flip signs—i.e., the optimal quantity of public debt is negative (implying positive private debt held by the government) and taxes become subsidies, or transfers.4

(ii) With uninsurable idiosyncratic income uncertainty, the Ramsey steady state is not a first-best allocation—it achieves the MGR but not IAE (not even in the sense of the expected marginal utilities of old household across states), and the tax rates on capital and old households’ labor income are non-zero despite lump-sum taxes/transfers to old households. Furthermore, the labor tax/subsidy for young households is zero unless lump-sum taxes/transfers to old households are not feasible. The sign of the quantity of public debt is positive/negative if and only if the laissez faire competitive equilibrium is dynamically inefficient/efficient. However, under a binding constraint on the government’s capacity to issue debt (e.g., \( B = 0 \)), neither the MGR nor IAE (even in the sense of expected value) can be achieved in the Ramsey steady state regardless of the availability of lump-sum taxes/transfers to old households.

Hence, our theoretical analysis also connects and contributes to the large literature on optimal taxation. Pivotal work by Judd (1985) and Chamley (1986) has shown that the best way for the government to finance its expenditures in the long run is to tax labor but not capital when the option of a lump-sum tax is not available. This zero-capital taxation result is surprising and has attracted many studies to examine its robustness.5 However, this important taxation problem

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4 Notice that the lack of private credit markets by no means imply that the government cannot lend to the private sector or purchase privately issued debt.

5 For example, in the introduction of Straub and Werning (2020): “One may even say that the result (zero capital tax) remains downright puzzling, as witnessed by the fact that economists have continued to take turns putting forth various intuitions to interpret it, none definitive nor universally accepted.”
is relatively less explored in the HAIM framework than in the representative-agent framework. Consequently, despite several important attempts such as Aiyagari (1995), Conesa, Kitao, and Krueger (2009), and Dávila, Hong, Krusell, and Ríos-Rull (2012), some issues still remain unsettled.

In our view, there are two major issues in studying optimal tax policies in the HAIM framework that cannot be adequately addressed in a representative-agent infinite-horizon model: (i) What is the role of dynamic (in)efficiency in determining the sign of the optimal quantity of public debt in a HAIM economy? (ii) How does the sign of the optimal quantity of public debt influence/determine the sign and path of optimal taxes?

Regarding the first question, in a representative-agent framework, since the Ricardian equivalent principle holds and government bonds are neither net wealth nor net liabilities of the private sector (in the language of Barro (1974)), the choice between debt or taxes financing government expenditure is irrelevant. However, in an OLG framework and especially in OLG-HAIM models, government debt is no longer neutral for the economy and the key consideration of the Ramsey planner is thus fundamentally different, especially when precautionary saving is no longer equivalent to oversaving. Therefore, government bonds are not necessarily net wealth of the private sector despite incomplete credit markets and uninsurable idiosyncratic risk under borrowing constraints (a la Barro (1974) and Woodford (1990)). An important criterion for the determination of public debt becomes dynamic (in)efficiency. Consequently, optimal tax policies are also changed in a fundamental way, depending both on the sign and on the magnitude of the optimal quantity of public debt. This interdependence of public debt and tax policies is of great importance because on the one hand the bulk of theoretical literature justifies a positive quantity of public debt due to incomplete financial markets (e.g., Barro (1974), Woodford (1990), and, Aiyagari and McGrattan (1998)) and on the other hand the bulk of the empirical literature claims that the U.S. economy and the economies of other major OECD nations are dynamically efficient (e.g., Abel, Mankiw, Summers, and Zeckhauser (1989)).

Regarding the second question, analyzing the Ramsey problem in HAIM models is often technically challenging because of analytical tractability. In general, the Ramsey equilibrium in a HAIM model (such as Aiyagari (1994)) does not have closed-form solutions and the equilibrium allocation depends critically on the endogenous distribution of wealth, which makes the Ramsey problem non-trackable. Without tractability, it is hard to analyze and even understand the long-term property of the Ramsey outcome. Consequently, a Ramsey steady state is often assumed rather than proven in the existing literature. Yet, the optimal fiscal policies drawn from the analysis may hinge critically on the validity of such an assumption (see the analyses of Chen, Chien, and Yang (2019) and Chien and Wen (2019b)).

To overcome the tractability problem in determining the optimal debt and tax policy in an OLG-
HAIM framework, we adopt the model of Krueger and Ludwig (2018), which features uninsurable idiosyncratic uncertainty in the marginal product of labor of the old generation. In this model, a Ramsey steady state can be proven to exist, thanks to the model’s tractability. However, these authors do not consider the pivotal role of government bonds and the important issue raised by Barro (1974) and Woodford (1990), because their focus is exclusively on optimal capital taxation (in the absence of labor taxes and government debt). Our paper goes beyond their scope of analysis by studying the net-wealth effect of government debt in conjunction with optimal tax policies. In doing so, our analysis also sheds light on the role of government bonds in determining and shaping optimal tax policies, especially optimal capital taxation. In other words, whether the optimal capital taxation is zero or whether capital should be taxed or subsidized may depend on the availability of government bonds and on whether the optimal quantity of public debt is positive or negative.

The rest of the paper is organized as follows. Section 2 presents the model and characterizes the competitive equilibrium. Section 3 solves the Ramsey problem. The Ramsey equilibrium is characterized in two scenarios: one without idiosyncratic shocks and one with idiosyncratic shocks. Section 4 provides a more detailed literature review of closely related works. Section 5 concludes the paper.

2 The Model

2.1 Firms

Time is discrete and indexed by \( t = 0, 1, 2, 3, \ldots \). A representative firm produces output according to the constant-returns-to-scale Cobb-Douglas technology, \( Y_t = F(K_t, N_t) = K_t^\alpha N_t^{1-\alpha}, \) where \( Y, K, \) and \( N \) denote aggregate output, capital, and labor, respectively. The firm rents capital and hires labor from households by paying a competitive rental rate \( q_t \) and real wage \( w_t \). The firm’s optimal conditions for profit maximization at time \( t \) thus satisfy

\[
\begin{align*}
w_t &= \frac{\partial F(K_t, N_t)}{\partial N_t} \equiv MP_{N_t}, \\
q_t &= \frac{\partial F(K_t, N_t)}{\partial K_t} \equiv MP_{K_t}.
\end{align*}
\]

2.2 Households

In each period \( t \), a new generation of households is born and lives for two periods: \( t \) and \( t + 1 \). An old generation also exists in the very beginning of time \( t = 0 \). Thus, at any point in time \( t \geq 0 \),
there is a young and an old generation of households. The population size of each generation remains constant over time and is normalized to unity. The labor productivity of the young and the old are different, which are denoted by $\kappa^y$ and $\kappa^o$, respectively. Given the competitive wage rate $w_t$, while all households have the same constant labor productivity $\kappa^y$ (or wage income $\kappa^y w_t$ per hour) when young, there is an idiosyncratic shock to labor productivity when old, denoted by $\theta \sim \Psi (\theta)$, such that the wage income per hour for the old is given by $\theta_t \kappa^o w_t$. The mean value of the idiosyncratic shocks is normalized to unity:

$$\int \theta d\Psi = 1.$$  

Each new-born generation $t$ of households has the lifetime expected utility:

$$V_t = u(c^y_t) - v(n^y_t) + \beta \int (u(c^o_{t+1}) - v(n^o_{t+1}))d\Psi,$$

where $\beta \in (0,1)$ is the discount factor, $c^y_t$ and $n^y_t$ denote consumption and raw labor supply when young, respectively; similarly, $c^o_{t+1}$ and $n^o_{t+1}$ denote consumption and raw labor supply, respectively, when old in period $t+1$. Note that the optimal decisions of old households are affected by the idiosyncratic shock $\theta$. The budget constraints of generation-$t$ households when young and old are given, respectively, by

$$c^y_t + a_{t+1} = (1 - \tau^y_{n,t}) w_t \kappa^y n^y_t,$$  

$$c^o_{t+1} = a_{t+1} r_{t+1} + (1 - \tau^o_{n,t+1}) w_{t+1} \theta_{t+1} \kappa^o n^o_{t+1} + T_{t+1},$$

where $a_{t+1}$ is the savings of the young, and $\tau^y_{n,t}$ and $\tau^o_{n,t}$ denote the labor tax rates of young and old households, respectively. In addition, $r_t$ is the gross risk-free rate of the return to savings and $T_{t+1}$ denotes a lump-sum transfer (or tax if negative) received from the government when old. Notice that we allow lump-sum taxes/transfers only to the old and rule out for the young, which is equivalent to eliminating bequest motives in Barro’s original model to break the Ricardian equivalence.

Notice the tremendous consumption risk that the idiosyncratic shock $\theta$ imposes on households: No matter how much they opt to save when young, their savings $a_{t+1}$ will never be adequate to fully smooth their consumption when old—given that $t+1$ is the end period of life, consumption of the old must fluctuate together with total income because there are no savings or bequests to act as a buffer stock to absorb the income shock $\theta_{t+1}$ when old. Therefore, we allow for a lump-sum transfer $T_{t+1}$ to old households because it provides a certain amount of insurance to buffer the idiosyncratic income risk from labor productivity, which may help reduce the need for precautionary saving by the young. We will show how this idiosyncratic risk and the lump-sum insurance device $T_{t+1}$ affect
the net wealth of government bonds.

