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Contagious Switching

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Contagious Switching*

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Abstract

In this paper, we analyze the propagation of recessions across countries. We construct a model with multiple qualitative state variables that evolve in a VAR setting. The VAR structure allows us to include country-level variables to determine whether policy also propagates across countries. We consider two different versions of the model. One version assumes the discrete state of the economy (expansion or recession) is observed. The other assumes that the state of the economy is unobserved and must be inferred from movements in economic growth. We apply the model to Canada, Mexico, and the U.S. to test if spillover effects were similar before and after NAFTA. We find that trade liberalization has increased the degree of business cycle propagation across the three countries.

JEL Codes: C32; E32

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1 Introduction

The study of trade liberalization's effect on business cycle synchronization offers two competing theories with opposite implications.[1] On one hand, trade liberalization can be synchronizing if the spillover from domestic shocks suddenly affect trading partners [Frankel and Rose (1998); Baxter and Kouparitsis (2005); and Kose and Yi (2006) to name just a few]. On the other hand, trade liberalization can spur industrial specialization, which may prevent or mitigate such spillovers [Imbs (2004)]. The empirical literature remains inconclusive as to the direction of the synchronization post-liberalization. On one hand, Stock and Watson (2005) and Kose, Otrok, Whiteman (2008) estimate factor models and show that the importance of the global factor has increased over time. On the other hand, Doyle and Faust (2005) conclude that correlation between GDP growth rates in Canada and the U.S. has remained unchanged since the 1960s. Heathcote and Perri (2003) argue the U.S. is less correlated with Europe, Canada, and Japan over the same period.

Measuring synchronization is another unsettled issue. While computationally simple, correlations or corellograms may not take into account all of the information available to the econometrician. A number of other papers that consider the alternative methods of measuring the interaction between business cycles. Kose, Otrok, and Whiteman (2003, 2008) estimate factor models with global and regional factors, where an increase in the variance share explained by the global or regional factors suggests higher synchronization. Hamilton and Owyang (2012) collect similar cycles into clusters (regions) that move together. This reduces the dimensionality of the problem but forces the cycles of series in the same cluster to be essentially identical. Leiva-Leon (2017) considers pairwise series of binaries where a third binary switches the interaction from fully synchronized to fully unsynchronized. Perhaps the papers most closely related to ours are models with time-varying transition probabilities that depend on the states for the other series [Kaufmann (2010); Billio, Casarin, Ravazzolo, and van Dijk (2016); and Agudze, Billio, Casarin, and

[1] Business cycle synchronization is distinct from growth rate convergence. The former measures the correlation between cycles (and possibly their leads and lags) and the timing of countries' turning points. The latter studies potential declines in the differences between countries' growth rates.

Ravazzolo (2018)].

In this paper, we continue the study of business cycle synchronization and propagation, proposing a model in which the state of the business cycle in one country can affect the current and future state of business cycles in other countries. Our model extends the literature on time-varying transition probabilities in multivariate Markov-switching models on two levels. First, we allow for the interaction of country business cycle phases, which allows us to ask whether one country's business cycle phase propagate to other countries or whether information about one country's business cycle phase help predict future changes in other countries' business cycle phases. Second, our model structure nests a VAR, which allows us to identify shocks to policy variables and determine their effect on the probability of changing future country-level business cycle phases.

We demonstrate how to estimate the model whether or not the business cycle phases are observed. In the former case, one could use outside measures of the business cycle—e.g., the NBER dates. The latter case can be particularly useful when the model is applied to sub-national cycles (regions or industries) or for countries (e.g., emerging markets) where "official" business cycle dates are unknown. In this latter case, we have to estimate both a discrete latent and a continuous latent, which depend on each other. We propose a Metropolis step that allows us to form a joint proposal that combines steps from a standard Kalman filter and a Bayesian modification of the Hamilton (1989) filter.

We apply our model to the U.S., Canada, and Mexico to determine whether the North American Free Trade Agreement (NAFTA) altered the propagation of business cycles across these countries. For the full sample, we find that an increase in the probability of recession in Canada or the U.S. leads to a statistically significant increase in the recession probabilities in its neighbors. An adverse shock to Mexico, on the other hand, has a subsequent but statistically insignificant increase in the recession probabilities for its neighbors.

In subsample analysis, we find a relatively low degree of recession spillovers prior to the introduction of NAFTA. However, since NAFTA was adopted in 1994, we find that recession shocks in any one of the three major North American economies leads to a significantly higher recession probability in the other two. Therefore, our paper adds to the evidence that trade liberalization increases the degree of business cycle synchronization across countries.

The balance of the paper is laid out as follows. Section 2 describes the model with both observed and unobserved states. Section 3 outlines the Bayesian estimation of the model. We describe in detail the sampler block required to obtain the joint draw of the discrete and continuous latent states. This section also describes the data and VAR identification. Section 4.1 presents the empirical results for the observed states. We also present the computation of the dynamic marginal effects. Section 4.2 presents the results with unobserved states. Section 5 introduces a break at the implementation of NAFTA and re-estimates the model for the pre- and post-break periods. Section 6 offers some conclusions.

2 Empirical Setup

Consider the interaction between the business cycles of $n = 1, ..., N$ countries over $t = 1, ..., T$ periods. Let $S_{nt} = \{0, 1\}$ represent the discrete business cycle phase for country n at time t, where $S_{nt} = 0$ represents an expansionary phase and $S_{nt} = 1$ represents a recessionary phase. Collect the business cycle phases into a vector $S_t = [S_{1t}, ..., S_{Nt}]'$.

2.1 Observed Regimes

To model the interdependence of business cycle phases across countries, we must specify how S_{nt} affects S_{mt}, $m \neq n$. Assume, initially, that each S_{nt} is observed. Further, suppose that the business cycle phases propagate across countries with a lag. Let z_{nt} represent a continuous latent variable related to the binary observed variable S_{nt} through the deterministic relationship:

$$S_{nt} = \begin{cases} 1 & \text{if} \quad z_{nt} \geq 0 \\ 0 & \quad \text{otherwise} \end{cases}.$$

Through the latent variable z_{nt}, we can study how other variables—both macro variables and the business cycle phases of other countries—affect the future business cycle phase of country n. Let $y_t = [y_{1t}, ..., y_{Jt}]'$ represent a $(J \times 1)$ vector of macro variables that could include country-specific policy variables or other economic indicators and let $z_t = [z_{1t}, ..., z_{Nt}]'$ collect the continuous latent business cycle indicators.

Define $Y_t = [z_t', y_t']'$, where the relationship between the contemporaneous Y_t and its lags follows a vector autoregression (VAR):

$$Y_t = B_0 + B(L) Y_{t-1} + u_t, \tag{1}$$

where $u_t = [u_{1t}^z, ..., u_{Nt}^z, u_{1t}^y, ..., u_{Jt}^y]'$ and $E_t[u_t u_t'] = \Sigma$. For exposition, we write (1) in a more detailed form:

$$\begin{bmatrix} z_t \\ y_t \end{bmatrix} = \begin{bmatrix} B_0^z \\ B_0^y \end{bmatrix} + \begin{bmatrix} B^{zz}(L) & B^{zy}(L) \\ B^{yz}(L) & B^{yy}(L) \end{bmatrix} \begin{bmatrix} z_{t-1} \\ y_{t-1} \end{bmatrix} + \begin{bmatrix} u_t^z \\ u_t^y \end{bmatrix},$$

where $B^{ij}(L)$ represents the lagged effect of j on i. Because the z_t are latent, we make scale assumptions by restricting their variances for identification. In particular, we assume that

$$\Sigma_{zz} = E_t[u_t^z u_t^{z\prime}]$$

has unit diagonal elements. In subsequent sections, we can impose additional restrictions on the decomposition of the VAR variance-covariance matrix that identify the structural form of the VAR from its reduced form.

The current model has a form similar to a multi-binary-variable extension of Dueker's (2005) Qual-VAR. In that paper, a single binary variable indicates the state of the economy and can be affected—and, importantly, can affect—a vector of macroeconomic variables at lags. This version of our model with observed S_t collapses to the Qual-VAR when S_t is a scalar.[2] Because of the assumption that the reduced-form VAR errors are

[2] The Prob-VAR outlined in Fornari and Lemke (2010) is a more restricted version of the Qual-VAR, where they assume $B^{zz}(L)$ and $B^{yz}(L)$ are both identically zero. Their model, then, is essentially a

multivariate normal, the z_t equations in the VAR resemble a multivariate extension of the dynamic probit outlined in Eichengreen, Grossman, and Watson (1986). Chib and Greenberg (1998) develop methods of estimating the static multivariate probit, which is equivalent to the z_t equations in the VAR in our model with observed S_t with the additional assumption that $B^{zz}(L) = 0$. This observed-S_t version of our model is perhaps most similar to the multivariate dynamic probit of Candelon, Dumitrescu, Hurlin, and Palm (2013) in which we add a propagation mechanism for the covariates that allows the latents to affect macro variables at lags.

