ECONOMIC RESEARCH
FEDERAL RESERVE BANK OF ST. LOUIS

WORKING PAPER SERIES

Optimal Fiscal Policies under Market Failures

Authors	YiLi Chien, and Yi Wen
Working Paper Number	2020-002C
Revision Date	August 2021
Citable Link	https://doi.org/10.20955/wp.2020.002
Suggested Citation	Chien, Y., Wen, Y., 2021; Optimal Fiscal Policies under Market Failures, Federal Reserve Bank of St. Louis Working Paper 2020-002. URL https://doi.org/10.20955/wp.2020.002

Federal Reserve Bank of St. Louis, Research Division, P.O. Box 442, St. Louis, MO 63166

The views expressed in this paper are those of the author(s) and do not necessarily reflect the views of the Federal Reserve System, the Board of Governors, or the regional Federal Reserve Banks. Federal Reserve Bank of St. Louis Working Papers are preliminary materials circulated to stimulate discussion and critical comment.

Optimal Fiscal Policy under Capital Overaccumulation

Yili Chien* Yi Wen†

August 30, 2021

Abstract

In a canonical model of heterogeneous agents with precautionary saving motives, Aiyagari (1995) breaks the classical result of zero capital tax obtained in representative-agent models. Aiyagari argues that with capital overaccumulation the optimal long-run capital tax should be strictly positive in order to achieve aggregate allocative efficiency suggested by the modified golden rule (MGR). In this paper, we find that, depending on the sources of capital overaccumulation, capital taxation may not be the most efficient means to restore the MGR when government debt is feasible. To demonstrate our point, we study optimal policy mix in achieving the socially optimal (MGR) level of aggregate capital stock in an infinite horizon heterogeneous-agents incomplete-markets economy where capital may be overaccumulated for two distinct reasons: (i) precautionary savings and (ii) production externalities. By solving the Ramsey problem analytically along the entire transitional path, we reveal that public debt and capital taxation play very distinct roles in dealing with the overaccumulation problem. The Ramsey planner opts neither to use a capital tax to correct the overaccumulation problem if it is caused solely by precautionary saving—regardless of the feasibility of public debt—nor to use debt (financed by consumption tax) to correct the overaccumulation problem if it is caused solely by production externality (such as pollution)—regardless of the feasibility of a capital tax. The key is that the MGR has two margins: an intratemporal margin pertaining to the marginal product of capital (MPK) and an intertemporal margin pertaining to the time discount rate. To achieve the MGR, the Ramsey planner needs to equate not only the private MPK with the social MPK but also the interest rate with the time discount rate—neither of which is equalized in a competitive equilibrium. Yet public debt and a capital tax are each effective only in calibrating one of the two margins, respectively, but not both.

JEL Classification: E13; E62; H21; H30

Key Words: Optimal Quantity of Debt, Capital Taxation, Ramsey Problem, Heterogeneous Agents, Incomplete Markets, Pollution, Production Externalities.

*Research Division, Federal Reserve Bank of St. Louis, P.O. Box 442, St. Louis, MO 63166-0442; Email: yilichien@gmail.com

†Antai School of Economics and Management, Shanghai Jiaotong University; and Research Division, Federal Reserve Bank of St. Louis, P.O. Box 442, St. Louis, MO 63166-0442; Email: yiwen08082008@outlook.com

1 Introduction

Whether capital should be taxed or not is a long-standing question in political economy and the history of economic thought. The classical answer to the question is "No," based on analysis with representative-agent models (eg. Chamley (1986)). However, Aiyagari (1995) shows that with heterogeneous households under borrowing constraints the classical result of zero capital tax is overturned. The intuition offered by Aiyagari is that borrowing constraints with idiosyncratic risks entice households to engage in precautionary saving behaviors that would lead to overaccumulation of capital. Hence, taxing capital in the steady state is optimal to restore aggregate allocative efficiency in light of the modified golden rule (MGR).

However, several papers have revisited these capital taxation issues in heterogenous-agents framework. For example, Chien and Wen (2021) show analytically that in the absence of any redistribution effects the optimal capital tax should still be zero despite precautionary saving motives that can lead to capital overaccumulation and the failure of the MGR. A similar result is also obtained by Bassetto and Cui (2020) based on a model with financial frictions on the firm side where the optimal long-run capital tax could be zero despite underaccumulation of capital.[1]

This paper intents to reveal and highlight the roles that MGR plays in shaping optimal fiscal policies when capital is overaccumulated for two distinct reasons: (i) precautionary savings and (ii) production externalities. We would like to understand under what conditions the failure of the MGR calls for non-zero capital taxation in the presence of heterogeneous agents and incomplete markets (HAIM) and why. Specifically, the MGR can fail for many reasons; for example, precautionary saving under financial frictions can result in overaccumulation of capital (á la Aiyagari (1994)); negative production externalities (such as environmental pollution) at the firm level can also lead to overaccumulation of capital (or overinvestment and overproduction) at the aggregate level (Cropper and Oates (1992)). In both cases the MGR does not hold in a competitive equilibrium. It is not yet clear if capital tax is the most effective and efficient policy tool to correct the overaccumulation problem regardless of the source of the market failures.

We introduce production externalities (e.g., pollution) into a version of the Aiyagari

[1] As shown by Bassetto and Cui (2020), there is another possible Ramsey steady state featuring positive capital tax if the consumption intertemporal elasticity of substitution is very high.

model to demonstrate why the MGR alone is not a sufficient criterion for capital taxation when the distribution of individual consumption matters for social welfare. Yet the MGR is the key criterion that Aiyagari (1995) relies on to derive his seminal result. The choice of production externality is by design since it is the simplest way to highlight the issue at stake without losing the analytical tractability of our model. Clearly, both precautionary saving and negative production externalities (such as pollution) can result in capital overaccumulation from a social viewpoint and thus the failure of the MGR.

In addition, the non-zero capital taxation result of Aiyagari (1995) and its explanations are not clearly understood in the existing literature. This is partly due to the model's intractability, which means that Aiyagari could not use the entire set of the Ramsey first-order conditions to check the internal consistency of his analysis and had to instead rely on the assumption of the existence of a Ramsey steady state with finite Lagrangian multipliers (without proof) in order to derive the result. But such an assumption might not be innocuous because the constraint set of the Ramsey problem could be non-convex. In contrast, by solving the Ramsey problem analytically in an infinite horizon general equilibrium HAIM model, we are able to utilize the full set of the Ramsey first-order conditions to prove the existence of the Ramsey steady state and derive our results; hence we can show clearly that public debt and capital taxation have very different functions and effectiveness in dealing with capital overaccumulation problems, depending on the root cause of the overaccumulation. Debt is shown to be more effective in improving welfare under pecuniary externalities (due to incomplete financial markets), while capital tax is more effective in improving welfare under production externalities (due to pollution or incomplete goods markets). In other words, the Ramsey planner will not use a capital tax to correct the overaccumulation problem if it is caused solely by precautionary saving under borrowing constraints—regardless of the feasibility of public debt—nor use debt to correct the overaccumulation problem if it is caused solely by pollution—regardless of the feasibility of capital taxation.

The intuition is that precautionary saving generates a pecuniary externality by depressing (distorting) the interest rate, while pollution generates a production externality by depressing (distorting) firms' total factor productivity. These two types of externalities can each lead to capital overaccumulation but are distinct in nature; hence, they call for very different fiscal policies or policy mixes to address them. The key insight is that the MGR in HAIM economies involves two distinct margins: the MPK (the first margin) and the time discount rate (the second margin). The first margin pertains to capital's intratemporal price, and the second margin pertains to capital's intertemporal price. To achieve the MGR, therefore, the Ramsey planner needs to equate not only the private MPK with the social MPK intratemporally, but also the interest rate (net of the liquidity premium of savings) with the time discount rate intertemporally. From a social viewpoint, neither margin is equalized in a laissez-faire competitive equilibrium. Yet a capital tax is more effective in dealing with the intratemporal (first) margin, while public debt is more effective in dealing with the intertemporal (second) margin (by eliminating the liquidity premium). When it comes to the issue of how to best finance public debt, we show that in the absence of a lump-sum tax the Ramsey planner opts to use a consumption tax instead of a capital tax. This result is reminiscent of the classical result obtained by Chamley (1986) that taxing capital is not optimal when other forms of distortionary taxes are available.

To focus attention on the core issues, we defer literature reviews to a latter section. So the remainder of the paper is organized as follows. Section 2 presents the model and solves its competitive equilibrium in closed forms. Section 3 uses the primal approach to analytically solve the Ramsey problem and optimal fiscal policies. Section 4 studies the robustness of our results by conducting two experiments: (i) ruling out government bonds—to show that with precautionary saving but without production externalities the optimal capital tax is still zero even if the planner cannot issue debt to crowd out capital; and (ii) ruling out a capital tax—to show that with pollution but without precautionary saving the optimal quantity of debt is indeterminate even if the planner cannot levy a capital tax to crowd out capital. Section 5 provides a brief literature review and Section 6 concludes the paper.

2 The Model

2.1 The Environment

Firms. A representative firm produces output according to the socially decreasing (or increasing) returns to scale but privately constant-returns-to-scale Cobb-Douglas technology,

$$Y_t = F(Z_t, K_t, N_t) = Z_t K_t^{\alpha} N_t^{1-\alpha}, \tag{1}$$

where Y, K, and L denote output, capital, and labor, respectively, and

$$Z_t = \overline{K}_t^{\varphi} \tag{2}$$

denotes production externalities (spill-over effects) arising from the average capital stock $\overline{K}_t$ in the economy, which is taken as given by the private sector, and where the elasticity parameter $\varphi \lesseqgtr 0$ implies that the externality (spillover effect) is negative, zero, or positive. Therefore, the production technology exhibits social decreasing (increasing) returns to scale if $\varphi < (>)\ 0$.[2]

The representative firm rents capital and hires labor from households by paying a competitive rental rate q_t and real wage rate w_t. Taking Z_t as given, the firm's optimal conditions for profit maximization at time t are

$$w_t = \frac{\partial F(Z_t, K_t, N_t)}{\partial N_t} \equiv MP_{N,t}, \tag{3}$$

$$q_t = \frac{\partial F(Z_t, K_t, N_t)}{\partial K_t} \equiv MP_{K,t}. \tag{4}$$

On the other hand, the social MPK takes $Z_t = K_t^{\varphi}$ endogenously and is defined as

$$\widetilde{q}_t \equiv \frac{\partial F\left(K_t^{\varphi}, K_t, N_t\right)}{\partial K_t} \tag{5}$$

In addition, $\widetilde{q}_t < q_t$ if $\varphi < 0$.

Households. There is a unit measure (continuum) of *ex ante* identical but *ex post* heterogeneous households that face idiosyncratic preference shock θ_t. The shock is identically and independently distributed (iid) over time and across households with mean $\overline{\theta}$ and the cumulative distribution $\mathbf{F}(\theta)$, which has the support $[\theta_L, \theta_H]$, where $\theta_H > \theta_L > 0$. Time is discrete and indexed by $t = 1, 2, ...\infty$.

There are two subperiods within each period t. The idiosyncratic preference shock θ_t is realized only in the second subperiod, and the labor supply decision must be made in the first subperiod before observing θ_t. Consumption and saving decisions are made in the second subperiod after the realization of θ_t. Namely, the idiosyncratic preference shock is uninsurable by labor income even if leisure enters the utility function linearly. Let $\theta^t \equiv$

[2] This paper mainly focuses on the case $\varphi < 0$.

$(\theta_1, ..., \theta_t)$ denote the history of idiosyncratic shocks. All households are endowed with the same asset holdings a_1 at the beginning of period 1.

Households are infinitely lived with a quasi-linear utility function and face the borrowing constraints: $a_{t+1}(\theta^t) \geq 0$. A household's lifetime expected utility is given by

$$V = E_1 \sum_{t=1}^{\infty} \beta^t \left[\theta_t \ln c_t(\theta^t) - n_t\left(\theta^{t-1}\right)\right], \tag{6}$$

where $\beta \in (0,1)$ is the discount factor and $c_t(\theta^t)$ and $n_t(\theta^{t-1})$ denote consumption and the labor supply, respectively, for a household with history θ^t at time t. Note that the labor supply in period t is only measurable with respect to θ^{t-1}, reflecting the assumption that the labor supply decision is made in the first subperiod before observing the preference shock θ_t.

Government. The government needs to finance an exogenous stream of purchases $\{G_t \geq 0\}_{t=1}^{\infty}$, and it can issue bonds and levy time-varying consumption and capital taxes at flat rates τ_t^c and τ_t^k, respectively. The flow government budget constraint in period t is

$$\tau_t^c C_t + \tau_t^k q_t K_t + B_{t+1} \geq G_t + r_t B_t, \tag{7}$$

where B_{t+1} is the level of government debt chosen in period t and r_t is the gross risk-free rate.

There is no aggregate uncertainty in our economy, and hence government bonds and productive capital are perfect substitutes as stores of value for households. As a result, the after-tax gross rate of return to capital must equal the gross risk-free rate:

$$r_t = 1 + (1 - \tau_t^k) q_t - \delta,$$

which constitutes a no-arbitrage condition for capital and bonds.

2.2 The Household Problem

Given the sequences of the interest rates $\{r_t\}_{t=1}^{\infty}$, consumption tax rates $\{\tau_t^c\}_{t=1}^{\infty}$, and wage rates $\{w_t\}_{t=1}^{\infty}$, a household chooses a plan of consumption, labor, and asset holdings, $\{c_t(\theta^t),$

$n_t(\theta^{t-1})$, $a_{t+1}(\theta^t)\}_{t=1}^{\infty}$, to solve

$$\max_{\{c_t(\theta^t), n_t(\theta^{t-1}), a_{t+1}(\theta^t)\}} E_1 \sum_{t=1}^{\infty} \beta^t \left\{\theta_t \ln c_t\left(\theta^t\right) - n_t\left(\theta^{t-1}\right)\right\}$$

subject to

$$\left(1+\tau_t^c\right) c_t\left(\theta^t\right) + a_{t+1}\left(\theta^t\right) \leq w_t n_t\left(\theta^{t-1}\right) + r_t a_t\left(\theta^{t-1}\right), \tag{8}$$

$$a_{t+1}\left(\theta^t\right) \geq 0, \tag{9}$$

with $a_1 > 0$ given and $n_t\left(\theta^{t-1}\right) \in \left[0, \overline{N}\right]$.