Hence, the utility-maximization problem of generation \( t \) is given by

\[
\max_{\{c_t^y, n_t^y, c_{t+1}^o, n_{t+1}^o, a_{t+1}\}} u(c_t^y) - v(n_t^y) + \beta \int (u(c_{t+1}^o) - v(n_{t+1}^o))d\Psi
\]

subject to (3) and (4). Notice that the aggregate effective labor units in the entire economy in each period \( t \) is given by

\[
N_t = \kappa^y n_t^y + \kappa^o \int \theta_t n_t^o d\Psi,
\]

which will be called (with slight abuse of terminology) the labor market clearing condition in the model.

The first-order conditions (FOCs) of generation-\( t \) households for \( \{c_t^y, n_t^y, c_{t+1}^o, n_{t+1}^o, a_{t+1}\} \) can be summarized by three equations:

\[
\begin{align*}
v_{n,t}^y &= u_{c,t}^y (1 - \tau_{n,t}^y) \kappa^y w_t, \\
v_{n,t+1}^o &= u_{c,t+1}^o (1 - \tau_{n,t+1}^o) \kappa^o \theta_{t+1} w_{t+1}, \\
u_{c,t}^y &= \beta r_{t+1} \int u_{c,t+1}^o d\Psi;
\end{align*}
\]

where \( u_{i,t}^i \) and \( v_{n,t}^i \) denote, respectively, the marginal utility of consumption and marginal disutility of labor for the young (\( i = y \)) cohort and old (\( i = o \)) cohort in period \( t \). The first two equations indicate that the marginal rate of substitution along the indifference curve for consumption and leisure equals the marginal rate of transformation along the budget constraint for the young and the old, respectively. The last equation indicates the households’ intertemporal optimality condition for saving—namely, the marginal-utility cost of forgone consumption when young equals the expected future marginal-utility gains under the stochastic discounting rate \( \beta r_{t+1} \). The Ramsey planner will respect the households’ FOCs when designing policies for public debt and taxes to improve the market allocations.

### 2.3 Government

The government in each period \( t \) can issue bonds, \( B_{t+1} \), collect lump-sum taxes or provide transfers to the currently old households in the amount \( T_t \), or levy a time-varying capital tax \( \tau_{k,t} \) and generation-specific labor taxes \( \{\tau_{n,t}^y, \tau_{n,t}^o\} \) to improve welfare. The budget constraint for the government in period \( t \) is given by

\[
\tau_{k,t} q_t K_t + \tau_{n,t}^y w_t \kappa^y n_t^y + \tau_{n,t}^o w_t \kappa^o \int \theta_t n_t^o d\Psi + B_{t+1} \geq T_t + r_t B_t, \text{ for all } t \geq 0,
\]
where the left-hand side is the source of government revenue and the right-hand side is the outlay of government expenditures.

There is no aggregate uncertainty in the economy, and government bonds and productive capital are perfect substitutes as stores of value for households. As a result, the after-tax gross rate of return to capital must equal the gross risk-free rate:

\[ r_t = 1 + (1 - \tau_t^k)q_t - \delta, \]

which constitutes a no-arbitrage condition for capital and bonds.

Note that the asset holdings of the initially old households in period 0 is exogenously given by \( a_0 = K_0 + B_0 \). In addition, for simplicity we assume that the generation-0 old households are not subject to any lump-sum taxes or transfers \( (T_0 = 0) \) and do not supply labor \( (n_0^o = 0) \). Therefore, their consumption is pinned down by their budget constraints: \( c_0^o = a_0 r_0 \).

### 2.4 Competitive Equilibrium

**Definition 1.** Given the initial capital \( K_0 \), the initial government bonds \( B_0 \), and the sequence of government policies \( \{\tau_{k,t}, \tau_{n,t}, \tau_{n,t+1}^o, T_{t+1}, B_{t+1}\}_{t=0}^\infty \), a competitive equilibrium is defined as the sequences of market prices \( \{w_t, r_t, q_t\}_{t=0}^\infty \), aggregate allocations \( \{C_t, N_t, K_{t+1}\}_{t=0}^\infty \), and individual allocation plans \( \{c_t^y, c_t^o, n_t^y, n_t^o, a_{t+1}\}_{t=0}^\infty \), such that the following hold:

1. Given the sequences of government policies \( \{\tau_{k,t}, \tau_{n,t}, \tau_{n,t+1}^o, T_{t+1}, B_{t+1}\}_{t=0}^\infty \) and market prices \( \{w_t, r_t\}_{t=0}^\infty \), the sequence \( \{c_t^y, c_{t+1}^o, n_t^y, n_{t+1}^o, a_{t+1}\}_{t=0}^\infty \) solves the household problem for all generations, given the initial conditions \( n_0^o = 0 \) and \( c_0^o = (K_0 + B_0)r_0 \).

2. Given the sequences of government policies \( \{\tau_{k,t}, \tau_{n,t}, \tau_{n,t+1}^o, T_{t+1}, B_{t+1}\}_{t=0}^\infty \) and market prices \( \{w_t, q_t\}_{t=0}^\infty \), the sequence of aggregate labor and capital \( \{N_t, K_t\} \) solves the representative firm’s problem for all period \( t \).

3. The households’ no-arbitrage condition holds: \( r_t = (1 - \tau_{k,t})q_t + 1 - \delta \) for all \( t \geq 0 \).

4. The government budget constraint holds for all \( t \geq 0 \):

\[ \tau_{k,t} q_t K_t + \tau_{n,t}^y w_t r_t^n y + \tau_{n,t}^o w_t r_t^o + \int \theta n_t^o d\Psi + B_{t+1} \geq T_t + r_t B_t, \]
5. All markets clear for all \( t \geq 0 \):

\[
\begin{align*}
\dot{a}_t & = B_{t+1} + K_{t+1} \\
N_t &= \kappa^y n_t^y + \kappa^o \int \theta n_t^o d\Psi \\
C_t &= c_t^y + \int c_t^o d\Psi \\
F(K_t, N_t) &= C_t + K_{t+1} - (1 - \delta) K_t,
\end{align*}
\]

where the first equation is the asset market clearing condition, the second is the effective labor’s market clearing condition, the third the identity of aggregate consumption, and the last the aggregate goods market clearing condition.

3 Ramsey Problem

Note that the competitive equilibrium defined above is a function of the path of government policies \( \{\tau_{k,t}, \tau_{n,t}^y, \tau_{n,t+1}^o, T_{t+1}, B_{t+1}\}_{t=0}^\infty \). Namely, each different path of government policies corresponds to a different competitive equilibrium. The Ramsey problem is to select a particular path of government policies such that the corresponding competitive equilibrium yields the maximum social welfare.

We use the primal approach to solve the Ramsey problem. Under the primal approach, we first substitute out all market prices and policy variables by using a subset of the competitive equilibrium’s FOC conditions, and then choose the path of resource allocation \( \{c_t^y, c_{t+1}^o, n_t^y, n_{t+1}^o, C_t, N_t, K_{t+1}\}_{t=0}^\infty \) to maximize social welfare subject to the rest of the equilibrium conditions. This implies that the more policy tools the Ramsey planner has, the smaller the number of constraints the planner is subject to in solving the Ramsey problem. The solution under such a primal approach is called a Ramsey allocation (outcome) or a Ramsey plan.\(^6\)

3.1 Conditions to Support a Competitive Equilibrium

To ensure that a Ramsey plan constitutes a competitive equilibrium, we must show first that all possible allocations in the choice set of the Ramsey planner, \( \{c_t^y, c_{t+1}^o, n_t^y, n_{t+1}^o, C_t, N_t, K_{t+1}\}_{t=0}^\infty \) (after substituting out all market prices and policy variables but before solving the Ramsey maximization problem), constitute a competitive equilibrium. The following proposition states the

\(^6\)For example, we have five policy variables and a total of eleven equilibrium conditions to solve for a competitive equilibrium. After using up eight equilibrium conditions to substitute out the five policy variables and three market prices, we have three conditions left (plus one government budget constraint) to serve as constraints for the Ramsey planner in determining the Ramsey plan.
conditions that the constructed allocations \(\{c^y_t, c^o_{t+1}, n^y_t, n^o_{t+1}, C_t, N_t, K_{t+1}\}_{t=0}^\infty\) must satisfy in order to constitute a competitive equilibrium.

