The Determination of Public Debt under both Aggregate and Idiosyncratic Uncertainty*

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Abstract

We analyze the Ramsey planner’s decisions to finance stochastic public expenditures under incomplete insurance markets for idiosyncratic risk. We show analytically that whenever the market interest rate lies below the time discount rate, the Ramsey planner has a dominant incentive to increase debt to meet the private sector’s demand for full self-insurance regardless of the relative size of aggregate shocks—suggesting a departure from tax smoothing. However, if a full self-insurance Ramsey allocation is infeasible in the absence of a government debt limit, an interior or bounded Ramsey equilibrium does not exist. The strong incentives for the Ramsey planner to smooth both individual consumption (via increasing public debt) and aggregate consumption (via tax smoothing) imply that (i) the long-run Ramsey equilibrium is characterized by full self-insurance and constant taxes if state-contingent bonds are available and (ii) when state-contingent bonds are not available, the government’s attempt to balance the competing incentives between tax smoothing and individual consumption smoothing—even at the cost of extra tax distortion—implies a bounded stochastic unit root component in optimal taxes and in the bond supply. In all cases considered in this paper, a sufficiently high average level of public debt (financed by distortionary taxation) to support full self-insurance is desirable and welfare improving. Therefore, adding a liquidity premium into the value of government bonds via incomplete financial markets can bring the theory of public finance into closer conformity with reality.

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1 Introduction

How to best finance unpredictable government spending is a long-standing issue in the history of political economic thought. Hume (1777) observed that it has been a common practice since antiquity that a government opts to engage in precautionary savings to insure itself against unforeseen expenditures (such as wars) to avoid volatile spontaneous taxation. However, without modern economic tools, Hume could not quantitatively investigate the optimal relationships among taxation, public debt, and government spending.

Barro (1979) is among the first to use an optimizing economic-agent model to rationalize Hume’s insight. Barro shows that an optimizing benevolent government opts to rely on public debt and distortionary taxes to finance random government expenditures so that changes in public debt comove with government spending shocks, while taxes are (imperfectly) smoothed like a random walk—analogue to consumption smoothing in a permanent-income consumption model.1

This permanent-income analogy of tax smoothing is validated by the seminal work of Lucas and Stokey (1983) (LS hereafter), who cast Barro’s public financing problem as a standard Ramsey problem in a simple representative-consumer model without capital. They show that while the representative consumer chooses consumption, leisure, and bond holdings to maximize lifetime utility subject to labor-income taxation, the Ramsey planner indeed finds it optimal to smooth taxes by setting the tax rate close to a constant, provided that it can issue (or hold) state-contingent bonds to insure against random government expenditures. This tax-smoothing result is indeed analogous to the permanent income model in which consumption would be close to a constant if the representative consumer can use state-contingent assets to buffer income shocks.

A drawback of the LS model is that the serial correlations in optimal tax rates are tied closely to those for government expenditures, belied the empirical work of Sargent and Velde (1995) as well as Barro’s earlier result that taxes should be a random walk for any stochastic process of government expenditures. For this reason, Aiyagari, Marcet, Sargent, and Seppala (2002) (AMSS henceforth) extend the LS model to a setting with only risk-free government bonds and confirm Barro’s conjecture that the optimal path of taxes has a unit-root component—fully analogous to consumption behavior in a standard permanent-income consumption model with non-state contingent financial assets. The intuition is that when the government is unable to issue (or hold) state-contingent bonds to insure against unanticipated government spending shocks, optimal tax rates must adjust for multiple periods and also more violently to balance the government

1Specifically, if we reinterpret taxation as (negative) consumption, government spending shocks as (negative) income shocks, and government debt as (negative) savings, then the optimal path of taxation (consumption) would indeed follow a random walk, provided that the objective function is concave in taxes. This is why Barro (1979) derives his results by putting taxes into a firm’s production function.
budgets in response to a one-time spending shock.

Nonetheless, this literature has abstracted from distributional questions and issues of possible conflict between the saving motives of the government and those governed. As a result, one possible outcome in the models of LS and AMSS is that optimal government debt could be frequently negative (unless an ad hoc lower debt limit is imposed on the government), which is not observed in the long historical data. Such predictions arise because in a representative-agent model, like those of LS and AMSS, the equilibrium interest rate must equal the private sector’s time discount rate, thus a benevolent government may opt to lend to (rather than borrow from) households—so that it can smooth taxes and finance future government expenditures using accumulated tax revenues and the private sector’s interest payments to the government. In other words, the incentive to smooth taxes entices the government to trade off between the mean and the variance of taxes by holding privately issued debt to prevent abrupt and volatile changes in taxation in the event of wars. But throughout recorded human history, not only have taxes been highly variable and persistent but the public debt-to-GDP ratio has always been positive and the growth rate of public debt strongly positively correlated with government expenditures (Barro (1979)).

This paper takes the private sector’s precautionary liquidity-demand motives seriously by introducing heterogeneous agents with uninsurable idiosyncratic risk into the models of LS and AMSS. Heterogeneity under incomplete individual insurance markets implies that the equilibrium interest rate can be lower than the private sector’s time discount rate. Unlike the original Aiyagari (1994) model, however, our model has the important property that the demand for financial assets remains finite even when the interest rate equals the time discount rate. This property guarantees the existence of an interior Ramsey steady state (regardless of government debt limits) and renders the Ramsey problem analytically tractable despite incomplete insurance markets and aggregate shocks.

We use this model to show that when government debt is not only a means to smooth taxes in financing random government expenditures but also a critical form of liquidity (store of value) to smooth individual consumption, an interesting transitional dynamic path emerges such that a departure from tax smoothing becomes desirable; and, consequently, the long-term dynamic behaviors of optimal government debt and taxes are in closer conformity with reality than the representative-agent models of LS and AMSS. The reason is that the demand for self-insurance by the private sector builds into an endogenous force that supports debt growth and thus a “soft” lower bound on the debt limit so that the Ramsey planner has little incentive to reduce the bond supply below this endogenously determined lower bound.

In other words, we show that whenever the equilibrium interest rate lies below the private sector’s time discount rate, a benevolent government’s attempt to smooth taxes and (at the same
time) to mitigate the idiosyncratic insurance risk for households—even at the cost of extra tax distortion—necessarily implies a rising average level of public debt in the transition period (starting from a low-enough initial debt level), regardless of the relative variance of aggregate spending shocks.

Specifically, we analyze whether there exists any trade-off between the government’s own precautionary saving motive to buffer aggregate spending shocks and the private sector’s precautionary saving motive to buffer idiosyncratic spending shocks when the market interest rate lies below the time discount rate because of a liquidity premium for public debt under incomplete financial markets. We show first that with state-contingent bonds and log-linear preferences (as in the indivisible labor literature), a benevolent government is willing to issue a sufficiently large amount of public debt to fully meet the self-insurance demand of households such that taxes are monotonically increasing in the transitional period and constant in the long run despite random government spending shocks, reminiscent of the classic result obtained by LS.

However, either if there exists an upper debt limit on the government’s capacity to issue state-contingent debt or if financial assets (bonds) are risk free (non-state contingent), then optimal taxes will follow a bounded stochastic unit-root (STUR) process, while the average level of public debt will remain strictly increasing in the transition period (starting from a low initial level) and its growth rate will positively correlate with government spending shocks, as anticipated by Barro (1979).

The intuition is as follows. With state-contingent debt, if households are borrowing constrained in a competitive equilibrium, then the market interest rate lies strictly below the time discount rate and, consequently, from the Ramsey planner’s viewpoint, the marginal benefit of relaxing the government budget constraint by issuing additional public debt to enhance households’ insurance position strictly dominates the discounted next-period marginal cost of tax distortion (for interest payments). Hence, the Ramsey planner has a dominant incentive to keep increasing the debt level to meet consumers’ precautionary demand for safe assets until the market interest rate equals the time discount rate—at which point the distortions in the individual insurance markets are fully eliminated and, consequently, no household is borrowing constrained. Once households are fully self-insured with enough holdings of public debt, the model starts to behave like a representative-agent LS model with complete markets for aggregate risk. Hence, the Ramsey planner’s incentive to improve households’ self-insurance positions across idiosyncratic states generates a natural endogenous “soft” lower bound on the debt limit or a positive force for the growth of public debt along the transitional path (if the initial debt level is too low to permit a full self-insurance allocation). On the other hand, if the initial debt level is already sufficiently high to guarantee the full self-insurance allocation in a competitive equilibrium, then the positive force does not present
unless the public debt supply falls below the critical level for full self-insurance. This also suggests
that if the government faces an upper borrowing limit such that a full self-insurance allocation is
impossible, then optimal taxes will be stochastic and serially correlated (even if government spend-
ing shocks are iid)—because the incentives for both tax smoothing and consumption smoothing
imply that the Ramsey planner opts to take multiple periods to balance the government budget
after a one-time shock to government spending when the debt supply is constrained from above.

The same insight applies when state-contingent government debt is not available or when fi-
nancial assets are risk free. In this case, although a full self-insurance allocation and a constant
tax rate cannot be maintained in the long run by any finite amount of government bonds—because
of the government’s inability to fully insure itself against aggregate spending shocks—the Ramsey
planner can nonetheless achieve the full self-insurance allocation stochastically (or with a positive
probability) at any point in time. Optimal income taxes thus also become stochastic and serially
correlated.

Therefore, regardless of the availability of state-contingent bonds, the Ramsey planner always
opts to issue a sufficiently large amount of bonds (up to any upper debt limit) to make consumers
as fully self-insured against idiosyncratic risk as possible despite (regardless of) aggregate spending
shocks.

This dominant incentive for providing full self-insurance to consumers—even at the cost of
extra tax distortion—derives from an intertemporal arbitrage opportunity for debt accumulation
when the market interest rate lies below the time discount rate, since at the margin discounted
future interest payments are always less than the current market price of debt. This suggests
that if a full self-insurance allocation is impossible to achieve (in the absence of any upper debt
limit)—either because of a fat-tailed distribution of idiosyncratic shocks in our model or because
households’ Euler equations follow a martingale process (as in the well-known Aiyagari (1994)
model) such that the demand for safe assets goes to infinity when the interest rate approaches
the time discount rate, then an interior Ramsey steady state (without aggregate uncertainty) or
a bounded stochastic Ramsey equilibrium (with aggregate uncertainty) does not exist unless an
upper debt limit (such as the natural debt limit or an ad hoc one) exists and is necessarily binding
in a Ramsey equilibrium.

Whenever an upper debt limit exists and binds in a Ramsey equilibrium, the optimal taxes
must be stochastic and serially correlated (imperfectly smoothed) even if the government spending
shock is iid and government bonds are state contingent. In such cases the Ramsey planner is unable
to achieve and maintain the full self-insurance allocation almost surely—because the government’s
ability to smooth taxes is severely hindered. In contrast to the model of AMSS, an ad hoc lower
debt-limit constraint on the government bond supply is not needed to ensure convergence to sta-
tionary (bounded) distributions of government debt and taxes because the desire to issue enough debt for self-insurance automatically keeps the optimal debt level from falling for too long and too low.

The assumption of heterogeneous agents and incomplete insurance markets thus offers not only a rationale for the stochastic property of taxes and the positive correlation between the growth rate of public debt and government spending shocks observed by Barro (1979), but also a rationale (consistent with the conventional wisdom) of the large supply of public debt across space and time. The introduction of non-state contingent bonds (as studied by AMSS) serves only to increase the variance and autocovariance of taxes—because in this situation government bonds are less capable of buffering aggregate spending shocks for any given level of idiosyncratic risk and, consequently, the incentive for tax smoothing results in more serially correlated taxes.

In addition to the seminal works of Barro (1979), LS, and AMSS discussed above, our paper is also related to a large and growing literature on optimal fiscal policies and debt financing. Below we discuss only the most closely related work and defer more detailed literature reviews to Section 5.

Using a two-agent two-period model with incomplete markets, Shin (2006) shows that heterogeneous agents with uninsurable idiosyncratic risk can alter the Ramsey planner’s public financing decisions and increase the optimal level of public debt. Specifically, he shows that when the idiosyncratic risk is sufficiently large relative to aggregate risk, the Ramsey planner chooses to issue debt and facilitate the precautionary saving of the private sector instead of holding IOUs issued by the households, even at the cost of extra tax distortion. Shin (2006) interprets these outcomes in terms of the trade-off between two competing insurance motives that concern the Ramsey planner: aggregate tax smoothing and individual consumption smoothing.

However, even with the simplifying two-agent and two-period assumptions, Shin relies either on local approximation or numerical methods to obtain his results. This approach raises several questions. First, when the number of agents is too small, it is difficult to distinguish idiosyncratic risk from aggregate risk. Second, when the horizon is too short (with only two periods), the dynamic and strategic considerations of the Ramsey planner cannot be fully revealed and, most importantly, the interesting transitional dynamics crucial for welfare analysis are lost, which may lead to biased characterization of the Ramsey allocation.

We extend Shin’s (2006) framework into a more general setting with an infinite horizon and a continuum of agents. Even with double infinity, we are able to characterize the Ramsey problem analytically using the primal approach and obtain the Ramsey first-order conditions in closed form, which allows us to derive our results analytically. In particular, we can analytically characterize the fundamental law of motion that governs the Ramsey planner’s decisions for the optimal supply
of public debt under aggregate uncertainty both in the transition period and in the long run.

The extension to a setting with an infinite horizon and a continuum of agents is important and nontrivial for several reasons. First, even under the simpler setting without aggregate uncertainty, many public-financing issues remain unsettled in the existing literature based on infinite-horizon heterogenous-agent incomplete markets (HAIM hereafter) models—to the best of our knowledge the existence of a Ramsey steady state in such models is often assumed instead of proven. Without this critical assumption, the Ramsey allocation is hard to analyze because of the models’ intractability; yet optimal fiscal policies drawn from the analyses may hinge critically on the validity of this assumption. Second, without the existence of a Ramsey steady state (in the absence of aggregate shocks), the accuracy and reliability of the results based on local approximation and numerical methods for solving models with aggregate shocks become dubious. Third, the existing literature has shown that not taking into full consideration the transitional dynamics can lead to very different conclusions and even the opposite welfare result. Last but not the least, it needs to be known under what conditions a trade-off can exist between the government’s precautionary saving motive and individuals’ precautionary saving motive. Does the trade-off depend on the magnitude of the market interest rate relative to the time discount rate?

This paper designs an analytically tractable HAIM model with an infinite horizon and a continuum of agents to shed light on these important issues. In particular, we analytically investigate the dynamic behaviors of taxes and public debt under the two competing motives concerning the Ramsey planner: aggregate tax smoothing and individual consumption smoothing. The tractability of our infinite-horizon model enables us to prove the existence of an interior Ramsey steady state (in the absence of aggregate uncertainty and a government debt limit) and derive both the competitive equilibrium and the stochastic Ramsey allocation under aggregate uncertainty in closed forms along the entire transition path. Our model has the desirable properties that a Ramsey steady state exists under proper parameter values and that a particular Ramsey steady state is one for which the expected market interest rate equals the time discount rate. These properties facilitate our understanding of the joint dynamic behaviors of taxes and public debt when aggregate uncertainty is present, as well as the important role played by government borrowing limits in determining these joint behaviors. In addition, we also show analytically the effects of (non)state-contingent bonds on Ramsey allocation by pointing out explicitly that the random-walk property of optimal taxes (as discovered by AMSS) originates precisely from the equality between the market interest rate and the time discount rate—meaning that the Ramsey planner is indifferent between issuing debt and holding privately issued IOUs only when the interest rate equals the time discount rate; thus the random-walk dynamics necessarily disappear in the transition when the interest rate lies

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2See, for instance, Domeij and Heathcote (2004) and Rohrs and Winter (2017)
below the time discount rate.

The rest of the paper is organized as follows. Section 2 extends the model of LS to a setting with heterogeneous agents and incomplete insurance markets for idiosyncratic risk and studies this benchmark model’s competitive-equilibrium properties. Section 3 shows how to solve the Ramsey problem analytically in the benchmark model using the primal approach and compares the results with those of LS. Section 4 relaxes the assumption of state-contingent bonds, shows how to solve the Ramsey problem analytically under aggregate uncertainty, and compares the results with those of AMSS. In both Section 3 and 4, numerical analyses are also provided to confirm our theoretical results. Section 5 provides a brief literature review. Finally, Section 6 concludes the paper.

2 Benchmark Model with State-Contingent Debt

We study first a benchmark model with state-contingent debt by introducing heterogeneous agents with uninsurable idiosyncratic risk into the model of LS. Our heterogeneous-agent approach follows Bewley (1980), Huggett (1993), Aiyagari (1994), and especially Wen (2009, 2015).³ An important property of our model is its analytical tractability with closed-form solutions, which enables us to use the primal approach to solve the Ramsey allocation analytically. The analytical tractability of our model is the consequence of two assumptions: (i) Household utility is log-linear (as in the case of indivisible labor) and labor choice is made before observing any idiosyncratic shocks, and (ii) the source of idiosyncratic shocks is from the marginal utility of consumption (instead of labor productivity), as in the heterogeneous-agent cash-in-advance model of Lucas (1980).

An important implication of these assumptions is that households’ marginal utility is no longer a martingale process; hence, as long as the support of idiosyncratic shocks is bounded, household demand for government bonds is always finite even when the interest rate equals the time discount rate—this is in sharp contrast to the original Aiyagari (1994) model where the demand for bonds goes to infinity when the interest rate approaches the time discount rate. This property renders our model more suitable for the analysis of a Ramsey problem under aggregate uncertainty, because it ensures the existence of an interior Ramsey steady state in the absence of aggregate shocks. Without the existence of a well-defined Ramsey steady state, it is difficult (if not impossible) to characterize a Ramsey allocation in an infinite-horizon model because the optimal supply of government bonds may diverge to infinity under heterogeneous agents and incomplete insurance markets.⁴

³In a companion paper, Chien and Wen (2019), we use a similar framework with capital accumulation to analyze the Ramsey problem of optimal capital taxation.

⁴See Chien and Wen (2019).
In a HAIM economy with aggregate uncertainty, these model-specific properties not only allow us to analytically characterize the Ramsey allocation when government bonds are state-contingent, but also help us understand tax smoothing and the determination of public debt when government bonds are not state contingent.

2.1 Environment

Time is discrete and indexed by $t = 0, 1, 2, \ldots$. Aggregate government spending $G_t(z^t)$ is stochastic, with the period-$t$ aggregate state of the shock denoted by $z_t$ and the history of the shock denoted by $z^t = (z_0, \ldots, z_t)$. The spending shock is assumed to be covariance stationary and follow a Markov process with a finite number of states. The probability of a shock event, $z_t$, is denoted by $\pi(z_t)$, and the probability of the history of events, $z^t$, is denoted by $\pi(z^t)$.

There is a unit measure of *ex ante* identical households that face idiosyncratic preference shocks $\theta_t$ in period $t$. The preference shock is iid over time and across households with the cumulative distribution $F(\theta)$ and support $[\theta_L, \theta_H]$, where $0 < \theta_L < \theta_H < \infty$, and the mean is denoted by

$$\bar{\theta} \equiv \int \theta dF(\theta).$$

Within each period $t$ there are two subperiods. The aggregate spending shock is realized in the beginning of the first subperiod, and the idiosyncratic preference shock $\theta_t$ is realized only in the beginning of the second subperiod. However, the decision for the labor supply must be made in the first subperiod before observing $\theta_t$, and the decisions for consumption and saving must be made in the second subperiod after the realization of $\theta_t$. Let $\theta^t \equiv (\theta_1, \ldots, \theta_t)$ denote the history of preference shocks including that in period $t$.

As in LS, a representative firm produces output according to a linear production technology in labor, $Y_t(z^t) = N_t(z^t)$, where $N_t(z^t)$ denote the aggregate labor input. The firm hires labor from households by paying a competitive real wage rate denoted by $w_t(z^t)$. Perfect competition implies

$$w_t(z^t) = \frac{\partial Y_t(z^t)}{\partial N_t(z^t)} = 1.$$

Initially, all households are endowed with the same amount of bond holdings in period 0. Households are infinitely lived with a log-linear utility function, so their lifetime expected utility is given by

$$V = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \theta_t \log c_t(\theta^t, z^t) - n_t(\theta^{t-1}, z^t) \right\},$$

(1)

where $\beta \in (0, 1)$ is the discount factor, $c_t(\theta^t, z^t)$ and $n_t(\theta^{t-1}, z^t)$ denote consumption and the labor
supply, respectively, for the household and for given shock histories at time $t$.

Note that the labor supply in period $t$ is only measurable with respect to $(\theta^t, z^t)$, reflecting the assumption that the labor supply decision in period $t$ is made before observing the preference shock $\theta_t$. This special timing arrangement is designed to make the idiosyncratic preference shock uninsurable despite the constant marginal utility cost of labor. In other words, this timing assumption is not needed if the utility function is strictly concave in leisure.\footnote{The model becomes intractable when the utility function is strictly concave in both consumption and leisure.}

In our benchmark economy, the asset markets are complete in terms of aggregate risk but incomplete in terms of idiosyncratic risk. In other words, households and the government can trade bonds contingent on the aggregate states but not on individual states. Later in the paper, we consider the environment where there is no state-contingent claim for either aggregate or idiosyncratic states.

The stochastic government spending $G_t(z^t)$ is financed by a flat labor-income tax rate, $\tau_t(z^t)$, and state contingent government bonds, $B_{t+1}(z_{t+1}|z^t)$. The period-$t$ price of the state-contingent bonds is $Q_{t+1}(z_{t+1}|z^t)$, which pays one unit of consumption good in period $t + 1$ conditional on the history $z^t$. The flow government budget constraint is given by

$$\tau_t(z^t)N_t(z^t) + \sum_{z_{t+1}} Q_{t+1}(z_{t+1}|z^t)B_{t+1}(z_{t+1}|z^t)\pi(z_{t+1}|z^t) \geq G_t(z^t) + B_t(z_t|z^{t-1}) \tag{2}$$

for all $t \geq 0$, where $B_0(z_0) = B_0$ for all $z_0$. The initial level of government bonds, $B_0$, is exogenously given.

### 2.2 Household Problem

Given sequences of the after-tax wage rates, $\{\overline{w}_t(z^t) \equiv 1 - \tau_t(z^t)\}_{t=0}^{\infty}$, and state-contingent bond prices, $\{Q_{t+1}(z_{t+1}|z^t)\}_{t=0}^{\infty}$, households solve their utility-maximizing problem by choosing a plan of consumption, labor, and state-contingent asset holdings as follows:

$$\max_{\{c_t(\theta^t,z^t), n_t(\theta^t-1,z^t), a_{t+1}(z_{t+1}|\theta^t,z^t)\}} E_0 \sum_{t=0}^{\infty} \beta^t \{ \theta_t \log c_t(\theta^t,z^t) - n_t(\theta^t-1,z^t) \}$$

subject to the budget constraint,

$$c_t(\theta^t,z^t) + \sum_{z_{t+1}} Q_{t+1}(z_{t+1}|z^t)a_{t+1}(z_{t+1}|\theta^t,z^t)\pi(z_{t+1}|z^t) \leq a_t(z_t|\theta^{t-1},z^{t-1}) + \overline{w}_t(z^t)n_t(\theta^{t-1},z^t) \tag{3}$$
and the state-dependent borrowing constraint,

\[ a_{t+1}(z_{t+1}|\theta^t, z^t) \geq 0, \]  \hspace{1cm} (4)

where \( a_0(z_0) = a_0 \) for all \( z_0 \). The initial asset holdings \( a_0 \geq 0 \) are given and assumed to be identical across households. The individual labor supply \( n_t(\theta^{t-1}, z^t) \) is bounded in the close interval of \([0, \overline{N}]\), where \( \overline{N} < \infty \).

### 2.3 Competitive Equilibrium

Denote aggregate consumption and the labor supply as \( C_t(z^t) \) and \( N_t(z^t) \), respectively. A competitive equilibrium is defined as follows:

**Definition 1.** Given a sequence of the labor tax, government spending shocks, and government bond supply, \( \{\tau_t(z^t), G_t(z^t), B_{t+1}(z_{t+1}|z^t)\}_{t=0}^{\infty} \), and the initial bond level \( B_0 \), a competitive equilibrium is a sequence of prices \( \{w_t(z^t), Q_{t+1}(z_{t+1}|z^t)\}_{t=0}^{\infty} \); and a sequence of individual and aggregate allocations \( \{c_t(\theta^t, z^t), n_t(\theta^{t-1}, z^t), a_{t+1}(z_{t+1}|\theta^t, z^t), N_t(z^t), C_t(z^t)\}_{t=0}^{\infty} \), such that

1. given the sequence \( \{w_t(z^t), Q_{t+1}(z_{t+1}|z^t), \tau_t(z^t)\}_{t=0}^{\infty} \), the household allocation \( \{c_t(\theta^t, z^t), n_t(\theta^{t-1}, z^t), a_{t+1}(z_{t+1}|\theta^t, z^t)\}_{t=0}^{\infty} \) solve the household problem;

2. given the sequence of real wages \( \{w_t(z^t) = 1\}_{t=0}^{\infty} \), the aggregate labor allocation \( \{N_t(z^t)\}_{t=0}^{\infty} \) solves the firm problem;

3. the government budget constraint (2) holds for all period \( t \); and

4. all markets clear for all history \( z^t \) and \( t \geq 0 \):

\[
B_{t+1}(z_{t+1}|z^t) = \int a_{t+1}(z_{t+1}|\theta^t, z^t)dF(\theta^t) \]  \hspace{1cm} (5)

\[
N_t(z^t) = \int n_t(\theta^{t-1}, z^t)dF(\theta^{t-1}) \]  \hspace{1cm} (6)

\[
C_t(z^t) = \int c_t(\theta^t, z^t)dF(\theta^t) \]  \hspace{1cm} (7)

\[
C_t(z^t) + G_t(z^t) = N_t(z^t). \]  \hspace{1cm} (8)

\[6\] In the online Appendix (Appendix B), we study the robustness of the results by relaxing the borrowing constraint in equation (4).
2.4 Characterization of the Competitive Equilibrium

Denote a household’s gross income (liquidity on hand) by

\[ x_t(\theta^{t-1}, z^t) \equiv a_t(z_t|\theta^{t-1}, z^{t-1}) + \bar{w}_t(z^t)n_t(\theta^{t-1}, z^t) \]  

(9)

and define the “effective” household savings as

\[ \hat{a}_t(\theta^t, z^t) \equiv \sum_{z_t+1} Q_{t+1}(z_{t+1}|z^t) a_{t+1}(z_{t+1}|z^t, \theta^t) \pi(z_{t+1}|z^t). \]

In addition, denote \( X_t \) and \( \hat{A}_t \) as the aggregate liquidity on hand and aggregate effective savings, respectively. The following proposition characterizes the competitive equilibrium and shows that the distribution of household choices is fully characterized by the cutoff \( \theta^*_t(z^t) \in [\theta_L, \theta_H] \).

