Optimal Ramsey Taxation in Heterogeneous Agent Economies with Quasi-Linear Preferences

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Optimal Ramsey Taxation in Heterogeneous Agent Economies with Quasi-Linear Preferences

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May 18, 2020

Abstract

We build a tractable heterogeneous-agent incomplete-markets model with quasi-linear preferences to address a set of long-standing issues in the optimal Ramsey taxation literature. The tractability of our model enables us to analytically prove the existence of a Ramsey steady state and establish several novel results: (i) In the absence of any redistributional effects of capital taxation or lump-sum transfers, the optimal capital tax is exclusively zero in a Ramsey steady state regardless of the modified golden rule (MGR) and government debt limits. (ii) Whether the MGR holds or not depends critically on the government’s capacity to issue debt but has no bearing on the planner’s long-run capital tax scheme. (iii) The optimal debt-to-GDP ratio, however, is determined by a positive wedge times the MGR saving rate: The wedge is decreasing in the strength of individuals’ self-insurance positions and approaches zero when the idiosyncratic risk vanishes or markets are complete. (iv) The assumption of the existence of a Ramsey steady state commonly made in the existing literature is not innocuous: When a Ramsey steady state does not exist but is erroneously assumed to exist, the MGR always appears to “hold” and the implied “optimal” long-run capital tax is strictly positive. (v) Along the transition path toward a Ramsey steady state, the optimal capital tax depends positively on the elasticity of intertemporal substitution. The key insight behind our results is that in the absence of any redistributional effects, taxing capital in the steady state permanently hinders individuals’ self-insurance positions and thus the Ramsey planner opts to issue debt rather than impose a steady-state capital tax to correct the capital-overaccumulation problem. However, if the demand for debt approaches infinity when the interest rate approaches the time discount rate, a Ramsey steady state may not exist; thus, the MGR can fail to hold in a Ramsey equilibrium whenever the government encounters a binding debt limit.

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1 Introduction

The seminal work of Aiyagari (1995) has inspired a large literature. However, despite several important revisits, such as Chamley (2001), Conesa, Kitao, and Krueger (2009), Dávila, Hong, Krusell, and Ríos-Rull (2012), and many others, many issues regarding optimal capital taxation in a heterogeneous-agent incomplete-markets (HAIM) economy (à la Aiyagari (1994)) remain unsettled.

For example, is a positive tax levied on capital by the Ramsey planner in an infinite-horizon HAIM economy motivated mainly by the modified golden rule (MGR) in light of capital overaccumulation, by wealth redistribution in light of inequality under borrowing constraints, or both? How does optimal capital taxation relate to the optimal level of public debt and the existence of a Ramsey steady state?

These issues are intertwined because government bonds not only serve as a self-insurance device for individuals, they also alleviate the overaccumulation of the aggregate capital stock and thus reduce the need for a distortionary capital tax.\(^1\) In the meantime, capital taxation (in addition to a labor tax) may be necessary in order to finance the interest payments on public debt and to redistribute wealth from the rich to the poor. Moreover, whenever the interest rate lies below the time discount rate, the marginal social cost of public debt may be strictly smaller than its marginal benefit; consequently, a Ramsey steady state may not exist if the optimal quantity of debt diverges to infinity when the interest rate approaches the time discount rate. Thus, the existence of a Ramsey steady state necessarily implies a finite level of government debt (in the absence of any ad hoc debt limits).\(^2\)

Yet, to the best of our knowledge, the existence of a Ramsey steady state in infinite-horizon HAIM models is often assumed instead of proven in the existing literature. Without such an assumption, the Ramsey allocation is hard to analyze because of these models’ intractability; but optimal tax policies drawn from the analyses may hinge critically on the validity of such an assumption.

The issue at stake is a trade-off under capital taxation between (i) aggregate allocative efficiency (AAE, in terms of the MGR) and (ii) individual allocative efficiency (IAE, in terms of self-insurance). A positive capital tax improves AAE by equalizing the after-tax liquidity-

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\(^1\) Aiyagari (1995) (pp. 1160-1161) acknowledged the importance of government debts in alleviating capital overaccumulation. However, the critical issue of government debt limits is ignored and a non-binding natural limit of government debt is implicitly assumed in his analysis as well as in the existing literature.