**Proposition 2.** Given the initial capital \(K_0\), government bonds \(B_0\), and capital tax \(\tau_{k,0}\), the sequence of allocations \(\{c^y_t, c^o_{t+1}, n^y_t, n^o_{t+1}, C_t, N_t, K_{t+1}\}_{t=0}^\infty\) can be supported as a competitive equilibrium if and only if the following conditions hold:

1. The aggregate resource constraint holds \((t \geq 0)\):

\[
F(K_t, \kappa^y n^y_t + N^o_t \kappa^o) - c^y_t - C^o_t - K_{t+1} + (1 - \delta)K_t \geq 0. \tag{5}
\]

2. The budget constraint for each old household is satisfied \((t > 0)\):

\[
\frac{v^o_n t+1}{u^o c, t+1} n^o_{t+1} - \int \frac{v^o n, t+1}{u^o c, t+1} d\Psi \times N^o_{t+1} \geq c^o_{t+1} - C^o_{t+1}. \tag{6}
\]

3. The old households’ FOCs are satisfied \((t > 0)\):

\[
v^o_n t+1 = u^o c, t+1 \theta_{t+1} \int \frac{v^o n, t+1}{u^o c, t+1} d\Psi. \tag{7}\]

4. The implementability condition holds \((t \geq 0)\):

\[
u^y c, t+1 - v^y n, t+1 n^y_t + \beta U^o C, t+1 \left[ C^o_{t+1} - \int \frac{v^o n, t+1}{u^o c, t+1} d\Psi \times N^o_{t+1} \right] \geq T_{t+1} \beta U^o C, t+1. \tag{8}\]

where \(N^o_t, C^o_t, \) and \(U^o C, t\) are defined as \(N^o_t \equiv \int \theta n^o_t d\Psi, C^o_t \equiv \int c^o_t d\Psi, \) and \(U^o C, t \equiv \int u^o c, t d\Psi, \) respectively. In addition, the aggregate consumption \(C_t\) and labor \(N_t\) can be chosen to satisfy the equilibrium conditions: \(N_t = \kappa^y n^y_t + N^o_t \kappa^o \) and \(C_t = c^y_t + C^o_t.\)

5. Finally, the equilibrium conditions also imply that the choice of government bonds must satisfy the asset market clearing condition \((t \geq 0)\):

\[
B_{t+1} = \frac{v^y n, t}{u^y c, t} n^y_t - c^y_t - K_{t+1}. \tag{9}\]

**Proof.** See Appendix A.1. \(\Box\)

\(^7\)The initial capital tax rate, \(\tau_{k,0}\), should be a choice variable for the Ramsey planner. For the sake of simplicity, we restrict the planner’s ability to choose \(\tau_{k,0}\) in the Ramsey problem.
Note that the last two conditions (8) and (9) are redundant because under the primal approach we have already used these conditions to substitute out the policy variables \( T_{t+1} \) and \( B_{t+1} \). This implies that, in the Ramsey problem (studied in the next subsection), the Lagrangian multipliers associated with these two conditions will be exclusively zero. As we will show later, however, adding these two conditions into the set of constraints in the Ramsey problem has an advantage when we study the net wealth of government bonds in special cases where lump-sum taxes and/or transfer are not available, or when the government’s capacity to issue debt or to lend is limited. In such special cases (\( T_{t+1} = 0 \) or \( T_{t+1} \geq 0 \), and \( B_{t+1} = 0 \) or \( B_{t+1} \leq 0 \)), the conditions (8) and (9) must be imposed on (respected by) the Ramsey planner and hence their Lagrangian multipliers are no longer zero. So for convenience, we replace the condition (9) by the following two-sided constraints:

\[
B \geq B_{t+1} \geq v_{n,t} - c_t - K_{t+1} \geq B
\]  

(10)

where \( B \geq 0 \) and \( B \leq 0 \). Notice that when government bonds are negative (\( B_{t+1} < 0 \)), it implies that the government is lending credit to the private sector (or holding privately issued debt).

Moreover, if there is no idiosyncratic risk, conditions (6) and (7) are trivially satisfied (they become nil identity \( 0 = 0 \)) and do not impose any restrictions on the Ramsey planner. In short, the setup in Proposition 2 is general enough and offers the convenience of investigating the Ramsey outcomes under several special scenarios.

3.2 Optimal Ramsey FOCs

Since \( C_t = c_t^y + \int c_t^y d\Psi \) and \( N_t = \kappa^y n_t^y + \kappa^o \int \theta n_t^o d\Psi \), the choice set of the Ramsey planner can be simplified from a sequence of seven variables to a sequence five variables: \( \{K_{t+1}, c_t^y, n_t^y, c_{t+1}^o, n_{t+1}^o\} \). Then by Proposition 2, the Ramsey problem under the primal approach is given by

\[
\max_{\{K_{t+1}, c_t^y, n_t^y, c_{t+1}^o, n_{t+1}^o\}} \sum_{t=0}^{\infty} \omega_t \left( u(c_t^y) - v(n_t^y) + \beta \int (u(c_{t+1}^o) - v(n_{t+1}^o)) d\Psi \right) 
\]

subject to the four constraints (5)-(8) listed in Proposition 2 together with the debt-limit constraints (10). In addition, \( \omega_t \) in the planner’s social-welfare function denotes the weight of generation \( t \). The solution to this maximization problem is called a Ramsey plan or a Ramsey allocation (outcome).

Let \( \beta^t \mu_t, \beta^{t+1} \lambda^1_{t+1}, \beta^{t+1} \lambda^2_{t+1}, \) and \( \beta^t \phi_t \) denote, respectively, the Lagrangian multipliers for the first four conditions in Proposition 2—which represent (i) the resource constraint, (ii) old

---

8By our simplifying assumption, the period-0 old households supply no labor and their consumption is exogenously given to the Ramsey planner by \( c_0^y = (K_0 + B_0)(1 - \tau_{k,0})q_0 + 1 - \delta \).
households’ budget constraint, (iii) old households’ FOCs, and (iv) the implementability condition. Denote $\beta_t \eta^H_t$ and $\beta_t \eta^L_t$ as the Lagrangian multipliers for the upper and lower debt-limit constraints in condition (10). Hence, a Ramsey allocation is characterized by these constraints together with the following five Ramsey FOCs with respect to $\{K_{t+1}, c_{t+1}^y, n_{t+1}^y, c_{t+1}^o, n_{t+1}^o\}$:

\[
\mu_t + \eta_t^L - \eta_t^H = \beta_{t+1}(MP_{K,t+1} + 1 - \delta),
\]

\[
\omega_t u_{c,t}^y + \beta_t \phi_t (u_{c,t}^y c_{t}^y + u_{c,t}^y) + \beta_t (\eta_t^H - \eta_t^L)(1 + \frac{v_{n,t}^y n_{t}^y}{(u_{c,t}^y)^2 u_{c,t}^y}) = \beta_t \mu_t,
\]

\[
\omega_t v_{n,t}^y + \beta_t \phi_t (v_{n,t}^y n_{t}^y + v_{n,t}^y) + \beta_t \phi_t \left(\frac{\eta_t^H - \eta_t^L}{u_{c,t}^y}\right) (v_{n,t}^y n_{t}^y + v_{n,t}^y) = \beta_t \mu_{t+1} MP_{N,t},
\]

\[
\omega_t \beta u_{c,t+1}^o + \beta_t^1 \phi_t u_{c,t+1}^o \left[ C_{t+1}^o - \int \frac{v_{n,t+1}^o}{u_{c,t+1}^o} d\Psi \times N_{t+1}^o \right] + \beta_t^1 \phi_t U_{C,t+1}^o \left[ 1 + \frac{v_{n,t+1}^o}{u_{c,t+1}^o} \right] \times N_{t+1}^o \times N_{t+1}^o
\]

\[
= \beta_t^1 \mu_{t+1} + \beta_t^1 u_{c,t+1}^o \theta_{n,t+1} \zeta_{t+1},
\]

and

\[
\omega_t \beta v_{n,t+1}^o + \beta_t^1 \phi_t U_{C,t+1}^o \left( \frac{v_{n,t+1}^o n_{t+1}^o}{u_{c,t+1}^o} N_{t+1}^o + \int \frac{v_{n,t+1}^o}{u_{c,t+1}^o} d\Psi \times \theta_{t+1} \right)
\]

\[
= \beta_t^1 \mu_{t+1} MP_{N,t+1} \theta_{t+1} + \beta_t^1 v_{n,t+1}^o \zeta_{t+1},
\]

where $\zeta_{t+1}$ is defined as

\[
\zeta_{t+1} \equiv \left[ \lambda_{t+1}^1 \left( n_{t+1}^o - \frac{1}{\theta_{t+1}} \int \theta n_{t+1}^o d\Psi \right) + \lambda_{t+1}^2 (1 - \theta_{t+1}) \right].
\]

Recall that the multipliers $\{\lambda^1, \lambda^2, \phi\}$ are possibly zero in the different scenarios considered below.

### 3.3 Characterization of the Ramsey Allocation

With some abuse of terminology and repetition, we define aggregate allocative efficiency (AAE) as a competitive-equilibrium allocation in which the MGR holds, and individual allocative efficiency (IAE) as a competitive-equilibrium allocation in which a young household’s marginal utility of consumption equals his/her marginal utility of consumption when old—which also implies that the market interest rate $r$ equals the time discount rate $1/\beta$ if there is no idiosyncratic uncertainty.
Similarly, we define the expected IAE (EIAE) as a competitive-equilibrium allocation in which the marginal utility of consumption when young equals the expected marginal utility of consumption when old across idiosyncratic states.