Define a household's gross income (cash on hand) as

$$x_t(\theta^{t-1}) \equiv r_t a_t(\theta^{t-1}) + w_t n_t(\theta^{t-1}) \tag{10}$$

and the aggregate (average) cash on hand as

$$X_t \equiv \int x_t(\theta^{t-1}) d\mathbf{F}(\theta_t) = r_t A_t + w_t N_t, \tag{11}$$

where $A_t \equiv \int a_t\left(\theta^{t-1}\right) d\mathbf{F}\left(\theta\right)$ and $N_t \equiv \int n_t\left(\theta^{t-1}\right) d\mathbf{F}\left(\theta\right)$ denote aggregate asset holdings and aggregate labor, respectively. The solution of the household problem can be characterized analytically by a cutoff strategy in the following proposition:

Proposition 1. *The optimal household decisions for cash on hand $x_t(\theta^{t-1})$, consumption $c_t\left(\theta^t\right)$, savings $a_{t+1}\left(\theta^t\right)$, and the labor supply $n_t\left(\theta^{t-1}\right)$ are given, respectively, by the following policy rules:*[3]

$$x_t\left(\theta^{t-1}\right) = x_t = \begin{cases} w_t L(\theta_t^*)\theta_t^* & \text{if } \theta_t^* < \theta_H \\ X_t & \text{if } \theta_t^* = \theta_H \end{cases}, \tag{12}$$

$$c_t(\theta^t) = \begin{cases} \min\left\{1, \frac{\theta_t}{\theta_t^*}\right\} \frac{x_t}{(1+\tau_t^c)} & \text{if } \theta_t^* < \theta_H \\ \theta_t \frac{w_t}{(1+\tau_t^c)} & \text{if } \theta_t^* = \theta_H \end{cases}, \tag{13}$$

$$a_{t+1}(\theta^t) = \begin{cases} \max\left\{1 - \frac{\theta_t}{\theta_t^*}, 0\right\} x_t & \text{if } \theta_t^* < \theta_H \\ x_t - \theta_t \frac{w_t}{(1+\tau_t^c)} & \text{if } \theta_t^* = \theta_H \end{cases}, \tag{14}$$

[3]The cutoff policy rules hold if the individual labor decision is an interior one; namely, $n_t \in (0, \overline{N})$. We discuss the conditions that ensure the interior solution of n in the proof of this proposition (Appendix A.1).

$$n_t(\theta^{t-1}) = \frac{1}{w_t}\left[x_t - r_t a_t(\theta^{t-1})\right], \tag{15}$$

where the cutoff θ_t^* is independent of individual history and is determined by the following Euler equation:

$$\frac{1}{r_{t+1}} = \beta \frac{w_t}{w_{t+1}} L(\theta_t^*), \tag{16}$$

where the function $L\left(\theta_t^*\right)$ captures the liquidity premium of savings and is given by

$$L(\theta_t^*) \equiv \int_{\theta \leq \theta_t^*} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) \geq 1. \tag{17}$$

Note the liquidity premium vanishes ($L\left(\theta_t^*\right) = 1$) when $\theta_t^* = \theta_H$. In addition, by summing up equations (13) and (14), aggregate consumption C_t and aggregate saving A_{t+1} are given, respectively, by

$$C_t = \begin{cases} D\left(\theta_t^*\right) \frac{x_t}{(1+\tau_t^c)} & \text{if } \theta_t^* < \theta_H \\ \overline{\theta} \frac{w_t}{(1+\tau_t^c)} & \text{if } \theta_t^* = \theta_H \end{cases} \tag{18}$$

and

$$A_{t+1} = \begin{cases} \left[1 - D\left(\theta_t^*\right)\right] x_t & \text{if } \theta_t^* < \theta_H \\ X_t - \overline{\theta} \frac{w_t}{(1+\tau_t^c)} & \text{if } \theta_t^* = \theta_H \end{cases}, \tag{19}$$

where the function $D(\theta_t^*)$ denotes the aggregate marginal propensity to consume and is defined as

$$D(\theta_t^*) \equiv \int_{\theta \leq \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} d\mathbf{F}(\theta) \in (0, 1]. \tag{20}$$

Proof. See Appendix A.1. □

Note that, as shown in the proof (Appendix A.1), the distribution of x_t is degenerated and hence all households choose the same x_t if $\theta_t^* < \theta_H$. However, if $\theta_t^* = \theta_H$, then the distribution of x_t becomes indeterminate so long as each household holds enough cash on hand to ensure non-binding borrowing constraints across all states of θ. In this latter case, since the distribution has no impact on the Ramsey allocation, we assume (without loss of generality) that in the case of $\theta^* = \theta_H$, x_t is degenerated such that $x_t = X_t$.

2.3 Competitive Equilibrium

Our discussion involves two different notions of the steady state: a "competitive equilibrium steady state" for a given set of government policies and a "Ramsey steady state" under optimal policies.

Denote C_t, N_t, and K_{t+1} as the levels of aggregate consumption, aggregate labor, and the aggregate capital stock, respectively. A competitive equilibrium allocation can be defined as follows:

Definition 1. *Given initial aggregate capital K_1 and bonds B_1, as well as a sequence of taxes, government spending, and government debt, $\{\tau_t^c, \tau_t^k, G_t, B_{t+1}\}_{t=1}^{\infty}$, a competitive equilibrium is a sequence of prices $\{w_t, q_t\}_{t=1}^{\infty}$ and allocations $\{c_t(\theta^t), n_t(\theta^{t-1}), a_{t+1}(\theta^t), K_{t+1}, N_t\}_{t=1}^{\infty}$ such that the following hold:*

1. *The average capital stock equals the private capital stock in the production function: $\overline{K}_t = K_t$.*
2. *The no-arbitrage condition holds for each period: $r_t = 1 + (1 - \tau_t^k)q_t - \delta$ for all $t \geq 1$.*
3. *Given the sequence $\{w_t, r_t, \tau_t^c\}_{t=1}^{\infty}$, the sequence $\{c_t(\theta^t), a_{t+1}(\theta^t), n_t(\theta^{t-1})\}_{t=1}^{\infty}$ solves the household problem.*
4. *Given the sequence $\{w_t, q_t\}_{t=1}^{\infty}$, the sequence $\{N_t, K_t\}_{t=1}^{\infty}$ solves the firm's problem.*
5. *The government budget constraint in equation (7) holds for each period.*
6. *All markets clear for all $t \geq 1$:*

$$K_{t+1} = \int a_{t+1}(\theta_t) d\mathbf{F}(\theta_t) - B_{t+1}, \tag{21}$$

$$N_t = \int n_t(\theta_{t-1}) d\mathbf{F}(\theta_{t-1}), \tag{22}$$

$$\int c_t(\theta_t) d\mathbf{F}(\theta_t) + G_t + K_{t+1} \leq F(K_t^{\varphi}, K_t, N_t) + (1 - \delta)K_t. \tag{23}$$

In the following proposition, we provide the condition that the steady-state capital stock in a laissez-faire competitive equilibrium is higher than that implied by the MGR such that there exists overaccumulation of capital even if the production externality parameter $\varphi = 0$.

Proposition 2. *If the upper bound θ_H of the preference shock is sufficiently large relative to the mean $E(\theta) \equiv \overline{\theta}$ such that the following condition holds:*

$$\frac{\alpha\beta\theta_H}{\theta_H - \overline{\theta}} + \beta(1-\alpha)(1-\delta) < 1, \tag{24}$$

then in a laissez-faire competitive equilibrium the steady-state risk-free rate is lower than the time discount rate, $r < 1/\beta$, with a positive liquidity premium $L(\theta^*) > 1$ and overaccumulated capital stock.

Proof. See Appendix A.2. □

Notice that when $\theta_H \to \infty$, as in the case of a Pareto distribution, the above condition is clearly satisfied. The intuition of Proposition 2 is straightforward. Since labor income is determined (*ex ante*) before the realization of the idiosyncratic preference shock θ_t, a household's total income may be insufficient *ex post* to provide full self-insurance for large enough preference shocks under condition (24). In this case, precautionary saving leads to overaccumulation of capital at the aggregate level, which reduces the equilibrium interest rate to a level below the time discount rate, regardless of the production externality φ. This outcome may seem inefficient from a social point of view but it emerges because of the negative pecuniary externality household savings have on the interest rate (due to diminishing MPK), as noted by Aiyagari (1994).

Note that when a negative production externality is allowed ($\varphi < 0$), the capital-overaccumulation problem further intensifies. In other words, both the pecuniary externality under precautionary saving and the production externality under $\varphi < 0$ lead to overaccumulation of capital. One of our interests in this paper is to understand how optimal fiscal policies react to the two types of externalities.

As explained by Chien and Wen (2021), in the absence of production externalities ($\varphi = 0$), a competitive equilibrium can achieve the MGR if the idiosyncratic risk is sufficiently small (e.g., the upper bound θ_H is close enough to the mean $\overline{\theta}$) such that condition (24) is violated. In this case, household savings can become sufficiently large to fully buffer preference shocks and, as a result, household borrowing constraints will never bind. Clearly, with full self-insurance, it must be true that the optimal cutoff is a corner solution at $\theta^* = \theta_H$ with a vanishing liquidity premium ($L(\theta^*) = 1$) and an interest rate that equals the time discount rate ($r = 1/\beta$). We will see in this paper if the same results hold when $\varphi \neq 0$.

But a competitive equilibrium with full self-insurance is impossible in the Aiyagari model (regardless of production externalities) because every household's marginal utility of consumption follows a supermartingale in his model when $r = 1/\beta$. This implies that household consumption and savings (or asset demand) diverge to infinity in the long run, which cannot constitute an equilibrium.[4]

In contrast, because the household utility function is quasi-linear in this paper, the expected shadow price of consumption goods is thus the same across agents and given by $\int \lambda_t \mathbf{F}(\theta) = \frac{1}{w_t}$, which kills the supermartingale property of the household marginal utility of consumption. As a result, household savings (or asset demand) are bounded away from infinity even at the point of $r = 1/\beta$. This property not only renders our model analytically tractable but also sheds great light on the issues of optimal quantity of public debt and optimal taxation.

More specifically, equations (12) and (14) show that when $r = 1/\beta$ household asset demand is always bounded above by $(\theta_H - \theta_t) w_t$ for any shock $\theta_t \in [\theta_L, \theta_H]$. This endogenous upper bound on asset demand is finite as long as the support $[\theta_L, \theta_H]$ of θ_t is bounded (a counter example is a Pareto distribution where $\theta_H = \infty$). This special property renders our model analytically tractable with closed-form solutions (provided that θ_t is iid)—despite incomplete markets and aggregate production externalities—and it implies that the Ramsey planner has the potential to use government debt to achieve the MGR in this economy when the competitive equilibrium is not socially optimal.

2.4 Conditions to Support a Competitive Equilibrium

Given that government policies are inside the aggregate state space of the full set of competitive equilibria and they affect the endogenous distributions (including the average) of all household variables, the Ramsey problem is to pick a competitive equilibrium (through policies) that attains the maximum of the expected household lifetime utility V defined in (6). Since V depends on the endogenous distributions (characterized by the sequence of the cutoff $\{\theta_t^*\}_{t=1}^{\infty}$), the Ramsey planner needs also to pick a particular time path (sequence) of distributions to achieve the maximum.

[4] See Ljungqvist and Sargent (2012, Chapter 17) for details.

Proposition 3. *Given initial capital K_1, initial government bonds B_1, and the initial risk-free rate r_1, the sequences of aggregate allocations $\{C_t, N_t, K_{t+1}, B_{t+1}\}_{t=1}^{\infty}$ and distribution statistics $\{\theta_t^*\}_{t=1}^{\infty}$ can be supported as a competitive equilibrium if and only if the following are true:*

1. *The resource constraint (23) holds with $\overline{K}_t = K_t$.*

2. *The wage rate condition holds with*

$$MP_{N,t} = \begin{cases} \frac{K_{t+1}+B_{t+1}}{\theta_t^* L(\theta_t^*)[1-D(\theta_t^*)]} & \text{if } \theta_t^* < \theta_H \\ MP_{N,t} & \text{if } \theta_t^* = \theta_H \end{cases}. \tag{25}$$

3. *The implementability conditions hold:*

$$N_t \leq \begin{cases} L(\theta_t^*)\theta_t^* - \frac{1}{\beta}\theta_{t-1}^* \left(1 - D(\theta_{t-1}^*)\right) & \text{if } \theta_t^* < \theta_H \\ \bar{\theta} + \frac{K_{t+1}+B_{t+1}}{MP_{N,t}} - \frac{1}{\beta}\frac{K_t+B_t}{MP_{N,t-1}} & \text{if } \theta_t^* = \theta_H \end{cases} \quad \text{for } t \geq 2 \tag{26}$$

$$N_1 \leq \begin{cases} L(\theta_1^*)\theta_1^* - r_1 C_1^{-1} D(\theta_1^*) L(\theta_1^*)\theta_1^*(K_1 + B_1) & \text{if } \theta_1^* < \theta_H \\ \bar{\theta} + \frac{K_2+B_2}{MP_{N,1}} - \frac{r_1(K_1+B_1)}{MP_{N,1}} & \text{if } \theta_1^* = \theta_H \end{cases} \quad \text{for } t = 1. \tag{27}$$

Proof. See Appendix A.3. □

This proposition demonstrates that the Ramsey planner can construct a competitive equilibrium by simply choosing the sequences of aggregate allocations $\{C_t, N_t, K_{t+1}, B_{t+1}\}_{t=1}^{\infty}$ and the distribution statistics $\{\theta_t^*\}_{t=1}^{\infty}$ to maximize welfare subject to the aggregate resource constraint, the asset market-clearing condition, and the implementability condition, as shown explicitly below.

