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Authors	YiLi Chien, and Yi Wen
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Federal Reserve Bank of St. Louis, Research Division, P.O. Box 442, St. Louis, MO 63166

The Determination of Public Debt under both Aggregate and Idiosyncratic Uncertainty*

YiLi Chien Yi Wen

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Abstract

We use an analytically tractable, heterogeneous-agent incomplete-markets model to show that the Ramsey planner's decision to finance stochastic public expenditures implies a departure from tax smoothing and an endogenous mean-reverting force to support positive debt growth despite the government's precautionary saving motives. Specifically, the government's attempt to balance the competing incentives between its own precautionary saving (tax smoothing) and households' precautionary saving (individual consumption smoothing)—even at the cost of extra tax distortion—implies an endogenous, soft lower bound on the stochastic unit-root dynamics of optimal taxes and public debt.

JEL Classification: E13; E62; H21; H30

Key Words: Optimal Public Debt, Tax Smoothing, Ramsey Problem, Incomplete Markets

*The views expressed are those of the individual authors and do not necessarily reflect the official positions of the Federal Reserve Bank of St. Louis, the Federal Reserve System, or the Board of Governors. Yili Chien is affiliated with the Federal Reserve Bank of St. Louis, P.O. Box 442, St. Louis, MO 63166-0442; email: yilichien@gmail.com. Yi Wen is affiliated with Antai College of Economics and Management, Shanghai Jiaotong University, Shanghai, China; email: wenyi001@sjtu.edu.cn.

1 Introduction

How to best finance unpredictable government spending is a long-standing issue in the history of political economy. Hume (1777) observed that it has been common practice since antiquity for a government to engage in precautionary saving to insure itself against unforeseen expenditures (such as wars) to avoid volatile and spontaneous taxation.

Barro (1979) argues that a rational and benevolent government should issue public debt in a way such that changes in public debt comove with government spending shock and taxes are smoothed like a random walk—analogous to consumption smoothing in a permanent-income consumption model. Lucas and Stokey (1983) (LS hereafter) cast this public-financing issue as a standard Ramsey problem in a simple representative-consumer model and show that it is optimal to smooth taxes in such a way that they inherit only the serial correlation structure of government expenditures when the financial asset is state-contingent; consequently, the optimal tax rate should be a constant if the government spending shock is iid or the household utility function exhibits constant elasticity of substitution (or constant relative risk aversion). Aiyagari, Marcet, Sargent, and Seppala (2002) (AMSS henceforth) extend the LS model to a setting with risk-free government bonds and confirm Barro's conjecture that an optimal tax rate has a unit-root component regardless of the serial correlation structure of government expenditures—fully analogous to optimal consumption behavior in a standard permanent-income consumption model with non-state-contingent financial assets. However, they also show that without ad hoc limits on the government's minimum asset holdings, the Ramsey outcome can diverge in important ways from Barro's.

This literature, nonetheless, has abstracted from the issue of possible conflict between the saving motives of the government and those governed. From the work of AMSS, we see that the government (typically) has an incentive to reduce government debt by accumulating private assets to finance random government expenditures. Thus, it may happen that the government accumulates so many assets in the long run (building a "war chest") so that all government expenditures are financed from the interest income on private assets, without having to resort to distortionary taxes at all. In other words, the optimal quantity of government debt can be negative in the long run, which is not observed in the long historical data. Throughout recorded human history, the public debt-to-GDP ratio has always been positive and the growth rate of public debt strongly positively correlated with government expenditures (Barro (1979)).

In this paper we argue that the precautionary saving of the government can interact with the precautionary saving of heterogeneous households in incomplete-market economies such that the two incentives counteract each other, generating a balancing force to support public debt growth and a departure from tax smoothing—a fundamental insight that can be traced back to the work of Shin (2006).

Specifically, we introduce heterogeneous agents and uninsurable idiosyncratic risk into the models of LS and AMSS and show that under quasi-linear preference (as in the model of indivisible labor), the Ramsey problem is analytically tractable despite both aggregate and idiosyncratic uncertainty. We use the tractable model to show explicitly that when government debt is not only a means to smooth taxes when financing random government expenditures but also a critical form of public liquidity (store of value) to smooth the private consumption of individual citizens, an interesting dynamic path emerges such that a departure from tax smoothing becomes optimal; and more importantly, the long-term dynamic behaviors of optimal government debt and taxes exhibit mean-reverting behaviors that render the random-walk dynamics in the representative-agent model of AMSS endogenously bounded below by the private sector's self-insurance demand, thus bringing the predicted time series of taxes and the bond-to-output ratio into closer conformity with the data without the need to impose ad hoc limits on the government's minimum capacity to issue debt.

The key reason is that the demand for self-insurance by the private sector builds into an endogenous force to support public debt growth, so the Ramsey planner has little incentive to reduce the government bond supply below an optimal (stochastic) target on the debt-to-output ratio. More specifically, when the market interest rate lies below the time discount rate—because of a liquidity premium under incomplete financial markets, the Ramsey planner has a dominant incentive to keep issuing public debt to provide full self-insurance (FSI) to consumers even at the cost of frequent deviations from tax smoothing.[1] Such a dominant incentive of the government derives from an intertemporal arbitrage opportunity for debt accumulation when the discounted future interest payment is less than the current market price of debt. This suggests that while the introduction of non-state-contingent bonds into the LS model (as studied by AMSS) serves to increase the variance and autocovariance of taxes and debt by adding a stochastic unit-root component—because in this situation government bonds are no longer capable of perfectly buffering aggregate spending shocks for any serial correlation structure of government expenditures—the introduction of idiosyncratic uncertainty serves to render the long-run level of public debt endogenously bounded below despite the presence of a unit-root component in the dynamics. Consequently, the tension between tax smoothing and a deviation from tax smoothing results in "mean-reverting" dynamics of optimal taxes and the debt-to-output ratio.[2] Thus, our analysis offers not only a rationale for the stochastic property of taxes observed by Barro (1979), but also a rationale (consistent with conventional

[1]FSI is defined as the situation where no one is borrowing constrained and everyone is fully insured against idiosyncratic risk.

[2]Our model features not only a well-defined interior Ramsey steady state (in the absence of aggregate uncertainty), but also an interior steady state that is stable in the sense that a deviation from it will trigger endogenous changes in the optimal supply of public debt to push the allocation back to the steady state. We call this a "mean-reverting" behavior in this paper.

wisdom) of the large quantity of public debt across nations and over time in the face of random government expenditures.

The rest of the paper is organized as follows. Section 2 extends the models of LS and AMSS to a setting with heterogeneous agents and studies the model's competitive-equilibrium properties. Section 3 shows how to solve the Ramsey problem analytically using the primal approach and compares the results with those of LS and AMSS, respectively. Section 4 provides numerical analyses to confirm our theoretical results and demonstrates transitional dynamics. Section 5 contains a brief literature review. Finally, Section 6 concludes the paper.

2 The Model

Our heterogeneous-agent approach follows Bewley (1980), Huggett (1993), and Aiyagari (1994). To make our model with both aggregate and idiosyncratic uncertainty tractable, we adopt the modeling strategy of Wen (2015) by assuming quasi-linear preferences; so the analytical tractability of our model is the consequence of three assumptions: (i) Household utility is log-linear (as in the case of indivisible labor); (ii) the source of idiosyncratic risk is from the marginal utility of consumption (as in the heterogeneous-agent cash-in-advance model of Lucas (1980)); and (iii) household decisions are made after any aggregate shock is realized, but the labor supply must be chosen before observing the idiosyncratic preference shock so that the precautionary saving motive is well preserved despite quasi-linear preferences.

An important implication of these assumptions is that households' marginal utility is no longer a martingale process; hence, as long as the support of idiosyncratic shocks is bounded, household demand for government bonds is always finite even when the interest rate equals the time discount rate; this is in sharp contrast to the original Aiyagari (1994) model, where the demand for bonds goes to infinity when the interest rate approaches the time discount rate. This property renders our model more suitable for the analysis of a Ramsey problem under aggregate uncertainty, not only because an interior Ramsey steady state can be ensured to exist, but also because the Ramsey first-order conditions (FOCs) have closed-form expressions. Without the existence of a well-defined Ramsey steady state and closed-form FOCs, it is difficult (if not impossible) to fully characterize a Ramsey allocation in an infinite-horizon heterogeneous-agent incomplete-markets (HAIM) model under aggregate uncertainty.

2.1 Environment

Time is discrete and indexed by $t = 0, 1, 2, \ldots$. Aggregate government spending $G_t(z^t)$ is stochastic, where z_t denotes the period-t aggregate state of the shock and $z^t = (z_0, \ldots, z_t)$ denotes the history of the shock. The spending shock is assumed to be covariance stationary and follow a Markov process with a finite number of states. The probability of a shock event, z_t, is denoted by $\pi(z_t)$, and the probability of the history of events, z^t, is denoted by $\pi(z^t)$.

There is a unit measure of ex ante identical households that face idiosyncratic preference shocks θ_t in period t. Let $\theta^t \equiv (\theta_0, \ldots, \theta_t)$ denote the history of preference shocks up to period t. The preference shock is iid over time and across households with the cumulative distribution $\mathbf{F}(\theta)$ and support $[\theta_L, \theta_H]$, where $0 < \theta_L < \theta_H < \infty$, and the mean is denoted by $\overline{\theta} \equiv \int \theta d\mathbf{F}(\theta)$.

Within each period t there are two subperiods. The aggregate spending shock is realized in the beginning of the first subperiod, and the idiosyncratic preference shock θ_t is realized only in the beginning of the second subperiod. The decision for the labor supply must be made in the first subperiod before observing θ_t, and the decisions for consumption and saving are made in the second subperiod after the realization of θ_t.

As in LS, a representative firm produces output according to a linear production technology in labor,

$$Y_t(z^t) = N_t(z^t),$$

where $N_t(z^t)$ denotes the aggregate labor input. The firm hires labor from households by paying a competitive real wage rate denoted by $w_t(z^t)$. Perfect competition implies

$$w_t(z^t) = \frac{\partial Y_t(z^t)}{\partial N_t(z^t)} = 1. \tag{1}$$

Households are infinitely lived with lifetime expected utility

$$V = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \theta_t \log c_t\left(\theta^t, z^t\right) - n_t\left(\theta^{t-1}, z^t\right) \right\}, \tag{2}$$

where $\beta \in (0,1)$ is the discount factor, and $c_t(\theta^t, z^t)$ and $n_t(\theta^{t-1}, z^t)$ denote consumption and the labor supply, respectively, for given shock histories at time t. Initially, all households are endowed with the same amount of bond holdings in period 0.

Note that the labor supply in period t is only measurable with respect to (θ^{t-1}, z^t), reflecting the assumption that the labor supply decision in period t is made before observing the preference shock θ_t. This special timing arrangement is designed to make the idiosyncratic preference shock uninsurable despite the constant marginal utility cost of labor. In other words, this timing assumption is not needed if the utility function is strictly concave in leisure.[3]

In this economy, the asset markets may be incomplete not only in terms of aggregate risk but also in terms of idiosyncratic risk. In other words, households and the government can trade risk-free bonds that are contingent neither on the aggregate states nor on individual states. Stochastic government spending $G_t(z^t)$ is financed by a flat labor-income tax rate $\tau_t(z^t)$ and risk-free government bonds $B_{t+1}(z^t)$. The period-t price of bonds is denoted by $Q_{t+1}(z^t) \leq 1$, which pays one unit of the consumption good in period $t+1$ regardless of the realization of z_{t+1}. The flow government budget constraint is given by

$$\tau_t(z^t)N_t(z^t) + Q_{t+1}(z^t)B_{t+1}(z^t) \geq G_t(z^t) + B_t(z^{t-1}) \tag{3}$$

for all $t \geq 0$, where the initial level of government bonds B_0 is exogenously given.

2.2 Household Problem

Given sequences of the after-tax wage rates, $\{\overline{w}_t(z^t) \equiv 1 - \tau_t(z^t)\}_{t=0}^{\infty}$, and bond prices, $\{Q_{t+1}(z^t)\}_{t=0}^{\infty}$, households solve their utility-maximizing problem by choosing a plan of consumption, labor, and asset holdings as follows:

$$\max_{\{c_t(\theta^t,z^t),n_t(\theta^{t-1},z^t),a_{t+1}(\theta^t,z^t)\}} E_0 \sum_{t=0}^{\infty} \beta^t \left\{\theta_t \log c_t\left(\theta^t, z^t\right) - n_t\left(\theta^{t-1}, z^t\right)\right\}$$

subject to the household budget constraint,

$$c_t\left(\theta^t, z^t\right) + Q_{t+1}(z^t)a_{t+1}(\theta^t, z^t) \leq a_t(\theta^{t-1}, z^{t-1}) + \overline{w}_t(z^t)n_t\left(\theta^{t-1}, z^t\right), \tag{4}$$

and the borrowing constraint,

$$a_{t+1}(\theta^t, z^t) \geq 0, \tag{5}$$

where the initial asset holdings $a_0 \geq 0$ are given and assumed to be identical across households. The individual labor supply $n_t\left(\theta^{t-1}, z^t\right)$ is bounded in the close interval of $[0, \overline{N}]$ with $\overline{N} < \infty$.

Denoting a household's cash on hand by

$$x_t(\theta^{t-1}, z^t) \equiv a_t(\theta^{t-1}, z^{t-1}) + \overline{w}_t(z^t)n_t(\theta^{t-1}, z^t), \tag{6}$$

the following proposition shows that the household-decision rules are characterized by a cutoff

[3]However, the model becomes intractable when the utility function is strictly concave in both consumption and leisure.

$\theta_t^*(z^t)$ strategy:

Proposition 1. ***(Household-Decision Rules)***

The optimal consumption $c_t(\theta^t, z^t)$ *and gross savings* $a_{t+1}(\theta^t, z^t)$ *are proportional to cash on hand* $x_t(z^t)$*:*

$$c_t(\theta_t, z^t) = \min\left\{1, \frac{\theta_t}{\theta_t^*(z^t)}\right\} x_t(z^t), \tag{7}$$

$$a_{t+1}\left(\theta_t, z^t\right) = \frac{1}{Q_{t+1}(z^t)} \max\left\{0, \frac{\theta_t^*(z^t) - \theta_t}{\theta_t^*(z^t)}\right\} x_t(z^t), \tag{8}$$

where the marginal propensity to consume, $\min\left\{1, \frac{\theta_t}{\theta_t^*(z^t)}\right\}$*, is a kinked concave function characterized by the cutoff* $\theta_t^*(z^t)$*, which is independent of the history of idiosyncratic shocks* θ^t*; and similarly the optimal cash on hand* $x_t(z^t)$ *depends on only the aggregate state:*

$$x_t(z^t) = \overline{w}_t(z^t) L(\theta_t^*(z^t)) \theta_t^*(z^t), \tag{9}$$

where the function $L(\theta_t^*)$ *denotes the liquidity premium of government bonds:*

$$L(\theta_t^*(z^t)) \equiv \int_{\theta_L}^{\theta_t^*(z^t)} d\mathbf{F}(\theta) + \int_{\theta_t^*(z^t)}^{\theta_H} \frac{\theta_t}{\theta_t^*(z^t)} d\mathbf{F}(\theta) \geq 1. \tag{10}$$

In addition, the bond price $Q_{t+1}(z^t)$ *is determined by the Euler equation*

$$Q_{t+1}(z^t) = \beta E_t \frac{\overline{w}_t(z^t)}{\overline{w}_{t+1}(z^{t+1})} L(\theta_t^*(z^t)), \tag{11}$$

and the cutoff $\theta_t^*(z^t)$ *can be expressed as*

$$\theta_t^*(z^t) = \beta \frac{x_t(z^t)}{Q_{t+1}(z^t)} E_t \frac{1}{\overline{w}_{t+1}(z^{t+1})}. \tag{12}$$

Proof. See Appendix A.1 □

Notice that the cutoff $\theta_t^*(z^t) \in [\theta_L, \theta_H]$ is unique and is a sufficient statistic to characterize the distribution of household allocations in the economy. This property hinges on our assumptions of (i) a log-linear utility function, (ii) iid idiosyncratic preference shocks, and (iii) the labor supply being predetermined in the first subperiod of t. These assumptions make the optimal distribution of cash on hand $x_t(z^t)$ in each period independent of the history of idiosyncratic shocks θ^t, as revealed in equation (9), which states that $x_t(z^t)$ is independent of θ^t if and only if $\theta_t^*(z^t)$ is independent of θ^t.

The intuition is that cash on hand $x_t(z^t)$ is predetermined by the optimal labor choice in each period t without observing θ_t. Since the marginal cost of leisure is constant, labor supply $n_t(\theta_{t-1}, z^t)$ is chosen such that the target level of cash on hand is optimal based on the distribution $\mathbf{F}(\theta)$; consequently, the optimal amount of cash on hand is the same for all individuals regardless of their idiosyncratic history θ^{t-1} and the realization of θ_t. This optimal target level of cash on hand is thus degenerate with respect to idiosyncratic shocks θ_t. In other words, a universal target policy (uniform to all households) is obtained because under quasi-linear preferences, total labor income $(\overline{w}_t(z^t)n_t(\theta_{t-1}, z^t))$ can be adjusted elastically to meet any target level of cash on hand $x_t(z^t)$ such that this target income level is ex ante optimal based on the distribution of θ_t for any given initial asset holdings $a_t(\theta_{t-1}, z^{t-1})$. Thus, given the target $x_t(z^t)$ in equation (9), the optimal labor supply can be determined by

$$n_t(\theta_{t-1}, z^t) = \frac{1}{\overline{w}_t(z^t)} \left[x_t(z^t) - a_t(\theta_{t-1}, z^{t-1}) \right]. \tag{13}$$

The optimal cash on hand $x_t(z^t)$ also simultaneously implies (pins down) the optimal cutoff $\theta_t^*(z^t)$, or vice versa (see equation (12)). To see the intuition behind this property, notice that since θ_t is realized only after $x_t(z^t)$ is determined, given the optimal target, consumption and saving decisions are then made after the realization of the shock θ_t, depending on whether θ_t is above or below the cutoff $\theta_t^*(z^t)$. Hence, in the first subperiod of any $t > 0$, all households will choose the same cutoff value regardless of their initial asset holdings $a_{t-1}(\theta_{t-1}, z^{t-1})$ and history θ^{t-1}. Thus, the cutoff $\theta_t^*(z^t)$ uniquely and fully characterizes the distributions of consumption $c_t(\theta_t, z^t)$ and asset holdings $a_{t+1}(\theta_t, z^t)$ in the economy.

2.3 Competitive Equilibrium

The competitive equilibrium is defined in a standard manner when the following market-clearing conditions hold for asset, labor, consumption, and output, respectively, at the aggregate level:

$$B_{t+1}(z^t) = \int a_{t+1}(\theta_t, z^t) d\mathbf{F}(\theta_t), \tag{14}$$

$$N_t(z^t) = \int n_t(\theta_{t-1}, z^t) d\mathbf{F}(\theta_{t-1}), \tag{15}$$

$$C_t(z^t) = \int c_t(\theta_t, z^t) d\mathbf{F}(\theta_t), \tag{16}$$

$$C_t(z^t) + G_t(z^t) = N_t(z^t), \tag{17}$$

where $C_t(z^t)$ is aggregate consumption and the last equation is the aggregate resource constraint.