\(^2\) A Ramsey steady state may also exist under a binding debt-limit constraint, but this may have different implications for the optimal capital tax. This is not a trivial issue and has not been investigated thoroughly by the existing literature.
premium-adjusted private marginal product of capital (MPK) and socially optimal MPK but worsens IAE by increasing the (liquidity) premium of returns to household savings (or worsening individuals’ self-insurance positions).

Specifically, the competitive market equilibrium in a HAIM model may appear to be dynamically inefficient due to overaccumulation of capital under precautionary saving motives caused by idiosyncratic risks and borrowing constraints, which results in (i) inequality across households, (ii) a liquidity premium on the rate of return to household savings, and (iii) an excessively low aggregate MPK. Consequently, the MGR is violated in a competitive equilibrium. This observation provides the key intuition of Aiyagari (1995) that the Ramsey planner should tax capital income to restore AAE. However, a capital tax also has a strong redistributional effect on individual wealth (see, e.g., Mirrlees (1971)). In a HAIM economy, even a uniform linear capital tax can redistribute income and wealth from the rich to the poor under borrowing constraints, thus improving social welfare.

In addition, the conventional wisdom for relying solely on the MGR to justify a positive capital tax in an infinite-horizon HAIM model is counterintuitive, at least from a micro viewpoint. By taxing capital income and thus reducing each individual’s optimal buffer stock of savings, the government is hampering and effectively destroying individuals’ ability to self-insure against idiosyncratic risks when lump-sum transfers are not available (as in Aiyagari (1995)). Since taxing capital per se does not directly address the lack-of-insurance problem for households (if anything, it intensifies the problem), why would taxing capital be always optimal for the Ramsey planner, especially when government bonds are less distortionary than capital taxes and thus more effective in addressing the problem of capital overaccumulation without hindering individuals’ self-insurance positions?

These questions are intriguing because the lack of full self-insurance is the root cause of the capital overaccumulation problem in HAIM economies and should hence be the ultimate concern of a benevolent government. In other words, in the absence of any redistributional effect of a capital tax, eliminating the inefficiency of overaccumulation through capital taxation does not help at all to alleviate the primal friction in the model—the lack of full self-insurance under borrowing constraints—thus, using only the MGR to justify a positive capital income tax regardless of its effect on IAE is not at all clear.

In addition, if taxing capital is optimal in a Ramsey steady state when government bonds are available, would it imply that the equilibrium level of government debt is too low to equate the interest rate with the time discount rate? But if other forms of distortionary taxation (such as labor taxation) are available and there is no debt limit, why would the
Ramsey planner not increase the bond supply to further reduce the aggregate capital stock and avoid the need for a distortionary capital tax?

In short, intuition tells us that in the absence of any redistributional concerns, a Ramsey planner can improve welfare more likely through improving individuals’ self-insurance positions (such as issuing enough bonds to substitute for capital) than through taxing individuals’ buffer stock (capital) when labor taxes are available to finance the interest payments on public debt.

This trade-off of capital taxation between AAE and IAE thus pertains intrinsically to the determination of the optimal level of public debt, which in turn depends critically on the existence of a Ramsey steady state. That is, the greater the supply of government bonds, the greater the self-insurance position individuals can achieve, creating less of a need to impose a distortionary capital tax. Hence, optimality seems to require the bond interest rate to equal the time discount rate. Yet the demand for government bonds in a standard infinite-horizon HAIM model approaches infinity as the interest rate approaches the time discount rate, as pointed out by Aiyagari (1994, 1995). Hence, without any borrowing limits, the assumption of the existence of a Ramsey steady state necessarily implies that the optimal level of the bond supply is finite. But the question is why? And how is such an optimal quantity of debt in the original Aiyagari model determined?

Therefore, daunting challenges in the determination of an optimal capital tax in the class of infinite-horizon HAIM models lie (critically) in analyzing the trade-off problem under capital taxation and in proving the existence of a Ramsey steady state, which in turn dictate the determination of the optimal level of public debt. In other words, since a capital tax can always be substituted by a labor tax and the capital tax rate can always be lowered further by increasing the bond supply, the policy mix must be simultaneously determined in a Ramsey equilibrium by investigating the full set of the Ramsey planner’s first-order optimal conditions for the bond supply, capital tax, and labor tax.

These issues are intrinsically related to the classical question studied in the representative-agent Ramsey literature; namely, if the government’s only option is to tax factor income, should the government tax capital income or labor income to finance a given level of government expenditures? The pivotal work by Judd (1985) and Chamley (1986) argued that the government should tax only labor income and not capital income in the long run regardless of the ratio of government spending to gross domestic product (GDP).