3 Ramsey Allocations

3.1 The Ramsey Problem

The Ramsey planner treats $\overline{K}_t = K_t$ as endogenous. So the aggregate production function for the Ramsey planner becomes $Y_t = F(K_t^{\varphi}, K_t, N_t) = K_t^{\alpha+\varphi} N_t^{1-\alpha}$, and the social MPK is given by $\widetilde{q}_t = (\alpha + \varphi)\frac{Y_t}{K_t}$. Armed with Proposition 3, we are ready to write down the Ramsey planner's problem and derive the first-order Ramsey conditions analytically. Appendix A.4 shows that the lifetime utility function, V, can be rewritten as a function of θ_t^* and aggregate variables:

$$V = \sum_{t=1}^{\infty} \beta^t \left\{ W(\theta_t^*) + \overline{\theta} \ln C_t - N_t \right\}, \tag{28}$$

where $W(\theta_t^*)$ is defined as

$$W(\theta_t^*) \equiv \overline{\theta} \ln \frac{1}{D(\theta_t^*)} + \int_{\theta \leq \theta_t^*} \theta \ln \left(\frac{\theta}{\theta_t^*} \right) d\mathbf{F}. \tag{29}$$

Thus, the Ramsey problem can be represented as maximizing the welfare function (28) by choosing the sequence of $\{\theta_t^*, N_t, C_t, K_{t+1}, B_{t+1}\}_{t=1}^{\infty}$ subject to the resource constraint (23), the wage-rate condition (25), and the implementability conditions (26) and (27). Therefore, the Lagrangian of the Ramsey problem is given by

$$\begin{aligned} L \quad = \quad & \max_{\{\theta_t^*, N_t, C_t, K_{t+1}, B_{t+1}\}} \sum_{t=1}^{\infty} \beta^t \left\{ W(\theta_t^*) + \overline{\theta} \ln C_t - N_t \right\} \\ & + \sum_{t=1}^{\infty} \beta^t \mu_t \left\{ F(K_t^{\varphi}, K_t, N_t) + (1-\delta) K_t - G_t - C_t - K_{t+1} \right\} \\ & + \beta^1 \lambda_1 \left\{ \begin{array}{c} \left(L(\theta_1^*)\theta_1^* - N_1 - \frac{D(\theta_1^*)L(\theta_1^*)\theta_1^*}{C_1} r_1 (K_1 + B_1) \right) \times \mathbf{1}_{\theta_1^* < \theta_H} \\ + \left(\bar{\theta} - N_1 - \frac{r_1(K_1+B_1)}{MP_{N,1}} + \frac{K_2+B_2}{MP_{N,1}} \right) \times \mathbf{1}_{\theta_1^* = \theta_H} \end{array} \right\} \\ & + \sum_{t=2}^{\infty} \beta^t \lambda_t \left\{ \begin{array}{c} \left(L(\theta_t^*)\theta_t^* - N_t - \frac{1}{\beta} \theta_{t-1}^* \left[1 - D(\theta_{t-1}^*) \right] \right) \times \mathbf{1}_{\theta_t^* < \theta_H} \\ + \left(\bar{\theta} - N_t - \frac{1}{\beta} \frac{K_t + B_t}{MP_{N,t-1}} + \frac{K_{t+1}+B_{t+1}}{MP_{N,t}} \right) \times \mathbf{1}_{\theta_t^* = \theta_H} \end{array} \right\} \\ & + \sum_{t=1}^{\infty} \beta^t \phi_t \left\{ \left(MP_{N,t} - \frac{K_{t+1} + B_{t+1}}{\theta_t^* L(\theta_t^*) \left[1 - D(\theta_t^*) \right]} \right) \times \mathbf{1}_{\theta_t^* < \theta_H} \right\}, \end{aligned} \tag{30}$$

where $\{\mu_t, \lambda_t, \phi_t\}$ denote the multipliers for the resource constraints, the implementability conditions, and wage condition, respectively. In addition, the index function $\mathbf{1}$ takes the value of 1 or 0 conditional on the state of the cutoff θ_t^*. To conserve space, the first-order Ramsey conditions as well as several useful lemmas for the upcoming proofs are relegated to the Appendix A.5.

3.2 Characterization of Optimal Steady-State Capital Taxation

Definition 2. *A Ramsey steady state is a Ramsey allocation where the parameter restriction $\theta_H < \frac{\theta_L}{1-\beta} < \infty$ (to ensure positive labor $n > 0$ for all individuals in all states) is satisfied and aggregate variables $\{K_t, N_t, C_t, \theta_t^*\}$ converge to finite positive values.*

The condition $\theta_H < \frac{\theta_L}{1-\beta}$ (or equivalently $\beta > \frac{\theta_H - \theta_L}{\theta_H}$) is required to ensure that all household labor decisions are positive—a necessary condition for Proposition 1. The intuition is that if the variance (support) of θ is too large (spread out), some agents may end up with too much savings in the last period and thus opt not to work this period. Our model becomes intractable if the constraint $N_t \geq 0$ binds occasionally, so $N_t = 0$ must be ruled out.

Proposition 4. *There exists a unique Ramsey steady state with the following properties:*

1. *The optimal quantity of debt is such that the cutoff $\theta^* = \theta_H$, the liquidity premium $L(\theta^*) = 1$, and no households are borrowing constrained: $a > 0$.*

2. *The social MPK equals the after-tax private MPK: $\widetilde{q} = (1 - \tau_k^*)\, q$.*

3. *The optimal capital tax τ_k^* is determined by the following equation:*

$$(1 - \tau_k^*) = \frac{\alpha + \varphi}{\alpha}. \tag{31}$$

Proof. See Appendix A.6. □

Equation (31) implies the following:

(i) If the quantity-spillover effect is zero ($\varphi = 0$), then the optimal capital tax $\tau_k^* = 0$. This result replicates that in Chien and Wen (2021) despite the fact that social returns to scale and private returns to scale do not equal. (ii) If the quantity-spillover effect is negative ($\varphi < 0$), then the steady-state optimal capital tax $\tau_k^* > 0$. (iii) If the quantity-spillover effect is positive ($\varphi > 0$), then the steady-state optimal capital tax $\tau_k^* < 0$.

Therefore, in the absence of any government debt-limit constraints, the Ramsey planner achieves the MGR without the need to tax/subsidize capital in the steady state unless production externalities are present—because capital taxation in the absence of quantity-spillover effects would decrease the steady-state household saving rate and thus permanently hampering households' self-insurance positions. Instead, the Ramsey planner opts to provide enough incentives for households to save through bond holdings by picking a sufficiently high interest rate ($= 1/\beta$) on government bonds, such that all households are fully self-insured in the long run, with zero probability of encountering a binding liquidity constraint.

Furthermore, even if capital taxation/subsidization is justifiable in the case of $\varphi \neq 0$, it is meant only to correct the production externalities and not to be a source of revenues to pay for interest on government bonds—which instead is financed by a distortionary consumption tax. In other words, if the overaccumulation of capital is caused by the pecuniary externality from precautionary saving of households, then there is no reason for the Ramsey planner to levy a capital tax because overaccumulation is the consequence of incomplete insurance markets and not of incomplete goods markets (if $\varphi = 0$).

Therefore, it is the role of government debt in improving self-insurance, not its role in crowding out capital or mitigating the distortionary effects of a capital tax, that determines the optimal quantity of public debt in this model.

In the next section, we will also show that even when the government cannot issue debt ($B_{t+1} = 0$ for all $t \geq 1$), the optimal capital tax rate is still given by equation (31)—in which case the full self-insurance allocation is no longer feasible and $\theta^* < \theta_H$ and $L(\theta^*) > 1$ in the Ramsey steady state. Therefore, optimal capital taxation is independent of the households' self-insurance positions and the optimal level of public debt depends only on the distortions from the financial markets (from the price-spillover effect).

To see how the debt supply and capital taxation can operate independently, we study in the next section two extreme cases: (i) the government cannot issue debt and (ii) the government cannot tax capital. In the case with no government bonds, we show that the optimal capital tax policy remains the same as in equation (31); namely, it is feasible but not optimal to levy a capital tax to correct the overaccumulation problem when the pecuniary externality is the only cause of the problem and government debt is not available to mitigate the problem. In the case with no capital tax, we show that government debt is an ineffective tool to address capital overaccumulation caused by production externalities.

4 Robustness Analyses

In what follows, we will consider two special cases: Case A where the only available policy tool is a capital tax and a lump-sum transfer that redistributes the government revenue back to households and Case B where the only available policy tool is public debt and a lump-sum tax to finance the public debt. For simplicity and without loss of generality, we assume in each case that government spending G is zero.

4.1 Case A: A Capital Tax Only

This special case is particularly illuminating on the role of the MGR in determining the optimal quantity of debt in a HAIM economy. It would appear that if it is optimal to tax capital to restore the MGR in the presence of pecuniary externalities even when government bonds and other forms of distortionary taxes are available, then it would be even more desirable to tax capital to restore the MGR when a consumption tax is not available and the government cannot issue any debt—since a capital tax is now the only tool available to restore the MGR. But we will show that this intuition is false and that if there is any reason to tax capital at all, it must be because of the production externality under pollution ($\varphi < 0$), not because of the pecuniary externality under precautionary savings.

This analysis also suggests that our quasi-linear preference structure is not essential for the previous result in equation (31). If it is not optimal to tax capital in the absence of debt, it should remain optimal not to tax capital after government bonds are reintroduced to the model if public debt can be financed by other forms of distortionary taxes (such as a labor tax or a consumption tax). Consequently, the optimal quantity of debt cannot be influenced by the trade-off between the provision of household self-insurance and the crowding out of capital under a capital tax. This result reinforces the findings in the zero-capital-taxation literature based on representative-agent models (see, e.g., Chamley (1986), Chari, Christiano, and Kehoe (1996), Chari and Kehoe (1999), and Chari, Atkeson, and Kehoe (1999)).

To show our results under Case A, since the primal approach becomes more involved when fewer government tools are available, we will take the following (dual) approach: The Ramsey planner directly chooses the infinite sequence of capital taxes $\left\{\tau_t^k\right\}_{t=2}^{\infty}$ that maximize the welfare V based on the competitive-equilibrium allocation $\{C_t, K_{t+1}, N_t, \theta_t^*\}_{t=1}^{\infty}$, in which each competitive-equilibrium quantity at time t is a function of the sequence $\left\{\tau_{t+j}^k\right\}_{j=1}^{\infty}$.

Specifically, for simplicity and without loss of generality, let $\delta = 1$ and $G_t = 0$. Assume that any government revenues from capital taxes are lump-sum transferred back to households:

$$T_t = \tau_t^k q_t K_t,$$

so the household budget constraint becomes

$$c_t + a_{t+1} \leq \left(1 - \tau_t^k\right) q_t a_t + w_t n_t + T_t \equiv x_t,$$

where the market rental rate q_t (private MPK) and wage rate w_t are the same as before and given in equilibrium, respectively, by

$$q_t = \alpha \frac{F\left(K_t^{\varphi}, K_t, N_t\right)}{K_t}, \tag{32}$$

$$w_t = (1 - \alpha) \frac{F\left(K_t^{\varphi}, K_t, N_t\right)}{N_t}. \tag{33}$$

Moreover, the private constant-returns-to-scale production function and the aggregate household resource constraint imply the aggregate goods market-clearing condition

$$x_t = C_t + K_{t+1} = Y_t.$$

Then, the competitive equilibrium is characterized by equations (32) and (33) together with the following aggregate decision rules:

$$1 = \beta \left(1 - \tau_{t+1}^k\right) q_{t+1} L(\theta_t^*) \frac{w_t}{w_{t+1}}, \tag{34}$$

$$C_t = D\left(\theta_t^*\right) Y_t, \tag{35}$$

$$K_{t+1} = \left[1 - D\left(\theta_t^*\right)\right] Y_t, \tag{36}$$

$$Y_t = w_t \theta_t^* L(\theta_t^*), \tag{37}$$

$$Y_t = F\left(K_t^{\varphi}, K_t, N_t\right) = K_t^{\alpha+\varphi} N_t^{1-\alpha}. \tag{38}$$

Given capital-tax policies, this system of seven equations uniquely solves for the paths of seven aggregate variables, $\{C_t, K_{t+1}, N_t, Y_t, w_t, q_t, \theta_t^*\}$, as functions of $\left\{\tau_t^k\right\}_{t=1}^{\infty}$.[5]

[5]The uniqueness of the equilibrium can be confirmed by the eigenvalue method that shows the steady state is saddle-path stable.

To establish these functions in closed forms, notice that equations (37) and (33) imply the equilibrium labor supply

$$N_t = (1-\alpha)\,\theta_t^* L(\theta_t^*). \tag{39}$$

Substituting the real wage and interest rate in equation (34) using their competitive-equilibrium definitions and rearranging, we have

$$\theta_t^*\left[1 - D(\theta_t^*)\right] = \beta\alpha\left(1-\tau_{t+1}^k\right)\theta_{t+1}^* L(\theta_{t+1}^*). \tag{40}$$

By the definitions of $L(\theta^*)$ and $D(\theta^*)$, these two functions are related by the identity

$$\theta_t^*\left[1 - D\left(\theta_t^*\right)\right] = \theta_t^* L\left(\theta_t^*\right) - \bar{\theta}. \tag{41}$$

Hence, equation (40) can be rewritten in the present-value form:

$$\begin{aligned}
\theta_t^* L(\theta_t^*) &= \bar{\theta} + \beta\alpha\left(1-\tau_{t+1}^k\right)\theta_{t+1}^* L\left(\theta_{t+1}^*\right) \\
&= \bar{\theta} + \sum_{j=0}^{\infty}\left[\prod_{h=0}^{j}\left[\beta\alpha\left(1-\tau_{t+1+h}^k\right)\right]\right]\bar{\theta},
\end{aligned} \tag{42}$$

which is a convergent sequence with the stochastic discounting factor: $0 < \beta\alpha\left(1-\tau_{t+j}^k\right) < 1$. Equation (42) implies that the cutoff θ_t^* in any period t depends only on the future tax rate $\left\{\tau_{t+1}^k, \tau_{t+2}^k, ..., \tau_{t+\infty}^k\right\}$, but not on the past history (including the current period) of the tax rate, $\left\{\tau_1^k, \tau_2^k, ..., \tau_t^k\right\}$, or any other endogenous state variable in the economy.

Therefore, the solution for the equilibrium cutoff in period t can be expressed implicitly as

$$\theta_t^* = \theta\left(\tau_{t+1}^k, \tau_{t+2}^k, ..., \tau_{t+\infty}^k\right). \tag{43}$$

Since $\theta_t^* L\left(\theta_t^*\right)$ is increasing in θ_t^*, an immediate implication of equation (42) is that

$$\frac{\partial\theta_t^*}{\partial\tau_{t+1+h}^k} < 0 \text{ for } h \geq 0. \tag{44}$$

Namely, higher future tax rates reduce the current cutoff. Since $\frac{\partial[1-\mathbf{F}(\theta^*)]}{\partial\theta^*} < 0$, the probability of a biding borrowing constraint $(1-\mathbf{F}\left(\theta^*\right))$ also increases with higher future tax rates. This suggests that capital income tax destroys households' self-insurance positions by tightening their borrowing constraints.

Once the equilibrium cutoff is solved as a function of future tax rates, equations (35)-(38) pin down the other endogenous variables completely and uniquely as functions of $\left\{\tau_{t+j}^k\right\}_{j\geq 1}$. The welfare function is then given by

$$V\left(\left\{\tau_t^k\right\}_{t=1}^{\infty}\right)=\sum_{t=1}^{\infty}\beta^t\left\{W\left(\theta_t^*\left(\{\tau_j^k\}_{j=t+1}^{\infty}\right)\right)+\bar{\theta}\log Y_t(\{\tau_j^k\}_{j=t+1}^{\infty})-N_t(\{\tau_j^k\}_{j=t+1}^{\infty})\right\},$$

where W is redefined as

$$W(\theta_t^*\left(\{\tau_j^k\}_{j=t+1}^{\infty}\right))\equiv\int_{\theta\leq\theta_t^*\left(\{\tau_j^k\}_{j=t+1}^{\infty}\right)}\theta\ln\left(\frac{\theta}{\theta_t^*\left(\{\tau_j^k\}_{j=t+1}^{\infty}\right)}\right)d\mathbf{F}.$$

Proposition 5. *Taking the first-period τ_1^k as given, the optimal steady-state capital tax rate is given by*

$$\tau_k^*=-\frac{\varphi}{\alpha}.$$

In the absence of production externalities ($\varphi=0$) and starting from any arbitrary initial date $t=1$, it is optimal to set $\tau_t^k=0$ for all $t>1$ regardless of the variance of idiosyncratic risk θ or the severeness of dynamic inefficiency due to precautionary savings (measured by the liquidity premium $L\left(\theta^*\right)>1$).

