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Abstract

The aggregate capital stock in a nation can be overaccumulated for many different reasons. This paper studies which policy or policy mix is more effective in achieving the socially optimal (modified golden rule) level of aggregate capital stock in an infinite-horizon heterogeneous-agents incomplete-markets economy where capital may be over-accumulated for two distinct reasons: (i) precautionary savings and (ii) production externalities. By solving the Ramsey problem analytically along the entire transitional path, we show that public debt and capital taxation play very distinct roles in dealing with the overaccumulation problem. The Ramsey planner opts neither to use a capital tax to correct the overaccumulation problem if it is caused solely by precautionary saving—regardless of the feasibility of public debt—nor use debt (financed by consumption tax) to correct the overaccumulation problem if it is caused solely by pollution—regardless of the feasibility of a capital tax. The key is that the modified golden rule has two margins: an intratemporal margin pertaining to the marginal product of capital (MPK) and an intertemporal margin pertaining to the time discount rate. To achieve the MGR, the Ramsey planner needs to equate not only the private MPK with the social MPK but also the interest rate with the time discount rate—neither of which is equalized in a competitive equilibrium. Yet public debt and a capital tax are each effective only in calibrating one of the two margins, respectively, but not both.

JEL Classification: E13; E62; H21; H30

Key Words: Optimal Quantity of Debt, Capital Taxation, Ramsey Problem, Heterogeneous Agents, Incomplete Markets, Pollution, Production Externalities.

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1 Introduction

The aggregate capital stock in a nation can overaccumulate for many reasons. For example, precautionary saving under financial frictions can result in overaccumulation of capital (Aiyagari (1994)). Negative production externalities (such as environmental pollution) at the firm level can also lead to overaccumulation of capital (or overinvestment and overproduction) at the aggregate level (Cropper and Oates (1992)). Capital overaccumulation is an important concern of the government because it implies that the economy’s marginal benefit of capital is lower than its marginal cost, leading to aggregate allocative (or dynamic) inefficiency and thus calling for government intervention.

It is well known in public finance that the government can use many types of policies to mitigate the capital-overaccumulation problem, such as discouraging households to consume/save or firms to produce/invest. Such polices include issuing government bonds to crowd out capital via households’ (or firms’) portfolio choices and levying taxes on consumption or the rate of return to capital to discourage households from consuming/saving and firms from producing/investing. Clearly, multidimensional impacts exist for each policy; e.g., government bonds affect the quantity of the capital stock through the intertemporal price of capital (the interest rate), while capital-income taxation changes the interest rate through its affect on the quantity of capital. These policy choices are reminiscent of the classic problem studied by Weitzman (1974).

The goal of this paper is to study which policy or policy mix is more effective in achieving the socially optimal (golden rule) level of aggregate capital stock in the presence of dynamic inefficiency in an Aiyagari-type economy where capital is overaccumulated for two distinct reasons: (i) precautionary saving and (ii) environmental pollution. Both types of market distortions involve externalities—the former a pecuniary externality that causes quantity to spill over prices (the interest rate) and the latter a production externality that causes prices (the marginal product of capital, MPK) to spill over quantities (output or investment).

In this study we focus on the optimality of three types of fiscal policies to address the problem of dynamic inefficiency: the optimal quantity of debt, the optimal consumption tax, and the optimal capital tax.
By solving the Ramsey problem analytically in an infinite-horizon general-equilibrium heterogeneous-agents incomplete-markets (HAIM) model, we are able to show that public debt and capital taxation have very different functions and effectiveness in dealing with the capital overaccumulation problem, depending on the root cause of the overaccumulation. Debt is more effective in improving welfare under pecuniary externalities (due to incomplete financial markets), while capital tax is more effective in improving welfare under production externalities (due to pollution or incomplete goods markets). In other words, the Ramsey planner will neither use a capital tax to correct the overaccumulation problem if it is caused solely by precautionary saving under borrowing constraints—regardless of the feasibility of public debt—nor use debt to correct the overaccumulation problem if it is caused solely by pollution—regardless of the feasibility of capital taxation. On the other hand, a consumption tax is needed to mainly finance the interest payments of public debt.

The intuition is that precautionary saving generates a pecuniary externality by depressing (distorting) the interest rate, while pollution generates a production externality by depressing (distorting) firms’ total factor productivity. These two types of externalities can each lead to capital overaccumulation but are distinct in nature; hence, they call for very different fiscal policies or policy mixes to address them. In either cases, the right policy or policy mix should aim at dealing with the root cause of the phenomenon: The first (a pecuniary or price externality) is the consequence of borrowing constraints and the lack of self-insurance; thus the best way to address it is to issue plenty of public debt so that households are willing to save and become less borrowing constrained. The second (a production or quantity externality) is the consequence of spill-over effects from pollution; thus the best way to address it is to reduce the rate of return to capital investment (or to household savings) by taxing capital income so that firms/households have fewer incentives to invest or save. In other words, debt policy is best for correcting a pecuniary externality while tax policy is best for correcting a production externality—although both policies reduce the aggregate capital stock the former encourages households to save, while the latter discourages them from saving (under the no-arbitrage assumption that bonds and capital yield the same rate of return). On the other hand, since a consumption tax does not directly address the root causes of the overaccumulation problem, it is best used to finance the costs (interest payments) of debt.
Therefore, if the government is not careful in distinguishing the different causes of the capital overaccumulation problem and the different functions of the policies, its interventions may be ineffective and likely welfare reducing (counter-productive). For example, when capital is overaccumulated through precautionary saving, taxing capital income will reduce welfare—because this policy effectively tightens household borrowing constraints, thus destroying households’ self-insurance positions. On the other hand, when capital is overaccumulated because of pollution, increasing public debt will increase the economy’s burden of debt payments without effectively raising firms’ total factor productivity.

The key insight behind these results is that the “modified golden rule” (MGR) in HAIM economies involves two distinct margins: the MPK (the first margin) and the time discount rate (the second margin). The first margin pertains to capital’s intratemporal price, and the second margin pertains to capital’s intertemporal price. To achieve the MGR, therefore, the Ramsey planner needs to equate not only the private MPK with the social MPK intratemporally, but also the interest rate (net of the liquidity premium of savings) with the time discount rate intertemporally. From a social viewpoint, neither margin is equalized in a laissez-faire competitive equilibrium. Yet a capital tax is more effective in dealing with the intratemporal (first) margin, while public debt is more effective in dealing with the intertemporal (second) margin (by eliminating liquidity premium).

When it comes to how to best finance public debt, we show that the Ramsey planner opts to use a consumption tax instead of a capital tax. This result is reminiscent of the classical result obtained by Judd (1985) and Chamley (1986) that taxing capital is not optimal when other forms of distortionary taxes are available.

Therefore, our work touches on important issues in the classical literature of optimal taxation and the determination of the optimal quantity of debt, especially in an environment with inequalities and heterogeneous agents. The bulk of the literature that studies optimal environmental taxes does not consider inequality or distributional issue, while the bulk of the literature that studies optimal fiscal policies in HAIM economies does not consider capital taxation and government debt jointly as complementary and competitive policy tools. For example, several important papers studying optimal capital taxation in HAIM models do not consider the important role of government debt; see, e.g., Conesa, Kitao, and Krueger.
(2009), Krueger and Ludwig (2018), and Dávila, Hong, Krusell, and Ríos-Rull (2012). On the other hand, the frontier research that studies the role of government debt in HAIM economies often does not focus on the distinct and specific role of capital tax. For example, Aiyagari and McGrattan (1998) study the optimal quantity of debt in steady state and restrict a capital tax and a labor tax to take the same rate. Bhandari, Evans, Golosov, and Sargent (2017) study the issue of public debt in a model without capital. Chien and Wen (2019a) also study the issue of public debt in a model without capital.

This paper intends to fill in the gap by exploring the complementary and competitive roles played by government debt, consumption taxation, and capital taxation in a HAIM economy featuring capital overaccumulation and dynamic inefficiency. Specifically, two important issues regarding optimal taxation and debt determination remain open in the literature based on HAIM models: (i) When capital is overaccumulated for different reasons or under different types of market failures, how will outcomes differ between taxing capital and increasing the quantity of public debt? (ii) What is the precise mechanism of the trade-offs that determines the optimal quantity of debt when multiple causes of capital overaccumulation exist, and how to finance public debt by distortionary taxes?

These intriguing questions are not easy to analyze because infinite-horizon HAIM models are highly intractable; thus to the best of our knowledge the trade-offs in benefits and costs of varying the quantity of public debt and the rate of capital/consumption taxes have never been analytically established.1 Analyzing these issues by numerical methods often relies critically on the assumption of the existence of a Ramsey steady state. Yet the validity of such an assumption is unfortunately never proven in the existing literature (see Chien and Wen (2019b) and Chen, Chien, and Yang (2019)).

Our analysis builds on the insight and the analytically tractable framework of Chien and Wen (2019b). We show in the current paper that the crowding-out effect of public debt on capital does not by itself automatically address the capital-overaccumulation problem arising from production externalities—because government bonds deal with the intertemporal (second) margin in the MGR but not the intratemporal (first) margin. Conversely, a capital tax

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1As acknowledged by Aiyagari and McGrattan (1998, p.445), they are unable to analytically determine the trade-offs and the effect of an increase in the quantity of debt on welfare.
does not by itself automatically address the capital-overaccumulation problem from precautionary saving under incomplete insurance markets—because it deals directly with the first margin in the MGR but not the second one. On the other hand, a consumption tax can do neither and is thus best used to finance public debt. Consequently, debt, capital-taxation, and consumption-taxation policies each have a distinct role to play in addressing market failures involving capital overaccumulation.

More specifically, the main function of public debt is to provide an additional means of smoothing consumption (besides capital) by effectively relaxing borrowing constraints and raising the interest rate, as noted by Aiyagari and McGrattan (1998). However, the overaccumulation of capital from the pecuniary externality of precautionary saving is not a distortion that capital taxation can effectively address even if government bonds are in limited supply or not available at all—because a capital tax does not address the root cause of the overaccumulation problem (arising from the lack of adequate self-insurance) and is thus counterproductive. So the best way to finance public debt is through a consumption tax. On the other hand, public debt is not a desirable tool for correcting the capital overaccumulation problem from production externalities because it does not mitigate the root cause of the problem (pollution). While a consumption tax can mitigate pollution by discouraging aggregate demand, it is not as effective as a capital tax in restoring the MGR along the intratemporal margin. The choice of policy therefore depends critically on the sources of distortions, even though all three polices can potentially reduce the aggregate capital stock to a desired level through affecting market prices—but the MGR has two price-margins that cannot be fully calibrated simply by adjusting one price alone.

Our main contributions are thus three-fold: First, on the technical side, we prove that a unique Ramsey steady state exists in our HAIM model under a general functional form of idiosyncratic risk and show that the Ramsey problem remains analytically tractable despite multiple sources of market failures such as incomplete insurance markets and the difference between private and social returns to scale in the production function.

Second, despite incomplete financial markets, we show analytically that the Ramsey planner will not levy a capital tax in the steady state unless the overaccumulation of capital is caused by production externalities—i.e., unless the spillover effects are from quantities
instead of prices—and that the sign of capital taxation is determined solely by the sign of production externalities regardless of the variance of idiosyncratic risk.

Third, we show that the optimal level of debt is determined solely by the Ramsey planner’s desire to provide full self-insurance despite the need to tax consumption, and that the optimal capital tax is independent of the quantity of public debt despite capital overaccumulation under borrowing constraints, thanks to the feasibility of a consumption tax.

These results reinforce the findings of Chien and Wen (2019b) and further suggest that optimal debt and capital-taxation policies are independent of each other in dealing with the failure of the MGR—the former addresses the intertemporal wedge (the second margin) in the MGR due to the lack of adequate self-insurance, while the latter addresses the intratemporal wedge (the first margin) in the MGR arising from production externalities (the difference between the social MPK and the private MPK). That is, the Ramsey planner will not use public debt to address the capital overaccumulation problem caused by production externalities and, conversely, the Ramsey planner will not use a capital tax to address the pecuniary externality problem due to precautionary saving (despite capital overaccumulation under pecuniary externalities)—because the two margins in the MGR are affected differently by the different policies. The two distinctive margins in the MGR explain why Chien and Wen (2019b) find that the Ramsey planner opts not to tax capital to restore the MGR even when the government’s capacity to issue debt is limited—so the failure of the MGR along the second margin cannot be adequately addressed by using tools best for the first margin and vice versa.

To focus attention on the core issues, we defer literature reviews to a latter section. So the remainder of the paper is organized as follows. Section 2 presents the model and solves its competitive equilibrium in closed forms. Section 3 uses the primal approach to analytically solve the Ramsey problem and optimal fiscal policies. Section 4 studies the robustness of our results by conducting two experiments: (i) ruling out government bonds—to show that with precautionary saving but without production externalities the optimal capital tax is still zero even if the planner cannot issue debt to crowd out capital and (ii) ruling out a capital tax—to show that with pollution but without precautionary saving the optimal quantity of debt is indeterminate even if the planner cannot levy a capital tax to crowd out capital.
Section 5 provides a brief literature review and Section 6 concludes the paper.

2 The Model

2.1 The Environment

Firms. A representative firm produces output according to the socially decreasing (or increasing) returns to scale but privately constant-returns-to-scale Cobb-Douglas technology,

\[ Y_t = F(Z_t, K_t, N_t) = Z_t K_t^\alpha N_t^{1-\alpha}, \]

where \( Y, K, \) and \( L \) denote output, capital, and labor, respectively, and

\[ Z_t = \overline{K}_t^\varphi \]

denotes production externalities (spill-over effects) arising from the average capital stock \( \overline{K}_t \) in the economy, which is taken as given by the private sector, and where the elasticity parameter \( \varphi \leq 0 \) implies that the externality (spillover effect) is negative, zero, or positive. Therefore, the production technology exhibits social decreasing (increasing) returns to scale if \( \varphi < (>) 0 \).

The representative firm rents capital and hires labor from households by paying a competitive rental rate \( q_t \) and real wage rate \( w_t \). Taking \( Z_t \) as given, the firm’s optimal conditions for profit maximization at time \( t \) are

\[ w_t = \frac{\partial F(Z_t, K_t, N_t)}{\partial N_t} \equiv MP_{N,t}, \]

\[ q_t = \frac{\partial F(Z_t, K_t, N_t)}{\partial K_t} \equiv MP_{K,t}. \]

On the other hand, the social MPK takes \( Z_t = K_t^\varphi \) endogenously and is defined as

\[ \tilde{q}_t = \frac{\partial F(K_t^\varphi, K_t, N_t)}{\partial K_t} \]

\footnote{This paper mainly focuses on the case \( \varphi < 0 \).}
In addition, $\tilde{q}_t < q_t$ if $\varphi < 0$.