Notice that in general equilibrium the aggregate bond demand must equal the aggregate bond supply and this market-clearing condition influences the optimal cutoff. Thus, if the aggregate bond supply $B_{t+1}(z^t)$ is high enough such that the equilibrium cutoff is $\theta_t^*(z^t) = \theta_H$ for any z^t, then no household will be borrowing constrained. That is, our model has an interesting property: It is possible to have a competitive equilibrium where all households are fully self-insured by holding enough bonds if the level of the aggregate bond supply is sufficiently large. Therefore, whenever $\theta_t^*(z^t) = \theta_H$, the liquidity premium vanishes ($L(\theta_H) = 1$) and the supply of bonds trivially pins down the competitive equilibrium such that the asset price Q_{t+1} becomes exogenously determined by the time discount factor β, and the asset distribution no longer imposes a constraint on the Ramsey planner for choosing an allocation; consequently, the bond market-clearing condition degenerates to the one in a representative-agent economy. On the other hand, if the supply of government bonds is low enough, then in a competitive equilibrium, a positive fraction of the households' borrowing constraint (5) will always be binding in every period $t \geq 0$ and the liquidity premium will always be positive: $L(\theta_t^*) > 1$; in such a case, the aggregate demand for bonds is a function of the distribution statistic θ_t^* such that the level of the bond supply influences the cutoff. To make our analyses interesting, we will assume throughout the paper that the parameter space and the initial bond level B_0 are such that if the future bond supply $B_{t+1}(z^t)$ remains at the initial level B_0 for all t and z^t, then the competitive equilibrium exhibits only partial self-insurance with $\theta_t^*(z^t) < \theta_H$.

Notice that the liquidity premium satisfies $\frac{\partial L(\theta^*)}{\partial \theta^*} < 0$ for $\theta^* \in (\theta_L, \theta_H)$. The intuition is as follows: The higher the cutoff value, the smaller the mass of constrained agents, and therefore the smaller the "liquidity premium" of the store of value (government bonds).

By aggregating equation (7), we have

$$C_t(z^t) = D(\theta_t^*(z^t))x_t(z^t), \tag{18}$$

where the function

$$D(\theta_t^*(z^t)) \equiv \int_{\theta_L}^{\theta_t^*(z^t)} \frac{\theta}{\theta_t^*(z^t)} d\mathbf{F}(\theta) + \int_{\theta_t^*(z^t)}^{\theta_H} d\mathbf{F}(\theta) \in (0,1], \tag{19}$$

denotes the aggregate marginal propensity to consume.

2.4 Conditions to Support a Competitive Equilibrium

To facilitate the discussion, define $U_{C,t}(z^t)$ as the marginal utility of aggregate consumption,

$$U_{C,t}(z^t) \equiv C_t(z^t)^{-1}. \tag{20}$$

Also define

$$Z(\theta_t^*(z^t)) \equiv D(\theta_t^*(z^t))\theta_t^*(z^t)L(\theta_t^*(z^t)), \tag{21}$$

which together with equations (18) and (9) imply

$$C_t\left(z^t\right) = \left(1 - \tau_t(z^t)\right) Z\left(\theta_t^*(z^t)\right). \tag{22}$$

Hence, $Z\left(\theta_t^*(z^t)\right)$ can be interpreted as a measure of gross aggregate income such that aggregate consumption equals the after-tax income. The following lemma helps to characterize the properties of $Z(\theta^*)$:

Lemma 1.

1. $\frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} > 0$ if θ_t^* is close to the upper corner θ_H;
2. $\frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} < 0$ if θ_t^* is close to the lower corner θ_L; and
3. $\frac{\partial Z(\theta_t^*)}{\partial \theta_t^*} = 0$ if $\theta_t^* = \{\theta_L, \theta_H\}$.

Proof. The proof is straightforward by using the definition of $Z(\theta_t^*)$. □

That is, the function $Z(\theta^*)$ has the shape of an inverted (upside down) hat and its derivative vanishes at the two boundary points. Equation (22) is very useful because it can be used to substitute out the tax rate in the following analysis and simplify the Ramsey problem under the primal approach.

Proposition 2. ***(Conditions to Support a Competitive Equilibrium)***

Given the initial government bond supply B_0 and the sequence of distribution $\{\theta_t^*(z^t)\}_{t=0}^{\infty}$, the aggregate allocation $\{N_t(z^t), C_t(z^t), B_{t+1}(z^t)\}_{t=0}^{\infty}$ can be supported as a competitive equilibrium if and only if

1. *the resource constraint (17) holds;*
2. *the government budget constraint (3) holds for all z^t with $t \geq 0$, which can be transformed into the following implementability constraint after substituting out the tax rate $\tau_t(z^t)$ and bond price $Q_{t+1}(z^t)$:*

$$\begin{aligned} & Z(\theta_t^*(z^t)) - N_t(z^t) + \beta E_t\left\{U_{C,t+1}(z^{t+1})Z(\theta_{t+1}^*(z^{t+1}))\right\} L(\theta_t^*(z^t))B_{t+1}(z^t) \\ \geq\ & U_{C,t}(z^t)Z(\theta_t^*(z^t))B_t(z^{t-1}), \end{aligned} \tag{23}$$

where $B_0 > 0$ is given; and

3. the following condition holds for all $t \geq 0$:

$$E_t \left\{ U_{C,t+1}(z^{t+1}) Z(\theta^*_{t+1}(z^{t+1})) \right\} B_{t+1}(z^t) = \frac{1}{\beta} \theta^*_t(z^t) \left[1 - D(\theta^*_t(z^t)) \right]. \tag{24}$$

Proof. See Appendix A.2. □

Notice that equation (24) is transformed directly from the original asset market-clearing condition:

$$Q_{t+1}(z^t) B_{t+1}(z^t) = \left[1 - D(\theta^*_t(z^t)) \right] x_t(z^t),$$

where the left-hand side is the aggregate bond supply (in terms of goods) and the right-hand side is the aggregate bond demand with $[1 - D(\theta^*_t(z^t))]$ as the households' aggregate propensity to save. Using equation (11) and definition (22) to substitute out the asset price and the tax rate $\{Q_{t+1}(z^t), \tau_t(z^t)\}$ in the above bond market-clearing condition gives equation (24). This transformed asset market-clearing condition (24) reveals that the Ramsey planner must consider the impact of its bond supply and taxes on asset prices (through the liquidity premium) and the distribution of asset holdings through the policies' impact on the cutoff $\theta^*_t(z^t)$. This is in contrast to the representative-agent model that has a degenerated distribution.

Moreover, by utilizing equations (7) and (18), the welfare function V can be expressed as a function of only the aggregate allocation as shown by the following proposition.

Proposition 3. ***(Welfare Function in Terms of Aggregate Variables)***

The welfare function V can be expressed as a function of only the aggregate allocation $\{C_t, N_t\}$ and the cutoff $\{\theta^*_t\}$:

$$V = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ W(\theta^*_t(z^t)) + \overline{\theta} \log C_t(z^t) - N_t(z^t) \right\}, \tag{25}$$

where the function $W(\theta^*_t)$ reflects the welfare effects of inequality (distribution) and is given explicitly by

$$W(\theta^*_t(z^t)) \equiv \overline{\theta} \log \frac{1}{D(\theta^*_t(z^t))} + \int_{\theta_t \leq \theta^*_t(z^t)} \theta_t \log \frac{\theta_t}{\theta^*_t(z^t)} d\mathbf{F}(\theta). \tag{26}$$

The function $W(\theta^*_t)$ has the following properties:

$$\left. \frac{\partial W(\theta^*_t)}{\partial \theta^*_t} \right|_{\theta^*_t \in (\theta_L, \theta_H)} > 0 \text{ and } \left. \frac{\partial W(\theta^*_t)}{\partial \theta^*_t} \right|_{\theta^*_t = \{\theta_L, \theta_H\}} = 0.$$

That is, $W(\theta^*_t)$ is S-shaped, monotonically increasing in the open interval $\theta^* \in (\theta_L, \theta_H)$ and flattens *out toward the two end points $\{\theta_L, \theta_H\}$.*

Proof. See Appendix A.3. $\square$

Armed with Propositions 2 and 3, the primal approach of the Ramsey problem becomes not only feasible but also analytically tractable in our model economy. In general, the adoption of a primal Ramsey approach in a HAIM economy faces the daunting challenge of formulating the implementability condition analytically. This challenge is overcome here by the property that government policies and intertemporal prices of the competitive equilibrium can all be expressed analytically as a function of the aggregate quantity variables $\{C_t(z^t), N_t(z^t), B_{t+1}(z^t)\}$, together with the cutoff variable $\theta_t^*(z^t)$, without the need to specify the functional form of the distribution $F(\theta)$, thanks to the fact that the cutoff $\theta_t^*(z^t)$ is independent of the history θ^t and hence a sufficient statistic for describing the liquidity premium and distributions of individual choice variables.

3 Ramsey Allocation

To simplify notations, from now on we suppress the aggregate state variables z_t and z^t unless confusion may arise.

3.1 The Ramsey Problem

Now, the Ramsey problem can be represented as maximizing the expected lifetime utility (25) by choosing the sequences of aggregate quantities (including the distribution), $\{N_t, C_t, B_{t+1}, \theta_t^*\}$, subject to a list of constraints that includes the aggregate resource constraint (17), the implementability condition (23), and the transformed asset market-clearing condition (24) listed in Proposition 2.

Proposition 4. ***(Dynamic Property of the Ramsey Allocation)***

Let ϕ_t denote the Lagrangian multiplier associated with the implementability condition (23) in the Ramsey problem; then under both aggregate and idiosyncratic uncertainty, the key dynamic property of the Ramsey allocation is characterized by the following equation:

$$E_t q_{t+1} \phi_{t+1} = \phi_t + \frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{1}{\mathbf{F}(\theta_t^*)} + (E_{t-1} q_t \phi_t - \phi_t) \frac{U_{C,t} B_t}{\mathbf{F}(\theta_t^*)} \frac{\partial Z(\theta_t^*)}{\partial \theta_t^*} \tag{27}$$

for $t > 0$, where the STUR (coefficient) q_{t+1} satisfies $E_t q_{t+1} = 1$ and is defined as

$$q_{t+1} \equiv \frac{U_{C,t+1} Z(\theta_{t+1}^*)}{E_t U_{C,t+1} Z(\theta_{t+1}^*)}. \tag{28}$$

The multiplier ϕ_t is related to the aggregate marginal utility (of consumption) by

$$\frac{\overline{\theta}}{C_t} = 1 + \phi_t + (E_{t-1}q_t\phi_t - \phi_t)\, U_{CC,t} Z(\theta_t^*) B_t, \tag{29}$$

and is in turn related to the optimal tax rate by

$$\tau_t = 1 - \frac{C_t}{Z(\theta_t^*)} = 1 - \frac{\overline{\theta}}{Z(\theta_t^*)} \frac{1}{(1 + \phi_t + (E_{t-1}q_t\phi_t - \phi_t)\, U_{CC,t} B_t Z(\theta_t^*))}. \tag{30}$$

Proof. See Appendix A.4. □

Notice that either in the absence of aggregate uncertainty or if there is perfect foresight on government expenditures, then $q_t = 1$ and $E_t\phi_{t+1} = \phi_{t+1}$; so equation (27) becomes $\phi_{t+1} = \phi_t + \frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{1}{\mathbf{F}(\theta_t^*)}$, which is a deterministic process for the multiplier ϕ_t and is thus much easier to analyze and understand. The same result can also be achieved by assuming that government debt is state-contingent. Thus, to help understand the meaning of equation (27) and the implied Ramsey allocation under both aggregate and idiosyncratic uncertainty, it is helpful to consider first the case where government bonds are state-contingent, as in LS (1983).

3.1.1 State-Contingent Debt

With state-contingent debt, the Ramsey planner is able to fully hedge against aggregate risk, we thus have the following corollary:

Corollary 1. ***(Ramsey Allocation with State-Contingent Debt)***

If government bonds are state-contingent, then with both aggregate and idiosyncratic uncertainty equation (27) simplifies to

$$\phi_{t+1} = \phi_t + \frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{1}{\mathbf{F}(\theta_t^*)}, \tag{31}$$

where the right-hand side captures the marginal benefit of one additional unit of debt and the left-hand side captures the marginal cost of debt financing in terms of distortionary taxation. The multiplier ϕ_t is related to the aggregate marginal utility (of consumption) by

$$\frac{\overline{\theta}}{C_t} = 1 + \phi_t \tag{32}$$

and is in turn related to the optimal tax rate by

$$\tau_t = 1 - \frac{\overline{\theta}}{(1 + \phi_t) Z(\theta_t^*)} = 1 - \frac{C_t}{Z(\theta_t^*)} \tag{33}$$

such that under a given distribution θ_t^*, a larger multiplier ϕ_t (or higher marginal cost of debt) is associated with a lower aggregate consumption C_t and a higher tax rate τ_t.

Proof. See Appendix A.5. □

Equation (31) captures the paper's key contribution, which governs the transitional dynamics of the optimal quantity of debt in the Ramsey allocation, as well as any deviations from the Ramsey steady state. This equation summarizes the most important information regarding the marginal benefit of debt in terms of welfare $W(\theta_t^*)$—the distribution of households' self-insurance positions—and the marginal cost of debt in terms of future distortionary taxes (see Appendix A.5 for a more detailed derivation of this important equation).

With state-contingent debt, equations (32) and (33) imply that if the multiplier is constant under a constant distribution θ^*, then aggregate consumption and the labor tax rate will be constant as well, suggesting perfect tax/consumption smoothing. Hence, considering the scenario of state-contingent debt is illuminating. Given that (i) the cutoff θ_t^* (as the measure of borrowing-unconstrained agents) is endogenously time-varying and (ii) the derivative $\frac{\partial W(\theta^*)}{\partial \theta^*} > 0$ is the marginal welfare gain from an improved distribution when the cutoff (or bond supply) rises, equation (31) indicates that the multiplier ϕ_t is strictly increasing with the bond supply during the transition period as long as $\theta_t^* < \theta_H$, and until an increase in the cutoff value is not beneficial anymore, i.e., until $\theta_t^* = \theta_H$ and no single household is borrowing constrained regardless of the preference shock and government spending shock. Thus, the planner opts to keep increasing the debt level and sustaining more distortions in order to improve households' self-insurance positions and reduce consumption inequality—as long as the market interest rate lies below the time discount rate. At the long-run Ramsey allocation, we have $\frac{\partial W(\theta^*)}{\partial \theta^*} = 0$ and $\phi_t = \phi$. Hence, a Ramsey steady state with a positive $\phi > \phi_0$ exists whenever $\theta_t^* = \theta_H$. A Ramsey allocation with $\theta_t^* = \theta_H$ is an FSI allocation, at which point every household holds enough assets such that none of them is borrowing constrained.

Notice that the counterpart of equation (31) in the LS model is given by $\phi_{t+1} = \phi_t$ for all $t > 0$. So the Ramsey allocation in the corresponding LS model (without idiosyncratic uncertainty) is characterized by a constant tax rate and constant consumption after the initial period 0, which are independent of the history of government spending shocks except the first-period shock G_0. Therefore, the long-run Ramsey allocation in our model (with $\theta_t^* = \theta_H$) looks similar to that in the LS model in the sense that the optimal paths of consumption and taxes are both constant—but constant in the LS model for $t \geq 1$ and in our model only for $t \geq T$ (where $T > 1$ is the end of the transition period in our model).

Also, notice that in the initial period, the cutoff θ_0^* in our model cannot be pinned down without knowing the terminal condition in the future by solving the entire transitional path of the dynamic system. The number of periods taken to reach the long-run Ramsey equilibrium in our model depends on the initial condition B_0 and the history of aggregate shocks $\{G_t\}_{t=0}^{\infty}$. Thus, the key difference in policy implications between our model under state-contingent debt and the LS model lie in (i) the transitional dynamics for the multiplier ϕ_t in equation (31), (ii) the long-run level of the tax rate τ, and (iii) the optimal quantity of the bond supply $\lim_{t\to\infty} B_{t+1}$. In other words, given the same process of government spending shocks $\{G_t\}_{t=0}^{\infty}$ and same initial bond holdings B_0 in our model and in the LS model, we have the following similarities and differences:

1. If the initial bond supply B_0 is sufficiently high to achieve FSI in a competitive equilibrium, then the optimal paths of the future bond supply $\{B_t\}_{t=1}^{\infty}$ and the tax rate $\{\tau_t\}_{t=1}^{\infty}$ in our model are identical to those in the LS model.

2. If the initial bond supply B_0 is below the level required for FSI in a competitive equilibrium, then the optimal long-run level of the bond supply in our model is strictly higher than that in the LS model, and the long-run multiplier ϕ_t is greater than that in the LS model. In other words, the optimal long-run tax rate in our model must be higher than that in the LS model and the optimal aggregate consumption level in our model must be lower than that in the LS model. The latter reflects the cost of distortionary taxes due to a permanently higher level of the bond supply, but with a strictly improved distribution θ_t^*, suggesting that the welfare gains from a higher level of public debt in our model come mainly from the improved distribution across households' insurance positions in the steady state as well as from front-loading consumption (thus a departure from tax smoothing) during the transition.

The above discussion leads to the following corollary.

Corollary 2. ***(Long-run Ramsey Allocation with State-Contingent Debt)***

Under state-contingent debt the long-run Ramsey equilibrium is characterized by the following properties: Given B_0, there exists a sufficiently large $T > 0$ such that for $t \geq T$, (i) the Ramsey allocation exhibits FSI with $\theta_t^* = \theta_H$ and (ii) aggregate consumption and labor tax are constant: $\{C_t, \tau_t\}_{t=T}^{\infty} = \{\overline{C}, \overline{\tau}\}$.

3.1.2 Non-State-Contingent Debt

Now we can return to our original model without state-contingent debt. In the case of non-state-contingent debt, the Ramsey planner is unable to perfectly hedge against the aggregate G_t shock, so the law of motion for the multiplier ϕ_{t+1} in equation (31) changes to equation (27). To facilitate discussion, we can rewrite equation (27) as

$$E_t\left[\left(1+\varepsilon^q_{t+1}\right)\phi_{t+1}\right]=\phi_t+\frac{\partial W(\theta^*_t)}{\partial\theta^*_t}\frac{1}{\mathbf{F}(\theta^*_t)}+\wp(\theta^*_t), \tag{34}$$

where the forecast error, ε^q_{t+1}, and the residual term, $\wp_t$, are defined, respectively, as

$$\varepsilon^q_{t+1}\equiv q_{t+1}-1=\frac{U_{C,t+1}Z(\theta^*_{t+1})-E_tU_{C,t+1}Z(\theta^*_{t+1})}{E_tU_{C,t+1}Z(\theta^*_{t+1})}$$

and

$$\wp_t\equiv\left(E_{t-1}q_t\phi_t-\phi_t\right)\frac{U_{C,t}B_t}{\mathbf{F}\left(\theta^*_t\right)}\frac{\partial Z(\theta^*_t)}{\partial\theta^*_t}.$$

This equation captures the marginal benefits and costs of a change in the debt level B_{t+1} subject to the government budget (implementability) constraint through the viewpoint of the change's effect on the distribution θ^*_t.

Compared with the representative-agent model of LS, introducing idiosyncratic uncertainty and risk-free debt brings about three changes to the dynamics of the multiplier ϕ_{t+1}: (i) a STUR $q_{t+1}\equiv 1+\varepsilon^q_{t+1}$ coming from risk-free debt (as in AMSS), (ii) a welfare term $\frac{\partial W(\theta^*_t)}{\partial\theta^*_t}\frac{1}{\mathbf{F}(\theta^*_t)}$ coming from the objective (welfare) function due to a change in the distribution θ^*_t, and (iii) the residual term $\wp_t$ coming from the implementability condition and the asset market-clearing condition. The forecast error ε^q in both term (i) and term (iii) is missing under state-contingent debt; and terms (ii) and (iii) will be missing in the absence of idiosyncratic uncertainty (as in the model of AMSS).