Proof. See Appendix A.7. □

This proposition states that in the Ramsey steady state, despite overaccumulation of capital, the optimal capital tax is zero if there is no production externality ($\varphi=0$), and the optimal capital tax is positive if and only if there is a production externality ($\varphi<0$). In addition, in the absence of production externalities and given any initial tax rate τ_1^k, it is optimal to immediately set future tax rates to zero: $\tau_t^k=0$ for all $\tau\geq 2$. These results are robust to the distribution $\mathbf{F}\left(\theta\right)$ of idiosyncratic shocks or the tightness of household borrowing constraints.

Therefore, despite overaccumulation of capital under precautionary saving, the Ramsey planner will not use a capital tax to restore the MGR or reduce the capital stock unless capital is overaccumulated for an entirely different reason, such as pollution. The fundamental rationale is that when the MGR fails to hold along the intertemporal margin ($L\left(\theta^*\right)>1$), the effective tool to restore the MGR is issuing government bonds instead of taxing capital. Since issuing government bonds is not feasible, the Ramsey planner opts to leave the intertemporal margin as it is and only use a capital tax to take care of the intratemporal margin if $\varphi \neq 0$.

4.2 Case B: Government Debt Only

The purpose of studying Case B is to show whether the Ramsey planner is able and willing to use the quantity of debt to correct capital overaccumulation when a full self-insurance position is already reached such that $\theta_t^* = \theta_H$ and no households are borrowing constrained ($L(\theta_t^*) = 1$). In this case the MGR fails only along the intratemporal margin where the social MPK does not equal the private MPK. Hence, Case B considers the situation where (i) $\theta_t^* = \theta_H$ for all period t, (ii) the Ramsey planner is equipped with only government bonds and a lump-sum tax as policy tools, and (iii) there is no government spending, $G_t = 0$. The condition $\theta_t^* = \theta_H$ can be justified by assuming a sufficiently large initial debt level B_1 so that the economy is under full self-insurance initially.

This special case is very informative on the role of government debt in addressing the failure of the MGR due to a negative production externality. If it is optimal in general to use debt to crowd out capital to restore the MGR in the presence of a production externality (financed by a distortionary consumption tax), then it would be even more desirable to do so in Case B. This is so because (i) the public debt's interest payment can now be financed by a non-distortionary lump-sum tax and (ii) government bonds are now the only possible tool to restore the MGR. We will show, surprisingly, that it is optimal for the Ramsey planner to do nothing in Case B, leaving the quantity of debt simply determined by its initial level B_1. The fundamental reason is that public debt is not the right tool to restore the MGR when MGR fails only along the intratemporal margin with $\varphi \neq 0$.

Notice that to study the Ramsey plan in Case B, several corresponding changes need to be made in the definition of a competitive equilibrium and in the construction of the Ramsey problem. The details of these changes are provided in Appendix A.8, which also proves the following proposition that describes the conditions needed to support the competitive equilibrium of Case B.

Proposition 6. *Given initial capital K_1, initial government bonds B_1, and the initial risk-free rate r_1, the sequence of Ramsey allocations $\{C_t, N_t, K_{t+1}, B_{t+1}\}_{t=1}^{\infty}$ can be supported as a competitive equilibrium if and only if the following are true:*

1. *The aggregate resource constraint (23) holds with* $\overline{K}_t = K_t$*:* $C_t + K_{t+1} = F\left(K_t^{\varphi}, K_t, N_t\right) + (1-\delta) K_t$.

2. *The competitive-equilibrium consumption function holds:* $C_t = \overline{\theta}\,(1-\alpha)\, K_t^{\alpha+\varphi} N_t^{-\alpha}$.

3. *The no-arbitrage condition* $\frac{1}{\beta}\frac{MP_{N,t-1}}{MP_{N,t}} = 1 - \delta + q_t$ *holds, where* $q_t = \alpha K_t^{\alpha+\varphi-1} N_t^{1-\alpha}$ *is the private MPK.*

Proof. See Appendix A.8. □

There are two important messages implied by the above proposition. First, notice that the optimal bond supply B_{t+1} does not enter the three constraints (1)-(3) listed in Proposition 6. This suggests that the Ramsey planner's use of government bonds is irrelevant for the Ramsey outcome—because it cannot influence the competitive equilibrium. Moreover, the three constraints in (1)-(3) of Proposition 6 exactly pin down the three unknowns, $\{C_t, N_t, K_{t+1}\}_{t=1}^{\infty}$, which implies that the Ramsey planner's allocation is identical to the allocation of the competitive equilibrium (except the levels of B_{t+1} and T_t).

The above discussions and Proposition 6 imply the following corollary:

Corollary 1. *The competitive equilibrium with policy* $B_{t+1} = B_1$ *and* $T_t = (r_t - 1)B_1$ *for all* $t \geq 1$ *is a Ramsey equilibrium.*

The fundamental reason is that the quantity of government bonds is not an effective tool to address the capital-overaccumulation problem caused by production externalities ($\varphi \neq 0$). To crowd out capital, the supply of debt has to alter the equilibrium interest rate—the intertemporal price of capital. However, once the economy is in full self-insurance, $\theta_t^* = \theta_H$, there is no liquidity premium and hence any additional supply of government bonds can no longer raise the market interest rate (determined by the time discount rate $1/\beta$) and thus cannot crowd out capital. This makes the quantity of government bonds ineffective in addressing the failure of the MGR along the intratemporal margin due to production externalities.

In short, despite overaccumulation of capital caused by the negative production externality ($\varphi < 0$), the Ramsey planner is powerless to restore the MGR by reducing the aggregate capital stock, unless the government is equipped with the right tool (such as a capital tax).

5 A Brief Literature Review

Our work follows and extends the traditional literature of optimal taxation based on representative-agent models. That literature has shown that if the government's only option is to tax factor income to finance government expenditures, then it should tax labor income instead of capital income; see e.g., Chamley (1986), Chari, Christiano, and Kehoe (1996), Chari and Kehoe (1999), and Chari, Atkeson, and Kehoe (1999).[6]

The literature on optimal fiscal policies in the HAIM framework is still developing and under-researched. Here we review the most-relevant papers in this area. The work of Aiyagari (1995) is the first attempt at investigating optimal Ramsey taxation in HAIM economies. Under the assumption of the existence of an interior Ramsey state steady, Aiyagari (1995) shows that the Ramsey planner opts to restore the MGR by taxing capital in the steady state even though a labor tax as well as government bonds are also available. The key intuition of taxing capital is to correct the production inefficiency caused by households' precautionary saving motives.

Chien and Wen (2021) show that the above intuition for justifying positive capital income taxation is counterintuitive and not necessarily correct in general. More specifically, they demonstrate that since government bonds can be used to eliminate the borrowing constraints, optimal capital tax should be zero in the absence of any wealth-redistribution effects of a capital tax. Hence, Chien and Wen (2021) argue in that paper that Aiyagari's explanations for a positive capital tax based on the MGR is not a robust feature of all HAIM models with capital overaccumulation caused by precautionary savings. So even if the Ramsey steady state does exist, a positive capital tax may have more to do with wealth redistribution than with capital overaccumulation caused by precautionary savings judged by the MGR. This paper further illuminates this issue by introducing another source of capital overaccumulation (such as a production externality) and showing that only if the failure of the MGR comes from the intratemporal margin instead of the intertemporal margin does optimal capital tax become non-zero.

Our paper is also closely related to a recent study by Bassetto and Cui (2020), who

[6]In addition, Lucas (1990) shows that even in a two-sector endogenous growth model with both fiscal and human capital it is still optimal to tax labor/human capital income and not capital income. However, Chen and Lu (2013) obtain exactly the opposite result of Lucas (1990) by using a slightly different two-sector growth model.

analyze optimal fiscal policies in an environment where the capital stock tends to be underaccumulated due to frictions in firms' financing constraints. They derive a result very similar to ours: In the absence of other types of market failures other than borrowing constraints, the optimal capital tax/subsidy rate could be zero in the steady state despite the failure of the MGR due to capital underaccumulation. Such a result is consistent with our findings despite the striking difference that the capital stock in our model tends to be overaccumulated due to households' precautionary (excessive) savings.

In contrast, Angeletos, Collard, and Dellas (2020) find in a HAIM model that the long-run optimal debt level may not necessarily be one to completely alleviate the borrowing constraints for all households even if this is feasible. Their result thus appears to be not fully consistent with ours and that obtained by Bassetto and Cui (2020). The explanation for such a difference could be due to the completeness of tax system. Both our paper and the work of Bassetto and Cui (2020) consider a complete tax system at the macroeconomic level. In contrast, the tax system of Angeletos, Collard, and Dellas (2020) is not complete since there is a non-taxable consumption good. As a result, the optimal level of government debt could be different in their model as it plays an additional role in altering the price of the untaxed good, which creates a constraint on the optimal level of government debt. What we show in this paper is that under a complete tax system the optimal level of government debt should reach the satiation point to relax all borrowing constraints and, as a result, the MGR is fully restored and there is no need to tax capital.

Aiyagari and McGrattan (1998) study the optimal quantity of public debt by considering the trade-off in benefits and costs of varying the quantity of debt. On the benefit side, they argue that government debt enhances the liquidity of households by providing an additional means of smoothing consumption (in addition to capital) and by effectively relaxing their borrowing constraints. On the cost side, they argue that (i) the implied taxes to finance public debt have adverse incentive and wealth-distribution effects and (ii) government debt crowds out capital via higher interest rates and thus lowers per capita consumption in the steady state. However, Aiyagari and McGrattan (1998) obtain their results through numerical methods under two critical assumptions: (i) the Ramsey planner considers only steady-state welfare; and (ii) the tax rates on labor and capital income are levied proportionally to each other. Our study complements theirs by relaxing both assumptions since the first assumption excludes the welfare during transition and the second assumption rules out the possibility of financing debt only through labor-income taxation.

Our paper is also connected to the role of government debt in the tax smoothing literature, such as Barro (1979) and Lucas and Stokey (1983). In our model, the Ramsey planner intends to keep increasing the supply of public debt in the transition period to the Ramsey steady state so that the borrowing constraints of all households are slack in the long run. This transitional dynamic implies a deviation from tax smoothing, which is a classical result found in the representative-agent framework.

By introducing uninsurable human-capital risks, Gottardi, Kajii, and Nakajima (2015) revisit optimal Ramsey taxation in a HAIM model. As in our model, tractability in their model enables them to provide transparent analysis on Ramsey taxation and facilitates intuitive interpretations for their results. When government spending and the bond supply are both set to zero, they find that the Ramsey planner should tax human capital and subsidize physical capital, despite the overaccumulation of physical capital. The purpose or the benefit of taxing human capital is to reduce uninsurable risk from human-capital returns; and the rationale for subsidizing physical capital despite overaccumulation is to satisfy the household demand for a buffer stock, similar to our finding.

6 Conclusion

In a canonical model of heterogeneous agents with precautionary saving motives, Aiyagari (1995) breaks the classical result of zero capital tax obtained in representative-agent models. Aiyagari argues that with capital overaccumulation the optimal long-run capital tax should be strictly positive in order to achieve aggregate allocative efficiency suggested by the MGR. We show in this paper that the Aiyagari's argument is not robust with respect to HAIM models with precautionary saving and capital overaccumulation. In particular, we argue that the sources of capital overaccumulation matter for optimal tax policy and that in general capital taxation may not be the most efficient means to restore the MGR when government debt is feasible. To demonstrate our point, this paper studies which policy or policy mix is more effective in achieving the socially optimal (MGR) level of aggregate capital stock in an infinite-horizon heterogeneous-agents incomplete-markets economy where capital may be overaccumulated for two distinct reasons: (i) precautionary savings and (ii) production externalities.

By solving the Ramsey problem analytically along the entire transitional path, we reveal that public debt and capital taxation play very distinct roles in dealing with the overaccumulation problem. The Ramsey planner opts neither to use a capital tax to correct the overaccumulation problem if it is caused solely by precautionary saving—regardless of the feasibility of public debt—nor to use debt (financed by consumption tax) to correct the overaccumulation problem if it is caused solely by production externality (such as pollution)—regardless of the feasibility of a capital tax.

The key insight behind our findings is that the MGR has two margins: an intratemporal margin pertaining to the wedge between social MPK and private MPK, and an intertemporal margin pertaining to the wedge between the market interest rate and the time discount rate. To achieve the MGR, the Ramsey planner needs to equate not only the private MPK with the social MPK but also the interest rate with the time discount rate—neither of which is equalized in a competitive equilibrium. Yet government debt is effective and desirable only in addressing the intertemporal wedge, while a capital tax is effective and desirable only in addressing the intratemporal wedge.

References

AIYAGARI, S. R. (1994): "Uninsured Idiosyncratic Risk and Aggregate Saving," *Quarterly Journal of Economics*, 109(3), 659–684.

AIYAGARI, S. R. (1995): "Optimal Capital Income Taxation with Incomplete Markets, Borrowing Constraints, and Constant Discounting," *Journal of Political Economy*, 103(6), 1158–75.

AIYAGARI, S. R., AND E. R. MCGRATTAN (1998): "The optimum quantity of debt," *Journal of Monetary Economics*, 42(3), 447–469.

ANGELETOS, G.-M., F. COLLARD, AND H. DELLAS (2020): "Public Debt as Private Liquidity: Optimal Policy," CEPR Discussion Papers 15488, C.E.P.R. Discussion Papers.

BARRO, R. J. (1979): "On the Determination of the Public Debt," *Journal of Political Economy*, 87(5), 940–71.

BASSETTO, M., AND W. CUI (2020): "A Ramsey Theory of Financial Distortions," Working Papers 775, Federal Reserve Bank of Minneapolis.

CHAMLEY, C. (1986): "Optimal Taxation of Capital Income in General Equilibrium with Infinite Lives," *Econometrica*, 54(3), 607–622.

CHARI, V., AND P. J. KEHOE (1999): "Optimal fiscal and monetary policy," in *Handbook of Macroeconomics*, ed. by J. B. Taylor, and M. Woodford, chap. 26, pp. 1671–1745. Elsevier.

CHARI, V. V., A. ATKESON, AND P. J. KEHOE (1999): "Taxing capital income: a bad idea," *Quarterly Review*, pp. 3–17.