Two key features differentiate our model from a set of independent time-varying transition probability switching models. First, there is a lagged cross-regime effect that is embedded in the off-diagonal elements of $B^{zz}(L)$. The lagged effect represents the contagious switching, where a regime change in one country can spill over into the regimes of its neighbors. Further notice that the regime cross-series dependence is a function of the continuous latent rather than the binary latent. This means that z_{nt} may be thought of as representing the strength of the cycle. Second, there is a contemporaneous cross-regime effect that is embedded in the tetrachoric correlation term in Σ. The tetrachoric effect can represent either simultaneity of shocks that cross country borders or within-quarter contagion effects. The model allows us to test for the presence of cyclical contagion, the speed at which it acts, and the degree to which countries affect each other. In addition, countercyclical or prophylactic policy can be included in the y_j's to determine whether, say, changes in fiscal or monetary policy can reduce the probability of recessions.

2.2 Unobserved Regimes

While we previously assumed that the S_t are observed, we can relax this assumption by including a vector of economic indicators whose means depends on the discrete regimes. Unobserved regimes can be relevant for a number of reasons. For example, one simply might not have the data available as all countries to not construct or announce business

VAR in the macro variables y and a probit where lags of y determine a scalar z. Their application is to forecasting S using iterative multistep methods. The VAR forms forecasts for y, which in turn informs the forecast of S at longer horizons.

cycle turning points. On the other hand, some countries have more than one set of turning point dates, suggesting some uncertainty over the timing of the events. In the U.S., the NBER Business Cycle Dating Committee dates are widely accepted as the "official" business cycle turning points. However, these dates are not revised even in the presence of new or revised data. Moreover, other measures such as the OECD Recession Indicators may vary slightly from the NBER in the timing and definition of the turning points. In some of these cases, it may be advantageous to estimate the regime changes directly from the data.

Suppose, then, that each of the N countries can be characterized by a period$-t$ business cycle indicator, x_{nt}. While x_{nt} could be scalar or a vector, for the purposes of exposition, we can call x_{nt} the output growth rate but it could be, in principle, any contemporaneous indicator of the cycle. Collect the period$-t$ output growth rates into a vector $x_t = [x_{1t}, ..., x_{Nt}]'$. We assume that output growth is a stochastic sampling from a mixture of normals, where μ_{n0} and μ_{n1} are the means of the two normal distributions and we impose $\mu_{n0} > \mu_{n1}$ for identification. Note that the mixtures can be potentially different for each country, as evidenced by the index n. The interpretation of our assumption is that each country's economy moves between two business cycle phases, a relatively high mean (expansion) and a relatively low mean (recession). Note that we do not impose that the mean recession growth rate is negative, but it must be less than the mean growth rate in expansion.

During each period, a country n's business cycle phase is represented by the latent variable S_{nt} that determines which of the two distributions x_{nt} is drawn from that period. The process can be summarized by

$$x_{nt} = \mu_{n0} + \Delta\mu_n S_{nt} + \phi_n (L) x_{n,t-1} + \varepsilon_{nt}, \tag{2}$$

where we can define $\Delta\mu_n = \mu_{n1} - \mu_{n0}$, $\Delta\mu_n < -\mu_{n0}$, as implied by our identifying assumption, and $\varepsilon_{nt} \sim N(0, \sigma_n^2)$. We impose that the output volatility is time invariant and that the output shocks are uncorrelated across countries, serially uncorrelated, and uncorrelated with the shocks to the variables in the VAR. In the current application, we suppress the autoregressive dynamics, $\phi_n(L) = 0$.[3]

3 Estimation and Data

3.1 The Sampler

We estimate the model using the Gibbs sampler, a Markov-Chain Monte Carlo algorithm that draws a block of the model parameters—including the underlying continuous states–conditional on the remaining parameters and the data. Let Ω_t represent the data available at time t. We specify a standard set of priors for the model with observed regimes. The parameters in B are multivariate normal and we assume a standard Minnesota prior. We assume similar priors to Chan and Jeliazkov (2009) on the parameters in D and L of the decomposition $\Sigma = L'^{-1}DL^{-1}$. For the case with unobserved regimes, we also need to set priors for the intercepts, the AR terms, and the innovation variances in the x_t equation. We assume that the parameters in the x_t equation have a Normal-inverse Gamma prior. Table 1 contains the parameterization of the prior for the more general model with unobserved regimes; the model with observed regimes has the same priors without the parameters governing the process for x (i.e., μ and σ).

Table 1: Prior Specifications for Estimation

Parameter	*Prior Distribution*	*Hyperparameters*	
$b = vec(B)$	$N\left(\bar{b}_0, \bar{B}_0\right)$	Minnesota Prior (See Appendix)	
a_k	$N\left(\boldsymbol{a}_0, \boldsymbol{A}_0\right)$	$\boldsymbol{a}_0 = \boldsymbol{0},\ \boldsymbol{A}_{k0} = (0.15^2) * \boldsymbol{I}$	$\forall k$
λ_k^{-1}	$\Gamma\left(\frac{\nu_0}{2}, \frac{\delta_0}{2}\right)$	$\nu_0 = 1,\ s_0 = 5$	$\forall k$
μ_n	$N\left(m_0, \sigma_n^2 M_0\right)$	$m_0 = [1, -1]',\ M_0 = 2 * \boldsymbol{I}_2$	$\forall n$
σ_n^{-2}	$\Gamma\left(\frac{v_0}{2}, \frac{\tau_0}{2}\right)$	$v_0 = 1,\ \tau_0 = 1$	$\forall n$

We divide the exposition of the sampler into two parts. In the first part, we outline the sampler for the case where S_t is observed. In this case, there are three blocks for estimation: (1) the coefficient matrices for the VAR, $B = \{B_0, ..., B_P\}$; (2) the VAR variance-covariance matrix, Σ; and (3) the latent states, $\{z_t\}_{t=1}^T$. The first block is conju-

[3] These assumptions are made for expositional clarity and are generally consistent with those in the business cycle identification literature. They are straightforward to relax.

gate normal. Because of the restrictions on the latent variances, the second block requires a Metropolis step, which is a modification of the algorithm outlined in Chan and Jeliazkov (2009). The third block is executed by drawing the continuous latent state variable recursively from smoothed Kalman posterior distributions.[4] The Technical Appendix outlines the state space of the model and each of the draws.

Aside from two additional blocks to sample the additional parameters in the x_t equation, the case of unobserved regimes adds a wrinkle that warrants more explanation. Because the sign of z_{nt} is determined by the value of S_{nt} and the past z_t determine the transition probabilities for S_{nt}, these two values must be sampled simultaneously. Thus, the sampler for the unobserved state case has five blocks: (1) the coefficient matrices for the VAR, $B = \{B_0, ..., B_P\}$; (2) the VAR variance-covariance matrix, Σ; (3) the coefficients for the measurement equation, $\Psi = \{\mu_0', \mu_1', \phi'\}$; (4) the measurement innovation variances, $\{\sigma_n^2\}_{n=1}^N$; and (5) the latent states, $\{z_t,\}_{t=1}^T$ and $\{S_t\}_{t=1}^T$. The two additional blocks (3) and (4) yield conjugate posterior distributions. We outline the filter used to obtain draws of block (5) below; other draws are detailed in the Technical Appendix.