To help understand the effects of these changes, notice that the bond price under state-contingent debt is given by (see Appendix A.5)

$$Q_{t+1}(z_{t+1}|z^t)=\beta\frac{U_{C_{t+1}}(z^{t+1})}{U_{C_t}(z^t)}\frac{Z(\theta^*_{t+1})}{Z(\theta^*_t)}L(\theta^*_t(z^t));$$

thus, in the case of risk-free debt we can express q_{t+1} in equation (28) as

$$q_{t+1}\equiv\frac{U_{C_{t+1}}Z\left(\theta^*_{t+1}\right)}{E_tU_{C_{t+1}}Z\left(\theta^*_{t+1}\right)}=\frac{Q_{t+1}}{E_tQ_{t+1}}=1+\underbrace{\frac{Q_{t+1}-E_tQ_{t+1}}{E_tQ_{t+1}}}_{\varepsilon^q_{t+1}},$$

where the "forecast error" ε^q_{t+1} satisfies $E_t\varepsilon^q_{t+1}=0$, which originates from predicting the future (state-contingent) bond price $Q_{t+1}\left(z_{t+1}|z^t\right)$ based on period-t information on government spending shocks, or it is the difference between the state-contingent bond price $Q_{t+1}\left(z_{t+1}|z^t\right)$ and the expected non-state-contingent bond price $E_tQ_{t+1}\left(z^t\right)$. Under rational expectations, this forecasting error must be zero on average so that $E_t q_{t+1} = 1 + E_t \varepsilon^q_{t+1} = 1$.

The forecasting error ε^q_{t+1} is a necessary wedge for adjusting the expected future marginal cost ϕ_{t+1} of distortionary taxation because the next-period intertemporal price, Q_{t+1}, may change due to new information about government spending—as if the Ramsey planner is issuing bonds in period t based on the expected bond price in $t+1$. This forward-looking (anticipation) behavior is rational because the interest rate payment (the inverse of the current bond price) is not due until the next period. Hence, the forecasting error for predicting the change in the bond price must be incorporated into the expected next-period marginal cost. Since both q_{t+1} and the multiplier ϕ_{t+1} are affected by the next-period government spending shock G_{t+1}, they are correlated in general, suggesting that $E_t q_{t+1}\phi_{t+1} \neq E_t q_{t+1} E_t \phi_{t+1} = E_t \phi_{t+1}$.

Notice that this forecasting error remains even if we eliminate idiosyncratic uncertainty, in which case we have

$$Q_{t+1}(z_{t+1}|z^t) = \beta \frac{U_{C_{t+1}}(z^{t+1})}{U_{C_t}(z^t)};$$

and thus we still have

$$q_{t+1} \equiv \frac{U_{C_{t+1}}}{E_t U_{C_{t+1}}} = \frac{Q_{t+1}}{E_t Q_{t+1}} = 1 + \underbrace{\frac{Q_{t+1} - E_t Q_{t+1}}{E_t Q_{t+1}}}_{\varepsilon^q_{t+1}},$$

as in the AMSS model. In other words, in the absence of idiosyncratic uncertainty, equation (34) is reduced to its counterpart in the AMSS model. To see this, assume a degenerate distribution $\mathbf{F}(\theta)$ with $\theta_L = \theta_H = \overline{\theta}$ such that agents become identical; we then have $\frac{\partial W(\theta^*)}{\partial \theta^*} = 0$ and $Z(\theta^*) = \overline{\theta}$. So equation (34) is reduced to

$$E_t \left(\frac{U_{C_{t+1}}}{E_t U_{C_{t+1}}} \right) \phi_{t+1} = \phi_t, \tag{35}$$

which is identical to equation (17) in AMSS (2002).

Since the meaning and influence of the welfare term $\frac{\partial W(\theta^*_t)}{\partial \theta^*_t}$ on the dynamics of ϕ_{t+1} are already discussed in the previous subsection under state-contingent debt—the welfare term generates a "mean-reverting" force to push the multiplier upward as long as $\theta^*_t < \theta_H$—we now discuss the meaning and influence of the residual term $\wp(\theta^*_t)$. Notice that this term is the product of two parts: the forecasting error, $(E_{t-1} q_t \phi_t - \phi_t)$, and the value of a marginal change in gross income, $\frac{U_{C,t} B_t}{\mathbf{F}(\theta^*_t)} \frac{\partial Z(\theta^*_t)}{\partial \theta^*_t}$, which by $C_t = (1-\tau_t) Z(\theta^*_t)$ can be expressed as $\frac{B_t}{\mathbf{F}(\theta^*_t)} \frac{\partial \left(\frac{1}{1-\tau_t}\right)}{\partial \theta^*_t}$. So this residual term $\wp(\theta^*_t)$ captures the forecasting error of a change in the value of the multiplier caused by an income effect from a change in the tax rate through a change in distribution—holding aggregate consumption C_t constant. This term appears in addition to the welfare term $\frac{\partial W(\theta^*)}{\partial \theta^*}$ because any increase in the debt level must be financed by taxes (via the government budget constraint or the implementability condition), and the optimal tax rate consistent with the government budget constraint must depend on the distribution. Notice that the income effect $\frac{\partial Z(\theta_t^*)}{\partial \theta_t^*}$ through taxation would have no impact on future multiplier ϕ_{t+1} if aggregate uncertainty is perfectly predictable or can be perfectly hedged by state-contingent debt such that the forecasting-error term is zero: $E_{t-1}q_t\phi_t - \phi_t = 0$; consequently, the income effect of distortionary taxation through the distribution cannot influence the dynamics of the multiplier (as in the LS model under idiosyncratic uncertainty). In addition, there would be no income effect of distortionary taxation through the government budget constraint if the optimal tax is independent of the distribution; specifically, the forecasting error is not zero with risk-free debt, but in the absence of idiosyncratic uncertainty we would still have $\wp(\theta_t^*) = 0$ because in this case, $\partial Z(\theta_t^*)/\partial \theta_t^* = 0$ since the optimal tax τ_t is independent of the distribution (as in the AMSS model or any representative-agent model). In short, the residual term $\wp(\theta_t^*)$ in equation (34) is a product of the two effects: It would be zero if either (i) debt is state-contingent or (ii) there is no idiosyncratic uncertainty.

To illustrate the dynamic effects of the residual term $\wp_t$ on the multiplier, we can assume $q_t = 1$ for a moment (as a first-order approximation); then, $\wp_t$ is simplified to $(E_{t-1}\phi_t - \phi_t)\frac{U_{C,t}B_t}{\mathbf{F}(\theta_t^*)}\frac{\partial Z(\theta_t^*)}{\partial \theta_t^*}$. Furthermore, ignoring the mean-reverting force $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*}\frac{1}{\mathbf{F}(\theta_t^*)}$ for a moment, equation (34) is simplified to

$$E_t\phi_{t+1} = \phi_t + (E_{t-1}\phi_t - \phi_t)\,\alpha_t, \tag{36}$$

where $\alpha_t \equiv \frac{U_{C,t}B_t}{\mathbf{F}(\theta_t^*)}\frac{\partial Z(\theta_t^*)}{\partial \theta_t^*}$. Since the value of ϕ_t can be defined or rewritten as $E_{t-1}\phi_t = \phi_t - \varepsilon_t$ with $E_{t-1}\varepsilon_t = 0$, the above equation can be rewritten as

$$E_t\phi_{t+1} = \phi_t - \alpha_t\varepsilon_t \tag{37}$$

or as

$$\phi_{t+1} = \phi_t - \alpha_t\varepsilon_t + \varepsilon_{t+1}, \tag{38}$$

which is a unit-root process with an additional lagged error term (or lagged impulse) $\alpha_t\varepsilon_t$.

However, this lagged impulse term $\alpha_t\varepsilon_t$ changes the random-walk dynamics of the multiplier by introducing an endogenous history-dependent force to push the multiplier downward if the cutoff is close to the upper corner θ_H and upward if the cutoff is close to the bottom corner θ_L. So it keeps the multiplier in the open interval (θ_L, θ_H) and thus introduces strong history-dependence into the multiplier.

To see these effects, we can use a one-period lagged version of equation (38) to substitute out

ε_t in equation (38) and obtain

$$\begin{aligned}\phi_{t+1} &= \phi_t + (\phi_{t-1} - \alpha_{t-1}\varepsilon_{t-1} - \phi_t)\,\alpha_t + \varepsilon_{t+1} \\ &= (1-\alpha_t)\,\phi_t + \alpha_t\phi_{t-1} - \alpha_t\alpha_{t-1}\varepsilon_{t-1} + \varepsilon_{t+1},\end{aligned} \tag{39}$$

which is a second-order difference equation with two roots (eigenvalues): $\{\lambda_1, \lambda_2\} = \{1, -\alpha_t\}$, plus a further lagged error term $\alpha_t\alpha_{t-1}\varepsilon_{t-1}$. Since one of the roots is λ_1=1, the residual term $\wp(\theta_t^*)$ does not change the unit-root property of the multiplier but has brought in one additional autoregressive root $\lambda_2 = -\alpha_t$ into the dynamics of ϕ_{t+1}, making the process more history-dependent with higher-order autoregressive dynamics. In particular, since $\alpha_t < 0$ for θ_t^* close to θ_L, it adds an additional mean-reverting positive force (positive growth rate $\lambda_2 > 0$) into the dynamics of ϕ_{t+1} to keep the multiplier away from zero (i.e., it makes $\phi_{t+1} > \phi_t$), reinforcing the growth effect of the welfare term $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*}\frac{1}{\mathbf{F}(\theta_t^*)} > 0$. On the other hand, for θ_t^* close to θ_H, we have $\alpha_t > 0$, so the second eigenvalue $\lambda_2 = -\alpha_t$ is negative. This implies that whenever the distribution or the cutoff is sufficiently close to FSI, the positive growth dynamics in the multiplier is mitigated by the negative root $\lambda_2 < 0$, not only slowing down the speed of convergence (or mean-reverting tendency) toward FSI but also pulling the multiplier downward and away from the FSI allocation, generating more serial correlations in the multiplier within the interval $\theta_t^* \in (\theta_L, \theta_H)$.

Obviously, in contrast to the case of state-contingent debt, under risk-free debt the dynamic effects of $\wp(\theta_t^*)$ in conjunction with the STUR q_t imply that the FSI allocation $\theta_t^* = \theta_H$ can no longer be maintained with probability 1 in a long-run Ramsey equilibrium, unlike the case of state-contingent debt. Instead, the model economy may deviate (downward) from the FSI allocation stochastically with history-dependent persistence, although the tendency to approach (or revert back to) the FSI allocation from *below* always exists, which makes the law of motion for ϕ_{t+1} a mean-reverting error-correction process despite the STUR component; but this unit-root process is stochastically bounded below by the positive growth forces toward FSI. In other words, the Ramsey equilibrium in our model not only has a random-walk component, but the random-walk dynamics are *endogenously* bounded—below by the "mean-reverting" forces of $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*}\frac{1}{\mathbf{F}(\theta_t^*)} > 0$ and $\alpha(\theta_t^*) > 0$ (when θ_t^* is sufficiently away from θ_H) and above (implicitly) by the government's natural borrowing limit. Hence, it is impossible for the multiplier to diverge to infinity or zero. The fact that $\alpha(\theta_t^*) < 0$ only when θ_t^* is sufficiently close to θ_H simply means that any deviation away from the FSI will have a certain degree of persistence such that the "mean-reverting" behavior is not instantaneous or immediate.

In sharp contrast, without idiosyncratic uncertainty (as in the model of AMSS), the multiplier ϕ_{t+1} may approach zero in the long run, at which point the government is accumulating so much private debt that all government expenditures are financed from the interest income on assets issued by the private sector, without having to resort to distortionary taxes at all.

With the above discussions, we are ready to state the following proposition that characterizes the stochastic-bounded Ramsey allocation under incomplete markets for both aggregate and idiosyncratic risks.

Proposition 5. ***(Long-run Ramsey Allocation with Non-State-Contingent Debt)***

When G_t is a stationary random process, a Ramsey equilibrium exists and it features bounded STUR dynamics where all variables are strictly positive with a finite variance.

Proof. See Appendix A.6. $\square$

4 Numerical Examples

This section uses numerical examples to confirm our theoretical analysis and reveal the stochastic behaviors of the model economy under specific parameter values. Our purpose is to illustrate numerically not only the transitional dynamics of the model but also how the presence of the welfare term $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*}\frac{1}{\mathbf{F}(\theta_t^*)}$ and the residual term $\wp(\theta_t^*)$ in equation (27) (or equation (34)) change the dynamic behavior of the Ramsey allocation through the multiplier ϕ_t in comparison to the AMSS model.

Consider the following parameter values: The government spending process is an iid two-state, first-order Markov process:

$$G = \begin{cases} G_L = 0.1 \\ G_H = 0.3 \end{cases}$$

with transition probability

$$\pi_G = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}.$$

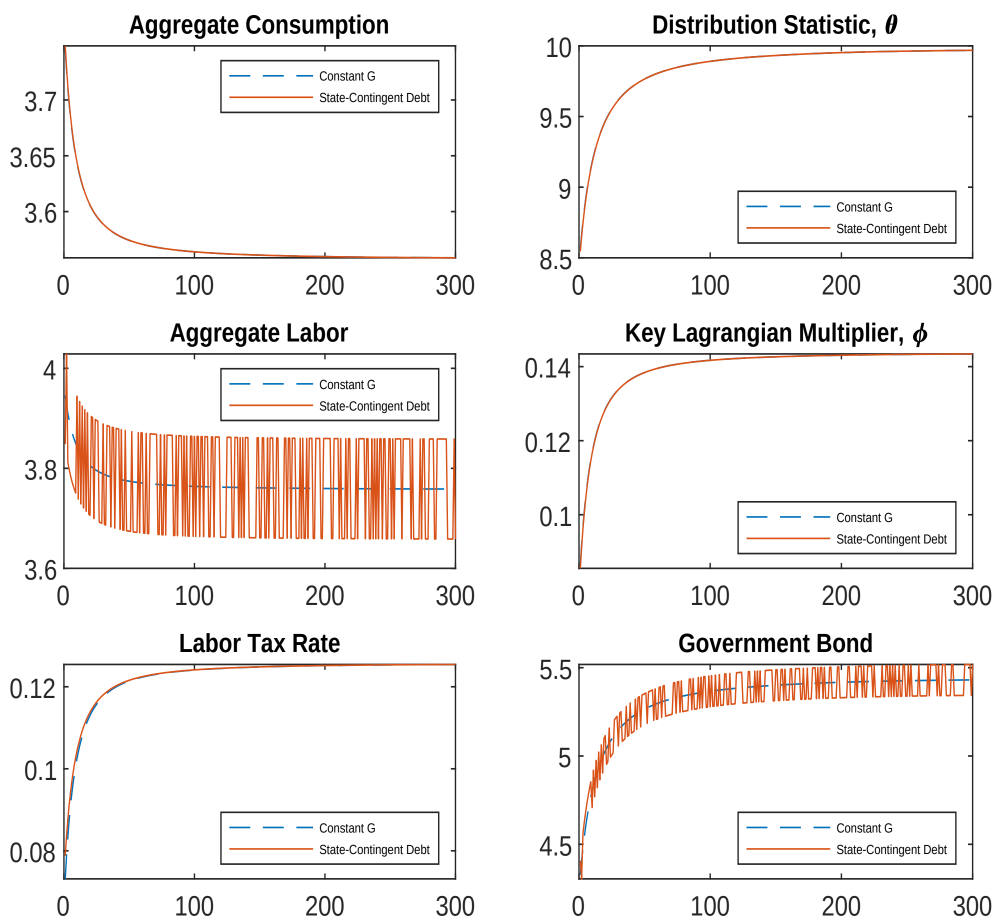
This process is used in all numerical simulations in this paper. In addition, the time discount rate is $\beta = 0.95$; the distribution of preference shock θ follows a power function $\mathbf{F}(\theta) = \frac{\theta^\gamma - \theta_L^\gamma}{\theta_H^\gamma - \theta_L^\gamma}$, where $\theta_L = 1$, $\theta_H = 10$ and $\gamma = 0.1$; and the initial debt level B_0 is chosen to be lower than the optimal debt level B^* implied by the FSI steady-state equilibrium without aggregate uncertainty (e.g., $B^* > B_0 = 0$). The results are qualitatively similar for other choices of the parameter values as long as $B_0 < B^*$.

These parameter values imply the following: (i) In the steady state (without aggregate uncertainty), the condition for a positive individual labor choice, $\theta_H < \frac{\theta_L}{(1-\beta)} < \infty$, is satisfied. (ii) In the transition, the condition for a positive labor supply, $n_t > 0$ for all $t \geq 0$, is satisfied and verified numerically in the simulation.

4.1 State-Contingent Debt

In this subsection we first solve numerically the version of our model with state-contingent debt. Given the closed-form Ramsey FOCs, the model can be solved using a global method instead of a local approximation method. Figure 1 plots the Ramsey transition path. As a reference point, we also plot the case without aggregate uncertainty ($G_t = \overline{G}$). The dashed lines represent the reference model without aggregate uncertainty; the solid lines represent the model with aggregate uncertainty. The counterparts of the corresponding LS model (without idiosyncratic uncertainty) are not shown in the figure, but they could serve as another reference point because they remain constant for $t \geq 1$ (except for labor and the bond supply) and feature no interesting transitional dynamics.

Figure 1: Ramsey Transition Path with State-Contingent Debt



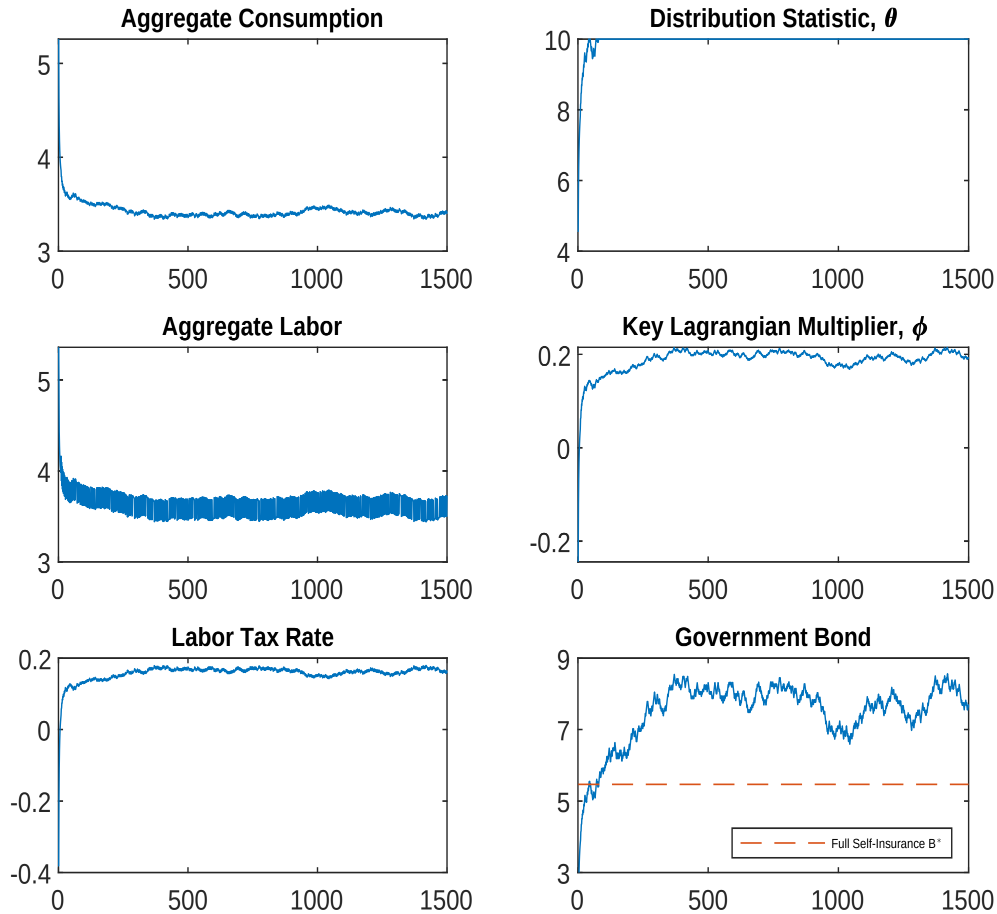
In the figure, when the two paths are identical, we can see only the solid lines, because they completely override the dashed lines. For example, the paths of consumption C_t (top-left panel), the cutoff θ_t^* (top-right panel), the multiplier ϕ_t (middle-right panel), and the tax rate τ_t (bottom-left panel) in Figure 1 are all monotonically approaching their respective steady state and are all identical to their counterparts in the reference model without aggregate uncertainty, suggesting the power of state-contingent debt in buffering aggregate shocks and smoothing consumption and taxes. These transitional dynamics also indicate a significant departure from the perfect tax smoothing in the corresponding LS model, thanks to the lengthy transitional dynamic movement in the distribution θ_t^*. Recall that in the corresponding LS model there are no transitional dynamics after the initial period $t = 0$, and all variables except labor and the bond supply remain constant after $t \geq 1$.