The goal of this paper is to design a tractable HAIM model to analytically investigate the trade-off between AAE and IAE when a Ramsey steady state can be proven to exist—in
the absence of any redistributitional effects of capital taxation or lump-sum transfers. Our infinite-horizon model follows the spirit of Aiyagari (1995), but with two key differences: Individuals in our model face idiosyncratic shocks to the marginal utility of consumption, and their preferences are quasi-linear. This preference structure completely eliminates any redistributitional effects of a capital tax and enables us to solve both the competitive equilibrium and the Ramsey allocation in closed forms. Our model has the desirable properties that a Ramsey steady state exists only in a proper parameter space and that a particular Ramsey steady state is one in which the interest rate equals the time discount rate \((1/\beta)\). These properties allow us to use counterfactual analyses to address some important questions regarding the critical role of the MGR in determining optimal capital taxes in the absence of income/wealth redistribution.

Our analytical approach shares a similar spirit to the recent work of Heathcote and Perri (2018). In their model households can reshuffle asset holdings at the end of each period such that the distribution of households’ end-of-period wealth is degenerated. This feature allows their HAIM model to be analytically tractable despite idiosyncratic risk and precautionary saving motives. In our model, however, the degenerated distribution of wealth is an endogenous outcome of individuals’ rational choices.

Our contributions are thus five-fold: First, we provide an analytically tractable model in which necessary and sufficient conditions for the existence of a Ramsey steady state can be explicitly stated.

Second, we show analytically that (i) the Ramsey planner will never tax capital in a Ramsey steady state, despite capital overaccumulation, and (ii) when a Ramsey steady state (featuring a non-binding government debt limit) does not exist but is erroneously assumed to exist, the MGR always appears to “hold” and the implied “optimal” long-run capital tax is strictly positive regardless of the model’s structural parameters and the level of government debt—suggesting that the assumption of the existence of a Ramsey steady state is not innocuous in HAIM models.\(^3\)

Third, our analysis provides clarification of the critical role of government debt in achieving the MGR and influencing the trade-offs of a capital tax between AAE and IAE. Specifically, we show that whether the MGR should hold or not depends critically on the Ramsey planner’s ability to issue bonds as an alternative store of value (aside from capital) for households to buffer idiosyncratic risks. In particular, the MGR would hold in our model if and

\(^3\)A Ramsey steady state refers to a Ramsey allocation where all aggregate variables and the moments of the distribution converge to constant and finite non-zero values.
only if the government can amass a sufficiently large stock of bonds to enable households to achieve full self-insurance. Once households are fully self-insured, the equilibrium interest rate in our model equals the time discount rate \((1/\beta)\). In such a case, AAE and IAE are simultaneously achieved by the Ramsey planner. However, when it is impossible to equalize the interest rate and the time discount rate—either because of unbounded variance of the idiosyncratic shocks or due to an ad hoc debt-limit constraint on the government’s capacity to issue bonds—the MGR does not hold. Despite the failure of the MGR, however, the optimal capital tax rate is still zero in the steady state (even in the case where the government cannot issue bonds).\(^4\) Hence, the MGR appears to have no bearing on the planner’s long-run capital tax scheme.

Fourth, we use numerical analysis to show that the Ramsey planner nonetheless opts to tax capital along the transition path—so as to front-load consumption and discourage capital accumulation. In particular, the larger the elasticity of intertemporal substitution, the higher the short-run tax rate. In fact, the levels of aggregate consumption, the capital stock, and hours worked in a Ramsey steady state are lower than their counterparts in the laissez-faire competitive equilibrium. Hence, the steady-state welfare gains under optimal policies derive primarily from the improved distribution of household self-insurance positions. Notice that in our numerical analyses the Ramsey transition path is calculated based on a proven Ramsey steady state, in contrast to the existing numerical literature that calculates the Ramsey transition path based on an unproven Ramsey steady state that may or may not exist. Our numerical exercises also serve as independent verification of our theoretical analyses.