In addition, we define the first-best allocation as the optimal solution by a social planner that maximize the social-welfare function (11) subject only to the aggregate resource constraint (5). Notice that the social planner’s solution may not constitute a competitive equilibrium because it ignores the other constraints in Proposition 2. We can, however, show easily that the first-best allocation implies AAE and IAE (as well as EIAE).

To facilitate our analysis, we impose a set of standard simplifying assumptions below:

**Assumption 3.** Assume the following:

1. The social welfare weight of generation \( t \) is set to be \( \beta^t \); that is, \( \omega_t = \beta^t \).

2. The utility function exhibits constant relative risk aversion:

\[
\frac{u_{cc}}{u_c} = \gamma_c < 0 \\
\frac{v_{nn}}{v_n} = \gamma_n > 0.
\]

With these conceptual definitions in mind, we can define a Ramsey steady state as follows:

**Definition 4.** A Ramsey steady state is a long-run Ramsey allocation where all variables \( \{K, N, C, c^y, c^o, n^y, n^o\} > 0 \) and the optimal level of government debt \( B \) converge to finite constants.

In what follows, we derive the Ramsey allocation under several special scenarios in order to gain insights and intuitions behind the net-wealth effects of government bonds and implied taxes. We consider two scenarios: (i) with no idiosyncratic uncertainty and (ii) with idiosyncratic uncertainty. In each scenario, we investigate how the net wealth of government bonds and optimal tax scheme would change according to the condition of dynamic (in)efficiency and the availability of different government policy tools.

### 3.3.1 Scenario 1: Ramsey Allocation without Idiosyncratic Uncertainty

Scenario 1 is also the benchmark model studied by Barro (1974) without bequest motives and imperfect capital markets. In this scenario the optimal Ramsey FOCs are dramatically simplified since the conditions (6) and (7) are trivially satisfied and do not impose any restrictions on the Ramsey planner; consequently, the multipliers \( \lambda^1_{t+1} \) and \( \lambda^2_{t+1} \) are zero. Also, as discussed previously, the multipliers associated with the conditions (8) and (10), \( \{\phi_t, \eta^H_t, \eta^L_t\} \), are zero if there are no restrictions on \( T_{t+1} \) and \( B_{t+1} \) and non-zero if there are restrictions.
Proposition 5. The Ramsey steady state in the absence of idiosyncratic uncertainty features the following properties:

1. If the debt constraints (10) do not bind (the absolute values of $|B|$ and $|\bar{B}|$ are sufficiently large) and a lump-sum tax/transfer to old households is feasible and unconstrained, the Ramsey steady state achieves the first-best allocation, which exhibits both AAE and IAE. All distortionary taxes (capital and labor taxes for the young and old) are zero. In addition, if the capital elasticity parameter satisfies $\alpha > 1/\beta - 1 + \delta$ (a sufficient condition for dynamic efficiency), the optimal quantity of public debt (or the net wealth of government bonds) is negative ($B < 0$) and the lump-sum transfer is positive ($T > 0$).

2. If the debt constraints (10) do not bind (the absolute values of $|B|$ and $|\bar{B}|$ are sufficiently large) and $T = 0$ (lump-sum taxes/transfers are not feasible), the Ramsey steady state still achieves both AAE and IAE. Moreover, the optimal quantity of public debt (net wealth of government bonds) is negative ($B < 0$) if the capital elasticity parameter satisfies $\alpha > 1/\beta - 1 + \delta$ (a sufficient condition for dynamic efficiency). The optimal labor taxes are negative and identical across age cohorts ($\tau^y_n = \tau^0_n < 0$), but the capital tax remains zero ($\tau^k = 0$).

3. If the government debt limit binds on either end of the debt-limit constraints (10), in the Ramsey steady state neither the MGR nor IAE holds regardless of the availability of lump-sum taxes/transfers $T$. The steady-state capital tax is not zero. Notice that even though it is feasible for the Ramsey planner to use a capital tax (subsidy) to achieve the MGR, it is not optimal to do so.

Proof. See Appendix A.2. □

Result (1) in Proposition 5 should not be confused with Ricardian equivalence. Although the Ramsey planner can achieve the first-best allocation featuring MGR and IAE, it is because government bonds have played the role of lump-sum taxes/transfers to young households in addition to lump-sum taxes/transfers ($T$) to old households. In other words, the demand for (supply of) bonds by young households acts as a lump-sum tax (or transfer) that transfers resources without distortions between the young and the old. In the Ramsey steady state, the interest cost of public debt is financed also by a lump-sum tax on the old ($T < 0$). If the public debt is negative, then the government receives interest payments from young households and provides lump-sum transfers to the old ($T > 0$).

Precisely because of the need to finance public debt (or lend credit to young households if $B < 0$), the second result in Proposition 5 shows that the labor tax is positive/negative for the
old if a lump-sum tax/transfer to the old is not feasible. This result is reminiscent of the classic result found in the literature (see, e.g., Judd (1985) and Chamley (1986)) that it is not optimal to tax capital when taxing labor is feasible. The key difference here is that labor taxation is a way to finance the interest on government bonds instead of exogenous government expenditures. Of course, when the public debt is negative, the government uses the interest payments received from the private sector to subsidize labor.

The third result in Proposition 5 clearly demonstrates the importance of government bonds (or credits) in improving social welfare. It highlights the critical role of government bonds (or government credit) in achieving the MGR and IAE in an OLG framework under dynamic (in)efficiency. It shows that whenever the Ramsey planner cannot fully utilize government bonds as a tool to achieve IAE, the MGR must also fail to hold, although it is still feasible to achieve the MGR by taxing or subsiding capital. Hence, the conventional wisdom (see Aiyagari (1995)) that it is optimal to tax capital to achieve the MGR does not apply in our model (also see the closely related work of Chien and Wen (2019b)).

Armed with these results and insights gained from Proposition 5, we are now ready to study the Ramsey plan under idiosyncratic uncertainty and incomplete financial markets.

### 3.3.2 Scenario 2: Ramsey Allocation with Idiosyncratic Uncertainty

**Proposition 6.** *Under uninsurable idiosyncratic risk for labor productivity in old age, a Ramsey steady state has the following properties:*

1. *Without any constraints on the government’s capacity to issue debt (or lend credit), if a lump-sum tax/transfer to old households is feasible and unconstrained, the Ramsey steady state achieves the MGR but not IAE (nor EIAE), and the tax/subsidize rates on capital and old-household labor income are non-zero despite lump-sum taxes/transfers to old households. Furthermore, the labor tax/subsidize on young households is zero. In addition, the optimal quantity of public debt is negative if the capital elasticity parameter satisfies* \( \alpha > \frac{1}{\beta} - 1 + \delta \) *(a sufficient condition for dynamic efficiency).*

2. *Without any constraints on the government’s capacity to issue debt (or lend credit), if a lump-sum tax/transfer to old households is not feasible, the Ramsey steady state achieves the MGR but not IAE (nor EIAE), and the taxes on capital and households’ labor income are all non-zero regardless of age. In addition, the optimal quantity of public debt is negative if the capital elasticity parameter satisfies* \( \alpha > \frac{1}{\beta} - 1 + \delta \) *(a sufficient condition for dynamic efficiency).*
3. Under a binding constraint on the government’s capacity to issue debt (i.e., $B = 0$), neither the MGR nor IAE or EIAE can be achieved in the Ramsey steady state regardless of the availability of lump-sum taxes/transfers to old households.

Proof. See Appendix A.3.

This proposition shows that precautionary saving motives do not necessarily lead to overaccumulation of capital from the viewpoint of the Ramsey planner. For the benevolent planner the aggregate capital stock is overaccumulated if and only if the competitive equilibrium is dynamically inefficient. Therefore, the optimal quantity of public debt can be negative despite shortages of private liquidity under imperfect credit markets and uninsurable income risk. In other words, in contrast to the claims of Barro (1974) and Woodford (1990), government bonds are not necessarily net wealth of the private sector (but are instead their net liability) even if financial markets are incomplete and idiosyncratic income risk cannot be effectively diversified by private credit markets.

This result, nonetheless, is fully consistent with the conventional wisdom (see Aiyagari and McGrattan (1998)) that the optimal level of government debt can be sufficiently high in an infinite-horizon HAIM economy. The reason is that in such an economy the market interest rate lies strictly below the time discount rate and the laissez faire equilibrium is dynamically inefficient.

Our result also suggests that dynamic inefficiency is not necessarily the correct rationale to justify a positive capital tax, as long as the government’s capacity to issue debt is unconstrained, in contrast to the argument of Aiyagari (1995) but consistent with the argument of Chien and Wen (2019b).