CHARI, V. V., L. J. CHRISTIANO, AND P. J. KEHOE (1996): "Optimality of the Friedman rule in economies with distorting taxes," *Journal of Monetary Economics*, 37(2-3), 203–223.

CHEN, B.-L., AND C.-H. LU (2013): "Optimal factor tax incidence in two-sector human capital-based models," *Journal of Public Economics*, 97(C), 75–94.

Chien, Y., and Y. Wen (2021): "Optimal Ramsey Taxation in Heterogeneous Agent Economies with Quasi-Linear Preferences," *Review of Economic Dynamics*, Forthcoming.

Cropper, M., and W. Oates (1992): "Environmental Economics: A Survey," *Journal of Economic Literature*, 30(2), 675–740.

Gottardi, P., A. Kajii, and T. Nakajima (2015): "Optimal Taxation and Debt with Uninsurable Risks to Human Capital Accumulation," *American Economic Review*, 105(11), 3443–3470.

Ljungqvist, L., and T. J. Sargent (2012): *Recursive Macroeconomic Theory*, MIT Press Books. The MIT Press.

Lucas, R. E. (1990): "Supply-Side Economics: An Analytical Review," *Oxford Economic Papers*, 42(2), 293–316.

Lucas, R. J., and N. L. Stokey (1983): "Optimal Fiscal and Monetary Policy in an Economy without Capital," *Journal of Monetary Economics*, 12(1), 55–93.

A Appendix

A.1 Proof of Proposition 1

A.1.1 Household Optimal Conditions

Denoting $\{\beta^t \lambda_t(\theta^t), \beta^t \mu_t(\theta^t)\}$ as the Lagrangian multipliers for constraints (8) and (9), respectively, the first-order conditions for $\{c_t(\theta^t), n_t(\theta^{t-1}), a_{t+1}(\theta^t)\}$ are given, respectively, by

$$\frac{\theta_t}{(1+\tau_t^c)\, c_t(\theta^t)} = \lambda_t(\theta^t) \tag{45}$$

$$1 = w_t \int \lambda_t(\theta^t)\, d\mathbf{F}(\theta_t) \tag{46}$$

$$\lambda_t(\theta^t) = \beta r_{t+1} \int \lambda_{t+1}(\theta^{t+1})\, d\mathbf{F}(\theta) + \mu_t(\theta^t), \tag{47}$$

where equation (46) reflects the fact that the labor supply $n_t(\theta^{t-1})$ must be chosen before the idiosyncratic taste shocks (and hence before the value of $\lambda_t(\theta^t)$) are realized. By the law of iterated expectations and the iid assumption of idiosyncratic shocks, equation (47) can be written (using equation (46)) as

$$\lambda_t(\theta^t) = \beta \frac{r_{t+1}}{w_{t+1}} + \mu_t(\theta^t), \tag{48}$$

where $\frac{1}{w}$ is the marginal utility of consumption in terms of labor income.

We characterize the competitive equilibrium in two cases. One is with full self-insurance and the other is not.

A.1.2 No Full Self-Insurance Case

We adopt a guess-and-verify strategy to derive the decision rules. The decision rules for an household's consumption and savings are characterized by a cutoff strategy, taking as given the aggregate states (such as the interest rate and real wage) given that there always exists a positive measure of households with a binding borrowing constraint (no full self-insurance). Anticipating that the optimal cutoff θ_t^* is independent of an household's history of shocks, consider two possible cases:

Case A. $\theta_t \leq \theta_t^*$. In this case the urge to consume is low. It is hence optimal to save so as to prevent possible liquidity constraints in the future. So $a_{t+1}(\theta^t) \geq 0$, $\mu_t(\theta^t) = 0$, and the shadow value is

$$\lambda_t(\theta^t) = \beta \frac{r_{t+1}}{w_{t+1}} \equiv \Lambda_t,$$

where Λ_t depends only on aggregate states. In this case, λ_t is independent of the history of idiosyncratic shocks. Equation (45) implies that consumption is given by $c_t(\theta^t) = \frac{\theta_t}{(1+\tau_t^c)}\Lambda_t^{-1}$. Defining $x_t(\theta^{t-1}) \equiv r_t a_t(\theta^{t-1}) + w_t n_t(\theta^{t-1})$ as the gross income of a household, the budget identity (8) then implies $a_{t+1}(\theta^t) = x_t(\theta^{t-1}) - \theta_t \Lambda_t^{-1}$. The requirement $a_{t+1}(\theta^t) \geq 0$ then implies

$$\theta_t \leq \Lambda_t x_t \equiv \theta_t^*, \tag{49}$$

which defines the cutoff θ_t^*.

We conjecture that the cutoff is independent of the idiosyncratic state, making the optimal gross income x_t also independent of the idiosyncratic state. The intuition is that x_t is determined before the realization of θ_t and all households face the same distribution of idiosyncratic shocks. Since the utility function is quasi-linear, a household is able to adjust labor income to meet any target level of liquidity on hand. As a result, the distribution of x_t is degenerate. This property simplifies the model tremendously.

Case B. $\theta_t > \theta_t^*$. In this case the urge to consume is high. It is then optimal not to save, so $a_{t+1}(\theta^t) = 0$ and $\mu_t(\theta^t) > 0$. By the resource constraint (8), we have $c_t(\theta^t) = \frac{x_t}{(1+\tau_t^c)}$, which by equation (49) implies $c_t(\theta^t) = \frac{\theta_t^*}{(1+\tau_t^c)}\Lambda_t^{-1}$. Equation (45) then implies that the shadow value is given by $\lambda_t(\theta^t) = \frac{\theta_t}{\theta_t^*}\Lambda_t$. Since $\theta_t > \theta^*$, equation (48) implies $\mu_t(\theta^t) = \Lambda_t \left[\frac{\theta_t}{\theta_t^*} - 1\right] > 0$. Notice that the shadow value of goods (the marginal utility of income), $\lambda_t(\theta^t)$, is higher under Case B than under Case A because of binding borrowing constraints.

By Cases A and B, the decision rules of household consumption and saving can then be summarized by equations (13) and (14), respectively. Finally, the decision rule of the household labor supply, equation (15), is decided residually to satisfy the household budget constraint.

The above analyses imply that the expected shadow value of income, $\int \lambda_t(\theta) d\mathbf{F}(\theta)$, and hence the optimal cutoff value θ^*, is determined by equation (46) by plugging in the expressions for $\lambda_t(\theta^t)$ into Cases A and B, which immediately gives equation (16). Specifically, combining Case A and Case B, we have

$$\begin{aligned}
\lambda_t(\theta^t) &= \beta \frac{r_{t+1}}{w_{t+1}} \text{ for } \theta \leq \theta_t^* \\
\lambda_t(\theta^t) &= \frac{\theta_t}{\theta_t^*} \beta \frac{r_{t+1}}{w_{t+1}} \text{ for } \theta_t \geq \theta_t^*.
\end{aligned}$$

The aggregate Euler equation is therefore given by

$$\frac{1}{w_t} = \int \lambda_t(\theta) d\mathbf{F}(\theta) = \beta \frac{r_{t+1}}{w_{t+1}} \left[\int_{\theta \leq \theta_t^*} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} \frac{\theta}{\theta^*} d\mathbf{F}(\theta) \right] = \beta \frac{r_{t+1}}{w_{t+1}} L(\theta_t^*),$$

which is equation (16). This equation reveals that the optimal cutoff depends only on aggregate states and is independent of the household's history.

Using equation (49) together with the aggregate Euler equation and the definition of Λ_t, we can solve for x_t :

$$x_t = \theta_t^* \left(\beta \frac{r_{t+1}}{w_{t+1}} \right)^{-1} = \theta_t^* L(\theta_t^*) w_t,$$

which is the first line in equation (12).

A.1.3 Full Self-Insurance Case

Next, we consider the full self-insurance case where all household borrowing constraints are non-binding. This is possible in our model if the initial aggregate bond supply is high enough. As we show below, the individual saving choices and their distribution become indeterminate so long as each household's cash on hand is sufficiently large (because of a large aggregate bond supply). In this case, we impose an additional assumption that all households still choose the same x_t. Under this assumption, $x_t = X_t = r_t A_t + w_t N_t$. Moreover, the first order conditions (46) and (47) imply

$$\lambda_t(\theta^t) = \beta \frac{r_{t+1}}{w_{t+1}} = \frac{1}{w_t},$$

which suggests that

$$\frac{1}{r_{t+1}} = \beta \frac{w_t}{w_{t+1}}.$$

The FOC (45) together with $\lambda_t(\theta^t) = w_t^{-1}$ give the second line in equation (13). The second line in equation (14) immediately follows since $x_t = X_t$.

A.1.4 Condition to Ensure Interior Labor

Finally, to ensure that the above proof and hence the associated cutoff policy rules are consistent with the assumption of interior choices of labor, namely, $n_t \in (0, \overline{N})$, we need to consider the following two cases.

First, to ensure that $n_t(\theta^{t-1}) > 0$, consider the worst case where $n_t(\theta^{t-1})$ takes its minimum possible value. Given $x_t = r_t a_t(\theta^{t-1}) + w_t n_t(\theta^{t-1})$, $n_t(\theta^{t-1})$ is at its minimum possible value if $\mu_t = 0$ and $a_t(\theta^{t-1})$ takes its maximum possible value of $a_t(\theta^{t-1}) = \left[1 - \left(\frac{\theta_L}{\theta^*_{t-1}}\right)\right] x_{t-1}$. So $n_t(\theta^{t-1}) > 0$ if

$$x_t - r_t \left[1 - \left(\frac{\theta_L}{\theta^*_{t-1}}\right)\right] x_{t-1} > 0, \tag{50}$$

which is independent of the shock θ_t. This condition in the steady state becomes $1 - r\left[1 - \left(\frac{\theta_L}{\theta^*}\right)\right] > 0$, or equivalently (by using equation (16)),

$$\beta L\left(\theta^*\right) > 1 - \left(\frac{\theta_L}{\theta^*}\right). \tag{51}$$

Given that $L\left(\theta^*\right)$ is a monotonic decreasing function in θ^* with a lower bound of 1, the necessary condition to satisfy (51) in the steady state is $\beta > 1 - \left(\frac{\theta_L}{\theta^*}\right)$, which is clearly true since the optimal cutoff $\theta^* > \theta_L$. This condition is further ensured by the requirement $\beta > 1 - \frac{\theta_L}{\theta_H}$. Therefore, as long as the condition $\beta > 1 - \frac{\theta_L}{\theta_H}$ is met, the condition (50) is assumed to hold throughout the paper.

Second, to ensure that $n_t < \overline{N}$, consider agents who encounter the borrowing constraint last period such that $a_t(\theta^{t-1}) = 0$. Their labor supply reaches the maximum value at $n_t(\theta^{t-1}) = \frac{x_t}{w_t} = \theta^*_t L(\theta^*_t) < \theta_H$. Given a finite steady-state value of θ^*, the value of $\overline{N}$ can be chosen such that

$$\overline{N} > \theta_H > \theta^* L(\theta^*). \tag{52}$$

A.2 Proof of Proposition 2

In the laissez-faire economy, the capital tax, the labor tax, government spending, and government bond, are all equal to zero. In this laissez-faire competitive equilibrium, the capital-to-labor ratio $\frac{K_t}{N_t}$ satisfies two conditions. The first condition is derived from the resource constraint (23), which can be expressed as

$$F(\overline{K}_t^{\varphi}, K_t, N_t) + (1-\delta)K_t = C_t + K_{t+1} = x_t,$$

where the last equality utilizes the definition of x_t. Dividing both sides of the equation by K_t gives

$$\overline{K}_t^{\varphi} \left(\frac{K_t}{N_t}\right)^{\alpha-1} + (1-\delta) = \frac{1}{1-D(\theta_t^*)}, \tag{53}$$

where x_t/K_t is substituted out by $\frac{1}{1-D(\theta_t^*)}$.

The second condition is derived by combining equation (16) and the no-arbitrage condition, $r_t = q_t + 1 - \delta$, which gives

$$1 = \beta \left(\alpha \overline{K}_t^{\varphi} \left(\frac{K_t}{N_t}\right)^{\alpha-1} + 1 - \delta \right) L(\theta_t^*), \tag{54}$$

where the MPK q_t is replaced by $\alpha \overline{K}_t^{\varphi} \left(\frac{K_t}{N_t}\right)^{\alpha-1}$. Since the capital-to-labor ratio must be the same in both equations, conditions (53) and (54) imply the following equation in the steady state:

$$\frac{\alpha\beta}{(1-D(\theta^*))} + \beta(1-\alpha)(1-\delta) = \frac{1}{L(\theta^*)}, \tag{55}$$

which solves for the steady-state value of θ^*. It can be shown easily that both $L(\theta^*)$ and $D(\theta^*)$ are monotonically decreasing in θ^*, thus the right-hand side (RHS) of equation (55) increases monotonically in θ^* and the left-hand side (LHS) of equation (55) decreases monotonically in θ^*. Hence, if a steady-state cutoff exists, it must be unique.

It remains to be shown if the RHS and the LHS cross each other at an interior value of $\theta^* \in [\theta_L, \theta_H]$. The RHS of equation (55) reaches its minimum value of 1 when $\theta^* = \theta_H$ and its maximum value of $\bar{\theta}/\theta_L > 1$ when $\theta^* = \theta_L$. The LHS of equation (55) takes the maximum value of infinity when $\theta^* = \theta_L$ and the minimum value of $\frac{\alpha\beta\theta_H}{\theta_H - \bar{\theta}} + \beta(1-\alpha)(1-\delta)$ when $\theta^* = \theta_H$. Thus, an interior solution exists if and only if

$$\frac{\alpha\beta\theta_H}{\theta_H - \overline{\theta}} + \beta(1-\alpha)(1-\delta) < 1.$$

Clearly, $\theta^* = \theta_L$ cannot constitute a solution for any positive value when $\theta_L > 0$. On the other hand, $\theta^* = \theta_H$ may constitute a solution if the above condition is violated. For example, if θ_H is small and close enough to the $\overline{\theta}$, then the above condition does not hold since its LHS approaches infinity when $\theta_H \to \overline{\theta}$. Therefore, an interior solution for θ^* exists if the upper bound of the idiosyncratic shock is large enough. Otherwise, we have the corner solution $\theta^* = \theta_H$. Finally, if θ^* is an interior solution, then $L(\theta^*) > 1$ and $r < 1/\beta$ by equation (16).

A.3 Proof of Proposition 3

A.3.1 The "Only If" Part

Assume that we have the allocation $\{\theta_t^*, C_t, N_t, K_{t+1}, B_{t+1}\}_{t=1}^{\infty}$ and the initial risk-free rate r_1. We then can directly construct the prices, taxes, and individual allocations in the competitive equilibrium in the following steps:

1. $\overline{K}_t$ is set to be K_t.