**Proposition 2.** (i) The optimal individual allocations for liquidity on hand \( x_t(\theta^{t-1}, z^t) \), consumption \( c_t(\theta^t, z^t) \), effective savings \( \hat{a}_t(\theta^t, z^t) \), and the labor supply \( n_t(\theta^{t-1}, z^t) \) are given, respectively, by the following decision rules:

\[
x_t(z^t) = \begin{cases} 
\bar{w}_t(z^t)L(\theta^*_t(z^t)) \theta^*_t(z^t) & \text{if } \theta^*_t(z^t) < \theta_H \\
X_t(z^t) & \text{if } \theta^*_t(z^t) = \theta_H 
\end{cases}
\]

(10)

\[
c_t(\theta_t, z^t) = \begin{cases} 
\min \left\{ 1, \frac{\theta_t}{\theta_t(z^t)} \right\} x_t(z^t) & \text{if } \theta^*_t(z^t) < \theta_H \\
\theta_t \bar{w}_t(z^t) & \text{if } \theta^*_t(z^t) = \theta_H 
\end{cases}
\]

(11)

\[
\hat{a}_t(\theta^t, z^t) = \begin{cases} 
\max \left\{ 0, \frac{\theta_T(z^t) - \theta_t}{\theta_t(z^t)} \right\} x_t(z^t) & \text{if } \theta^*_t(z^t) < \theta_H \\
X_t(z^t) - \theta_t \bar{w}_t(z^t) & \text{if } \theta^*_t(z^t) = \theta_H 
\end{cases}
\]

(12)

\[
n_t(\theta_{t-1}, z^t) = \frac{1}{\bar{w}_t(z^t)} \left[ x_t(z^t) - a_t(z_t|\theta_{t-1}, z^{t-1}) \right], \]

(13)

where the function \( L(\theta^*_t) \) denotes the bond liquidity premium and is defined as

\[
L(\theta^*_t(z^t)) \equiv \int_{\theta_L}^{\theta^*_t(z^t)} dF(\theta) + \int_{\theta^*_t(z^t)}^{\theta_H} \frac{\theta_t}{\theta^*_t(z^t)} dF(\theta) \geq 1.
\]

(14)

(ii) Aggregate liquidity in hand, aggregate consumption and aggregate effective savings are given,
respectively, by

\[
X_t(z^t) = \begin{cases} 
\sum_{z_{t+1}} Q_{t+1}(z_{t+1}|z^t) B_{t+1}(z_{t+1}|z^t) \pi(z_{t+1}|z^t) + \omega(z^t) & \text{if } \theta^*_t(z^t) < \theta_H \\
\theta^*_t(z^t) L(\theta^*_t(z^t)) \theta^*_t(z^t) & \text{if } \theta^*_t(z^t) = \theta_H 
\end{cases}
\]  
(15)

\[
C_t(z^t) = \begin{cases} 
D(\theta^*_t(z^t)) x_t(z^t) & \text{if } \theta^*_t(z^t) < \theta_H \\
\omega(z^t) & \text{if } \theta^*_t(z^t) = \theta_H 
\end{cases}
\]  
(16)

\[
\hat{A}_t(z^t) = \begin{cases} 
[1 - D(\theta^*_t(z^t))] x_t(z^t), & \text{if } \theta^*_t(z^t) < \theta_H \\
x_t(z^t) - \omega(z^t) & \text{if } \theta^*_t(z^t) = \theta_H 
\end{cases}
\]  
(17)

where the function \( D(\theta^*_t) \) denotes the aggregate marginal propensity to consume and is defined as

\[
D(\theta^*_t(z^t)) \equiv \int_{\theta_L}^{\theta^*_t(z^t)} \frac{\theta}{\theta^*_t(z^t)} dF(\theta) + \int_{\theta^*_t(z^t)}^{\theta_H} dF(\theta) > 0, \]
(18)

(iii) The state-contingent bond price \( Q_{t+1}(z_{t+1}|z^t) \) is given by

\[
Q_{t+1}(z_{t+1}|z^t) = \beta \frac{\omega(z^t)}{\omega(z_{t+1})} L(\theta^*_t(z^t)),
\]  
(19)

(iv) Finally, the following condition ensures \( n_t(\theta^{t-1}, z^t) > 0 \) for all \( t > 0 \):

\[
x_t(z^t) = \frac{\theta^*_t(z^{t-1}) - \theta_L}{\theta^*_t(z^{t-1}) - Q_t(z_{t-1})} > 0,
\]  
(20)

which is assumed to hold throughout the paper under properly calibrated parameter values. Notice that in the steady state (without aggregate uncertainty) the above condition for positive labor is ensured by the sufficient condition \( \beta > \frac{\theta_H - \theta_L}{\theta_H} \), which guarantees the condition \( \beta L(\theta^*) > \frac{\theta^* - \theta_L}{\theta^*} \), where \( \beta L(\theta^*) = Q \).

Proof. See Appendix A.1

Notice that the cutoff \( \theta^*_t(z^t) \in [\theta_L, \theta_H] \) is unique and sufficient to characterize the distribution of household allocations in the economy. This property hinges on the assumptions of a log-linear utility function and iid idiosyncratic preference shocks, as well as the assumption that the labor supply in each period \( t \) is predetermined in the first subperiod of \( t \). These assumptions make the optimal distribution of gross income \( x_t(z^t) \) in each period independent of the history of idiosyncratic shocks \( \theta^t \), as revealed in equation (10), which states that \( x_t \) is independent of \( \theta^t \) if and only if \( \theta^*_t \) is independent of \( \theta^t \).
The intuition is as follows. Gross income $x_t$ is determined by the optimal labor choice in each period $t$ without observing $\theta_t$. Since the marginal cost of leisure is constant, labor supply $n_t$ is chosen such that the level of liquidity on hand is the same for all individuals regardless of their idiosyncratic history $\theta_t^{t-1}$ and the realization of $\theta_t$. This optimal target level of liquidity on hand is thus degenerate with respect to idiosyncratic shocks $\theta_t$. In other words, this target policy (uniform to all households) obtains because labor income $(\pi_t(z^t) n_t(\theta_{t-1}, z^t))$ can be adjusted elastically to meet an optimal target level of income $x_t$ such that this income level is ex ante optimal based on the distribution of $\theta_t$ for any given initial asset holdings $a_t(z_t|\theta_{t-1}, z_{t-1})$.

The optimal target income (liquidity on hand) also simultaneously pins down the optimal cutoff $\theta^*_t(z^t)$, or vice versa. Since $\theta_t$ is realized only after $x_t$ is determined, given this optimal target income, consumption and saving decisions are then made after the realization of the shock $\theta_t$, depending on whether $\theta_t$ is above or below $\theta^*_t$. Hence, in the first subperiod of any $t > 0$, all households will choose the same level of gross income $x_t(z^t)$ and $\theta^*_t(z^t)$ regardless of their initial asset holdings $a_{t-1}(z_t|\theta_{t-1}, z_{t-1})$ and the history $\theta^{t-1}$. Thus, the cutoff, $\theta^*_t(z^t)$, uniquely and fully characterizes the distributions of consumption $c_t(z^t, \theta_t)$ and savings $\hat a_t(\theta^t, z^t)$ in the economy.

However, notice that in general equilibrium, aggregate bond demand must equal aggregate bond supply. Thus clearly, if the aggregate bond supply $B_{t+1}(z_{t+1}|z^t)$ is high enough such that the equilibrium cutoff is $\theta^*_t(z^t) = \theta_H$ for any $z_t$, then no household will be borrowing constrained. In this case, the average effective saving rate is shown in the second line in equation (17).

The above discussions and Proposition 2 imply the following corollary:

**Corollary 3.** In the absence of aggregate uncertainty (e.g., $G_t = \overline{G}$), the model has two types of competitive equilibria, depending on the level of government debt: (i) a partial self-insurance equilibrium with $\theta^*_t < \theta_H$ if the bond supply is sufficiently low and (ii) a full self-insurance equilibrium with $\theta^*_t = \theta_H$ if the bond supply is sufficiently high.

### 2.5 Conditions to Support a Competitive Equilibrium

The following proposition describes the conditions needed to support a competitive equilibrium in terms of the aggregate variables, $C_t(z^t)$, $N_t(z^t)$, $B_{t+1}(z_{t+1}|z^t)$, and the cutoff $\theta^*_t(z^t)$.

**Proposition 4.** Given the initial government bond supply $B_0$, the sequence of aggregate allocation $\{N_t(z^t), C_t(z^t), B_{t+1}(z_{t+1}|z^t)\}_{t=0}^{\infty}$ and the sequence of distribution $\{\theta^*_t(z^t)\}_{t=0}^{\infty}$ can be supported as a competitive equilibrium if and only if

- the resource constraint (8) holds;
2. the implementability constraint

\[ Z(\theta^*_t(z^t)) - N_t(z^t) + \sum_{z_{t+1}} \beta U_C(z^{t+1}) Z(\theta^*_{t+1}(z^{t+1})) L(\theta^*_t(z^t)) B_{t+1}(z_{t+1}|z^t) \pi(z_{t+1}|z^t) \]

\[ \geq U_C(z^t) Z(\theta^*_t(z^t)) B_t(z_t|z^{t-1}) \]

holds for all \( z^t \) with \( t \geq 0 \), where (i) \( B_0(z^0) = B_0 \) for all \( z_0 \), and (ii) \( U_C(z^t) \) and \( Z(\theta^*_t(z^t)) \) are defined as

\[ U_C(z^t) \equiv C_t(z^t) - 1 \]

and

\[ Z(\theta^*_t(z^t)) \equiv D(\theta^*_t(z^t)) \theta^*_t(z^t) L(\theta^*_t(z^t)), \]

respectively.

3. the following bond market-clearing conditions hold for all \( t \geq 0 \):

\[
\begin{cases}
\sum_{z_{t+1}} U_{C,t+1} Z(\theta^*_{t+1}) B_{t+1}(z_{t+1}|z^t) \pi(z_{t+1}|z^t) = \beta \theta^*_t [1 - D(\theta^*_t)], & \text{if } \theta^*_t < \theta_H \\
B_{t+1}(z_{t+1}|z^t) = B_{t+1}(z_{t+1}|z^t), & \text{if } \theta^*_t = \theta_H 
\end{cases}
\]  

(24)

Proof. See Appendix A.2.

Notice that the first equation in constraint (24) is derived from the equation of aggregate household saving (see Appendix A.2):

\[
\sum_{z_{t+1}} Q_{t+1}(z_{t+1}|z^t) B_{t+1}(z^{t+1}|z^t) \pi(z_{t+1}|z^t) = [1 - D(\theta^*_t(z^t))] x_t(z^t),
\]

where \([1 - D(\theta^*_t(z^t))]\) is the households’ aggregate propensity to save. This asset market-clearing condition reveals that the Ramsey planner must consider the impact of its bond policy on asset prices and the distribution of asset holdings through the policy’s impact on the cutoff \( \theta^*_t(z^t) \), as long as \( \theta^*_t(z^t) < \theta_H \). This is in contrast to the representative-agent model that has a degenerated asset distribution.

However, our model has an interesting property: It is possible to have a competitive equilibrium where all households are fully self-insured by holding enough bonds if the level of the aggregate bond supply is sufficiently large. In such a case, \( \theta^*_t(z^t) = \theta_H \), implying that no household is borrowing constrained (since the probability of drawing a shock greater than \( \theta_H \) is zero), as in a representative-agent economy. Therefore, whenever \( \theta^*_t = \theta_H \), the supply of bonds determines the equilibrium and hence the asset distribution does not impose a constraint on the Ramsey planner.
for choosing an equilibrium; consequently the bond market-clearing condition degenerates to the second equation in condition (24), which is identical to that in a representative-agent economy.

Armed with Proposition 4, the primal approach of the Ramsey problem becomes feasible in our economy. In general, the adoption of a primal Ramsey approach in a HAIM economy faces this main challenge: to formulate the implementability condition analytically. This challenge is overcome here by the property that individual allocations and intertemporal prices of competitive equilibrium can all be expressed as a function of the aggregate quantity variables \( \{C_t(z^t), N_t(z^t), B_{t+1}(z_{t+1}|z^t) \} \), together with the cutoff variable \( \theta^*_t(z^t) \), thanks to the fact that the cutoff \( \theta^*_t(z^t) \) is a sufficient statistic for describing the distributions of individual choice variables.

To make our analyses interesting, we will assume from now on that the parameter space and the initial bond level \( B_0 \) are such that if the future bond supply \( B_{t+1}(z_{t+1}|z^t) \) remains at the initial level \( B_0 \) for all \( t \) and \( z^t \), then the competitive equilibrium exhibits only partial self-insurance with \( \theta^*_t(z^t) < \theta_H \). We study the Ramsey allocation under such parameter restrictions.

### 3 Ramsey Allocation in the Benchmark Model

#### 3.1 The Ramsey Problem

By utilizing equations (11) and (16), the welfare function \( V \) can be expressed as a function of the aggregate allocation \( \{C_t(z^t), N_t(z^t)\} \) and the cutoff \( \{\theta^*_t(z^t)\} \) as follows:

\[
V = \sum_{t=1}^{\infty} \sum_{z^t} \beta^t \left\{ W(\theta^*_t(z^t)) + \bar{\theta} \log C_t(z^t) - N_t(z^t) \right\} \pi(z^t),
\]

where the function \( W(\theta^*_t) \) captures the welfare effects of inequality (distribution) and is given by

\[
W(\theta^*_t(z^t)) \equiv \bar{\theta} \log \frac{1}{D(\theta^*_t(z^t))} + \int_{\theta \leq \theta^*_t(z^t)} \theta \log \frac{\theta}{\theta^*_t(z^t)} dF(\theta),
\]

which has the properties summarized in the following lemma:

**Lemma 5.** \( W(\theta^*_t) \) is S-shaped and strictly increasing in the open interval \( \theta^* \in (\theta_L, \theta_H) \), and it becomes flat toward the two end points \( \{\theta_L, \theta_H\} \). Namely,

\[
\left. \frac{\partial W(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \right|_{\theta^*_t \in (\theta_L, \theta_H)} > 0 \quad \text{and} \quad \left. \frac{\partial W(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \right|_{\theta^*_t = \{\theta_L, \theta_H\}} = 0.
\]

**Proof.** See Appendix A.3

\( \square \)
Now, the Ramsey problem can be represented as maximizing the lifetime utility (25) by choosing the sequences of aggregate quantities and the distribution: \( \{N_t(z^t), C_t(z^t), B_{t+1}(z_{t+1}|z^t), \theta_t^*(z^t)\} \) subject to the constraints listed in Proposition 4. As a result, the Lagrangian of the Ramsey problem is given by

\[
\begin{align*}
\max_{\{\theta_t^*(z^t), N_t(z^t), C_t(z^t), B_{t+1}(z_{t+1}|z^t)\}} \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \left[ W(\theta_t^*(z^t)) + \bar{\theta} \log C_t(z^t) - N_t(z^t) \right] \pi(z^t) \\
+ \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \psi_t(z^t) \pi(z^t) \left( N_t(z^t) - C_t(z^t) - G_t(z^t) \right)
\end{align*}
\]

\[
\begin{align*}
+ \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \phi_t(z^t) \pi(z^t) \left( Z(\theta_t^*(z^t)) - N_t(z^t) \right) \\
+ \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \mu_t^B(z^t) \pi(z^t) \left( \sum_{z_{t+1}} \beta B_{t+1}(z_{t+1}|z^t) U_C(z_{t+1}) Z(\theta_{t+1}^*(z_{t+1})) \pi(z_{t+1}|z^t) \right) \\
= 0 \quad \text{if} \quad \theta_t^* = \theta_H.
\end{align*}
\]

where \( \beta^t \psi_t(z^t) \pi(z^t), \beta^t \phi_t(z^t) \pi(z^t), \) and \( \beta^t \mu_t^B(z^t) \pi(z^t) \) denote, respectively, the Lagrangian multipliers for (i) the resource constraint in the second row, (ii) the implementability condition in the third row, and (iii) the bond market-clearing condition in the fourth row (note that \( \mu_t^B = 0 \) if \( \theta_t^* = \theta_H \)).

### 3.2 First-Order Conditions for Ramsey Allocation

The system of first-order conditions that characterize the Ramsey allocation are provided in the following proposition:

**Proposition 6.** The Ramsey allocation \( \{C_t, N_t, B_{t+1}, \theta_t^*, \pi_t, \phi_t, \mu_t^B, Q_{t+1}\}_{t=0}^{\infty} \) is characterized by the following equations:

\[
\begin{align*}
\bar{\theta} U_C(z^0) - \phi_0(z^0) U_{CC}(z^0) Z(\theta_0^*(z^0)) B_0 &= 1 + \phi_0(z^0), \quad (28) \\
\bar{\theta} U_{C,t}(z^t) &= 1 + \phi_t(z^t), \quad \text{for } t > 0, \quad (29) \\
\phi_{t+1}(z^{t+1}) &= \phi_t(z^t) + \frac{\partial W(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \frac{1}{F(\theta_t^*(z^t))}, \quad (30) \\
\tau_t(z^t) &= 1 - \frac{1}{U_C(z^t) Z(\theta_t^*(z^t))}, \quad (31)
\end{align*}
\]

\(^{7}\text{Note that the multiplier } \mu_t^B \text{ can take both positive and negative values.}\)
\[ Q_{t+1}(z_{t+1}|z^t) = \beta \frac{U_{C,t+1}(z_{t+1})Z(\theta^*_{t+1}(z_{t+1}))}{U_{C,t}(z^t)Z(\theta^*_t(z^t))} L(\theta^*_t(z^t)), \]  

(32)

plus the implementability condition (21), the asset market-clearing condition (24), the goods market-clearing condition (8), and the following equation for the multiplier \( \mu^B_t \):

\[
\mu^B_t(z^t) = \begin{cases} 
\phi_{t+1}(z^{t+1}) - \phi_t(z^t)L(\theta^*_t(z^t)) & \text{if } \theta^*_t < \theta_H \\
0 & \text{if } \theta^*_t = \theta_H 
\end{cases}
\]  

(33)

**Proof.** See Appendix A.4

To help understand the dynamic properties of the Ramsey allocation implied by the system of equations in the above proposition, we first consider two special cases: the model without aggregate uncertainty (case A), and the model without idiosyncratic uncertainty by setting \( \theta_L = \theta_H = \bar{\theta} \) (case B).

**Proposition 7.** Case A has a Ramsey steady state that exhibits full self-insurance characterized by \( \theta^*_t = \theta_H \) (even though the competitive equilibrium features partial self-insurance in the initial period \( t = 0 \)).

**Proof.** See Appendix A.5.

This proposition says that there exists an optimal level of debt such that if the initial debt level is below the optimal target, the supply of debt \( B_{t+1} \) will keep increasing over time until the full self-insurance allocation characterized by \( \theta^*_t = \theta_H \) is achieved. On the other hand, if the initial debt level is already high enough to support the full self-insurance allocation with \( \theta^*_0 = \theta_H \), then the Ramsey planner may not have incentives to deviate the future bond supply from this initial condition. As an example, suppose \( B_0 = 0 \) and \( G_t = 0 \) for all \( t \geq 0 \); then the model exhibits a transition period toward the long-run Ramsey steady state characterized by \( Q^* = \beta \), \( \tau^* = \frac{1-\beta}{\beta} (\theta_H - \bar{\theta}) > 0 \), \( B^* = \frac{1-\tau^*}{1-\beta} > B_0 \), \( C^* = N^* = 1 - \tau^* \), and \( \frac{B^*_C}{C^*} = \frac{1}{\beta} [\theta_H - \bar{\theta}] \). This example implies that even if government spending is zero, the long-run bond supply is positive; hence, the long-run tax rate is also positive in order to finance the interest payments. However, in the early phase of the transition period, the optimal tax rate is below its steady-state value and can be even negative but will monotonically increase over time—which implies that the rising level of government borrowing (the bond supply) is welfare improving despite a departure from perfect tax smoothing in the short run.
Proposition 8. In case B, the Ramsey allocation, \( \{C_t, N_t, B_{t+1}, \tau_t, \phi_t, Q_{t+1}\}_{t=0}^{\infty} \), is characterized by the following conditions:

\[
\bar{\theta} U_C(z^0) - \phi_0(z^0) U_{CC}(z^0) \bar{\theta} B_0 = 1 + \phi_0(z^0), \tag{34}
\]

\[
\bar{\theta} U_{C,t}(z^t) = 1 + \phi_0(z^0) \text{ for } t > 0, \tag{35}
\]

\[
\phi_{t+1}(z^{t+1}) = \phi_t(z^t) = \phi_0(z^0), \tag{36}
\]

\[
\tau_t(z^t) = 1 - \frac{1}{\bar{\theta} U_{C,t}(z^t)}, \tag{37}
\]

\[
N_t = G_t + C_t \tag{38}
\]

\[
Q_{t+1}(z_{t+1}|z^t) = \beta \text{ for } t > 0 \tag{39}
\]

\[
B_0 = \tau_0(z^0) N_0(z^0) - G_0(z^0) + \beta B_1(z_1|z^0) \tag{40}
\]

\[
B_t(z_t|z^{t-1}) = \frac{\bar{\theta} \phi_0}{(1 + \phi_0)^2} \frac{1}{1 - \beta} - \frac{PV_t^G(z^t)}{1 + \phi_0}, \text{ for } t > 0; \tag{41}
\]

where \( PV_t^G \) denotes the present value of government spending defined as

\[
PV_t^G(z^t) \equiv G_t(z_t) + E_t \sum_{\tau=t+1}^{\infty} \beta^{\tau-t} G_\tau(z^\tau); \tag{42}
\]

and the initial-period multiplier \( \phi_0 \) is pinned down by the initial bond supply \( B_0 \) using equation (41).

Proof. See Appendix A.6.

This proposition shows that without idiosyncratic uncertainty, our model reduces to the LS model with log-linear preferences. In particular, equation (36) indicates that the multiplier \( \phi_t \) of the government budget constraint is constant after \( t = 0 \) and is independent of the history of government spending shocks—except the initial shock \( G_0 \) because the initial bond level \( B_0 \) is exogenously given and cannot respond to \( G_0 \). The property of a constant multiplier \( \phi_t \) is the hallmark of the complete-markets LS model under aggregate shocks, regardless of the preference structure, since the Ramsey planner can utilize state-contingent bonds to hedge against government spending shocks, leaving the multiplier of the government implementability condition or government budget constraint unaffected by the shock history after the initial period. This property implies perfect tax smoothing.\(^8\) Given our log-liner utility assumption, the constant multiplier \( \phi \) implies that

\(^8\)In this paper, perfect tax smoothing is referred to as the Ramsey outcome featuring constant \( \phi \). Note that a labor tax is still distortionary under quasi-linear preferences since it alters the relative price between consumption and leisure. Hence, it is optimal to eliminate the distortionary effects of taxation. But linear utility in leisure
the optimal tax rate and the aggregate consumption are both constant for \( t > 0 \) since they are only functions of \( \phi_t \). Moreover, a constant \( \phi_t \) also implies that the price of state-contingent bonds is constant with \( Q_{t+1}(z_{t+1}|z^t) = \beta \). Perfect tax smoothing also implies that labor responds to government spending shocks one for one.

For example, we can show easily that the following is true in the LS model (under log-linear preferences):

(i) If \( B_0 = 0 \) and \( G_t = 0 \) for all \( t \geq 0 \), then \( \tau_t = 0 \) and \( B_t = 0 \) for all \( t \geq 0 \).

(ii) If \( B_0 = 0 \), \( G_t = 0 \) for \( t \neq T \) and \( G_T > 0 \), then \( \tau_t = \tau_0 = \frac{\beta^{T-1}}{1+\beta+\beta^2+...}G_T > 0 \); and for \( t < T \) we have \( B_t = \frac{\beta^t}{1+\phi_0}G_T < 0 \); for \( t = T \) we have \( B_T = \frac{\beta^{T-1}}{1+\phi_0}G_T < B_{T-1} \); and for \( t > T \) we have \( B_T = \frac{\beta^t}{1+\phi_0}G_T > 0 \). So, tax is positive and constant (completely smoothed) for all \( t \geq 0 \), while government debt is negative before \( T \), even more negative in period \( T \), and then becomes positive after \( T \), suggesting that the Ramsey planner opts to save tax revenues before the spending period and use total savings to finance spending \( G_T \) in period \( T \) and then issue debt (borrow) to finance the deficits—so that the tax rate remains the same for all \( t \). Notice that the constant tax rate

\[
\tau = \frac{\beta^T}{1+\beta+\beta^2+...}G_T
\]

is essentially the discounted average of future spending: the more remote the spending is, the lower the tax rate; specifically, the tax rate approaches zero if the spending takes place only in the infinite future. On the other hand, if the spending takes place in period \( t = 0 \) (similar to a MIT shock), then the tax rate is given by \( \tau_t = \tau_0 = \frac{1}{1+\beta+\beta^2+...}G_0 \) and the bond supply \( B_t = (1-\tau_0)G_0 > 0 \) for all \( t > 0 \). This implies that an unexpected spending shock always triggers a permanently higher tax rate and a permanently higher level of debt.