3.1.1 Drawing $\{z_t\}_{t=1}^T, \{S_t\}_{t=1}^T$ conditional on B, Σ, $\{\sigma_n^2\}_{n=1}^N$, Ψ

Unfortunately, when $\{S_t\}_{t=1}^T$ is unknown, we cannot draw the sequences of the two latent variables in separate blocks. The value of S_{nt} is directly related to the sign of z_{nt}. One might posit a draw in which the full sequence $\{S_\tau\}_{\tau=1}^T$ is drawn, conditional on the past iteration of $\{z_\tau\}_{\tau=1}^T$; then, a draw of the full sequence of $\{z_\tau\}_{\tau=1}^T$, conditional on the new draw of $\{S_\tau\}_{\tau=1}^T$, where each S_{nt} determines the direction of the truncation of z_{nt}. However, any draw that changes S_{nt} across Gibbs iteration invalidates the last draw of z_{nt}, as the truncation would be improper. Drawing the full sequence $\{z_\tau\}_{\tau=1}^T$ first also would be invalid. While we can obtain a Kalman posterior for z_{nt}, the exact conditional distribution will be truncated. Simply drawing z_{nt} from the Kalman posterior and then assigning S_{nt} based on the sign of z_{nt} would ignore information in the x's that inform S_{nt}.

[4] This differs from Dueker's original sampler. In this sampler, Dueker used exact conditional distributions for the interior $T - 2P$ periods. The first P periods were drawn using Metropolis-Hastings. The last P periods are drawn by iterating forward the mean of the exact conditional distribution for the $T - P - 1$ period.

We adopt an alternative approach that takes advantage of both the Kalman filter and Hamilton's Markov switching filter to draw candidates for a Metropolis-in-Gibbs step. Because we need to use the draws of lagged z_t to form the transition probabilities for the Hamilton filter, we cannot draw the candidates using smoothed probabilities. Instead, for each t, we draw a candidate S_t^*, conditional on lags of z_t, using the forward component of the Hamilton filter. We then draw a candidate z_t^* from the posterior obtained by the forward component of the Kalman filter.

Specifically, start with set of initialization probabilities, $\Pr[S_{n0}]$, which could be the steady state regime probability, and initialize the vector of latents, z_0 and the state covariance matrix $P^z_{0|0}$. The goal is to obtain (jointly) a candidate pair of vectors (z_t^*, S_t^*) for each $t = 1, ..., T$. We can form the joint proposal density as

$$p\left(z_t^*, S_t^*|\Omega_t\right) = p\left(z_t^*|\Omega_t, S_{nt}^*, \{z_\tau\}_{\tau=1}^{t-1}\right)\prod_{n=1}^{N} p\left(S_{nt}^*|\Omega_t, \{z_\tau\}_{\tau=1}^{t-1}\right).$$

We draw the candidate S_{nt}^* from

$$\Pr\left[S_{nt}^* = 1|\Omega_t\right] = \frac{\sum_{S_{n,t-1}} \ell\left(S_{nt}^* = 1, S_{n,t-1}|\Omega_t, \{z_\tau\}_{\tau=1}^{t-1}\right)\Pr\left[S_{nt}^* = 1|S_{n,t-1}, \{z_\tau\}_{\tau=1}^{t-1}\right]\Pr\left[S_{n,t-1}|\Omega_{t-1}, \{z_\tau\}_{\tau=1}^{t-1}\right]}{\sum_{S_{nt}}\sum_{S_{n,t-1}} \ell\left(S_{nt}^*, S_{n,t-1}|\Omega_t, \{z_\tau\}_{\tau=1}^{t-1}\right)\Pr\left[S_{n,t}^*|S_{n,t-1}, \{z_\tau\}_{\tau=1}^{t-1}\right]\Pr\left[S_{n,t-1}|\Omega_{t-1}, \{z_\tau\}_{\tau=1}^{t-1}\right]},$$

where $\ell(.,.|.)$ is the likelihood and

$$\Pr\left[S_{nt}^* = 1|S_{n,t-1}, \{z_\tau\}_{\tau=1}^{t-1}\right] = \Pr\left[z_{nt} > 0|z_{t-1}\right]$$

are the transition probabilities, which depend on the lagged continuous latent variable for all n.

The conditional distributions for the z_{nt}'s can be obtained by the forward component of the Kalman filter. Based on the state equation, (1), the Kalman filter obtains the forecast density for the vector z_t, conditional on its lags. Then, the filter updates the forecast density using information from the current realization of y_t to obtain

$$p\left(z_t|\Omega_t\right) \sim N\left(\widehat{z}_{t|t}, P^z_{t|t}\right),$$

where $\widehat{z}_{t|t}$ is the mean of the conditional distribution and $P^z_{t|t}$ is the covariance matrix.[5] Then, conditional on S^*_t, we can draw the candidate from the truncated normal, where the truncation direction depends on S^*_t:

$$p\left(z_t|\Omega_t, S_t\right) \sim TN\left(\hat{z}_{t|t}, P^z_{t|t}, S_t\right).$$

Finally, we validate the candidate (z^*_t, S^*_t)—drawn jointly for all n—using standard MH acceptance probabilities. The candidate (z^*_t, S^*_t) is accepted with probability α, where

$$\alpha = \min\left[1, \frac{\pi\left(S^*_t, z^*_t\right) f(x_t, y_t|S^*_t, z^*_t) q\left(S^{[i-1]}_t | z^{[i-1]_t}\right)}{\pi\left(S^{[i-1]}_t, z^{[i-1]}_t\right) f(x_t, y_t|S^{[i-1]}_t, z^{[i-1]}_t) q\left(S^*_t, z^*_t\right)}\right],$$

where $\pi\left(.,.\right)$ is the prior, $f(x, y|.,.)$ is the joint likelihood, and $q\left(.|.\right)$ are the move probabilities. Because we have an independence chain, the ratio of the move probabilities collapses to 1. Using the fact that y does not depend on S and the identity $P(S_t|z_t) = 1$, the posterior likelihood is

$$\pi\left(S_t, z_t\right) f(x_t, y_t|S_t, z_t) = f(y_t, z_t|S_t) \prod_{n=1}^{N} f(x_{nt}|S_{nt}),$$

where

$$f(y_t, z_t|S_t) = \frac{1}{|2\pi\Sigma|^{1/2}} \exp\left\{-\frac{1}{2} u'_t \Sigma^{-1} u_t\right\}$$

and

$$f(x_{nt}|S_{nt}) = \frac{1}{\sqrt{2\pi\sigma^2_n}} \exp\left\{-\frac{1}{2\sigma^2_n} \varepsilon'_{nt} \varepsilon_{nt}\right\}.$$

[5] Each of these quantities will be a subvector and submatrix, respectively, of the output of the Kalman filter.

3.2 Data

We apply the model to the NAFTA member countries (Canada, Mexico, and the U.S.). We perform two experiments. First, we consider the model with observed recessions. This model requires two sets of data: (1) the recession indicator S_{nt}, and (2) the macroeconomic variables in y_t. For the recession indicators S_{nt}, we use NBER dates for the U.S., recession dates from the C.D. Howe Institute for Canada, and recession dates obtained from the quarterly application of the Bry-Boschan (BBQ) method for Mexico.[6] For the macroeconomic variables, we use the U.S. effective federal funds rate or the Wu-Xia (2016) shadow short rate obtained from the Atlanta Fed. The effective federal funds rate comes from FRED and the shadow short rate is available from the Atlanta Fed. In order to properly identify shocks to monetary policy, we add two additional series. The first is the difference in the log of the PCE price level. The second is the change in log commodity prices obtained from the Commodity Research Board.

Next, we consider the model with unobserved regimes. This model also requires two sets of data: (1) x_t, the variable that informs S_t, and (2) the macroeconomic variables in y_t. For the latter, we use the same macroeconomic variables as in the previous experiment For x_t, we use the first principle component across four series for each country, including real GDP growth, employment growth, industrial production growth, and retail trade growth. The real GDP growth data comes from the OECD Quarterly National Accounts, employment growth comes from the OECD Short-Term Labour Market Statistics, and industrial production growth and retail trade growth come from the OECD Monthly Economic Indicators.

All of the data are available for the U.S. and Canada for the period 1980:Q1-2018:Q1. For Mexico, real GDP growth and industrial production growth are available from 1980:Q1; retail trade growth is available starting in 1986:Q1; and employment growth is available starting in 2000:Q1.

[6] The BBQ algorithm is described in detail in Harding and Pagan (2002).

3.3 Identifying the VAR

We identify shocks to both the fed funds rate and each country's business cycle indicator. We impose ordering restrictions on the VAR and identify the structural shocks using the Cholesky decomposition. The ordering of the variables in this case is

$$Y_t = [z_{US,t}, z_{CA,t}, z_{MX,t}, PCE_t, FFR_t, PCOM_t]',$$

which implies that the fed funds rate responds contemporaneously to inflation and the business cycle indicators but not vice versa. Moreover, the causal ordering assumes that shocks to the business cycle variable in the U.S. affect other countries contemporaneously but not vice versa.