**Households.** There is a unit measure (continuum) of *ex ante* identical but *ex post* heterogeneous households that face idiosyncratic preference shock $\theta_t$. The shock is identically and independently distributed (iid) over time and across households with mean $\bar{\theta}$ and the cumulative distribution $F(\theta)$, which has the support $[\theta_L, \theta_H]$, where $\theta_H > \theta_L > 0$. Time is discrete and indexed by $t = 1, 2, \ldots, \infty$.

There are two subperiods within each period $t$. The idiosyncratic preference shock $\theta_t$ is realized only in the second subperiod, and the labor supply decision must be made in the first subperiod before observing $\theta_t$. Consumption and saving decisions are made in the second subperiod after the realization of $\theta_t$. Namely, the idiosyncratic preference shock is uninsurable by labor income even if leisure enters the utility function linearly. Let $\theta^t \equiv (\theta_1, \ldots, \theta_t)$ denote the history of idiosyncratic shocks. All households are endowed with the same asset holdings $a_1$ at the beginning of period 1.

Households are infinitely lived with a quasi-linear utility function and face the borrowing constraints: $a_{t+1}(\theta^t) \geq 0$. A household’s lifetime expected utility is given by

$$V = E_1 \sum_{t=1}^{\infty} \beta^t \left[ \theta_t \ln c_t(\theta^t) - n_t(\theta^{t-1}) \right],$$

where $\beta \in (0, 1)$ is the discount factor and $c_t(\theta^t)$ and $n_t(\theta^{t-1})$ denote consumption and the labor supply, respectively, for a household with history $\theta^t$ at time $t$. Note that the labor supply in period $t$ is only measurable with respect to $\theta^{t-1}$, reflecting the assumption that the labor supply decision is made in the first subperiod before observing the preference shock $\theta_t$.

**Government.** The government needs to finance an exogenous stream of purchases $\{G_t \geq 0\}_{t=1}^{\infty}$, and it can issue bonds and levy time-varying consumption and capital taxes at flat rates $\tau^c_t$ and $\tau^k_t$, respectively. The flow government budget constraint in period $t$ is

$$\tau^c_t C_t + \tau^k_t q_t K_t + B_{t+1} \geq G_t + r_t B_t,$$

where $B_{t+1}$ is the level of government debt chosen in period $t$ and $r_t$ is the gross risk-free rate.
There is no aggregate uncertainty in our economy, and hence government bonds and productive capital are perfect substitutes as stores of value for households. As a result, the after-tax gross rate of return to capital must equal the gross risk-free rate:

\[ r_t = 1 + (1 - \tau_t^{\kappa})q_t - \delta, \]

which constitutes a no-arbitrage condition for capital and bonds.

### 2.2 The Household Problem

Given the sequences of the interest rates \( \{r_t\}_{t=1}^{\infty} \), consumption tax rates \( \{\tau_t^c\}_{t=1}^{\infty} \), and wage rates \( \{w_t\}_{t=1}^{\infty} \), a household chooses a plan of consumption, labor, and asset holdings, \( \{c_t(\theta^t), n_t(\theta^{t-1}), a_{t+1}(\theta^t)\}_{t=1}^{\infty} \), to solve

\[
\max_{\{c_t(\theta^t), n_t(\theta^{t-1}), a_{t+1}(\theta^t)\}} E_1 \sum_{t=1}^{\infty} \beta^t \left\{ \theta_t \ln c_t(\theta^t) - n_t(\theta^{t-1}) \right\}
\]

subject to

\[
(1 + \tau_t^c)c_t(\theta^t) + a_{t+1}(\theta^t) \leq w_t n_t(\theta^{t-1}) + r_t a_t(\theta^{t-1}), \quad (8)
\]

\[
a_{t+1}(\theta^t) \geq 0, \quad (9)
\]

with \( a_1 > 0 \) given and \( n_t(\theta^{t-1}) \in [0, N] \).

Define a household’s gross income (cash on hand) as

\[
x_t(\theta^{t-1}) \equiv r_t a_t(\theta^{t-1}) + w_t n_t(\theta^{t-1}) \quad (10)
\]

and the aggregate (average) cash on hand as

\[
X_t \equiv \int x_t(\theta^{t-1}) dF(\theta_t) = r_t A_t + w_t N_t, \quad (11)
\]

where \( A_t \equiv \int a_t(\theta^{t-1}) dF(\theta) \) and \( N_t \equiv \int n_t(\theta^{t-1}) dF(\theta) \) denote aggregate asset holdings and aggregate labor, respectively. The solution of the household problem can be characterized
analytically by a cutoff strategy in the following proposition:

**Proposition 1.** The optimal household decisions for cash on hand $x_t(\theta^{t-1})$, consumption $c_t(\theta^t)$, savings $a_{t+1}(\theta^t)$, and the labor supply $n_t(\theta^{t-1})$ are given, respectively, by the following policy rules:

\[ x_t(\theta^{t-1}) = x_t = \begin{cases} \frac{w_t L_1(\theta^*_t)}{\theta^*_t} & \text{if } \theta^*_t < \theta_H \\ X_t & \text{if } \theta^*_t = \theta_H \end{cases}, \tag{12} \]

\[ c_t(\theta^t) = \begin{cases} \min \left\{ 1, \frac{\theta^*_t}{\theta_t} \right\} \frac{x_t}{(1+\tau^c_t)} & \text{if } \theta^*_t < \theta_H \\ \frac{\theta_t}{(1+\tau^c_t)} & \text{if } \theta^*_t = \theta_H \end{cases}, \tag{13} \]

\[ a_{t+1}(\theta^t) = \begin{cases} \max \left\{ 1 - \frac{\theta^*_t}{\theta_t}, 0 \right\} x_t & \text{if } \theta^*_t < \theta_H \\ x_t - \frac{\theta_t}{(1+\tau^c_t)} & \text{if } \theta^*_t = \theta_H \end{cases}, \tag{14} \]

\[ n_t(\theta^{t-1}) = \frac{1}{w_t} [x_t - r_t a_t(\theta^{t-1})], \tag{15} \]

where the cutoff $\theta^*_t$ is independent of individual history and is determined by the following Euler equation:

\[ \frac{1}{r_{t+1}} = \beta \frac{w_t}{w_{t+1}} L_1(\theta^*_t), \tag{16} \]

where the function $L_1(\theta^*_t)$ captures the liquidity premium of savings and is given by

\[ L_1(\theta^*_t) \equiv \int_{\theta \leq \theta_t} dF(\theta) + \int_{\theta > \theta_t} \frac{\theta}{\theta^*_t} dF(\theta) \geq 1. \tag{17} \]

Note the liquidity premium vanishes ($L_1(\theta^*_t) = 1$) when $\theta^*_t = \theta_H$. In addition, by summing up equations (13) and (14), aggregate consumption $C_t$ and aggregate saving $A_{t+1}$ are given, respectively, by

\[ C_t = \begin{cases} D(\theta^*_t) \frac{x_t}{(1+\tau^c_t)} & \text{if } \theta^*_t < \theta_H \\ \frac{\theta_t}{(1+\tau^c_t)} & \text{if } \theta^*_t = \theta_H \end{cases}, \tag{18} \]

and

\[ A_{t+1} = \begin{cases} [1 - D(\theta^*_t)] x_t & \text{if } \theta^*_t < \theta_H \\ X_t - \frac{x_t}{(1+\tau^c_t)} & \text{if } \theta^*_t = \theta_H \end{cases}. \tag{19} \]

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3The cutoff policy rules hold if the individual labor decision is an interior one; namely, $n_t \in (0, N)$. We discuss the conditions that ensure the interior solution of $n$ in the proof of this proposition (Appendix A.1).
where the function $D(\theta^*_t)$ denotes the aggregate marginal propensity to consume and is defined as

$$D(\theta^*_t) \equiv \int_{\theta \leq \theta^*_t} \frac{\theta}{\theta^*_t} dF(\theta) + \int_{\theta > \theta^*_t} dF(\theta) \in (0, 1].$$  \hspace{1cm}(20)$$

**Proof.** See Appendix A.1. \hfill \Box

Note that, as shown in the proof (Appendix A.1), the distribution of $x_t$ is degenerated and hence all households choose the same $x_t$ if $\theta^*_t < \theta_H$. However, if $\theta^*_t = \theta_H$, then the distribution of $x_t$ becomes indeterminate so long as each household holds enough cash on hand to ensure non-binding borrowing constraints across all states of $\theta$. In this latter case, since the distribution has no impact on the Ramsey allocation, we assume (without loss of generality) that in the case of $\theta^* = \theta_H$, $x_t$ is degenerated such that $x_t = X_t$.

### 2.3 Competitive Equilibrium

Our discussion involves two different notions of the steady state: a “competitive equilibrium steady state” for a given set of government policies and a “Ramsey steady state” under optimal policies.

Denote $C_t, N_t,$ and $K_{t+1}$ as the levels of aggregate consumption, aggregate labor, and the aggregate capital stock, respectively. A competitive equilibrium allocation can be defined as follows:

**Definition 1.** Given initial aggregate capital $K_1$ and bonds $B_1$, as well as a sequence of taxes, government spending, and government debt, $\{\tau^c_t, \tau^k_t, G_t, B_{t+1}\}_{t=1}^\infty$, a competitive equilibrium is a sequence of prices $\{w_t, q_t\}_{t=1}^\infty$ and allocations $\{c_t(\theta^t), a_{t+1}(\theta^t), K_{t+1}, N_t\}_{t=1}^\infty$ such that the following hold:

1. The average capital stock equals the private capital stock in the production function:
   $$\overline{K}_t = K_t.$$

2. The no-arbitrage condition holds for each period: $r_t = 1 + (1 - \tau^k_t)q_t - \delta$ for all $t \geq 1$.

3. Given the sequence $\{w_t, r_t, \sigma_t^c\}_{t=1}^\infty$, the sequence $\{c_t(\theta^t), a_{t+1}(\theta^t), n_t(\theta^{t-1})\}_{t=1}^\infty$ solves the household problem.
4. Given the sequence \( \{w_t, q_t\}_{t=1}^{\infty} \), the sequence \( \{N_t, K_t\}_{t=1}^{\infty} \) solves the firm’s problem.

5. The government budget constraint in equation (7) holds for each period.

6. All markets clear for all \( t \geq 1 \):

\[
K_{t+1} = \int a_{t+1}(\theta_t)dF(\theta_t) - B_{t+1},
\]
\[
N_t = \int n_t(\theta_{t-1})dF(\theta_{t-1}),
\]
\[
\int c_t(\theta_t)dF(\theta_t) + G_t + K_{t+1} \leq F(K_t^\varphi, K_t, N_t) + (1 - \delta)K_t.
\]

In the following proposition, we provide the condition that the steady-state capital stock in a laissez-faire competitive equilibrium is higher than that implied by the MGR such that there exists overaccumulation of capital even if the production externality parameter \( \varphi = 0 \).

**Proposition 2.** If the upper bound \( \theta_H \) of the preference shock is sufficiently large relative to the mean \( E(\theta) \equiv \bar{\theta} \) such that the following condition holds:

\[
\frac{\alpha \beta \theta_H}{\theta_H - \bar{\theta}} + \beta (1 - \alpha) (1 - \delta) < 1,
\]

then in a laissez-faire competitive equilibrium the steady-state risk-free rate is lower than the time discount rate, \( r < 1/\beta \), with a positive liquidity premium \( L(\theta^*) > 1 \) and overaccumulated capital stock.

**Proof.** See Appendix A.2. \( \square \)

Notice that when \( \theta_H \rightarrow \infty \), as in the case of a Pareto distribution, the above condition is clearly satisfied. The intuition of Proposition 2 is straightforward. Since labor income is determined (ex ante) before the realization of the idiosyncratic preference shock \( \theta_t \), a household’s total income may be insufficient (ex post) to provide full self-insurance for large enough preference shocks under condition (24). In this case, precautionary saving leads to overaccumulation of capital at the aggregate level, which reduces the equilibrium interest rate to a level below the time discount rate, regardless of the production externality \( \varphi \). This
outcome may seem inefficient from a social point of view but it emerges because of the negative pecuniary externality household savings have on the interest rate (due to diminishing MPK), as noted by Aiyagari (1994).

Note that when a negative production externality is allowed ($\varphi < 0$), the capital-overaccumulation problem further intensifies. In other words, both the pecuniary externality under precautionary saving and the production externality under $\varphi < 0$ lead to overaccumulation of capital. One of our interests in this paper is to understand how optimal fiscal policies react to the two types of externalities.

As explained by Chien and Wen (2019b), in the absence of production externalities ($\varphi = 0$), a competitive equilibrium can achieve the MGR if the idiosyncratic risk is sufficiently small (e.g., the upper bound $\theta_H$ is close enough to the mean $\bar{\theta}$) such that condition (24) is violated. In this case, household savings can become sufficiently large to fully buffer preference shocks and, as a result, household borrowing constraints will never bind. Clearly, with full self-insurance, it must be true that the optimal cutoff is a corner solution at $\theta^* = \theta_H$ with a vanishing liquidity premium ($L(\theta^*) = 1$) and an interest rate that equals the time discount rate ($r = 1/\beta$). We will see in this paper if the same results hold when $\varphi \neq 0$.

But a competitive equilibrium with full self-insurance is impossible in the Aiyagari model (regardless of production externalities) because every household’s marginal utility of consumption follows a supermartingale in his model when $r = 1/\beta$. This implies that household consumption and savings (or asset demand) diverge to infinity in the long run, which cannot constitute an equilibrium.\(^4\)

In contrast, because the household utility function is quasi-linear in this paper, the expected shadow price of consumption goods is thus the same across agents and given by $\int \lambda_t F(\theta) \, d\theta = \frac{1}{\omega_t}$, which kills the supermartingale property of the household marginal utility of consumption. As a result, household savings (or asset demand) are bounded away from infinity even at the point of $r = 1/\beta$. This property not only renders our model analytically tractable but also sheds great light on the issues of optimal quantity of public debt and optimal taxation.

More specifically, equations (12) and (14) show that when $r = 1/\beta$ household asset

\(^4\)See Ljungqvist and Sargent (2012, Chapter 17) for details.
demand is always bounded above by \((\theta_H - \theta_t) w_t\) for any shock \(\theta_t \in [\theta_L, \theta_H]\). This endogenous upper bound on asset demand is finite as long as the support \([\theta_L, \theta_H]\) of \(\theta_t\) is bounded (a counter example is a Pareto distribution where \(\theta_H = \infty\)). This special property renders our model analytically tractable with closed-form solutions (provided that \(\theta_t\) is iid)—despite incomplete markets and aggregate production externalities—and it implies that the Ramsey planner has the potential to use government debt to achieve the MGR in this economy when the competitive equilibrium is not socially optimal.