**Proposition 9.** Under both aggregate and idiosyncratic uncertainty, the long-run Ramsey equilibrium is characterized by the following properties: For any given \( B_0 \), there always exists a sufficiently large \( T > 0 \) such that for \( t \geq T \), (i) the Ramsey allocation exhibits full self-insurance with \( \theta_t^*(z^t) = \theta_H \) and (ii) \( \{\phi_t(z^t), C_t(z^t), \tau_t(z^t)\}_{t=T}^\infty = \{\bar{\phi}, \bar{C}, \bar{\tau}\} \) are all constant.

**Proof.** See Appendix A.7.

Notice that the initial-period cutoff \( \theta_0 \) cannot be pinned down without knowing the terminal condition in the future by solving the entire transitional path of the dynamic system. Also, the number of periods taken to reach the long-run equilibrium depends on the initial condition \( B_0 \) and the history of aggregate shocks \( \{G_t\}_{t=0}^\infty \).

Recall that the Ramsey allocation in the corresponding LS model (with log-linear preference) is characterized by a constant tax rate and constant consumption after the initial period 0, which are independent of the history of government spending shocks except the initial value of \( G_0 \). Therefore, together with perfect tax smoothing implies that it is optimal to adjust labor income to fully buffer government spending shocks.
the long-run Ramsey allocation in our model (with heterogeneous agents) looks identical to that in the LS model in the sense that the optimal paths of consumption and taxes are both constant—but constant in the LS model for \( t > 0 \) and in our model only for \( t \geq T \).

Thus, the key difference in policy implications between our benchmark model and the LS model lie in (i) the transitional dynamics, (ii) the long-run level of the tax rate, and (iii) the optimal level of the bond supply. In particular, equation (30) indicates that the multiplier \( \phi_t \) in our model increases over time in the transition period as long as \( \theta_t^* < \theta_H \) (because \( \frac{\partial W(\theta_t^*)}{\partial \theta_t^*} > 0 \)); whereas in the LS model \( \phi_t = \phi_0(z^0) \) for all \( t \geq 1 \). This implies that given the same process of government spending shocks \( \{G_t\}_{t=0}^\infty \) and same initial bond holdings \( B_0 \) in our model and in the LS model, we have the following:

1. If the initial bond supply \( B_0 \) is sufficiently high to achieve full self-insurance in our model, then the optimal path of the future bond supply \( \{B_t\}_{t=1}^\infty \) in our model is identical to that in the LS model.

2. If the initial bond supply \( B_0 \) is below the level required for full self-insurance, then the long-run level of the bond supply in our model is strictly higher than that in the LS model, and the long-run multiplier \( \phi_t \) is greater than that in the LS model. Namely, the optimal long-run tax rate in our model is higher than that in the LS model and the optimal aggregate consumption level in our model is lower than that in the LS model. The latter reflects the cost of distortionary taxes due to a permanently higher level of the bond supply, but with a strictly improved \( \theta_t^* \), suggesting that the welfare gains from a higher level of public debt in our model come mainly from the improved distribution across households’ insurance positions or equivalently, from perfectly smoothed household consumption across both space and time.

### 3.3 Intuition for Full Self-Insurance

An intriguing question is this: Giving that a permanently higher debt level requires a larger amount of distortionary taxes to finance it, why is a persistently rising debt level (in the transition period) always beneficial (or always outweighs the cost of extra tax distortion) as long as \( \theta_t^* < \theta_H \)?

The answer lies in the inequality between the interest rate and the time discount rate. To see the intuition behind the answer, recall that in the LS model the state-contingent bond price \( Q_{t+1}(z_{t+1}|z^t) = \beta \). Thus, based on the government budget constraint (2) and holding everything else equal, the (marginal) benefit of raising one additional unit of debt is \( Q_{t+1}(z_{t+1}|z^t)\phi_t(z^t) \) (bond price times the Lagrangian multiplier \( \phi_t \) of the budget constraint or the shadow value of debt); while the discounted (present-value) next-period marginal cost to pay for the matured debt is \( \beta \phi_{t+1}(z^{t+1}) \).
At the optimum we must have \( Q_{t+1}(z_{t+1}|z^t) \phi_t(z^t) = \beta \phi_{t+1}(z^{t+1}) \), or \( \phi_{t+1}(z^{t+1}) = \phi_t(z^t) = \phi_0 \), which implies a constant shadow value of debt and a constant tax rate \( \tau_t = \tau_0 \) for \( t > 0 \).

In contrast, in our model along the transitional path, the state-contingent bond price is given by \( Q_{t+1}(z_{t+1}|z^t) = \beta \frac{1-\tau_t(z^t)}{1-\tau_{t+1}(z^{t+1})} L(\theta_t^*(z^t)) \), which is approximately \( \beta L(\theta_t^*(z^t)) \) (for the sake of argument, holding the tax rate constant and ignoring the effect of changes in the cutoff for the moment); this value is greater than \( \beta \) because the liquidity premium is greater than one, \( L(\theta_t^*(z^t)) > 1 \). However, the Ramsey planner discounts the future by \( \beta \) instead of by \( Q_{t+1}(z_{t+1}|z^t) > \beta \). Hence, the marginal benefit of issuing one additional unit of state-contingent debt is \( \beta L(\theta_t^*(z^t)) \phi_t(z^t) \), while the discounted marginal cost to finance the debt payment the next period is \( \beta \phi_{t+1}(z^{t+1}) \).

Since \( \beta L(\theta^*_t(z^t)) > \beta \), at the optimum we must have \( \phi_{t+1}(z^{t+1}) > \phi_t(z^t) \) (with equality if and only if \( L(\theta_t^*(z^t)) = 1 \) or \( \theta_t^*(z^t) = \theta_H \)); hence, the optimal debt level will keep increasing in our model until the full self-insurance position is reached (at which point \( L(\theta_H^*) = 1 \) and \( \phi_{t+1}(z^{t+1}) = \phi_t(z^t) \)).

As the debt level rises, not only the bond price \( Q_{t+1}(z_{t+1}|z^t) \) but also the tax rate \( \tau_t(z^t) \) increase; so \( \frac{1-\tau_t(z^t)}{1-\tau_{t+1}(z^{t+1})} > 1 \) whenever \( \tau_{t+1}(z^{t+1}) > \tau_t(z^t) \); consequently, \( Q_{t+1}(z_{t+1}|z^t) = \beta \frac{1-\tau_t(z^t)}{1-\tau_{t+1}(z^{t+1})} L(\theta_t^*(z^t)) > \beta L(\theta_t^*(z^t)) \), which reinforces our arguments above. On the other hand, an increase in the bond supply improves welfare through a higher cutoff \( \theta_t^* \), which slows down the rise of the liquidity premium: \( \frac{\partial L(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} < 0 \). But as long as the liquidity premium \( L(\theta_t^*) > 1 \), this negative effect does not alter the inequality between the marginal benefit and the marginal cost of increasing the bond supply but simply indicates that the marginal benefit is diminishing. Therefore, the Ramsey planner always finds it optimal to keep increasing the bond supply until a full self-insurance allocation is achieved.

This general-equilibrium positive force to push up the bond supply \( B_{t+1}(z_{t+1}|z^t) \) and the multiplier \( \phi_{t+1}(z^{t+1}) \) can be seen more clearly in equation (30), which is derived by combining the Ramsey first-order conditions with respect to \( \theta_t^* \) and \( B_t^* \), where the second term \( \frac{\partial W(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \frac{1}{F(\theta_t^*(z^t))} \) on the right-hand side is precisely the net marginal benefits of increasing government debt:

\[
\frac{\partial W(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \frac{1}{F(\theta_t^*(z^t))} = \phi_{t+1}(z^{t+1}) - \phi_t(z^t) > 0,
\]

which is strictly positive and fully captures the welfare gains from the improved distribution of household wealth when the bond supply rises (recall that \( \frac{\partial W(\theta^*)}{\partial \theta^*} > 0 \) for \( \theta_t^* < \theta_H \) by lemma 5).

In contrast, the net marginal benefit \( (\phi_{t+1}(z^{t+1}) - \phi_t(z^t)) \) in the LS model is exclusively zero—because there the interest rate exactly equals the time discount rate (or the market price of state-contingent bonds \( Q_{t+1}(z_{t+1}|z^t) \) is exactly equal to \( \beta \)). Hence, in a representative-agent economy there is no net welfare gain for the Ramsey planner to exploit by deviating the future path of the bond supply \( \{B_{t+1}\}_{t=0}^\infty \) away from the initial level \( B_0 \), regardless of \( B_0 \) being positive or negative.
But in our model with heterogeneous agents and incomplete insurance markets, if the initial bond supply is too low to meet the households’ full self-insurance demand, the Ramsey planner will raise it until the liquidity premium \( L(\theta^*_t(z^*)) - 1 \) vanishes (or until \( Q_{t+1}(z_{t+1}|z^t) = \beta \)).\(^9\)

Therefore, the fundamental reason for the existence of a positive general-equilibrium force behind government-debt growth is that the interest rate lies below the time discount rate (as in most of the HAIM models, such as Aiyagari (1994)). Consequently, the marginal benefit of raising the debt level is always larger than the present-value next-period marginal cost.

Since the Ramsey planner opts to supply a larger and larger amount of bonds to meet the full self-insurance demand of households, everything else equal, the optimal tax rate in our model must also increase to cover the additional interest payments on bond financing—which is always worth the while because the interest rate is below the time discount rate as long as some households are still borrowing constrained. Consequently, the average (aggregate) consumption level must decline along the transition. Nonetheless, the welfare losses from a lower average consumption level are fully compensated by better household self-insurance positions because it is “cheap” to provide public liquidity and finance public debt when \( Q_{t+1} > \beta \).

This important property of HAIM models has gone largely unnoticed in the existing literature—maybe because the commonly used Aiyagari-type models are not analytical tractable and may not even have an interior Ramsey steady state (—see, e.g., the recent analyses of Chen, Chien, and Yang (2019) and Chien and Wen (2019)). This is why in this paper we opt to use a modified and well-structured HAIM model in which a well-defined Ramsey steady state can be proven to exist and a full self-insurance Ramsey allocation is feasible. Using this model as a benchmark, robustness and counterfactual analyses can be conducted by relaxing the assumptions in our model such that a Ramsey steady state or bounded stochastic Ramsey equilibrium no longer exist.

3.4 The Role of an Upper Debt Limit

We can show that the optimal level of public debt in our model depends positively on the parameter value of \( \theta_H \). If \( \theta_H \) goes to infinity (as in the case of a Pareto distribution for \( \theta_t \)), a stationary Ramsey allocation with a finite level of public debt may no longer exist. So, given that the optimal long-run level of the bond supply is higher in our model than in the corresponding LS model, it would be interesting to consider the situation when there exists an upper debt limit on the level of government bonds and this debt limit binds (stochastically) in the long run:

\[
B_{t+1} (z_{t+1}|z^t) \leq \overline{B}.
\]

\(^9\)As we show later, the mechanism and intuition for increasing \( B_t \) remain valid in the environment without state-contingent debt.
Denoting the corresponding Lagrangian multiplier for this debt-limit constraint as \( v_t^H (z_{t+1} | z^t) \), we have the following proposition:

**Proposition 10.** With a sufficiently low government debt limit such that the full self-insurance Ramsey allocation is not feasible, the multiplier \( \phi_t \) follows a stochastic stationary process with the following law of motion:

\[
\phi_{t+1}(z_{t+1}^t) = \phi_t(z^t) + e_{t+1}(z_{t+1} | z^t),
\]

where the error term is defined as

\[
e_{t+1}(z_{t+1} | z^t) \equiv \frac{\partial W(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \frac{1}{F(\theta_t^*(z^t))} - \frac{v_t^H(z_{t+1} | z^t)}{U_C(z_{t+1} | z^t) Z(\theta_{t+1}^*(z_{t+1} | z^t))} \frac{\partial W(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \frac{U_C(z_{t+1} | z^t)}{U_C(z_t | z^{t-1}) Z(\theta_t^*(z_t))}.
\]

Consequently, optimal taxes also follow a stationary stochastic process.

**Proof.** See Appendix A.8.

Proposition (10) implies that the long-run Ramsey equilibrium must exhibit stochastic allocation instead of constant allocation for optimal taxes. The reason is as follows: Given that the debt limit \( B \) is too low for the Ramsey planner to achieve a full self-insurance allocation, it is no longer possible to keep the cutoff \( \theta_t^* \) constant. To see this point, denote its highest possible value as \( \bar{\theta}^* \), where the borrowing constraint binds. Suppose the cutoff stays at \( \bar{\theta}^* \), then \( \frac{\partial W(\bar{\theta}^*)}{\partial \bar{\theta}^*} > 0 \) and remains constant, while the multiplier \( v_t^H \) must respond to government spending shocks and must be stochastic. Consequently, the error term becomes negative whenever the desire to increase \( B_{t+1} \) is sufficiently strong. But a negative value of \( e_t \) implies that \( \phi_{t+1} \) will decline, thus pushing the cutoff \( \theta_t^* \) to fall below the highest possible value of \( \bar{\theta}^* \), therefore making it impossible to stay at \( \bar{\theta}^* \). More rigorous proof of the proposition can be found in Appendix A.8.

### 3.5 Numerical Examples

In this subsection we solve our model numerically to confirm our theoretical analyses above. The model can be solved using a global method instead of a local approximation method.

Consider the following parameter values: The government spending process is a two-state first-order Markov process:

\[
G = \begin{cases} 
G_L = 0.1 \\
G_H = 0.3 
\end{cases}
\]

\(^{10}\)Note that the value of the multiplier for the state-contingent debt is also state contingent.
with transition probability
\[
\pi_G = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}.
\]

This process is used in all numerical simulations in this paper, including those in the next section. In addition, the time discount rate is \( \beta = 0.95 \); the distribution of preference shock \( \theta \) follows a power function \( F(\theta) = \frac{\theta^\gamma - \theta_L^\gamma}{\theta_H^\gamma - \theta_L^\gamma} \), where \( \theta_L = 1 \), \( \theta_H = 10 \) and \( \gamma = 0.1 \); and the initial \( B_0 \) is chosen to be lower than the value implied by the full self-insurance equilibrium (say \( B_0 = 0 \)). The results are qualitatively similar for other choices of parameter values.

These parameter values imply the following: (i) In the steady state (without aggregate risks), the condition for positive individual labor choice, \( \theta_H < \frac{\theta_L}{(1-\beta)} < \infty \), is satisfied. (ii) In the transition, the condition for a positive labor supply \( n_t > 0 \) for all \( t \geq 0 \) is satisfied and verified numerically in the simulation.

We show the transitional dynamics of our model under two scenarios: Scenario A assumes that the government debt limit is high enough such that the full self-insurance equilibrium is feasible; scenario B sets the government debt limit low enough to make the full self-insurance equilibrium infeasible. Under each scenario, we also compute the case without aggregate uncertainty (\( G_t = \overline{G} \)) as a reference point.

Figure 1 plots the Ramsey transition path in scenario A. The blue lines in each panel represent the reference model without aggregate uncertainty; the red lines represent the model with aggregate uncertainty. The counterparts of the corresponding LS model are not shown in the figure, because they remain constant for \( t \geq 1 \) (except for labor and bond supply) and feature no interesting transitional dynamics. In the figure, when two paths are identical, we can see only the red lines, because they completely override the blue lines. For example, the paths of consumption \( C_t \) (top-left panel), the cutoff \( \theta_t^* \) (top-right panel), the multiplier \( \phi_t \) (middle-right panel), and the tax rate \( \tau_t \) (bottom-left panel) in Figure 1 are all monotonically approaching their respective steady state and are all identical to their counterparts in the reference model without aggregate uncertainty, suggesting complete consumption smoothing and tax smoothing (in the sense of being immune to aggregate uncertainty).

Recall that in the corresponding LS model there are no transitional dynamics after the initial period \( t = 0 \) and all variables except labor and the bond supply remain constant after \( t \geq 1 \). In contrast, in our heterogeneous-agent model, the paths of all variables except labor and the bond supply exhibit a smooth and gradual transition toward their respective long-run steady-state values. Although labor and the bond supply are stochastic, their average values also exhibit monotonic transitions and these average values are identical to their respective counterparts (the blue lines) in the reference economy (in both the short run and the long run).
In particular, during the transition, the average level of the bond supply $B_{t+1}$, the multiplier $\phi_t$, the tax rate $\tau_t$, and the cutoff $\theta^*_t$ all increase over time—indicating a departure from perfect tax smoothing; whereas aggregate consumption $C_t$ and the average level of hours worked $N_t$ decrease over time. Consequently, the steady-state values of the average bond supply $\overline{B}$ and tax rate $\tau$ are much higher than their respective initial values $\{B_0, \tau_0\}$; and steady-state consumption $\overline{C}$ and average labor $\overline{N}$ are much lower than their initial values $\{C_0, N_0\}$. However, despite $\overline{C} < C_0$, the social welfare in the long run is much higher than in the initial period because (i) the distribution is much improved when the cutoff $\theta^*_t$ approaches $\theta_H$, and (ii) the average leisure level is much higher in the long run than initially. That is, in the long run, the full self-insurance allocation enables households to work less (on average) while saving more and receiving a higher rate of return.

Figure 1: Ramsey Outcome of Benchmark Model Scenario A

An even larger departure from tax smoothing arises in scenario B—namely, a departure from
tax smoothing both in the short run and in the long run, where the smooth and constant long-run paths of consumption and taxes seen in scenario A disappear. The new Ramsey paths under scenario B are plotted in Figure 2, where the blue lines represent the reference paths under scenario A without a debt limit and aggregate uncertainty, and the red paths represent the Ramsey paths in scenario B with a debt limit and an aggregate uncertainty. In contrast to scenario A and the reference paths therein, the panels in Figure 2 show that when there exists a government borrowing limit such that a full self-insurance Ramsey allocation is not feasible, the paths of taxes $\tau_t$, consumption $C_t$, the cutoff $\theta^*_t$, and the multiplier $\phi_t$ all become stochastic in the long run (only after the transition period ends). In particular, in the long run after the transition, consumption fluctuates above its reference line, while taxes, the cutoff, the multiplier, and the bond supply all fluctuate below their respective reference lines. Interestingly, consumption, the multiplier, and taxes occasionally overshoot their respective reference paths, while the cutoff never reaches its reference path—because the debt limit is too low to make a full self-insurance allocation possible, so $\theta^*_t < \theta_H$ for all $t$. However, during the transition the paths of $\{C_t, \theta^*_t, \phi_t, \tau_t\}$ are as smooth as their respective counterparts because the debt-limit constraint does not bind in transition.

Intuitively, when the Ramsey planner is unable to supply enough bonds to achieve the full self-insurance allocation, the planner opts for an average level of the bond supply that is smaller than the constant debt limit $\bar{B}$ and has room to fluctuate upward whenever the government spending shocks are high. This precautionary bond-supply behavior implies that (i) the debt-limit constraint binds only occasionally and (ii) the optimal tax rate must also fluctuate to partially absorb the impact of government spending shocks. The lack of a fully and freely adjustable bond position and a stochastic tax process imply that consumption is also stochastic in the long run because perfect consumption smoothing is no longer possible at either the individual level or the aggregate level despite state-contingent bonds. In such a case, the partial derivative $\frac{\partial W(\theta_t)}{\partial \theta_t} > 0$ is strictly positive but the error term $e_{t+1}$ can take either positive or negative values in equation (44), so that the multiplier $\phi_{t+1}$ behaves like a bounded random-walk process. However, the welfare loss due to stochastic aggregate consumption and a stochastic distribution is partially compensated by the higher average consumption level shown in the top-left panel of Figure 2.

Under the ad hoc debt limit in scenario B, optimal taxes are not only stochastic but also serially correlated, even though government spending shocks are iid. Table 1 reports the statistics based on simulations under scenario A (second column) and scenario B (third column). In scenario B with a binding debt limit, taxes are no longer smooth, but the standard deviation is only 0.124%, compared with a standard deviation of 10% for $G_t$. The serial correlation of taxes $(\tau_{N,t})$ is 0.626, much greater than the assumed zero autocorrelation of $G_t$. Meanwhile, under both scenario A and scenario B, the state-contingent debt $B_{t+1}$ is negatively correlated with $G_{t+1}$ and positively
Figure 2: Ramsey Outcome of Benchmark Model Scenario B

Table 1: Result Statistics of State-Contingent Debt Case

<table>
<thead>
<tr>
<th>Cases</th>
<th>Scenario A</th>
<th>Scenario B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full Self-Insurance</td>
<td>Achieved</td>
<td>Infeasible</td>
</tr>
<tr>
<td><strong>Statistics</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$std(G)(%)$</td>
<td>10.000</td>
<td>10.000</td>
</tr>
<tr>
<td>$std(\tau_N)(%)$</td>
<td>0</td>
<td>0.124</td>
</tr>
<tr>
<td>$\rho(\tau_{N,t}, \tau_{N,t-1})$</td>
<td>NA</td>
<td>0.626</td>
</tr>
<tr>
<td>$\rho(B_{t+1}, G_{t+1})$</td>
<td>-1.000</td>
<td>-0.996</td>
</tr>
<tr>
<td>$\rho(B_{t+1}, G_{t})$</td>
<td>0.009</td>
<td>0.050</td>
</tr>
</tbody>
</table>

Notes: These statistics are generated by a simulation with 10,000 periods.
correlated with $G_t$ (but the cross-correlation is slightly weaker in scenario B than in scenario A), which also implies that the growth rate of debt $(B_{t+1} - B_t)$ positively correlates with government spending shocks $G_t$ under both scenarios. The significantly large serial correlation in taxes partially captures the random-walk property of taxes emphasized by Barro (1979).

Notice that in our model the optimal level of government debt depends positively on the variance of the idiosyncratic shocks $\theta$. That is, as the upper support $\theta_H$ increases or goes to infinity (such as in a Pareto distribution), the optimal supply of bonds also increases accordingly or approaches infinity, making the government natural debt limit more and more likely to bind. Hence, scenario B becomes more realistic than scenario A. In addition, given any value of $\theta_H$, the larger the variance of government spending shocks (such as in the event of wars), the more likely the government natural debt limit is to bind. Hence, the observed high serial correlation of taxes in the real world could be a natural outcome of our model.

However, as the bottom-right panel in Figure 2 shows, the optimal bond supply has little serial correlation when government spending shocks are iid. To generate a random-walk component in both taxes and the bond supply under iid government spending shocks requires modifying our benchmark model in the following manner, as suggested by AMSS.

4 Model Without State-Contingent Debt

We now turn to the case where government bonds are risk-free (or non-state contingent). In this case, several corresponding changes need to be made in the household problem, the definition of competitive equilibrium, and hence the construction of the Ramsey problem (the details are provided in Appendix 11, i.e., contained in the proof of Proposition 11).

With risk-free bonds, the period-$t$ bond price depends only on $z^t$, so the bond price in equation (19) must be modified to become

$$\bar{Q}_{t+1}(z^t) = \beta E_t \frac{1 - \tau_t}{1 - \tau_{t+1}} L(\theta^*_t) = \beta E_t \frac{U_{C_{t+1}} Z(\theta^*_{t+1})}{U_{C_{t+1}} Z(\theta^*_t)} L(\theta^*_t) = E_t Q_{t+1}(z_{t+1}|z^t);$$

namely, the non-state-contingent bond price equals the expected value of the state-contingent bond price. To conserve space, we suppress the $z^t$ notation in this section for the case without state-contingent bonds.

**Proposition 11.** In the absence of any debt limit, the following system of equations fully charac-
terize the dynamic properties of the Ramsey allocation:

\[
\begin{align*}
\bar{\theta}U_{C,0} - \phi_0 U_{CC}Z(\theta_0^*)B_0 &= 1 + \phi_0, \\
\bar{\theta}U_{C,t} + (\phi_{t-1}L(\theta_{t-1}^*) - \phi_t + \mu_{t-1}^B) U_{CC,t}Z(\theta_t^*)B_t &= 1 + \phi_t, \text{ for } t > 0,
\end{align*}
\]

(47)

\[
\mu_t^B = \begin{cases} 
E_t q_{t+1} \phi_{t+1} - \phi_t L(\theta_t^*) & \text{if } \theta_t^* < \theta_H \\
0 & \text{if } \theta_t^* = \theta_H
\end{cases},
\]

(48)

\[
E_t q_{t+1} \phi_{t+1} = \phi_t + \frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{1}{F(\theta_t^*)} + (E_{t-1} q_t \phi_t - \phi_t) \frac{U_{C,t}B_t}{F(\theta_t^*)} \frac{\partial Z(\theta_t^*)}{\partial \theta_t^*}, \text{ for } t > 0,
\]

(49)

\[
\tau_t = 1 - \frac{1}{U_{C,t}Z(\theta_t^*)},
\]

(50)

where the pricing kernel \( q_{t+1} \equiv \frac{U_{C,t+1}Z(\theta_{t+1}^*)}{q_t U_{C,t}Z(\theta_t^*)} \) satisfies \( E_t q_{t+1} = 1 \), plus the risk-free bond price equation (46) and the constraints listed in Proposition 18, which include the resource constraint (8), the implementability condition (80), and the asset market-clearing condition (81).