4 Empirical Application

While our model, in principle, can be used to track the propagation of any set of binary outcomes (e.g., bank failures, etc), our application is to international business cycles. Others have previously studied the transmission of U.S. shocks to other countries [see, among many, Kim (2001) and Feldkircher and Huber (2016), who consider the international transmission of U.S. monetary shocks (and others) in VARs]. Here, we examine the transmission of business cycle shocks—which may be induced by monetary contractions or other shocks included in the VAR—across the NAFTA countries. We set the lag order of the VAR to $p = 1$.

4.1 S_t Observed Results

We first consider the model with observed recessions to determine the extent to which business cycle phases might spread across countries. Again, for this experiment, we take recession values from sources external to the model and treat these as given. Figure **1** shows the posterior median for the latent continuous recession variables z_{nt}, along with the values of the recession indicators shaded in gray. Because the timings of the recessions

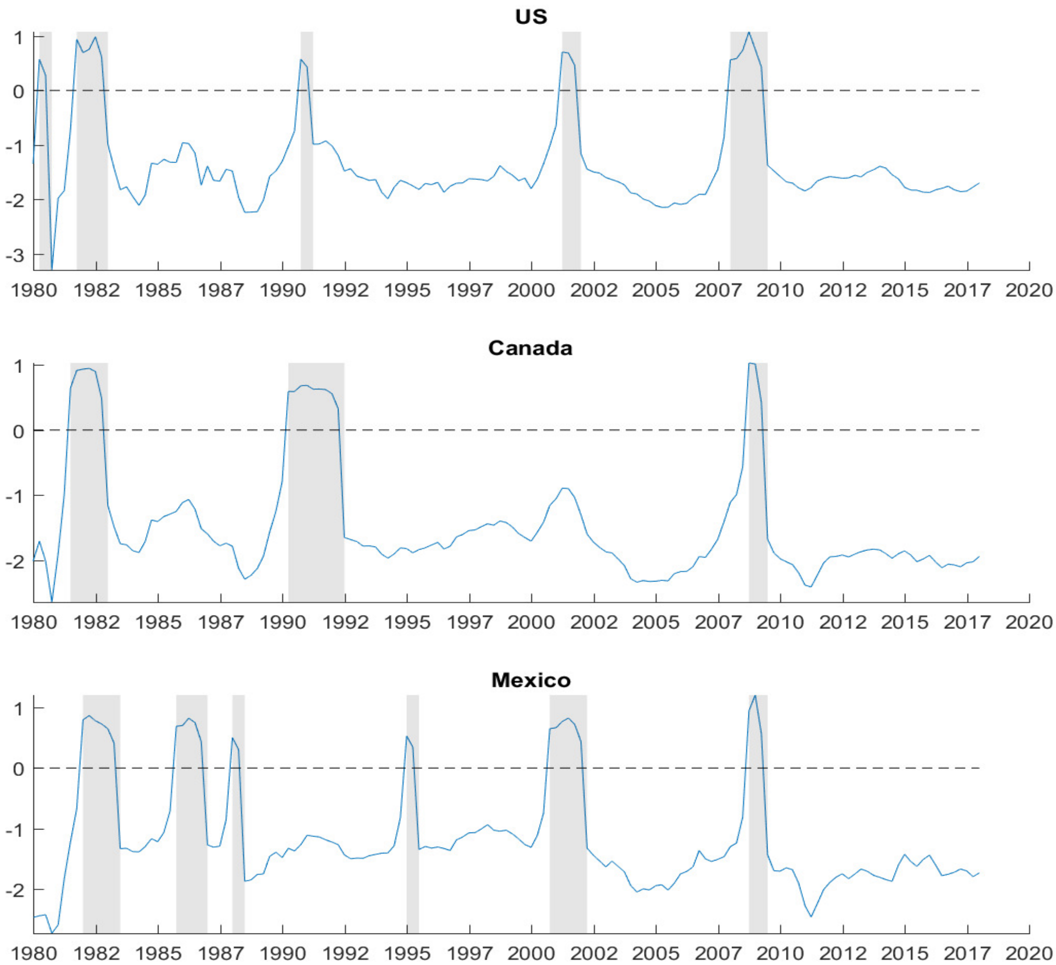

Figure 1: *Continuous Recession Variables.* This figure shows the posterior median of the continuous recession variable z_{nt} when the discrete regime S_{nt} is observed. Gray shading reflect recession dating for each country (U.S. dates from NBER; Canada dates from C.D. Howe Institute; Mexico dates from BBQ estimation).

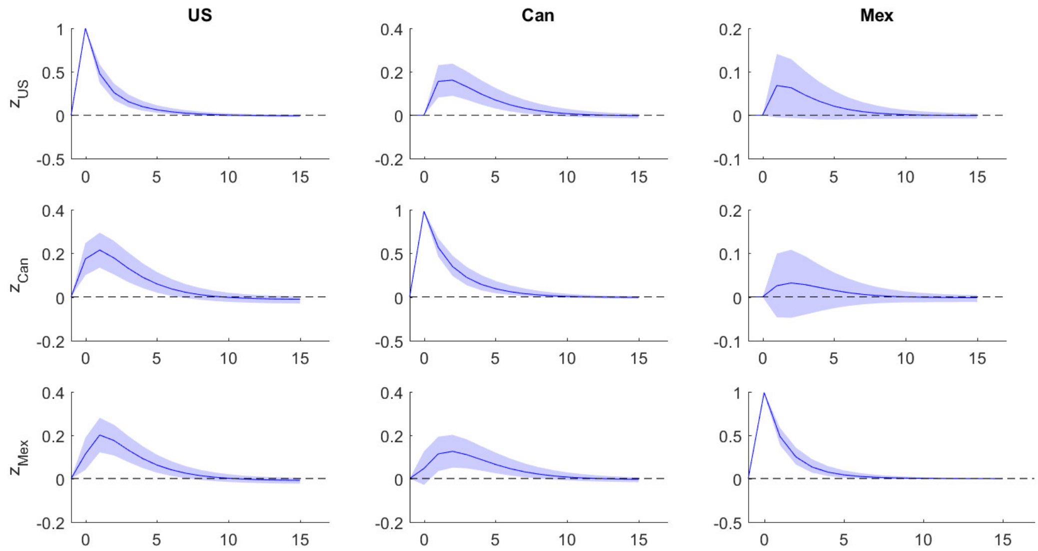

Figure 2: *Impulse Response Functions of z_{mt} to shocks to z_{nt}.* This figure shows the response of each country's response to a shock to the continuous recession variable. The first column shows the response of each of the three countries to a recessionary shock to the U.S. The second and third columns show the individual country responses to Canada and Mexico, respectively. The solid line shows the posterior median response and the shaded region shows the 67% HPD interval.

are taken as data, the signs of the z_{nt}'s are deterministic (there are no false positives, etc); however, the dynamics of the z_{nt}'s are produced by the dynamics of the model.

4.1.1 Impulse Responses

Figure **2** shows the responses of latent business cycle variables to each other. The columns show, respectively, the effects of one-standard-deviation contractionary shocks to the U.S., Canadian, and Mexican latent business cycle indicators. Recall that the shock—a one-standard-deviation increase in z_{nt}—-is an adverse shock, pushing the economy closer to or into recession. The rows show the shocks' effects on the U.S., Canadian, and Mexican latent business cycle indicator, respectively.