2.4 Conditions to Support a Competitive Equilibrium

Given that government policies are inside the aggregate state space of the full set of competitive equilibria and they affect the endogenous distributions (including the average) of all household variables, the Ramsey problem is to pick a competitive equilibrium (through policies) that attains the maximum of the expected household lifetime utility \(V\) defined in (6). Since \(V\) depends on the endogenous distributions (characterized by the sequence of the cutoff \(\{\theta^*_t\}_{t=1}^\infty\)), the Ramsey planner needs also to pick a particular time path (sequence) of distributions to achieve the maximum.

Proposition 3. Given initial capital \(K_1\), initial government bonds \(B_1\), and the initial risk-free rate \(r_1\), the sequences of aggregate allocations \(\{C_t, N_t, K_{t+1}, B_{t+1}\}_{t=1}^\infty\) and distribution statistics \(\{\theta^*_t\}_{t=1}^\infty\) can be supported as a competitive equilibrium if and only if the following are true:

1. The resource constraint (23) holds with \(\overline{K}_t = K_t\).

2. The wage rate condition holds with

\[
MP_{N,t} = \begin{cases} 
\frac{K_{t+1} + B_{t+1}}{\theta^*_t L(\theta^*_t) [1-D(\theta^*_t)]} & \text{if } \theta^*_t < \theta_H \\
MP_{N,t} & \text{if } \theta^*_t = \theta_H
\end{cases}
\]  

(25)
3. The implementability conditions hold:

\[ N_t \leq \begin{cases} 
L(\theta_t^*) \theta_t^* - \frac{1}{\beta} \theta_t^* \left( 1 - D(\theta_t^*) \right) & \text{if } \theta_t^* < \theta_H \\
\frac{K_{t+1} + B_{t+1}}{MPN_t} - \frac{1}{\beta} \frac{K_{t+1} + B_{t+1}}{MPN_{t-1}} & \text{if } \theta_t^* = \theta_H
\end{cases} \quad \text{for } t \geq 2 \quad (26) \]

\[ N_1 \leq \begin{cases} 
L(\theta_1^*) \theta_1^* - r_1 C_1^{-1} D(\theta_1^*) L(\theta_1^*) \theta_1^* (K_1 + B_1) & \text{if } \theta_1^* < \theta_H \\
\frac{K_2 + B_2}{MPN_1} - r_1 (K_1 + B_1) & \text{if } \theta_1^* = \theta_H
\end{cases} \quad \text{for } t = 1. \quad (27) \]

Proof. See Appendix A.3.

This proposition demonstrates that the Ramsey planner can construct a competitive equilibrium by simply choosing the sequences of aggregate allocations \( \{C_t, N_t, K_{t+1}, B_{t+1}\}_{t=1}^{\infty} \) and the distribution statistics \( \{\theta_t^*\}_{t=1}^{\infty} \) to maximize welfare subject to the aggregate resource constraint, the asset market-clearing condition, and the implementability condition, as shown explicitly below.

3 Ramsey Allocations

3.1 The Ramsey Problem

The Ramsey planner treats \( \overline{\mathbf{K}}_t = K_t \) as endogenous. So the aggregate production function for the Ramsey planner becomes \( Y_t = F(K_t^\varphi, K_t, N_t) = K_t^{\alpha + \varphi} N_t^{1-\alpha} \), and the social MPK is given by \( \tilde{q}_t = (\alpha + \varphi) \frac{Y_t}{K_t} \). Armed with Proposition 3, we are ready to write down the Ramsey planner’s problem and derive the first-order Ramsey conditions analytically. Appendix A.4 shows that the lifetime utility function, \( V \), can be rewritten as a function of \( \theta_t^* \) and aggregate variables:

\[ V = \sum_{t=1}^{\infty} \beta^t \left\{ W(\theta_t^*) + \overline{\theta} \ln C_t - N_t \right\}, \quad (28) \]

where \( W(\theta_t^*) \) is defined as

\[ W(\theta_t^*) \equiv \overline{\theta} \ln \frac{1}{D(\theta_t^*)} + \int_{\theta < \theta_t^*} \theta \ln \left( \frac{\theta}{\theta_t^*} \right) dF. \quad (29) \]

Thus, the Ramsey problem can be represented as maximizing the welfare function (28) by
choosing the sequence of \( \{\theta^*_t, N_t, C_t, K_{t+1}, B_{t+1}\}_{t=1}^{\infty} \) subject to the resource constraint (23), the wage-rate condition (25), and the implementability conditions (26) and (27). Therefore, the Lagrangian of the Ramsey problem is given by

\[
L = \max_{\{\theta^*_t, N_t, C_t, K_{t+1}, B_{t+1}\}} \sum_{t=1}^{\infty} \beta^t \left\{ W(\theta^*_t) + \bar{\theta} \ln C_t - N_t \right\} 
+ \sum_{t=1}^{\infty} \beta^t \mu_t \left\{ F(K^\phi_t, K_t, N_t) + (1 - \delta)K_t - G_t - C_t - K_{t+1} \right\} 
+ \beta^1 \lambda_1 \left\{ \left( L(\theta^*_1)\theta^*_1 - N_1 - \frac{D(\theta^*_1)L(\theta^*_1)\theta^*_1}{C_1}r_1(K_1 + B_1) \right) \times 1_{\theta^*_1 < \theta_H} \right\} 
+ \sum_{t=2}^{\infty} \beta^t \lambda_t \left\{ \left( L(\theta^*_t)\theta^*_t - N_t - \frac{1}{\beta} \theta^*_{t-1} \left[ 1 - D(\theta^*_{t-1}) \right] \right) \times 1_{\theta^*_t < \theta_H} \right\} 
+ \sum_{t=1}^{\infty} \beta^t \phi_t \left\{ \left( MP_{N,t} - \frac{K_{t+1} + B_{t+1}}{\theta^*_t L(\theta^*_t) \left[ 1 - D(\theta^*_t) \right]} \right) \times 1_{\theta^*_t < \theta_H} \right\},
\]

where \( \{\mu_t, \lambda_t, \phi_t\} \) denote the multipliers for the resource constraints, the implementability conditions, and wage condition, respectively. In addition, the index function \( 1 \) takes the value of 1 or 0 conditional on the state of the cutoff \( \theta^*_t \). To conserve space, the first-order Ramsey conditions as well as several useful lemmas for the upcoming proofs are relegated to the Appendix A.5.

### 3.2 Characterization of Optimal Steady-State Capital Taxation

**Definition 2.** A Ramsey steady state is a Ramsey allocation where the parameter restriction \( \theta_H < \frac{\theta_L}{1 - \beta} < \infty \) (to ensure positive labor \( n > 0 \) for all individuals in all states) is satisfied and aggregate variables \( \{K_t, N_t, C_t, \theta^*_t\} \) converge to finite positive values.

The condition \( \theta_H < \frac{\theta_L}{1 - \beta} \) (or equivalently \( \beta > \frac{\theta_H - \theta_L}{\theta_H} \)) is required to ensure that all household labor decisions are positive—a necessary condition for Proposition 1. The intuition is that if the variance (support) of \( \theta \) is too large (spread out), some agents may end up with too much savings in the last period and thus opt not to work this period. Our model becomes intractable if the constraint \( N_t \geq 0 \) binds occasionally, so \( N_t = 0 \) must be ruled out.
Proposition 4. There exists a unique Ramsey steady state with the following properties:

1. The optimal quantity of debt is such that the cutoff \( \theta^* = \theta_H \), the liquidity premium \( L(\theta^*) = 1 \), and no households are borrowing constrained: \( a > 0 \).

2. The social MPK equals the after-tax private MPK: \( \bar{q} = (1 - \tau_k^*) q \).

3. The optimal capital tax \( \tau_k^* \) is determined by the following equation:

\[
(1 - \tau_k^*) = \frac{\alpha + \varphi}{\alpha}.
\]  

(31)

Proof. See Appendix A.6.

Equation (31) implies the following:

(i) If the quantity-spillover effect is zero \( (\varphi = 0) \), then the optimal capital tax \( \tau_k^* = 0 \). This result replicates that in Chien and Wen (2019b) despite the fact that social returns to scale and private returns to scale do not equal. (ii) If the quantity-spillover effect is negative \( (\varphi < 0) \), then the steady-state optimal capital tax \( \tau_k^* > 0 \). (iii) If the quantity-spillover effect is positive \( (\varphi > 0) \), then the steady-state optimal capital tax \( \tau_k^* < 0 \).

Therefore, in the absence of any government debt-limit constraints, the Ramsey planner achieves the MGR without the need to tax/subsidize capital in the steady state unless production externalities are present—because capital taxation in the absence of quantity-spillover effects would decrease the steady-state household saving rate and thus permanently hampering households’ self-insurance positions. Instead, the Ramsey planner opts to provide enough incentives for households to save through bond holdings by picking a sufficiently high interest rate \( (= 1/\beta) \) on government bonds, such that all households are fully self-insured in the long run, with zero probability of encountering a binding liquidity constraint.

Furthermore, even if capital taxation/subsidization is justifiable in the case of \( \varphi \neq 0 \), it is meant only to correct the production externalities and not to be a source of revenues to pay for interest on government bonds—which instead is financed by a distortionary consumption tax. In other words, if the overaccumulation of capital is caused by the pecuniary externality from precautionary saving of households, then there is no reason for the Ramsey planner
to levy a capital tax because overaccumulation is the consequence of incomplete insurance markets and not of incomplete goods markets (if $\varphi = 0$).

Therefore, it is the role of government debt in improving self-insurance, not its role in crowding out capital or mitigating the distortionary effects of a capital tax, that determines the optimal quantity of public debt in this model.

In the next section, we will also show that even when the government cannot issue debt ($B_{t+1} = 0$ for all $t \geq 1$), the optimal capital tax rate is still given by equation (31)—in which case the full self-insurance allocation is no longer feasible and $\theta^* < \theta_H$ and $L(\theta^*) > 1$ in the Ramsey steady state. Therefore, optimal capital taxation is independent of the households’ self-insurance positions and the optimal level of public debt depends only on the distortions from the financial markets (from the price-spillover effect).

To see how the debt supply and capital taxation can operate independently, we study in the next subsection two extreme cases: (i) the government cannot issue debt and (ii) the government cannot tax capital. In the case with no government bonds, we show that the optimal capital tax policy remains the same as in equation (31); namely, it is feasible but not optimal to levy a capital tax to correct the overaccumulation problem when the pecuniary externality is the only cause of the problem and government debt is not available to mitigate the problem. In the case with no capital tax, we show that government debt is an ineffective tool to address capital overaccumulation caused by production externalities.

4 Robustness Analyses

In what follows, we will consider two special cases: Case A where the only available policy tool is a capital tax and a lump-sum transfer that redistributes the government revenue back to households and Case B where the only available policy tool is public debt and a lump-sum tax to finance the public debt. For simplicity and without loss of generality, we assume in each case that government spending $G$ is zero.
4.1 Case A: A Capital Tax Only

This special case is particularly illuminating on the role of the MGR in determining the optimal quantity of debt in a HAIM economy. It would appear that if it is optimal to tax capital to restore the MGR in the presence of pecuniary externalities even when government bonds and other forms of distortionary taxes are available, then it would be even more desirable to tax capital to restore the MGR when a consumption tax is not available and the government cannot issue any debt—since a capital tax is now the only tool available to restore the MGR. But we will show that this intuition is false and that if there is any reason to tax capital at all, it must be because of the production externality under pollution ($\varphi < 0$), not because of the pecuniary externality under precautionary savings.

This analysis also suggests that our quasi-linear preference structure is not essential for the previous result in equation (31). If it is not optimal to tax capital in the absence of debt, it should remain optimal not to tax capital after government bonds are reintroduced to the model if public debt can be financed by other forms of distortionary taxes (such as a labor tax or a consumption tax). Consequently, the optimal quantity of debt cannot be influenced by the trade-off between the provision of household self-insurance and the crowding out of capital under a capital tax. This result reinforces the findings in the zero-capital-taxation literature based on representative-agent models (see, e.g., Judd (1985), Chamley (1986), Chari, Christiano, and Kehoe (1996), Chari and Kehoe (1999), and Chari, Atkeson, and Kehoe (1999)).

To show our results under Case A, since the primal approach becomes more involved when fewer government tools are available, we will take the following (dual) approach: The Ramsey planner directly chooses the infinite sequence of capital taxes \( \{\tau^k_t\}_{t=2}^{\infty} \) that maximize the welfare \( V \) based on the competitive-equilibrium allocation \( \{C_t, K_{t+1}, N_t, \theta^*_t\}_{t=1}^{\infty} \), in which each competitive-equilibrium quantity at time \( t \) is a function of the sequence \( \{\tau^k_{t+j}\}_{j=1}^{\infty} \).

Specifically, for simplicity and without loss of generality, let $\delta = 1$ and $G_t = 0$. Assume that any government revenues from capital taxes are lump-sum transferred back to households:

\[
T_t = \tau^k_t q_t K_t,
\]
so the household budget constraint becomes

\[ c_t + a_{t+1} \leq (1 - \tau^k_t) q_t a_t + w_t n_t + T_t \equiv x_t, \]

where the market rental rate \( q_t \) (private MPK) and wage rate \( w_t \) are the same as before and given in equilibrium, respectively, by

\[ q_t = \alpha F(K_t^\varphi, K_t, N_t) \]

\[ w_t = \frac{(1 - \alpha) F(K_t^\varphi, K_t, N_t)}{N_t}. \]

Moreover, the private constant-returns-to-scale production function and the aggregate household resource constraint imply the aggregate goods market-clearing condition

\[ x_t = C_t + K_{t+1} = Y_t. \]

Then, the competitive equilibrium is characterized by equations (32) and (33) together with the following aggregate decision rules:

\[ 1 = \beta \left( 1 - \tau^k_{t+1} \right) q_{t+1} L(\theta^*_t) \frac{w_t}{w_{t+1}}, \]

\[ C_t = D(\theta^*_t) Y_t, \]

\[ K_{t+1} = [1 - D(\theta^*_t)] Y_t, \]

\[ Y_t = w_t \theta^*_t L(\theta^*_t), \]

\[ Y_t = F(K_t^\varphi, K_t, N_t) = K_t^{\alpha + \varphi} N_t^{1-\alpha}. \]

Given capital-tax policies, this system of seven equations uniquely solves for the paths of seven aggregate variables, \( \{C_t, K_{t+1}, N_t, Y_t, w_t, q_t, \theta^*_t\} \), as functions of \( \{\tau^k_t\}_{t=1}^{\infty} \).