Proof. See Appendix A.9. \qed

Making government debt non-state contingent means that the Ramsey planner is unable to perfectly hedge against the aggregate \( G \) shock. So the most important changes after introducing non-state-contingent bonds are that (i) the multiplier \( \mu_t^B \) for the bond market-clearing condition changes from \( \mu_t^B = \phi_{t+1} - \phi_t L(\theta_t^*) \) (i.e., equation (33) in Proposition 6) to \( \mu_t^B = E_t q_{t+1} \phi_{t+1} - \phi_t L(\theta_t^*) \) and (ii) consequently, the law of motion for the multiplier \( \phi_{t+1} \) in equation (30) now changes to equation (50).

To help understand the effects of these changes, let us revisit the LS model and our benchmark model with state-contingent debt. In our benchmark model the value of the multiplier \( \mu_t^B \) is given by \( \mu_t^B = \phi_{t+1} - \phi_t L(\theta_t^*) \), while in the LS model it is given by \( \mu_t^B = \phi_{t+1} - \phi_t \). In the benchmark model, \( \phi_t L(\theta_t^*) \) reflects the marginal benefit of having one additional unit of debt and \( \phi_{t+1} \) reflects the future marginal cost. Since we have argued above in the previous section that the marginal benefit exceeds the marginal cost when \( \theta_t^* < \theta_H \), it must be true that \( \phi_{t+1} > \phi_t \) in our benchmark model during the transition period and \( \mu_t^B = 0 \) in the LS model for \( t > 0 \).

Now, with non-state-contingent bonds, although the marginal benefit of increasing the debt supply is still \( \phi_t L(\theta_t^*) \), the expected next-period marginal cost becomes \( E_t q_{t+1} \phi_{t+1} \) instead of \( \phi_{t+1} \),
where by equation (46) we can express $q_{t+1}$ as

$$q_{t+1} \equiv \frac{U_{C_t+1} Z(\theta_{t+1}^*)}{E_t U_{C_t+1} Z(\theta_{t+1}^*)} = \frac{Q_{t+1}}{E_t Q_{t+1}} = 1 + \frac{Q_{t+1} - E_t Q_{t+1}}{E_t Q_{t+1}},$$ \quad (52)$$

where the “forecast error” $\varepsilon^q_{t+1}$ originates from predicting the future (state-contingent) bond price based on period-$t$ information on government spending shocks, or the difference between the state-contingent bond price and non-state-contingent bond price. Under rational expectations, this forecasting error must be zero on average so that $E_t q_{t+1} = 1 + E_t \varepsilon^q_{t+1} = 1$.

The forecasting error $\varepsilon^q_{t+1}$ is a necessary wedge for adjusting the expected future marginal cost $\phi_{t+1}$ because the next-period interest rate may change due to new information about government spending—as if the Ramsey planner is issuing bonds in period $t$ based on the expected bond price in $t+1$. This forward looking (anticipation) behavior is rational because the interest rate payment (the inverse of the current bond price) is not due until the next period. Hence, the forecasting error for predicting the change in the bond price must be incorporated into the expected next-period marginal cost. Since both $q_{t+1}$ and the multiplier $\phi_{t+1}$ are affected by the next-period government spending shock $G_{t+1}$, they are correlated in general, suggesting that $E_t q_{t+1} \phi_{t+1} \neq E_t q_{t+1} E_t \phi_{t+1} = E_t \phi_{t+1}$.

If we define another forecasting error $\varepsilon^\phi_t \equiv E_{t-1} \phi_t - \phi_t$ with $E_{t-1} \varepsilon^\phi_t = 0$, then the coefficient of the last term on the right-hand side of equation (50) can be expressed as

$$E_{t-1} q_t \phi_t - \phi_t = \varepsilon^\phi_t + E_{t-1} \varepsilon^q_t \phi_t;$$ \quad (53)$$

so equation (50) can be rewritten as

$$E_t \left[(1 + \varepsilon^q_{t+1}) \phi_{t+1}\right] = \phi_t + \frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{1}{F(\theta_t^*)} + \varphi_t,$$ \quad (54)$$

where the residual $\varphi_t$ is a term involving zero-mean forecasting errors as its coefficient:

$$\varphi_t \equiv \left(\varepsilon^\phi_t + E_{t-1} \varepsilon^q_t \phi_t\right) \frac{U_{C_t} B_t \partial Z(\theta_t^*)}{F(\theta_t^*)} \frac{1}{\partial \theta_t^*}.$$

Notice that (whenever $\theta_t^* < \theta_H$) $\varphi_t$ vanishes only under two conditions: (i) $\phi_t = 0$ for all $t$ or (ii) the variance of government spending shocks $\sigma^2$ goes to zero so that the forecast errors $\varepsilon^\phi_t = \varepsilon^q_t = 0$ for all $t$. However, condition (i) can never be true under incomplete insurance markets, because $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{1}{F(\theta_t^*)} > 0$; in other words, as long as the liquidity premium of public debt is positive, an
allocation with \( \lim_{t \to \infty} \phi_t = 0, \lim_{t \to \infty} \tau_t = 0 \), and \( \lim_{t \to \infty} B_{t+1} < 0 \) cannot be supported as a Ramsey equilibrium, in sharp contrast to the results obtained by AMSS in a representative-agent model.

Therefore, (whenever \( \theta^* \) is sufficiently below \( \theta_H \)) the residual \( \phi_t \) vanishes if and only if \( \sigma^2 \to 0 \). On the other hand, the magnitude of \( \frac{\partial W(\theta^*)}{\partial \theta^*} \frac{1}{F(\theta^*)} \) does not depend on \( \sigma^2 \) but depends only on the distance between the cutoff \( \theta^* \) and \( \theta_H \); hence, this partial derivative term does not vanish when \( \sigma^2 \to 0 \). This implies that the positive force \( \frac{\partial W(\theta^*)}{\partial \theta^*} \frac{1}{F(\theta^*)} \) for bond growth stochastically dominates the residual term \( \phi_t \) (at least for relatively small variance of government spending shocks) whenever the cutoff \( \theta^* \) is sufficiently away from \( \theta_H \), suggesting that the insight obtained in the benchmark model continues to apply here under non-state contingent bonds. But the force of convergence toward full self-insurance is weakened by switching from state-contingent bonds to risk-free bonds—because \( \lim_{\theta \to 0} \frac{\partial Z(\theta)}{\partial \theta} \frac{1}{F(\theta)} = \lim_{\theta \to 0} \frac{\partial W(\theta)}{\partial \theta} \frac{1}{F(\theta)} = 0 \), as \( \theta^* \to \theta_H \) the force of convergence disappears and equation (54) becomes a pure STUR process without drift: \( E_t \left[ (1 + \varepsilon_{t+1}^q) \phi_{t+1} \right] = \phi_t \). This pure STUR process is identical to that in the AMSS model and will be discussed in more detail below.

In other words, the random-walk property emerges (only locally) in our model around the corner \( \theta_H \) because the mean-reverting force \( \frac{\partial W(\theta^*)}{\partial \theta^*} \frac{1}{F(\theta^*)} \) toward full self-insurance disappears as \( \theta^* \to \theta_H \), in which case the law of motion in equation (50) becomes the STUR process: \( E_t q_{t+1} \phi_{t+1} = \phi_t \), which does not converge and will drift either upward until the government natural borrowing limit binds or downward until the cutoff \( \theta^* \) is sufficiently below \( \theta_H \). This happens because the Ramsey planner becomes indifferent between issuing more debt, less debt, or holding privately-issued IOUs, given that the expected market interest rate is exactly equal to the time discount rate (whenever \( \theta^* = \theta_H \)).

Therefore, compared to the benchmark model, aggregate uncertainty under non-state contingent bonds introduces two related changes or effects to the law of motion of the multiplier \( \phi_{t+1} \): It introduces (i) a stochastic unit root \( q_{t+1} = 1 + \varepsilon_{t+1}^q \) into the law of motion of the multiplier \( \phi_{t+1} \)\(^{11}\) and (ii) an additional (additive) error term \( \varepsilon_t \) to the law of motion of \( \phi_{t+1} \).

The first effect adds long memories (or highly persistent effects) of government spending shocks into the Ramsey planner’s decisions—not only by introducing a unit root into the dynamic system but also by making the unit root stochastic. The second effect adds an error term \( \varepsilon_t \), and since it depends on the sizes of the government spending shocks, it weakens the positive force \( \frac{\partial W(\theta^*)}{\partial \theta^*} \frac{1}{F(\theta^*)} \) behind the positive growth of the bond supply in the benchmark model. That is, \( \varepsilon_t \) acts as a

\(^{11}\)The time-series properties of STUR process are analyzed by Granger and Swanson (1997), where they show that a STUR process has no finite first moment and second moment, similar to a unit-root process. We will argue below that the model exhibits bounded random-walk dynamics in the sense that a Ramsey equilibrium has a finite variance in the long run despite the STUR component.
stochastic drift in the STUR process and the coefficient term \((\varepsilon_t^\phi + E_{t-1}\varepsilon_t^q\phi_t)\) in the drift can take either positive or negative signs.

These two effects imply that the full self-insurance allocation \(\theta_t^* = \theta_H\) can no longer be maintained with probability 1 in a long-run Ramsey equilibrium. Instead, the model economy may deviate from the full self-insurance allocation stochastically, although the tendency to approach (or revert to) the full self-insurance allocation from below always exists, which makes the law of motion for \(\phi_{t+1}\) an error-correction process with a stochastic unit root. Consequently, the Ramsey equilibrium not only exhibiting random walk dynamics but the random-walk process is endogenously bounded—below by the “mean-reverting” force \(\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{1}{F(\theta_t^*)}\) toward full self-insurance and above by the government’s natural borrowing limit.

The requirement for the government’s natural debt limit to support a bounded Ramsey equilibrium is discussed by AMSS. But in contrast to AMSS, an ad hoc lower debt limit is no longer needed here because the positive force for bond growth, \(\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{1}{F(\theta_t^*)} > 0\), is always operative as long as \(\theta_t^* < \theta_H\).

To make the above arguments more formal and to build up intuition, we introduce an ad hoc borrowing limit

\[ B_{t+1} \leq \overline{B} \tag{55} \]

and consider several special cases where this debt-limit constraint does not bind (such as in the absence of aggregate uncertainty). Under this debt-limit constraint, the law of motion for \(\phi_{t+1}\) in equation (54) becomes (see the proof of Proposition 11 in the Appendix):

\[ E_t \left[ \left(1 + \varepsilon_{t+1}^q\right) \phi_{t+1} \right] = \phi_t + \frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{1}{F(\theta_t^*)} + \varphi_t + \tilde{e}_t, \tag{56} \]

where the new error term \(\tilde{e}_t\) is a function of the Lagrangian multiplier \(v_t^H\) for the debt-limit constraint (55) and is given by

\[ \tilde{e}_t \equiv -\frac{v_t^H}{\beta E_t U_{C,t+1} Z(\theta_{t+1}^*)} + \frac{v_{t-1}^H}{\beta E_{t-1} U_{C,t} Z(\theta_t^*)} B_t U_{C,t} \frac{\partial Z(\theta_t^*)}{\partial \theta_t^*}. \]

When there is no aggregate uncertainty, then whether the government debt is state-contingent or not makes no difference. More specifically, it is straightforward to verify that the optimal conditions listed in Proposition 6 and Proposition 11 become identical if there is no aggregate uncertainty. Proposition 7 implies that the long-run Ramsey allocation must exhibit full self-insurance characterized by \(\theta_t^* = \theta_H\), as stated formally in each of the following lemmas.

**Lemma 12.** If \(G_t\) is constant with \(G_t = \overline{G} > 0\) and the debt-limit constraint does not bind, then
the long-run Ramsey allocation exhibits full self-insurance characterized by \( \theta_t^* = \theta_H \), as well as a positive constant tax rate, denoted by \( \bar{\tau} > 0 \) and a positive constant bond supply, denoted by \( \bar{B} > 0 \).

**Proof.** See Appendix A.10.

**Lemma 13.** With a time-varying deterministic \( G_t \) process with average value \( \bar{G} \) (e.g., featuring two alternating values between \( G_L \) and \( G_H \) with \( G_H > G_L > 0 \)) and a debt-limit constraint that does not bind, the long-run Ramsey allocation exhibits full self-insurance characterized by \( \theta_t^* = \theta_H \), as well as a positive constant tax rate \( \bar{\tau} > \hat{\tau} \), and a time-varying bond supply \( B_{t+1} \) with average bond value, denoted by \( \tilde{B} \), larger than \( \bar{B} \).

**Proof.** See Appendix A.10.

In this second special case, since \( G_t \) is fully predictable, it is as if government bonds are made state-contingent, so the expectation operator drops off from all the Ramsey first-order conditions (FOCs). Consequently, the case is again identical to the no-aggregate-shock case in the benchmark model. The reason that the optimal bond supply is higher under a time-varying deterministic process of \( G_t \) than under a constant \( G_t \) (with the same average value \( \bar{G} \)) is this: To ensure full self-insurance, the average level of the bond supply must be high enough to avoid the worst possible case when \( B_{t+1} \) is too low to provide full self-insurance. Namely, besides providing a sufficiently high level of bonds \( \bar{B} \) to ensure full self-insurance against a constant \( \bar{G} \), additional bonds \( \Delta B \) (\( \equiv \tilde{B} - \bar{B} > 0 \)) must be provided to buffer against the deterministic fluctuations in \( G_t \) around \( \bar{G} \).

**Lemma 14.** If \( G_t \) is one-time shock in period \( j > 0 \) (e.g., \( G_t = \rho G_{t-1} + (1 - \rho) \bar{G} + \varepsilon_t \) with \( \varepsilon_j = 1 \) and \( \varepsilon_{t \neq j} = 0 \)) and the debt-limit constraint does not bind, then the long-run Ramsey allocation exhibits full self-insurance characterized by \( \theta_t^* = \theta_H \) as well as a constant tax rate \( \bar{\tau} \) (for \( t \) large enough).

**Proof.** See Appendix A.10.

In this third special case, since \( G_t \) is predictable after the impact period, in the long run this case is identical to the case with a stationary deterministic government spending process.

Notice that the Lagrangian multiplier \( \phi_t \) for the government budget constraint and the public debt \( B_t \) are positively correlated in the sense that a higher level of existing debt tightens the government budget constraint. Therefore, an increasing/decreasing \( \phi_t \) implies an increasing/decreasing debt level \( B_t \) and vice versa. With this property and the above three special cases in mind, we are ready to state the following main proposition on the optimal Ramsey allocation when \( G_t \) is random.

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Define a stochastic bounded Ramsey equilibrium (SBRE) as a Ramsey allocation where all variables are strictly positive and with finite variance. The following proposition characterizes SBRE under incomplete markets for both aggregate and idiosyncratic risks:

**Proposition 15.** When $G_t$ is a stationary random process with a finite variance $\sigma^2$ (e.g., $G_t = \bar{G} + \varepsilon_t$ with mean $\bar{G}$ and positive support), if a Ramsey equilibrium exists, it must feature a (local) STUR. Furthermore, a SBRE is ensured if there exists an upper debt limit (such as the government’s natural borrowing limit) on the bond supply.

**Proof.** See Appendix A.11.

### 4.1 Numerical Examples

This subsection uses numerical methods to confirm our theoretical results in the case without state-contingent bonds. The government shock process and the parameter values are identical to those in Subsection 3.5 except the ad hoc debt limit (55) is imposed.

There are two scenarios to be considered here: In scenario A the upper debt limit $\bar{B} > B^*$ is sufficiently high such that full self-insurance is possible, where $B^*$ is the optimal debt level without aggregate uncertainty ($G_t = \bar{G}$); in scenario B the upper debt limit $\bar{B} < B^*$ is sufficiently low such that full self insurance is impossible. We will show that the dynamics of taxes and bond supply are quite different between the two scenarios.

The simulation results for scenario A are plotted in Figure 3 and demonstrate the optimal dynamic paths of these variables for 3000 periods. Clearly, except labor $N_t$ (middle-left panel) and the cutoff $\theta^*_t$ (top-right panel), all variables such as consumption (top-left panel), the tax rate (bottom-left panel), and the bond supply (bottom-right panel) behave like a bounded random walk or a "mean-reverting" process with high serial correlations, despite the government spending shocks being iid. The iid property of the government spending shocks is only visible the most for labor (middle-left panel).

The behavior of the cutoff $\theta^*_t$ (top-right panel) is most interesting: It has a strong tendency to stay at the full self-insurance level ($\theta_H$) with high probability but occasionally falls below the optimal target. So the full self-insurance point $\theta_H$ is clearly the center of "gravity" that the cutoff tries to revert back to over time. To ensure the highest probability of staying at the full self-insurance allocation, the bond supply (bottom-right panel) adjusts very violently but at the same time is highly serially correlated—as if following a random-walk process. However, this random-walk process is clearly bounded: The upper red line in the bottom-right panel of Figure 3 is the ad hoc debt limit $\bar{B} > B^*$—which serves as a proxy for the government’s natural borrowing limit; the lower yellow line is a reference path for the optimal bond supply without aggregate uncertainty.
(which supports full self-insurance). Clearly, under aggregate uncertainty the Ramsey planner opts to issue (on average) more bonds than required for full self-insurance without aggregate uncertainty. So the fact that the average level of the bond supply lies above the reference line reveals the Ramsey planner’s dominant incentive to mitigate the financial frictions and household borrowing constraints—even at the cost of extra tax distortion.

The simulation results for scenario B are plotted in Figure 4 and show the behavior of the Ramsey allocation when the upper debt limit $\overline{B} < B^*$ becomes so tight such that full self-insurance allocation is infeasible. The main differences between scenario B and scenario A are as follows: (i) The Ramsey planner is less able to smooth consumption and taxes in scenario B than in scenario A; consequently, the variances of $\theta^*_t$, $C_t$, and $\tau_t$ are larger and the variance of $B_{t+1}$ is smaller than their counterparts in scenario A, reflecting the worsened government capacity to adjust bond supply to improve welfare by smoothing consumption and taxes. (ii) Nonetheless, the serial correlation in taxes is weaker in scenario B than in scenario A, reflecting a more restrictive random-walk component in $\tau_t$. (iii) Most interestingly, the dynamic path of the bond supply (bottom-right panel) continues to exhibit the upward “mean-reverting” or “error-correction” behavior of public debt: It rises stochastically toward the borrowing limit and tends to stay there stochastically with stronger serial correlation but fewer high-frequency jumps than in scenario A, so as to uphold the cutoff $\theta^*_t$ (top-right panel) at a level as close to $\theta_H (= 10)$ as possible, but never reaching it nor remaining at a consistent distance. Consequently, the distribution of household allocations $(\theta^*_t)$ also behaves like a bounded random walk, with much stronger serial correlation and a longer memory than in scenario A.

The statistics associated with these two scenarios are reported in Table 2, where the middle column pertains to scenario A and the right column pertains to scenario B. The table shows that the variance of taxes is about 10 times smaller than that of government spending shocks in both scenarios, indicating the Ramsey planner’s ability to smooth taxes; but the variance of taxes increases slightly from 0.932 to 1.015 when the debt limit is lowered to rule out full self-insurance. The autocorrelation of taxes is much weaker in scenario B (about 0.62) than in scenario A (about 0.9). The autocorrelation of the bond supply is closer to a unit root in scenario A (0.98) than in scenario B (0.867). The growth rate of the bond supply ($B_{t+1}/B_t$) is nearly insensitive to future shock $G_{t+1}$ but highly sensitive (nearly a one-for-one response) to current shock $G_t$ in both scenarios (but it is slightly weaker in scenario B).

These results suggest that the Ramsey allocation with risk-free bonds is more likely to exhibit a random-walk behavior if the upper debt limit $\overline{B}$ is more relaxed. Nonetheless, the random-walk process is endogenously bounded—below by the growth of public debt to support full self-insurance and above by the government’s natural debt limit (even though for simplicity and tractability we
do not model the natural debt limit in this paper).

**Robustness Analysis.** So far in our analyses we have assumed that households cannot borrow under the constraint $a_{t+1} \geq 0$. Presumably, if households can borrow up to a positive limit $\bar{a}$, $a_{t+1} \geq -\bar{a}$, the optimal level of public debt can be negative in the long run if $\bar{a}$ is sufficiently large, despite incomplete insurance markets. Therefore, the assumption $a_{t+1} \geq 0$ implicitly rules out the possibility that the government can finance its spending purely from interest income generated by a negative debt position.

Nonetheless, we show in our online Appendix that relaxing the households’ borrowing constraints does not change the fundamental insight gained from the previous analyses—that incomplete markets for individual risk sharing will entice a benevolent government to keep increasing the level of public debt to eliminate or mitigate the distortions in the insurance markets as much
as possible to support a full self-insurance allocation. Therefore, an endogenous lower bound on public debt always exists such that the optimal bond supply exhibits “mean-reverting” or “error-correction” behaviors despite the existence of a STUR in the law of motion of the Lagrangian multiplier $\phi_t$.

5 Brief Literature Review

Angeletos, Collard, and Dellas (2016) study the Ramsey policy problem in the Lagos and Wright (2005) framework with heterogeneous agents and incomplete insurance markets. They show that when risk-free government bonds contribute to the supply of liquidity to alleviate private agents’ borrowing constraints, issuing more debt raises welfare by improving the allocation of resources.
Table 2: Result Statistics for the Non-State-Contingent Debt Case

<table>
<thead>
<tr>
<th>Cases</th>
<th>Scenario A</th>
<th>Scenario B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fall Self-Insurance</td>
<td>Feasible</td>
<td>Infeasible</td>
</tr>
<tr>
<td>std(G)(%)</td>
<td>10.000</td>
<td>10.000</td>
</tr>
<tr>
<td>std(τN)(%)</td>
<td>0.932</td>
<td>1.015</td>
</tr>
<tr>
<td>ρ(τN,t, τN,t−1)</td>
<td>0.894</td>
<td>0.617</td>
</tr>
<tr>
<td>ρ(Bt, Bt−1)</td>
<td>0.981</td>
<td>0.867</td>
</tr>
<tr>
<td>ρ(Bt+1/Bt, Gt+1)</td>
<td>−0.012</td>
<td>−0.010</td>
</tr>
<tr>
<td>ρ(Bt+1/Bt, Gt)</td>
<td>0.977</td>
<td>0.930</td>
</tr>
</tbody>
</table>

Notes: These statistics are generated by a simulation with 10,000 periods.

Similar to our findings, they show that the HAIM structure justifies an optimal level of public debt and introduces an interesting transitional path toward a Ramsey steady state—along which a departure from tax smoothing becomes desirable. In contrast to our model, their model has multiple Ramsey steady states and they focus on one where the optimal debt level preserves financial frictions in order to depress the interest rate on public debt. Using numerical methods, they study optimal policy responses to various types of aggregate shocks, including government spending shocks, technology shocks, and financial shocks.

Bassetto (2014) uses a model with heterogeneous agents to show that when tax liabilities are unevenly spread across the population, deviations from tax smoothing lead to welfare-improving interest rate changes that redistribute wealth. In particular, when a “bad shock” hits the economy, the optimal policy will call for smaller or larger deficits depending on the political power of different groups. For example, Bassetto shows that his model can explain why England relied heavily on debt to finance its wars, while France made heavy use of temporary tax increase: England had tax-paying merchants on political power, while France had landlords with political power.

Bassetto and Kocherlakota (2004) show that the paths of government debt can be irrelevant under distortionary taxes. In particular, they show that if the government collects taxes in a given period based only on incomes earned in previous periods, then the government can adjust its tax policy so as to attain any debt path without affecting equilibrium allocations or prices.

Bhandari, Evans, Golosov, and Sargent (2017b) study the determination of public debt and optimal taxation by extending the representative-agent model of AMSS to a setting with an HAIM structure similar to ours (i.e., with both aggregate and idiosyncratic uncertainties). Under the assumption that the government can set a lump-sum transfer and a linear tax on labor income, they show that (i) if households can borrow freely up to their natural borrowing limits, then the Ricar-
dian equivalence holds and the optimal level of public debt is indeterminate; and (ii) if households are subject to more-stringent borrowing limits (including the case where they cannot borrow at all), the Ricardian equivalence breaks down and social welfare can be further improved by decreasing the level of public debt. They interpret this improvement as a consequence of the government’s monopoly rents from issuing public debt without facing competing private borrowers—that is, a monopolistic government is able to lower the expected market interest rate below the time discount rate by restricting the bond supply. In other words, their analysis suggests that the government should decrease the aggregate supply of public liquidity by honoring ad hoc borrowing limits (i.e., by limiting the enforcement of private debt contracts) and using market power thereby acquired to extract monopoly rents from providing liquidity. This result does not seem to offer a rational for the observed high debt-to-GDP ratio in the U.S. economy and is the opposite to ours. This difference may come from their focus on the redistributive role of public debt under the assumption of unrestricted lump-sum transfers and *ex ante* heterogeneity, whereas we focus on the insurance role of public debt under *ex post* heterogeneity. Our result is consistent with that obtained by Shin (2006), who also focus on the insurance role of debt by ruling out the possibility of lump-sum transfers.

Bhandari, Evans, Golosov, and Sargent (2017a) introduce portfolio choices into the LS model without state-contingent bonds, as in AMSS. They develop a solution method that uses second-order approximations of Ramsey policies to obtain formulas for conditional and unconditional moments of government debt and taxes that include means and variances of the invariant distribution as well as speeds of mean reversion. They show that asymptotically the planner’s portfolio minimizes a measure of fiscal risk. For U.S. data, they find that the optimal target debt level is negative but close to zero, that the invariant distribution of debt is very dispersed, and that mean reversion is slow (with a half-life of nearly 250 years).