4 A Brief Literature Review

In a seminal paper, Barro (1974) studied the conditions under which government bonds are perceived by the private sector as net wealth—i.e., situations where the value of government bonds exceeds the cost of financing them in terms of the future tax burden on the private sector. In particular, Barro (1974) studied three scenarios in which government bonds are net wealth to the private sector because they can improve welfare: (i) the consumer life span is finite (as in an OLG economy) and voluntary intergenerational transfers of resources are ruled out or hindered; (ii) consumers within the same age-cohort are heterogeneous in their time discounting factor and the credit market is imperfect for competitive lending and borrowing; and (iii) government bonds can provide better liquidity services than private debt. Our results in this paper provide an explanation for Barro’s finding because he implicitly assumes that the competitive equilibrium in his model is dynamically inefficient. Otherwise, he would be likely to find government bonds to be the net
liability of the private sector instead of net wealth even if private credit markets are imperfect or
government bonds can provide better liquidity services than private debt.

In a similar spirit, Woodford (1990) shows in an infinite-horizon model with heterogeneous
agents and uninsurable individual income risk that public debt can improve welfare if the market
interest rate lies below the time discount rate because of liquidity constraints. But similar to
Barro (1974), Woodford (1990) does not study optimal tax policy in conjunction with optimal
debt policy.

The work of Aiyagari (1995) is the first attempt at investigating optimal Ramsey taxation
in HAIM economies. Under the assumption of the existence of an interior Ramsey state steady,
Aiyagari (1995) shows that the Ramsey planner opts to restore the MGR by taxing capital in the
steady state even though a labor tax is also available. However, Chien and Wen (2019b) show
that the above intuition for justifying positive capital income taxation is counterintuitive and not
necessarily correct in general. They argue that by taxing capital income in the steady state and
thus permanently reducing individuals’ optimal buffer stock of savings, the government is effectively
destroying their ability to self-insure against idiosyncratic risks when lump-sum transfers are not
feasible (as is assumed in Aiyagari’s (1995) analysis). Since taxing capital per se does not directly
address the lack-of-insurance problem for households (if anything, it intensifies the problem), Chien
and Wen (2019b) argue that a positive capital tax in the steady state is never optimal if other forms
of distortionary taxation (such as a labor tax) are feasible. The result of this paper shows that
government bonds play the most critical role in addressing the capital overaccumulation problem,
which is consistent with the findings of Chien and Wen (2019b).

Aiyagari and McGrattan (1998) build on the Aiyagari (1994) model to determine the optimal
quantity of debt by studying the trade-offs in benefits and costs of varying the quantity of debt.
Their analysis mainly focus on steady-state welfare through numerical methods under a critical
assumption that the proportional tax rates on labor and capital income are levied equally across
households to finance public debt. This assumption rules out the possibility that the optimal
capital tax may be zero. Our results provide an explanation of their finding because in their model
the interest rate lies below the time discount rate and a competitive equilibrium is dynamically
inefficient.

Barro (1979), Lucas and Stokey (1983), and Aiyagari, Marcet, Sargent, and Seppala (2002)
study the determination of optimal debt under aggregate uncertainty in representative-agent mod-
els without capital. Chien and Wen (2019a) follow this literature by introducing aggregate uncer-
tainty into the model of Chien and Wen (2019b) and analytically study the problem of optimal
debt determination without capital. They show in closed forms that the optimal tax rate as well
as the level of risk-free debt would follow an endogenously bounded stochastic unit-root process—
bounded below by the Ramsey planner’s desire to provide full self-insurance and above by the
government’s natural borrowing limit. This endogenous lower bound on optimal debt emerges be-
cause of the Ramsey planner’s dominant incentive to issue more debt whenever the interest rate lies
below the time discount rate—until a full self-insurance allocation is reached unless prevented by a
government borrowing limit. However, if government bonds are state contingent, then the long-run
Ramsey equilibrium exhibits constant taxes in the absence of a government borrowing limit and
stochastic taxes if the borrowing limit binds. In either case, the Ramsey planner’s dominant incen-
tive is to keep increasing public liquidity (debt) to meet the self-insurance needs of households even
though it may create labor-tax distortions and crowd out capital—precisely because an interest
rate below the time discount rate renders the marginal benefit of increasing the debt larger than
the marginal cost of doing so when households are borrowing constrained.

Angeletos, Collard, and Dellas (2016) study the Ramsey policy problem in the Lagos and
Wright (2005) framework with HAIM properties. They show that when risk-free government bonds
contribute to the supply of liquidity to alleviate private agents’ borrowing constraints, issuing more
debt raises welfare by improving the allocation of resources. Our results provide an explanation of
their finding because in their model the market interest rate lies below the time discount rate and
hence a competitive equilibrium is dynamically inefficient.

Bassetto (2014) uses a model with heterogeneous agents to show that when tax liabilities are
unevenly spread across the population, deviations from tax smoothing lead to welfare-improving
interest rate changes that redistribute wealth. In particular, when a “bad shock” hits the economy,
the optimal policy will call for smaller or larger deficits depending on the political power of different
groups. For example, Bassetto shows that his model can explain why England relied heavily on
debt to finance its wars, while France made heavy use of temporary tax increases: England had
tax-paying merchants with political power, while France had landlords with political power.

Bassetto and Kocherlakota (2004) show that the paths of government debt can be irrelevant
under distortionary taxes. In particular, they show that if the government collects taxes in a given
period based only on incomes earned in previous periods, then the government can adjust its tax
policy so as to attain any debt path without affecting equilibrium allocations or prices.

Bhandari, Evans, Golosov, and Sargent (2017) study the optimal portfolio choices of the Ramsey
planner under aggregate uncertainty. They develop a solution method that uses second-order
approximations of Ramsey policies to obtain formulas for conditional and unconditional moments
of government debt and taxes that include means and variances of the invariant distribution as
well as speeds of mean reversion. They show that asymptotically the planner’s portfolio minimizes
a measure of fiscal risk. In the model calibrated to U.S. data, the optimal target debt level is
slightly below zero, the invariant distribution of debt is very dispersed, and the mean reversion is
slow (with a half-life of nearly 250 years).

Erosa and Gervais (2002) and Garriga (2019) both study the Ramsey problem in OLG models and focus mainly on the issue of optimal capital taxation. They show that the classical result of a zero capital tax still holds in life-cycle economies provided that labor tax rates can be varied according to age.

Compared with the aforementioned literature, our model framework is more closely related to Krueger and Ludwig (2018). But that paper also focuses exclusively on optimal capital taxation (in the absence of labor taxes and government debt).

5 Conclusion

The rapidly growing national debt in the U.S. since the 1970s has alarmed and intrigued the academic world. Consequently, the concept of dynamic (in)efficiency in an OLG world and the importance of the HAIM hypothesis to justify a high debt-to-GDP ratio have been extensively studied. Two important consensus views have emerged from this literature: (i) The optimal quantity of public debt is positive if private liquidity is insufficient to support private saving and investment demand, ans (ii) the optimal capital tax is positive—because of precautionary saving and the consequent failure of the modified golden rule. But these two views are seldom derived jointly in the same model; consequently, the dynamic relationship between optimal debt and optimal tax remains unclear, especially considering that the optimal quantity of debt must be judged by the golden-rule saving rate and any debt must be financed by future taxes.

In this paper we use a primal Ramsey approach to analytically determine the net-wealth effect of government bonds in an OLG-HAIM model with capital. We show that even if private liquidity is insufficient for providing an adequate buffer to smooth consumption and precautionary holding of public debt can improve self-insurance against idiosyncratic income risk, government bonds are still not necessarily net wealth to households. The reason is that precautionary saving and oversaving are not necessarily the same thing, and they may have very different policy implications—i.e., the Ramsey planner may opt to issue bonds to crowd out capital or private savings if and only if the economy is dynamically inefficient regardless of precautionary saving motives and market incompleteness. In other words, optimal debt can be negative even if households cannot insure themselves against idiosyncratic risk due to borrowing constraints and would opt to hold government bonds in a competitive equilibrium, so long as the market interest rate exceeds the time discount rate (dynamic efficiency).

More importantly, since the sign and magnitude of the optimal quantity of debt dictate the sign and magnitude of optimal taxes as well as the priority order of tax tools such as a labor tax vs.
capital tax, we argue that optimal tax policy and optimal debt policy must be studied jointly. Yet the existing literature often studies these two types of fiscal policy separately.

A formidable challenge to studying optimal debt policy and optimal tax policy jointly in a HAIM framework is the intractability of Ramsey problem. We overcome this difficulty by using a two-period OLG-HAIM model that allows us to solve the Ramsey problem analytically. Our result reinforces the recent finding of Chien and Wen (2019b) that an interior Ramsey steady state may not exist in an infinite-horizon HAIM economy, because a benevolent government has the incentive to keep increasing the level of public debt indefinitely as long as the market interest rate lies below the time discount rate (or as long as the economy remains dynamically inefficient).