To establish these functions in closed forms, notice that equations (37) and (33) imply

\[ \text{5The uniqueness of the equilibrium can be confirmed by the eigenvalue method that shows the steady state is saddle-path stable.} \]
the equilibrium labor supply

\[ N_t = (1 - \alpha) \theta_t^* L(\theta_t^*). \]  \hspace{1cm} (39)

Substituting the real wage and interest rate in equation (34) using their competitive-equilibrium definitions and rearranging, we have

\[ \theta_t^* [1 - D(\theta_t^*)] = \beta \alpha \left(1 - \tau_{t+1}^k\right) \theta_{t+1}^* L(\theta_{t+1}^*). \]  \hspace{1cm} (40)

By the definitions of \( L(\theta^*) \) and \( D(\theta^*) \), these two functions are related by the identity

\[ \theta_t^* [1 - D(\theta_t^*)] = \theta_t^* L(\theta_t^*) - \bar{\theta}. \]  \hspace{1cm} (41)

Hence, equation (40) can be rewritten in the present-value form:

\[
\theta_t^* L(\theta_t^*) = \bar{\theta} + \beta \alpha \left(1 - \tau_{t+1}^k\right) \theta_{t+1}^* L(\theta_{t+1}^*) \\
= \bar{\theta} + \sum_{j=0}^{\infty} \left[ \prod_{h=0}^{j} \left[ \beta \alpha \left(1 - \tau_{t+1+h}^k\right) \right] \right] \bar{\theta},
\]

which is a convergent sequence with the stochastic discounting factor: \( 0 < \beta \alpha \left(1 - \tau_{t+j}^k\right) < 1 \).

Equation (42) implies that the cutoff \( \theta_t^* \) in any period \( t \) depends only on the future tax rate \( \{\tau_{t+1}^k, \tau_{t+2}^k, ..., \tau_{t+\infty}^k\} \), but not on the past history (including the current period) of the tax rate, \( \{\tau_1^k, \tau_2^k, ..., \tau_t^k\} \), or any other endogenous state variable in the economy.

Therefore, the solution for the equilibrium cutoff in period \( t \) can be expressed implicitly as

\[ \theta_t^* = \theta \left(\tau_{t+1}^k, \tau_{t+2}^k, ..., \tau_{t+\infty}^k\right). \]  \hspace{1cm} (43)

Since \( \theta_t^* L(\theta_t^*) \) is increasing in \( \theta_t^* \), an immediate implication of equation (42) is that

\[ \frac{\partial \theta_t^*}{\partial \tau_{t+1+h}^k} < 0 \text{ for } h \geq 0. \]  \hspace{1cm} (44)

Namely, higher future tax rates reduce the current cutoff. Since \( \frac{\partial [1 - F(\theta^*)]}{\partial \theta^*} < 0 \), the probability of a biding borrowing constraint \((1 - F(\theta^*))\) also increases with higher future tax rates. This
suggests that capital income tax destroys households’ self-insurance positions by tightening their borrowing constraints.

Once the equilibrium cutoff is solved as a function of future tax rates, equations (35)-(38) pin down the other endogenous variables completely and uniquely as functions of \( \{\tau^k_{t+j}\}_{j \geq 1} \).

The welfare function is then given by

\[
V(\{\tau^k_t\}_{t=1}^\infty) = \sum_{t=1}^\infty \beta^t \left\{ W(\theta^*_t(\{\tau^k_j\}_{j=t+1}^\infty)) + \bar{\theta} \log Y_t(\{\tau^k_j\}_{j=t+1}^\infty) - N_t(\{\tau^k_j\}_{j=t+1}^\infty) \right\},
\]

where \( W \) is redefined as

\[
W(\theta^*_t(\{\tau^k_j\}_{j=t+1}^\infty)) = \int_{\theta \leq \theta^*_t(\{\tau^k_j\}_{j=t+1}^\infty)} \theta \ln \left( \frac{\theta}{\theta^*_t(\{\tau^k_j\}_{j=t+1}^\infty)} \right) dF.
\]

**Proposition 5.** Taking the first-period \( \tau^k_1 \) as given, the optimal steady-state capital tax rate is given by

\[
\tau^*_k = -\frac{\varphi}{\alpha}.
\]

In the absence of production externalities \( (\varphi = 0) \) and starting from any arbitrary initial date \( t = 1 \), it is optimal to set \( \tau^k_t = 0 \) for all \( t > 1 \) regardless of the variance of idiosyncratic risk \( \theta \) or the severeness of dynamic inefficiency due to precautionary savings (measured by the liquidity premium \( L(\theta^*) > 1 \)).

**Proof.** See Appendix A.7. \( \square \)

This proposition states that in the Ramsey steady state, despite overaccumulation of capital, the optimal capital tax is zero if there is no production externality \( (\varphi = 0) \), and the optimal capital tax is positive if and only if there is a production externality \( (\varphi < 0) \). In addition, in the absence of production externalities and given any initial tax rate \( \tau^k_1 \), it is optimal to immediately set future tax rates to zero: \( \tau^k_t = 0 \) for all \( t \geq 2 \). These results are robust to the distribution \( F(\theta) \) of idiosyncratic shocks or the tightness of household borrowing constraints.

Therefore, despite overaccumulation of capital under precautionary saving, the Ramsey planner will not use a capital tax to restore the MGR or reduce the capital stock unless capital
is overaccumulated for an entirely different reason, such as pollution. The fundamental rationale is that when the MGR fails to hold along the intertemporal margin \((L (\theta^*) > 1)\), the effective tool to restore the MGR is issuing government bonds instead of taxing capital. Since issuing government bonds is not feasible, the Ramsey planner opts to leave the intertemporal margin as it is and only use a capital tax to take care of the intratemporal margin if \(\varphi \neq 0\).

### 4.2 Case B: Government Debt Only

The purpose of studying Case B is to show whether the Ramsey planner is able and willing to use the quantity of debt to correct capital overaccumulation when a full self-insurance position is already reached such that \(\theta_t^* = \theta_H\) and no households are borrowing constrained \((L (\theta_t^*) = 1)\). In this case the MGR fails only along the intratemporal margin where the social MPK does not equal the private MPK. Hence, Case B considers the situation where (i) \(\theta_t^* = \theta_H\) for all period \(t\), (ii) the Ramsey planner is equipped with only government bonds and a lump-sum tax as policy tools, and (iii) there is no government spending, \(G_t = 0\). The condition \(\theta_t^* = \theta_H\) can be justified by assuming a sufficiently large initial debt level \(B_1\) so that the economy is under full self-insurance initially.

This special case is very informative on the role of government debt in addressing the failure of the MGR due to a negative production externality. If it is optimal in general to use debt to crowd out capital to restore the MGR in the presence of a production externality (financed by a distortionary consumption tax), then it would be even more desirable to do so in Case B. This is so because (i) the public debt’s interest payment can now be financed by a non-distortionary lump-sum tax and (ii) government bonds are now the only possible tool to restore the MGR. We will show, surprisingly, that it is optimal for the Ramsey planner to do nothing in Case B, leaving the quantity of debt simply determined by its initial level \(B_1\). The fundamental reason is that public debt is not the right tool to restore the MGR when MGR fails only along the intratemporal margin with \(\varphi \neq 0\).

Notice that to study the Ramsey plan in Case B, several corresponding changes need to be made in the definition of a competitive equilibrium and in the construction of the Ramsey problem. The details of these changes are provided in Appendix A.8, which also proves
the following proposition that describes the conditions needed to support the competitive equilibrium of Case B.

**Proposition 6.** Given initial capital $K_1$, initial government bonds $B_1$, and the initial risk-free rate $r_1$, the sequence of Ramsey allocations $\{C_t, N_t, K_{t+1}, B_{t+1}\}_{t=1}^{\infty}$ can be supported as a competitive equilibrium if and only if the following are true:

1. The aggregate resource constraint (23) holds with $\overline{K}_t = K_t$: $C_t + K_{t+1} = F(K_t^\varphi, K_t, N_t) + (1 - \delta) K_t$.

2. The competitive-equilibrium consumption function holds: $C_t = \theta(1 - \alpha)K_t^{\alpha + \varphi}N_t^{1 - \alpha}$.

3. The no-arbitrage condition $\frac{1}{\beta} \frac{MP_t}{MP_{N,t}} = 1 - \delta + q_t$ holds, where $q_t = \alpha K_t^{\alpha + \varphi - 1} N_t^{1 - \alpha}$ is the private MPK.

**Proof.** See Appendix A.8.

There are two important messages implied by the above proposition. First, notice that the optimal bond supply $B_{t+1}$ does not enter the three constraints (1)-(3) listed in Proposition 6. This suggests that the Ramsey planner’s use of government bonds is irrelevant for the Ramsey outcome—because it cannot influence the competitive equilibrium. Moreover, the three constraints in (1)-(3) of Proposition 6 exactly pin down the three unknowns, $\{C_t, N_t, K_{t+1}\}_{t=1}^{\infty}$, which implies that the Ramsey planner’s allocation is identical to the allocation of the competitive equilibrium (except the levels of $B_{t+1}$ and $T_t$).

The above discussions and Proposition 6 imply the following corollary:

**Corollary 1.** The competitive equilibrium with policy $B_{t+1} = B_1$ and $T_t = (r_t - 1)B_1$ for all $t \geq 1$ is a Ramsey equilibrium.

The fundamental reason is that the quantity of government bonds is not an effective tool to address the capital-overaccumulation problem caused by production externalities ($\varphi \neq 0$). To crowd out capital, the supply of debt has to alter the equilibrium interest rate—the intertemporal price of capital. However, once the economy is in full self-insurance, $\theta_t^* = \theta_H$, there is no liquidity premium and hence any additional supply of government bonds
can no longer raise the market interest rate (determined by the time discount rate $1/\beta$) and thus cannot crowd out capital. This makes the quantity of government bonds ineffective in addressing the failure of the MGR along the intratemporal margin due to production externalities.

In short, despite overaccumulation of capital caused by the negative production externality ($\varphi < 0$), the Ramsey planner is powerless to restore the MGR by reducing the aggregate capital stock, unless the government is equipped with the right tool (such as a capital tax).

5 A Brief Literature Review

The literature related to optimal environmental taxation is as vast as the literature on optimal capital taxation. In the first strand of the literature, Parry (1995) studies the optimal level of environmental taxation in second-best economies and shows that environmental taxes tend to reduce GDP by raising private marginal production costs. Goulder (1995) reviews the debate about whether the revenue-neutral substitution of environmental taxes for ordinary income taxes might offer a double dividend in improving the environment and reducing certain costs of the tax system. The paper connects the double-dividend issue with principles of optimal environmental taxation in a second-best setting.

Bovenberg and Goulder (1996) argue that the consequence of pollution taxes depend fundamentally on the levels of other distortionary taxes, such as income and consumption taxes, yet economists generally analyze environmental taxes without taking into account other distortionary taxes. They thus analyze in general equilibrium how optimal environmental tax rates deviate from rates implied by the Pigovian principle in a second-best setting where other distortionary taxes are present.

In a general equilibrium setting, Sandmo (1975) and Bovenberg and van der Ploeg (1994) demonstrate how the well-known Ramsey formula for optimal consumption taxes is altered when one of the consumption goods generates an externality.

Simpson (1995) studies the relative magnitude of an optimal pollution tax and the marginal damage inflicted by the pollution. He argues that the optimal pollution tax on a monopoly is less than the marginal damage, while the optimal tax may exceed the marginal
damage for a Cournot duopoly. This is so because the tax may be an effective instrument for allocating production from the less to the more-efficient firm.

Our work is also closely related to the traditional literature of optimal taxation in representative-agent models. That literature has shown that if the government’s only option is to tax factor income, then it should tax labor income instead of capital income; see e.g., Judd (1985), Chamley (1986), Chari, Christiano, and Kehoe (1996), Chari and Kehoe (1999), and Chari, Atkeson, and Kehoe (1999).

In addition, Lucas (1990) shows that even in a two-sector endogenous growth model with both fiscal and human capitals it is still optimal to tax labor/human capital income and not capital income. However, Chen and Lu (2013) obtain exactly the opposite result of Lucas (1990) by using a slightly different two-sector growth model.

The literature on optimal fiscal policies in the HAIM framework is still developing and under-researched. Here we review the most-relevant papers in this area. The work of Aiyagari (1995) is the first attempt at investigating optimal Ramsey taxation in HAIM economies. Under the assumption of the existence of an interior Ramsey state steady, Aiyagari (1995) shows that the Ramsey planner opts to restore the MGR by taxing capital in the steady state even though a labor tax is also available.

Chien and Wen (2019b) show that the above intuition for justifying positive capital income taxation is counterintuitive and not necessarily correct in general. They argue that by taxing capital income in the steady state and thus permanently reducing individuals’ optimal buffer stock of savings, the government is effectively destroying their ability to self-insure against idiosyncratic risks when lump-sum transfers are not feasible. Since taxing capital per se does not directly address the lack-of-insurance problem for households (if anything, it intensifies the problem), Chien and Wen (2019b) argue that government bond might be more suitable than capital tax for addressing the production inefficiency purely caused by the precautionary savings in HAIM models.

Chen, Chien, and Yang (2019) show in a standard Aiyagari (1994)-type model that the optimal Ramsey allocation may feature no interior steady state. Hence, optimal fiscal policies in Aiyagari-type models remain an unsolved issue. In this regard, the most important contribution of Chien and Wen (2019b) is to show analytically that when a Ramsey steady
state can be proven to exist the optimal capital tax could still be zero, despite the overaccumulation of capital.

Aiyagari and McGrattan (1998) build on the Aiyagari (1995) model to determine the optimal quantity of debt by studying the trade-offs in benefits and costs of varying the quantity of debt. On the benefit side, they argue that government debt enhances the liquidity of households by providing an additional means of smoothing consumption (in addition to capital) and by effectively relaxing their borrowing constraints. When the interest rate is raised, government debt makes assets both less costly to hold and more effective in smoothing consumption. On the cost side, they argue that (i) the implied taxes to finance public debt have adverse incentive and wealth-distribution effects and (ii) government debt crowds out capital via higher interest rates and thus lowers per capita consumption in the steady state. However, Aiyagari and McGrattan (1998) obtain their results through numerical methods under two assumptions: (i) the Ramsey planner considers only steady-state welfare; and (ii) the tax rates on labor and capital income are levied proportionally to each other. The first assumption ignores the welfare during transition and the second assumption rules out the possibility of financing debt only through labor-income taxation.

Barro (1979), Lucas and Stokey (1983), and Aiyagari, Marcet, Sargent, and Seppala (2002) study the determination of optimal debt under aggregate uncertainty in representative-agent models without capital. Chien and Wen (2019a) follow this literature by introducing aggregate uncertainty into the model of Chien and Wen (2019b) and analytically study the problem of optimal debt determination without capital. They show in closed forms that the optimal tax rate as well as the level of risk-free debt would follow an endogenously bounded stochastic unit root process—bounded below by the Ramsey planner’s desire to provide full self-insurance and above by the government’s natural borrowing limit. This endogenous lower bound on optimal debt emerges because of the Ramsey planner’s dominant incentive to issue more debt whenever the interest rate lies below the time discount rate—until a full self-insurance allocation is reached unless prevented by a government borrowing limit. However, if government bonds are state contingent, then the long-run Ramsey equilibrium exhibits constant taxes in the absence of a government borrowing limit and stochastic taxes if the borrowing limit binds. In either case, the incentives of the Ramsey planner to keep
increasing public liquidity (debt) to meet the self-insurance demand of households dominate the costs of labor-tax distortions and of crowding-out capital—precisely because an interest rate below the time discount rate renders the marginal benefit of increasing the debt larger than the marginal cost of doing so when households are borrowing constrained.