Bhandari, Evans, Golosov, and Sargent (2018) study optimal monetary and fiscal policy in a model with heterogeneous agents, incomplete markets, and nominal rigidities. They develop numerical techniques to approximate Ramsey plans in the very short run and apply them to a calibrated economy to compute optimal responses of nominal interest rates and labor tax rates to aggregate shocks. They find the responses differ qualitatively from those in a representative-agent economy and are an order of magnitude larger. They also find that Taylor rules poorly approximate the Ramsey optimal nominal interest rate. In their model, conventional price stabilization motives are swamped by an across-person insurance motive that arises from heterogeneity and incomplete markets.

Karantounias, Hansen, and Sargent (2009) study an optimal fiscal policy problem of the LS model but in a situation in which the representative agent’s distrust of the probability model
for government expenditures puts model-uncertainty premia into history-contingent prices. They show that such a situation gives rise to a motive for expectation management that is absent within rational expectations and a novel incentive for the planner to smooth the shadow value of the agent’s subjective belief to manipulate the equilibrium price of government debt. They find that unlike in the LS model, the optimal allocation, tax rate, and debt become history dependent despite complete markets and Markov government expenditures.

Azzimonti, de Francisco, and Quadrini (2014) propose a multi-country model with incomplete markets and show that governments may choose higher levels of debt when financial markets become internationally integrated. Similar to our findings, they also show that public debt increases with the volatility of uninsurable idiosyncratic income risk. Therefore, to the extent that the increase in income inequality observed in some industrialized countries has been associated with higher idiosyncratic risk, their paper suggests another potential mechanism for the rise in public debt. In this regard, our work complements theirs by offering a rigorous Ramsey approach to the same problem.

Martin (2009) studies optimal fiscal policies in a Lagos-Wright model with nominal debt. He shows that a government that cannot commit to future policy choices faces a trade-off between tax distortion and inflation that explains the level of debt. On the one hand, there is an incentive to increase debt and delay taxation, so as to reduce current distortions. On the other hand, inflating current prices lowers the real value of nominal debt, so there is a motive to reduce it now. The size of long-run debt will depend on the interaction of these two opposing incentives. The critical determinant in his model is the willingness of households to substitute away from goods being taxed by inflation. He also uses numerical simulations to show that the model matches some qualitative and quantitative properties of U.S. policy variables, including the fact that wars are frequently financed with a mix of instruments.

6 Conclusion

In this paper we analyzed the Ramsey planner’s decisions to finance stochastic public expenditures under incomplete insurance markets for idiosyncratic risk. We showed that as long as the market interest rate lies below the time discount rate, the Ramsey planner has a dominant incentive to increase the debt supply to meet the private sector’s demand for full self-insurance—even at the cost of extra tax distortion, regardless of the relative size of aggregate shocks. Therefore, a departure from tax smoothing is optimal whenever the market interest rate lies below the time discount rate. However, in the long run, if the full self-insurance Ramsey allocation is impossible (or infeasible) to achieve, an interior or bounded Ramsey equilibrium may not exist.
The strong incentives for the Ramsey planner to smooth both individual consumption and income taxes imply that (i) when state-contingent bonds are available, the long-run Ramsey equilibrium is characterized by full self-insurance and constant taxes, unless an upper government debt limit exists and binds; and (ii) when state-contingent debt is not available, the government’s attempt to balance the competing incentives between tax smoothing and consumption smoothing in the long run implies not only a bounded STUR component in optimal taxes but also a sufficiently high level of public debt such that the probability of staying at the full self-insurance allocation is strictly positive (unless a government borrowing limit exists and is binding).

Therefore, adding a liquidity premium into the value of government bonds via incomplete financial markets can bring the theory of public finance into closer conformity with reality. To derive our results, we have deliberately chosen a well-structured model—not only to make the Ramsey problem analytically tractable and transparent, but also to shed light on the formidable challenges in the determination of public debt in infinite-horizon HAIM models.


A Appendix

A.1 Proof of Proposition 2

We prove the decision rules in Proposition 2 by a guess-and-verify strategy. Denote the Lagrangian multipliers for constraints (3) and (4) as \( \beta^t \lambda_t(\theta^t, z^t) \pi(z^t, \theta^t) \) and \( \beta^t \mu_t(\theta^t, z^{t+1}) Q_{t+1}(z_{t+1}|z^t) \pi(z_{t+1}|z^t) \pi(z^t, \theta^t) \), respectively. The first-order conditions for \( c_t(\theta^t, z^t), n_t(\theta^{t-1}, z^t) \), and \( a_{t+1}(z^{t+1}|\theta^t, z^t) \) are given, respectively, by

\[
\frac{\theta^t}{c_t(\theta^t, z^t)} = \lambda_t(\theta^t, z^t),
\]

\[
1 = \overline{w_t}(z^t) E_{\theta_t} \lambda_t(\theta^t, z^t) = \overline{w_t}(z^t) \int \lambda_t(\theta^t, z^t) dF(\theta_t),
\]

\[
\lambda_t(\theta^t, z^t) = \frac{\beta}{Q_{t+1}(z_{t+1}|z^t)} \int \lambda_{t+1}(\theta^{t+1}, z^{t+1}) dF(\theta_{t+1}) + \mu_t(\theta^t, z^{t+1}),
\]

where equation (58) reflects the fact that labor supply \( n_t(\theta^{t-1}, z^t) \) must be chosen before the idiosyncratic taste shocks (and hence the value of \( \lambda_t(\theta^t, z^t) \)) are realized. By the iid assumption of \( \theta_t \), equation (59) can be written as (using equation (58))

\[
\lambda_t(\theta^t, z^t) = \Lambda_t(z^{t+1}) + \mu_t(\theta^t, z^{t+1}),
\]

where \( \Lambda_t(z^{t+1}) \) is defined as \( \beta (Q_{t+1}(z_{t+1}|z^t) \overline{w}_{t+1}(z^{t+1})^{-1} \); notice that \( \overline{w}_{t+1}(z^{t+1})^{-1} \) by equation (58) is the expected marginal utility of consumption.

Without loss of generality, assume that in each period \( t \) there are \( N \) states (realizations) of aggregate shock \( z_t \), denoted by \( \{z_{1t}, \ldots, z_{Nt}\} \in z_t \). In addition, without loss of generality, let the realized value of \( \Lambda_t(z^{t+1}) \) across the \( N \) states satisfy

\[
\Lambda_t([z^t, z_{1t+1}]) \geq \Lambda_t([z^t, z_{2t+1}]) \geq \ldots \geq \Lambda_t([z^t, z_{Nt+1}]).
\]

In the rest of the proof, we show that the decision rules for each individual’s consumption and savings are characterized by a unique cutoff strategy in the competitive equilibrium. We adopt a guess-and-verify strategy to derive the decision rules. Define \( x_t(\theta^{t-1}, z^t) \equiv a_t(z^t|\theta^{t-1}, z^{t-1}) + \overline{w}_t(z^t)n_t(\theta^{t-1}, z^t) \) as the gross wealth (cash on hand) of a household. We consider the following three cases where in case A and case B at least some households (with positive measure) are borrowing-constrained in all states. We will then discuss in case C where no households are borrowing constrained in all states. These cases establish that the cutoff \( \theta^t_t(z^t) \) is unique and independent of the individual history \( \theta^t \).

Case A. In this case the urge to consume (the realization of \( \theta_t \)) is high enough that it is optimal
Therefore, under the assumption in equation (61), a cutoff \( \theta_t^* \) is pinned down by the following equation:

\[
\theta_t^* (z^t) \equiv x_t(z^t) \max_{z_{t+1}} \Lambda_t(z_{t+1}^t) = x_t(z^t) \Lambda_t(z^t, z_{t+1}).
\]

Namely, Case A occurs for any household with \( \theta_t \geq \theta_t^* (z^t) \).

Case B: In this case the urge to consume is not high enough for a household, so it is optimal to save: \( \hat{a}_t(\theta^t, z^t) > 0 \). However, the portfolio choice of households in terms of \( a_{t+1}(z^t+1|\theta^t, z^t) \) is indeterminate. Namely, any portfolio choice of \( a_{t+1}(z^t+1|\theta^t, z^t) \) is optimal given the level of \( \hat{a}_t(\theta^t, z^t) \) so long as \( \sum_{z_{t+1}} Q_{t+1}(z_{t+1}|z^t)a_{t+1}(z_{t+1}|z^t, \theta^t)\pi(z_{t+1}|z^t) = \hat{a}_t(\theta^t, z^t) \) is satisfied. We therefore make a portfolio choice assumption that \( Q_{t+1}(z_{t+1}|z^t)a_{t+1}(z_{t+1}|\theta^t, z^t) = \sum_{z_{t+1}} Q_{t+1}(z_{t+1}|z^t)a_{t+1}(z_{t+1}|\theta^t, z^t)\pi(z_{t+1}|z^t) \) for all states of \( z_{t+1} \) and all \( \theta^t \). Given the portfolio assumption, \( \hat{a}_t(\theta^t, z^t) > 0 \) implies that \( a_{t+1}(z_{t+1}|\theta^t, z^t) > 0 \) and \( \mu_t(\theta^t, z^t+1) = 0 \) for all \( z_{t+1} \). Hence, the FOC (60) implies that

\[
\lambda_t(\theta^t, z^t) = \Lambda_t(z^t, z_{t+1}) \text{ for all } z_{t+1},
\]

suggesting that \( \Lambda_t(z^t) \) must be identical across all states of \( z_{t+1} \). The FOCs (57) and (60) together imply

\[
\frac{\theta_t}{c_t(\theta^t, z^t)} = \lambda_t(\theta^t, z^t) = \Lambda_t(z^t, z_{t+1}) \text{ for all } z_{t+1}
\]

This implies that \( \lambda_t(\theta^t, z^t) \) in case B equals \( \Lambda_t(z^t, z_{t+1}) \), which is independent of \( \theta^t \). Given this fact, equation (57) implies that consumption is given by \( c_t(\theta^t, z^t) = \theta_t \Lambda_t(z^t, z_{t+1})^{-1} \), and the budget identity (3) then implies

\[
a_{t+1}(z_1|\theta^t, z^t)Q_{t+1}(z_1|z^t)\pi(z_1|z^t) = x_t(\theta_{t-1}^t, z^t) - \theta_t \Lambda_t(z^t, z_{t+1})^{-1}.
\]

The requirement \( a_{t+1}(z_{t+1}|\theta^t, z^t) > 0 \) implies

\[
\theta_t < \Lambda_t(z^t, z_{t+1})x_t(z^t) \equiv \theta_t^* (z^t),
\]

\( 62 \)
which is the same cutoff $\theta_t^*(z^t)$ derived in case A. In short, case B occurs for any household with $\theta_t < \theta_t^*(z^t)$.

By the discussion of cases A and B, we see that the cutoff $\theta_t^*(z^t)$ is unique and it fully characterizes the distribution of household allocations. Most importantly, this cutoff is independent of the history of idiosyncratic shocks. In addition, we know that $\Lambda_t(z^t)$ must be identical for all $z_{t+1}$ in equilibrium. This property implies that the borrowing constraints of households either bind across all aggregate states of $z_{t+1}$ conditioned on $z_t$ or do not bind at all in any aggregate state. That is, $\tilde{a}_t(\theta^t, z^t) = 0$ if and only if each component $a_{t+1}(z^t+1 | z^t, \theta^t) = 0$; and $\tilde{a}_t(\theta^t, z^t) > 0$ if and only if each component $a_{t+1}(z^t+1 | z^t, \theta^t) > 0$. Consequently, the consumption and saving decision rule can be written as in the first line of equation (11) and (12), respectively.

For the state-contingent bond price, the FOC (58) can be rewritten as

$$1 = \overline{w}_t(z^t) \left[ \int_{\theta_t}^{\theta_t^*(z^t)} \frac{\theta_t^*(z^t)}{x_t(z^t)} d\mathcal{F}(\theta_t) + \int_{\theta_t^*(z^t)}^{\theta_H} \frac{\theta_t}{x_t(z^t)} d\mathcal{F}(\theta_t) \right]$$

$$\equiv \overline{w}_t(z^t) \frac{\theta_t^*(z^t)}{x_t(z^t)} L(\theta_t^*(z^t)),$$

where the liquidity premium, $L(\theta_t^*(z^t))$, is defined as

$$L(\theta_t^*(z^t)) \equiv \int_{\theta_L}^{\theta_t^*(z^t)} d\mathcal{F}(\theta_t) + \int_{\theta_t^*(z^t)}^{\theta_H} \frac{\theta_t}{\theta_t^*(z^t)} d\mathcal{F}(\theta_t).$$

This gives equation (19). Hence, we can also obtain the state-contingent price as (using equation (63))

$$Q_{t+1}(z_{t+1} | z^t) = \beta \frac{\overline{w}_t(z^t)}{\overline{w}_{t+1}(z^{t+1})} L(\theta_t^*(z^t))$$

for all states of $z_{t+1}$.

We then solve for $x_t(z^t)$ as follows. By solving for an agent with a binding borrowing constraint in period $t$, the agent’s consumption equals cash on hand, $c_t(\theta^t, z^t) = x_t(\theta^{t-1}, z^t)$, and the equilibrium cash on hand $x_t$ is obtained by

$$x_t(z^t) = \theta_t^*(z^t) \left[ \frac{1}{Q_{t+1}(z_{t+1} | z^t)} \frac{1}{\overline{w}_{t+1}(z^{t+1})} \right]^{-1} = \theta_t^*(z^t) \overline{w}_t(z^t) L(\theta_t^*(z^t)),$$

which leads to the first line in equation (10) and (15). This also suggests that the state-contingent bond price responds to a next-period government spending shock so that $Q_{t+1}(z_{t+1} | z^t)\overline{w}_{t+1}(z^{t+1})$ is the same across all states of $z_{t+1}$.

Case C: The above discussions in cases A and B implicitly assume that the aggregate bond supply is low enough that there always exists a positive measure of households with a binding
borrowing constraints across all states of $z_{t+1}$. However, if the aggregate bond supply is high enough such that no household is borrowing constrained—i.e., $\theta_t^* = \theta_H$—then the distribution of individual saving stocks becomes indeterminate so long as each household’s cash on hand is sufficiently large (because of a large enough aggregate bond supply). In this case, we can simply assume that all households still choose the same cash on hand $x_t = X_t$. The second line of equation (12) immediately follows. Under the portfolio choice assumption, $\tilde{\alpha}_t(\theta^t, z^t) > 0$ for all $\theta_t$ implies that $\mu_t(\theta^t, z^{t+1}) = 0$ for all $z_{t+1}$ and all $\theta_t$. Hence, $\lambda_t(\theta_t, z^t)$ is equalized across all households according to (59). Thus, household consumption by (57) and (58) is given by

$$c_t(\theta_t, z^t) = \theta_t \overline{w}_t(z^t),$$

which is the second line in equation (11). Summing up the second line of (11) across agents gives

$$C_t(z^t) = \overline{\theta} \overline{w}_t(z^t).$$

Moreover, summing up $x_t(z^t)$ across households together with the asset-market clearing condition, $\int a_{t+1}(z^{t+1}|\theta_t, z^t) dF(\theta_t) = B_{t+1}(z^{t+1}|z^t)$, gives

$$X_t(z^t) = \sum_{z_{t+1}} Q_{t+1}(z_{t+1}|z^t) B_{t+1}(z^{t+1}|z^t, \theta^t) \pi(z_{t+1}|z^t) + \overline{\theta} \overline{w}_t(z^t),$$

which is the second line in equation (15). Finally, (58), (59) together with identical $\lambda_t(\theta_t, z^t)$ imply

$$Q_{t+1}(z_{t+1}|z^t) = \beta \frac{\overline{w}_t(z^t)}{\overline{w}_{t+1}(z^{t+1})} \text{ for all } z_{t+1},$$

which is identical to equation (19) if $\theta_t^*(z^t) = \theta_H$.

By aggregating (11) and (12), it is straightforward to obtain the aggregate consumption (16) and aggregate effective savings (17). Hence, the decision rules of household can then be summarized by equations (10)-(13) and the aggregate variables are given by equations (15)-(17).

Note that the decision rule of the household labor supply in equation (13) is derived residually to satisfy the household budget constraint. To ensure that the above proof and hence the associated cut-off policies are consistent with the requirement of interior choices of labor, namely, $n_t \in (0, \overline{N})$, we need to consider the following two cases:

(i) First, to ensure a nonnegative $n_t$, consider the worst situation where $n_t$ takes its minimum value. Given the definition of $x_t(z^t)$, $n_t$ reaches its minimum if $\mu_{t+1}(\theta^t, z^{t+1}) = 0$ and hence $\sum_{z_{t+1}} Q_{t+1}(z_{t+1}|z^t) a_{t+1}(z^{t+1}|\theta^t, z^t) \pi(z_{t+1}|z^t)$ takes the maximum possible value. Given the house-
hold portfolio choice assumption, \( a_{t+1}(z_{t+1} | \theta^t, z^t) = \frac{\theta_{t-1}(z_{t-1}^t - \theta_L x_{t-1}(z_{t-1}^t))}{Q_t(z_{t-1}^t)} \). So \( n_t(\theta^{t-1}, z^t) > 0 \) if

\[
x_t(z^t) - \frac{\theta^*_t(z_{t-1}^t) - \theta_L x_{t-1}(z_{t-1}^t)}{Q_t(z_{t-1}^t)} > 0. \tag{64}
\]

The parameter values required to ensure condition (64) are assumed to hold throughout the paper.

(ii) Second, to ensure that \( n_t < \bar{N} \), consider those agents who encounter the borrowing constraint in the immediate past period such that \( a_t(z_t | \theta^{t-1}, z_{t-1}^t) = 0 \) for all states of \( z_t \). Their labor supply reaches the maximum value at \( n_t(\theta_{t-1}, z^t) = \frac{\theta^*_t(z^t)}{\theta^*_t(z^t)} = \theta^*_t(z^t) | L(\theta^*_t(z^t)) \). Given that the cutoff \( \theta^*_t(z^t) \) is finite and that \( \theta^*_t(z^t) | L(\theta^*_t(z^t)) \leq \theta_H \), the value of \( \bar{N} \) can be chosen such that

\[
\bar{N} > \theta_H. \tag{65}
\]

A.2 Proof of Proposition 4

A.2.1 The “Only If” Part

Assume that we have an allocation \( \{ \theta^*_t(z^t), N_t(z^t), C_t(z^t), B_{t+1}(z_{t+1}^t) \}_{t=0}^\infty \). With this allocation, we can directly construct prices, taxes, and all household allocations in the competitive equilibrium in the following steps.

1. According to the optimal condition of the firm’s problem, \( w_t(z^t) \) is set to 1.

2. Depending on the value of \( \theta^*_t \), we consider two cases below

   (a) The case with \( \theta^*_t(z^t) < \theta_H \). Given \( C_t(z^t) \) and \( \theta^*_t(z^t) \), we can compute \( x_t(z^t) \) by utilizing (11) and consumption goods market-clearing condition:

   \[
   C_t(z^t) = \int c_t(\theta_t, z^t) dF(\theta_t) = D(\theta^*_t(z^t)) x_t(z^t),
   \]

   which implies \( x_t(z^t) = \frac{C_t(z^t)}{D(\theta^*_t(z^t))} = \frac{1}{U_C(z^t) D(\theta^*_t(z^t))} \). By the first line of equation (10) and \( w_t(z^t) = 1 \), the labor tax rate \( \tau_t(z^t) \) is given by

   \[
   \overline{\tau_t(z^t)} = (1 - \tau_t(z^t)) w_t(z^t) = \frac{x_t(z^t)}{L(\theta^*_t(z^t)) \theta^*_t(z^t)} = \frac{1}{U_C(z^t) Z(\theta^*_t(z^t))}, \tag{66}
   \]

   where \( Z(\theta^*_t(z^t)) \equiv L(\theta^*_t(z^t)) \theta^*_t(z^t) D(\theta^*_t(z^t)) \).
(b) $\theta^*_i(z^t) = \theta_H$ case: The wage rate is chosen by the second line of (16):

$$\overline{w}_t(z^t) = (1 - \tau_t(z^t))w_t(z^t) = \frac{1}{U_C(z^t)\theta},$$

which is the same as (66) under $\theta^*_i = \theta_H$. Hence, the $Q_{t+1}$ and labor tax $\tau_t(z^t)$ can be backed out by using (66) and (67), respectively, by setting $\theta^*_i(z^t) = \theta_H$. Finally, the aggregation of $x_t$ and asset market clearing condition imply

$$X_t(z^t) = \sum_{z_{t+1}} Q_{t+1}(z_{t+1}|z^t)B_{t+1}(z_{t+1}|z^t)\pi(z_{t+1}|z^t) + C_t(z^t).$$

The individual consumption and asset holdings, $c_t(\theta_t, z^t)$ and $a_{t+1}(z_{t+1}|\theta_t, z^t)$, are given by equations (11) and (12). In addition, the state-contingent bond prices can be expressed as

$$Q_{t+1}(z_{t+1}|z^t) = \beta \frac{\overline{w}_t(z^t)}{w_{t+1}(z_t^{t+1})} L(\theta^*_i(z^t)) = \beta \frac{U_C(z^{t+1}) Z(\theta^*_{t+1}(z^{t+1}))}{U_C(z^t) Z(\theta^*_i(z^t))} L(\theta^*_i(z^t)). \quad (67)$$

Finally, $n_t(\theta_{t-1})$ is set to satisfy the following condition implied by the individual household budget constraint:

$$n_t(\theta_{t-1}, z^t) = \frac{1}{w_t(z^t)} \left[ x_t(z^t) - a_t(z_t|\theta_{t-1}, z^t) \right].$$

3. By equation (17) and the asset market-clearing condition, we can derive:

$$\sum_{z_{t+1}} Q_{t+1}(z_{t+1}|z^t)B_{t+1}(z_{t+1}|z^t)\pi(z_{t+1}|z^t) = \begin{cases} [1 - D(\theta^*_i(z^t))] x_t(z^t), & \text{if } \theta^*_i(z^t) < \theta_H \smallskip \\
X_t(z^t) - \overline{w}_t(z^t) & \text{if } \theta^*_i(z^t) = \theta_H \end{cases}.$$  

Using the expression of $Q_{t+1}(z_{t+1}|z^t)$ and equation (10), we arrive at two cases. If $\theta^*_i = \theta_H$, then the above equation always holds since it becomes an identity. If $\theta^*_i < \theta_H$, then

$$\sum_{z_{t+1}} \beta B_{t+1}(z_{t+1}|z^t) \frac{U_C(z^{t+1})}{U_C(z^t)} \frac{Z(\theta^*_{t+1}(z^{t+1}))}{Z(\theta^*_i(z^t))} L(\theta^*_i(z^t)) \pi(z_{t+1}|z^t)$$

$$= (1 - D(\theta^*_i(z^t)))\overline{w}_t(z^t)L(\theta^*_i(z^t))\theta^*_i(z^t),$$

which could be rewritten as the following constraint:

$$\sum_{z_{t+1}} \beta B_{t+1}(z_{t+1}|z^t) U_C(z^{t+1}) Z(\theta^*_i(z^{t+1})) \pi(z_{t+1}|z^t) = \theta^*_i(z^t) (1 - D(\theta^*_i(z^t))) \quad (68),$$

which is listed in equation (24).
4. Now we derive the implementability condition. Replacing \( G_t(z^t) \) with \( N_t(z^t) - C_t(z^t) \) in the flow government budget constraint gives

\[
C_t(z^t) - (1 - \tau_t(z^t))w_t(z^t)N_t(z^t) + \sum_{z_{t+1}} Q_{t+1}(z_{t+1}|z^t)B_{t+1}(z_{t+1}|z^t)\pi(z_{t+1}|z^t) \geq B_t(z_t|z^{t-1}).
\]

By Steps 3 and 2, both \((1 - \tau_t(z^t))w_t(z^t)\) and \(Q_{t+1}(z_{t+1}|z^t)\) can be expressed as functions of \(C_t\) and \(\theta_t^*\):

\[
C_t(z^t) - \frac{C_t(z^t)}{Z(\theta_t^*(z^t))}N_t(z^t) + \sum_{z_{t+1}} \beta \frac{U_C(z^{t+1})}{U_C(z^t)} \frac{Z(\theta_t^*(z^{t+1}))}{Z(\theta_t^*(z^t))} L(\theta_t^*(z^t))B_{t+1}(z_{t+1}|z^t)\pi(z_{t+1}|z^t) \geq B_t(z^t).
\]

Multiply the above equation with \(U_C(z^t)Z(\theta_t^*(z^t))\) gives

\[
U_C(z^t)C_t(z^t) - U_C(z^t)Z(\theta_t^*(z^t)) - U_C(z^t)C_t(z^t)N_t(z^t) + \sum_{z_{t+1}} \beta U_C(z^{t+1})Z(\theta_t^*(z^{t+1}))L(\theta_t^*(z^t))B_{t+1}(z_{t+1}|z^t)\pi(z_{t+1}|z^t) \geq U_C(z^t)Z(\theta_t^*(z^t))B_t(z_t|z^{t-1}).
\]

In short, Step 1 ensures that the representative firm’s problem is solved. Steps 2 ensures the individual household problem is solved. Step 3 and 4 ensure the asset market-clearing condition and government budget constraints are satisfied, respectively. The labor market clearing condition is satisfied by the Walras law.