Since the criterion for determining the public debt in the long run is dynamic (in)efficiency, optimal tax policies must also be studied accordingly, as they depend both on the sign and on the magnitude of the optimal quantity of public debt. Calling attention to this interdependence of public debt and tax policies is of great importance: On the one hand the bulk of the theoretical literature justifies a positive quantity of public debt by incomplete financial markets (e.g., Barro (1974), Woodford (1990), Aiyagari and McGrattan (1998)). On the other hand the bulk of the empirical literature claims that the U.S. economy and the economies of other major OECD nations are dynamically efficient (e.g., Abel, Mankiw, Summers, and Zeckhauser (1989)).
References


A Appendix

A.1 Proof of Proposition 2

The “If” Part: Given the initial $B_0$ and $K_0$ as well as the allocation $\{c_t^y, c_t^o, n_t^y, n_t^o, K_{t+1}\}_{t=0}^\infty$, a competitive equilibrium can be constructed by using the 5 conditions in Proposition 2 and following the steps below that uniquely back up the sequences of the 11 variables, $\{N_t, C_t, a_{t+1}\}_{t=0}^\infty$, $\{w_t, q_t, r_t\}_{t=0}^\infty$, and $\{\tau_{k,t}, \tau_{n,t}, \tau_{o,t}, T_{t+1}, B_{t+1}\}_{t=0}^\infty$, respectively:

1. Given $n_0^y = 0$, the aggregate labor $N_t$ is determined by the labor market clearing condition:

$$N_t = \kappa^y n_t^y + \kappa^o \int \theta n_t^o d\Psi.$$

2. The aggregate consumption $C_t$ is determined by the consumption market clearing condition:

$$C_t = c_t^y + \int c_t^o d\Psi.$$

3. Given $\{N_t, K_t\}_{t=0}^\infty$, the prices $w_t$ and $q_t$ are determined by the firm’s FOCs for all $t \geq 0$:

$$w_t = MP_{N,t}$$
$$q_t = MP_{K,t}.$$

4. Given $\{c_t^y, c_{t+1}^o\}$, the two variables $\{r_{t+1}, \tau_{k,t+1}\}$ are determined by

$$r_{t+1} = \frac{\beta \int u_c(c_{t+1}^o) d\Psi}{\kappa^y u_c(c_t^y)}$$
$$\tau_{k,t+1} = 1 - \frac{r_{t+1} - 1 + \delta}{q_{t+1}}.$$

In addition, $r_0$ and $c_0^y$ are determined by $r_0 = (1 - \tau_{k,0})q_0 + 1 - \delta$ and $c_0^y = (K_0 + B_0)r_0$, respectively.

5. The labor tax rates are determined by

$$(1 - \tau_{n,t}^y) = \frac{v_n(n_t^y)}{u_c(c_t^y)w_t^y\kappa^y},$$
$$(1 - \tau_{n,t+1}^o) = \frac{1}{w_{t+1}^y\kappa^o} \int \frac{v_n(n_{t+1}^o)}{u_c(c_{t+1}^o)} d\Psi.$$
6. Household savings \( a_{t+1} \) are determined by the young households’ budget constraint:

\[
a_{t+1} = \frac{v_n(n_t^y)}{u_c(c_t^y)} n_t^y - c_t^y.
\]

7. Bond holdings \( B_{t+1} \) are determined by the asset market clearing condition: \( B_{t+1} = a_{t+1} - K_{t+1} = \frac{v_n(n_t^y)}{u_c(c_t^y)} n_t^y - c_t^y - K_{t+1} \).

8. The lump-sum tax/transfer to old households \( T_t \) is determined by the implementability condition. We derive the implementability condition by plugging the aggregate resource constraint into the government’s budget constraint:

\[
-(1 - \tau_{k,t})q_t K_t - (1 - \tau_{n,t}^y)w_t \kappa^y n_t^y - (1 - \tau_{n,t}^o)w_t \kappa^o \int \theta_t n_t^o d\Psi + B_{t+1} - r_t B_t + c_t^y + \int c_t^o(\theta)d\Psi + K_{t+1} - (1 - \delta)K_t \geq T_t.
\]

We then substitute out taxes and prices in the above equation to get

\[
-(1 - \tau_{k,t})q_t K_t - \frac{v_{n,t}^y}{u_{c,t}^y} n_t^y - \int \frac{v_n(n_t^o)}{u_c(c_t^o)} d\Psi \int \theta_t n_t^o d\Psi + \frac{v_{n,t}^y}{u_{c,t}^y} n_t^y - c_t^y - K_{t+1} - \frac{u_{c,t}^y}{\beta} \int u_{c,t}^o d\Psi (\frac{v_{n,t-1}^y}{u_{c,t-1}^y} n_{t-1}^y - c_{t-1}^y - K_t) + c_t^y + \int c_t^o(\theta)d\Psi + K_{t+1} - (1 - \delta)K_t \geq T_t,
\]

which can be simplified to

\[
\frac{u_{c,t-1}^y}{\beta} \int u_{c,t}^o d\Psi c_{t-1}^y - \frac{v_{n,t-1}^y}{\beta} \int u_{c,t}^o d\Psi n_{t-1}^y + \int c_t^o d\Psi - \int \frac{v_n(n_t^o)}{u_c(c_t^o)} d\Psi \int \theta_t n_t^o d\Psi \geq T_t.
\]

Rearranging terms, the above equation becomes

\[
u_{c,t-1}^y c_{t-1}^y - v_{n,t-1}^y n_{t-1}^y + \beta \int u_{c,t}^o d\Psi \int c_t^o d\Psi - \beta \int u_{c,t}^o d\Psi \int \frac{v_n(n_t^o)}{u_c(c_t^o)} d\Psi \int \theta_t n_t^o d\Psi \geq T_t \beta \int u_{c,t}^o d\Psi,
\]

which gives equation (8) by updating to period \( t \).

9. The above steps leave the following three constraints to the Ramsey planner in determining the optimal policy mix:

i. The aggregate resource constraint can be rewritten as equation (5) by using the labor and consumption market clearing conditions.
ii. The old households’ FOC is expressed as equation (7) after substituting out the labor tax for old households.

iii. By substituting out the lump-sum transfer $T_{t+1}$ using the government budget constraint, the old households’ budget constraint is rewritten as

$$
\tau_{k,t+1}q_{t+1}K_{t+1} + \tau_{n,t+1}^y w_{t+1}^y \kappa_{t+1}^y n_{t+1}^y + \tau_{n,t+1}^o w_{t+1}^o + \int \theta_{t+1} n_{t+1}^o d\Psi + B_{t+2} - r_{t+1} B_{t+1}
$$

which can be further rewritten as

$$
q_{t+1} K_{t+1} + (1 - \delta) K_{t+1} + w_{t+1} N_{t+1} - (1 - \tau_{n,t+1}^y) \kappa_{t+1}^o w_{t+1} n_{t+1}^y
$$

$$
- (1 - \tau_{n,t+1}^o) w_{t+1} n_{t+1}^o \int n_{t+1}^o d\Psi + B_{t+2}
$$

$$
\geq c_{t+1}^o - a_{t+1} r_{t+1} - \frac{v_n(n_{t+1}^o)}{u_c(c_{t+1}^o)} n_{t+1}^o,
$$

By substituting out labor taxes and using the aggregate resource constraint in the above expression, we arrive at the old households’ budget constraint:

$$
c_{t+1}^y + \int c_{t+1}^o(\theta) d\Psi + K_{t+2} - \frac{v_n(n_{t+1}^y)}{u_c(c_{t+1}^o)} n_{t+1}^y
$$

$$
- \int \frac{v_n(n_{t+1}^o)}{u_c(c_{t+1}^o)} d\Psi \int \eta n_{t+1}^o d\Psi + B_{t+2}
$$

$$
\geq c_{t+1}^o - \frac{v_n(n_{t+1}^o)}{u_c(c_{t+1}^o)} n_{t+1}^o.
$$

Finally, the above equation becomes (6) by using the young households’ budget constraint.

Notice that without idiosyncratic uncertainty, the old households’ budget constraint and the old households’ FOCs are trivially satisfied, leaving the aggregate resource constraint as the only constraint on the Ramsey problem. Since this situation is equivalent to a social planner’s problem, the resulting Ramsey plan must be the first-best allocation.

**The “Only If” Part:** The constraints listed in Proposition 2 are trivially satisfied because they are part of the competitive-equilibrium conditions.
A.2 Proof of Proposition 5

A.2.1 With Unconstrained $B$ and $T$

With both $B_{t+1}$ and $T_t$ unconstrained and without idiosyncratic uncertainty, the Ramsey problem becomes the social planner’s problem since the only constraint left in Proposition 2 is the aggregate resource constraint. In this case, it is straightforward to verify that IAE is achieved in the steady state according to the simplified FOCs with respect to $c^y_t$ and $c^o_{t+1}$ after setting the Lagrangian multipliers $\{\lambda^1, \lambda^2, \phi, \eta^L, \eta^H\}$ to zero:

$$u^y_{c,t} = \mu_t$$
$$u^o_{c,t+1} = \mu_{t+1}.$$

Since $\mu_t = \mu_{t+1}$ in the steady state, we must have $u^y_c = u^o_c$. In addition, the MGR holds according to the simplified Ramsey FOC with respect to $K_{t+1}$:

$$\mu_t = \beta \mu_{t+1}(MP_{K,t+1} + 1 - \delta),$$

which implies that the capital tax is zero not only in the steady state but also in the entire transition period. Also, the simplified Ramsey FOCs with respect to $n^y_t$ and $n^o_{t+1}$ are given by

$$v^y_{n,t} = \mu_t MP_{N,t}k^y$$

and

$$v^o_{n,t+1} = \mu_{t+1} MP_{N,t+1}k^o,$$

which suggest that labor taxes for both young and old are zero:

$$1 - \tau^y_{n,t} = \frac{v_n(n^y_t)}{u_c(c^y_t)k^yw_t} = \frac{\mu_t}{\mu_t} = 1$$
$$1 - \tau^o_{n,t+1} = \frac{v^o_{n,t+1}}{u_c(k^o_{t+1}w_{t+1})} = \frac{\mu_{t+1}}{\mu_{t+1}} = 1.$$

Hence, the interest payments on government bonds must be financed by lump-sum taxes on the old: $T_{t+1} < 0$.