Last but not the least, in the absence of a binding debt limit, the optimal debt-to-GDP ratio in our model is determined by a positive wedge times the aggregate saving rate implied by the MGR. This wedge is an increasing function of the extent of individual allocative inefficiency and would vanish only if idiosyncratic risk approaches zero or markets become complete. Namely, the optimal debt-to-GDP ratio in our model is zero if (and only if) households are fully self-insured and no longer borrowing constrained in a competitive equilibrium. This result suggests (again) that the single most important role of government debt is to improve individuals’ self-insurance positions through which the MGR is achieved (if possible). In other words, in the absence of any redistributional effects, the MGR does not appear to be the primal concern of the Ramsey planner, and it is never optimal to tax capital

\(^4\)This result is striking yet very intuitive: If it is optimal to set the capital tax to zero without the tool of government debt, then it should be optimal to maintain a zero capital tax when government bonds become available. See the proofs in the next section.
in the steady state simply to achieve AAE even though it is feasible to do so (as long as other forms of distortionary taxes such as a labor income tax are available).

These results are intuitive. On the one hand, it is the lack of an insurance market that induces agents to overaccumulate capital to self-insure against idiosyncratic consumption risk. In the absence of any redistributational effects, taxing capital in the steady state would permanently hamper individuals’ self-insurance—the lack of which is the root cause of aggregate allocative inefficiency—and is thus not a desirable tool to restore the MGR.

On the other hand, government bonds meet individuals’ demand for precautionary saving without creating pecuniary externalities on the marginal product of capital. Hence, by substituting for (or crowding out) capital, government debt can satisfy the household buffer-stock saving motives and, at the same time, correct aggregate inefficiency due to over-accumulation of capital. This is why the Ramsey planner opts to flood the asset market with a sufficient amount of bonds to ensure (as much as possible) full self-insurance for all households across all states. However, when debt limits exist, the Ramsey planner is unable to issue enough bonds to ensure full self-insurance; but, in spite of this, the planner will not levy a permanent tax on capital simply to achieve the MGR. Instead, the planner opts to tax capital in the short run to reduce as much as possible the steady-state capital stock.

In addition, with certain parameter values such as unbounded idiosyncratic shocks (as in the case of a Pareto distribution) and unbounded government debt limits (which are implicitly assumed in Aiyagari (1995)), a Ramsey steady state (featuring non-binding government debt-limit constraints) may not exist in our model. In such a case, the interest rate always lies below the time discount rate and the Ramsey planner could improve social welfare and individuals’ self-insurance positions by issuing more and more government bonds until a government borrowing limit kicks in or else destroying the existence of the Ramsey steady state.

Although our model is just a special case of standard infinite-horizon HAIM models, it serves to demonstrate that (i) the classical result of zero capital taxation obtained in the representative-agent literature can still hold in infinite-horizon HAIM economies with over-accumulated capital stock—as long as the redistributational channel of capital taxation is shut down—and (ii) the assumption of the existence of a Ramsey steady state in the absence of a government debt limit may not be innocuous. This second point is also supported independently by the recent work of Chen, Chien, and Yang (2019), which shows that in the Aiyagari model the assumption of a Ramsey steady state (featuring a non-binding government debt limit) is inconsistent with some of the Ramsey planner’s first-order conditions.
But the intractability of the model prohibits them from drawing any conclusions on optimal taxation, the necessary and sufficient condition for the existence of a Ramsey steady state, and the optimal quantity of public debt.\footnote{Aiyagari and McGrattan (1998) build on the Aiyagari (1994) model to determine the optimal quantity of debt by studying the trade-offs in benefits and costs of varying the quantity of debt. Their analysis assumes the existence of a Ramsey steady state, and they focus on steady-state welfare through numerical methods under another critical assumption that the proportional tax rates on labor and capital income are levied equally across households to finance public debt. This latter assumption rules out the possibility that the optimal capital tax may be zero. Last but not the least, they are not able to isolate the roles of the MGR and wealth redistribution in determining the optimal capital tax rate.}

The remainder of the paper is organized as follows. Section 2 describes the model, derives the competitive equilibrium, and provides sufficient conditions for the Ramsey planner to support a competitive equilibrium. Section 3 shows how to solve for the Ramsey allocation analytically, to prove the existence of a Ramsey steady state, and to derive the optimal capital tax in a Ramsey steady state. Section 4 performs numerical exercises to study transition dynamics and also to confirm our theoretical analyses. Section 5 provides a brief literature review. The last section concludes the paper with remarks for future research.