Gottardi, Kajii, and Nakajima (2015) revisit optimal Ramsey taxation in an incomplete-markets model with uninsurable human-capital risk. As in our model, tractability in their model enables them to provide transparent analysis on Ramsey taxation and facilitates intuitive interpretations for their results. When government spending and the bond supply are both set to zero, they find that the Ramsey planner should tax human capital and subsidize physical capital, despite the overaccumulation of physical capital. The purpose or the benefit of taxing human capital is to reduce uninsurable risk from human-capital returns; and the rationale for subsidizing physical capital despite overaccumulation is to satisfy the household demand for a buffer stock, similar to our finding but in contrast to Aiyagari’s results. However, the authors solve the Ramsey problem indirectly, and they can characterize analytically the properties of optimal taxes only in a neighborhood of zero government bonds and zero government spending. In contrast, we can solve the Ramsey problem analytically and directly along the entire dynamic path of the model, which permits transparent examination of how the Ramsey planner takes into account how policies affect the dynamic distribution of household decisions and aggregate productive efficiency when production technology exhibits social increasing/decreasing returns to scale.

6 Conclusion

The aggregate capital stock in a nation can be overaccumulated for many reasons, such as precautionary saving and pollution. Capital overaccumulation is an important concern of the government because it leads to dynamic inefficiency and thus calls for government intervention.

Many policies can mitigate the overaccumulation problem, such as issuing government bonds to crowd out capital, levying environmental taxes on production and consumption, or taxing capital income to discourage households/firms from saving/investing.
In this paper we focus mainly on three popular types of policies—public debt, a consumption tax, and a capital tax—and study which policy or policy mix is most effective in achieving the socially optimal (golden rule) level of aggregate capital stock in an infinite-horizon HAIM economy, where capital is overaccumulated for two distinct reasons: (i) precautionary saving and (ii) a production externality.

By solving the Ramsey problem analytically along the entire transitional path, we show that public debt and capital taxation play very distinct roles in dealing with the overaccumulation problem. Debt is far more effective in improving welfare under pecuniary externalities (due to financial frictions), while a capital tax is far more effective in improving welfare under production externalities (due to pollution). On the other hand, a consumption tax should be used (instead of a capital tax) to finance the interest payments of public debt. More importantly, we show that using the wrong tool to address capital overaccumulation can lead to welfare losses or be ineffective even if the right tool is not available.

In other words, the Ramsey planner will not opt to use a capital tax to correct the overaccumulation problem if it is caused solely by precautionary saving—regardless of the feasibility of public debt—nor use debt to correct the overaccumulation problem if it is caused solely by pollution—regardless of the feasibility of a capital taxation.

The key insight behind our findings is that the MGR has two margins: an intratemporal margin pertaining to the wedge between social MPK and private MPK, and an intertemporal margin pertaining to the wedge between the market interest rate and the time discount rate. To achieve the MGR, the Ramsey planner needs to equate not only the private MPK with the social MPK but also the interest rate with the time discount rate—neither of which is equalized in a competitive equilibrium. Yet government debt is effective and desirable only in addressing the intertemporal wedge, while a capital tax is effective and desirable only in addressing the intratemporal wedge.
References


A Appendix

A.1 Proof of Proposition 1

A.1.1 Household Optimal Conditions

Denoting \( \{ \beta^t \lambda_t(\theta^t), \beta^t \mu_t(\theta^t) \} \) as the Lagrangian multipliers for constraints (8) and (9), respectively, the first-order conditions for \( \{ c_t(\theta^t), n_t(\theta^{t-1}), a_{t+1}(\theta^t) \} \) are given, respectively, by

\[
\frac{\theta_t}{(1 + \tau_t^c) c_t(\theta^t)} = \lambda_t(\theta^t) \tag{45}
\]

\[
1 = w_t \int \lambda_t(\theta^t) \, dF(\theta_t) \tag{46}
\]

\[
\lambda_t(\theta^t) = \beta r_{t+1} \int \lambda_{t+1}(\theta^{t+1}) \, dF(\theta) + \mu_t(\theta^t), \tag{47}
\]

where equation (46) reflects the fact that the labor supply \( n_t(\theta^{t-1}) \) must be chosen before the idiosyncratic taste shocks (and hence before the value of \( \lambda_t(\theta^t) \)) are realized. By the law of iterated expectations and the iid assumption of idiosyncratic shocks, equation (47) can be written (using equation (46)) as

\[
\lambda_t(\theta^t) = \beta r_{t+1} \frac{n_t(\theta^{t-1})}{w_{t+1}} + \mu_t(\theta^t), \tag{48}
\]

where \( \frac{1}{w} \) is the marginal utility of consumption in terms of labor income.

We characterize the competitive equilibrium in two cases. One is with full self-insurance and the other is not.

A.1.2 No Full Self-Insurance Case

We adopt a guess-and-verify strategy to derive the decision rules. The decision rules for an household’s consumption and savings are characterized by a cutoff strategy, taking as given the aggregate states (such as the interest rate and real wage) given that there always exists a positive measure of households with a binding borrowing constraint (no full self-insurance). Anticipating that the optimal cutoff \( \theta^*_t \) is independent of an household’s history of shocks,
consider two possible cases:

**Case A.** $\theta_t \leq \theta^*_t$. In this case the urge to consume is low. It is hence optimal to save so as to prevent possible liquidity constraints in the future. So $a_{t+1}(\theta^t) \geq 0$, $\mu_t(\theta^t) = 0$, and the shadow value is

$$\lambda_t(\theta^t) = \beta \frac{r_{t+1}}{w_{t+1}} \equiv \Lambda_t,$$

where $\Lambda_t$ depends only on aggregate states. In this case, $\lambda_t$ is independent of the history of idiosyncratic shocks. Equation (45) implies that consumption is given by $c_t(\theta^t) = x_t(\theta^t-1) - \theta_t \Lambda_t^{-1}$. The requirement $a_{t+1}(\theta^t) \geq 0$ then implies

$$\theta_t \leq \Lambda_t x_t \equiv \theta^*_t,$$

which defines the cutoff $\theta^*_t$.

We conjecture that the cutoff is independent of the idiosyncratic state, making the optimal gross income $x_t$ also independent of the idiosyncratic state. The intuition is that $x_t$ is determined before the realization of $\theta_t$ and all households face the same distribution of idiosyncratic shocks. Since the utility function is quasi-linear, a household is able to adjust labor income to meet any target level of liquidity on hand. As a result, the distribution of $x_t$ is degenerate. This property simplifies the model tremendously.

**Case B.** $\theta_t > \theta^*_t$. In this case the urge to consume is high. It is then optimal not to save, so $a_{t+1}(\theta^t) = 0$ and $\mu_t(\theta^t) > 0$. By the resource constraint (8), we have $c_t(\theta^t) = \frac{x_t}{1+\tau_c}$, which by equation (49) implies $c_t(\theta^t) = \frac{\theta^*_t}{(1+\tau_c)} \Lambda_t^{-1}$. Equation (45) then implies that the shadow value is given by $\lambda_t(\theta^t) = \frac{\theta^*_t}{\theta^*_t-1} \Lambda_t$. Since $\theta_t > \theta^*$, equation (48) implies $\mu_t(\theta^t) = \Lambda_t \left[ \frac{\theta_t}{\theta^*_t} - 1 \right] > 0$. Notice that the shadow value of goods (the marginal utility of income), $\lambda_t(\theta^t)$, is higher under Case B than under Case A because of binding borrowing constraints.

By Cases A and B, the decision rules of household consumption and saving can then be summarized by equations (13) and (14), respectively. Finally, the decision rule of the household labor supply, equation (15), is decided residually to satisfy the household budget constraint.

The above analyses imply that the expected shadow value of income, $\int \lambda_t(\theta) dF(\theta)$, and
hence the optimal cutoff value $\theta^*$, is determined by equation (46) by plugging in the expressions for $\lambda_t(\theta^t)$ into Cases A and B, which immediately gives equation (16). Specifically, combining Case A and Case B, we have

$$
\lambda_t(\theta^t) = \frac{\beta r_{t+1}}{w_{t+1}} \quad \text{for} \quad \theta \leq \theta_t^*
$$

$$
\lambda_t(\theta^t) = \frac{\theta_t^*}{\theta_t^* \beta} \frac{r_{t+1}}{w_{t+1}} \quad \text{for} \quad \theta \geq \theta_t^*.
$$

The aggregate Euler equation is therefore given by

$$
\frac{1}{w_t} = \int \lambda_t(\theta) dF(\theta) = \beta \frac{r_{t+1}}{w_{t+1}} \left[ \int_{\theta \leq \theta_t^*} dF(\theta) + \int_{\theta > \theta_t^*} \frac{\theta}{\theta_t^*} dF(\theta) \right] = \beta \frac{r_{t+1}}{w_{t+1}} L(\theta_t^*),
$$

which is equation (16). This equation reveals that the optimal cutoff depends only on aggregate states and is independent of the household’s history.

Using equation (49) together with the aggregate Euler equation and the definition of $\Lambda_t$, we can solve for $x_t$:

$$
x_t = \theta_t^* \left( \frac{\beta r_{t+1}}{w_{t+1}} \right)^{-1} = \theta_t^* L(\theta_t^*) w_t,
$$

which is the first line in equation (12).

### A.1.3 Full Self-Insurance Case

Next, we consider the full self-insurance case where all household borrowing constraints are non-binding. This is possible in our model if the initial aggregate bond supply is high enough. As we show below, the individual saving choices and their distribution become indeterminate so long as each household’s cash on hand is sufficiently large (because of a large aggregate bond supply). In this case, we impose an additional assumption that all households still choose the same $x_t$. Under this assumption, $x_t = X_t = r_t A_t + w_t N_t$. Moreover, the first order conditions (46) and (47) imply

$$
\lambda_t(\theta^t) = \beta \frac{r_{t+1}}{w_{t+1}} = \frac{1}{w_t}.
$$
which suggests that
\[ \frac{1}{r_{t+1}} = \beta \frac{w_t}{w_{t+1}}. \]
The FOC (45) together with \( \lambda_t(\theta^t) = w_t^{-1} \) give the second line in equation (13). The second line in equation (14) immediately follows since \( x_t = X_t \).

### A.1.4 Condition to Ensure Interior Labor

Finally, to ensure that the above proof and hence the associated cutoff policy rules are consistent with the assumption of interior choices of labor, namely, \( n_t \in (0, N) \), we need to consider the following two cases.

First, to ensure that \( n_t(\theta^{t-1}) > 0 \), consider the worst case where \( n_t(\theta^{t-1}) \) takes its minimum possible value. Given \( x_t = r_t a_t(\theta^{t-1}) + w_t n_t(\theta^{t-1}) \), \( n_t(\theta^{t-1}) \) is at its minimum possible value if \( \mu_t = 0 \) and \( a_t(\theta^{t-1}) \) takes its maximum possible value of \( a_t(\theta^{t-1}) = \left[ 1 - \left( \frac{\theta_L}{\theta^{t-1}} \right) \right] x_{t-1} \). So \( n_t(\theta^{t-1}) > 0 \) if
\[ x_t - r_t \left[ 1 - \left( \frac{\theta_L}{\theta^{t-1}} \right) \right] x_{t-1} > 0, \tag{50} \]
which is independent of the shock \( \theta_t \). This condition in the steady state becomes \( 1 - r \left[ 1 - \left( \frac{\theta_L}{\theta^*} \right) \right] > 0 \), or equivalently (by using equation (16)),
\[ \beta L(\theta^*) > 1 - \left( \frac{\theta_L}{\theta^*} \right). \tag{51} \]
Given that \( L(\theta^*) \) is a monotonic decreasing function in \( \theta^* \) with a lower bound of 1, the necessary condition to satisfy (51) in the steady state is \( \beta > 1 - \left( \frac{\theta_L}{\theta^*} \right) \), which is clearly true since the optimal cutoff \( \theta^* > \theta_L \). This condition is further ensured by the requirement \( \beta > 1 - \frac{\theta_L}{\theta_H} \). Therefore, as long as the condition \( \beta > 1 - \frac{\theta_L}{\theta_H} \) is met, the condition (50) is assumed to hold throughout the paper.

Second, to ensure that \( n_t < N \), consider agents who encounter the borrowing constraint last period such that \( a_t(\theta^{t-1}) = 0 \). Their labor supply reaches the maximum value at \( n_t(\theta^{t-1}) = \frac{w_t}{w_t} = \theta^*_t L(\theta^*_t) < \theta_H \). Given a finite steady-state value of \( \theta^* \), the value of \( N \) can be chosen such that
\[ N > \theta_H > \theta^* L(\theta^*). \tag{52} \]
A.2 Proof of Proposition 2

In the laissez-faire economy, the capital tax, the labor tax, government spending, and government bond, are all equal to zero. In this laissez-faire competitive equilibrium, the capital-to-labor ratio $\frac{K_t}{N_t}$ satisfies two conditions. The first condition is derived from the resource constraint (23), which can be expressed as

$$F(\bar{K}_t^\phi, K_t, N_t) + (1 - \delta)K_t = C_t + K_{t+1} = x_t,$$

where the last equality utilizes the definition of $x_t$. Dividing both sides of the equation by $K_t$ gives

$$\bar{K}_t^\phi \left( \frac{K_t}{N_t} \right)^{\alpha - 1} + (1 - \delta) = \frac{1}{1 - D(\theta_t^*)},$$

where $x_t/K_t$ is substituted out by $\frac{1}{1 - D(\theta_t^*)}$.

The second condition is derived by combining equation (16) and the no-arbitrage condition, $r_t = q_t + 1 - \delta$, which gives

$$1 = \beta \left( \alpha \bar{K}_t^\phi \left( \frac{K_t}{N_t} \right)^{\alpha - 1} + 1 - \delta \right) L(\theta_t^*),$$

where the MPK $q_t$ is replaced by $\alpha \bar{K}_t^\phi \left( \frac{K_t}{N_t} \right)^{\alpha - 1}$. Since the capital-to-labor ratio must be the same in both equations, conditions (53) and (54) imply the following equation in the steady state:

$$\frac{\alpha \beta}{(1 - D(\theta^*))} + \beta(1 - \alpha)(1 - \delta) = \frac{1}{L(\theta^*)},$$

which solves for the steady-state value of $\theta^*$. It can be shown easily that both $L(\theta^*)$ and $D(\theta^*)$ are monotonically decreasing in $\theta^*$, thus the right-hand side (RHS) of equation (55) increases monotonically in $\theta^*$ and the left-hand side (LHS) of equation (55) decreases monotonically in $\theta^*$. Hence, if a steady-state cutoff exists, it must be unique.