From the discussion above, we know that the Ramsey steady state features the following properties: (1) $c^y = c^o = C_t / 2$, (2) $\tau^y_n = \tau^o_n = 0$, (3) $\beta r = 1$, and (4) $MP_K = \frac{1}{\beta} - 1 + \delta$, which is the MGR. Hence, we can solve for $\frac{K}{Y}$ with the MGR:

$$\frac{K}{Y} = \frac{\alpha \beta}{1 - \beta + \beta \delta}.$$
In addition, the resource constraint implies that \( \frac{C}{Y} = 1 - \delta \frac{K}{Y} \). The steady-state \( B/Y \) ratio can be solved by using the budget constraint of young households:

\[
\frac{B}{Y} = \frac{wK^y n^y}{Y} - \frac{c_y}{Y} - \frac{K}{Y} < 1 - \alpha - \frac{1}{2} - \frac{1}{2} - \frac{\delta}{2} - 1 - \delta \frac{K}{Y} - \frac{K}{Y},
\]

where the last inequality utilizes the fact that the labor share of young households has to be less than the aggregate labor share of both young and old households, which is \( 1 - \alpha \). Hence,

\[
\frac{B}{Y} < \frac{1}{2} - \alpha + \frac{\delta}{2} - 1 - \alpha \left( \frac{\alpha \beta}{1 - \beta + \beta \delta} \right),
\]

which is negative unless \( \alpha \) is sufficiently close to zero. It is thus easy to check that a sufficient condition for a negative \( B/Y \) ratio is \( \alpha > 1/\beta - 1 + \delta \).

**A.2.2 With \( B \) Unconstrained and \( T = 0 \)**

With unconstrained government debt limits but the constrained \( T = 0 \), the multipliers \( \eta^H_t \) and \( \eta^L_t \) are zero but \( \phi_t \neq 0 \). Under Assumption 3, the FOCs with respect to \( c^y_t \) and \( c^{o}_{t+1} \) are reduced to

\[
(1 + \phi_t (1 + \gamma_c)) u_{c,t}^y = \mu_t
\]

\[
(1 + \phi_t (1 + \gamma_c)) u_{c,t+1}^o = \mu_{t+1}
\]

and the Euler equation of the Ramsey planner is given by

\[
1 = \beta \frac{\mu_{t+1}}{\mu_t} (MP_{K,t+1} + 1 - \delta).
\]

Therefore, it must be true that (i) the capital tax is zero for all period \( t \):

\[
\tau_{k,t+1} = 1 - \frac{\mu_c(c^y_{t+1})}{\beta u_c(c^{o}_{t+1})} - 1 + \delta = 0,
\]

and (ii) in the Ramsey steady state (where \( \mu_t = \mu_{t+1} \)) both the MGR and IAE (\( u^y_n = u^o_n \)) hold.

We then discuss the tax rates for labor. The young and old households’ FOCs with respect to labor can be rewritten as

\[
(1 + \phi_t (1 + \gamma_n)) u_{n,t}^y = \mu_t MP_{N,t} K^y,
\]

\[
(1 + \phi_t (1 + \gamma_n)) u_{n,t+1}^o = \mu_{t+1} MP_{N,t+1} K^y.
\]
and 

$$(1 + \phi_t(1 + \gamma_n))v_{n,t+1} = \mu_{t+1}MP_{N,t+1}K^\alpha,$$

which together with the FOCs for young and old households’ consumption give

$$\tau_{n,t}^y = 1 - \frac{v_{n}(n_{t}^y)}{u_{c}(c_{t}^y)\kappa_{n}^y w_{t}} = 1 - \frac{(1 + \phi_t(1 + \gamma_c))}{(1 + \phi_t(1 + \gamma_n))}$$

and

$$\tau_{n,t+1}^o = 1 - \frac{v_{n,t+1}^o}{u_{c,t+1}^o \kappa_{t+1}^o} = 1 - \frac{(1 + \phi_t(1 + \gamma_c))}{(1 + \phi_t(1 + \gamma_n))}.$$ 

So the labor tax rates for young and old households are identical. The sign of the labor tax depends on the value of $\phi_t$, which relates to the need for government revenue or the sign of public debt. The steady-state government budget constraint is given by

$$\tau_{n}^y w^y n^y + \tau_{n}^o w^o = (1 - \beta)B.$$ 

Hence, the sign of $\tau_n$ is decide by the sign of $B$.

Next, we show that the Ramsey steady-state level of government bonds is negative given the condition that $\alpha > 1/\beta - 1 + \delta$. First, the MGR holds and can be used to solve for $K/Y$:

$$\frac{K}{Y} = \frac{\alpha \beta}{1 - \beta + \beta \delta}.$$ 

In addition, the resource constraint implies that $\frac{C}{Y} = 1 - \delta \frac{K}{Y}$. Suppose $B/Y > 0$, then $\tau_{n}^y > 0$. The budget constraint of young households in the steady state implies:

$$\frac{B}{Y} = \frac{(1 - \tau_{n}^y)w^y n^y}{Y} - \frac{c_{Y}}{Y} - \frac{K}{Y} < 1 - \alpha - \frac{K}{Y},$$

where the last inequality stands for two facts: (1) the labor share of young households has to be less than the total labor share in the aggregate economy, which is $1 - \alpha$, and (2) $\frac{C}{Y} > 0$. Hence, there is a upper bound on $B/Y$ given by

$$\frac{B}{Y} < 1 - \alpha - \frac{\alpha \beta}{1 - \beta + \beta \delta},$$

which is negative if $\alpha > 1/\beta - 1 + \delta$, which implies the $K/Y$ ratio is no less than 1. Since $B/Y < 0$, the government receives interest payment from the private sector and it in turn spends it on labor subsidy to balance the government budget.
A.2.3 With Binding Debt Limits on $B_{t+1}$

In this case, either $\eta^H_t > 0$ or $\eta^L_t > 0$. Under Assumption 3, the FOCs with respect to $c^y_t$ and $c^o_{t+1}$ are given, respectively, by

$$(1 + \phi_t(1 + \gamma_c))u^y_{c,t} + \gamma_c \eta^H_t \eta^H_t u^y_{n,t} = \mu_t - \eta^H_t + \eta^L_t$$

and

$$(1 + \phi_t(1 + \gamma_c))u^o_{c,t+1} = \mu_{t+1}.$$ 

The above two FOCs lead to

$$\frac{\mu_{t+1}}{\mu_t - \eta^H_t + \eta^L_t} = \frac{(1 + \phi_t(1 + \gamma_c))u^o_{c,t+1}}{(1 + \phi_t(1 + \gamma_c))u^y_{c,t} + \gamma_c(\eta^H_t - \eta^L_t)\eta^H_t u^y_{n,t}u^y_{c,t}} > \frac{u^y_{c,t+1}}{u^y_{c,t}} = \beta r_{t+1} \text{ if } \eta^H_t > 0$$

which suggests that IAE fails to hold in the Ramsey steady state. In addition, the Ramsey FOC with respect to $K_{t+1}$ is

$$1 = \beta \frac{\mu_{t+1}}{\mu_t - \eta^H_t + \eta^L_t} (MP_{K,t+1} + 1 - \delta),$$

hence the MGR must fail in the Ramsey steady state as well since either $\eta^H_t > 0$ or $\eta^L_t > 0$. In addition, by comparing the equation above with

$$1 = \beta \frac{u^o_{c,t+1}}{u^y_{c,t}}((1 - \tau_{k,t+1})MP_{K,t+1} + 1 - \delta),$$

we can see that the sign of the capital tax is negative if $\eta^H_t > 0$ and positive if $\eta^L_t > 0$. This means that a capital tax (subsidy) is optimal if the level of government bonds hits its lower (upper) limit.

The implication for the old households’ labor tax is identical to that in subsection A.2.2 since the Ramsey FOCs with respect to $n^o_{t+1}$ and $c^o_{t+1}$ remain unchanged. Hence, the old households’ labor tax is given by

$$\tau^o_{n,t+1} = 1 - \frac{u^o_{n,t+1}}{u^o_{c,t+1}w_{t+1}K^o} = 1 - \frac{(1 + \phi_t(1 + \gamma_n))}{(1 + \phi_t(1 + \gamma_o))},$$

which equals zero if and only if $\phi_t = 0$, namely if and only if the lump-sum policy tool $T_{t+1}$ is available and unconstrained.