Overall, a shock to the business cycle indicator produces the expected response: An increase in z_{nt} pushes the domestic economy toward or into recession. Similar results could be obtained from a univariate Markov-switching model; however, our model also allows us to investigate the cross-country effects of a domestic recession shock. Adverse shocks in the U.S. have statistically important effects on both the Canadian and Mexican business

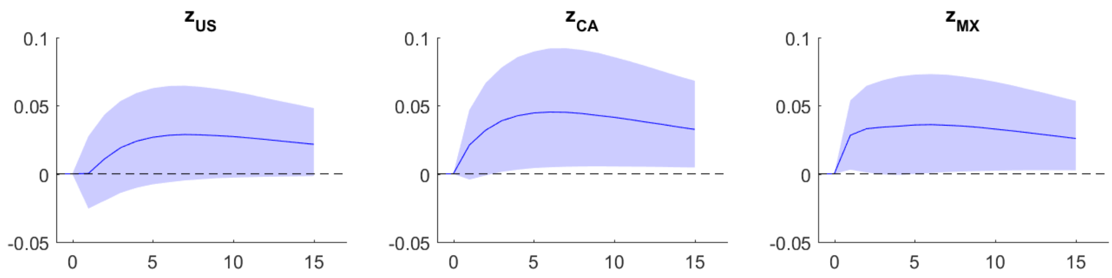

Figure 3: *Impulse Response Functions of z_{mt} to shocks to FFR_t.* This figure shows the response of each country's recession variable to a shock to the shadow rate. The solid line shows the posterior median response and the dashed lines show the 67% HPD interval.

cycles, raising the likelihood of recessions in both countries. Similarly, an adverse shock in Canada significantly increases the probability of a recession in both the U.S. and Mexico. However, an adverse economic shock in Mexico does not affect the business cycles in its neighbors in a statistically relevant way—the uncertainty bands contain zero throughout the response horizon.[7]

These results, taken as a whole, are perhaps not surprising. In particular, Feldkircher and Huber (2016) find the trade channel to be an important element to transmitting shocks internationally. Over the sample period, trade between Canada and Mexico is relatively small compared to trade between the U.S. and Canada.[8] Also, trade between the U.S. and Mexico is a small fraction of U.S. GDP but a large fraction of Mexican GDP.[9]

We find that U.S. business cycle conditions spill over to its neighbors. A next logical question is whether contractionary U.S. policy (i) affects U.S. business cycle conditions and (ii) spills over into its neighbor's business cycle conditions. Figure **3** shows the responses of business cycle conditions, the z_{nt}'s, to a one-standard-deviation increase in the federal funds rate.[10] Here, we consider only the effect of U.S. monetary policy as previous studies have focused on its international transmission. As expected, the U.S. recession

[7] In results not shown here, a shock to each country's latent business cycle indicator reduces the fed funds rate but does not affect inflation or commodity prices. These results are avialable upon request.

[8] Across our sample period 1980 - 2018, the average total trade between Canada and Mexico is 0.13% of Candian GDP whereas total trade between Canada and the U.S. averages 9.15% of Canadian GDP.

[9] The average total trade between the U.S. and Mexico over the period 1980 - 2018 was 0.46% of U.S. GDP compared to 4.01% of Mexican GDP.

[10] The interest rate equation has elements of the Taylor rule, including both a measure of economic activity z_{nt} and the inflation rate.

variable increases as the policy rate rises, however the effect is not significant since zero is in the relatively wide HPD interval. Additionally, we find that the U.S. monetary policy shocks spill over to both the Canadian and Mexican economies, increasing each of their recession variables in a statisticall significant manner.

4.1.2 Quantifying the Spillover Effect

The fact that an adverse shock in one country produces an adverse response in its neighbors is, perhaps, not surprising. Our model allows us to quantify this response in terms of the change in the probability of a future turning point. However, unlike standard impulse responses, computing these marginal effects requires knowing the conditions at the time of the shock. Moreover, our model differ from the typical probit model because the marginal effects from our model are dynamic. Thus, we cannot simply choose the initial conditions at the time of the shock; we need to account for how sequences of shocks could alter the recession probabilities.

To compute the dynamic marginal effects, we use a techniques similar to generalized impulse responses [Koop, Pesaran, and Potter (1996)], where reduced-form shocks for the period subsequent to the structural shock in question are drawn by Monte Carlo methods. Specifically, let $\overline{\mathbf{Y}}_t^R$ denote the sample averages for the vector Y_t and its lags subject to the set of restriction $S_t = R$. The probability of a recession in country n when starting in state R is

$$\Pr\left[S_{n,t+h} = 1|R\right] = \frac{1}{Q}\sum_{q}^{Q}\Phi\left(z_{t+h}^{n}|\overline{\mathbf{Y}}_t^R, \Theta, u_{nt}^{z} = \delta, \left\{u_{t+l}^{[q]}\right\}_{l=1}^{h}\right),$$

where $u_{nt}^{z} = \delta$ is the structural shock of interest, $\left\{u_{t+l}^{[q]}\right\}_{l=1}^{h}$ is the qth draw of the future reduced-form shocks, and Θ are the set of model parameters. We can obtain a measure of uncertainty by integrating over both the Monte Carlo reduced-form shocks and over the Gibbs iterations to obtain a density. The change in the probability resulting from the structural shock can be obtained by subtracting the same quantity for $u_{nt}^{z} = 0$ and using the same post-shock reduced-form shocks.

We compute the marginal effects of country n experiencing an adverse shock when all countries are in expansion. That is, we consider the change in the recession probability produced by a three-standard deviation increase in z_{nt} when all three countries are initially in expansion. Thus, we compute $\overline{\mathbf{Y}}_t^R$ as the average for Y_t during all periods in which the three countries are in expansion.[11]

Because the dynamic marginal effects have similar shapes as the linear impulse responses, we do not illustrate them here; however, they can have asymmetric magnitudes depending on the starting conditions. Because the starting values are set to the expansion average, the initial probability of recession is relatively low for each country when they start in expansion. Thus, the scenario we consider starts with countries securely in expansion and subjects them to large adverse shocks that essentially guarantee a subsequent recession in the domestic economy. We then assess how this shock affects the probability of a recession in the other countries. We are more interested in whether a foreign recession is likely than whether it is more likely to occur in a particular period. Thus, we compute the change in the probability of a recession over the next four periods.

A three-standard-deviation shock to z_{US} increases the probability of a U.S. recession in that quarter from 9.87 percent to 91.3 percent. In that same quarter, the probability of recession rises 6.6 and 6.5 percentage points for Canada and Mexico, respectively. However, the shock to z_{US} propagates across future horizons. The probability of a recession over the next year rises by 22.3 percentage points for Canada and 21.34 percentage points for Mexico.

We find similar effects for an increase in Canada's recession variable. A three-standard-deviation shock to z_{CAN} increases the probability of a recession over the next four quarters by 20.2 percentage points for the U.S. and 16.6 percentage points for Mexico. A shock to Mexico's recession variable propagates relatively less than shocks to z_{US} and z_{CAN}. A three-standard-deviation shock to z_{MEX} leads to a 6.9 percentage point increase in the probability of recession over the next year for the U.S. and a 3.8 percentage point increase for Canada.

[11] We also considered an alternative scenario where all three are countries are in recession and country n experiences an expansionary shock (negative shock to z_{nt}). These results are available upon request.

4.2 S_t Unobserved Results

In this section, we report the results of the estimation with the S_{nt}'s unobserved. Because business cycles are typically measured by more than just the GDP growth rate, which can also be noisy for some countries, we define x_{nt} as the first principal component of real GDP growth, industrial growth, employment growth, and retail trade growth.[12] Many of the underlying results are similar to those obtained with the S_{nt}'s observed. For example, the impulse responses are, as expected, qualitatively similar in both cases. In the interest of brevity, we do not report these results but they are available upon request from the authors.

The central issue for S_{nt} unobserved is how well the estimated states compare to the observed states. Obviously, in many applications, this comparison would not be available. However, for our application, we do have an objective measure of the states to compare, keeping in mind that the methods and variables used to identify the external recession dates may differ substantially from ours.

Figure 4 shows the estimates of the continuous latent variable z_{nt} for the three countries, along with the shaded recession dates for each country.[13] One noticeable difference between the results with unobservable S_{nt} is the volatility of the z_{nt} variable compared with the observed S_{nt}. Because the states are not predetermined, the filter picks up a fair number of false positives and a few false negatives in the middle of recessions.[14]

Another way to evaluate the regime inference in the unobserved S_{nt} model is to compute the area under the receiver operator characteristic curves (AUROC), which measures accuracy by weighing both false positives and false negatives.[15] For reference, a pure coin flip would have an AUROC of 0.5 and larger AUROC suggests more accurate regime inference. Comparing the unobserved S_{nt} model with the observed S_{nt} model is

[12] These are four of the variables that the NBER Business Cycle Dataing Committee says they focus on when determining the U.S. turning points (https://www.nber.org/cycles/recessions.html).

[13] We use the same recession dates for each country from the application to when S_t is observed.

[14] One solution to these problems could be to allow the intercept term in the VAR to switch as a function of S_{nt}. While this would introduce a number of complications, it would allow the persistence of the recession and expansion regimes to be different.

[15] The receiver operator characteristic curve plots the true positive rate against the false positive rate. Because the model output is a posterior recession probability, the ROC curve varies the threshold at which the probability is classified as a positive outcome. See Berge and Jordà for more details (2011).