\section{The Model}

This model is based on Bewley (1980), Lucas (1980), Huggett (1993), Aiyagari (1994), and especially Wen (2009, 2015). To fix notions, for this paper (with some abuse of terminology) we define AAE (aggregate allocative efficiency) as a competitive equilibrium allocation in which the (after-tax) liquidity-premium-adjusted private MPK equals the socially optimal MPK, and we define IAE (individual allocative efficiency) as a competitive equilibrium allocation in which all households are fully self-insured (given government bonds) and their borrowing constraints do not bind in all idiosyncratic states.

Obviously, in a competitive equilibrium without government intervention, IAE implies AAE but not necessarily vice versa. Also, the Ramsey planner may design policies that achieve IAE without achieving AAE, or vice versa.

We also define the “golden-rule allocation” (GRA) as a Ramsey steady state allocation in which both AAE and IAE are achieved under optimal government policies. Accordingly, our discussions involve two different notions of steady state: the “competitive equilibrium steady state” for a given set of government policies, and the “Ramsey steady state” under optimal policies.
2.1 Environment

A representative firm produces output according to the constant-returns-to-scale Cobb-Douglas technology, \( Y_t = F(K_t, N_t) = K_t^\alpha N_t^{1-\alpha} \), where \( Y, K, \) and \( N \) denote aggregate output, capital, and labor, respectively. The firm rents capital and hires labor from households by paying a competitive rental rate and real wage, denoted by \( q_t \) and \( w_t \), respectively. The firm’s optimal conditions for profit maximization at time \( t \) satisfy

\[
\begin{align*}
  w_t &= \frac{\partial F(K_t, N_t)}{\partial N_t}, \\
  q_t &= \frac{\partial F(K_t, N_t)}{\partial K_t}.
\end{align*}
\]

There is a unit measure of \textit{ex ante} identical households that face idiosyncratic preference shocks, denoted by \( \theta \). The shocks are identically and independently distributed (iid) over time and across households, and have the mean \( \bar{\theta} \) and the cumulative distribution \( F(\theta) \) with support \([\theta_L, \theta_H]\), where \( \theta_H > \theta_L > 0 \).

Time is discrete and indexed by \( t = 1, 2, ..., \infty \). There are two subperiods in each period \( t \). The idiosyncratic preference shock \( \theta_t \) is realized only in the second subperiod, and the labor supply decision must be made in the first subperiod before observing \( \theta_t \). Namely, the idiosyncratic preference shock is uninsurable by wage income even when leisure enters the utility function linearly. Let \( \theta^t \equiv (\theta_1, ..., \theta_t) \) denote the history of idiosyncratic shocks. All households are endowed with the same asset holdings at the beginning of time 1.

Households are infinitely lived with a quasi-linear utility function and face borrowing constraints. Their lifetime expected utility is given by

\[
V = E \sum_{t=1}^{\infty} \beta^{t-1} \left[ \theta_t c_t(\theta^t)^{1-\sigma} - \sigma - n_t(\theta^t)^{1-\sigma} \right],
\]

where \( \beta \in (0, 1) \) is the discount factor; \( \sigma \in (0, \infty) \) is a parameter that determines the elasticity of intertemporal substitution (EIS) and risk aversion of the household; \( c_t(\theta^t) \) and \( n_t(\theta^{t-1}) \) denote consumption and the labor supply, respectively, for a household with history \( \theta^t \) at time \( t \). Note that the labor supply in period \( t \) is only measurable with respect to \( \theta^{t-1} \), reflecting the assumption that the labor-supply decision is made in the first subperiod before observing the preference shock \( \theta_t \).

The government needs to finance an exogenous stream of purchases, denoted by \( G_t \geq 0 \) for all \( t \), and it can issue bonds and levy time-varying labor and capital taxes at flat rates.
\( \tau_{n,t} \) and \( \tau_{k,t} \), respectively. The flow government budget constraint in period \( t \) is

\[
\tau_{n,t}w_tN_t + \tau_{k,t}q_tK_t + B_{t+1} \geq G_t + r_tB_t, \tag{4}
\]

where \( B_{t+1} \) is the level of government bonds chosen in period \( t \), and \( r_t \) is the gross risk-free rate.

### 2.2 Household Problem

We assume there is no aggregate uncertainty and that government bonds and capital are perfect substitutes for store of value for households. As a result, the after-tax gross rate of return to capital must equal the gross risk-free rate:

\[
1 + (1 - \tau_{k,t})q_t - \delta = r_t, \tag{5}
\]

which constitutes a no-arbitrage condition for capital and bonds.