In addition, suppose we assume a lump-sum tax is not available (namely, $T_{t+1} \geq 0$), which implies $\phi_t \geq 0$. We can then show that $\tau^y_{n,t} > 0$ if $\eta^H_t > 0$. We prove this by contradiction: Suppose this is not true, then $v_n(n^y_t) \geq \kappa^y w_t u(c^y_t)$ by the young households’ FOC. The Ramsey FOC with
respect to $n^y_t$ can be written as

$$
\mu_tw_t\kappa^y = (1 + \phi_t(1 + \gamma_n))v^y_{n,t} + \eta^H_t v^y_{n,t} u^{y}_{c,t}(1 + \gamma_n) \\
\geq (1 + \phi_t(1 + \gamma_n))\kappa^y w_t u^{y}_{c,t} + \eta^H_t \kappa^y w_t(1 + \gamma_n),
$$

which implies

$$
\mu_t \geq (1 + \phi_t(1 + \gamma_n))u^y_{c,t} + \eta^H_t (1 + \gamma_n) \\
> (1 + \phi_t(1 + \gamma_c))u^y_{c,t} + \eta^H_t \left(1 + \gamma_c\frac{v^y_{n,t} n^y_t}{u^y_{c,t} c^y_t}\right),
$$

where the last inequality utilizes the fact that $\gamma_n > 0 > \gamma_c$. However, the last inequality leads to a contradiction since it violates the Ramsey FOC with respect to $c^y_t$.

Similarly, it can be shown that $\tau^y_{n,t} < 0$ if $\eta^L_t > 0$. Suppose this is not true, then $v_n(n^y_t) \leq \kappa^y w_t u^y_{c,t}(c^y_t)$ by the young households’ FOC. The Ramsey FOC with respect to $n^y_t$ can be written as

$$
\mu_tw_t\kappa^y = (1 + \phi_t(1 + \gamma_n))v^y_{n,t} - \eta^L_t v^y_{n,t} u^{y}_{c,t}(1 + \gamma_n) \\
\leq (1 + \phi_t(1 + \gamma_n))\kappa^y w_t u^{y}_{c,t} - \eta^L_t \kappa^y w_t(1 + \gamma_n),
$$

which implies

$$
\mu_t \leq (1 + \phi_t(1 + \gamma_n))u^y_{c,t} - \eta^L_t (1 + \gamma_n) \\
< (1 + \phi_t(1 + \gamma_c))u^y_{c,t} - \eta^L_t \left(1 + \gamma_c\frac{v^y_{n,t} n^y_t}{u^y_{c,t} c^y_t}\right),
$$

where the last inequality utilizes the fact that $\gamma_n > 0 > \gamma_c$. However, the last inequality leads to a contradiction since it violates the Ramsey FOC with respect to $c^y_t$.

### A.3 Proof of Proposition 6

Obviously, IAE fails in an environment with uninsurable idiosyncratic risks because IAE requires infinite tools to be achieved if the idiosyncratic shock has a continuum of realizations. However, it is possible to achieve EIAE. The implications with respect to the MGR and EIAE under different policy mix are discussed in the following cases.
A.3.1 With Both B and T Unconstrained

Under Assumption 3 as well as the conditions $\phi_t = \eta_t = 0$, the MGR is implied by combining the steady-state version of Ramsey FOC with respect to $K_{t+1}$.

Summing the Ramsey FOC with respect to $c^o_{t+1}$ across old households gives

$$\int u^o_{c,t+1} d\Psi = \mu_{t+1} + \gamma_c \int \frac{v^o_{n,t+1}}{c^o_{t+1}} \zeta_{t+1} d\Psi,$$

which together with

$$\tau_{k,t+1} = 1 - \frac{\beta}{\beta} \int \frac{u_c(c^y)}{c^o_{t+1}} d\Psi - 1 + \delta$$

implies that the capital tax is non-zero unless $\int \frac{v^o_{n,t+1}}{c^o_{t+1}} \zeta_{t+1} d\Psi = 0$.

By the household FOCs, the young households’ labor tax rate is

$$1 - \tau^y_{n,t} = \frac{v_n(n^y)}{u_c(c^y)\kappa^y w_t},$$

which together with the Ramsey FOCs with respect to $n^o_{t+1}$ and $c^y_{t+1}$ implies a zero labor tax for the young.

The old households’ labor tax rate is non-zero. We can show this by contradiction. Suppose $\tau^o_{n,t+1} = 0$, then the old households’ FOC with respect to $n^o_{t+1}$ becomes

$$v^o_{n,t+1} = u^o_{c,t+1} \kappa^o \theta_{t+1} w_{t+1}. \quad (19)$$

In addition, the Ramsey FOCs with respect to $c^o_{t+1}$ and $n^o_{t+1}$ can be rewritten as

$$u^o_{c,t+1} = \mu_{t+1} + \frac{v^o_{c,t+1}}{u^o_{c,t+1}} v^o_{n,t+1} \zeta_{t+1}$$

and

$$v^o_{n,t+1} = \mu_{t+1} MP_{N,t+1} \theta_{t+1} + v^o_{n,n,t+1} \zeta_{t+1}.$$

Hence, for equation (19) to be valid, it must be the case that (by using the two FOCs above)

$$MP_{N,t+1} \theta_{t+1} \frac{\gamma_c}{c^o_{t+1}} \frac{n^o_{t+1}}{\gamma_n} \zeta_{t+1} = \zeta_{t+1},$$

which cannot be true since $MP_{N,t+1} \theta_{t+1} \frac{\gamma_c}{c^o_{t+1}} \frac{n^o_{t+1}}{\gamma_n} < 0$ and $\zeta_{t+1} \neq 0$ for all $\theta_{t+1}$.

We now turn to the optimal level and sign of government bonds in the Ramsey steady state.
Given that the MGR still holds and \( \tau_n^y = 0 \), the \( B/Y \) ratio faces the same upper bound as in subsection A.2.2, which is given by
\[
\frac{B}{Y} < 1 - \alpha - \frac{K}{Y},
\]
which is negative if \( \alpha > 1/\beta - 1 + \delta \) (since the \( K/Y \) ratio is no less than 1).

### A.3.2 With \( B \) Unconstrained and \( T = 0 \)

Under Assumption 3 and the condition \( \eta_t = 0 \), the MGR is implied by the steady-state version of Ramsey FOC with respect to \( K_{t+1} \).

The Ramsey FOCs with respect to \( n_t^y \) and \( c_t^y \) are given by
\[
(1 + \phi_t(1 + \gamma_c))u_{c_t, t}^y = \mu_t
\]
and
\[
(1 + \phi_t(1 + \gamma_n))v_{n_t, t}^y = \mu_tMP_{N_t}^{K_y},
\]
which imply a non-zero labor tax for the young:
\[
\tau_{n_t, t}^y = 1 - \frac{v_{n_t}(n_t^y)}{u_c(c_t^y)K^yw_t} = 1 - \frac{(1 + \phi_t(1 + \gamma_c))(1 + \phi_t(1 + \gamma_n))}{(1 + \phi_t(1 + \gamma_n))} \neq 0,
\]
where the last inequality utilizes the property that \( \phi_t \neq 0 \) that is implied from the strictly binding constraint \( T = 0 \).

Integrating the Ramsey FOC with respect to \( c_{t+1}^o \) across old households gives
\[
\int u_{c, t+1}^o d\Psi + \int u_{cc, t+1}^o d\Psi \left[ C_{t+1}^o - \int \frac{v_{n_t, t+1}^o}{u_{c, t+1}^c} d\Psi \times N_t^o \right] \nonumber \\
+ \phi_t U_{C, t+1}^o \left[ 1 + \gamma_c \int \frac{1}{c_{t+1}^o} u_{c, t+1}^c d\Psi \times N_t^o \right] 
= \mu_{t+1} + \gamma_c \int \frac{v_{n_t, t+1}^o}{c_{t+1}^o} \zeta_{t+1} d\Psi,
\]
which together with the capital tax formula implies that the capital tax is non-zero in general. Similarly, without the lump-sum tax/transfer (i.e., \( \phi_t \neq 0 \)), the labor tax for the old is also non-zero.

We now turn to the optimal level of bonds in the Ramsey steady state. Given that the MGR still holds and \( \tau_n^y \geq 0 \) in the Ramsey steady state, the \( B/Y \) ratio faces the same upper bound as
in subsection A.2.2, which is given by

\[ \frac{B}{Y} < 1 - \alpha - \frac{K}{Y}, \]

which is negative if \( \alpha > \frac{1}{\beta} - 1 + \delta \) (since \( K/Y \) is no less than 1).

A.3.3 With Binding Debt Limits on \( B_{t+1} \)

Under Assumption 3, the Ramsey FOC with respect to \( K_{t+1} \) shows that the MGR fails in the Ramsey steady state.