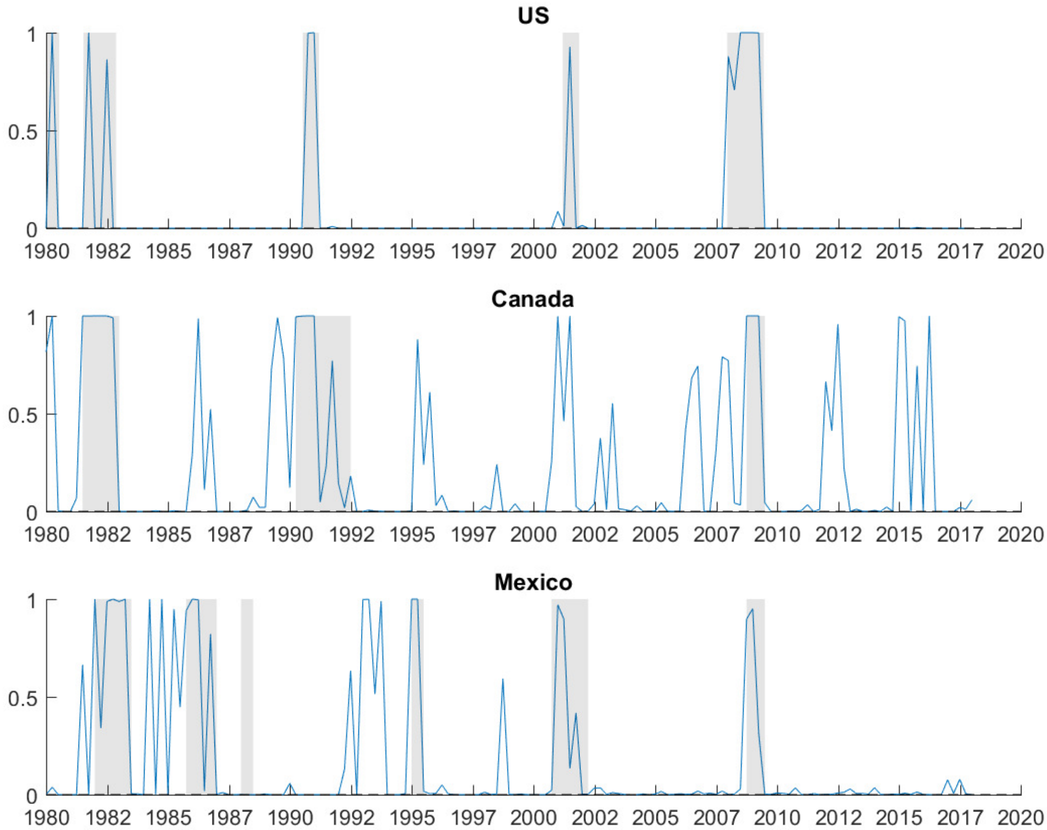

Figure 4: *Posterior Probability of Recession.* This figure shows the posterior probability of recession for each country. Gray shading reflect recession dating for each country (U.S. dates from NBER; Canada dates from C.D. Howe Institute; Mexico dates from BBQ estimation).

uninformative; however, we can compare our model with a simple univariate, constant transition probability Markov-switching model. Table 2 displays the AUROC for both our model (OPS) and the univariate Markov-switching model (MS). For each country, the contagious switching model correctly identifies a large proportion of the business cycle dates correctly represented by an AUROC of greater than 0.85. While the univariate model does marginally better than our model for the U.S. and Canada, our model does marginally better for Mexico. This suggests that accounting for information about U.S. recessions that may propagate to Mexico helps identify recessions south of the border.

Table 2: AUROC

Country	OPS	MS
U.S.	0.926	0.990
Canada	0.919	0.951
Mexico	0.881	0.870

5 The Effect of NAFTA

Armed with a model of cross-country business cycle propagation, we investigate whether trade liberalization altered the transmission of business cycles across borders [see also Kose, Prasad, and Terrones (2003)]. A number of studies [for example, Burfisher, Robinson, and Thierfelder (2001) and Miles and Vijverberg (2011) among others] have attempted to evaluate the effects of NAFTA, which liberalized trade beween the U.S., Canada, and Mexico. From 1990 (before NAFTA) to 2015 (well after NAFTA was enacted), total trade volume between the three countries rose from $333 billion to $2.137 trillion. In 1990 before NAFTA, the correlations between the U.S. and Canada, the U.S. and Mexico, and Mexico and Canada GDP growth rates were 0.87, -0.02, and 0.12, respectively; in 2015, those correlations were 0.78, 0.63, and 0.54, respectively.

To account for NAFTA, we re-estimate the model imposing a break in 1994, the time of NAFTA's implementation. We present results from the model with an unobserved business cycle state; results with S_t is observed are qualitatively similar but with sharper inference because of reduced uncertainty.[16]

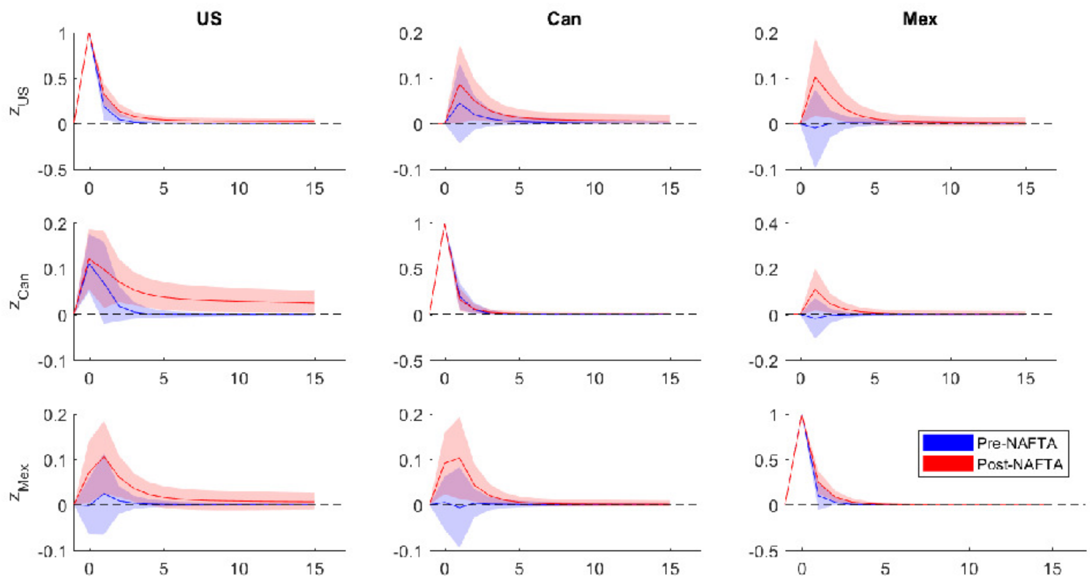

Figure 5: *Impulse Response Functions of z_{mt} to shocks to z_{nt} Using Pre- and Post-NAFTA Samples.* This figure shows the response of each country's response to a shock to the continuous recession variable based on pre-NAFTA sample (1980:Q1 - 1993:Q4) and post-NAFTA sample (1994:Q1 - 2018:Q1). The solid line shows the posterior median response and the shaded regions show the 67% HPD interval.

Similar to Figure 2, each row of Figure 5 shows the response of a country's business cycle indicator to a one-standard-deviation increase in the business cycle indicator for the country indicated by the column. The figure shows the responses for both the pre-NAFTA and NAFTA periods in blue and red, respectively.[17] Before the trade agreement, there are no significant spillover effects across any of the three countries. We find some evidence that recession propagate across the U.S.-Canada border as reflected by the positive posterior medians, but zero is included in the IRFs.

After the trade agreement, we find that recessions spread across all three nations. The magnitude of the recession pass-through from the U.S. to Canada in the two sub-samples is virtually identical, but statistically insignificant in the pre-NAFTA sample but

[16]The reduced uncertainty in the S_t-observed case also has larger magnitude spillovers across countries.