### A.2.2 The “If” Part

Note that the resource constraint and asset market-clearing condition are trivially implied by a competitive equilibrium since they are part of the definition. The implementability condition is constructed as follows. First, by using \(w_t(z^t) = 1\), we rewrite the government budget constraint as

\[
G_t(z^t) \leq N_t(z^t) - (1 - \tau_t(z^t))N_t(z^t) + \sum_{z_{t+1}} Q_{t+1}(z_{t+1}|z^t)B_{t+1}(z_{t+1}|z^t)\pi(z_{t+1}|z^t) - B_t(z_t|z^{t-1}).
\]

Combining this equation with the resource constraint (8) and equation (19) implies

\[
(1 - \tau_t(z^t))N_t(z^t) + B_t(z_t|z^{t-1}) \leq C_t(z^t) + \beta \sum_{z_{t+1}} \frac{w_t(z^t)}{w_{t+1}(z^{t+1})}B_{t+1}(z_{t+1}|z^t)\pi(z_{t+1}|z^t).
\]
Multiply the above equation with $1/w_t(z^t)$ gives

$$N_t(z^t) + \frac{B_t(z_t|z^{t-1})}{w_t(z^t)} \leq \frac{C_t(z^t)}{w_t(z^t)} + \beta \sum_{z_{t+1}} \frac{1}{w_{t+1}(z_{t+1})} B_{t+1}(z_{t+1}|z^t) \pi(z_{t+1}|z^t). \quad (70)$$

In addition, combining equation (10) and (11) gives $1/w_t(z^t) = U_C(z^t)Z(\theta_t^*(z^t))$, which implies that equation (70) can be expressed as (21).

### A.3 Proof of lemma 5

We first show that $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} = 0$ if $\theta_t^* = \theta_L$ or $\theta_H$:

$$\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} = -\frac{\partial D(\theta_t^*)}{\partial \theta_t^*} \frac{\bar{\theta}}{D(\theta_t^*)} \int_{\theta < \theta_t^*} \theta \frac{dF(\theta)}{\theta_t^*} - \int_{\theta \geq \theta_t^*} \theta \frac{dF(\theta)}{\theta_t^*} = \left\{ \begin{array}{ll} 1 - \frac{\bar{\theta}}{\theta_{HA} - \theta_{L}} & \bar{\theta} = 0 \text{ if } \theta_t^* = \theta_H \\ 0 & \bar{\theta} = 0 \text{ if } \theta_t^* = \theta_L. \end{array} \right.$$

Next, we show that $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} > 0$ for any $\theta_t^* \in (\theta_L, \theta_H)$. Note that

$$D(\theta_t^*) \theta_t^* = \int_{\theta < \theta_t^*} \theta dF(\theta) + \theta_t^* \int_{\theta > \theta_t^*} dF(\theta) = \bar{\theta} - \int_{\theta > \theta_t^*} (\theta - \theta_t^*) dF(\theta) < \bar{\theta}$$

$$\rightarrow \frac{\bar{\theta}}{D(\theta_t^*) \theta_t^*} > 1.$$

Hence,

$$\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} = \left\{ \begin{array}{ll} \frac{\bar{\theta}}{D(\theta_t^*) \theta_t^*} - 1 \int_{\theta < \theta_t^*} \theta \frac{dF(\theta)}{\theta_t^*} & > 0. \end{array} \right.$$

### A.4 Proof of Proposition 6

For our benchmark economy, the first-order conditions of the Ramsey planner’s problem (27) with respect to $N_t(z^t)$, $C_t(z^t)$, $\theta_t^*(z^t)$ and $B_{t+1}^*(z_{t+1})$ for $t \geq 0$ are given, respectively, by

$$N_t(z^t) : 1 + \phi_t(z^t) = \psi_t(z^t) \quad (71)$$

$$C_0(z^0) : \bar{\theta}U_C(z^0) - \phi_0(z^0)U_{CC}(z^0)Z(\theta_0^*(z^0))B_0 = \psi_0(z^0) \quad (72).$$
\[
\begin{align*}
C_t(z^t) : & \quad \partial U_t(z^t) + \left( \phi_{t-1}(z^{t-1})L(\theta^*_t(z^{t-1})) - \phi_t(z^t) + \mu^B_{t-1}(z^{t-1}) \right) U_{CC}(z^t)Z(\theta^*_t(z^t))B_t(z^t | z^{t-1}) \quad (73) \\
& = \psi_t(z^t)
\end{align*}
\]

\[
\begin{align*}
\theta^*_t(z^t) : & \quad \frac{\partial W(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} + \phi_{t-1}(z^{t-1})U_t(z^t) \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} L(\theta^*_t(z^{t-1}))B_t(z_t | z^{t-1}) \quad \quad (74) \\
& \quad + \phi_t(z^t) \left[ \left(1 - U_t(z^t)B_t(z_t^t) \right) \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \right] \\
& \quad + \sum_{z_{t+1}} \beta U_t(z^{t+1})Z(\theta^*_t(z^{t+1})) \frac{\partial L(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} B_{t+1}(z_{t+1} | z^t) \pi(z_{t+1} | z^t) \\
& \quad = J(\theta^*_t(z^t)) \mu^B_t(z^t) - \mu^B_{t-1}(z^{t-1})B_t(z_t^t | z^{t-1})U_t(z^t) \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)}, \quad (75)
\end{align*}
\]

where function \( J(\theta^*_t(z^t)) \) is defined in Lemma 16. Plugging equation (71) into (72) and (73) gives (28) and (29). For the optimal condition (30), we first plug (75) and (24) into (74), which gives

\[
\begin{align*}
\frac{\partial W(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} & \quad + \phi_{t-1}(z^{t-1})U_t(z^t) \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} L(\theta^*_t(z^{t-1}))B_t(z_t | z^{t-1}) \\
& \quad + \phi_t(z^t) \left[ \left(1 - U_t(z^t)B_t(z_t^t) \right) \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \right] \\
& \quad + \sum_{z_{t+1}} \beta U_t(z^{t+1})Z(\theta^*_t(z^{t+1})) \frac{\partial L(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} B_{t+1}(z_{t+1} | z^t) \pi(z_{t+1} | z^t) \\
& \quad = J(\theta^*_t(z^t)) \left( \phi_{t+1}(z^{t+1}) - \phi_t(z^t) \right) L(\theta^*_t(z^t)) + \frac{v^H_t(z_{t+1} | z^t) - v_t^L(z_{t+1} | z^t)}{U_t(z^{t+1})Z(\theta^*_t(z^{t+1}))} \\
& \quad - B_t(z_{t+1} | z^t)U_t(z^t) \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \left( \phi_t(z^t) - \phi_{t-1}(z^{t-1})L(\theta^*_t(z^{t-1})) + \frac{v^H_{t-1}(z_t | z^{t-1}) - v^L_{t-1}(z_t | z^{t-1})}{U_t(z^{t-1})Z(\theta^*_t(z^{t-1}))} \right).
\end{align*}
\]

The equation above can be simplified as the follows:

\[
\begin{align*}
\frac{\partial W(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} & \quad + \phi_t(z^t) \left[ \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} + \frac{\partial L(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \theta^*_t(z^t) \left(1 - D(\theta^*_t(z^t)) \right) \right] \\
& \quad \quad \quad \quad = J(\theta^*_t(z^t)) \left( \phi_{t+1}(z^{t+1}) - \phi_t(z^t) \right) L(\theta^*_t(z^t)) + \frac{v^H_{t+1}(z_t | z^{t+1}) - v^L_{t+1}(z_t | z^{t+1})}{U_t(z^{t+1})Z(\theta^*_t(z^{t+1}))} \\
& \quad \quad \quad \quad - B_t(z_{t+1} | z^t)U_t(z^t) \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \left( \frac{v^H_{t-1}(z_t | z^{t-1}) - v^L_{t-1}(z_t | z^{t-1})}{U_t(z^{t-1})Z(\theta^*_t(z^{t-1}))} \right),
\end{align*}
\]

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which can be further simplified to

$$\frac{\partial W(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} + \phi_t(z^t) \left[ \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} + \frac{\partial L(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \theta^*_t(z^t) (1 - D(\theta^*_t(z^t))) \right] - J(\theta^*_t(z^t)) \phi_{t+1}(z^{t+1})$$

$$= J(\theta^*_t(z^t)) \left( \frac{v^H_t(z_{t+1}|z^t) - v^L_t(z_{t+1}|z^t)}{U_C(z^{t+1})Z(\theta^*_{t+1}(z^{t+1}))} \right)$$

$$- B_t(z_t|z^{t-1}) U_C(z^t) \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \left( \frac{v^H_{t-1}(z_t|z^{t-1}) - v^L_{t-1}(z_t|z^{t-1})}{U_C(z^t)Z(\theta^*_t(z^t))} \right),$$

which can be rewritten as the following equation by using lemma 16:

$$\frac{\partial W(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} + F(\theta^*_t) \phi_t(z^t) - F(\theta^*_t) \phi_{t+1}(z^{t+1}) = F(\theta^*_t) \left( \frac{v^H_t(z_{t+1}|z^t) - v^L_t(z_{t+1}|z^t)}{U_C(z^{t+1})Z(\theta^*_{t+1}(z^{t+1}))} \right)$$

$$- B_t(z_t|z^{t-1}) U_C(z^t) \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \left( \frac{v^H_{t-1}(z_t|z^{t-1}) - v^L_{t-1}(z_t|z^{t-1})}{U_C(z^t)Z(\theta^*_t(z^t))} \right),$$

which is identical to equation (30) (without debt-limit constraints) if $v^H_t = v^L_t = 0$ for all $t$.

The labor tax formula (31) is pinned down by the third step of the proof in Proposition 4. Furthermore, equation (32) is identical to equation (19). Equation (75) implies equation (33).

**Lemma 16.** First, let $H(\theta^*_t)$ and $J(\theta^*_t)$ be defined as

$$H(\theta^*_t) = \frac{\partial (L(\theta^*_t) \theta^*_t)}{\partial \theta^*_t} = \left( L(\theta^*_t) + \frac{\partial L(\theta^*_t)}{\partial \theta^*_t} \theta^*_t \right)$$

$$J(\theta^*_t) = \frac{\partial (1 - D(\theta^*_t)) \theta^*_t}{\partial \theta^*_t} = \left( 1 - D(\theta^*_t) - \theta^*_t \frac{\partial D(\theta^*_t)}{\partial \theta^*_t} \right)$$

1. By the following derivation, we reach the result $H(\theta^*_t) = J(\theta^*_t) = F(\theta^*_t)$:

$$J(\theta^*_t) = \left( 1 - D(\theta^*_t) - \theta^*_t \frac{\partial D(\theta^*_t)}{\partial \theta^*_t} \right)$$

$$= 1 - \left[ \int_{\theta < \theta^*_t} \frac{\theta}{\theta^*_t} dF(\theta) + \int_{\theta > \theta^*_t} dF(\theta) \right] + \int_{\theta < \theta^*_t} \frac{\theta}{\theta^*_t} dF(\theta)$$

$$= 1 - \int_{\theta > \theta^*_t} dF(\theta) = F(\theta^*_t)$$
\[ H(\theta^*_t) \equiv \left( L(\theta^*_t) + \frac{\partial L(\theta^*_t)}{\partial \theta^*_t} \right) \]
\[ = \int_{\theta \leq \theta^*_t} dF(\theta) + \int_{\theta > \theta^*_t} \frac{\theta}{\theta^*_t} dF(\theta) - \int_{\theta > \theta^*_t} \frac{\theta}{\theta^*_t} dF(\theta) \]
\[ = \int_{\theta \leq \theta^*_t} dF(\theta) = F(\theta^*_t). \]

2. In addition,
\[ \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} + \frac{\partial L(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \theta^*_t(z^t) (1 - D(\theta^*_t(z^t))) + J(\theta^*_t(z^t)) L(\theta^*_t(z^t)) \]
\[ = \frac{\partial (L(\theta^*_t(z^t)) D(\theta^*_t))}{\partial \theta^*_t(z^t)} + \frac{\partial L(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \theta^*_t(z^t) (1 - D(\theta^*_t(z^t))) + \frac{\partial (1 - D(\theta^*_t)) \theta^*_t}{\partial \theta^*_t} L(\theta^*_t) \]
\[ = \frac{\partial (L(\theta^*_t(z^t)) D(\theta^*_t))}{\partial \theta^*_t(z^t)} + \frac{\partial L(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \theta^*_t(z^t) (1 - D(\theta^*_t(z^t))) \]
\[ = \frac{\partial (L(\theta^*_t(z^t)) D(\theta^*_t))}{\partial \theta^*_t(z^t)} = H(\theta^*_t). \]

### A.5 Proof of Proposition 7

According to the Ramsey optimal condition (29), the Ramsey steady state must feature a constant \( \phi \) since there is no aggregate shock and constant \( C \) in the long run (steady state). A constant \( \phi \) together with Ramsey optimal condition (30) implies that \( \theta^*_t = \theta_H \) is a possible steady state. This is true regardless of the initial value of \( B_0 \). As shown by Corollary 3, a sufficiently low \( B_0 \) may result a partial self-insurance equilibrium.

### A.6 Proof of Proposition 8

To eliminate idiosyncratic risks from the model, we can simply let the distribution of the idiosyncratic shock \( \theta_t \) in our model degenerate such that \( \theta_H = \theta_L = \bar{\theta} \) and hence \( \theta^*_t \to \bar{\theta} \). Then by definition, we have \( \theta^*_t(z^t) \to \bar{\theta}, L(\theta^*_t(z^t)) \to 1, D(\theta^*_t(z^t)) \to 1, \) and \( W(\theta^*_t(z^t)) \to 0 \). Also, the asset market-clearing condition (24) degenerates into an identity: \( B_{t+1}(z_{t+1}|z^t) = B_{t+1}(z_{t+1}|z^t) \); consequently, the constraint in the fourth row of the maximization problem (27) drops out and the associated multiplier \( \mu^P_t = 0 \). In addition, the FOC with respect to the distribution \( \theta^*_t(z^t) \) in equation (30) drops out.

Given the degenerated \( \theta \) distribution, equation (36) is implied by equation (29). Equations (34), (35), and (37) are reduced from equations (28), (29) and (31). Equation (19) implies equation (39). Finally, (40) and (41) are the time-zero government budget constraint and the period-\( t \) present value budget constraint in this case.
A.7 Proof of Proposition 9

By equation (30) and lemma 5, we know that \( \phi_t(z^t) \) has to be an non-decreasing stochastic process. If the Ramsey equilibrium is stationary, then \( \phi_t(z^t) \) has to converge to a positive nonzero number, which implies that \( \theta^*_t(z^t) = \theta_H \) for \( t \geq T \), given that \( T \) is sufficiently large. Given \( \theta^*_t(z^t) = \theta_H \), then \( \frac{\partial W(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} = 0 \), \( Z(\theta^*_t(z^t)) = \overline{\theta} \), and \( L(\theta^*_t) = 1 \). According to Proposition 6, \( \phi_t(z^t) \), \( C_t(z^t) \), and \( \tau_t(z^t) \) all become constant.

A.8 Proof of Proposition 10

We first construct equation (44). If the upper debt limit binds, then the optimal Ramsey condition (76) listed in the Appendix A.4 becomes

\[ \phi_{t+1}(z^{t+1}) = \phi_t(z^t) + e_{t+1}(z_{t+1}|z^t) \]  

which is identical to (30) by defining

\[
e_{t+1}(z_{t+1}|z^t) \equiv \frac{\partial W(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \frac{1}{\theta^*_t(z^t)} - \frac{v^H_t(z_{t+1}|z^t)}{U_C(z_{t+1})Z(\theta^*_t(z^{t+1}))}
\]

\[+ B_t(z_t|z^{t-1}) \frac{U_C(z^t)}{\theta^*_t(z^t)} \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \frac{v^H_{t-1}(z_{t}|z^{t-1})}{U_C(z^t)Z(\theta^*_t(z^t))}.\]

We next prove that \( \phi \) follows a stochastic stationary process by the following steps. The prove is built on the following lemma (the proof of this lemma is straightforward by using the definitions of \( Z(\theta^*) \) and \( q \)):

**Lemma 17.**  
1. \( \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} > 0 \) and \( \frac{\partial q_t(z^t)}{\partial \theta^*_t(z^t)} > 0 \) if \( \theta^*_t \) is sufficiently close to but not equal to \( \theta_H \);  
2. \( \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} < 0 \) and \( \frac{\partial q_t(z^t)}{\partial \theta^*_t(z^t)} < 0 \) if \( \theta^*_t \) is sufficiently close to but not equal to \( \theta_L \);  
3. If \( \theta^*_t = \{\theta_L, \theta_H\} \), then \( \frac{\partial Z(\theta^*_t)}{\partial \theta^*_t} \big|_{\theta^*_t=\{\theta_L, \theta_H\}} = 0 \).

First, suppose \( v^H_t = v^H_{t-1} = 0 \), so the debt limit does not bind currently or in the past; then, the multiplier \( \phi_{t+1} \) will be increasing until the debt-limit constraint binds with \( B_{t+1} = \overline{B} \) and \( v^H_t > 0 \). In particular, if the binding debt-limit constraint is not tight enoug h, the multiplier will keep increasing at the debt level \( B_{t+1} = \overline{B} \) until the multiplier \( v^H_t \) eventually becomes sufficiently high to render the error term negative, \( e_{t+1} < 0 \); after that the multiplier \( \phi_{t+1} \) will start to decrease until the error term becomes positive again. Second, if \( v^H_{t-1} > 0 \) and the debt-limit constraint was binding in period \( t - 1 \), \( B_t(z_t|z^{t-1}) = \overline{B} \). Each element in the last term on the right-hand side of
the above equation is positive except the derivative \( \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \), which is positive if the cutoff is close to \( \theta_H \) and negative if close to \( \theta_L \). Suppose first in the worst case where the term \( \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \) is positive, then the multiplier \( \phi_{t+1} \) will be increasing until the period-\( t \) debt-limit constraint is not only binding but also sufficiently tight with \( v^H_t \to \infty \), or so large that the error term changes sign with \( e_{t+1} < 0 \). Hence, a forever increasing \( \phi_{t+1} \) is impossible under the debt-limit constraint. The fact that the last-period \( v^H_{t-1} > 0 \) brings in a positive force proportional to \( \overline{B} \) simply indicates that the multiplier \( \phi_{t+1} \) has a tendency to remain at the point where \( B_t = \overline{B} \) or converge to the point where the debt-limit constraint binds. Now consider the case where \( \overline{B} \) is close to zero such that \( \theta^*_t \) is close to \( \theta_L \) and \( \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} < 0 \), then the force to push down the multiplier \( \phi_{t+1} \) is reinforced if the debt-limit constraint has been binding in the past and in the present, implying that bond supply \( B_{t+1} \) may start to fall whenever the debt-limit constraint binds. However, as soon as the bond supply \( B_{t+1} \) falls below \( \overline{B} \) toward zero, the negative forces in the error term disappear and consequently, the multiplier will start to increase again toward the point where \( B_{t+1} = \overline{B} \). Also notice that when \( \overline{B} \) is small or close to zero, the importance of the last term in equation (45) is proportionally reduced.

Finally, we show that \( B_{t+1} = \overline{B} \) for all \( t \geq T \) in the long run is not a Ramsey equilibrium. For this case to be an equilibrium, it requires the multiplier \( \phi_{t+1} \) to be constant such that \( \phi_{t+1} = \phi_t \) or the error term \( e_{t+1} = 0 \) for \( t \geq T \), which implies that the aggregate consumption \( C_t \), the debt limit \( \overline{B} \), the distribution \( \theta^*_t \), and the multipliers \( v^H_t \) must all be constant with exactly the right values such that \( e_{t+1} = 0 \). This cannot be true since at the least \( \overline{B} \) is arbitrary and the value of \( \theta^*_t \) depends on the distribution \( F(\theta) \). This implies that \( \phi_{t+1} \) will not converge to zero and must be stochastic with a (stochastically) mean-reverting property. Finally, optimal taxes inherit the property of \( \phi \) according to equation (31).

**A.9 Proof of Proposition 11**

**A.9.1 Conditions to Support a Competitive Equilibrium without State-Contingent Bonds**

Notice that, without state-contingent bonds, the discounted price of risk-free bonds is given by equation (46). Then, it is straightforward to verify that Proposition A.2 is modified slightly into the following proposition given the assumption of non-state-contingent government bonds:

**Proposition 18.** Given the initial government bond supply \( B_0 \), the sequence of aggregate allocations \( \{N_t(z^t), C_t(z^t), B_{t+1}(z_t)\}_{t=0}^\infty \) and the sequence of distribution \( \{\theta^*_t(z^t)\}_{t=0}^\infty \) can be supported as a competitive equilibrium if and only if the following occur:

1. The resource constraint (8) holds.
2. The implementability constraint,

\[
Z(\theta_t^*(z^t)) - N_t(z^t) + B_{t+1}(z^t) \sum_{z_{t+1}} \beta U_{C}(z^{t+1})Z(\theta_{t+1}^*(z^{t+1}))L(\theta_t^*(z^t))\pi(z_{t+1}|z^t) \geq U_C(z^t)Z(\theta_t^*(z^t))B_t(z^{t-1}),
\]

holds for all \(z^t\) with \(t \geq 0\), where \(B_0(z^0) = B_0\) for all \(z_0\); \(Z(\theta_t^*)\), \(D(\theta_t^*)\), and \(U_C(z^t)\) are defined as before in Proposition 4.

3. The bond constraint,

\[
B_{t+1}(z^t) \sum_{z_{t+1}} \beta U_{C}(z^{t+1})Z(\theta_{t+1}^*(z^{t+1}))\pi(z_{t+1}|z^t) = \theta_t^*(z^t) (1 - D(\theta_t^*(z^t))),
\]

holds for all \(z^t\) with \(t \geq 0\) if \(\theta_t^*(z^t) < \theta_H\).

Moreover, we can impose the following ad hoc government-debt limits to facilitate our analysis of the role of government debt (as in AMSS)\(^{12}\):

\[
\underline{B} \leq B_{t+1}(z^t) \leq \overline{B} \quad \text{for all } t \geq 0 \text{ and } z^t.
\]

\(^{12}\)Notice that the government also faces a natural borrowing limit, which is defined as the present value of the maximum amount of tax revenue that the government can collect. For simplicity, we assume that the natural limit is above the ad hoc upper debt limit.

A.9.2 Ramsey Problem Without State-Contingent Bonds

The Ramsey problem can be represented as maximizing life-time utility (25) by choosing the sequences of \(\{N_t(z^t), C_t(z^t), \theta_t^*(z^t)\}\) subject to the constraints listed in Proposition 18. As a result,
the Lagrangian of the Ramsey problem is given by

\[
\max_{\{\theta_t^*(z^t), N_t(z^t), C_t(z^t), B_{t+1}(z^t)\}} \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \left[ W(\theta_t^*(z^t)) + \bar{\theta} \log C_t(z^t) - N_t(z^t) \right] \pi(z^t)
\]

(83)

\[+ \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \psi_t(z^t) \pi(z^t) \left( N_t(z^t) - C_t(z^t) - G_t(z^t) \right)\]

\[+ \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \phi_t(z^t) \pi(z^t) \left( B_{t+1}(z^t) \sum_{z_{t+1}} \beta U_C(z^{t+1}) Z(\theta_{t+1}^*(z^{t+1})) L(\theta_t^*(z^t)) \pi(z_{t+1}|z^t) \right) \]

\[+ \sum_{t=0}^{\infty} \sum_{z^t} \beta^t \mu_t^B(z^t) \pi(z^t) \left( B_{t+1}(z^t) \sum_{z_{t+1}} \beta U_C(z^{t+1}) Z(\theta_{t+1}^*(z^{t+1})) \pi(z_{t+1}|z^t) \right) \]

\[+ \sum_{t=0}^{\infty} \sum_{z^t} \beta^t v_t^L(z^t) \pi(z^t) (B_{t+1}(z^t) - B)\]

\[+ \sum_{t=0}^{\infty} \sum_{z^t} \beta^t v_t^H(z^t) \pi(z^t) (\bar{B} - B_{t+1}(z^t)),\]

where \(\beta^t \psi_t(z^t) \pi(z^t), \beta^t \phi_t(z^t) \pi(z^t), \beta^t v_t^L(z^t) \pi(z^t),\) and \(\beta^t v_t^H(z^t) \pi(z^t)\) denote the multipliers for the resource constraints, implementability conditions, and low and high boundary condition of \(\theta_t^*\), respectively.

### A.9.3 Ramsey Optimal Conditions without State-Contingent Bonds

The first-order conditions of Ramsey-planter problem (83) with respect to \(N_t(z^t), C_t(z^t), B_{t+1}(z^t),\) and \(\theta_t^*(z^t)\) for \(t \geq 0\) are given, respectively, by

\[1 + \phi_t(z^t) = \psi_t(z^t) \text{ for all } t \geq 0 \text{ and } z^t \]

(84)

\[\bar{\theta} U_C(z^0) - \phi_0(z^0) U_{CC}(z^0) Z(\theta_0^*(z^0)) B_0 = \psi_0(z^0) \text{ for all } z^0 \text{ at period 0} \]

(85)

\[\bar{\theta} U_C(z^t) + (\phi_{t-1}(z^{t-1}) L(\theta_{t-1}^*(z^{t-1})) - \phi_t(z^t) + \mu_{t-1}^B(z^{t-1}) U_{CC}(z^t) Z(\theta_t^*(z^t)) B_t(z^{t-1}) = \psi_t(z^t) \text{ for all } t \geq 1 \text{ and } z^t \]

(86)

\[\mu_t^B(z^t) = E_t \phi_{t+1}(z^{t+1}) U_C(z^{t+1}) Z(\theta_{t+1}^*(z^{t+1})) \frac{L(\theta_t^*(z^t))}{E_t U_C(z^{t+1}) Z(\theta_{t+1}^*(z^{t+1}))} - \phi_t(z^t) + \frac{v_t^H(z^t) - v_t^L(z^t)}{E_t \beta U_C(z^{t+1}) Z(\theta_{t+1}^*(z^{t+1}))} \]

(87)
\[ \frac{\partial W(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} + \phi_t(z^t) \left[ (1 - U_C(z^t) B_t(z^{t-1})) \frac{\partial Z(\theta^*_t(z^t))^\top}{\partial \theta^*_t(z^t)} + \frac{\partial L(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \theta^*_t(z^t) \right] = J(\theta^*_t(z^t)) - \mu^B_t(z^{t-1}) B_t(z^{t-1}) \frac{\partial Z(\theta^*_t(z^t))^\top}{\partial \theta^*_t(z^t)}, \]

where \( H(\theta^*_t) \) and \( J(\theta^*_t) \) are defined as in Lemma 16.