2. w_t and q_t are set by (3) and (4), which are $w_t = MP_{N,t}$ and $q_t = MP_{K,t}$, respectively.

3. Depending on the value of θ_t^*, we consider two cases below.

   (a) Consider the case in which $\theta_t^* < \theta_H$. A_{t+1} is set by the asset market clearing condition, $A_{t+1} = K_{t+1} + B_{t+1}$. x_t is chosen as $x_t = \frac{A_{t+1}}{1-D(\theta_t^*)} = \frac{K_{t+1}+B_{t+1}}{1-D(\theta_t^*)}$ according to the first line of equation (19). By the first line of (12), the $w_t = \frac{x_t}{\theta_t^* L(\theta_t^*)} = \frac{A_{t+1}}{\theta_t^* L(\theta_t^*)[1-D(\theta_t^*)]} = \frac{K_{t+1}+B_{t+1}}{\theta_t^* L(\theta_t^*)[1-D(\theta_t^*)]}$, which together with $w_t = MP_{N,t}$ imply the first line of condition (25). τ_c is set by the first line of equation (18):

   $$1 + \tau_{c,t} = \frac{D(\theta_t^*)\, x_t}{C_t} = \frac{D(\theta_t^*)}{C_t} \frac{K_{t+1} + B_{t+1}}{[1 - D(\theta_t^*)]}.$$

   Hence, r_t is implied by equation (16)

   $$\frac{1}{r_t} = \beta \frac{w_{t-1}}{w_t} L(\theta_{t-1}^*) = \beta \frac{A_t}{A_{t+1}} \frac{\theta_t^*}{\theta_{t-1}^*} \frac{[1 - D(\theta_t^*)]}{\left[1 - D\left(\theta_{t-1}^*\right)\right]} L(\theta_t^*) \qquad (56)$$

   (b) Suppose $\theta_t^* = \theta_H$. Use the asset-market-clearing condition to set $A_{t+1} = K_{t+1} +$

B_{t+1}. By aggregating the second line of equation (18), $1+\tau_t^c$ is determined by

$$1+\tau_t^c = \frac{\overline{\theta} w_t}{C_t} = \frac{\bar{\theta} MP_{N,t}}{C_t},$$

and hence X_t is chosen according its definition:

$$X_t = A_{t+1} + (1+\tau_t^c)\, C_t = K_{t+1} + B_{t+1} + w_t \bar{\theta}.$$

The interest rate is set as

$$r_t = \frac{1}{\beta}\frac{w_{t-1}}{w_t}.$$

Given r_1 and the expression $\{r_{t+1}\}_{t=1}^{\infty}$, the capital tax $\{\tau_{t+1}^k\}_{t=0}^{\infty}$ is chosen to satisfy the no-arbitrage condition: $r_t = 1-\delta+(1-\tau_t^k) MP_{K,t}$ for all $t \geq 1$. $c_t(\theta_t)$ and $a_{t+1}(\theta_t)$, are pinned down by the second lines in equations (13) and (14), respectively. Finally, set $n_t(\theta_{t-1})$ to satisfy equation (15), which is implied by the individual household budget constraint.

4. There are two cases.

(a) First consider the case $\theta_t^* < \theta_H$. The implementability conditions are

$$L(\theta_1^*)\theta_1^* \geq N_1 + r_1 C_1^{-1} D(\theta_1^*) L(\theta_1^*)\theta_1^*(K_1+B_1)$$

and

$$L(\theta_t^*)\theta_t^* \geq N_t + \frac{1}{\beta}\theta_{t-1}^* \left[1 - D\left(\theta_{t-1}^*\right)\right]$$

for $t=1$ and $t \geq 2$, respectively. Multiplying both sides of the above equations by $\frac{C_t}{D(\theta_t^*)L(\theta_t^*)\theta_t^*}$ leads to

$$\frac{C_1}{D(\theta_1^*)} \geq \frac{C_1}{D(\theta_1^*)L(\theta_1^*)\theta_1^*} N_1 + r_1(K_1+B_1),$$

$$\frac{C_t}{D(\theta_t^*)} \geq \frac{C_t}{D(\theta_t^*)L(\theta_t^*)\theta_t^*} N_t + \frac{\theta_{t-1}^*}{D(\theta_t^*)L(\theta_t^*)\theta_t^*}\frac{1}{\beta}\left[1 - D(\theta_{t-1}^*)\right] C_t.$$

Inserting the relationships constructed in steps 3 for x_t, C_t, A_{t+1}, τ_t^c, w_t, and r_t

into the above equation gives

$$(1+\tau_t^c)\,C_t + A_{t+1} \geq w_t N_t + r_t A_t \text{ for all } t \geq 1,$$

which together with the aggregate resource constraint and the identity $Y_t = q_t K_t + w_t N_t$ give the government budget constraint.

(b) Next consider the case $\theta_t^* = \theta_H$. Plugging the relationships constructed in steps 2 and 3 for w_t, x_t, C_t, A_{t+1}, $1+\tau_t^c, w_t$ and r_t into the implementability conditions gives

$$(1+\tau_t^c)\,C_t + A_{t+1} \geq w_t N_t + r_t A_t \text{ for all } t \geq 1,$$

which together with the aggregate resource constraint and the identity $Y_t = q_t K_t + w_t N_t$ gives the government budget constraint.

A.3.2 The "If" Part

Note that the aggregate resource constraint is trivially implied by a competitive equilibrium, since it is part of the definition. The implementability condition is constructed as follows. First, rewrite the government budget constraint as

$$G_t \leq \tau_k^k q_t K_t + \tau_t^c C_t + B_{t+1} - r_t B_t.$$

Combining this equation with the resource constraint (23), the no-arbitrage condition, and the identity $Y_t = q_t K_t + w_t N_t$ implies

$$(1+\tau_t^c)\,C_t + A_{t+1} \geq w_t N_t + r_t A_t. \tag{57}$$

We then consider two cases below.

The $\theta_t^* < \theta_H$ Case. For $t \geq 2$, the aggregate consumption function, saving function, and equations (12) and (16) suggest that $\{w_t, r_t, 1-\tau_t^c\}$ can be expressed, respectively, as

$$w_t = \frac{A_{t+1}}{[1-D(\theta_t^*)]\,L(\theta_t^*)\theta_t^*},$$

$$r_t = \frac{A_{t+1}\left[1 - D(\theta^*_{t-1})\right]\theta^*_{t-1}}{\beta A_t\left[1 - D(\theta^*_t)\right]\theta^*_t L(\theta^*_t)},$$

and

$$1 + \tau^c_t = \left[\frac{1 - D\left(\theta^*_t\right)}{D\left(\theta^*_t\right)}\frac{C_t}{A_{t+1}}\right]^{-1}.$$

Substituting the above equations into (57) and rearranging terms, we get the first line of implementability condition (26):

$$L(\theta^*_t)\theta^*_t \geq N_t + \frac{1}{\beta}\theta^*_{t-1}\left[1 - D\left(\theta^*_{t-1}\right)\right].$$

For the first period, B_1, K_1, and $\tau_{k,1}$ are given, which implies that $r_1 = 1+(1-\tau^k_1)MP_{K,1}-\delta$ is also given. Therefore, the first-period implementability condition could be rewritten as

$$L(\theta^*_1)\theta^*_1 \geq N_1 + r_1 C_1^{-1} D(\theta^*_1)L(\theta^*_1)\theta^*_1(K_1 + B_1).$$

The $\theta^*_t = \theta_H$ Case. The aggregate consumption function, aggregate saving function, and equations (12) and (16) suggest that $\{r_t, 1 - \tau^c_t\}$ can be expressed, respectively, as

$$r_t = \frac{MP_{N,t}}{\beta MP_{N,t-1}}$$

and

$$1 + \tau^c_t = \left[\frac{C_t}{MP_{N,t}\bar{\theta}}\right]^{-1}.$$

Substituting the above equations into (57) and rearranging terms, we get the second line of implementability condition (26):

$$\bar{\theta} \geq N_t + \frac{1}{\beta}\frac{K_t + B_t}{MP_{N,t-1}} - \frac{K_{t+1} + B_{t+1}}{MP_{N,t}}.$$

For the first period, r_1 is given. Therefore, the first-period implementability condition could be rewritten as

$$\bar{\theta} \geq N_1 + \frac{r_1(K_1 + B_1)}{MP_{N,1}} - \frac{K_2 + B_2}{MP_{N,1}}.$$

A.4 The Ramsey Objective Function

By equation (13) and the third step of Appendix A.3, the individual objective function can be rewritten as

$$c_t(\theta_t) = \begin{cases} \min\left\{1, \frac{\theta_t}{\theta_t^*}\right\} \frac{C_t}{D(\theta_t^*)} & \text{if } \theta_t^* < \theta_H \\ \frac{\theta_t}{\theta} C_t & \text{if } \theta_t^* = \theta_H \end{cases}.$$

Conditional on the value of $\theta_t^* < \theta_H$, the consumption part of the objective function at period t becomes

$$\begin{aligned} \int\limits_{\theta > \theta_t^*} \theta \log c_t(\theta) d\mathbf{F}(\theta) &= \int\limits_{\theta > \theta_t^*} \theta \ln \frac{C_t}{D(\theta_t^*)} d\mathbf{F}(\theta) + \int\limits_{\theta \leq \theta_t^*} \theta (\ln \frac{\theta}{\theta_t^*} + \ln \frac{C_t}{D(\theta_t^*)}) d\mathbf{F}(\theta) \\ &= W(\theta_t^*) + \overline{\theta} \ln C_t, \end{aligned} \tag{58}$$

where $W(\theta_t^*)$ is defined as

$$W(\theta_t^*) \equiv -\overline{\theta} \ln D(\theta_t^*) + \int\limits_{\theta \leq \theta_t^*} \theta \ln \left(\frac{\theta}{\theta_t^*} \right) d\mathbf{F}(\theta).$$

We then consider the case in which $\theta_t^* = \theta_H$:

$$\int \theta_t \ln c_t(\theta^t) d\mathbf{F}(\theta) = \int \theta \ln \theta d\mathbf{F}(\theta) - \bar{\theta} \ln \bar{\theta} + \overline{\theta} \ln C_t,$$

which is equal to equation (58) when $\theta_t^* = \theta_H$. As a result, the Ramsey objective function is written as equation (28).

A.5 Ramsey Optimal Conditions

The Ramsey planner treats $\overline{K} = K$ as endogenous. So the first-order Ramsey conditions for $\{B_{t+1}, \theta_t^*, N_t, C_t, K_{t+1}\}_{t=2}^{\infty}$ are given, respectively, by

$$-\frac{\lambda_t}{MP_{N,t}} \mathbf{1}_{\theta_t^* = \theta_H} + \frac{\lambda_{t+1}}{MP_{N,t}} \mathbf{1}_{\theta_{t+1}^* = \theta_H} - \frac{\phi_t}{\theta_t^* L(\theta_t^*) [1 - D(\theta_t^*)]} \mathbf{1}_{\theta_t^* < \theta_H} = 0, \tag{59}$$

$$\lambda_{t+1} J(\theta_t^*) \mathbf{1}_{\theta_{t+1}^* < \theta_H} = \lambda_t H(\theta_t^*) \mathbf{1}_{\theta_t^* < \theta_H} + \frac{\partial W(\theta_t^*)}{\partial \theta_t^*} - \phi_t \frac{\partial [\theta_t^* L(\theta_t^*) [1 - D(\theta_t^*)]]^{-1}}{\partial \theta_t^*} \mathbf{1}_{\theta_t^* < \theta_H}, \tag{60}$$

$$
\begin{aligned}
1+\lambda_t &= \mu_t \frac{\partial F(K_t^{\varphi}, K_t, N_t)}{\partial N_t} + (\lambda_t \mathbf{1}_{\theta_t^*=\theta_H} - \lambda_{t+1}\mathbf{1}_{\theta_{t+1}^*=\theta_H}) \frac{\partial MP_{N,t}^{-1}}{\partial N_t} \\
&\quad + \phi_t MP_{NN,t} \mathbf{1}_{\theta_t^*<\theta_H},
\end{aligned} \tag{61}
$$

$$
\frac{\bar{\theta}}{C_t} = \mu_t, \tag{62}
$$

$$
\begin{aligned}
\mu_t &= \beta \mu_{t+1} \left[\frac{\partial F(K_{t+1}^{\varphi}, K_{t+1}, N_{t+1})}{\partial K_{t+1}} + 1 - \delta \right] \\
&\quad + (\lambda_t \mathbf{1}_{\theta_t^*=\theta_H} - \lambda_{t+1}\mathbf{1}_{\theta_{t+1}^*=\theta_H}) \frac{1}{MP_{N,t}} \\
&\quad + \left(\lambda_{t+1} \mathbf{1}_{\theta_{t+1}^*=\theta_H} - \lambda_{t+2}\mathbf{1}_{\theta_{t+2}^*=\theta_H} \right) \left(\beta (K_{t+2} + B_{t+2}) \frac{\partial MP_{N,t+1}^{-1}}{\partial K_{t+1}} \right) \\
&\quad + \beta \phi_{t+1} \frac{\partial MP_{N,t+1}}{\partial K_{t+1}} \mathbf{1}_{\theta_{t+1}^*<\theta_H} - \frac{\phi_t}{\theta_t^* L(\theta_t^*) \left[1 - D(\theta_t^*)\right]} \mathbf{1}_{\theta_t^*<\theta_H},
\end{aligned} \tag{63}
$$

where $M(\theta_t^*) \equiv \frac{\partial[L(\theta_t^*)\theta_t^*]}{\partial \theta_t^*}$, $J(\theta_t^*) \equiv \frac{\partial[\theta_t^*(1-D(\theta_t^*))]}{\partial \theta_t^*}$, and these functions satisfy $M(\theta_t^*) = J(\theta_t^*) = \mathbf{F}(\theta_t^*) > 0$ for $\theta_t^* \in (\theta_L, \theta_H]$ (as shown in Lemma 1).

We then consider the following cases:

1. Suppose $\theta_t^* < \theta_H$ and $\theta_{t+1}^* < \theta_H$. In this case, $\phi_t = 0$ and we get (by using lemma (1))

$$
\lambda_{t+1} = \lambda_t + \frac{1}{\mathbf{F}(\theta_t^*)} \frac{\partial W(\theta_t^*)}{\partial \theta_t^*}. \tag{64}
$$

   This suggests that the λ_t is a monotonic increasing sequence if $\theta_t^* < \theta_H$.

2. Suppose $\theta_t^* = \theta_H$ and $\theta_{t+1}^* < \theta_H$. FOCs (59) and (60) imply $\lambda_{t+1} = 0$ and $\lambda_t = 0$, which suggests that the government budget constraint does not bind at period t and $t+1$. This case is impossible assuming $\theta_1^* < \theta_H$ and $\lambda_1 > 0$.