[17]In Figure 5, the NAFTA period includes the Great Recession. Previous studies have suggested that the Great Recession substantally increased business cycle synchronization. We considered a NAFTA period that excluded the Great Recession and found qualitatively similar results for the relationship between the U.S. and Mexico. However, excluding the Great Recession means results in no NAFTA period recessions for Canada. These results are avalable upon request.

statistically significant after NAFTA is enacted. NAFTA's effect on the pass-through of recession from the U.S. to Mexico is substantially larger. The differences in the changes in magnitudes for the transmission of U.S. shocks to Canada and Mexico are consistent with the effect of trade liberalization. While NAFTA increased the trade volume between the U.S. and Canada, its effects on U.S.-Mexico trade volume was on the order of three times larger over the same period. After the enactment of NAFTA, shocks originating in Canada and Mexico are more effectively transmitted to the U.S. Similarly to shocks originating in the U.S., the differences are more substantial for the U.S.-Mexico relationship than for the U.S.-Canada relationship.

In most of the other cases, the median response of z_{mt} to a shock to z_{nt} is larger after NAFTA went into effect. In particular, we find that Mexican shocks transmit to Canada and Canadian shocks transmit to Mexico in the NAFTA period. While both responses have periods in which zero is outside the HPD, the error bands are relatively large. While NAFTA did substantially increase the trade volume between the two countries, the total trade volume between Canada and Mexico is still only a small fraction of the trade between the U.S. and its neighbors.

In order to interpret this change in the degree of spillovers pre- and post-NAFTA, we contextualize them by calculating the dynamic marginal effects as we did in section 4.1.2. Table 3 displays the marginal effects of a three-standard-deviation shock to z_n on the probability of recession over the next four quarters for the other two countries. Before NAFTA, the cross-country spillovers of a U.S. recessionary shock are trivial, with only a 1.38 percentage point increase for Canada and a 0.27 percentage point increase for Mexico. A negative shock to Canada has similar effects on the other two countries. Shocks to Mexico's recessionary variable have essentially no marginal effect on the one-year probability of recession for the U.S. or Canada.

After NAFTA went into effect, we find a substantial change in the degree of propagation. A three-standard-deviation shock to z_{US} increases the probability of recession over the next year in Canada and Mexico by 8.24 percentage points and 7.59 percentage points, respectively. Similarly, adverse shocks to the Canadian economy have similar

Table 3: Marginal Effects Pre- and Post-NAFTA

	Pre-NAFTA	Post-NAFTA
Shock to z_{US}		
Canada	1.38	8.24
Mexico	0.27	7.59
Shock to z_{CAN}		
U.S.	1.69	6.21
Mexico	0.12	1.05
Shock to z_{MEX}		
U.S.	-0.02	7.66
Canada	0.00	4.78

spillover effects on the probability of recession in the U.S. or Mexico. Lastly, a shock to z_{MEX} affects the recession probabilities for both the U.S. and Canada after the trade agreement.

Consistent with a number of previous studies, we conclude that trade liberalization has affected how business cycles transmit across borders. However, our results suggest that the transmission between the U.S., Canada, and Mexico does not occur contemporaneously. Thus, computing simple correlations between measures of GDP may not tell the whole story behind business cycle synchronization. Moreover, the effect is bilateral, suggesting that transmission is not influenced only by the size of the U.S. economy.

6 Conclusions

In this paper, we construct a model to track the cross-country propagation of recessions. We show how the model can be estimated using both observed or unobserved business cycle phases. We apply the model to the U.S., Canada, and Mexico and find that there is a propensity for cycles to propagate across borders. Additionally, we find that U.S. monetary policy shocks affect the recession probabilities of both Canada and Mexico.

We then consider whether trade liberalization has affect the propagation of business cycles. We estimate the model for subsamples before and after a predetermined NAFTA break and find that recessions did not propagate pre-NAFTA. However, after NAFTA, adverse shocks originating in any of the three nations spread to the other two. This provides evidence that trade liberalization significantly increased the degree of business cycle synchronization.

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A Technical Appendix

The following appendix describes in detail the draws for the parameters. We first outline the state space representation of the VAR. We then describe the two draws that are invariant to whether we observe the regime. These draws condition only on the continuous latent state, z_t. We then describe the draw for the continuous latent variable z_t when S_t is observed. Finally, we describe the draws for the parameters in the measurement equation that relates the discrete regime to the growth variable, x_t.

A.1 The State Space Representation

Recall that z_t is $(N \times 1)$ and y_t is $(J \times 1)$ and let $K = N + J$. Define $Y_t = [z_t', y_t']'$ and $\varsigma_t = \left[Y_t', Y_{t-1}', \cdots, Y_{t-P+1}'\right]'$ as the state in the state-space representation of the model with measurement equation:

$$y_t = H\varsigma_t,$$

and transition equation:

$$\varsigma_t = M + F\varsigma_{t-1} + e_t,$$

where

$$e_t \sim N(\mathbf{0}_{KP\times 1}, Q).$$

The parameters of the state space are defined as follows:

$$Q = \begin{bmatrix} \Sigma & \mathbf{0}_{K\times K(P-1)} \\ \mathbf{0}_{K(P-1)\times K(P)} & \end{bmatrix},$$

$$H = \begin{bmatrix} \mathbf{0}_{J\times N} & \boldsymbol{I}_J & \mathbf{0}_{J\times K(P-1)} \end{bmatrix},$$

$$M = \begin{bmatrix} B_0 \\ \mathbf{0}_{K(P-1)\times 1} \end{bmatrix},$$

and

$$F = \begin{bmatrix} B_1 & B_2 & \cdots & B_{P-1} & B_P \\ \boldsymbol{I}_K & \mathbf{0}_{K\times K} & \cdots & \mathbf{0}_{K\times K} & \mathbf{0}_{K\times K} \\ \mathbf{0}_{K\times K} & \boldsymbol{I}_K & \cdots & \mathbf{0}_{K\times K} & \mathbf{0}_{K\times K} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0}_{K\times K} & \mathbf{0}_{K\times K} & & \boldsymbol{I}_K & \mathbf{0}_{K\times K} \end{bmatrix}.$$

Notice that the measurement equation is deterministic.

A.2 Drawing B conditional on Σ, $\{z_\tau\}_{\tau=1}^{t-1}$

Conditional on Σ, the VAR parameters B are conjugate Normal. Define $\varsigma_t = \left[Y_t', Y_{t-1}', \cdots, Y_{t-P+1}'\right]'$. Then, the VAR can be written as:

$$Y_t = Bx_t + u_t,$$

where $x_t = \left[1, Y_{t-1}', ..., Y_{t-P}'\right]'$. Stacking the observations, we get:

$$Y = XB' + U$$

where $Y = [Y_{P+1}, Y_{P+2}, ..., Y_T]'$ and $X = [X_{P+1}, X_{P+2}, ..., X_T]'$. Let $\hat{B}' = (X'X')^{-1}(X'Y)$ be the OLS estimates for B'.

We assume a Minnesota prior for $B = [B_0, B_1, ..., B_P]$. The prior distribution for $b = vec(B')$ is $b \sim N(\bar{b}_0, \bar{B}_0)$ where $\bar{b}_0 = \mathbf{0}_{K(P+1)\times 1}$ and the diagonal elements of $\bar{B}_0$ are set according to:

$$var\left(B_{l,ij}\right) = \begin{cases} \left(\frac{\lambda_1}{p^{\lambda_3}}\right)^2 & \text{if } l > 0 \text{ and } i = j \\ \left(\frac{s_i \lambda_1 \lambda_2}{s_j p^{\lambda_3}}\right)^2 & \text{if } l > 0 \text{ and } i \neq j \\ \left(s_i \lambda_4\right)^2 & \text{if } l = 0 \end{cases}.$$

We set the hyperparameters to $\lambda_1 = 0.2$, $\lambda_2 = 0.5$, $\lambda_3 = 2$, $\lambda_4 = 0.20$. Define $\hat{b} = vec(\hat{B}')$.

The posterior distribution for b is:

$$b \sim N(\bar{b}_1, \bar{B}_1),$$

where

$$b_1 = B_1^{-1}(\bar{B}_0^{-1}\bar{b}_0 + \Sigma^{-1} \otimes X'X\hat{b}),$$

$$B_1 = (\bar{B}_0^{-1} + \Sigma^{-1} \otimes X'X)^{-1}.$$

We redraw B if the usual stationarity condition is violated.