4.2 Non-State-Contingent Debt

With risk-free bonds, the dynamic paths are quite different from the previous case (under state-contingent debt) as well as from the representative-agent AMSS model where the multiplier ϕ_t is a pure random walk with only an implicit upper bound implied by the government's natural borrowing limit. Figure 2 demonstrates the dynamic paths of the model for 1500 periods. It shows that all variables have a stochastic transition period for about 100 periods during which the deviation from tax smoothing is most visible (e.g., the optimal tax rate in the bottom-left panel). Second, the level of debt (bottom-right panel) increases stochastically during the transition, and as soon as it passes the FSI line indicated by the horizontal line of $B^* = 5.5$, it starts to wander around like a random walk above B^* and never falls below B^* within our simulation horizon. Third, after the transition period, except for labor N_t (middle-left panel) and the cutoff θ_t^* (top-right panel), all variables such as consumption (top-left panel), the tax rate (bottom-left panel), and the multiplier (middle-right panel) also behave like a random walk with only implicit bounds—from below by the FSI position and above by the government's natural debt limit; these variables also exhibit extremely high serial correlations despite the fact that government spending shocks are iid. Fourth, the behavior of the cutoff θ_t^* (top-right panel) is striking: After a brief upward transition period, it reaches the FSI allocation ($\theta_H = 10$) and then exhibits a tendency to permanently stay at the FSI target level and never falls below this target, because the corresponding level of debt supply wanders above the FSI level B^*.

These results suggest that the Ramsey allocation with risk-free bonds exhibits random-walk behavior with implicit bounds—below by the growth of public debt to support FSI and above by the government's natural debt limit. However, the endogenous natural borrowing limit may bind,

Figure 2: Ramsey Allocation with Non-State-Contingent Debt



but our analysis has ignored such a possibility under the assumption that it will never bind, which is clearly valid in the case of state-contingent debt. Nonetheless, because the unit-root component may make the multiplier diverge to infinity with positive probability, with a long-enough simulation period the debt limit may bind occasionally.

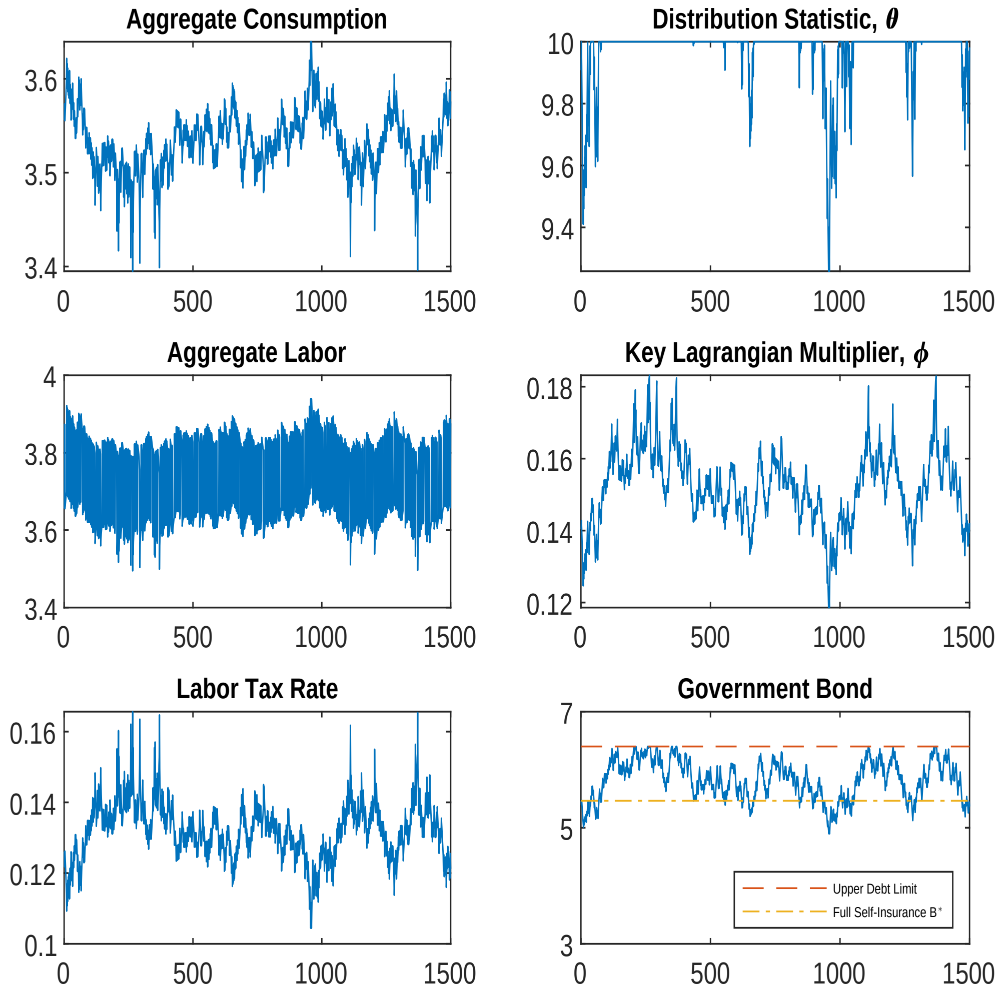
What would the Ramsey allocation look like if the government's natural borrowing limit binds occasionally? Since modeling the government's natural borrowing limit explicitly would make the model intractable (see AMSS), in the following experiment we impose an ad hoc upper limit to the government's borrowing capacity as follows:

$$B_{t+1} \leq \overline{B}, \tag{40}$$

where the limit $\overline{B}$ is a debt level above the FSI level B^* but still low enough such that there is a positive probability for the debt-limit constraint to bind in a relatively short horizon.

The simulation results under the ad hoc debt-limit constraint are shown in Figure 3, which

Figure 3: Ramsey Allocation with Non-State-Contingent Debt (ad hoc limit)



demonstrates the optimal paths of the six variables for 1500 periods of time. The results are interesting. First, the most striking difference in this case is that the ad hoc upper debt limit of the bond supply occasionally binds (bottom-right panel). The bottom-right panel shows two horizontal lines: The lower line is the optimal debt level B^* to ensure FSI in the Ramsey steady state without aggregate uncertainty, and the upper line is the ad hoc debt limit $\overline{B} > B^*$. It shows the following features: (i) Starting from an initial debt level below the FSI level B^*, the optimal quantity of debt increases stochastically during a transition period, and once it passes the FSI line B^* it starts wandering around above the FSI line. (ii) Whenever the debt level hits the upper debt limit $\overline{B}$ it immediately bounces back and wanders downward. (iii) It is now possible that the debt level can fall below the FSI level B^* frequently; however, whenever the debt level falls below the FSI line it reverts back eventually. It does not immediately bounce back but instead often struggles throughout multiple periods before reverting back to the FSI line and entering the region between the two reference lines.

Second, the cutoff (top-right panel) shows a pattern that mimics the behavior of the debt: Whenever the debt level falls below the FSI line, the cutoff also falls below the FSI position $\theta_H = 10$, and it takes several periods for the cutoff to revert back to the FSI position. Specifically, the more frequently the debt level falls below the FSI line in the lower-right panel, the more frequently the cutoff falls below the FSI position in the top-right panel; this is so because the cutoff lies below the FSI point if and only if the debt level is too low to support FSI.

Third, the reason the debt level frequently falls below the FSI line is because the upper debt limit line always prevents the debt level from wandering further upward and thus forces it to move downward frequently. Hence, the optimal debt level behaves exactly like a bounded random walk—bounded above by the ad hoc debt limit $\overline{B}$ and below by the FSI position B^*, although this lower bound is only a "soft" bound in the sense that the debt level can fall temporarily below this lower bound, albeit not for too long. Once the debt falls below the optimal level B^* there are two forces that push the debt level upward: The first is the welfare term $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{1}{\mathbf{F}(\theta_t^*)} > 0$, which generates a positive growth rate in debt; and the second is the residual term $\wp(\theta_t^*)$, which generates a positive growth effect on debt if the cutoff is close to the lower corner θ_L and generates a negative growth effect on debt if the cutoff is close to the upper corner θ_H. This is why it often takes multiple periods for the cutoff and the debt level to revert back to and move beyond the FSI allocation.

Fourth, as a consequence of the occasionally binding ad hoc debt limit on the optimal debt level, the other variables in Figure 3 all appear to be far more stationary without a clear sign of the "random walk" property, in contrast to the previous case in Figure 2. This exercise hence suggests that if the government's natural borrowing limit would be explicitly imposed, and if this limit occasionally binds, the behavior of the model would be similar to that in Figure 3 instead of Figure 2. However, regardless of the natural debt limit, the FSI allocation is clearly a center of "gravity" that the model tries to revert back to. This "mean-reverting" behavior in our model is absent in the representative-agent models of LS and AMSS.

Robustness Analysis. So far in our analyses we have assumed that households cannot borrow under the constraint $a_{t+1} \geq 0$. Presumably, if households can borrow up to a positive limit $\overline{\alpha}$ such that $a_{t+1} \geq -\overline{\alpha}$, the optimal level of public debt might be negative in the long run if $\overline{\alpha}$ is sufficiently large, despite incomplete insurance markets. Therefore, the assumption $a_{t+1} \geq 0$ implicitly rules out the possibility that the government can finance its spending purely from interest income generated by a negative debt position. Nonetheless, we show in our online appendix (Appendix B) that relaxing households' borrowing constraints does not change the fundamental insight gained from the previous analyses—that incomplete markets for individual risk sharing will entice a benevolent government to keep increasing the level of public debt to eliminate or mitigate the distortions in the insurance markets as much as possible to support a FSI allocation.[4] Therefore, an endogenous lower bound on public debt always exists such that the optimal bond supply exhibits "mean-reverting" or "error-correction" behaviors despite the existence of a STUR in the law of motion of the Lagrangian multiplier ϕ_t.

5 A Brief Literature Review

In addition to the seminal works of Barro (1979), LS, and AMSS discussed above, our paper is related to a large and growing literature on optimal fiscal policies and debt financing. Below we discuss the most closely related works.

Using a two-agent, two-period model with incomplete markets, Shin (2006) shows that heterogeneous agents with uninsurable idiosyncratic risk can alter the Ramsey planner's public-financing decisions and increase the optimal level of public debt. Specifically, he shows that when idiosyncratic risk is sufficiently large relative to aggregate risk, the Ramsey planner chooses to issue debt and facilitate the precautionary saving of the private sector instead of holding IOUs issued by households, even at the cost of extra tax distortion. Shin (2006) interprets these outcomes in terms of the trade-off between two competing insurance motives that concern the Ramsey planner: aggregate tax smoothing and individual consumption smoothing. However, even with the simplifying two-agent and two-period assumptions, Shin (2006) relies on either local approximation or numerical methods to obtain his results. Such an approach raises several questions. First, when the number of agents is too small, it is difficult to distinguish idiosyncratic risk from aggregate risk. Second, when the horizon is too short (with only two periods), the dynamic and strategic considerations of the Ramsey planner cannot be fully revealed, and, most importantly, the interesting transitional dynamics crucial for welfare analysis are lost, which may lead to a biased characterization of the Ramsey allocation. We extend Shin's (2006) framework into a more general setting with an infinite horizon and a continuum of agents. Even with the double infinity, we are able to characterize the Ramsey problem analytically using the primal approach and obtain the Ramsey FOCs in closed form, which allows us to derive our results analytically. In particular, we can analytically characterize the fundamental law of motion that governs the Ramsey planner's decisions for the optimal supply of public debt under both aggregate and idiosyncratic uncertainty, both in the transition period and in the long run, regardless of whether or not the public debt is state-contingent. The Ramsey planner's incentive for a departure from tax smoothing under both

[4]In addition, this behavior of mean-reverting back to the FSI allocation is not an artifact of our assumption of quasi-linear preference. In a companion paper (Chien and Wen (2021)), we use an entirely different model with non-linear preferences but without aggregate uncertainty to show that the Ramsey allocation still features FSI despite standard preferences with finite Frisch elasticity of labor supply.

aggregate and idiosyncratic uncertainty is thus revealed analytically.

Our result complements the findings of Bhandari, Evans, Golosov, and Sargent (2017) and Azzimonti and Yared (2017). We show that the optimal debt level under (only) ex post heterogeneity should approach FSI as long as the interest rate lies below the time discount rate, whereas in that literature the Ramsey planner opts to provide only partial insurance to rich agents and use a low interest rate tactic to redistribute income from the rich to the poor. Specifically, in the model of Bhandari, Evans, Golosov, and Sargent (2017), the Ramsey planner opts to exercise monopoly power to keep the interest rate sufficiently low so that they can borrow more cheaply from rich agents and subsidize poor agents. In other words, the Ramsey planner does not want to increase the bond supply to achieve FSI; instead, they opt to keep the interest rate sufficiently low so as to "tax" the rich (asset holders) and "subsidize" the poor (asset non-holders). Similarly, in the model of Azzimonti and Yared (2017), the Ramsey planner chooses not to issue a sufficient amount of debt to completely relax the borrowing constraints for all households and instead opts to monopolize the interest rate by borrowing more cheaply from the rich and subsidizing the poor through redistribution. The key difference between that literature and our model is that heterogeneity in our model comes entirely from unpredictable idiosyncratic uncertainty (as in standard Aiyagari-type models), whereas in that literature income/wealth difference across agents is fully or partially predictable (a given state in the beginning) due to ex ante differences in income/wealth (as in the two-class or overlapping generations models). In the case of only ex ante heterogeneity, public debt is no longer a critical device of self-insurance against future idiosyncratic risk, as in the case of ex post heterogeneity. Hence, income/wealth redistribution is a critical concern for the Ramsey planner in that literature, leading to their finding of the "under-supply" of public debt to provide rich agents with only minimal or partial self-insurance. Hence, one of our contributions is to show that under ex post heterogeneity the mechanism of debt determination is different from the models of Bhandari, Evans, Golosov, and Sargent (2017) and Azzimonti and Yared (2017); thus, our results will be totally different because of the prominent insurance role of government debt under ex post heterogeneity. This clear-cut difference between ex ante and ex post heterogeneity is why we are able to make progress on the contributions of LS and AMSS.

Angeletos, Collard, and Dellas (2020) study the Ramsey policy problem in the Lagos and Wright (2005) framework with heterogeneous agents and incomplete insurance markets. They show that when risk-free government bonds contribute to the supply of liquidity to alleviate private agents' borrowing constraints, issuing more debt raises welfare by improving the allocation of resources. Similar to our findings, they show that the heterogeneous-agents structure justifies a higher optimal level of public debt and introduces an interesting transitional path toward a Ramsey steady state—along which a departure from tax smoothing becomes desirable. In contrast to our model, their model does not necessarily feature FSI as the optimal target, and their optimal debt level preserves financial frictions in order to depress the interest rate on public debt. The key reason driving such a difference from ours is that their model does not have enough policy tools to match the number of goods (meaning that their model has an incomplete tax system at the macroeconomic level).

A recent work by Bassetto and Cui (2020) studies the optimal fiscal policy (e.g., capital tax) in an environment with capital accumulation but without aggregate uncertainty. Their analysis also indicates that the optimal debt policy is to provide a sufficient amount of public liquidity to completely alleviate firms' financial frictions whenever the fiscal capacity is not binding. Their result is similar to the finding in one of our related works, Chien and Wen (2019), despite the difference that in their model capital is underaccumulated but in our model capital is overaccumulated. They also show that if the consumption elasticity of intertemporal substitution is sufficiently large (as in the case of $\sigma < 1$ under the utility function $\frac{1}{1-\sigma}c^{1-\sigma}$), there can exist a different Ramsey steady state where the Lagrangian multiplier associated with the government budget constraint diverges to infinity. However, in our related work (Chien and Wen (2019)), we prove analytically that this alternative steady state can occur only if σ is strictly less than 1 and hence it does not apply to the model studied in this paper.

Bassetto (2014) uses a heterogeneous-agents model to show that when tax liabilities are unevenly spread across the population, deviations from tax smoothing lead to welfare improvement through redistributing wealth. In particular, when a "bad shock" hits the economy, the optimal policy will call for smaller or larger deficits depending on the political power of different groups. However, the result of deviation from tax smoothing found in our paper is not driven by redistribution but by the discrepancy between the market interest rate and the time discount rate, which is a hallmark feature embedded in Aiyagari-type models.

Aiyagari and McGrattan (1998) and Floden (2001) have studied optimal fiscal policies by maximizing the steady-state welfare of the competitive equilibrium and show that public debt can improve welfare but that the Ramsey allocation does not feature FSI. Our study complements that literature by taking into account the entire dynamic path of expected future welfare at time zero. More importantly, we show that the Ramsey planner has a dominant incentive to achieve FSI at "all costs" because of the wedge between the market interest rate and the time discount rate under precautionary saving motives. Yet, such a dynamic consideration of the Ramsey planner is missing in steady-state welfare maximization.[5]

In a model with informational friction, Gorton and Ordoñez (2022) study the role of government debt for intertemporal insurance against aggregate shocks and the role of credit in providing

[5] See Chien and Wen (2021) for more details on the difference between steady-state welfare analysis and the dynamic welfare analysis in the Ramsey literature.

insurance against idiosyncratic shocks, a topic closely related to our work. In contrast to our paper, their work emphasizes the role of information friction in the determination of optimal government debt.

Karantounias (2018) shows that even in a representative-agent model without any friction, the assumption of recursive preferences alone can lead to a departure from tax smoothing by adding a positive draft term in the dynamics of optimal debt.

Finally, Jiang, Lustig, Nieuwerburgh, and Xiaolan (2020) characterize a trade-off facing the government between insuring taxpayers or bondholders in an environment where asset pricing and debt dynamics are plausible but exogenously given.

6 Conclusion

In this paper we analyze the Ramsey planner's decision to finance stochastic public expenditures under a heterogeneous-agent incomplete-markets framework. Since models with heterogeneity are plagued with tractability issues, our paper makes substantial progress in this aspect by making the Ramsey problem analytically tractable despite having both aggregate and idiosyncratic uncertainty.

We show that as long as the market interest rate lies below the time discount rate (or the liquidity premium of debt is positive), the Ramsey planner has a dominant incentive to increase the debt supply to meet the private sector's demand for FSI—even at the cost of extra tax distortion and thus a departure from tax smoothing—regardless of whether or not bond returns are state-contingent. In particular, when financial assets are not state-contingent, the competing incentives for the Ramsey planner to smooth taxes and individual consumption imply that the long-run Ramsey allocation features a mean-reverting behavior, despite a unit-root component in optimal taxes and public debt, such that the probability of staying at the FSI allocation is strictly positive, albeit less than 1. In other words, when the precautionary saving motives of the government interact with the precautionary saving motives of heterogeneous households, the Ramsey allocation not only has a random-walk component, but the random-walk dynamics are *endogenously* bounded. We also use numerical analysis to confirm that the optimal quantity of public debt wanders around the FSI level and seldom falls below unless there exists an upper limit on government debt that binds occasionally.