3. Suppose $\theta_t^* = \theta_H$ and $\theta_{t+1}^* = \theta_H$. The first-order condition with respect to B_{t+1} suggests $\lambda_t = \lambda_{t+1}$ and $\phi_t = 0$. Given the discussion of previous cases, we know that λ_t is monotonic increasing until $\theta_t^* = \theta_H$. Once θ_t^* reaches θ_H, θ_t^* stays at θ_H and λ_t becomes constant.

4. Suppose $\theta_t^* < \theta_H$ and $\theta_{t+1}^* = \theta_H$. This case describe the last transition period before reaching the Ramsey steady state. The λ_{t+1}, λ_t, and θ_t have to satisfy the relationship

implied by the FOCs (59) and (60):

$$\frac{\theta_t^* L\left(\theta_t^*\right)\left[1-D\left(\theta_t^*\right)\right]}{MP_{N,t}} \frac{\partial[\theta_t^* L\left(\theta_t^*\right)\left[1-D\left(\theta_t^*\right)\right]]^{-1}}{\partial \theta_t^*} \lambda_{t+1}=\lambda_t \mathbf{F}(\theta_t^*)+\frac{\partial W\left(\theta_t^*\right)}{\partial \theta_t^*}.$$

Moreover, K_1 and B_1 as well as r_1 are taken as given, since the initial capital tax is assumed to be given. Assuming $\theta_1^* < \theta_H$, the first-order Ramsey conditions with respect to N_1, C_1, and θ_1^* are given, respectively, by

$$1+\lambda_1+\lambda_1 \frac{D(\theta_1^*)L(\theta_1^*)\theta_1^*}{C_1} \frac{\partial r_1}{\partial N_1}(K_1+B_1)=\mu_1 \frac{\partial F(K_1^{\varphi}, K_1, N_1)}{\partial N_1}, \tag{65}$$

$$\mu_1=\frac{\bar{\theta}}{C_1}+\lambda_1 \frac{D(\theta_1^*)L(\theta_1^*)\theta_1^*}{C_1^2} r_1(K_1+B_1), \tag{66}$$

$$\frac{\partial W(\theta_1^*)}{\partial \theta_1^*}+\lambda_1 H(\theta_1^*)-\lambda_2 J(\theta_1^*)=\lambda_1 \frac{\partial\left[D(\theta_1^*)L(\theta_1^*)\theta_1^*\right]}{\partial \theta_1^*} \frac{r_1(K_1+B_1)}{C_1}. \tag{67}$$

A.5.1 Several Lemmas

The following three lemmas are useful to characterize the optimal Ramsey allocation:

Lemma 1. $M\left(\theta_t^*\right)=J\left(\theta_t^*\right)=\mathbf{F}(\theta_t^*)$.

Proof.

$$\begin{aligned} M\left(\theta_t^*\right) &\equiv \left(L(\theta_t^*)+\frac{\partial L(\theta_t^*)}{\partial \theta_t^*}\theta_t^*\right)=\int_{\theta \leq \theta_t^*} d\mathbf{F}(\theta)+\int_{\theta>\theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta)-\int_{\theta>\theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) \\ &= \int_{\theta \leq \theta_t^*} d\mathbf{F}(\theta)=\mathbf{F}(\theta_t^*). \end{aligned}$$

$$\begin{aligned} J\left(\theta_t^*\right) &\equiv \left(1-D(\theta_t^*)-\theta_t^* \frac{\partial D\left(\theta_t^*\right)}{\partial \theta_t^*}\right)=1-\left[\int_{\theta \leq \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta)+\int_{\theta>\theta_t^*} d\mathbf{F}(\theta)\right]+\int_{\theta \leq \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) \\ &= 1-\int_{\theta>\theta_t^*} d\mathbf{F}(\theta)=\mathbf{F}(\theta_t^*) \end{aligned}$$

□

Lemma 2. $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*}>0$ *for all* $\theta_t^* \in (\theta_L, \theta_H)$*, and* $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*}=0$ *if* $\theta_t^*=\theta_L$ *or* $\theta_t^*=\theta_H$.

Proof. We first show that $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} = 0$ if $\theta_t^* = \theta_L$ or θ_H :

$$\begin{aligned}
\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} &= -\frac{\partial D\left(\theta_t^*\right)}{\partial \theta_t^*}\frac{\overline{\theta}}{D\left(\theta_t^*\right)} - \int_{\theta\leq\theta_t^*}\frac{\theta}{\theta_t^*}d\mathbf{F}(\theta) = \left[\frac{\overline{\theta}}{D\left(\theta_t^*\right)\theta_t^*} - 1\right]\int_{\theta\leq\theta_t^*}\frac{\theta}{\theta_t^*}d\mathbf{F}(\theta) \\
&= \begin{cases} \left[1 - \frac{\overline{\theta}}{\frac{\overline{\theta}}{\theta_H}\theta_H}\right]\frac{\overline{\theta}}{\theta_H} = 0 & \text{if } \theta_t^* = \theta_H \\ \left[1 - \frac{\overline{\theta}}{\theta_L}\right]0 = 0 & \text{if } \theta_t^* = \theta_L \end{cases}.
\end{aligned}$$

Next, we show that $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} > 0$ for any $\theta_t^* \in (\theta_L, \theta_H)$. Note that

$$\begin{aligned}
D\left(\theta_t^*\right)\theta_t^* &= \int_{\theta\leq\theta_t^*}\theta d\mathbf{F}(\theta) + \theta_t^*\int_{\theta>\theta_t^*}d\mathbf{F}(\theta) = \overline{\theta} - \int_{\theta>\theta_t^*}(\theta-\theta_t^*)d\mathbf{F}(\theta) < \overline{\theta} \\
&\rightarrow \frac{\overline{\theta}}{D\left(\theta_t^*\right)\theta_t^*} > 1.
\end{aligned}$$

Hence,

$$\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} = \left[\frac{\overline{\theta}}{D\left(\theta_t^*\right)\theta_t^*} - 1\right]\int_{\theta\leq\theta_t^*}\frac{\theta}{\theta_t^*}d\mathbf{F}(\theta) > 0$$

□

Lemma 3. $\frac{\partial D(\theta_t^*)}{\partial \theta_t^*} < 0$ *for all* $\theta_t^* \in (\theta_L, \theta_H)$.

Proof. The definition of D is given by

$$D\left(\theta_t^*\right) = \int_{\theta\leq\theta_t^*}\frac{\theta}{\theta_t^*}dF(\theta) + \int_{\theta>\theta_t^*}dF(\theta) < 1$$

and hence the derivative is

$$\frac{\partial D\left(\theta_t^*\right)}{\partial \theta_t^*} = -1 - \int_{\theta\leq\theta_t^*}\frac{\theta}{\theta_t^{*2}}dF(\theta) + 1 = -\int_{\theta\leq\theta_t^*}\frac{\theta}{\theta_t^{*2}}dF(\theta) < 0.$$

□

A.6 Proof of Proposition 4

A.6.1 Existence of the Ramsey Steady State

In what follows, we first sketch the proof that a Ramsey steady state featuring $\theta^* = \theta_H$ exists. We proceed by the following steps, which show that the conjecture $\theta^* = \theta_H$ satisfies all of the Ramsey FOCs and the FOC-implied steady-state values of the aggregate allocation $\{C, N, K, B\}$ and that the Lagrangian multipliers $\{\lambda, \mu\}$ are unique, mutually consistent, strictly positive, and finitely valued:

1. The FOC with respect to B_{t+1} in equation (59) implies that λ_t is constant at λ and $\phi_t = 0$.

2. The FOC with respect to θ_t^* in equation (60) is satisfied at $\theta_t^* = \theta_H$.

3. The FOC with respect to K in equation (63) is reduced to

$$1 = \beta \left(\frac{\alpha + \varphi}{\alpha} MP_K + 1 - \delta \right),$$

which implies $MP_K \equiv \alpha K^{\varphi} \left(\frac{K}{N}\right)^{\alpha-1} = \frac{\alpha}{\alpha+\varphi} \frac{1-\beta(1-\delta)}{\beta} \in (0, \infty)$ (i.e., the capital stock is unique, strictly positive, and bounded, given that $N \in \left(0, \overline{N}\right)$). Given the assumption of the production function, it must be true that the following ratios are unique, strictly positive, and finite: $\{K/N, Y/K, MP_N, Y/N\} \in (0, \infty)$. More specifically, the Y/N and Y/K ratios can be expressed, respectively, as

$$\frac{Y}{N} = K^{\varphi} \left(\frac{K}{N} \right)^{\alpha} = \left(\frac{1 - \beta (1 - \delta)}{\beta (\alpha + \varphi)} \right) \left(\frac{K}{N} \right) = K^{\frac{\varphi}{1-\alpha}} \left(\frac{\beta (\alpha + \varphi)}{1 - \beta (1 - \delta)} \right)^{\frac{\alpha}{1-\alpha}} \tag{68}$$

and

$$\frac{Y}{K} = K^{\varphi} \left(\frac{K}{N} \right)^{\alpha - 1} = \frac{1 - \beta (1 - \delta)}{\beta (\alpha + \varphi)}. \tag{69}$$

4. The resource constraint,

$$F(K, N) = K^{(\alpha+\varphi)} N^{1-\alpha} = C + \delta K + G,$$

together with a finite level of government spending G implies the unique ratio $C/K \in (0, \infty)$ is

$$\frac{C}{Y} = \left(1 - \delta \frac{K}{Y} - \frac{G}{Y} \right) = \left(1 - \frac{(\alpha + \varphi) \beta \delta}{1 - \beta (1 - \delta)} - \frac{G}{Y} \right), \tag{70}$$

where the last equality uses (69).

5. We know that under our parameter restrictions the level of labor is interior, $N \in$

$(0, \overline{N})$; hence, it must be true that the aggregate allocation is also unique and interior: $\{C, K, Y\} \in (0, \infty)$.

6. Next, we show that $\{\mu, \lambda\} \in (0, \infty)$ and that these steady-state values are unique. Given $\theta^* = \theta_H$ in the steady state, first order conditions (61) and (62) become $1 + \lambda = \mu \times MP_N$ and $\mu \times C = \bar{\theta}$, respectively. These two equations imply $\{\mu, \lambda\}$ are unique and $\in (0, \infty)$.

7. The Ramsey steady-state version of equation (63) reads as

$$1 = \beta \left[\frac{\partial F(K^{\varphi}, K, N)}{\partial K} + 1 - \delta \right],$$

which is the MGR with the social MPK.

8. The optimal capital tax is chosen such that the Euler equation in the competitive equilibrium (16) is consistent with the one chosen by the Ramsey planner in (63). Hence, $\tau_{k,t+1}$ is pinned down by

$$1 - \tau_{t+1}^{k} = \left(\frac{\varphi + \alpha}{\alpha} \right) \left(\frac{\frac{w_{t+1}}{w_t} \frac{1}{L(\theta_t^*)} - \beta(1-\delta)}{\frac{\mu_t}{\mu_{t+1}} - \beta(1-\delta)} \right),$$

which is the steady-state equation (31) since $L(\theta_H) = 1$.

This finishes the proof for the existence of the Ramsey steady state. To show uniqueness, we then show that there is no Ramsey steady state for $\theta^* \in [\theta_L, \theta_H)$.

A.6.2 Uniqueness of the Ramsey Steady State

Suppose there exists another Ramsey steady state with $\theta^* \in (\theta_L, \theta_H)$. From equation (59), we see that $\theta_t^* < \theta_H$ and $\theta_{t+1}^* < \theta_H$ imply $\phi_t = 0$, which together with FOCs (62) and (61) imply that both λ_t and μ_t have to be finite and positive in the Ramsey steady state. Moreover, the first-order condition with respect to θ_t^* is then reduced to

$$\lambda_{t+1} = \lambda_t + \frac{1}{\mathbf{F}(\theta_t^*)} \frac{\partial W(\theta_t^*)}{\partial \theta_t^*},$$

which leads to a contradiction since the above equation suggests an ever increasing λ_t.

Finally, we show that the case of $\theta_t^* = \theta_L$ cannot constitute a Ramsey equilibrium, although the necessary FOC with respect to θ_t^* is satisfied. The reason is that the first term of the Ramsey objective function (28), $W(\theta_t^*)$, is monotonically increasing in $\theta_t^* \in (\theta_L, \theta_H)$. Hence, for a global maximum, a cutoff θ_t^* at its lower corner cannot be a Ramsey equilibrium.

To ensure that $n \in \left(0, \overline{N}\right)$ (see Proposition 1), note that we have assumed $\theta_H < \frac{\theta_L}{1-\beta}$, which ensures that the minimum individual labor input remains positive, as shown in Appendix A.1. Moreover, by equation (52), the maximum value of n is less than $\overline{N}$ if $\overline{N} > \theta_H$ in this case.

In addition, we can show that the maximum individual asset demand remains finite in the steady state even if the risk-free rate is equal to the time discount rate, $r = 1/\beta$. Since $\theta_H < \frac{\theta_L}{1-\beta}$, we have

$$x_t = \frac{C_t}{D(\theta_H)} = \frac{C_t}{\overline{\theta}}\theta_H < \infty.$$

Given the finite value of x_t, the individual asset holding a_{t+1} is determined by the size of the idiosyncratic shock θ_t, and the agents with the largest asset holdings are those who receive the smallest shock $\theta_t = \theta_L$; i.e.,

$$a_{t+1}(\theta_L) = \left[1 - \frac{\theta_L}{\theta_H}\right] x_t,$$

which is strictly positive and finite.