It remains to be shown if the RHS and the LHS cross each other at an interior value of $\theta^* \in [\theta_L, \theta_H]$. The RHS of equation (55) reaches its minimum value of 1 when $\theta^* = \theta_H$ and its maximum value of $\bar{\theta}/\theta_L > 1$ when $\theta^* = \theta_L$. The LHS of equation (55) takes the
maximum value of infinity when $\theta^* = \theta_L$ and the minimum value of $\frac{\alpha \beta \theta_H}{\theta_H - \theta} + \beta (1 - \alpha) (1 - \delta)$ when $\theta^* = \theta_H$. Thus, an interior solution exists if and only if

$$\frac{\alpha \beta \theta_H}{\theta_H - \theta} + \beta (1 - \alpha) (1 - \delta) < 1.$$  

Clearly, $\theta^* = \theta_L$ cannot constitute a solution for any positive value when $\theta_L > 0$. On the other hand, $\theta^* = \theta_H$ may constitute a solution if the above condition is violated. For example, if $\theta_H$ is small and close enough to the $\theta$, then the above condition does not hold since its LHS approaches infinity when $\theta_H \to \theta$. Therefore, an interior solution for $\theta^*$ exists if the upper bound of the idiosyncratic shock is large enough. Otherwise, we have the corner solution $\theta^* = \theta_H$. Finally, if $\theta^*$ is an interior solution, then $L(\theta^*) > 1$ and $r < 1/\beta$ by equation (16).

### A.3 Proof of Proposition 3

#### A.3.1 The “Only If” Part

Assume that we have the allocation $\{\theta_t^*, C_t, N_t, K_{t+1}, B_{t+1}\}_{t=1}^{\infty}$ and the initial risk-free rate $r_1$. We then can directly construct the prices, taxes, and individual allocations in the competitive equilibrium in the following steps:

1. $\overline{K}_t$ is set to be $K_t$.

2. $w_t$ and $q_t$ are set by (3) and (4), which are $w_t = MP_{N,t}$ and $q_t = MP_{K,t}$, respectively.

3. Depending on the value of $\theta_t^*$, we consider two cases below.

   (a) Consider the case in which $\theta_t^* < \theta_H$. $A_{t+1}$ is set by the asset market clearing condition, $A_{t+1} = K_{t+1} + B_{t+1}$. $x_t$ is chosen as $x_t = \frac{A_{t+1}}{1 - D(\theta_t^*)} = \frac{K_{t+1} + B_{t+1}}{1 - D(\theta_t^*)}$ according to the first line of equation (19). By the first line of (12), the $w_t = \frac{x_t}{\theta_t^* L(\theta_t^*)} = \frac{A_{t+1}}{\theta_t^* L(\theta_t^*) [1 - D(\theta_t^*)]} = \frac{K_{t+1} + B_{t+1}}{\theta_t^* L(\theta_t^*) [1 - D(\theta_t^*)]}$, which together with $w_t = MP_{N,t}$ imply the first line of condition (25). $\tau_c$ is set by the first line of equation (18):

   $$1 + \tau_{c,t} = \frac{D(\theta_t^*) x_t}{C_t} = \frac{D(\theta_t^*) K_{t+1} + B_{t+1}}{C_t [1 - D(\theta_t^*)]}.$$
Hence, $r_t$ is implied by equation (16)

$$
\frac{1}{r_t} = \beta \frac{w_{t-1}}{w_t} L(\theta^*_{t-1}) = \beta \frac{A_t}{\theta^*_{t-1}} \frac{\theta^*_{t}}{A_{t+1}} \left[ \frac{1 - D(\theta^*_{t-1})}{1 - D(\theta^*_{t-1})} \right] L(\theta^*_{t})
$$

(b) Suppose $\theta^*_t = \theta_H$. Use the asset-market-clearing condition to set $A_{t+1} = K_{t+1} + B_{t+1}$. By aggregating the second line of equation (18), $1 + \tau^c_t$ is determined by

$$
1 + \tau^c_t = \frac{\bar{\theta} w_t}{C_t} = \frac{\bar{\theta} MP_{N,t}}{C_t},
$$

and hence $X_t$ is chosen according its definition:

$$
X_t = A_{t+1} + (1 + \tau^c_t) C_t = K_{t+1} + B_{t+1} + w_t \bar{\theta}.
$$

The interest rate is set as

$$
r_t = \frac{1}{\beta} \frac{w_{t-1}}{w_t}.
$$

Given $r_1$ and the expression $\{r_{t+1}\}_{t=1}^\infty$, the capital tax $\{\tau^c_{t+1}\}_{t=0}^\infty$ is chosen to satisfy the no-arbitrage condition: $r_t = 1 - \delta + (1 - \tau^c_t)MP_{K,t}$ for all $t \geq 1$. $c_t(\theta_t)$ and $a_{t+1}(\theta_t)$, are pinned down by the second lines in equations (13) and (14), respectively. Finally, set $n_t(\theta_{t-1})$ to satisfy equation (15), which is implied by the individual household budget constraint.

4. There are two cases.

(a) First consider the case $\theta^*_t < \theta_H$. The implementability conditions are

$$
L(\theta^*_t) \theta^*_t \geq N_t + r_1 C_t^{-1} D(\theta^*_t) L(\theta^*_t) \theta^*_t (K_1 + B_1)
$$

and

$$
L(\theta^*_t) \theta^*_t \geq N_t + \frac{1}{\beta} \theta^*_{t-1} \left[ 1 - D(\theta^*_t) \right]
$$

for $t = 1$ and $t \geq 2$, respectively. Multiplying both sides of the above equations
by \( \frac{C_t}{D(\theta^*_t) L(\theta^*_t) \theta^*_t} \) leads to

\[
\frac{C_t}{D(\theta^*_t)} \geq \frac{C_t}{D(\theta^*_t) L(\theta^*_t) \theta^*_t} N_t + r_1 (K_1 + B_1),
\]

\[
\frac{C_t}{D(\theta^*_t)} \geq \frac{C_t}{D(\theta^*_t) L(\theta^*_t) \theta^*_t} N_t + \frac{\theta^*_{t-1}}{D(\theta^*_t) L(\theta^*_t) \theta^*_t} \frac{1}{1 - D(\theta^*_{t-1})} C_t.
\]

Inserting the relationships constructed in steps 3 for \( x_t, C_t, A_{t+1}, \tau_c^t, w_t, \) and \( r_t \) into the above equation gives

\[
(1 + \tau_c^t) C_t + A_{t+1} \geq w_t N_t + r_t A_t \text{ for all } t \geq 1,
\]

which together with the aggregate resource constraint and the identity \( Y_t = q_t K_t + w_t N_t \) give the government budget constraint.

(b) Next consider the case \( \theta^*_t = \theta_H \). Plugging the relationships constructed in steps 2 and 3 for \( w_t, x_t, C_t, A_{t+1}, 1 + \tau_c^t, w_t \) and \( r_t \) into the implementability conditions gives

\[
(1 + \tau_c^t) C_t + A_{t+1} \geq w_t N_t + r_t A_t \text{ for all } t \geq 1,
\]

which together with the aggregate resource constraint and the identity \( Y_t = q_t K_t + w_t N_t \) gives the government budget constraint.

### A.3.2 The "If" Part

Note that the aggregate resource constraint is trivially implied by a competitive equilibrium, since it is part of the definition. The implementability condition is constructed as follows. First, rewrite the government budget constraint as

\[
G_t \leq \tau_c^t q_t K_t + \tau_c^t C_t + B_{t+1} - r_t B_t.
\]
Combining this equation with the resource constraint (23), the no-arbitrage condition, and the identity $Y_t = q_t K_t + w_t N_t$ implies

$$(1 + \tau_t^c) C_t + A_{t+1} \geq w_t N_t + r_t A_t.$$  

(57)

We then consider two cases below.

**The $\theta_t^* < \theta_H$ Case.** For $t \geq 2$, the aggregate consumption function, saving function, and equations (12) and (16) suggest that \{\(w_t, r_t, 1 - \tau_t^c\)\} can be expressed, respectively, as

$$w_t = \frac{A_{t+1}}{[1 - D(\theta_t^*)] L(\theta_t^*) \theta_t^*}$$

$$r_t = \frac{A_{t+1}[1 - D(\theta_{t-1}^*)]}{\beta A_t [1 - D(\theta_t^*)] \theta_t^* L(\theta_t^*)} \theta_{t-1}^*$$

and

$$1 + \tau_t^c = \left[\frac{1 - D(\theta_t^*)}{D(\theta_t^*)} \frac{C_t}{A_{t+1}}\right]^{-1}.$$  

Substituting the above equations into (57) and rearranging terms, we get the first line of implementability condition (26):

$$L(\theta_t^*) \theta_t^* \geq N_t + \frac{1}{\beta} \theta_{t-1}^* [1 - D(\theta_{t-1}^*)].$$

For the first period, $B_1, K_1, \text{ and } \tau_{k,1}$ are given, which implies that $r_1 = 1 + (1 - \tau_1^k) MP_{K,1} - \delta$ is also given. Therefore, the first-period implementability condition could be rewritten as

$$L(\theta_1^*) \theta_1^* \geq N_1 + r_1 C_1^{-1} D(\theta_1^*) L(\theta_1^*) \theta_1^*(K_1 + B_1).$$

**The $\theta_t^* = \theta_H$ Case.** The aggregate consumption function, aggregate saving function, and equations (12) and (16) suggest that \{\(r_t, 1 - \tau_t^c\)\} can be expressed, respectively, as

$$r_t = \frac{MP_{N,t}}{\beta MP_{N,t-1}}$$
and
\[ 1 + \tau_i^e = \left[ \frac{C_t}{MP_{N,t}\theta} \right]^{-1}. \]

Substituting the above equations into (57) and rearranging terms, we get the second line of implementability condition (26):
\[ \bar{\theta} \geq N_t + \frac{1}{\beta} \frac{K_t + B_t}{MP_{N,t-1}} - \frac{K_{t+1} + B_{t+1}}{MP_{N,t}}. \]

For the first period, \( r_1 \) is given. Therefore, the first-period implementability condition could be rewritten as
\[ \bar{\theta} \geq N_1 + \frac{r_1(K_1 + B_1)}{MP_{N,1}} - \frac{K_2 + B_2}{MP_{N,1}}. \]

### A.4 The Ramsey Objective Function

By equation (13) and the third step of Appendix A.3, the individual objective function can be rewritten as
\[
c_t(\theta_t) = \begin{cases} 
\min \left\{ 1, \frac{\theta_t}{\theta_t^*} \right\} & \text{if } \theta_t^* < \theta_H \\
\frac{\theta_t}{\theta_t^*} C_t & \text{if } \theta_t^* = \theta_H.
\end{cases}
\]

Conditional on the value of \( \theta_t^* < \theta_H \), the consumption part of the objective function at period \( t \) becomes
\[
\int_{\theta > \theta_t^*} \theta \log c_t(\theta) dF(\theta) = \int_{\theta > \theta_t^*} \theta \ln \frac{C_t}{D(\theta_t^*)} dF(\theta) + \int_{\theta \leq \theta_t^*} \theta (\ln \frac{\theta}{\theta_t^*} + \ln \frac{C_t}{D(\theta_t^*)}) dF(\theta) \quad (58)
\]

\[ = W(\theta_t^*) + \bar{\theta} \ln C_t, \]

where \( W(\theta_t^*) \) is defined as
\[
W(\theta_t^*) \equiv -\bar{\theta} \ln D(\theta_t^*) + \int_{\theta \leq \theta_t^*} \theta \ln \left( \frac{\theta}{\theta_t^*} \right) dF(\theta).
\]

We then consider the case in which \( \theta_t^* = \theta_H \):
\[
\int \theta_t \ln c_t(\theta^t) dF(\theta) = \int \theta \ln \theta dF(\theta) - \bar{\theta} \ln \bar{\theta} + \bar{\theta} \ln C_t,
\]

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which is equal to equation (58) when $\theta^*_t = \theta_H$. As a result, the Ramsey objective function is written as equation (28).

**A.5 Ramsey Optimal Conditions**

The Ramsey planner treats $K_t$ as endogenous. So the first-order Ramsey conditions for $\{B_{t+1}, \theta^*_t, N_t, C_t, K_{t+1}\}_{t=2}^\infty$ are given, respectively, by

\[
\lambda_t + \lambda_{t+1} 1_{\theta^*_t < \theta_H} = \lambda_t H(\theta^*_t) 1_{\theta^*_t < \theta_H} + \frac{\phi_t}{\theta^*_t L(\theta^*_t) [1 - D(\theta^*_t)]} 1_{\theta^*_t < \theta_H},
\]

\[
1 + \mu_t = \frac{\partial F(K^*_t, K_{t+1}, N_{t+1})}{\partial N_{t+1}} + \left( \lambda_t 1_{\theta^*_t < \theta_H} - \lambda_{t+1} 1_{\theta^*_t < \theta_H} \right) \frac{\partial M^{-1}_{N,t}}{\partial N_{t+1}}
\]

\[
\frac{\bar{\theta}}{C_t} = \mu_t,
\]

\[
\mu_t = \beta \mu_{t+1} \left[ \frac{\partial F(K^*_t, K_{t+1}, N_{t+1})}{\partial K_{t+1}} + 1 - \delta \right] + \left( \lambda_t 1_{\theta^*_t = \theta_H} - \lambda_{t+1} 1_{\theta^*_t = \theta_H} \right) \frac{1}{M_{N,t}}
\]

\[
+ \left( \lambda_{t+1} 1_{\theta^*_t = \theta_H} - \lambda_{t+2} 1_{\theta^*_t = \theta_H} \right) \left( \beta(K_{t+2} + B_{t+2}) \frac{\partial M^{-1}_{N,t+1}}{\partial K_{t+1}} \right)
\]

\[
+ \beta \phi_{t+1} \frac{\partial M_{N,t+1}}{\partial K_{t+1}} 1_{\theta^*_t < \theta_H} - \frac{\phi_t}{\theta^*_t L(\theta^*_t) [1 - D(\theta^*_t)]} 1_{\theta^*_t < \theta_H},
\]

where $M(\theta^*_t) \equiv \frac{\partial[L(\theta^*_t)\phi_t]}{\partial \theta^*_t}$, $J(\theta^*_t) \equiv \frac{\partial[M^{-1}((1-D(\theta^*_t))]}{\partial \theta^*_t}$, and these functions satisfy $M(\theta^*_t) = J(\theta^*_t) = F(\theta^*_t) > 0$ for $\theta^*_t \in (\theta_L, \theta_H)$ (as shown in Lemma 1).