Therefore, we believe that adding a liquidity premium into the value of government bonds via incomplete financial markets can bring the theory of public finance into closer conformity with reality.

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A Appendix

A.1 Proof of Proposition 1

Let $\beta^t \lambda_t(\theta^t, z^t)\pi(z^t, \theta^t)$ and $\beta^t \mu_t(\theta^t, z^{t+1})Q_{t+1}(z_{t+1})$ denote the Lagrangian multipliers for constraints (4) and (5), respectively. The first-order conditions for $c_t\left(\theta^t, z^t\right)$, $n_t\left(\theta^{t-1}, z^t\right)$, and $a_{t+1}(\theta^t, z^t)$ are given, respectively, by

$$\frac{\theta_t}{c_t\left(\theta^t, z^t\right)} = \lambda_t\left(\theta^t, z^t\right), \tag{41}$$

$$1 = \overline{w}_t(z^t) E_{\theta t} \lambda_t\left(\theta^t, z^t\right) = \overline{w}_t(z^t) \int \lambda_t\left(\theta^t, z^t\right) d\mathbf{F}(\theta_t), \tag{42}$$

$$\lambda_t\left(\theta^t, z^t\right) = \frac{\beta}{Q_{t+1}(z^t)} E_t E_{\theta t} \lambda_{t+1}\left(\theta^{t+1}, z^{t+1}\right) d\mathbf{F}(\theta_{t+1}) + \mu_t(\theta^t, z^t), \tag{43}$$

where equation (42) reflects the fact that labor supply $n_t(\theta^{t-1}, z^t)$ must be chosen before the idiosyncratic taste shocks (and hence the value of $\lambda_t(\theta^t, z^t)$) are realized. By the iid assumption of θ_t, equation (43) can be written as (using equation (42))

$$\lambda_t(\theta^t, z^t) = \Lambda_t(z^{t+1}) + \mu_t(\theta^t, z^{t+1}) \geq \Lambda_t(z^{t+1}), \tag{44}$$

where $\Lambda_t(z^{t+1})$ is defined as $\beta Q_{t+1}(z^t)^{-1} E_t \overline{w}_{t+1}(z^{t+1})^{-1}$; notice that $\overline{w}_{t+1}(z^{t+1})^{-1}$ by equation (42) pertains to the expected marginal utility of consumption.

The decision rules for an individual's consumption and savings are characterized by a cutoff strategy, taking as given the aggregate environment (such as interest rate and real wage). We adopt a guess-and-verify strategy to derive the decision rules. Anticipating that the optimal cutoff $\theta_t^*(z^t)$ is independent of an individual's history of shocks, consider the following two possible cases:

Case A. $\theta_t \leq \theta_t^*(z^t)$. In this case the urge to consume is low. It is hence optimal to save so as to prevent possible liquidity constraints in the future. From the household budget constraint, consider $a_{t+1}(\theta^t, z^{t+1}) \geq 0$, $\mu_t(\theta^t, z^{t+1}) = 0$ and the shadow value of good

$$\lambda_t(\theta^t, z^t) = \Lambda_t(z^t).$$

This implies that $\lambda_t(\theta^t, z^t)$ for this non-binding agent cannot depend on θ^t, which is defined as $\Lambda_t(z^t)$. Given that λ_t is independent of idiosyncratic shock, equation (41) implies that consumption is given by $c_t(\theta^t, z^t) = \theta_t \Lambda_t(z^t)^{-1}$. Defining $x_t(\theta^{t-1}, z^t) \equiv a_t(\theta^{t-1}, z^{t-1}) + \overline{w}_t(z^t) n_t(\theta^{t-1}, z^t)$ as the gross wealth (cash in hand) of a household, the budget identity (4) then implies

$$a_{t+1}(\theta^t, z^t) Q_{t+1}(z^t) = x_t(\theta^{t-1}, z^t) - \theta_t \Lambda_t(z^t)^{-1}.$$

The requirement of $a_{t+1}(\theta^t, z^t) \geq 0$ implies

$$\theta_t \leq \Lambda_t(z^t)x_t(z^t) \equiv \theta_t^*(z^t), \tag{45}$$

which defines the cutoff $\theta_t^*(z^t)$.

Notice that if the cutoff is independent of the idiosyncratic state, then the optimal cash in hand $x_t(z^t)$ is also independent of the idiosyncratic state. The intuition is that $x_t(z^t)$ is determined before the realization of θ_t and that all households face the same distribution of idiosyncratic shocks. Since the utility function is quasi-linear, the household is able to adjust labor income to meet any target level of cash in hand. As a result, the distribution of x is degenerate. This property simplifies the computation of the general equilibrium of the model tremendously.

Case B. $\theta_t > \theta_t^*(z^t)$. In this case the urge to consume is high;

$$\lambda_t(\theta^t, z^t) = \Lambda_t(z^t) + \mu_t(\theta^t, z^t).$$

It is then optimal not to save, so $a_{t+1}(\theta^t, z^t) = 0$ and $\mu_t(\theta^t, z^t) > 0$. By the resource constraint (4), we have $c_t(\theta^t, z^t) = x_t(z^t)$, which by equation (45) implies $c_t(\theta^t, z^t) = \theta_t^*(z^t)\Lambda_t(z^t)^{-1}$. Equation (41) then implies that the shadow value is given by $\lambda_t(\theta^t, z^t) = \frac{\theta_t}{\theta_t^*(z^t)}\Lambda_t(z^t)$. Since $\theta_t > \theta_t^*(z^t)$, equation (44) implies $\mu_t(\theta^t, z^{t+1}) = \Lambda_t(z^t)\left[\frac{\theta_t}{\theta_t^*(z^t)} - 1\right] > 0$. Notice that the shadow value of goods (the marginal utility of consumption), $\lambda_t(\theta^t, z^t)$, is higher under case B than under case A because of the binding borrowing constraint.

In addition, consider the case where the borrowing constraints just bind, which implies

$$c_t(\theta^t, z^t) = x_t(z^t) = \theta_t^*(z^t)\Lambda_t(z^t)^{-1},$$

which together with the discussions in cases A and B imply that the individual consumption and saving decision rules are given by equations (7) and (8), respectively.

The intertemporal price $Q_{t+1}(z^t)$ can be solved by the following steps. First, we know that

$$\lambda_t(\theta_t, z^t) = \begin{cases} \Lambda_t(z^t) \text{ if } \theta_t \leq \theta_t^*(z^t) \\ \frac{\theta_t}{\theta_t^*(z^t)}\Lambda_t(z^t) \text{ if } \theta_t > \theta_t^*(z^t) \end{cases}$$

and hence equation (42) can be rewritten as

$$\begin{aligned}
1 &= \overline{w}_t(z^t)\int \lambda_t(\theta, z^t)d\mathbf{F}(\theta) \\
&= \beta\frac{1}{Q_{t+1}(z^t)}E_t\frac{\overline{w}_t(z^t)}{\overline{w}_{t+1}(z^{t+1})}\left[\int_{\theta_L}^{\theta_t^*(z^t)} d\mathbf{F}(\theta) + \int_{\theta_t^*(z^t)}^{\theta_H} \frac{\theta}{\theta_t^*(z^t)} d\mathbf{F}(\theta)\right],
\end{aligned}$$

which implies

$$Q_{t+1}(z^t) = \beta E_t\frac{\overline{w}_t(z^t)}{\overline{w}_{t+1}(z^{t+1})}L(\theta_t^*(z^t)),$$

where $L(\theta_t^*(z^t))$ is defined as in equation (10).

The consumption of agents with a binding borrowing constraint in period t is given by $c_t(\theta^t, z^t) = x_t(\theta^{t-1}, z^t)$, so cash in hand x_t is obtained as

$$x_t(z^t) = \theta_t^*(z^t)\left[\beta\frac{1}{Q_{t+1}(z^t)}E_t\frac{1}{\overline{w}_{t+1}(z^{t+1})}\right]^{-1} = \theta_t^*(z^t)\overline{w}_t(z^t)L(\theta_t^*(z^t)),$$

which leads to equation (9).

Note that the decision rule of the household labor supply, equation (13), is decided residually to satisfy the household budget constraint. Finally, to ensure that the above proof and hence the associated cutoff-policy rules are consistent with the requirements of the interior choices of labor (namely $n_t \in (0, \overline{N})$), we need to consider the following two cases:

First, to ensure a non-negative n_t, consider the worst situation where n_t takes its minimum value. Given the definition of $x_t(z^t)$, n_t reaches its minimum if $\mu_t = 0$ and hence a_t takes the maximum possible value, $a_t(\theta^{t-1}, z^{t-1}) = \frac{\theta_{t-1}(z^{t-1})-\theta_L}{\theta_{t-1}(z^{t-1})}\frac{x_{t-1}(z^{t-1})}{Q_t(z^{t-1})}$. So $n_t(\theta^{t-1}) > 0$ if

$$x_t(z^t) - \frac{\theta_{t-1}^*(z^{t-1}) - \theta_L}{\theta_{t-1}^*(z^{t-1})}\frac{x_{t-1}(z^{t-1})}{Q_t(z^{t-1})} > 0, \tag{46}$$

The condition (46) is assumed to hold throughout the paper.

Second, to ensure that $n_t < \overline{N}$, consider those agents who encounter the borrowing constraint last period such that $a_t(\theta^{t-1}, z^{t-1}) = 0$. Their labor supply reaches the maximum value at $n_t(\theta_{t-1}, z^t) = \frac{x_t(z^t)}{\overline{w}_t(z^t)} = \theta_t^*(z^t)L(\theta_t^*(z^t))$. Given a finite value of $\theta_t^*(z^t)$, the value of $\overline{N}$ can be chosen such that

$$\overline{N} > \theta_t^*(z^t)L(\theta_t^*(z^t)) \text{ for all possible } z^t. \tag{47}$$

A.2 Proof of Proposition 2

A.2.1 The "Only If" Part

Assume that we have an allocation $\{\theta_t^*(z^t), N_t(z^t), C_t(z^t), B_{t+1}(z^t)\}_{t=0}^{\infty}$. With this allocation, we can directly construct prices, taxes, and all household allocations in the competitive equilibrium in the following steps.

1. According to the optimal condition of the firm's problem, $w_t(z^t)$ is set to 1.

2. Given $C_t(z^t)$ and $\theta_t^*(z^t)$, we can compute $x_t(z^t)$ by utilizing (18):

$$C_t(z^t) = D(\theta_t^*(z^t))x_t(z^t),$$

which implies $x_t(z^t) = \frac{C_t(z^t)}{D(\theta_t^*(z^t))} = \frac{1}{U_C(z^t)D(\theta_t^*(z^t))}$. By equation (9) and $w_t(z^t) = 1$, the labor tax rate $\tau_t(z^t)$ is given by

$$\overline{w}_t(z^t) = (1 - \tau_t(z^t))w_t(z^t) = \frac{x_t(z^t)}{L(\theta_t^*(z^t))\theta_t^*(z^t)} = \frac{1}{U_C(z^t)Z(\theta_t^*(z^t))}, \tag{48}$$

where $Z(\theta_t^*(z^t)) \equiv L(\theta_t^*(z^t))\theta_t^*(z^t)D(\theta_t^*(z^t))$.

3. Individual consumption and asset holdings, $c_t(\theta_t, z^t)$ and $a_{t+1}(\theta_t, z^t)$, are given by equations (7) and (8). In addition, the bond prices can be expressed as

$$Q_{t+1}(z^t) = \beta E_t \frac{\overline{w}_t(z^t)}{\overline{w}_{t+1}(z^{t+1})} L(\theta_t^*(z^t)) = \beta E_t \frac{U_C(z^{t+1})}{U_C(z^t)} \frac{Z(\theta_{t+1}^*(z^{t+1}))}{Z(\theta_t^*(z^t))} L(\theta_t^*(z^t)). \tag{49}$$

Finally, $n_t(\theta_{t-1})$ is set to satisfy the following condition implied by the individual household budget constraint:

$$n_t(\theta_{t-1}, z^t) = \frac{1}{\overline{w}_t(z^t)} \left[x_t(z^t) - a_t(\theta_{t-1}, z^t)\right].$$

4. By equation (8) and the asset market-clearing condition (14), we can derive:

$$Q_{t+1}(z^t)B_{t+1}(z^t) = \left[1 - D(\theta_t^*(z^t))\right] x_t(z^t),$$

which together with equation (49), equation (9) and equation (48) gives

$$\begin{aligned} & E_t\beta \frac{U_C(z^{t+1})}{U_C(z^t)} \frac{Z(\theta_{t+1}^*(z^{t+1}))}{Z(\theta_t^*(z^t))} L(\theta_t^*(z^t))B_{t+1}(z^t) \\ = \; & (1 - D(\theta_t^*(z^t))) \frac{1}{U_C(z^t)Z(\theta_t^*(z^t))} L(\theta_t^*(z^t))\theta_t^*(z^t). \end{aligned}$$

The equation above could be rewritten as the following constraint:

$$E_t U_C(z^{t+1}) Z(\theta^*_{t+1}(z^{t+1})) B_{t+1}(z^t) = \frac{1}{\beta}\theta^*_t(z^t)\left[1 - D(\theta^*_t(z^t))\right], \tag{50}$$

which is listed as equation (24).

5. Now we derive the implementability condition. Replacing $G_t(z^t)$ with $N_t(z^t) - C_t(z^t)$ in the flow government budget constraint gives

$$C_t(z^t) - (1-\tau_t(z^t))w_t(z^t)N_t(z^t) + \sum_{z_{t+1}} Q_{t+1}(z_{t+1}|z^t)B_{t+1}(z^t)\pi(z_{t+1}|z^t) \geq B_t(z^{t-1}).$$

By Steps 3 and 2, both $(1-\tau_t(z^t))w_t(z^t)$ and $Q_{t+1}(z^t)$ can be expressed as functions of C_t and θ^*_t:

$$C_t(z^t) - \frac{C_t(z^t)}{Z(\theta^*_t(z^t))}N_t(z^t) + E_t\beta\frac{U_C(z^{t+1})}{U_C(z^t)}\frac{Z(\theta^*_{t+1}(z^{t+1}))}{Z(\theta^*_t(z^t))}L(\theta^*_t(z^t))B_{t+1}(z^t) \geq B_t(z^{t-1}).$$

Multiplying the above equation with $U_C(z^t)Z(\theta^*_t(z^t))$ and using $U_C(z^t)C_t(z^t) = 1$ gives

$$\begin{aligned} & Z(\theta^*_t(z^t)) - N_t(z^t) + \beta E_t U_C(z^{t+1}) Z(\theta^*_{t+1}(z^{t+1})) L(\theta^*_t(z^t)) B_{t+1}(z^t) \\ \geq\ & U_C(z^t) Z(\theta^*_t(z^t)) B_t(z^{t-1}), \end{aligned} \tag{51}$$

which is a form of equation (23).

In short, Step 1 ensures that the representative firm's problem is solved. Step 2 ensures the individual household problem is solved. Steps 3 and 4 ensure the asset market-clearing condition and government budget constraints are satisfied, respectively. The labor market-clearing condition is satisfied by the Walras law.

A.2.2 The "If" Part

Note that the resource constraint and asset market-clearing condition are trivially implied by a competitive equilibrium since they are part of the definition. The implementability condition is constructed as follows. First, by using $w_t(z^t) = 1$, we rewrite the government budget constraint as

$$G_t(z^t) \leq N_t(z^t) - (1-\tau_t(z^t))N_t(z^t) + Q_{t+1}(z^t)B_{t+1}(z^t) - B_t(z^{t-1}).$$

Combining this equation with the resource constraint (17) and equation (11) implies

$$(1-\tau_t(z^t))N_t(z^t) + B_t(z^{t-1}) \leq C_t(z^t) + \beta E_t \frac{\overline{w}_t(z^t)}{\overline{w}_{t+1}(z^{t+1})} B_{t+1}(z^t).$$

Multiplying the above equation with $1/\overline{w}_t(z^t)$ gives

$$N_t(z^t) + \frac{B_t(z_t|z^{t-1})}{\overline{w}_t(z^t)} \leq \frac{C_t(z^t)}{\overline{w}_t(z^t)} + \beta E_t \frac{1}{\overline{w}_{t+1}(z^{t+1})} B_{t+1}(z^t). \tag{52}$$

In addition, combining equations (9) and (7) gives $1/\overline{w}_t(z^t) = U_C(z^t)Z(\theta_t^*(z^t))$, which implies that equation (51) can be expressed as (23).

A.3 Proof of Proposition 3

A.3.1 The Objective Function

Equations (7) and (18) imply that

$$c_t(\theta_t) = \min\left\{1, \frac{\theta_t}{\theta_t^*}\right\} x_t \text{ and } x_t = \frac{C_t}{D(\theta_t^*)}.$$

Hence, the objective function can be written as

$$\begin{aligned}
V &= E_0 \sum_{t=0}^{\infty} \beta^t \left[\theta_t \log c_t(\theta_t) - n_t\left(\theta_{t-1}\right)\right] \\
&= E_0 \sum_{t=0}^{\infty} \beta^t \left[\begin{array}{c} \int\limits_{\theta>\theta_t^*} \theta \log \frac{C_t}{D(\theta_t^*)} d\mathbf{F}(\theta) \\ + \int\limits_{\theta\leq\theta_t^*} \theta \log \frac{\theta}{\theta_t^*(z^t)} \frac{C_t}{D(\theta_t^*)} d\mathbf{F}(\theta) - N_t \end{array}\right] \\
&= E_0 \sum_{t=0}^{\infty} \beta^t \left[W(\theta_t^*) + \overline{\theta} \log C_t - N_t\right],
\end{aligned}$$

where $W(\theta_t^*)$ is defined as in equation (26).