Plugging (84) into (85) and (86) gives (47) and (48), respectively. Under the assumption of \( v^L_t(z^t) = v^L_t(z^t) = 0 \), equation (49) is implied by (87). To show the condition (50), we plug (87) into (50) and get

\[
\frac{\partial W(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} + \phi_t(z^t) \left[ (1 - U_C(z^t) B_t(z^{t-1})) \frac{\partial Z(\theta^*_t(z^t))^\top}{\partial \theta^*_t(z^t)} + \frac{\partial L(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \theta^*_t(z^t) \right] = J(\theta^*_t(z^t)) \left( E_t \phi_{t+1}(z^{t+1}) U_t(z^{t+1}) Z(\theta^*_t(z^{t+1})) - \phi_t(z^t) L(\theta^*_t(z^t)) + \frac{v^H_t(z^t) - v^L_t(z^t)}{E_t \beta U_C(z^{t+1}) Z(\theta^*_t(z^{t+1}))} \right) \\
- (E_{t-1} \phi_t(z^t) - U_{t-1}(z^{t-1})) \frac{\partial Z(\theta^*_t(z^t))^\top}{\partial \theta^*_t(z^t)} = \frac{\partial Z(\theta^*_t(z^t))^\top}{\partial \theta^*_t(z^t)} \frac{\partial Z(\theta^*_t(z^t))^\top}{\partial \theta^*_t(z^t)}
\]

Rearranging the equation above leads to

\[
\frac{\partial W(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} + \phi_t(z^t) \left[ \frac{\partial Z(\theta^*_t(z^t))^\top}{\partial \theta^*_t(z^t)} - U_C(z^t) B_t(z^{t-1}) \frac{\partial Z(\theta^*_t(z^t))^\top}{\partial \theta^*_t(z^t)} \right] = F(\theta^*_t(z^t)) E_t \phi_{t+1}(z^{t+1}) U_t(z^{t+1}) Z(\theta^*_t(z^{t+1})) + F(\theta^*_t(z^t)) (v^H_t(z^t) - v^L_t(z^t)) \\
- (E_{t-1} \phi_t(z^t) - U_{t-1}(z^{t-1})) \frac{\partial Z(\theta^*_t(z^t))^\top}{\partial \theta^*_t(z^t)} = \frac{\partial Z(\theta^*_t(z^t))^\top}{\partial \theta^*_t(z^t)} \frac{\partial Z(\theta^*_t(z^t))^\top}{\partial \theta^*_t(z^t)}
\]

Applying Lemma 16, which proves that

\[
H(\theta^*_t) = J(\theta^*_t) = F(\theta^*_t) = \frac{\partial Z(\theta^*_t(z^t))^\top}{\partial \theta^*_t(z^t)} + \frac{\partial L(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \theta^*_t(z^t) \left[ 1 - D(\theta^*_t(z^t)) \right] + J(\theta^*_t(z^t)) L(\theta^*_t(z^t)),
\]
into the equation above gives

\[
\frac{\partial W(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} + \phi_t(z^t) \left[ F(\theta_t^*(z^t)) - U_C(z_t^t) B_t(z_t^{t-1}) \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \right] = F(\theta_t^*(z^t)) E_t \phi_{t+1}(z_t^{t+1}) \frac{U_C(z_t^{t+1}) Z(\theta_t^{t+1}(z_t^{t+1}))}{E_t U_C(z_t^{t+1}) Z(\theta_t^{t+1}(z_t^{t+1}))} + \frac{F(\theta_t^*(z^t)) (v_t^H(z_t^t) - v_t^L(z_t^t))}{E_t U_C(z_t^{t+1}) Z(\theta_t^{t+1}(z_t^{t+1}))} - \left( E_{t-1} \phi_t(z^t) \frac{U_C(z_t^t) Z(\theta_t^*(z_t^t))}{E_{t-1} U_C(z_t^t) Z(\theta_t^*(z_t^t))} + \frac{v_{t-1}^H(z_t^{t-1}) - v_{t-1}^L(z_t^{t-1})}{E_{t-1} \beta U_C(z_t^t) Z(\theta_t^*(z_t^t))} \right) B_t(z_t^{t-1}) U_C(z_t^t) \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)}. \]

Rearranging the equation above leads to

\[
\frac{\partial W(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \frac{1}{F(\theta_t^*(z^t))} + \phi_t(z^t) = E_t \phi_{t+1}(z_t^{t+1}) \frac{U_C(z_t^{t+1}) Z(\theta_t^{t+1}(z_t^{t+1}))}{E_t U_C(z_t^{t+1}) Z(\theta_t^{t+1}(z_t^{t+1}))} \frac{U_C(z_t^t)}{F(\theta_t^*(z^t))} B_t(z_t^{t-1}) \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} + \left( -E_{t-1} \phi_t(z_t^t) \frac{U_C(z_t^t) Z(\theta_t^*(z_t^t))}{E_{t-1} U_C(z_t^t) Z(\theta_t^*(z_t^t))} \right) \frac{U_C(z_t^t)}{F(\theta_t^*(z^t))} B_t(z_t^{t-1}) \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} - \frac{(v_t^H(z_t^t) - v_t^L(z_t^t))}{E_t \beta U_C(z_t^{t+1}) Z(\theta_t^{t+1}(z_t^{t+1}))} \frac{U_C(z_t^t)}{F(\theta_t^*(z^t))} B_t(z_t^{t-1}) \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)}.
\]

The condition (50) immediately follows under the assumption of nonbinding debt limits.

A.10 Proofs of Lemma 12, 13 and 14

**Proof of Lemma 12:** The proof of Proposition 7 (Appendix A.5) indicates that \( \phi_t \) is constant and \( \theta_t^* = \theta_H \) in the long run. A constant tax rate immediately follows given equation (51). Without aggregate uncertainty, the constant \( \phi_t \), denoted by \( \bar{\phi}_1 \), implies that \( B_t \) is determined by the following equation:

\[
B_t = \frac{\bar{\phi}_1}{(1 + \bar{\phi}_1)^2 1 - \beta} \frac{1}{1 + \bar{\phi}_1} - \frac{PV_t^G}{1 + \bar{\phi}_1}, \tag{90}
\]

where the present value of government spending, \( PV_t^G \), is defined as

\[
PV_t^G = G_t + \sum_{\tau=t+1}^{\infty} \beta^\tau G_\tau.
\]

Given that \( PV_t^G \) is constant when \( G_t = \bar{G} \), the bond supply \( B_t \) is also constant for all \( t > 0 \) and given by

\[
\overline{B}_t = \frac{\bar{\phi}_1}{(1 + \bar{\phi}_1)^2 1 - \beta} \frac{1}{1 + \bar{\phi}_1} - \frac{\bar{G}}{1 + \bar{\phi}_1} 1 - \beta \equiv \hat{B}.
\]

60
Given that $\hat{B}$ is financed by distortionary taxes, the optimal $\hat{B}$ is chosen to be the lowest optimal level of the bond supply, which is denoted by $B^* > 0$, to ensure full self-insurance (given that $B_0 < B^*$).

**Proof of Lemma 13:** The first part of the proof is identical to the proof of Lemma 12, which implies $\theta_t^* = \theta_H$ and a constant tax rate in the long run. The bond supply $B_t$ is determined by

$$B_t = \frac{\bar{\phi}_2}{(1 + \bar{\phi}_2)^2} \frac{1}{1 - \beta} - \frac{PV_t^G}{1 + \bar{\phi}_2},$$

where $\bar{\phi}_2$ is the value of $\phi_t$ in this case. Given that $PV_t^G$ is higher in periods with $G_t = G_H$, the bond supply $B_t$ is lower in $G_H$-periods. However, $\theta_t^* = \theta_H$ has to hold regardless of the value of $G$, which suggests that the lowest value of $B_t$ has to be no less than the $B^*$. Hence, the average $B_t$, denoted by $\bar{B}$, has to be strictly larger than $B^* = \hat{B}$. Given the higher average $B_t$, the tax rate has to be higher in order to satisfy government budget constraints.

**Proof of Lemma 14:** The proof is identical to that of Lemma 12 above.

### A.11 Proof of Proposition 15

The proof takes several steps:

**Step 1** Any allocation with a constant Lagrangian multiplier $\phi \geq 0$ and a cutoff $\theta_t^* \in (\theta_L, \theta_H)$ cannot be a Ramsey equilibrium. A constant $\phi_t = \phi$ implies that equation (50) becomes:

$$0 = \frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{1}{F(\theta_t^*)},$$

which cannot be true because $\theta_t^* \in (\theta_L, \theta_H)$ implies $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{1}{F(\theta_t^*)} > 0$.

**Step 2** Any allocation featuring full self-insurance with $\theta_t^* = \theta_H$ for $t \geq T$ ($T$ large) cannot be a Ramsey equilibrium. Suppose instead we have $\theta^* = \theta_H$ for $t \geq T$, then in the long run the system of equations (48)-(48) is reduced to

$$\bar{\theta}U_{C,t} = 1 + \phi_t,$$

$$E_t q_{t+1} \phi_{t+1} = \phi_t;$$

which features a STUR process (since $E_t q_{t+1} = 1$) and is identical to the AMSS model, which shows that a constant Ramsey allocation with $\phi > 0$ is impossible. It is also impossible for $\phi = 0$, because in that case the level of the bond supply must decrease to a non-positive
value, which cannot support the full self-insurance allocation. Hence, the STUR property of \( \phi_t \) implies that a positive probability exists that \( \phi_t \) will decrease permanently without bound, which also implies that the bond supply \( B_t \) will decrease to a non-positive value and render a full self-insurance allocation impossible.

**Step 3** Any allocation featuring \( B_t \leq 0 \) and \( \theta^*_t = \theta_L \) cannot be a Ramsey equilibrium. Notice that we have implicitly imposed a lower debt limit of 0 on the model under the assumption \( a_{t+1} \geq 0 \) for the households, which implies that aggregate bond supply cannot be negative. So, when \( B_t = 0 \) and \( \theta^*_t = \theta_L \), we have \( Z(\theta_L) = \bar{\theta}, L(\theta_L) = \bar{\theta}, \lim_{\theta^* \to \theta_L} \frac{\partial Z(\theta^*_t)}{\partial \theta^*_t} \frac{1}{F(\theta^*_t)} = \frac{\theta_L - \bar{\theta}}{\bar{\theta}} < 0 \), and \( \lim_{\theta^* \to \theta_L} \frac{\partial W(\theta^*_t)}{\partial \theta^*_t} \frac{1}{F(\theta^*_t)} = \frac{\bar{\theta} - \theta_L}{\bar{\theta}} > 0 \); so equation (50) is reduced to

\[
E_t q_{t+1} \phi_{t+1} = \phi_t + \frac{\bar{\theta} - \theta_L}{\theta_L},
\]  

(94)

where \( E_t q_{t+1} = 1 \). Thus, \( \phi_t \) follows a (STUR process with a positive drift \( \frac{\bar{\theta} - \theta_L}{\theta_L} > 0 \). This implies that \( \phi_t \) and \( B_t \) will increase and diverge to infinity stochastically, making \( B_{t+1} > 0 \) and \( \theta^*_t > \theta_L \) almost surely. This also rules out the possibility for \( \phi_t \) to converge to zero in the long run, which would violate equation (94).

**Step 4** The above three steps rule out any constant allocation as a Ramsey equilibrium. Now we can consider the possibility of a bounded stochastic Ramsey equilibrium. First, we show that if \( \theta^*_t < \theta_H \), the Lagrangian multiplier \( \phi_t \) has a tendency to increase stochastically over time, which implies that the bond supply also increases stochastically over time as long as \( \theta^*_t < \theta_H \); hence, the model will eventually converge to \( \theta^*_t = \theta_H \). Second, we show that the economy cannot stay at the full self-insurance position indefinitely; instead, \( \phi_t \) and \( B_t \) will wonder around and eventually make \( \theta^*_t < \theta_H \) again with positive probability. Consider equation (50) when \( \theta^*_t \) is close but not equal to \( \theta_H \):

\[
E_t q_{t+1} \phi_{t+1} = \phi_t + \Delta_t,
\]  

(95)

where

\[
\Delta_t = \frac{\partial W(\theta^*_t)}{\partial \theta^*_t} \cdot \frac{1}{F(\theta^*_t)} + (E_{t-1} q_t \phi_t - \phi_t) \cdot \frac{U_{C,t} B_t \partial Z(\theta^*_t)}{F(\theta^*_t)}.
\]  

(96)

Since \( E_t q_{t+1} = 1 \) (which by the law of iterated expectations also implies the unconditional mean \( E q_t = 1 \)), the equation indicates that \( \phi_t \) follows a stochastic unit root (STUR) process with a stochastic drift \( \Delta_t \). Because \( \lim_{\theta^* \to \theta_H} \frac{\partial W(\theta^*_t)}{\partial \theta^*_t} \frac{1}{F(\theta^*_t)} = \lim_{\theta^* \to \theta_H} \frac{\partial Z(\theta^*_t)}{\partial \theta^*_t} \frac{1}{F(\theta^*_t)} = 0 \), the drift vanishes (\( \Delta_t = 0 \)) if and only if \( \theta^*_t = \theta_H \). A STUR process behaves very much like a unit root process (see Granger and Swanson (1997)), so a STUR process with a drift will increase
(decrease) stochastically over time if the drift is stochastically positive (negative). Since we
have already proved above that $\phi_t$ cannot possibly converge to zero, the drift term $\Delta_t$ must
be stochastically positive; otherwise no Ramsey equilibrium exists if $\Delta_t$ is stochastically
negative. In fact, notice that the first term in equation (96) is strictly positive; also, the
second term in the equation has two components: the coefficient $\gamma_t \equiv (E_{t-1} q_{t-1} \phi_t - \phi_t)$
and the term $\frac{U_{C,t} B_t}{F(\sigma_1^t)} \frac{\partial Z(\sigma_1)}{\partial \sigma_1^t} > 0$ (since $\frac{\partial Z(\sigma_1)}{\partial \sigma_1^t}$ is positive for $\theta_1^t$ close to $\theta_H$). If we expand the system
around its last-period values in period $t-1$, the following linear approximation holds:

$$ E_t q_{t+1} \phi_{t+1} \approx E_t \left[ q_t \phi_t + q_t (\phi_{t+1} - \phi_t) + \phi_t (q_{t+1} - q_t) \right] $$

$$ = q_t \left( E_t \phi_{t+1} - \phi_t \right) + \phi_t. $$

The approximation becomes more and more accurate as the length of the time period shrinks.
Therefore equation (95) can be closely approximated as

$$ E_t \phi_{t+1} \approx \phi_t + \frac{\Delta_t}{q_t}. \quad (97) $$

This is a random-walk process with a unit root and a stochastic drift $\frac{\Delta_t}{q_t}$. The drift $\Delta_t$ can
be decomposed as

$$ \Delta_t = \delta_{1t} + \gamma_t \delta_{2t} $$

where $\delta_{1t} \equiv \frac{\partial W(\sigma_1^t)}{\partial \sigma_1^t} \frac{1}{F(\sigma_1^t)} > 0$, $\delta_{2t} \equiv \frac{U_{C,t} B_t}{F(\sigma_1^t)} \frac{\partial Z(\sigma_1)}{\partial \sigma_1^t} > 0$, and the coefficient

$$ \gamma_t \equiv (E_{t-1} q_t \phi_t - \phi_t) \approx q_{t-1} \left( E_{t-1} \phi_t - \phi_{t-1} \right) + \phi_{t-1} - \phi_t $$

$$ \approx q_{t-1} \left( \frac{\Delta_{t-1}}{q_{t-1}} \right) + \phi_{t-1} - \phi_t \approx \Delta_{t-1} - \frac{\Delta_{t-1}}{q_{t-1}} $$

$$ = \left( \frac{q_{t-1} - 1}{q_{t-1}} \right) \Delta_{t-1} = \frac{\varepsilon_{t-1}^q}{q_{t-1}} \Delta_{t-1}, \text{ where } E\varepsilon_{t-1}^q = 0; $$

hence, we can rewrite the drift as

$$ \Delta_t = \delta_{1t} + \left( \frac{\varepsilon_{t-1}^q}{q_{t-1}} \delta_{2t} \right) \Delta_{t-1}. $$

Thus, $\Delta_t$ is a stochastic first-order autoregressive process with a positive intercept $\delta_{1t}$ and
an autoregressive coefficient $\left( \frac{\varepsilon_{t-1}^q}{q_{t-1}} \delta_{2t} \right)$. The possibility that the autoregressive coefficient
$\left( \frac{\varepsilon_{t-1}^q}{q_{t-1}} \delta_{2t} \right) \leq -1$ can be ruled out because a divergent $\Delta_t$ with negative infinity is inconsistent
with the arguments in Step 3 above. So $\Delta_t$ must be stochastically positive and consequently, $\phi_t$ increases stochastically over time. This implies that eventually the bond supply will
become high enough such that a full self-insurance position is reached with \( \theta_t^* = \theta_H \). At this point, we have \( \Delta_t = 0 \) and \( \phi_t \) becomes a random walk (STUR) process without drift, as in the AMSS model. If \( \phi_t \) wonders downward and results in \( \theta_t^* < \theta_H \), the convergence process described above will resume or be turned on again. However, if \( \phi_t \) wonders upward, the supply of bonds will eventually reach the government’s natural borrowing limit. We now turn to this issue.

**Step 5** Since the government faces a natural borrowing limit, the optimal debt level cannot go to infinity. For the sake of argument, consider two possible cases where an ad hoc debt-limit constraint on the government’s capacity to issue debt is imposed: \( B_{t+1} \leq \overline{B} \).

5.1. We consider first the case where the upper debt limit \( \overline{B} \) is greater than \( B^* \)—the optimal debt level under constant government spending \( G_t = \overline{G} \) to achieve a full self-insurance allocation. This implies that \( \theta_t^* = \theta_H \) with probability \( P \in (0, 1) \). Denoting the Lagrangian multiplier associated with the debt-limit constraint as \( v_H(t) \), the Ramsey FOCs become

\[
\bar{U}_{C,t} + \left( \phi_{t-1} L(\theta_t^*) - \phi_t + \mu^B_t \right) U_{C,t} Z(\theta_t^*) = 1 + \phi_t \tag{98}
\]

\[
\mu^B_t = E_t q_{t+1} \phi_{t+1} - \phi_t L(\theta_t^*) + \frac{v_H t}{E_t \beta U_{C,t+1} Z(\theta_t^*)} \tag{99}
\]

\[
E_t q_{t+1} \phi_{t+1} = \phi_t + \Delta_t - \frac{v_H t}{E_t \beta U_{C,t+1} Z(\theta_t^*)} + \frac{v_H}{U_{C,t} B_t} \frac{U_{C,t} B_t \partial Z(\theta_t^*)}{\partial \theta_t^*} \tag{100}
\]

Now, consider a Ramsey allocation that is constantly switching between two possible states or regimes: (i) \( \theta_t^* = \theta_H \) and \( v_H \geq 0 \); (ii) \( \theta_t^* < \theta_H \) and \( v_H = 0 \). Under state (i), equation (100) is reduced to

\[
E_t q_{t+1} \phi_{t+1} = \phi_t - \frac{v_H}{E_t \beta U_{C,t+1}},
\]

which is a STUR process with a negative drift. Hence, \( \phi_t \) decreases over time, suggesting that optimal bond supply \( B_{t+1} \) decreases stochastically over time and moves away from the upper bound \( \overline{B} \), then \( v_H = 0 \) and the above equation becomes

\[
E_t q_{t+1} \phi_{t+1} = \phi_t;
\]

hence, \( \phi_{t+1} \) follows a STUR process. So \( \phi_{t+1} \) and \( B_{t+1} \) will either increase together stochastically (with positive probability) back to the situation with a binding debt limit, or continue to drift downward together until the system swings back to state (ii) where \( \theta_t^* < \theta_H \) and
\( v^H_t = 0 \) (with positive probability). Once in state (ii), equation (100) becomes
\[
E_t q_{t+1} \phi_{t+1} = \phi_t + \Delta_t, \tag{101}
\]
which is identical to the case analyzed in step 4, so \{\phi_{t+1}, B_{t+1}\} will increase stochastically and swing back to state (i).

5.2. The upper debt limit \( \overline{B} \) is smaller than \( B^* \)—the optimal debt level under constant government spending \( G_t = \overline{G} \) to achieve a full self-insurance allocation. This implies that \( \theta^*_t < \theta_H \) almost surely. Equation (100) remains the same as
\[
E_t q_{t+1} \phi_{t+1} + \frac{v^H_t}{E_t \beta U_{C,t+1} Z(\theta^*_{t+1})} + \frac{v^H_{t-1}}{E_{t-1} \beta U_{C,t} \partial Z(\theta^*_t)} F(\theta_t) \frac{\partial Z(\theta^*_t)}{\partial \theta^*_t},
\]
where \( \Delta_t \equiv \frac{\partial W(\theta^*_t)}{\partial \theta^*_t} \frac{1}{F(\theta_t)} + (E_{t-1} \phi_t - \phi_t) \frac{U_{C,t} B_t}{\beta U_{C,t} \partial Z(\theta^*_t)} \) is the stochastically positive drift analyzed before. Notice that the last term is stochastically positive if \( \theta^*_t \) is close to \( \theta_H \), so besides \( \Delta_t \) there is now an additional positive force to reinforce the tendency for \( \phi \) to increase and for \( B_t \) to increase toward the full self-insurance position; hence, only if the debt limit is currently binding with \( v^H_t > 0 \) or \( -\frac{v^H_{t-1}}{E_{t-1} \beta U_{C,t} \partial Z(\theta^*_t)} < 0 \), will there exist a negative force to counter the upward movement and push both \( \phi_{t+1} \) and \( B_{t+1} \) downward; but such downward pressure would relax the debt-limit constraint and eventually render \( v^H_t = 0 \). Therefore, the speed of convergence toward the debt limit \( \overline{B} \) is stronger if the debt-limit constraint has been binding in the past \( (v^H_{t-1} > 0) \), but the tendency to converge will be weakened only if the debt-limit constraint is currently binding, in which case \( \phi_t \) and \( B_t \) tend to fall. Also, if the system is too far below the attractor of full self-insurance, then \( v^H_t = v^H_{t-1} = 0 \), which leaves \( \Delta_t \) as the only drift to dictate the dynamics of \( \phi_t \). Therefore, if the debt limit \( \overline{B} \) is lower than required for full self-insurance in any time period, then this debt limit itself becomes the new point of attraction or center of gravity (in the place of the full self-insurance position \( \theta^H \)) for the economy to revert to from below.

Step 6 The above analysis suggests that the full self-insurance position characterized by \( \theta^*_t = \theta_H \) is an attractor (or center of gravity) that the Ramsey planner opts to revert to and stay around, unless a sufficiently low debt-limit \( \overline{B} \) exists to hinder or prevent \( \theta^*_t \) from reaching \( \theta_H \), in which case the debt limit \( \overline{B} \) itself will serve as the new attractor (or center of gravity). Therefore, regardless of the debt limit, if a Ramsey equilibrium exists, it must be a stochastic bounded process that converges toward a region instead of a point. The intuition is that public debt serves two important functions: (i) to smooth taxes by buffering the impact of...
aggregate uncertainty and (ii) to provide self-insurance for households to buffer idiosyncratic uncertainty. This suggests that the Ramsey planner has the tendency to issue as much debt as possible while respecting any upper debt limit $\overline{B}$. 
This online Appendix shows that the main message in our paper is robust even if we relax the household borrowing constraint in equation (4). To take advantage of the tractability of our model, we can use a finite-horizon version (with $t = 0, 1, 2, \ldots, T$) of our model (with risk-free debt) to conduct the analysis, and then let $T \rightarrow \infty$ to study limiting behavior under an infinite horizon. In this way, we can also analyze the terminal-period effect or the turnpike phenomenon of the optimal supply of public debt. The key change is a new form of household borrowing constraints:

$$a_{t+1} \geq -\bar{\alpha},$$

where the initial asset position $a_0$ and the borrowing limit $\bar{\alpha} \geq 0$ are exogenously given and assumed to be the same across households.

In the anticipation that $T$ is the end-of-life (last) period, it must be true that the last-period savings are $a_{T+1} = 0$ for all households and the last-period optimal bond supply is $B_{T+1} \left( z^T \right) = 0$ with price $Q_{T+1} \left( z^T \right) = 0$. This implies that if government bonds are non-state contingent, not only is the Ramsey planner unable to issue new debt to buffer against the last-period government spending shock $G_T$ and smooth the last-period tax rate $\tau_T$, but also that households are unable to borrow ($a_{T+1} = 0$) to buffer against the last-period idiosyncratic preference shock $\theta_T$.