We then consider the following cases:
1. Suppose $\theta^*_t < \theta_H$ and $\theta^*_{t+1} < \theta_H$. In this case, $\phi_t = 0$ and we get (by using lemma (1))

$$\lambda_{t+1} = \lambda_t + \frac{1}{F(\theta^*_t)} \frac{\partial W(\theta^*_t)}{\partial \theta^*_t}.$$  \hspace{1cm} (64)

This suggests that the $\lambda_t$ is a monotonic increasing sequence if $\theta^*_t < \theta_H$.

2. Suppose $\theta^*_t = \theta_H$ and $\theta^*_{t+1} < \theta_H$. FOCs (59) and (60) imply $\lambda_{t+1} = 0$ and $\lambda_t = 0$, which suggests that the government budget constraint does not bind at period $t$ and $t+1$. This case is impossible assuming $\theta^*_1 < \theta_H$ and $\lambda_1 > 0$.

3. Suppose $\theta^*_t = \theta_H$ and $\theta^*_{t+1} = \theta_H$. The first-order condition with respect to $B_{t+1}$ suggests $\lambda_t = \lambda_{t+1}$ and $\phi_t = 0$. Given the discussion of previous cases, we know that $\lambda_t$ is monotonic increasing until $\theta^*_t = \theta_H$. Once $\theta^*_t$ reaches $\theta_H$, $\theta^*_t$ stays at $\theta_H$ and $\lambda_t$ becomes constant.

4. Suppose $\theta^*_t < \theta_H$ and $\theta^*_{t+1} = \theta_H$. This case describe the last transition period before reaching the Ramsey steady state. The $\lambda_{t+1}$, $\lambda_t$, and $\theta_t$ have to satisfy the relationship implied by the FOCs (59) and (60):

$$\frac{\theta^*_t L(\theta^*_t) [1 - D(\theta^*_t)]}{MP_{N,t}} \frac{\partial [\theta^*_t L(\theta^*_t) [1 - D(\theta^*_t)]]^{-1}}{\partial \theta^*_t} \lambda_{t+1} = \lambda_t F(\theta^*_t) + \frac{\partial W(\theta^*_t)}{\partial \theta^*_t}.$$  \hspace{1cm} (65)

Moreover, $K_1$ and $B_1$ as well as $r_1$ are taken as given, since the initial capital tax is assumed to be given. Assuming $\theta^*_1 < \theta_H$, the first-order Ramsey conditions with respect to $N_1$, $C_1$, and $\theta^*_1$ are given, respectively, by

$$1 + \lambda_1 + \lambda_1 \frac{D(\theta^*_1)L(\theta^*_1)\theta^*_1}{C_1} \frac{\partial r_1}{\partial N_1} (K_1 + B_1) = \mu_1 \frac{\partial F(K^*_1, K_1, N_1)}{\partial N_1},$$  \hspace{1cm} (65)

$$\mu_1 = \frac{\hat{\theta}}{C_1} + \lambda_1 \frac{D(\theta^*_1)L(\theta^*_1)\theta^*_1}{C_1^2} r_1 (K_1 + B_1),$$  \hspace{1cm} (66)

$$\frac{\partial W(\theta^*_1)}{\partial \theta^*_1} + \lambda_1 H(\theta^*_1) - \lambda_2 J(\theta^*_1) = \lambda_1 \frac{\partial [D(\theta^*_1)L(\theta^*_1)\theta^*_1]}{\partial \theta^*_1} \frac{\partial C_1}{\partial r_1} r_1 (K_1 + B_1).$$  \hspace{1cm} (67)
A.5.1 Several Lemmas

The following three lemmas are useful to characterize the optimal Ramsey allocation:

Lemma 1. \( M(\theta^*_t) = J(\theta^*_t) = F(\theta^*_t) \).

Proof.

\[
M(\theta^*_t) \equiv \left( L(\theta^*_t) + \frac{\partial L(\theta^*_t)}{\partial \theta^*_t} \theta^*_t \right) = \int_{\theta \leq \theta^*_t} dF(\theta) + \int_{\theta > \theta^*_t} \frac{\theta}{\theta^*_t} dF(\theta) - \int_{\theta > \theta^*_t} \frac{\theta}{\theta^*_t} dF(\theta) \\
= \int_{\theta \leq \theta^*_t} dF(\theta) = F(\theta^*_t).
\]

\[
J(\theta^*_t) \equiv \left( 1 - D(\theta^*_t) - \theta^*_t \frac{\partial D(\theta^*_t)}{\partial \theta^*_t} \right) = 1 - \left[ \int_{\theta \leq \theta^*_t} \frac{\theta}{\theta^*_t} dF(\theta) + \int_{\theta > \theta^*_t} dF(\theta) \right] + \int_{\theta \leq \theta^*_t} \frac{\theta}{\theta^*_t} dF(\theta) \\
= 1 - \int_{\theta > \theta^*_t} dF(\theta) = F(\theta^*_t)
\]

\( \square \)

Lemma 2. \( \frac{\partial W(\theta^*_t)}{\partial \theta^*_t} > 0 \) for all \( \theta^*_t \in (\theta_L, \theta_H) \), and \( \frac{\partial W(\theta^*_t)}{\partial \theta^*_t} = 0 \) if \( \theta^*_t = \theta_L \) or \( \theta^*_t = \theta_H \).

Proof. We first show that \( \frac{\partial W(\theta^*_t)}{\partial \theta^*_t} = 0 \) if \( \theta^*_t = \theta_L \) or \( \theta_H \):

\[
\frac{\partial W(\theta^*_t)}{\partial \theta^*_t} = -D(\theta^*_t) \frac{\bar{\theta}}{D(\theta^*_t)} - \int_{\theta \leq \theta^*_t} \frac{\theta}{\theta^*_t} dF(\theta) = \left[ \frac{\bar{\theta}}{D(\theta^*_t) \theta^*_t} - 1 \right] \int_{\theta \leq \theta^*_t} \frac{\theta}{\theta^*_t} dF(\theta)
\]

\[
= \begin{cases} 
1 - \frac{\bar{\theta}}{\theta^*_t} \frac{\theta_L}{\theta_H} & \text{if } \theta^*_t = \theta_H \\
1 - \frac{\bar{\theta}}{\theta^*_t} \frac{\theta_H}{\theta_L} & \text{if } \theta^*_t = \theta_L
\end{cases}
\]

Next, we show that \( \frac{\partial W(\theta^*_t)}{\partial \theta^*_t} > 0 \) for any \( \theta^*_t \in (\theta_L, \theta_H) \). Note that

\[
D(\theta^*_t) \theta^*_t = \int_{\theta \leq \theta^*_t} \theta dF(\theta) + \theta^*_t \int_{\theta > \theta^*_t} dF(\theta) = \bar{\theta} - \int_{\theta > \theta^*_t} (\theta - \theta^*_t) dF(\theta) < \bar{\theta}
\]

\[
\rightarrow \frac{\bar{\theta}}{D(\theta^*_t) \theta^*_t} > 1.
\]
Hence,

\[
\frac{\partial W(\theta^*_t)}{\partial \theta^*_t} = \left[ \frac{\theta}{D(\theta^*_t) \theta^*_t} - 1 \right] \int_{\theta \leq \theta^*_t} \frac{\theta}{\theta^*_t} dF(\theta) > 0
\]

\[\square\]

**Lemma 3.** \(\frac{\partial D(\theta^*_t)}{\partial \theta^*_t} < 0\) for all \(\theta^*_t \in (\theta_L, \theta_H)\).

*Proof.* The definition of \(D\) is given by

\[
D(\theta^*_t) = \int_{\theta \leq \theta^*_t} \frac{\theta}{\theta^*_t} dF(\theta) + \int_{\theta > \theta^*_t} dF(\theta) < 1
\]

and hence the derivative is

\[
\frac{\partial D(\theta^*_t)}{\partial \theta^*_t} = -1 - \int_{\theta \leq \theta^*_t} \frac{\theta}{\theta^*_t^2} dF(\theta) + 1 = - \int_{\theta \leq \theta^*_t} \frac{\theta}{\theta^*_t^2} dF(\theta) < 0.
\]

\[\square\]

**A.6 Proof of Proposition 4**

**A.6.1 Existence of the Ramsey Steady State**

In what follows, we first sketch the proof that a Ramsey steady state featuring \(\theta^* = \theta_H\) exists. We proceed by the following steps, which show that the conjecture \(\theta^* = \theta_H\) satisfies all of the Ramsey FOCs and the FOC-implied steady-state values of the aggregate allocation \(\{C, N, K, B\}\) and that the Lagrangian multipliers \(\{\lambda, \mu\}\) are unique, mutually consistent, strictly positive, and finitely valued:

1. The FOC with respect to \(B_{t+1}\) in equation (59) implies that \(\lambda_t\) is constant at \(\lambda\) and \(\phi_t = 0\).

2. The FOC with respect to \(\theta^*_t\) in equation (60) is satisfied at \(\theta^*_t = \theta_H\).

3. The FOC with respect to \(K\) in equation (63) is reduced to

\[
1 = \beta \left( \frac{\alpha + \varphi}{\alpha} MP_K + 1 - \delta \right)
\]

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which implies $MP_K \equiv \alpha K^\varphi \left( \frac{K}{N} \right)^{\alpha-1} = \frac{\alpha}{\alpha+\varphi} \frac{1-\beta(1-\delta)}{\beta} \in (0, \infty)$ (i.e., the capital stock is unique, strictly positive, and bounded, given that $N \in (0, \bar{N})$). Given the assumption of the production function, it must be true that the following ratios are unique, strictly positive, and finite: \{K/N, Y/K, MP_N, Y/N\} $\in (0, \infty)$. More specifically, the $Y/N$ and $Y/K$ ratios can be expressed, respectively, as

$$\frac{Y}{N} = K^\varphi \left( \frac{K}{N} \right)^{\alpha} = \left( \frac{1-\beta(1-\delta)}{\beta(\alpha+\varphi)} \right) \left( \frac{K}{N} \right) = K^{\frac{\varphi}{1-\alpha}} \left( \frac{\beta(\alpha+\varphi)}{1-\beta(1-\delta)} \right)^{\frac{1}{1-\alpha}} \tag{68}$$

and

$$\frac{Y}{K} = K^\varphi \left( \frac{K}{N} \right)^{\alpha-1} = \frac{1-\beta(1-\delta)}{\beta(\alpha+\varphi)}. \tag{69}$$

4. The resource constraint,

$$F(K, N) = K^{(\alpha+\varphi)}N^{1-\alpha} = C + \delta K + G,$$

together with a finite level of government spending $G$ implies the unique ratio $C/K \in (0, \infty)$ is

$$\frac{C}{Y} = \left( 1 - \delta \frac{K}{Y} - \frac{G}{Y} \right) = \left( 1 - \frac{(\alpha+\varphi)\beta\delta}{1-\beta(1-\delta)} - \frac{G}{Y} \right), \tag{70}$$

where the last equality uses (69).

5. We know that under our parameter restrictions the level of labor is interior, $N \in (0, \bar{N})$; hence, it must be true that the aggregate allocation is also unique and interior: \{C, K, Y\} $\in (0, \infty)$.

6. Next, we show that \{\mu, \lambda\} $\in (0, \infty)$ and that these steady-state values are unique. Given $\theta^* = \theta_H$ in the steady state, first order conditions (61) and (62) become $1 + \lambda = \mu \times MP_N$ and $\mu \times C = \bar{\theta}$, respectively. These two equations imply \{\mu, \lambda\} are unique and $\in (0, \infty)$.

7. The Ramsey steady-state version of equation (63) reads as

$$1 = \beta \left[ \frac{\partial F(K^\varphi, K, N)}{\partial K} + 1 - \delta \right],$$

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which is the MGR with the social MPK.

8. The optimal capital tax is chosen such that the Euler equation in the competitive equilibrium (16) is consistent with the one chosen by the Ramsey planner in (63). Hence, \( \tau_{k,t+1} \) is pinned down by

\[
1 - \tau_{k,t+1}^k = \left( \varphi + \frac{\alpha}{\alpha} \right) \left( \frac{\mu_{t+1} \frac{1}{L(\theta_t^*)} - \beta (1 - \delta) \mu_{t+1}}{\mu_{t+1} \beta (1 - \delta)} \right),
\]

which is the steady-state equation (31) since \( L(\theta_H) = 1 \).

This finishes the proof for the existence of the Ramsey steady state. To show uniqueness, we then show that there is no Ramsey steady state for \( \theta^* \in (\theta_L, \theta_H) \).

A.6.2 Uniqueness of the Ramsey Steady State

Suppose there exists another Ramsey steady state with \( \theta^* \in (\theta_L, \theta_H) \). From equation (59), we see that \( \theta_{t}^* < \theta_H \) and \( \theta_{t+1}^* < \theta_H \) imply \( \phi_t = 0 \), which together with FOCs (62) and (61) imply that both \( \lambda_t \) and \( \mu_t \) have to be finite and positive in the Ramsey steady state. Moreover, the first-order condition with respect to \( \theta^*_t \) is then reduced to

\[
\lambda_{t+1} = \lambda_t + \frac{1}{F(\theta^*_t)} \left( \frac{\partial W(\theta^*_t)}{\partial \theta^*_t} \right),
\]

which leads to a contradiction since the above equation suggests an ever increasing \( \lambda_t \).

Finally, we show that the case of \( \theta^*_t = \theta_L \) cannot constitute a Ramsey equilibrium, although the necessary FOC with respect to \( \theta^*_t \) is satisfied. The reason is that the first term of the Ramsey objective function (28), \( W(\theta^*_t) \), is monotonically increasing in \( \theta^*_t \in (\theta_L, \theta_H) \). Hence, for a global maximum, a cutoff \( \theta^*_t \) at its lower corner cannot be a Ramsey equilibrium.

To ensure that \( n \in (0, \bar{N}) \) (see Proposition 1), note that we have assumed \( \theta_H < \frac{\theta_L}{1 - \beta} \), which ensures that the minimum individual labor input remains positive, as shown in Appendix A.1. Moreover, by equation (52), the maximum value of \( n \) is less than \( \bar{N} \) if \( \bar{N} > \theta_H \) in this case.

In addition, we can show that the maximum individual asset demand remains finite in
the steady state even if the risk-free rate is equal to the time discount rate, \( r = 1 / \beta \). Since \( \theta_H < \frac{\theta_L}{1 - \beta} \), we have

\[
x_t = \frac{C_t}{D(\theta_H)} = \frac{C_t}{\theta} \theta_H < \infty.
\]

Given the finite value of \( x_t \), the individual asset holding \( a_{t+1} \) is determined by the size of the idiosyncratic shock \( \theta_t \), and the agents with the largest asset holdings are those who receive the smallest shock \( \theta_t = \theta_L \); i.e.,

\[
a_{t+1}(\theta_L) = \left[ 1 - \frac{\theta_L}{\theta_H} \right] x_t,
\]

which is strictly positive and finite.