A.3.2 The Property of W function

We first show that $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} = 0$ if $\theta_t^* = \theta_L$ or θ_H :

$$\begin{aligned}
\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} &= -\frac{\partial D\left(\theta_t^*\right)}{\partial \theta_t^*}\frac{\overline{\theta}}{D\left(\theta_t^*\right)} - \int_{\theta\le\theta_t^*}\frac{\theta}{\theta_t^*}d\mathbf{F}(\theta) \\
&= \left[\frac{\overline{\theta}}{D\left(\theta_t^*\right)\theta_t^*} - 1\right]\int_{\theta\le\theta_t^*}\frac{\theta}{\theta_t^*}d\mathbf{F}(\theta) \\
&= \begin{cases} \left[\frac{\overline{\theta}}{\overline{\theta}} - 1\right] \times \frac{\overline{\theta}}{\theta_H} = 0 & \text{if } \theta_t^* = \theta_H \\ \left[\frac{\overline{\theta}}{\theta_L} - 1\right] \times 0 = 0 & \text{if } \theta_t^* = \theta_L \end{cases}.
\end{aligned}$$

Next, we show that $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} > 0$ for any $\theta_t^* \in (\theta_L, \theta_H)$. Note that

$$\begin{aligned}
D\left(\theta_t^*\right)\theta_t^* &= \int_{\theta\le\theta_t^*}\theta d\mathbf{F}(\theta) + \theta_t^*\int_{\theta>\theta_t^*}d\mathbf{F}(\theta) = \overline{\theta} - \int_{\theta>\theta_t^*}(\theta-\theta_t^*)d\mathbf{F}(\theta) < \overline{\theta} \\
&\to \frac{\overline{\theta}}{D\left(\theta_t^*\right)\theta_t^*} > 1.
\end{aligned}$$

Hence,

$$\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} = \left[\frac{\overline{\theta}}{D\left(\theta_t^*\right)\theta_t^*} - 1\right]\int_{\theta\le\theta_t^*}\frac{\theta}{\theta_t^*}d\mathbf{F}(\theta) > 0.$$

A.4 Proof of Proposition 4

The Lagrangian of the Ramsey problem is then given by

$$\begin{aligned}
&\sum_{t=0}^{\infty}\sum_{z^t}\beta^t\left[W(\theta_t^*(z^t)) + \overline{\theta}\log C_t(z^t) - N_t(z^t)\right]\pi(z^t) \qquad (53)\\
&+\sum_{t=0}^{\infty}\sum_{z^t}\beta^t\psi_t(z^t)\pi(z^t)\left(N_t(z^t) - C_t(z^t) - G_t(z^t)\right) \\
&+\sum_{t=0}^{\infty}\sum_{z^t}\beta^t\phi_t(z^t)\pi(z^t)\left(\begin{array}{c} Z(\theta_t^*(z^t)) - N_t(z^t) \\ +B_{t+1}(z^t)\sum_{z_{t+1}}\beta U_C(z^{t+1})Z(\theta_{t+1}^*(z^{t+1}))L(\theta_t^*(z^t))\pi(z_{t+1}|z^t) \\ -U_C(z^t)Z(\theta_t^*(z^t))B_t(z^{t-1}) \end{array}\right) \\
&+\sum_{t=0}^{\infty}\sum_{z^{t+1}}\beta^t\mu_t^B(z^t)\pi(z^t)\left\{\begin{array}{c} B_{t+1}(z^t)\sum\limits_{z_{t+1}}\beta U_C(z^{t+1})Z(\theta_{t+1}^*(z^{t+1}))\pi(z_{t+1}|z^t) \\ -\theta_t^*(z^t)\left[1 - D(\theta_t^*(z^t))\right] \end{array}\right\},
\end{aligned}$$

where $\beta^t\psi_t(z^t)\pi(z^t)$, $\beta^t\phi_t(z^t)\pi(z^t)$, and $\beta^t\mu_t^B(z^t)\pi(z^t)$ denote the multipliers for the resource constraint, the implementability condition, and the bond market-clearing condition, respectively.

The FOCs with respect to $N_t(z^t)$, $C_t(z^t), B_{t+1}(z^t)$, and $\theta_t^*(z^t)$ for $t \geq 0$ are given, respectively, by

$$1+\phi_t(z^t) = \psi_t(z^t) \text{ for all } t \geq 0 \text{ and } z^t, \tag{54}$$

$$\overline{\theta} U_C(z^0) - \phi_0(z^0) U_{CC}(z^0) Z(\theta_0^*(z^0)) B_0 = \psi_0(z^0) \text{ for all } z^0 \text{ at period } 0, \tag{55}$$

$$\begin{aligned} & \overline{\theta} U_C(z^t) + (\phi_{t-1}(z^{t-1}) L(\theta_{t-1}^*(z^{t-1})) - \phi_t(z^t) + \mu_{t-1}^B(z^{t-1})) U_{CC}(z^t) Z(\theta_t^*(z^t)) B_t(z^{t-1}) \\ = \ & \psi_t(z^t) \text{ for all } t \geq 1 \text{ and } z^t, \end{aligned} \tag{56}$$

$$\mu_t^B(z^t) = E_t \frac{\phi_{t+1}(z^{t+1}) U_C(z^{t+1}) Z(\theta_{t+1}^*(z^{t+1}))}{E_t U_C(z^{t+1}) Z(\theta_{t+1}^*(z^{t+1}))} - \phi_t(z^t) L(\theta_t^*(z^t)), \tag{57}$$

$$\begin{aligned} & \frac{\partial W(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} + \phi_{t-1}(z^{t-1}) U_C(z^t) \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} L(\theta_{t-1}^*(z^{t-1})) B_t(z^{t-1}) \\ & + \phi_t(z^t) \left[\begin{array}{c} (1 - U_C(z^t) B_t(z^{t-1})) \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \\ + \frac{\partial L(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} B_{t+1}(z^t) E_t \beta U_C(z^{t+1}) Z(\theta_{t+1}^*(z^{t+1})) \end{array} \right] \\ = \ & J\left(\theta_t^*(z^t)\right) \mu_t^B(z^t) - \mu_{t-1}^B(z^{t-1}) B_t(z^{t-1}) U_C(z^t) \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)}, \end{aligned} \tag{58}$$

where the functions $H\left(\theta_t^*\right)$ and $J\left(\theta_t^*\right)$ are defined as

$$H\left(\theta_t^*\right) \equiv \frac{\partial (L(\theta_t^*)\theta_t^*)}{\partial \theta_t^*} = \left(L(\theta_t^*) + \frac{\partial L(\theta_t^*)}{\partial \theta_t^*} \theta_t^* \right), \tag{59}$$

$$J\left(\theta_t^*\right) \equiv \frac{\partial (1 - D(\theta_t^*))\theta_t^*}{\partial \theta_t^*} = \left(1 - D(\theta_t^*) - \theta_t^* \frac{\partial D\left(\theta_t^*\right)}{\partial \theta_t^*} \right). \tag{60}$$

To derive equation (27), we plug equation (57) into (58) to obtain

$$\begin{aligned} & \frac{\partial W(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} + \phi_t(z^t) \left[\left(1 - U_C(z^t) B_t(z^{t-1})\right) \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} + \frac{\partial L(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \theta_t^*(z^t) \left[1 - D(\theta_t^*(z^t))\right] \right] \\ = \ & J\left(\theta_t^*(z^t)\right) \left(E_t \phi_{t+1}(z^{t+1}) \frac{U_C(z^{t+1}) Z(\theta_{t+1}^*(z^{t+1}))}{E_t U_C(z^{t+1}) Z(\theta_{t+1}^*(z^{t+1}))} - \phi_t(z^t) L(\theta_t^*(z^t)) \right) \\ & - \left(\begin{array}{c} E_{t-1} \phi_t(z^t) \frac{U_C(z^t) Z(\theta_t^*(z^t))}{E_{t-1} U_C(z^t) Z(\theta_t^*(z^t))} \\ - \phi_{t-1}(z^{t-1}) L(\theta_{t-1}^*(z^{t-1})) \end{array} \right) B_t(z^{t-1}) U_C(z^t) \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \\ & - \phi_{t-1}(z^{t-1}) U_C(z^t) \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} L(\theta_{t-1}^*(z^{t-1})) B_t(z^{t-1}). \end{aligned}$$

Rearranging the above equation leads to

$$
\begin{aligned}
& \frac{\partial W(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} + \phi_t(z^t) \left[\begin{array}{c} \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} - U_C(z^t) B_t(z^{t-1}) \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \\ + \frac{\partial L(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \theta_t^*(z^t) \left[1 - D(\theta_t^*(z^t))\right] + J\left(\theta_t^*(z^t)\right) L(\theta_t^*(z^t)) \end{array} \right] \\
= \; & \mathbf{F}\left(\theta_t^*(z^t)\right) E_t \phi_{t+1}(z^{t+1}) \frac{U_C(z^{t+1}) Z(\theta_{t+1}^*(z^{t+1}))}{E_t U_C(z^{t+1}) Z(\theta_{t+1}^*(z^{t+1}))} \\
& - \left(E_{t-1} \phi_t(z^t) \frac{U_C(z^t) Z(\theta_t^*(z^t))}{E_{t-1} U_C(z^t) Z(\theta_t^*(z^t))} \right) B_t(z^{t-1}) U_C(z^t) \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)}. \qquad (61)
\end{aligned}
$$

Since Lemma 2 (see below) shows that

$$
H\left(\theta_t^*\right) = J\left(\theta_t^*\right) = \mathbf{F}(\theta_t^*) = \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} + \frac{\partial L(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \theta_t^*(z^t) \left[1 - D(\theta_t^*(z^t))\right] + J\left(\theta_t^*(z^t)\right) L(\theta_t^*(z^t)),
$$

equation (61) can be simplified to

$$
\begin{aligned}
& \frac{\partial W(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \frac{1}{\mathbf{F}\left(\theta_t^*(z^t)\right)} + \phi_t(z^t) = E_t \phi_{t+1}(z^{t+1}) \frac{U_C(z^{t+1}) Z(\theta_{t+1}^*(z^{t+1}))}{E_t U_C(z^{t+1}) Z(\theta_{t+1}^*(z^{t+1}))} \qquad (62) \\
& + \left(\begin{array}{c} \phi_t(z^t) \\ - E_{t-1} \phi_t(z^t) \frac{U_C(z^t) Z(\theta_t^*(z^t))}{E_{t-1} U_C(z^t) Z(\theta_t^*(z^t))} \end{array} \right) \frac{U_C(z^t)}{\mathbf{F}\left(\theta_t^*(z^t)\right)} B_t(z^{t-1}) \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)},
\end{aligned}
$$

which is equation (27).

Equation (29) can be obtained by plugging equations (57) and (54) into (56). The labor tax formula (30) is pinned down by the third step of the proof in Proposition 2.

Lemma 2. *1. The following derivation shows that $H\left(\theta_t^*\right) = J\left(\theta_t^*\right) = \mathbf{F}(\theta_t^*)$:*

$$
\begin{aligned}
J\left(\theta_t^*\right) & \equiv \left(1 - D(\theta_t^*) - \theta_t^* \frac{\partial D\left(\theta_t^*\right)}{\partial \theta_t^*}\right) \\
& = 1 - \left[\int_{\theta \le \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} d\mathbf{F}(\theta) \right] + \int_{\theta \le \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) \\
& = 1 - \int_{\theta > \theta_t^*} d\mathbf{F}(\theta) = \mathbf{F}(\theta_t^*),
\end{aligned}
$$

$$
\begin{aligned}
H\left(\theta_t^*\right) & \equiv \left(L(\theta_t^*) + \frac{\partial L(\theta_t^*)}{\partial \theta_t^*} \theta_t^*\right) \\
& = \int_{\theta \le \theta_t^*} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) - \int_{\theta > \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) \\
& = \int_{\theta \le \theta_t^*} d\mathbf{F}(\theta) = \mathbf{F}(\theta_t^*).
\end{aligned}
$$

2. In addition, we have

$$
\begin{aligned}
& \frac{\partial Z(\theta_t^*(z^t))}{\partial\theta_t^*(z^t)} + \frac{\partial L(\theta_t^*(z^t))}{\partial\theta_t^*(z^t)}\theta_t^*(z^t)\left(1 - D(\theta_t^*(z^t))\right) + J\left(\theta_t^*(z^t)\right) L(\theta_t^*(z^t) \\
= \; & \frac{\partial(L(\theta_t^*)\theta_t^* D(\theta_t^*))}{\partial\theta_t^*(z^t)} + \frac{\partial L(\theta_t^*(z^t))}{\partial\theta_t^*(z^t)}\theta_t^*(z^t)\left(1 - D(\theta_t^*(z^t))\right) + \frac{\partial(1 - D(\theta_t^*))\theta_t^*}{\partial\theta_t^*}L(\theta_t^*) \\
= \; & \frac{\partial(L(\theta_t^*)\theta_t^* D(\theta_t^*))}{\partial\theta_t^*(z^t)} + \frac{\partial L(\theta_t^*(z^t))\theta_t^*(z^t)\left(1 - D(\theta_t^*(z^t))\right)}{\partial\theta_t^*(z^t)} \\
= \; & \frac{\partial(L(\theta_t^*)\theta_t^*)}{\partial\theta_t^*(z^t)} = H\left(\theta_t^*\right).
\end{aligned}
$$

A.5 Proof of Corollary 1

It is straightforward to check that, with state-contingent bonds, the period-t bond price is modified to

$$Q_{t+1}(z_{t+1}|z^t) = \beta\frac{U_C(z^{t+1})}{U_C(z^t)}\frac{Z(\theta_{t+1}^*(z^{t+1}))}{Z(\theta_t^*(z^t))}L(\theta_t^*(z^t)).$$

A.5.1 Conditions to Support a Competitive Equilibrium with State-Contingent Bonds

With state-contingent bonds, it is straight forward to verify that Proposition 2 can be modified slightly into the following proposition:

Proposition 6. *Given the initial government bond supply B_0, the sequence of aggregate allocation $\{N_t(z^t), C_t(z^t), B_{t+1}(z_{t+1}|z^t)\}_{t=0}^{\infty}$ and the sequence of distribution $\{\theta_t^*(z^t)\}_{t=0}^{\infty}$ can be supported as a competitive equilibrium if and only if*

1. the resource constraint (17) holds;

2. the government budget constraint (3) holds for all z^t with $t \geq 0$, which can be transformed into the following implementability constraint after substituting out the tax rate $\tau_t(z^t)$ and state-contingent bond price $Q_{t+1}(z_{t+1}|z^t)$:

$$
\begin{aligned}
& Z(\theta_t^*(z^t)) - N_t(z^t) + \sum_{z_{t+1}} \beta U_C(z^{t+1}) Z(\theta_{t+1}^*(z^{t+1})) L(\theta_t^*(z^t)) B_{t+1}(z_{t+1}|z^t)\pi(z_{t+1}|z^t) \\
\geq \; & U_C(z^t) Z(\theta_t^*(z^t)) B_t(z_t|z^{t-1}),
\end{aligned} \tag{63}
$$

where $B_0(z^0) = B_0$ for all z_0; and

3. the following bond market-clearing conditions hold for all $t \geq 0$:

$$\sum_{z_{t+1}} U_{C,t+1} Z(\theta_{t+1}^*(z^{t+1})) B_{t+1}(z_{t+1}|z^t)\pi(z_{t+1}|z^t) = \frac{1}{\beta}\theta_t^*(z^t)\left[1 - D(\theta_t^*(z^t))\right]. \tag{64}$$

A.5.2 Ramsey Problem With State-Contingent Bonds

The Lagrangian of the Ramsey problem for the benchmark economy is given by

$$\begin{aligned}
&\sum_{t=0}^{\infty}\sum_{z^t}\beta^t\left[W(\theta_t^*(z^t))+\overline{\theta}\log C_t(z^t)-N_t(z^t)\right]\pi(z^t) \\
&+\sum_{t=0}^{\infty}\sum_{z^t}\beta^t\psi_t(z^t)\pi(z^t)\left(N_t(z^t)-C_t(z^t)-G_t(z^t)\right) \\
&+\sum_{t=0}^{\infty}\sum_{z^t}\beta^t\phi_t(z^t)\pi(z^t)\left(\begin{array}{c} Z(\theta_t^*(z^t))-N_t(z^t) \\ +\sum_{z_{t+1}}\beta U_C(z^{t+1})Z(\theta_{t+1}^*(z^{t+1}))L(\theta_t^*(z^t))B_{t+1}(z_{t+1}|z^t)\pi(z_{t+1}|z^t) \\ -U_C(z^t)Z(\theta_t^*(z^t))B_t(z_t|z^{t-1}) \end{array}\right) \\
&+\sum_{t=0}^{\infty}\sum_{z^t}\beta^t\mu_t^B(z^t)\pi(z^t)\left(\begin{array}{c} \sum\limits_{z_{t+1}}\beta B_{t+1}(z_{t+1}|z^t)U_C(z^{t+1})Z(\theta_{t+1}^*(z^{t+1}))\pi(z_{t+1}|z^t) \\ -\theta_t^*(z^t)\left[1-D(\theta_t^*(z^t))\right] \end{array}\right),
\end{aligned} \tag{65}$$

where $\beta^t\psi_t(z^t)\pi(z^t)$, $\beta^t\phi_t(z^t)\pi(z^t)$, and $\beta^t\mu^B(z^t)\pi(z^t)$ denote, respectively, the Lagrangian multipliers for (i) the resource constraint, (ii) the implementability condition, and (iii) the bond market-clearing condition.

The FOCs with respect to $N_t(z^t)$, $C_t(z^t)$, $\theta_t^*(z^t)$, and $B_{t+1}^*(z^{t+1})$ for $t \geq 0$ are given, respectively, by

$$N_t(z^t): 1+\phi_t(z^t)=\psi_t(z^t), \tag{66}$$

$$C_0(z^0): \overline{\theta}U_C(z^0)-\phi_0(z^0)U_{CC}(z^0)Z(\theta_0^*(z^0))B_0=\psi_0(z^0), \tag{67}$$

$$\begin{aligned}
C_t(z^t) \quad &: \quad \overline{\theta}U_C(z^t)+\left(\begin{array}{c} \phi_{t-1}(z^{t-1})L(\theta_{t-1}^*(z^{t-1})) \\ -\phi_t(z^t)+\mu_{t-1}^B(z^{t-1}) \end{array}\right)U_{CC}(z^t)Z(\theta_t^*(z^t))B_t(z^t|z^{t-1}) \\
&= \quad \psi_t(z^t),
\end{aligned} \tag{68}$$

$$\begin{aligned}
\theta_t^*(z^t) \quad &: \quad \frac{\partial W(\theta_t^*(z^t))}{\partial\theta_t^*(z^t)}+\phi_{t-1}(z^{t-1})U_C(z^t)\frac{\partial Z(\theta_t^*(z^t))}{\partial\theta_t^*(z^t)}L(\theta_{t-1}^*(z^{t-1}))B_t(z_t|z^{t-1}) \\
&+ \quad \phi_t(z^t)\left[\begin{array}{c} (1-U_C(z^t)B_t(z^t))\frac{\partial Z(\theta_t^*(z^t))}{\partial\theta_t^*(z^t)} \\ +\sum_{z_{t+1}}\beta U_C(z^{t+1})Z(\theta_{t+1}^*(z^{t+1}))\frac{\partial L(\theta_t^*(z^t))}{\partial\theta_t^*(z^t)}B_{t+1}(z_{t+1}|z^t)\pi(z_{t+1}|z^t) \end{array}\right] \\
&= \quad J\left(\theta_t^*(z^t)\right)\mu_t^B(z^t)-\mu_{t-1}^B(z^{t-1})B_t(z_t|z^{t-1})U_C(z^t)\frac{\partial Z(\theta_t^*(z^t))}{\partial\theta_t^*(z^t)},
\end{aligned} \tag{69}$$

$$B_{t+1}(z_{t+1}|z^t): \mu_t^B(z^t)=\phi_{t+1}(z^{t+1})-\phi_t(z^t)L(\theta_t^*(z^t)), \tag{70}$$

where the functions $H(\theta_t^*)$ and $J(\theta_t^*)$ are defined as equations (59) and (60), respectively.

Plugging equations (70) and (64) into (69) gives

$$\begin{aligned}
& \frac{\partial W(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} + \phi_{t-1}(z^{t-1}) U_C(z^t) \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} L(\theta_{t-1}^*(z^{t-1})) B_t(z_t|z^{t-1}) \\
& + \phi_t(z^t) \left[\begin{array}{c} (1 - U_C(z^t) B_t(z_t|z^{t-1})) \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \\ + \sum_{z_{t+1}} \beta U_C(z^{t+1}) Z(\theta_{t+1}^*(z^{t+1})) \frac{\partial L(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} B_{t+1}(z_{t+1}|z^t) \pi(z_{t+1}|z^t) \end{array} \right] \\
= \; & J\left(\theta_t^*(z^t)\right) \left(\phi_{t+1}(z^{t+1}) - \phi_t(z^t) L(\theta_t^*(z^t))\right) \\
& - B_t(z_t|z^{t-1}) U_C(z^t) \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \left(\phi_t(z^t) - \phi_{t-1}(z^{t-1}) L(\theta_{t-1}^*(z^{t-1}))\right).
\end{aligned}$$

The equation above can be simplified to

$$\begin{aligned}
& \frac{\partial W(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} + \phi_t(z^t) \left[\frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} + \frac{\partial L(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \theta_t^*(z^t) \left(1 - D(\theta_t^*(z^t))\right) \right] \\
= \; & J\left(\theta_t^*(z^t)\right) \left(\phi_{t+1}(z^{t+1}) - \phi_t(z^t) L(\theta_t^*(z^t)) + \frac{v_{t+1}^H(z^{t+1}) - v_{t+1}^L(z^{t+1})}{U_C(z^{t+1}) Z(\theta_{t+1}^*(z^{t+1}))} \right) \\
& - B_t(z_t|z^{t-1}) U_C(z^t) \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \left(\frac{v_{t-1}^H(z_t|z^{t-1}) - v_{t-1}^L(z_t|z^{t-1})}{U_C(z^t) Z(\theta_t^*(z^t))} \right),
\end{aligned}$$

which can be further simplified to

$$\frac{\partial W(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} + \phi_t(z^t) \left[\begin{array}{c} \frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} + \frac{\partial L(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \theta_t^*(z^t) \left(1 - D(\theta_t^*(z^t))\right) \\ + J\left(\theta_t^*(z^t)\right) L(\theta_t^*(z^t) \end{array} \right] = J\left(\theta_t^*(z^t)\right) \phi_{t+1}(z^{t+1}).$$

Using Lemma 2, the above equation can be further simplified to

$$\frac{\partial W(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} + \mathbf{F}(\theta_t^*) \phi_t(z^t) = \mathbf{F}(\theta_t^*) \phi_{t+1}(z^{t+1}),$$

which is equation (31).