A.7 Proof of Proposition 5

Define $H(\theta^*) \equiv [1 - D(\theta^*)]$. The production function and equation (36) imply that the logarithms of the capital stock and output can each be expressed as a moving-average process:

$$\begin{aligned}\log K_{t+1} &= \log H(\theta_t^*) + (\alpha+\varphi)\log K_t + (1-\alpha)\log N_t \\ &= (\alpha+\varphi)^t K_1 + \sum_{j=0}^{t-1} (\alpha+\varphi)^j \log H\left(\theta_{t-j}^*\right) + (1-\alpha)\sum_{j=0}^{t-1}(\alpha+\varphi)^j \log N_{t-j}.\end{aligned}$$

$$\begin{aligned}\log Y_t &= (\alpha+\varphi)\log K_t + (1-\alpha)\log N_t \\ &= (\alpha+\varphi)^t K_1 + \sum_{j=1}^{t-1} (\alpha+\varphi)^j \log H\left(\theta_{t-j}^*\right) + (1-\alpha)\sum_{j=0}^{t-1}(\alpha+\varphi)^j \log N_{t-j}.\end{aligned}$$

Hence, the welfare function can be written as

$$V\left(\{\tau_t\}_{t=1}^{\infty}\right)=\sum_{t=1}^{\infty}\beta^{t-1}\left\{\begin{array}{c}\int_{\theta_t\le\theta_t^*}\theta_t\left(i\right)\log\frac{\theta_t(i)}{\theta_t^*}d\mathbf{F}\left(\theta\right)-N_t+\\ \bar{\theta}\left[\begin{array}{c}\left(\alpha+\varphi\right)^t K_1+\sum_{j=1}^{t-1}\left(\alpha+\varphi\right)^j\log H\left(\theta_{t-j}^*\right)\\ +\left(1-\alpha\right)\sum_{j=0}^{t-1}\left(\alpha+\varphi\right)^j\log N_{t-j}\end{array}\right]\end{array}\right\}.$$

Since all period-t variables in the objective function depend on the tax rate through the

cutoff θ_t^*, and since the cutoff θ_t^* depends only on future taxes, the derivative of V with respect to τ_t (for $t\ge 2$) can be decomposed as the product of two terms:

$$\frac{\partial V}{\partial\tau_{t+1}}=\frac{\partial V}{\partial\theta_t^*}\frac{\partial\theta_t^*}{\partial\tau_{t+1}}.\tag{71}$$

Since equation (42) implies

$$\frac{\partial\theta_t^*}{\partial\tau_{t+1}}<0,\tag{72}$$

it remains to determine the magnitude and sign of $\frac{\partial V}{\partial\theta_t^*}$. Notice that equations (32)-(38) imply that all time-t aggregate quantities depend only on the current cutoff θ_t^*. Taking the derivative of $V\left(\{\tau_t\}_{t=0}^{\infty}\right)$ with respect to θ_t^* gives

$$\frac{\partial V}{\partial\theta_t^*}=\left\{\begin{array}{c}-\int_{\theta\le\theta_t^*}\frac{\theta(i)}{\theta_t^*}d\mathbf{F}-\frac{\partial N_t}{\partial\theta_t^*}+\beta\bar{\theta}\left(\alpha+\varphi\right)\left[1+\beta\left(\alpha+\varphi\right)+\beta^2\left(\alpha+\varphi\right)^2+...\right]\frac{1}{H_t}\frac{\partial H_t}{\partial\theta_t^*}\\ +\bar{\theta}\left(1-\alpha\right)\left[1+\beta\left(\alpha+\varphi\right)+\beta^2\left(\alpha+\varphi\right)^2+...\right]\frac{1}{N_t}\frac{\partial N_t}{\partial\theta_t^*}\end{array}\right\}.$$

By the definition of the functions $D\left(\theta^*\right)$ and $L\left(\theta^*\right)$, we have $\frac{\partial H_t}{\partial\theta_t^*}=-\frac{\partial D_t}{\partial\theta_t^*}=\int_{\theta\le\theta_t^*}\frac{\theta(i)}{\theta_t^{*2}}d\mathbf{F}$ and $\frac{\partial N_t}{\partial\theta_t^*}=\left(1-\alpha\right)\frac{\partial\left[\theta_t^*L(\theta_t^*)\right]}{\partial\theta_t^*}=\left(1-\alpha\right)\mathbf{F}\left(\theta^*\right)$, so

$$\begin{aligned}\frac{\partial V}{\partial\theta_t^*}&=-\int_{\theta\le\theta_t^*}\frac{\theta\left(i\right)}{\theta_t^*}d\mathbf{F}-\left(1-\alpha\right)\mathbf{F}\left(\theta^*\right)\\ &\quad+\frac{\beta\bar{\theta}\left(\alpha+\varphi\right)}{1-\beta\left(\alpha+\varphi\right)}\frac{1}{H\left(\theta_t^*\right)}\int_{\theta\le\theta_t^*}\frac{\theta\left(i\right)}{\theta_t^{*2}}d\mathbf{F}+\frac{\bar{\theta}\left(1-\alpha\right)}{1-\beta\left(\alpha+\varphi\right)}\frac{\mathbf{F}\left(\theta_t^*\right)}{\theta_t^*L\left(\theta_t^*\right)}.\end{aligned}$$

Multiplying both sides by $L(\theta_t^*)$ gives

$$
\begin{aligned}
L(\theta_t^*)\frac{\partial V}{\partial \theta_t^*} &= \left\{ \begin{array}{c} -L(\theta_t^*)\int_{\theta\le\theta_t^*}\frac{\theta(i)}{\theta_t^*}d\mathbf{F}-(1-\alpha)L(\theta_t^*)\mathbf{F}(\theta^*) \\ +\frac{\beta\bar{\theta}(\alpha+\varphi)}{1-\beta(\alpha+\varphi)}\frac{L(\theta_t^*)}{\theta_t^*H(\theta_t^*)}\int_{\theta\le\theta_t^*}\frac{\theta(i)}{\theta_t^*}d\mathbf{F}+\frac{\bar{\theta}(1-\alpha)}{1-\beta(\alpha+\varphi)}\frac{\mathbf{F}(\theta_t^*)}{\theta_t^*} \end{array} \right\} \\
&= \left\{ \begin{array}{c} \left[\beta(\alpha+\varphi)\frac{\bar{\theta}}{\theta_t^*}-(1-\beta(\alpha+\varphi))H(\theta_t^*)\right]\frac{L(\theta_t^*)}{(1-\beta(\alpha+\varphi))H(\theta_t^*)}\left[\int_{\theta\le\theta_t^*}\frac{\theta(i)}{\theta_t^*}d\mathbf{F}\right] \\ +\left[\frac{\bar{\theta}}{\theta_t^*}-(1-\beta(\alpha+\varphi))L(\theta_t^*)\right]\frac{(1-\alpha)}{(1-\beta(\alpha+\varphi))}\mathbf{F}(\theta_t^*) \end{array} \right\} \\
&= \left\{ \begin{array}{c} \left[\beta(\alpha+\varphi)L(\theta_t^*)-H(\theta_t^*)\right]\frac{L(\theta_t^*)}{(1-\beta(\alpha+\varphi))H(\theta_t^*)}\left[\int_{\theta\le\theta_t^*}\frac{\theta(i)}{\theta_t^*}d\mathbf{F}\right] \\ +\left[\beta(\alpha+\varphi)L(\theta_t^*)-H(\theta_t^*)\right]\frac{(1-\alpha)}{(1-\beta(\alpha+\varphi))}\mathbf{F}(\theta_t^*) \end{array} \right\},
\end{aligned}
$$

where the last equality is based on the identity in equation (41), $\frac{\bar{\theta}}{\theta_t^*}=L(\theta_t^*)-H(\theta_t^*)$.

Clearly, $\frac{\partial V}{\partial \theta_t^*}=0$ if and only if

$$\beta(\alpha+\varphi)L(\theta_t^*)-H(\theta_t^*)=0. \tag{73}$$

Notice that equation (40) can be rewritten as

$$H(\theta_t^*)=\beta\alpha L(\theta_t^*)(1-\tau_{t+1})\frac{\theta_{t+1}^*L(\theta_{t+1}^*)}{\theta^*L(\theta_t^*)}.$$

So

$$\frac{\partial V}{\partial \theta_t^*}=0 \quad \text{if and only if} \quad \frac{\alpha+\varphi}{\alpha}=\left(1-\tau_{t+1}^k\right)\frac{\theta_{t+1}^*L(\theta_{t+1}^*)}{\theta_t^*L(\theta_t^*)}, \tag{74}$$

which in the Ramsey steady state implies

$$1-\tau_k^*=\frac{\alpha+\varphi}{\alpha}.$$

Note $\theta^*L(\theta^*)=\int_{\theta\le\theta^*}\theta^*d\mathbf{F}+\int_{\theta>\theta^*}\theta(i)\,d\mathbf{F}$ is an increasing function of θ_t^*, thus $\frac{\partial[\theta^*L(\theta^*)]}{\partial\tau}<0$. Also, θ_t^* is constant if τ_t^k is constant. Therefore, we have the following possible cases to consider:

First, for any constant tax rate $\bar{\tau}$, it is easy to see that

$$\frac{\alpha+\varphi}{\alpha}\lesseqqgtr(1-\bar{\tau})\frac{\theta^*L(\theta^*)}{\theta^*L(\theta^*)} \quad \text{if and only if} \quad \bar{\tau}\lesseqqgtr 0;$$

hence, setting $\bar{\tau}=-\frac{\varphi}{\alpha}$ for all t maximizes $\bar{V}\left(\{\bar{\tau}\}_{t=0}^{\infty}\right)$ given that $\tau_t=\bar{\tau}$.

Now suppose $\varphi = 0$ and the optimal tax rate is time varying and contains an increasing sequence in the time interval $[t, T]$ with $\left\{\tau_t^k < \tau_{t+1}^k < \tau_{t+2}^k < ... < \tau_T^k\right\}$. This implies that we have a monotonically decreasing sequence for $\{\theta_t^* L_t\}$ up to $T > t$. So $1 > \frac{\theta_{t+1}^* L(\theta_{t+1}^*)}{\theta^* L(\theta_t^*)} > \left(1 - \tau_{t+1}^k\right) \frac{\theta_{t+1}^* L(\theta_{t+1}^*)}{\theta_t^* L(\theta_t^*)}$ and $\frac{\partial V}{\partial \theta_t^*} > 0$. By equations (71) and (72), we have $\frac{\partial V}{\partial \tau_{t+1}} < 0$, implying that increasing the tax rate is reducing welfare, thus not optimal. Hence, without production externality, any dynamic tax path with an increasing tax rate in any time interval is not optimal.

Now suppose $\varphi = 0$ and that the optimal tax rate is a monotonically decreasing sequence $\left\{\tau_t^k > \tau_{t+1}^k > \tau_{t+2}^k > ...\right\}$ for all $t > 1$ and that this sequence converges to a positive constant $\bar{\tau} > 0$. This implies that in the limit we have $\frac{\theta_{t+1}^* L(\theta_{t+1}^*)}{\theta^* L(\theta_t^*)} \to 1$ and $\left(1 - \tau_{t+1}^k\right) \frac{\theta_{t+1}^* L(\theta_{t+1}^*)}{\theta^* L(\theta_t^*)} \to (1 - \bar{\tau}) < 1$ in the long run. By equations (71) and (72), we have $\frac{\partial V}{\partial \bar{\tau}} < 0$, suggesting $\bar{\tau}$ should be zero in the long run—a contradiction.

Therefore, without production externalities any optimal path of the tax rate must either be monotonically converging to zero or constant at zero starting from $t > 1$. Now suppose τ_t^k monotonically converges to 0. Since along a declining tax path, the term $\theta_t^* L\left(\theta_t^*\right)$ is monotonically increasing, we must have $\frac{\theta_{t+1}^* L\left(\theta_{t+1}^*\right)}{\theta^* L(\theta_t^*)} > 1$ at any point of time $t \in (1, \infty)$. By equation (74), suppose $1 < \left(1 - \tau_{t+1}^k\right) \frac{\theta_{t+1}^* L(\theta_{t+1}^*)}{\theta_t^* L(\theta_t^*)}$ along the declining tax path in period $t \in (1, \infty)$, then we must have $\frac{\partial V}{\partial \theta_t^*} < 0$ and $\frac{\partial V}{\partial \tau_{t+1}} > 0$, which suggests that a higher (rather than lower) tax rate tomorrow in period $t + 1$ would maximize welfare. This contradicts the requirement that $\left\{\tau_t^k\right\}_{t=2}^{\infty}$ be a monotonically decreasing sequence. Hence, we must have $1 > \left(1 - \tau_{t+1}^k\right) \frac{\theta_{t+1}^* R(\theta_{t+1}^*)}{\theta_t^* R(\theta_t^*)}$ along the declining tax path. On this path, since $\frac{\partial V}{\partial \theta_t^*} > 0$ and $\frac{\partial V}{\partial \tau_{t+1}} < 0$ for all t, a decreasing sequence of the tax rate is optimal. This also implies that the welfare function $V\left(\left\{\tau_t^k\right\}_{t=0}^{\infty}\right)$ converges monotonically in the limit to the upper bound $V(\{0\})$.

Therefore, without production externalities, the welfare along a declining tax path must be strictly lower than the welfare along the constant (zero) tax path: $V\left(\left\{\tau_t^k\right\}_{t=0}^{\infty}\right) < V(0)$. Hence, without production externalities, setting τ_t^k immediately to zero for $t > 1$ is optimal.

A.8 Proof of Proposition 6

The definition of competitive equilibrium and the equilibrium property of Case B remain largely unchanged compared to the benchmark model (Proposition 1), except the following modifications: First, we consider only the case where $\theta_t^* = \theta_H$. Second, x_t is redefined as

$$x_t(\theta^{t-1}) \equiv r_t a_t(\theta^{t-1}) + w_t n_t(\theta^{t-1}) - T_t.$$

Third, the government budget constraint and household budget constraint are changed to

$$B_{t+1} + T_t \geq G_t + r_t B_t$$

and

$$c_t(\theta^t) + a_{t+1}(\theta^t) \leq r_t a_t(\theta^{t-1}) + w_t n_t(\theta^{t-1}) - T_t,$$

respectively. Finally, the no-arbitrage condition becomes

$$r_t = 1 - \delta + MP_{K,t}.$$

A.8.1 The "Only If" Part

Assume that we have the allocation $\{C_t, N_t, K_{t+1}, B_{t+1}\}_{t=1}^{\infty}$ and the initial risk-free rate r_1. We then can directly construct the prices, taxes, and individual allocations in the competitive equilibrium in the following steps:

1. $\overline{K}_t$ is set to equal K_t.

2. w_t and q_t are given by (3) and (4).

3. The asset-market-clearing condition is used to set $A_{t+1} = K_{t+1} + B_{t+1}$.

4. The interest rate is given by
$$r_t = \frac{1}{\beta}\frac{w_{t-1}}{w_t}.$$

5. T_t is chosen such that the government budget constraint holds.

6. X_t is chosen according to the second line of equation (19):
$$X_t = A_{t+1} + C_t.$$

7. $c_t(\theta_t)$ and $a_{t+1}(\theta_t)$ are pinned down by the second lines in equations (13) and (14),

respectively. Finally, $n_t(\theta_{t-1})$ is set to satisfy equation (15), which is implied by the individual household budget constraint.

Therefore, the allocation has to satisfy the following three conditions in order to construct a competitive equilibrium:

1. The no-arbitrage condition $r_t = 1 - \delta + MP_{K,t}$ holds for all $t \geq 1$.

2. The aggregate resource constraint $C_t + K_{t+1} = F\left(K_t^{\varphi}, K_t, N_t\right) + (1-\delta)\, K_t$ holds.

3. By the second line of equation (18), C_t has to satisfy $C_t = \overline{\theta} w_t = \overline{\theta} MP_{N,t}$.

A.8.2 The "If" Part

Note that the aggregate resource constraint and no-arbitrage condition are trivially implied by a competitive equilibrium, since it is part of the definition. The condition that $C_t = \overline{\theta} MP_{N,t}$ holds in competitive equilibrium according to the Case B version of Proposition 1.