### A.7 Proof of Proposition 5

Define \( H(\theta^*) \equiv [1 - D(\theta^*)] \). The production function and equation (36) imply that the logarithms of the capital stock and output can each be expressed as a moving-average process:

\[
\log K_{t+1} = \log H(\theta^*_t) + (\alpha + \varphi) \log K_t + (1 - \alpha) \log N_t
\]

\[
= (\alpha + \varphi)^t K_1 + \sum_{j=0}^{t-1} (\alpha + \varphi)^j \log H(\theta^*_{t-j}) + (1 - \alpha) \sum_{j=0}^{t-1} (\alpha + \varphi)^j \log N_{t-j}.
\]

\[
\log Y_t = (\alpha + \varphi) \log K_t + (1 - \alpha) \log N_t
\]

\[
= (\alpha + \varphi)^t K_1 + \sum_{j=1}^{t-1} (\alpha + \varphi)^j \log H(\theta^*_{t-j}) + (1 - \alpha) \sum_{j=0}^{t-1} (\alpha + \varphi)^j \log N_{t-j}.
\]

Hence, the welfare function can be written as

\[
V(\{\tau_t\}_{t=1}^{\infty}) = \sum_{t=1}^{\infty} \beta^{t-1} \left\{ \frac{\int_{\theta \leq \theta_t} \theta_t(i) \log \frac{\theta_t(i)}{\theta_t} d\mathbf{F}(\theta) - N_t}{\theta} \left[ \left( \alpha + \varphi \right)^t K_1 + \sum_{j=1}^{t-1} (\alpha + \varphi)^j \log H(\theta^*_{t-j}) \right] + (1 - \alpha) \sum_{j=0}^{t-1} (\alpha + \varphi)^j \log N_{t-j} \right\}.
\]

Since all period-\( t \) variables in the objective function depend on the tax rate through the
cutoff $\theta_t^*$, and since the cutoff $\theta_t^*$ depends only on future taxes, the derivative of $V$ with respect to $\tau_t$ (for $t \geq 2$) can be decomposed as the product of two terms:

$$\frac{\partial V}{\partial \tau_{t+1}} = \frac{\partial V}{\partial \theta_t^*} \frac{\partial \theta_t^*}{\partial \tau_{t+1}}.$$  \hfill (71)

Since equation (42) implies

$$\frac{\partial \theta_t^*}{\partial \tau_{t+1}} < 0,$$  \hfill (72)

it remains to determine the magnitude and sign of $\frac{\partial V}{\partial \theta_t^*}$. Notice that equations (32)-(38) imply that all time-$t$ aggregate quantities depend only on the current cutoff $\theta_t^*$. Taking the derivative of $V(\{\tau_t\}_{t=0}^\infty)$ with respect to $\theta_t^*$ gives

$$\frac{\partial V}{\partial \theta_t^*} = \left\{-\int_{\theta \leq \theta_t^*} \frac{\partial}{\partial \theta_t^*} dF - (1 - \alpha) F(\theta^*) + \frac{\beta \bar{\theta} (\alpha + \varphi)}{1 - \beta (\alpha + \varphi)} \frac{1}{H(\theta^*)} \int_{\theta \leq \theta_t^*} \frac{\partial}{\partial \theta_t^*} dF + \frac{\bar{\theta} (1 - \alpha)}{1 - \beta (\alpha + \varphi)} F(\theta^*) \right\}.$$  

By the definition of the functions $D(\theta^*)$ and $L(\theta^*)$, we have $\frac{\partial H_t}{\partial \theta_t^*} = -\frac{\partial D_t}{\partial \theta_t^*} = \int_{\theta \leq \theta_t^*} \frac{\partial}{\partial \theta_t^*} dF$ and $\frac{\partial N_t}{\partial \theta_t^*} = (1 - \alpha) \frac{\partial \varphi L(\theta^*)}{\partial \theta_t^*} = (1 - \alpha) F(\theta^*)$, so

$$\frac{\partial V}{\partial \theta_t^*} = -\int_{\theta \leq \theta_t^*} \frac{\partial}{\partial \theta_t^*} dF - (1 - \alpha) F(\theta^*) + \frac{\beta \bar{\theta} (\alpha + \varphi)}{1 - \beta (\alpha + \varphi)} \frac{1}{H(\theta^*)} \int_{\theta \leq \theta_t^*} \frac{\partial}{\partial \theta_t^*} dF + \frac{\bar{\theta} (1 - \alpha)}{1 - \beta (\alpha + \varphi)} F(\theta^*).$$

Multiplying both sides by $L(\theta_t^*)$ gives

$$L(\theta_t^*) \frac{\partial V}{\partial \theta_t^*} = \left\{-L(\theta_t^*) \int_{\theta \leq \theta_t^*} \frac{\partial}{\partial \theta_t^*} dF - (1 - \alpha) L(\theta_t^*) F(\theta^*) + \frac{\beta \bar{\theta} (\alpha + \varphi)}{1 - \beta (\alpha + \varphi)} L(\theta_t^*) \int_{\theta \leq \theta_t^*} \frac{\partial}{\partial \theta_t^*} dF + \frac{\bar{\theta} (1 - \alpha)}{1 - \beta (\alpha + \varphi)} L(\theta_t^*) \right\}$$

$$= \left\{\left[\beta (\alpha + \varphi) L(\theta_t^*) - H(\theta_t^*)\right] \frac{L(\theta_t^*)}{(1 - (1 - \beta (\alpha + \varphi)) H(\theta_t^*))} \int_{\theta \leq \theta_t^*} \frac{\partial}{\partial \theta_t^*} dF \right\}$$

$$= \left\{\left[\beta (\alpha + \varphi) L(\theta_t^*) - H(\theta_t^*)\right] \frac{L(\theta_t^*)}{(1 - (1 - \beta (\alpha + \varphi)) H(\theta_t^*))} \int_{\theta \leq \theta_t^*} \frac{\partial}{\partial \theta_t^*} dF \right\},$$

where the last equality is based on the identity in equation (41), $\frac{\partial}{\partial \theta_t^*} = L(\theta_t^*) - H(\theta_t^*)$.  

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Clearly, \( \frac{\partial V}{\partial \theta^*_t} = 0 \) if and only if

\[
\beta (\alpha + \varphi) L(\theta^*_t) - H(\theta^*_t) = 0.
\] (73)

Notice that equation (40) can be rewritten as

\[
H(\theta^*_t) = \frac{\alpha}{\beta} L(\theta^*_t) \left(1 - \tau_{t+1}^k\right) \frac{\theta^*_{t+1} L(\theta^*_{t+1})}{\theta^*_t L(\theta^*_t)}.
\]

So

\[
\frac{\partial V}{\partial \theta^*_t} = 0 \quad \text{if and only if} \quad \frac{\alpha + \varphi}{\alpha} = \left(1 - \tau_{t+1}^k\right) \frac{\theta^*_{t+1} L(\theta^*_{t+1})}{\theta^*_t L(\theta^*_t)},
\] (74)

which in the Ramsey steady state implies

\[
1 - \tau_{k}^* = \frac{\alpha + \varphi}{\alpha}.
\]

Note \( \theta^* L(\theta^*) = \int_{\theta < \theta^*} \theta^* dF + \int_{\theta > \theta^*} \theta^* (i) dF \) is an increasing function of \( \theta^*_t \), thus \( \frac{\partial [\theta^* L(\theta^*)]}{\partial \tau} < 0 \). Also, \( \theta^*_t \) is constant if \( \tau_{t}^k \) is constant. Therefore, we have the following possible cases to consider:

First, for any constant tax rate \( \bar{\tau} \), it is easy to see that

\[
\frac{\alpha + \varphi}{\alpha} \leq \left(1 - \bar{\tau}\right) \frac{\theta^* L(\theta^*)}{\theta^* L(\theta^*)} \quad \text{if and only if} \quad \bar{\tau} \leq 0;
\]

hence, setting \( \bar{\tau} = -\frac{\varphi}{\alpha} \) for all \( t \) maximizes \( \bar{V} (\{\bar{\tau}_t\}_{t=0}^\infty) \) given that \( \tau_t = \bar{\tau} \).

Now suppose \( \varphi = 0 \) and the optimal tax rate is time varying and contains an increasing sequence in the time interval \([t, T]\) with \( \{\tau_{t}^k < \tau_{t+1}^k < \tau_{t+2}^k < \ldots < \tau_{T}^k\} \). This implies that we have a monotonically decreasing sequence for \( \{\theta^*_t L_t\} \) up to \( T > t \). So \( \frac{\theta^*_{t+1} L(\theta^*_{t+1})}{\theta^*_t L(\theta^*_t)} > \left(1 - \tau_{t+1}^k\right) \frac{\theta^*_{t+1} L(\theta^*_{t+1})}{\theta^*_t L(\theta^*_t)} \) and \( \frac{\partial V}{\partial \tau_{t+1}^k} > 0 \). By equations (71) and (72), we have \( \frac{\partial V}{\partial \tau_{t+1}^k} < 0 \), implying that increasing the tax rate is reducing welfare, thus not optimal. Hence, without production externality, any dynamic tax path with an increasing tax rate in any time interval is not optimal.

Now suppose \( \varphi = 0 \) and that the optimal tax rate is a monotonically decreasing sequence
\(\{\tau^k_t > \tau^k_{t+1} > \tau^k_{t+2} > \ldots\}\) for all \(t > 1\) and that this sequence converges to a positive constant \(\bar{\tau} > 0\). This implies that in the limit we have \(\frac{\theta^*_{t+1}L(\theta^*_{t+1})}{\theta^*_{t}L(\theta^*_{t})} \to 1\) and \((1 - \tau^k_{t+1}) \frac{\theta^*_{t+1}L(\theta^*_{t+1})}{\theta^*_{t}L(\theta^*_{t})} \to (1 - \bar{\tau}) < 1\) in the long run. By equations (71) and (72), we have \(\frac{\partial V}{\partial \bar{\tau}} < 0\), suggesting \(\bar{\tau}\) should be zero in the long run—a contradiction.

Therefore, without production externalities any optimal path of the tax rate must either be monotonically converging to zero or constant at zero starting from \(t > 1\). Now suppose \(\tau^k_t\) monotonically converges to 0. Since along a declining tax path, the term \(\theta^*_{t}L(\theta^*_{t})\) is monotonically increasing, we must have \(\frac{\theta^*_{t+1}L(\theta^*_{t+1})}{\theta^*_{t}L(\theta^*_{t})} > 1\) at any point of time \(t \in (1, \infty)\). By equation (74), suppose \(1 < (1 - \tau^k_{t+1}) \frac{\theta^*_{t+1}L(\theta^*_{t+1})}{\theta^*_{t}L(\theta^*_{t})}\) along the declining tax path in period \(t \in (1, \infty)\), then we must have \(\frac{\partial V}{\partial \theta^*_{t}} < 0\) and \(\frac{\partial V}{\partial \tau_{t+1}} > 0\), which suggests that a higher (rather than lower) tax rate tomorrow in period \(t + 1\) would maximize welfare. This contradicts the requirement that \(\{\tau^k_t\}_{t=2}^{\infty}\) be a monotonically decreasing sequence. Hence, we must have \(1 > (1 - \tau^k_{t+1}) \frac{\theta^*_{t+1}R(\theta^*_{t+1})}{\theta^*_{t}R(\theta^*_{t})}\) along the declining tax path. On this path, since \(\frac{\partial V}{\partial \theta^*_{t}} > 0\) and \(\frac{\partial V}{\partial \tau_{t+1}} < 0\) for all \(t\), a decreasing sequence of the tax rate is optimal. This also implies that the welfare function \(V(\{\tau^k_t\}_{t=0}^{\infty})\) converges monotonically in the limit to the upper bound \(V(\{0\})\).

Therefore, without production externalities, the welfare along a declining tax path must be strictly lower than the welfare along the constant (zero) tax path: \(V(\{\tau^k_t\}_{t=0}^{\infty}) < V(0)\). Hence, without production externalities, setting \(\tau^k_t\) immediately to zero for \(t > 1\) is optimal.

### A.8 Proof of Proposition 6

The definition of competitive equilibrium and the equilibrium property of Case B remain largely unchanged compared to the benchmark model (Proposition 1), except the following modifications: First, we consider only the case where \(\theta^*_{t} = \theta_H\). Second, \(x_t\) is redefined as

\[x_t(\theta^{t-1}) \equiv r_t a_t(\theta^{t-1}) + w_t r_t(\theta^{t-1}) - T_t.\]

Third, the government budget constraint and household budget constraint are changed to

\[B_{t+1} + T_t \geq G_t + r_t B_t\]
and
\[ c_t(\theta^t) + a_{t+1}(\theta^t) \leq r_t a_t(\theta^{t-1}) + w_t n_t(\theta^{t-1}) - T_t, \]
respectively. Finally, the no-arbitrage condition becomes
\[ r_t = 1 - \delta + MP_{K,t}. \]

A.8.1 The “Only If” Part

Assume that we have the allocation \( \{C_t, N_t, K_{t+1}, B_{t+1}\}_{t=1}^{\infty} \) and the initial risk-free rate \( r_1 \). We then can directly construct the prices, taxes, and individual allocations in the competitive equilibrium in the following steps:

1. \( \overline{K}_t \) is set to equal \( K_t \).

2. \( w_t \) and \( q_t \) are given by (3) and (4).

3. The asset-market-clearing condition is used to set \( A_{t+1} = K_{t+1} + B_{t+1} \).

4. The interest rate is given by
\[ r_t = \frac{1}{\beta} \frac{w_{t-1}}{w_t}. \]

5. \( T_t \) is chosen such that the government budget constraint holds.

6. \( X_t \) is chosen according to the second line of equation (19):
\[ X_t = A_{t+1} + C_t. \]

7. \( c_t(\theta_t) \) and \( a_{t+1}(\theta_t) \) are pinned down by the second lines in equations (13) and (14), respectively. Finally, \( n_t(\theta_{t-1}) \) is set to satisfy equation (15), which is implied by the individual household budget constraint.

Therefore, the allocation has to satisfy the following three conditions in order to construct a competitive equilibrium:

1. The no-arbitrage condition \( r_t = 1 - \delta + MP_{K,t} \) holds for all \( t \geq 1 \).
2. The aggregate resource constraint $C_t + K_{t+1} = F(K_t^\phi, K_t, N_t) + (1 - \delta) K_t$ holds.

3. By the second line of equation (18), $C_t$ has to satisfy $C_t = \bar{\theta} w_t = \bar{\theta} MP_{N_t,t}$.

A.8.2 The “If” Part

Note that the aggregate resource constraint and no-arbitrage condition are trivially implied by a competitive equilibrium, since it is part of the definition. The condition that $C_t = \bar{\theta} MP_{N,t}$ holds in competitive equilibrium according to the Case B version of Proposition 1.