Now, plugging equations (66) and (70) into (68) gives (32). The labor tax formula (33) is pinned down by step 3 of the proof in Proposition 2.

A.6 Proof of Proposition 5

The proof takes several steps, and in each step we assume that the natural debt limit does not bind (or if there exists any ad hoc debt limit $\overline{B}$ we assume that it lies sufficiently above the optimal debt level B^* required for FSI). These assumptions are innocuous because binding debt limits would automatically imply a bounded variance.

Step 1 Any allocation with a constant Lagrangian multiplier $\phi \geq 0$ and a cutoff $\theta_t^* \in (\theta_L, \theta_H)$ cannot be a Ramsey equilibrium. A constant ϕ implies that equation (27) becomes

$$0 = \frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{1}{\mathbf{F}(\theta_t^*)},$$

which cannot be true, because $\theta_t^* \in (\theta_L, \theta_H)$ implies $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{1}{\mathbf{F}(\theta_t^*)} > 0$.

Step 2 Any allocation staying permanently at FSI with $\theta_t^* = \theta_H$ for $t \geq T$ (T large) cannot be a Ramsey equilibrium. We prove by contradiction. Suppose instead we have $\theta_t^* = \theta_H$ for $t \geq T$, then in the long run, equation (27) is reduced to

$$E_t q_{t+1} \phi_{t+1} = \phi_t,$$

where $q_{t+1} = \frac{U_{C_{t+1}}}{E_t U_{C_{t+1}}}$. In this case, a Ramsey allocation with constant $\phi > 0$ is impossible since it implies a constant C and N and hence violates the resource constant—given that G_t is stochastic. It is also impossible for $\phi = 0$, because the level of the bond supply must decrease to a non-positive value, which cannot support the FSI allocation. Hence, ϕ_t must feature a STUR process (since $E_t q_{t+1} = 1$) and is identical to the one in the AMSS model. This STUR process is implicitly bounded only above by the government's natural borrowing limit, but there is no lower bound to prevent divergence. Hence, the STUR property of ϕ_t implies a strictly positive probability that ϕ_t will decrease stochastically to a low level such that the bond supply B_t decreases below the one required by FSI and hence renders a FSI allocation impossible.

Step 3 Any allocation featuring $B_t \leq 0$ and $\theta_t^* = \theta_L$ cannot be a Ramsey equilibrium. Notice that we have implicitly imposed a lower debt limit of 0 on the model under the assumption $a_{t+1} \geq 0$, which implies that the aggregate bond supply cannot be negative.[6] So, we only need to consider the possible allocation with $\lim_{t \to \infty} B_t = 0$ and prove that it cannot be a Ramsey equilibrium. When the optimal debt level is so low ($B_t = 0$) such that $\theta_t^* = \theta_L$, we have $\lim_{\theta^* \to \theta_L} \frac{\partial Z(\theta_t^*)}{\partial \theta_t^*} \frac{1}{\mathbf{F}(\theta^*)} = 0$ and $\lim_{\theta^* \to \theta_L} \frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{1}{\mathbf{F}(\theta_t^*)} = 0$; so equation (27) is reduced to

$$E_t q_{t+1} \phi_{t+1} = \phi_t, \tag{71}$$

where $q_{t+1} = \frac{U_{C_{t+1}}}{E_t U_{C_{t+1}}}$ and $E_t q_{t+1} = 1$. Thus, as in Step 2, ϕ_t follows a STUR process with a strictly positive probability that ϕ_t and B_t will increase stochastically, thus making $B_{t+1} > 0$

[6]But see our robustness analysis for the case with positive household borrowing in the online Appendix.

and $\theta_t^* > \theta_L$ with positive probability. This also rules out the possibility for ϕ_t to converge to zero in the long run.

Step 4 The above three steps rule out any constant allocation as a Ramsey equilibrium. Now we can consider the possibility of a bounded stochastic Ramsey equilibrium. First, we show that if $\theta_t^* < \theta_H$ and is close to θ_L, the Lagrangian multiplier ϕ_t will always have a tendency to increase stochastically over time, which implies that the bond supply also increases stochastically over time as long as $\theta_t^* < \theta_H$ and is close to θ_L; hence, the model will stochastically tend to converge toward $\theta_t^* = \theta_H$. Second, we have already shown in Step 2 that the economy cannot stay at the FSI position indefinitely; hence, ϕ_t and B_t will wander around and eventually make $\theta_t^* < \theta_H$ again with positive probability (because it is bounded above by the government's natural borrowing limit). Since we assume that the government's natural debt limit creates an upper bound on the multiplier, to show that equation (34) is a bounded STUR process we only need to show that there exists an endogenous lower bound to prevent the multiplier from diverging to negative infinity or zero. Since the welfare term $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*}\frac{1}{\mathbf{F}(\theta_t^*)}$ always exerts a positive growth force on the multiplier as long as $\theta_t^* < \theta_H$, we can ignore the welfare term $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*}\frac{1}{\mathbf{F}(\theta_t^*)}$ in equation (34) and need only to consider the dynamic effects of the residual term $\wp_t \equiv (E_{t-1}q_t\phi_t - \phi_t)\frac{U_{C,t}B_t}{\mathbf{F}(\theta_t^*)}\frac{\partial Z(\theta_t^*)}{\partial \theta_t^*}$ on the multiplier. Since this residual term is correlated with the history of ϕ_t, it effectively creates long memory and history-dependence in the dynamics of the multiplier. To see this, we can consider the following analysis. Since any expected value of a variable x_t can be rewritten as $E_{t-1}x_t = x_t - \varepsilon_t$ with $E_{t-1}\varepsilon_t = 0$, we can define $E_{t-1}q_t\phi_t = q_t\phi_t - \varepsilon_t$. Also, define $\alpha_t \equiv \frac{U_{C,t}B_t}{\mathbf{F}(\theta_t^*)}\frac{\partial Z(\theta_t^*)}{\partial \theta_t^*}$, equation (34) is then simplified to

$$E_t q_{t+1}\phi_{t+1} = \phi_t + (q_t - 1)\phi_t\alpha_t - \varepsilon_t\alpha_t. \tag{72}$$

Lagging this equation by one period gives $E_{t-1}q_t\phi_t = \phi_{t-1} + (q_{t-1} - 1)\phi_{t-1}\alpha_{t-1} - \varepsilon_{t-1}\alpha_{t-1}$. Plugging this lagged expression back into the right-hand side of the original equation (34) and rearranging terms gives

$$E_t q_{t+1}\phi_{t+1} = (1 - \alpha_t)\phi_t + \phi_{t-1}\alpha_t(1 + (q_t - 1)\alpha_{t-1}) - \varepsilon_{t-1}\alpha_{t-1}\alpha_t. \tag{73}$$

Since q_t is the STUR in equation (34), it should remain the STUR in the above equation as well. To isolate the dynamic effects of α_t, we can simply let $q_t = 1$ so that the STUR in equation (34) becomes a deterministic unit root but remains responsible for the random-walk property of the multiplier ϕ_t. This simplification is accurate to a first-order approximation without loss of generality since the forecasting error $\varepsilon_t^q = q_t - 1$ is only a perturbation to both the deterministic unit root and the coefficient α_t. In this simplified case of $q_t = 1$ equation (72) becomes

$$\phi_{t+1} = \phi_t + \varepsilon_{t+1} - \varepsilon_t \alpha_t, \tag{74}$$

and equation (73) simplifies to

$$\phi_{t+1} = (1 - \alpha_t)\phi_t + \phi_{t-1}\alpha_t + \varepsilon_{t+1} - \alpha_{t-1}\alpha_t\varepsilon_{t-1}. \tag{75}$$

Clearly, the substitutions that transformed equation (72) into equation (73) are equivalent to transforming the first-order difference equation (74) into the second-order difference equation (75) under the simplifying assumption $q_t = 1$ and the identity $\varepsilon_t \equiv \phi_t - E_{t-1}\phi_t$. This second-order difference equation (75) has two roots (eigenvalues), $\{\lambda_1, \lambda_2\} = \{1, -\alpha_t\}$, and a lagged error term $\alpha_t\alpha_{t-1}\varepsilon_{t-1}$. Give that the lagged error term contains the effect of α_{t-1}, the above substitution can be iterated, generating more higher-order terms and eigenvalues. But stopping at the second-order expansion is already sufficiently illuminating. It shows that the error term $\wp(\theta_t^*)$ does not change the STUR q_t but has brought in one additional autoregressive root $\lambda_2 = -\alpha_t$ into the random-walk process of ϕ_t, making the STUR process more history-dependent. In particular, since $\alpha_t < 0$ for θ_t^* close to θ_L, it adds an additional mean-reverting force (positive growth rate λ_2) into the dynamics of ϕ_t to keep the multiplier away from zero and the cutoff away from its lower bound, reinforcing the growth effect of the welfare term $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*}\frac{1}{\mathbf{F}(\theta_t^*)} > 0$. On the other hand, for θ_t^* close to θ_H, we have $\alpha_t > 0$, so the second eigenvalue λ_2 is negative. This implies that whenever the distribution is sufficiently close to FSI, the positive growth dynamics of the multiplier is mitigated by the negative root $\lambda_2 = -\alpha_t < 0$, slowing down the speed of convergence (or mean-reverting tendency) toward FSI and pulling the debt supply downward, thus not only increasing the variances of optimal debt level and taxes but also creating an implicit force to reduce the probability of reaching the FSI allocation. Like the growth effect of the welfare term $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*}\frac{1}{\mathbf{F}(\theta_t^*)}$, this autoregressive root $\lambda_2 = -\alpha_t(\theta_t^*)$ also vanishes in the limit as θ_t^* approaches θ_H or θ_L. Since we have already proved that the cutoff cannot permanently stay at the two corners $\{\theta_L, \theta_H\}$, so it must fluctuate between these two corners and render the multiplier a bounded STUR process, thanks to the dynamic influence of the time-varying distribution θ_t^*. In other words, the STUR process in equation (34) is endogenously bounded—below by the mean-reverting forces created by the welfare term $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*}\frac{1}{\mathbf{F}(\theta_t^*)}$ and the residual term $\wp_t(\theta_t^*)$ and above by the government natural borrowing limit.

B Online Appendix (Not for Publication)

This online Appendix shows that the main message in our paper is robust even if we relax the household borrowing constraint in equation (5) to allow borrowing. To take advantage of the tractability of our model, we can use a finite-horizon version (with $t = 0, 1, 2, ..., T$) of our model (with risk-free debt) to conduct the analysis, and then let $T \to \infty$ to study the model's limiting behavior under an infinite horizon. In this way, we can also analyze the terminal-period effect or the turnpike phenomenon of the optimal supply of public debt. The key change is a new form of household borrowing constraints:

$$a_{t+1} \geq -\overline{\alpha}, \tag{76}$$

where the initial asset position a_0 and the borrowing limit $\overline{\alpha} \geq 0$ are exogenously given and assumed to be the same across households.

In the anticipation that T is the end-of-life (last) period, it must be true that the last-period savings are $a_{T+1} = 0$ for all households and the last-period optimal bond supply is $B_{T+1}\left(z^T\right) = 0$ with price $Q_{T+1}\left(z^T\right) = 0$. This implies that if government bonds are non-state contingent, not only is the Ramsey planner unable to issue new debt to buffer against the last-period government spending shock G_T and smooth the last-period tax rate τ_T, but households are also unable to borrow ($a_{T+1} = 0$) to buffer against the last-period idiosyncratic preference shock θ_T.

To facilitate the analysis, we consider two scenarios: In scenario A, government spending is time-varying but deterministic; in scenario B, government spending is stochastic. We will show that the insight gained for the effect of $\overline{\alpha}$ on debt policy in scenario A can be applied to scenario B with aggregate uncertainty. Under scenario A, we have the following proposition:

Proposition 7. *Without aggregate uncertainty, the dynamics of ϕ_{t+1} are given by the following equations:*

$$\phi_1 = \phi_0 \left[1 + \frac{\overline{\alpha}}{B_0 + \overline{\alpha}} \Omega^A\left(\theta_0^*\right)\right] + \Delta^A\left(\theta_0^*\right) - \frac{\phi_0 B_0 U_{C_0}}{\mathbf{F}\left(\theta_0^*\right)}, \tag{77}$$

$$\phi_{t+1} = \phi_t \left[1 + \frac{\overline{\alpha}}{B_t + \overline{\alpha}} \Omega^A\left(\theta_t^*\right)\right] + \Delta^A\left(\theta_t^*\right), \tag{78}$$

for $t \in 1, 2, ..., T-2$, and the dynamics of ϕ_T in the last period are given by

$$\phi_T = \phi_{T-1} \left[1 + \frac{\overline{\alpha}}{B_{T-1} + \overline{\alpha}} \Omega^A\left(\theta_{T-1}^*\right)\right] + \Delta^A\left(\theta_{T-1}^*\right) + \Theta^A\left(\theta_{T-2}^*, \theta_{T-1}^*\right), \tag{79}$$

where

$$\Omega^A(\theta_t^*) \equiv -\left(\frac{[1-D(\theta_t^*)]\,\theta_t^*}{\mathbf{F}(\theta_t^*)}\frac{\partial L(\theta_t^*)}{\partial\theta_t^*}\right) \geq 0,$$

$$\Delta^A(\theta_t^*) \equiv \frac{\partial W(\theta_t^*)}{\partial\theta_t^*}\frac{1}{\mathbf{F}(\theta_t^*)} \geq 0,$$

$$\Theta^A\left(\theta_{T-2}^*,\theta_{T-1}^*\right) \equiv \overline{\alpha}\frac{\left(\phi_{T-1}-\phi_{T-2}L\left(\theta_{T-2}^*\right)\right)}{\mathbf{F}\left(\theta_{T-1}^*\right)}\frac{U_{C_{T-1}}}{\mathbf{F}\left(\theta_{T-1}^*\right)}\frac{\partial Z\left(\theta_{T-1}^*\right)}{\partial\theta_{T-1}^*}.$$

In addition, if $\theta_t^* = \theta_H$*, then* $\Omega^A(\theta_t^*) = \Delta^A(\theta_t^*) = 0$*; also,* $\Theta^A\left(\theta_{T-2}^*,\theta_{T-1}^*\right) = 0$ *if* $\theta_{T-1}^* = \theta_H$*.*

Equation (78) is the counterpart of equation (31) in Corollary 1, except here with an additional positive term $\frac{\bar{\alpha}\phi_t}{B_t+\overline{\alpha}}\Omega^A(\theta_t^*) > 0$ (with equality if and only if $\theta_t^* = \theta_H$) in the auto-regressive coefficient of ϕ_t. It shows that (i) the marginal benefit of an improved distribution, $\frac{\partial W(\theta_t^*)}{\partial\theta_t^*}\frac{1}{\mathbf{F}(\theta_t^*)}$, due to an increase in public debt B_{t+1} still exerts a positive force on the next period multiplier ϕ_{t+1} as before, and (ii) allowing for household borrowing generates an additional positive force $\frac{\overline{\alpha}}{B_t+\overline{\alpha}}\Omega^A(\theta_t^*) > 0$ in the growth rate or the auto-regressive root of ϕ_{t+1}, thus greatly amplifying and reinforcing the original upward trend in ϕ_{t+1}.

Equation (77) shows that the initial debt level B_0 negatively affects debt growth in period $t = 1$, because the higher B_0 is, the less need there is to increase the future bond supply to reach the FSI allocation. This negative effect of the initial-period bond supply is short-lived and disappears in the law of motion for ϕ_{t+1} for $t \geq 1$.

The intuition is as follows: Suppose the initial bond supply B_0 is already high enough to achieve FSI in period $t = 1$, then $\theta_1^* = \theta_H$, $\frac{\partial W\left(\theta_1^*\right)}{\partial\theta_1^*}\frac{1}{\mathbf{F}\left(\theta_1^*\right)} + \frac{\bar{\alpha}\phi_1}{B_1+\bar{\alpha}}\Omega^A(\theta_1^*) = 0$; so there is no need to adjust the new bond supply B_2 in period $t = 1$. This also implies that B_2 is high enough to achieve FSI in period $t = 2$; hence, $\theta_2^* = \theta_H$ and equation (78) implies $\phi_3 = \phi_2$; consequently, we have $\phi_{t+1} = \phi_t = \phi_1$ and $\theta_t^* = \theta_H$ for $t \leq T-1$. This also implies that the additional term in equation (79) $\Theta^A\left(\theta_{T-2}^*,\theta_{T-1}^*\right) = 0$, because $\theta_{T-1}^* = \theta_{T-2}^* = \theta_H$.

However, if B_0 is sufficiently low such that $\theta_1^* < \theta_H$, then the Ramsey planner's incentive to reach a FSI allocation in $t = 2$ by increasing B_1 is stronger if it is easier for households to borrow—although this does not necessarily mean that it is optimal to adjust B_1 immediately to reach FSI within one period, as the decision must consider the dynamic adjustment costs and trade-offs.

In other words, a more relaxed borrowing constraint on households does not eliminate the Ramsey planner's incentive for providing FSI as long as the household borrowing constraints still bind with positive probability (or for some households). However, with the possibility of household borrowing ($\overline{\alpha} > 0$), the marginal benefit of increasing government bonds is greater than in the case of $\overline{\alpha} = 0$. This happens because as argued before, an increase in the debt supply has not only the direct benefit of $\phi_t L(\theta_t^*)$, but also an indirect benefit $\frac{\partial Q_{t+1}}{\partial B_{t+1}} > 0$ through a positive change in bond prices (as shown before). However, in both the LS model and our benchmark model, the gross benefit of a bond-price change is $\frac{\partial Q_{t+1}}{\partial B_{t+1}} B_{t+1}$, where B_{t+1} is the total bond supply. In the current model, the gross benefit is $\frac{\partial Q_{t+1}}{\partial B_{t+1}}(B_{t+1}+\overline{\alpha})$ because the higher bond price also increases the value of household borrowing since the gross savings of household are given by $(B_{t+1}+\overline{\alpha})$ (this can be seen by changing equation (76) to a gross borrowing constraint: $a_{t+1}+\overline{\alpha} \geq 0$). So the gross benefit of a bond-price change is $\frac{\partial Q_{t+1}}{\partial B_{t+1}}(B_{t+1}+\overline{\alpha})$, which is greater when $\overline{\alpha} > 0$. In this regard, the term $\frac{\bar{\alpha}\phi_t}{B_t+\bar{\alpha}}\Omega(\theta_t^*)$ in equation (78) can be interpreted as the increased marginal benefit of the bond supply due to the additional bond-price effect from household borrowing $\overline{\alpha}$.