To facilitate the analysis, we consider two scenarios: In scenario A, government spending is time-varying but deterministic; in scenario B, government spending is stochastic. We will show that the insight gained for the effect of $\bar{\alpha}$ on debt policy in scenario A can be applied to scenario B with aggregate uncertainty. Under scenario A, we have the following proposition:

**Proposition 19.** Without aggregate uncertainty, the dynamics of $\phi_{t+1}$ are given by the following equations:

$$\phi_1 = \phi_0 \left[ 1 + \frac{\bar{\alpha}}{B_0 + \bar{\alpha}} \Omega^A \left( \theta_0^* \right) \right] + \Delta^A \left( \theta_0^* \right) - \frac{\phi_0 B_0 U c_0}{F \left( \theta_0^* \right)},$$

$$\phi_{t+1} = \phi_t \left[ 1 + \frac{\bar{\alpha}}{B_t + \bar{\alpha}} \Omega^A \left( \theta_t^* \right) \right] + \Delta^A \left( \theta_t^* \right),$$

for $t \in 1, 2, \ldots, T - 2$, and the dynamics of $\phi_T$ in the last period are given by

$$\phi_T = \phi_{T-1} \left[ 1 + \frac{\bar{\alpha}}{B_{T-1} + \bar{\alpha}} \Omega^A \left( \theta_{T-1}^* \right) \right] + \Delta^A \left( \theta_{T-1}^* \right) + \Theta^A \left( \theta_{T-2}^*, \theta_{T-1}^* \right),$$
where

\[ \Omega^A (\theta^*_t) \equiv - \left( \frac{[1 - D(\theta^*_t)] \theta^*_t \partial L(\theta^*_t)}{F(\theta^*_t)} \right) \geq 0 \]

\[ \Delta^A (\theta^*_t) \equiv \frac{\partial W(\theta^*_t)}{\partial \theta^*_t} \frac{1}{F(\theta^*_t)} \geq 0 \]

\[ \Theta^A (\theta^*_{t-2}, \theta^*_{t-1}) \equiv \frac{\alpha \left( \phi_{T-1} - \phi_{T-2} L(\theta^*_{T-2}) \right)}{F(\theta^*_{T-1})} \frac{U_{C_{T-1}}}{F(\theta^*_{T-1})} \frac{\partial Z(\theta^*_{T-1})}{\partial \theta^*_{T-1}} \]

In addition, if \( \theta^*_t = \theta_H \), then \( \Omega^A (\theta^*_t) = \Delta^A (\theta^*_t) = 0 \); also, \( \Theta^A (\theta^*_{T-2}, \theta^*_{T-1}) = 0 \) if \( \theta^*_{T-1} = \theta_H \).

Equation (104) is the counterpart of equation (30) in Proposition 6, except here with an additional positive term \( \frac{\alpha \phi_t}{B_{t+1}} \Omega^A (\theta^*_t) > 0 \) (with equality if and only if \( \theta^*_t = \theta_H \)) in the autoregressive coefficient of \( \phi_t \). It shows that (i) the marginal benefit of an improved distribution, \( \frac{\partial W(\theta^*_t)}{\partial \theta^*_t} \frac{1}{F(\theta^*_t)} \), due to an increase in public debt \( B_{t+1} \) still exerts a positive force on the next period multiplier \( \phi_{t+1} \) as before, and (ii) allowing for household borrowing generates an additional positive force \( \frac{\alpha \phi_t}{B_{t+1}} \Omega^A (\theta^*_t) > 0 \) in the growth rate or the autoregressive root of \( \phi_{t+1} \), thus greatly amplifying and reinforcing the original upward trend in \( \phi_{t+1} \).

Equation (103) shows that the initial debt level \( B_0 \) negatively affects debt growth in period \( t = 1 \), because the higher \( B_0 \) is, the less need there is to increase the future bond supply to reach the full self-insurance allocation. This negative effect of the initial-period bond supply is short-lived and disappears in the law of motion for \( \phi_{t+1} \) for \( t \geq 1 \).

The intuition is as follows: Suppose the initial bond supply \( B_0 \) is already high enough to achieve full self-insurance in period \( t = 1 \), then \( \theta^*_1 = \theta_H \), \( \frac{\partial W(\theta^*_t)}{\partial \theta^*_t} \frac{1}{F(\theta^*_t)} + \frac{\alpha \phi_t}{B_{t+1}} \Omega^A (\theta^*_t) = 0 \); so there is no need to adjust the new bond supply \( B_2 \) in period \( t = 1 \). This also implies that \( B_2 \) is high enough to achieve full self-insurance in period \( t = 2 \); hence, \( \theta^*_2 = \theta_H \) and equation (104) implies \( \phi_3 = \phi_2 \); consequently, we have \( \phi_{t+1} = \phi_t = \phi_1 \) and \( \theta^*_t = \theta_H \) for \( t \leq T - 1 \). This also implies that the additional term in equation (105) \( \Theta^A (\theta^*_{T-2}, \theta^*_{T-1}) = 0 \), because \( \theta^*_{T-1} = \theta^*_{T-2} = \theta_H \).

However, if \( B_0 \) is sufficiently low such that \( \theta^*_1 < \theta_H \), then the Ramsey planner’s incentive to reach a full self-insurance allocation in \( t = 2 \) by increasing \( B_1 \) is stronger if it is easier for households to borrow—although this does not necessarily mean that it is optimal to adjust \( B_1 \) immediately to reach full self-insurance within one period, as the decision must consider the dynamic adjustment costs and trade offs.

In other words, a more relaxed borrowing constraint on households does not eliminate the Ramsey planner’s incentive for providing full self-insurance as long as the household borrowing constraints still bind with positive probability (or for some households). However, with the possibility of household borrowing (\( \overline{\pi} > 0 \)), the marginal benefit of increasing government bonds is
greater than in the case of $\overline{\alpha} = 0$. This happens because as argued before, an increase in the debt supply has not only the direct benefit of $\phi_t L(\theta^*_t)$, but also an indirect benefit $\frac{\partial Q_{t+1}}{\partial B_{t+1}} > 0$ through a positive change in bond prices (as shown before). However, in both the LS model and our benchmark model, the gross benefit of a bond-price change is $\frac{\partial Q_{t+1}}{\partial B_{t+1}} B_{t+1}$, where $B_{t+1}$ is the total bond supply. In the current model, the gross benefit is $\frac{\partial Q_{t+1}}{\partial B_{t+1}} (B_{t+1} + \overline{\alpha})$ because the higher bond price also increases the value of household borrowing since the gross savings of household are given by $(B_{t+1} + \overline{\alpha})$ (this can be seen by changing equation (102) to a gross borrowing constraint: $a_{t+1} + \overline{\alpha} \geq 0$). So the gross benefit of a bond-price change is $\frac{\partial Q_{t+1}}{\partial B_{t+1}} (B_{t+1} + \overline{\alpha})$, which is greater when $\overline{\alpha} > 0$. In this regard, the term $\frac{\partial \phi_t}{\partial B_{t+1}} \Omega (\theta^*_t)$ in equation (104) can be interpreted as the increased marginal benefit of the bond supply due to the additional bond-price effect from household borrowing $\overline{\alpha}$.

Therefore, equation (104) shows that when households can borrow $\overline{\alpha} > 0$, there still exists the need for more self-insurance so long as the borrowing constraints bind for some households (such that $\theta^*_t < \theta_H$). Consequently, the positive force behind debt growth (or the growth in $\phi_{t+1}$) not only exists but is even stronger and reinforced by household borrowing. This implies that the transition period toward a full self-insurance allocation can be reinforced and thus shortened by allowing households to borrow.

Nonetheless, the long-run equilibrium debt level can be negative if $\overline{\alpha}$ is large enough. This can be seen from the aggregate demand function of government bonds in this finite-horizon model:

$$Q_{t+1} B_{t+1} = \frac{1 - D(\theta^*_t)}{D(\theta^*_t)} C_t - Q_{t+1} \overline{\alpha}, \text{ for } t = 0, 1, 2, ..., T - 1, \quad (106)$$

which shows that, everything else equal, an increase in $\overline{\alpha}$ can eventually imply a negative level of public debt $B_{t+1}$ since $\frac{1-D(\theta^*_t)}{D(\theta^*_t)} C_t$ and $Q_{t+1} = \beta E_t \frac{1-n}{1-n+1} L(\theta^*_t)$ are both bounded. By the same argument, everything else equal, if the variance of idiosyncratic risk $\theta$ shrinks, then the same borrowing limit $\overline{\alpha}$ may imply a more negative optimal debt level $B_{t+1}$—because a smaller variance of $\theta$ implies it is easier to achieve full self-insurance with a lower debt level.

However, notice that these arguments are made based on the assumption that $\theta^*_0 < \theta_H$. If the borrowing limit $\overline{\alpha}$ is high enough such that any initial debt level $B_0$ can support a full self-insurance allocation (with $\theta^*_0 = \theta_H$), then the equilibrium bond supply $B_{t+1}$ will be determined by $B_0$ regardless it being positive or negative, as in the LS model.

Now consider the Ramsey planner’s decision for bond supply $B_T$ in the second to last period $T-1$ and the dynamics of $\phi_T$ between $T-1$ and $T$. There is now an additional term $\Theta_T^A \equiv \frac{\alpha}{(\phi_T - \phi_{T-2})} \frac{\partial V_C}{\partial (\theta^*_{T-1})} \frac{\partial Z}{\partial (\theta^*_{T-1})}$ appearing in equation (105) compared to equation (104). This additional term shows up in the FOC of $B_{T-1}$ because $T$ is the last period and no new bonds
\( B_{T+1} = 0 \) can be issued in the last period. This additional term \( \eta_T \) is positive (negative) if 
\[
\left[ \phi_{T-1} - \phi_{T-2}L(\theta_{T-2}^*) \right] > 0 \quad (< 0). 
\]
Since \( \left[ \phi_{T-1} - \phi_{T-2}L(\theta_{T-2}^*) \right] \) is precisely the net marginal utility cost of increasing the bond supply in the previous period \( T-2 \) when deciding \( B_{T-1} \), \( \Theta^A_T > 0 \) implies that it was not worth increasing the bond supply in the previous period; consequently, \( B_{T-1} \) is now too low to provide full self-insurance in period \( T-1 \); hence, if \( \Theta^A_T > 0 \), then the incentive to increase bond supply \( B_{T-1} \) is reinforced.

However, given that government spending in each period is fully anticipated, \( \Theta^A_T > 0 \) contradicts the logic that the Ramsey planner always opts to increase the bond supply whenever \( \theta^*_t < \theta_H \). Hence, it must be true that \( \Theta^A_T < 0 \). In this case, the implication is that it was optimal in the previous period \( T-2 \) to increase the bond supply \( B_{T-1} \); however, doing so would reduce the incentive to further increase public debt \( B_T \) in period \( T-1 \) because \( T \) is the last period and the terminal period bond level \( B_{T+1} \) must be zero. Therefore, equation (105) adds an additional consideration for the determination of public debt in period \( T-1 \), leading to a slowdown in the growth of public debt \( B_T \) (as well as the growth rate of \( \phi_T \)). But as \( T \to \infty \), this additional terminal concern vanishes and the law of motion for \( \phi_{t+1} \) will always be characterized by equation (104).

Now consider scenario B with aggregate uncertainty. For simplicity, assuming \( B_0 = 0 \), we have the following proposition:

**Proposition 20.** Under aggregate uncertainty, the dynamic path \( \{\phi_t\}_{t=0}^T \) is given by the following equations:

\[
E_0 q_1 \phi_1 = \phi_0 \left[ 1 + \frac{\alpha}{B_0 + \alpha} \Omega^B_0 \right] + \Delta^B_0 - \frac{\phi_0 B_0 U_C}{F(\theta^*_0)} \quad (107)
\]

\[
E_t q_{t+1} \phi_{t+1} = \phi_t \left[ 1 + \frac{\alpha}{B_t + \alpha} \Omega^B_t \right] + \Delta^B_t, \quad \text{for } t = 1, 2, ..., T-2 \quad (108)
\]

\[
E_{T-1} q_T \phi_T = \phi_{T-1} \left[ 1 + \frac{\alpha}{B_{T-1} + \alpha} \Omega^B_{T-1} \right] + \Delta^B_{T-1} + \Theta^B_T, \quad (109)
\]

where

\[
\Omega^B_t \left( z^t \right) \equiv - \left( \left[ 1 - D(\theta^*_t) \right] \frac{\theta^*_t}{F(\theta^*_t)} \right) \frac{\partial L(\theta^*_t)}{\partial \theta_t^*} \geq 0,
\]

\[
\Delta^B_t \left( z^t \right) \equiv \frac{\partial W(\theta^*_t)}{\partial \theta_t^*} \frac{1}{F(\theta^*_t)} + \left( E_{t-1} q_t \phi_t - \phi_t \right) \frac{U_{C,t} B_t}{F(\theta^*_t)} \frac{\partial Z(\theta^*_t)}{\partial \theta_t^*},
\]

\[
\Theta^B_T \equiv \frac{U_{C,T-1}}{F(\theta^*_T)} \frac{\partial Z(\theta^*_T)}{\partial \theta^*_T} \left[ E_{T-2} q_{T-1} \phi_{T-1} - \phi_{T-2} L(\theta^*_T) \right].
\]

Clearly, the law of motion for \( \phi_{t+1} \) under aggregate uncertainty is similar to that without aggregate uncertainty except that (i) there is now a STUR \( q_{t+1} \) and (ii) the terminal period has
an additional negative term $\Theta^B_T$ that is analogous to $\Theta^A_T$ in equation (105).

Therefore, we conclude that allowing households to borrow does not alter the basic insight from the previous analysis $(\alpha = 0)$: There exists a dominant positive force that induces the growth of public debt. The size of the variance of aggregate risk (for any given variance of idiosyncratic risk) only serves to make the path of convergence toward the full self-insurance allocation more stochastic and possibly more autocorrelated, but it does not by itself negatively affect the growth rate of public debt as long as the liquidity premium of public debt remains positive.

In this sense, there does not exist any competition between aggregate tax smoothing and individual consumption smoothing—suggesting a departure from tax smoothing—along the transition. In the model of AMSS, the Ramsey planner is shown to have a precautionary saving motive to hold private-sector issued bonds because there does not exist a bounded stationary Ramsey equilibrium except in the special case where $\lim_{t \to \infty} \phi_{t+1} = 0$, so that $\lim_{t \to \infty} t = 0$ and $\lim_{t \to \infty} B_{t+1} < 0$. Such Ramsey equilibrium is destroyed (eliminated) in our model by the Ramsey planner’s incentives to supply debt to provide full self-insurance.

**B.1 Proof of Proposition 19 and 20**

We start with the proof of Proposition 20 since Proposition 19 is just a special case of the former without aggregate risk.

**B.1.1 Competitive Equilibrium**

First, the household decision rules given in Proposition 2 are modified in the following way. For $t < T$, we have

\[
x_t(z^t) = \begin{cases} 
\bar{w}_t(z^t)L(\theta^*_t(z^t))\theta^*_t(z^t) - Q_{t+1}(z^{t+1}) & \text{if } \theta^*_t(z^t) < \theta_H \\
\frac{Q_{t+1}(z^{t+1})B_{t+1}(z^{t+1})}{1-D(\theta_H)} & \text{if } \theta^*_t(z^t) = \theta_H \end{cases},
\]

(110)

\[
c_t(\theta_t, z^t) = \min \left\{ 1, \frac{\theta_t}{\theta^*_t(z^t)} \right\} \left( x_t(z^t) + Q_{t+1}(z^{t+1}) \alpha \right),
\]

(111)

\[
Q_{t+1}(z^t) a_{t+1}(\theta^t, z^t) = \max \left\{ 0, \frac{\theta^*_t(z^t) - \theta_t}{\theta^*_t(z^t)} \right\} \left( x_t(z^t) + Q_{t+1}(z^{t+1}) \alpha \right),
\]

(112)

and

\[
n_t(\theta_{t-1}, z^t) = \frac{1}{\bar{w}_t(z^t)} \left[ x_t(z^t) - a_t(z_t) \theta_{t-1}, z^{t-1} \right].
\]

(113)

For $t = T$, $c_T(z^T) = x_T(z^T) = \bar{\theta} \bar{w}_T(z^T)$, $Q_{T+1}(z^T) = a_{T+1}(z^T) = 0$, and $n_T(\theta_{T-1}, z^T) = \frac{1}{\bar{w}_T(z^T)} \left[ \bar{\theta} \bar{w}_T(z^T) - a_T(\theta_{T-1}, z^{T-1}) \right]$. 

5
Given these individual decision rules, the aggregate quantities are given by

\[ C_t(z^t) = D \left( \theta_t^* (z^t) \right) \left( x_t(z^t) + Q_{t+1}(z^t) \right) \]
\[ = D(\theta_t^* (z^t)) \bar{w}(z^t) \theta_t^* (z^t) L \left( \theta_t^* (z^t) \right) = Z \left( \theta_t^* (z^t) \right) \bar{w}_t(z^t), \text{ for } t < T \]

\[ C_T(z^T) = \bar{\theta} \bar{w}_T(z^T) \]

\[ Q_{t+1}(z^t) B_{t+1}(z^t) = \left[ 1 - D \left( \theta_t^* (z^t) \right) \right] \left( x_t(z^t) + Q_{t+1}(z^t) \right) - Q_{t+1}(z^t) \bar{\alpha} \]
\[ = \frac{1 - D(\theta_t^* (z^t))}{D(\theta_t^* (z^t))} C_t(z^t) - Q_{t+1}(z^t) \bar{\alpha}, \text{ for } t < T. \]

Second, the equilibrium prices are given by

\[ Q_{t+1} \left( z^t \right) = \beta E_t \left( \frac{\bar{w}_t}{\bar{w}_{t+1}} \right) L(\theta_t^*), \text{ for } t < T, \]
\[ Q_{T+1}(z^T) = 0, \text{ for } t = T, \]

where the liquidity premium \( L(\theta_t^*) \) is the same as before, so we have

\[ \bar{w}_t = \frac{C_t}{D(\theta_t^*) \bar{w}(\theta_t^*)} \equiv \frac{C_t}{Z(\theta_t^*)}, \text{ for } t < T \]
\[ \bar{w}_T = \frac{C_T}{Z(\theta_T^*)}, \text{ for } t = T, \]

\[ Q_{t+1} \left( z^t \right) = \begin{cases} 
\beta E_t \frac{Z(\theta_{t+1}^*(z^{t+1}))}{Z(\theta_t^*(z^t))} \frac{C_{t+1}(z^{t+1})}{C_t(z^t)} L(\theta_t^* (z^t)), & \text{for } t < T - 1 \\
\beta E_t \frac{Z(\theta_{t+1}^*(z^{t+1}))}{Z(\theta_t^*(z^t))} \frac{C_{t+1}(z^{t+1})}{C_t(z^t)} L(\theta_t^* (z^t)), & \text{for } t = T - 1 
\end{cases}, \]

and \( Q_{T+1}(z^T) = 0. \)

**B.1.2 Conditions to Support Competitive Equilibrium**

The implementability conditions are modified to

\[ Z \left( \theta_t^* (z^t) \right) - N_t(z^t) + \beta E_t Z \left( \theta_{t+1}^* \right) U_C(z^{t+1}) L \left( \theta_t^* (z^t) \right) B_{t+1}(z^t) \]
\[ \leq B_t(z^{t-1}) Z \left( \theta_t^* (z^t) \right) U_C(z^t) \text{ for } t < T - 1, \]

\[ Z \left( \theta_t^* (z^t) \right) - N_t(z^t) + \beta E_t \bar{U}_C(z^{t+1}) L \left( \theta_t^* (z^t) \right) B_{t+1}(z^t) \]
\[ \leq B_t(z^{t-1}) Z \left( \theta_t^* (z^t) \right) U_C(z^t) \text{ for } t = T - 1, \]
and

\[ \bar{\theta} - N_T(z^T) \leq B_T(z^{T-1})\bar{\theta}U_C(z^T) \text{ for } t = T. \]

The asset market-clearing condition and resource constraint remain unchanged.

### B.1.3 Ramsey Solution

For simplicity, substituting \( N_t \) with \( C_t \), substituting \( G_t \) with the resource constraint, and assuming that the debt limit never binds, the Lagrangian of the Ramsey problem is given by

\[
\begin{align*}
\max_{\{\theta_t(z^t), N_t(z^t), C_t(z^t), B_{t+1}(z^t)\}} \quad & E_0 \sum_{t=0}^{T} \beta^t \left[ W(\theta^*_t(z^t)) + \bar{\theta} \log C_t(z^t) \right. \\
+ & \left. Z(\theta^*_t(z^t)) - C_t(z^t) - G_t(z^t) \right] \\
+ & \sum_{t=0}^{T-1} \sum_{z^t} \beta^t \phi_t(z^t) \pi(z^t) \left( E_{t+1}(z^{t+1}) + B_{t+1}(z^{t+1}) \right) \left( \sum_{z^t} \beta^t \phi_t(z^t) \pi(z^t) \right)
\end{align*}
\]

where \( Z(\theta^*_t(z^T)) = \bar{\theta} \).

The Ramsey FOCs with respect to \( C \) are given as follows. For \( t = 0 \),

\[ \bar{\theta}U_C(z^0) = 1 + \phi_0(z^0) + \phi_0(z^0)B_0 \left( \theta^*_0(z^0) \right) U_{CC}(z^0), \]

and for \( t = 1, \ldots, T \),

\[
\bar{\theta}U_C(z^t) = 1 + \phi_t(z^t) + \left[ \phi_t(z^t) \left( \frac{B_t(z^{t-1})}{(B_t(z^{t-1}) + \sigma)} \right) \right. \\
- \left. \phi_{t-1}(z^{t-1}) \left( \theta^*_t(z^{t-1}) \right) \frac{B_t(z^{t-1})}{(B_t(z^{t-1}) + \sigma)} \right] \left[ \frac{B_t(z^{t-1})}{(B_t(z^{t-1}) + \sigma)} \right] \\
\times Z(\theta^*_t(z^t)) \left( \theta^*_t(z^t) \right) U_{CC}(z^t) \left( B_t(z^{t-1}) + \sigma \right)
\]

The Ramsey FOCs with respect to \( B \) are as follows. For \( t = 0, 1, \ldots, T - 1 \), we have

\[
\phi_t(z^t) \left( \theta^*_t(z^t) \right) L(\theta^*_t(z^t)) - \frac{E_t\phi_{t+1}(z^{t+1})U_C(z^{t+1})Z(\theta^*_{t+1}(z^{t+1}))}{E_tU_C(z^{t+1})Z(\theta^*_{t+1}(z^{t+1}))} + \mu^B_t(z^t) = 0.
\]
The Ramsey FOCs with respect to $\theta^*_t$ are given as follows. For $t = 0$, we have

$$\frac{\partial W(\theta^*_0(z^0))}{\partial \theta^*_0(z^0)} + \phi_0(z^0) \left[ (1 - U_C(z^0)B_0) \frac{\partial Z(\theta^*_0(z^0))}{\partial \theta^*_0(z^0)} + \frac{\partial L(\theta^*_0(z^0))}{\partial \theta^*_0(z^0)}B_1(z^0)E_0 \beta U_C(z^1)Z(\theta^*_1(z^1)) \right] = J(\theta^*_0(z^0)) \mu^B_0(z^0);$$

and for period $t = 1, \ldots T - 1$, we have

$$\frac{\partial W(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} + \phi_t(z^t) \left[ (1 - U_C(z^t)B_t(z^{t-1})) \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)}L(\theta^*_t(z^{t-1}))B_t(z^{t-1}) + \frac{\partial L(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)}B_{t+1}(z^t)E_t \beta U_C(z^{t+1})Z(\theta^*_t(z^{t+1})) \right] = J(\theta^*_t(z^t)) \mu^B_t(z^t) - \mu^B_{t-1}(z^{t-1}) (B_t(z^{t-1}) + \alpha) U_C(z^t) \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)}.$$

Following the same steps as in Appendix A.9, combing equations (117) and (119) together by utilizing Lemma 16 gives

$$E_t \phi_{t+1}(z^{t+1}) \frac{U_C(z^{t+1})Z(\theta^*_t(z^{t+1}))}{E_t U_C(z^{t+1})Z(\theta^*_t(z^{t+1}))} = \frac{\partial W(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)} \left( \frac{1}{F(\theta^*_t(z^t))} + \left[ 1 + \frac{\alpha}{B_t + \alpha} \Omega(\theta^*_t) \right] \phi_t(z^t) \right) - \left( -E_{t-1} \phi_t(z^t) \frac{U_C(z^t)Z(\theta^*_t(z^t))}{E_{t-1} U_C(z^t)Z(\theta^*_t(z^t))} \right) \frac{U_C(z^t)}{F(\theta^*_t(z^t))} B_t(z^{t-1}) \frac{\partial Z(\theta^*_t(z^t))}{\partial \theta^*_t(z^t)},$$

which is equation (108). Similarly, equation (108) and (107) can be obtained by combing equations (117) and (119) or (118) in period $T - 1$ and period 0, respectively.

For the proof of Proposition 19, it is straight forward to verify that equations (107) to (109) are reduced to equations (103) and (105), respectively, in the case without aggregate uncertainty. Moreover, $\Omega^A(\theta^*_t) \geq 0$ given that $\frac{\partial L(\theta^*_t)}{\partial \theta^*_t} \leq 0$ and $D(\theta^*_t) \leq 1, \Delta^A(\theta^*_t) \geq 0$ is implied by Lemma 5. Also, if $\theta^*_t = \theta_H$, then $\frac{\partial L(\theta^*_t)}{\partial \theta^*_t} = 0$ and $\frac{\partial W(\theta^*_t)}{\partial \theta^*_t} = 0$. Hence, $\Omega^A(\theta^*_t) = 0$ and $\Delta^A(\theta^*_t) = 0$ if $\theta^*_t = \theta_H$. Finally, if $\theta^*_t = \theta_H$, $\frac{\partial Z(\theta^*_t)}{\partial \theta^*_t} = 0$ and, consequently, $\Theta^A(\theta^*_t) = 0.$