A.3 Drawing Σ conditional on B, $\{z_t\}_{t=1}^{T}$

The draw of Σ is nonstandard due to the restrictions placed on the variance parameters of the latent variables z_{nt} for $n = 1, ...N$. Specifically we restrict the variance of each z_{nt} to 1. We adopt the algorithm outlined by Chan and Jeliazkov (2009) for drawing restricted covariance matrices. This method takes advantage of the decomposition $\Sigma = L'^{-1}DL^{-1}$ where

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_{N+J} \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ a_{21} & 1 & 0 & \cdots & 0 \\ a_{31} & a_{32} & 1 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N+J1} & a_{N+J2} & \cdots & \cdots & 1 \end{bmatrix}.$$

As noted by Chan and Jeliazkov (2009), the first N diagonal parameters $\lambda_1, ..., \lambda_N$ are deterministic functions of the a_{ij} parameters of the lower triangular matrix due to the identification restrictions placed on z_{nt}. Therefore $\lambda_1, ..., \lambda_N$ and the a parameters must be drawn jointly via a Metropolis-Hastings step. However the remaining diagonal parameters $\lambda_{N+1}, ..., \lambda_{N+J}$ can be drawn unrestricted via a Gibbs sampling step.

We first outline the prior distributions. For for λ_k for $k = N+1, ..., N+J$, we assume the following prior:

$$\lambda_k \sim IG\left(\frac{v_{k0}}{2}, \frac{\delta_{k0}}{2}\right),$$

and for $\mathbf{a}_k = [a_{k1}, ..., a_{k,k-1}]'$ for $k = 2, ..., N+J$:

$$\mathbf{a}_k \sim N\left(\mathbf{a}_{k0}, \mathbf{A}_{k0}\right).$$

The algorithm for drawing all of the parameters governing Σ is:

Step 1: Draw $\lambda_{N+1}, ..., \lambda_{N+J}$ from equation (2.5) from Chan and Jeliazkov (2009):

$$\lambda_k \sim IG\left(\frac{v_{k0}+T}{2}, \frac{\delta_{k0}+s_k}{2}\right),$$

where s_k is the (k,k)-element of $\sum_{t=1}^{T} w_t w_t'$ and $w_t = Lu_t$. In our application, $u_t = Y_t - Bx_t$ where $x_t = \left[1, Y_{t-1}', ..., Y_{t-P}'\right]'$.

Step 2: We draw a candidate $\mathbf{a}_k^c$ from the multivariate t-distribution outlined in equation (3.7) of Chan and Jeliazkov (2009):

$$\mathbf{a}_k^c \sim MVT\left(\mathbf{a}_k, \tau\mathbf{A}_k, \kappa\right),$$

where $\mathbf{A}_k = \left(A_{k0}^{-1} + \lambda_k^{-1} X_k' X_k\right)^{-1}$, $\mathbf{a}_k = \mathbf{A}_k\left(A_{k0}^{-1} a_{k0} - \lambda_k^{-1} X_k' z_k\right)$, $z_k = [u_{1k}, ..., u_{Tk}]'$, $X_k = [z_1, ..., z_{k-1}]$, u_{tk} is the k-th element of u_t, τ is a tuning parameter, and κ represents the degrees of freedom. Based on the candidate draw $\mathbf{a}_k^c$ for $k = 2, ..., N+J$, we compute the associated candidates for the diagonal parameters $\lambda_1^c, ..., \lambda_N^c$ using (3.3) and (3.4) from Chan and Jeliazkov (2009):

$$\lambda_1^c = 1,$$
$$\lambda_k^c = 1 - \sum_{j=1}^{k-1} (a^{kj})^2 \lambda_j,$$

where a^{kj} is the (k,j)-element of the lower triangular matrix L^{-1}. The candidate draw is accepted with probability:

$$\alpha = \min\left\{1, \frac{l\left(Y \mid \Sigma^{c}\right) \prod_{k=2}^{N+J} f_T\left(\mathbf{a}_k^c | \mathbf{a}_{k0}, \tau \mathbf{A}_{k0}, \kappa\right) f_N\left(\mathbf{a}_k^{i-1} | \mathbf{a}_k, \mathbf{A}_k\right)}{l\left(Y \mid \Sigma^{i-1}\right) \prod_{k=2}^{N+J} f_T\left(\mathbf{a}_k^{i-1} | \mathbf{a}_{k0}, \tau \mathbf{A}_{k0}, \kappa\right) f_N\left(\mathbf{a}_k^c | \mathbf{a}_k, \mathbf{A}_k\right)}\right\}$$

where $f_T(\cdot)$ is the multivariate t density. If the candidate is accepted, we use $\lambda_1^c, ..., \lambda_N^c$ and $\mathbf{a}_2^c, ..., \mathbf{a}_{N+J}^c$ with the draw from Step 1 for $\lambda_{N+1}, ..., \lambda_{N+J}$ to calculate the new covariance matrix $\Sigma = L'^{-1} D L^{-1}$. Otherwise, we use the previous draw $\lambda_1^{i-1}, ..., \lambda_N^{i-1}$ and $\mathbf{a}_2^{i-1}, ..., \mathbf{a}_{N+J}^{i-1}$ to calculate the covariance matrix.

A.4 Drawing z_t conditional on B, Σ, and observed $\{S_\tau\}_{\tau=1}^T$

We implement the Kalman filter with smoothing to draw the vector z_t given the state vector $S_t = [S_{1t}, ..., S_{Nt}]'$. If the sign of the draw for z_t does not match the state implied by S_t, we redraw until the condition is met. Since Q is singular, we use the modification outlined by Kim and Nelson (1999) that simplifies the backwards smoother to only the relevant conditioning factors.

A.5 Drawing $\Psi = \{\mu_0', \mu_1', \phi'\}$ conditional on $\{S_\tau\}_{\tau=1}^T$ and $\left\{\sigma_n^2\right\}_{n=1}^N$

A.5.1 Drawing $\mu_n = [\mu_{n0}, \mu_{n1}]'$ given S_n, σ_n^2, and ϕ_n

We first define:

$$\tilde{x}_{nt} = \frac{x_{nt} - \phi_n(L) x_{nt-1}}{\sigma_n},$$

$$\tilde{S}_{nt} = \left[\frac{1 - S_{nt}}{\sigma_n}, \frac{S_{nt}}{\sigma_n}\right],$$

$$\tilde{S}_n = \left[\tilde{S}_{n1}^{‘}, ..., \tilde{S}_{nT}^{‘}\right]^{‘}.$$

Assuming a normal prior distribution $\mu_n \sim N\left(m_{n0}, M_{n0}\right)$, we draw the regime-specific growth parameters from

$$\mu_n \sim N\left(m_{n1}, M_{n1}\right),$$

where

$$M_{n1} = \left(M_{n0}^{-1} + \tilde{S}_n' \tilde{S}_n\right)^{-1},$$

$$m_{n1} = M_{n1}\left(M_{n0}^{-1} m_{n0} + \tilde{S}_n' x_n\right).$$

A.5.2 Drawing ϕ_n given S_n, σ_n^2, and μ_n

Similar to the draw for μ_n, we define:

$$\breve{x}_{nt} = \frac{x_{nt}}{\sigma_n} - \tilde{S}_{nt}\mu_n,$$

$$\breve{x}_n^T = \left[\breve{x}_{n,p+1}, .., \breve{x}_{n,T}\right],$$

and $\breve{X}_n^T$ as the $[p \times (T-p)]$ matrix containing the p lags of $\breve{x}_n^T$. Then assuming the prior distribution $\phi_n \sim N(p_{n0}, P_{n0})$ and the roots of $1 - \phi_n(L)$ fall outside the unit circle, we have the following posterior distribution:

$$\phi_n \sim N(p_{n1}, P_{n1}),$$

where

$$P_{n1} = \left(P_{n0}^{-1} + \breve{X}_n^{T\prime} \breve{X}_n^T\right)^{-1},$$

$$p_{n1} = P_{n1}\left(P_{n0}^{-1} p_{n0} + \breve{X}_n^{T\prime} \breve{x}_{nt}\right).$$

A.6 Drawing $\left\{\sigma_n^2\right\}_{n=1}^N$ conditional on $\left\{S_t\right\}_{\tau=1}^T$ and $\Psi = \{\mu_0', \mu_1', \phi'\}$

The error variance for the business cycle process is drawn from the following posterior distribution:

$$\sigma_n^{-2} \sim \Gamma\left(\frac{\upsilon_0 + T}{2}, \frac{\tau_0 + \sum_{t=1}^T \hat{\varepsilon}_{nt}^2}{2}\right),$$

where

$$\hat{\varepsilon}_{nt} = x_{nt} - \mu_{n0} - \Delta\mu_n S_{nt} - \phi(L) x_{nt-1}.$$