Therefore, equation (78) shows that when households can borrow $\overline{\alpha} > 0$, there still exists the need for more self-insurance so long as the borrowing constraints bind for some households (such that $\theta_t^* < \theta_H$). Consequently, the positive force behind debt growth (or the growth in ϕ_{t+1}) not only exists but is even stronger and reinforced by household borrowing. This implies that the transition period toward a FSI allocation can be reinforced and thus shortened by allowing households to borrow.

Nonetheless, the long-run equilibrium debt level can be negative if $\overline{\alpha}$ is large enough. This can be seen from the aggregate demand function of government bonds in this finite-horizon model:

$$\overline{Q}_{t+1} B_{t+1} = \frac{1-D(\theta_t^*)}{D(\theta_t^*)} C_t - \overline{Q}_{t+1}\overline{\alpha}, \text{ for } t = 0, 1, 2, ..., T-1, \tag{80}$$

which shows that, everything else equal, an increase in $\overline{\alpha}$ can eventually imply a negative level of public debt B_{t+1} since $\frac{1-D(\theta_t^*)}{D(\theta_t^*)} C_t$ and $\overline{Q}_{t+1} = \beta \frac{1-\tau_t}{1-\tau_{t+1}} L(\theta_t^*)$ are both bounded. By the same argument, everything else equal, if the variance of idiosyncratic risk θ shrinks, then the same borrowing limit $\overline{\alpha}$ may imply a more negative optimal debt level B_{t+1}—because a smaller variance of θ implies it is easier to achieve FSI with a lower debt level.

However, notice that these arguments are made based on the assumption that $\theta_0^* < \theta_H$. If the borrowing limit $\overline{\alpha}$ is high enough such that any initial debt level B_0 can support a FSI allocation (with $\theta_0^* = \theta_H$), then the equilibrium bond supply B_{t+1} will be determined by B_0 regardless it being positive or negative, as in the LS model.

Now consider the Ramsey planner's decision for bond supply B_T in the second to last period $T-1$ and the dynamics of ϕ_T between $T-1$ and T. There is now an additional term $\Theta_T^A \equiv \overline{\alpha}\frac{\left(\phi_{T-1}-\phi_{T-2}L\left(\theta_{T-2}^*\right)\right)}{\mathbf{F}\left(\theta_{T-1}^*\right)}\frac{U_{C_{T-1}}}{\mathbf{F}\left(\theta_{T-1}^*\right)}\frac{\partial Z\left(\theta_{T-1}^*\right)}{\partial\theta_{T-1}^*}$ appearing in equation (79) compared to equation (78). This additional term shows up in the FOC of B_{T-1} because T is the last period and no new bonds $B_{T+1} = 0$ can be issued in the last period. This additional term η_T is positive (negative) if $\left[\phi_{T-1} - \phi_{T-2} L\left(\theta_{T-2}^*\right)\right] > 0$ (< 0). Since $\left[\phi_{T-1} - \phi_{T-2} L\left(\theta_{T-2}^*\right)\right]$ is precisely the net marginal utility cost of increasing the bond supply in the previous period $T-2$ when deciding B_{T-1}, $\Theta_T^A > 0$ implies that it was not worth increasing the bond supply in the previous period; consequently, B_{T-1} is now too low to provide FSI in period $T-1$; hence, if $\Theta_T^A > 0$, then the incentive to increase bond supply B_{T-1} is reinforced.

However, given that government spending in each period is fully anticipated, $\Theta_T^A > 0$ contradicts the logic that the Ramsey planner always opts to increase the bond supply whenever $\theta_t^* < \theta_H$. Hence, it must be true that $\Theta_T^A < 0$. In this case, the implication is that it was optimal in the previous period $T-2$ to increase the bond supply B_{T-1}; however, doing so would reduce the incentive to further increase public debt B_T in period $T-1$ because T is the last period and the terminal period bond level B_{T+1} must be zero. Therefore, equation (79) adds an additional consideration for the determination of public debt in period $T-1$, leading to a slowdown in the growth of public debt B_T (as well as the growth rate of ϕ_T). But as $T \to \infty$, this additional terminal concern vanishes and the law of motion for ϕ_{t+1} will always be characterized by equation (78).

Now consider scenario B with aggregate uncertainty. For simplicity, assuming $B_0 = 0$, we have the following proposition:

Proposition 8. *Under aggregate uncertainty, the dynamic path $\{\phi_t\}_{t=0}^T$ is given by the following equations:*

$$E_0 q_1 \phi_1 = \phi_0 \left[1 + \frac{\overline{\alpha}}{B_0 + \overline{\alpha}} \Omega_0^B\right] + \Delta_0^B - \frac{\phi_0 B_0 U_C}{\mathbf{F}\left(\theta_0^*\right)} \tag{81}$$

$$E_t q_{t+1} \phi_{t+1} = \phi_t \left[1 + \frac{\overline{\alpha}}{B_t + \overline{\alpha}} \Omega_t^B\right] + \Delta_t^B, \text{ for } t = 1, 2, ..., T-2 \tag{82}$$

$$E_{T-1} q_T \phi_T = \phi_{T-1} \left[1 + \frac{\overline{\alpha}}{B_{T-1} + \overline{\alpha}} \Omega_{T-1}^B\right] + \Delta_{T-1}^B + \Theta_T^B, \tag{83}$$

where

$$\begin{aligned}
\Omega_t^B\left(z^t\right) &\equiv -\left(\frac{\left[1 - D\left(\theta_t^*\right)\right] \theta_t^*}{\mathbf{F}\left(\theta_t^*\right)} \frac{\partial L\left(\theta_t^*\right)}{\partial \theta_t^*}\right) \geq 0, \\
\Delta_t^B\left(z^t\right) &\equiv \frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{1}{\mathbf{F}(\theta_t^*)} + \left(E_{t-1} q_t \phi_t - \phi_t\right) \frac{U_{C,t} B_t}{\mathbf{F}\left(\theta_t^*\right)} \frac{\partial Z(\theta_t^*)}{\partial \theta_t^*}, \\
\Theta_T^B &\equiv \overline{\alpha} \frac{U_{C,T-1}}{\mathbf{F}\left(\theta_{T-1}^*\right)} \frac{\partial Z(\theta_{T-1}^*)}{\partial \theta_{T-1}^*} \left[E_{T-2} q_{T-1} \phi_{T-1} - \phi_{T-2} L\left(\theta_{T-2}^*\right)\right].
\end{aligned}$$

Clearly, the law of motion for ϕ_{t+1} under aggregate uncertainty is similar to that without aggregate uncertainty except that (i) there is now a STUR q_{t+1} and (ii) the terminal period has an additional negative term Θ_T^B that is analogous to Θ_T^A in equation (79).

Therefore, we conclude that allowing households to borrow does not alter the basic insight from the previous analysis (under $\overline{\alpha} = 0$): There exists a dominant positive force that induces the growth of public debt. The size of the variance of aggregate risk (for any given variance of idiosyncratic risk) only serves to make the path of convergence toward the FSI allocation more stochastic and possibly more auto-correlated, but it does not by itself negatively affect the growth rate of public debt as long as the liquidity premium of public debt remains positive.

In this sense, there does not exist any competition between aggregate tax smoothing and individual consumption smoothing—suggesting a departure from tax smoothing—along the transition. In the model of AMSS, the Ramsey planner is shown to have a precautionary saving motive to hold private-sector issued bonds because there does not exist a bounded stationary Ramsey equilibrium except in the special case where $\lim_{t\to\infty} \phi_{t+1} = 0$, so that $\lim_{t\to\infty} \tau_t = 0$ and $\lim_{t\to\infty} B_{t+1} < 0$. Such Ramsey equilibrium is destroyed (eliminated) in our model by the Ramsey planner's incentives to supply debt to provide FSI.

B.1 Proof of Propositions 7 and 8

We start with the proof of Proposition 8 since Proposition 7 is just a special case of the former without aggregate risk.

B.1.1 Competitive Equilibrium

First, the household decision rules given in Proposition 1 are modified in the following way. For $t < T$, we have

$$x_t(z^t) = \overline{w}_t(z^t) L(\theta_t^*(z^t))\theta_t^*(z^t) - Q_{t+1}(z^t)\overline{\alpha}, \tag{84}$$

$$c_t(\theta_t, z^t) = \min\left\{1, \frac{\theta_t}{\theta_t^*(z^t)}\right\} \left(x_t(z^t) + Q_{t+1}(z^t)\overline{\alpha}\right), \tag{85}$$

$$Q_{t+1}(z^t)a_{t+1}\left(\theta^t, z^t\right) = \max\left\{0, \frac{\theta_t^*(z^t) - \theta_t}{\theta_t^*(z^t)}\right\} \left(x_t(z^t) + Q_{t+1}(z^t)\overline{\alpha}\right), \tag{86}$$

and

$$n_t(\theta_{t-1}, z^t) = \frac{1}{\overline{w}_t(z^t)} \left[x_t(z^t) - a_t(z_t|\theta_{t-1}, z^{t-1})\right]. \tag{87}$$

For $t = T$, $c_T(z^T) = x_T(z^T) = \overline{\theta}\overline{w}_T(z^T)$, $Q_{T+1}(z^T) = a_{T+1}(z^T) = 0$, and $n_T(\theta_{T-1}, z^T) = \frac{1}{\overline{w}_T(z^T)}\left[\overline{\theta}\overline{w}_T(z^T) - a_T(\theta_{T-1}, z^{T-1})\right]$.

Given these individual decision rules, the aggregate quantities are given by

$$\begin{aligned} C_t(z^t) &= D\left(\theta_t^*(z^t)\right)\left(x_t(z^t) + Q_{t+1}(z^t)\overline{\alpha}\right) \\ &= D(\theta_t^*(z^t))\bar{w}_t(z^t)\theta_t^*(z^t)L\left(\theta_t^*(z^t)\right) = Z\left(\theta_t^*(z^t)\right)\bar{w}_t(z^t), \text{ for } t < T \end{aligned}$$

$$C_T(z^T) = \bar{\theta}\bar{w}_T(z^T)$$

$$\begin{aligned} Q_{t+1}(z^t)B_{t+1}(z^t) &= \left[1 - D\left(\theta_t^*(z^t)\right)\right]\left(x_t(z^t) + Q_{t+1}(z^t)\overline{\alpha}\right) - Q_{t+1}(z^t)\overline{\alpha} \\ &= \frac{1 - D\left(\theta_t^*(z^t)\right)}{D\left(\theta_t^*(z^t)\right)}C_t(z^t) - Q_{t+1}(z^t)\overline{\alpha}, \text{ for } t < T. \end{aligned}$$

Second, the equilibrium prices are given by

$$\begin{aligned} Q_{t+1}\left(z^t\right) &= \beta E_t\left(\frac{\overline{w}_t}{\overline{w}_{t+1}}\right)L(\theta_t^*), \text{ for } t < T, \\ Q_{T+1}(z^T) &= 0, \text{ for } t = T, \end{aligned}$$

where the equilibrium after tax wage rates are

$$\overline{w}_t = \begin{cases} \frac{C_t}{D(\theta_t^*)\theta_t^* L(\theta_t^*)} \equiv \frac{C_t}{Z(\theta_t^*)}, & \text{for } t < T \\ \frac{C_T}{\overline{\theta}} & \text{for } t = T \end{cases}.$$

B.1.2 Conditions to Support Competitive Equilibrium

The implementability conditions are modified to

$$\begin{aligned} & Z\left(\theta_t^*(z^t)\right) - N_t(z^t) + \beta E_t Z\left(\theta_{t+1}^*\right)U_C(z^{t+1})L\left(\theta_t^*(z^t)\right)B_{t+1}(z^t) \\ \leq \;& B_t(z^{t-1})Z\left(\theta_t^*(z^t)\right)U_C(z^t) \text{ for } t < T-1, \end{aligned}$$

$$\begin{aligned} & Z\left(\theta_t^*(z^t)\right) - N_t(z^t) + \beta E_t\overline{\theta}U_C(z^{t+1})L\left(\theta_t^*(z^t)\right)B_{t+1}(z^t) \\ \leq \;& B_t(z^{t-1})Z\left(\theta_t^*(z^t)\right)U_C(z^t) \text{ for } t = T-1, \end{aligned}$$

and

$$\overline{\theta} - N_T(z^T) \leq B_T(z^{T-1})\overline{\theta}U_C(z^T) \text{ for } t = T.$$

The asset market-clearing condition and resource constraint remain unchanged.

B.1.3 Ramsey Solution

For simplicity, substituting N_t with C_t, substituting G_t with the resource constraint, and assuming that the debt limit never binds, the Lagrangian of the Ramsey problem is given by

$$
\begin{aligned}
&E_0 \sum_{t=0}^{T} \beta^t \left[\begin{array}{c} W(\theta_t^*(z^t)) + \overline{\theta} \log C_t(z^t) \\ -C_t(z^t) - G_t(z^t) \end{array} \right] \\
&+ \sum_{t=0}^{T-1} \sum_{z^t} \beta^t \phi_t(z^t) \pi(z^t) \left(\begin{array}{c} Z(\theta_t^*(z^t)) - C_t(z^t) - G_t(z^t) \\ +B_{t+1}(z^t) E_t \beta U_C(z^{t+1}) Z(\theta_{t+1}^*(z^{t+1})) L(\theta_t^*(z^t)) \\ -U_C(z^t) Z(\theta_t^*(z^t)) B_t(z^{t-1}) \end{array} \right) \\
&+ \sum_{z^T} \beta^T \phi_T(z^t) \pi(z^T) \left(\overline{\theta} - C_T(z^T) - G_T(z^T) - U_C(z^T) \overline{\theta} B_T(z^{T-1}) \right) \\
&+ \sum_{t=0}^{T-1} \sum_{z^t} \beta^t \mu_t^B(z^t) \pi(z^t) \left\{ \begin{array}{c} \left(B_{t+1}(z^t) + \overline{\alpha} \right) E_t \beta U_C(z^{t+1}) Z(\theta_{t+1}^*(z^{t+1})) \\ -\theta_t^*(z^t) \left[1 - D(\theta_t^*(z^t)) \right] \end{array} \right\},
\end{aligned} \tag{88}
$$

where $Z(\theta_T^*(z^T)) = \overline{\theta}$.

The Ramsey FOCs with respect to C are given as follows. For $t = 0$,

$$
\bar{\theta} U_C(z^0) = 1 + \phi_0(z^0) + \phi_0(z^0) B_0 Z\left(\theta_0^*(z^0)\right) U_{CC}(z^0), \tag{89}
$$

and for $t = 1, ..T$,

$$
\begin{aligned}
\bar{\theta} U_C(z^t) \quad = \quad & 1 + \phi_t(z^t) + \left[\begin{array}{c} \phi_t(z^t) \frac{B_t(z^{t-1})}{(B_t(z^{t-1}) + \overline{\alpha})} \\ -\phi_{t-1}(z^{t-1}) L\left(\theta_{t-1}^*(z^{t-1})\right) \frac{B_t(z^{t-1})}{(B_t(z^{t-1}) + \overline{\alpha})} - \mu_{t-1}^B(z^{t-1}) \end{array} \right] \\
& \times Z\left(\theta_t^*(z^t)\right) U_{CC}(z^t) \left(B_t(z^{t-1}) + \overline{\alpha} \right)
\end{aligned} \tag{90}
$$

The Ramsey FOCs with respect to B are as follows. For $t = 0, 1, .., T-1$, we have

$$
\phi_t(z^t) L\left(\theta_t^*(z^t)\right) - \frac{E_t \phi_{t+1}(z^{t+1}) U_C(z^{t+1}) Z\left(\theta_{t+1}^*(z^{t+1})\right)}{E_t U_C(z^{t+1}) Z\left(\theta_{t+1}^*(z^{t+1})\right)} + \mu_t^B(z^t) = 0. \tag{91}
$$

The Ramsey FOCs with respect to θ_t^* are given as follows. For $t = 0$, we have

$$
\begin{aligned}
& \frac{\partial W(\theta_0^*(z^0))}{\partial \theta_0^*(z^0)} + \phi_0(z^0) \left[\begin{array}{c} \left(1 - U_C(z^0) B_0\right) \frac{\partial Z(\theta_0^*(z^0))}{\partial \theta_0^*(z^0)} \\ + \frac{\partial L(\theta_0^*(z^0))}{\partial \theta_0^*(z^0)} B_1(z^0) E_0 \beta U_C(z^1) Z(\theta_1^*(z^1)) \end{array} \right] \\
= \quad & J\left(\theta_0^*(z^0)\right) \mu_0^B(z^0);
\end{aligned} \tag{92}
$$

and for period $t = 1, ...T-1$, we have

$$\begin{aligned} & \frac{\partial W(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} + \phi_{t-1}(z^{t-1})U_C(z^t)\frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)}L(\theta_{t-1}^*(z^{t-1}))B_t(z^{t-1}) \\ & +\phi_t(z^t)\left[\begin{array}{c} (1-U_C(z^t)B_t(z^{t-1}))\frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)} \\ +\frac{\partial L(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)}B_{t+1}(z^t)E_t\beta U_C(z^{t+1})Z(\theta_{t+1}^*(z^{t+1})) \end{array}\right] \\ = \; & J\left(\theta_t^*(z^t)\right)\mu_t^B(z^t) - \mu_{t-1}^B(z^{t-1})\left(B_t(z^{t-1})+\overline{\alpha}\right)U_C(z^t)\frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)}. \end{aligned} \tag{93}$$

Following the same steps as in Appendix A.4, combing equations (91) and (93) together by utilizing Lemma 2 gives

$$\begin{aligned} & E_t\phi_{t+1}(z^{t+1})\frac{U_C(z^{t+1})Z(\theta_{t+1}^*(z^{t+1}))}{E_tU_C(z^{t+1})Z(\theta_{t+1}^*(z^{t+1}))} \\ = \; & \frac{\partial W(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)}\frac{1}{\mathbf{F}\left(\theta_t^*(z^t)\right)} + \left[1+\frac{\overline{\alpha}}{B_t+\overline{\alpha}}\Omega\left(\theta_t^*\right)\right]\phi_t(z^t) \\ & -\left(\begin{array}{c} \phi_t(z^t) \\ -E_{t-1}\phi_t(z^t)\frac{U_C(z^t)Z(\theta_t^*(z^t))}{E_{t-1}U_C(z^t)Z(\theta_t^*(z^t))} \end{array}\right)\frac{U_C(z^t)}{\mathbf{F}\left(\theta_t^*(z^t)\right)}B_t(z^{t-1})\frac{\partial Z(\theta_t^*(z^t))}{\partial \theta_t^*(z^t)}, \end{aligned}$$

which is equation (82). Similarly, equations (82) and (81) can be obtained by combing equations (91) and (93) or (92) in period $T-1$ and period 0, respectively.

For the proof of Proposition 7, it is straight forward to verify that equations (81) to (83) are reduced to equations (77) and (79), respectively, in the case without aggregate uncertainty. Moreover, $\Omega^A\left(\theta_t^*\right) \geq 0$ given that $\frac{\partial L(\theta_t^*)}{\partial \theta_t^*} \leq 0$ and $D\left(\theta_t^*\right) \leq 1$. $\Delta^A\left(\theta_t^*\right) \geq 0$ is implied by Appendix A.3. Also, if $\theta_t^* = \theta_H$, then $\frac{\partial L(\theta_t^*)}{\partial \theta_t^*} = 0$ and $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} = 0$. Hence, $\Omega^A\left(\theta_t^*\right) = 0$ and $\Delta^A\left(\theta_t^*\right) = 0$ if $\theta_t^* = \theta_H$. Finally, if $\theta_{T-1}^* = \theta_H$, $\frac{\partial Z\left(\theta_{T-1}^*\right)}{\partial \theta_{T-1}^*} = 0$ and, consequently, $\Theta^A\left(\theta_{T-2}^*, \theta_{T-1}^*\right) = 0$.