Given the sequence of interest rates, \( \{r_t\}_{t=1}^\infty \), and after tax wage rates, \( \{w_t \equiv (1 - \tau_{n,t})w_t\}_{t=1}^\infty \), a household maximizes (3) by choosing a plan of consumption, labor, and asset holdings, \( \{c_t(\theta^t), n_t(\theta^{t-1}), a_{t+1}(\theta^t)\} \) subject to

\[
c_t(\theta^t) + a_{t+1}(\theta^t) \leq \bar{w}_t n_t(\theta^{t-1}) + r_t a_t(\theta^{t-1}), \tag{6}
\]

\[
a_{t+1}(\theta^t) \geq 0, \tag{7}
\]

with \( a_1 > 0 \) given and \( n_t(\theta^{t-1}) \in [0, \bar{N}] \). The solution of the household problem can be characterized analytically by the following proposition.

**Proposition 1.** Denoting household gross income (or total liquidity on hand) by \( x_t(\theta^{t-1}) \equiv r_t a_t(\theta^{t-1}) + \bar{w}_t n_t(\theta^{t-1}) \), the optimal decisions for \( x_t(\theta^{t-1}) \), consumption \( c_t(\theta^t) \), savings \( a_{t+1}(\theta^t) \), and the labor supply \( n_t(\theta^{t-1}) \) are given, respectively, by the following cutoff-policy rules\(^6\) :

\[
x_t = \left[\bar{w}_t L(\theta^*) \theta^*\right]^{1/\sigma} \tag{8}
\]

\[
c_t(\theta_t) = \min \left\{ 1, \left( \frac{\theta_t}{\theta^*} \right)^{1/\sigma} \right\} x_t \tag{9}
\]

\(^6\)The cutoff-policy rules hold if the individual labor decision is an interior one, namely, \( n_t \in (0, \bar{N}) \). In the proof of this proposition (Appendix A.1), we show that with reasonable parameter values and a sufficiently large chosen \( \bar{N} \), individual hours worked are ensured to be interior.
\( a_{t+1}(\theta_t) = \max \left\{ 1 - \left( \frac{\theta_t}{\theta^*_t} \right)^{1/\sigma}, 0 \right\} x_t \) \tag{10}

\( n_t(\theta_{t-1}) = \frac{1}{\lambda_t} [x_t - r_t a_t(\theta_{t-1})] , \) \tag{11}

where the cutoff \( \theta^*_t \) is independent of individual history and determined by the Euler equation

\[
\frac{1}{\lambda_t} = \beta \frac{r_{t+1}}{w_{t+1}} L(\theta^*_t), \tag{12}
\]

and the function \( L(\theta^*_t) \geq 1 \) captures the (gross) liquidity premium of savings and is given by

\[
L(\theta^*_t) \equiv \int_{\theta \leq \theta^*_t} dF(\theta) + \int_{\theta > \theta^*_t} \frac{\theta}{\theta^*_t} dF(\theta). \tag{13}
\]

Proof. See Appendix A.1. \qed

Notice that the individual consumption function in equation (9) is reminiscent of that derived by Deaton (1991) under a numerical method, and the saving function in equation (10) exhibits a buffer-stock behavior: When the urge to consume is low (\( \theta_t < \theta^*_t \)), the individual opts to consume only a \( \frac{\theta_t}{\theta^*_t} < 1 \) fraction of total income and save the rest, anticipating that future consumption demand may be high. On the other hand, when the urge to consume is high (\( \theta_t \geq \theta^*_t \)), the agent opts to consume all gross income, up to the limit where the borrowing constraint binds, so the saving stock is reduced to zero. The function \( L(\theta^*_t) - 1 \geq 0 \) reflects the extra rate of return to savings due to the option value (liquidity premium) of the buffer stock.

Denote \( \Lambda_t \equiv \frac{1}{\lambda_t} \) as the expected marginal utility of income. Then the left-hand side of equation (12) is the average marginal cost of consumption in the current period, and the right-hand side is the discounted expected next-period return to savings (augmented by \( r_{t+1} \)), which takes two possible values in light of the two components for the liquidity premium in equation (13): The first is simply the discounted next-period marginal utility of consumption \( \Lambda_{t+1} \) in the case where the borrowing constraint does not bind, which has probability \( \int_{\theta \leq \theta^*_t} dF(\theta) \). The second is the discounted marginal utility of consumption \( \Lambda_{t+1} \frac{\theta_r}{\theta^*_t} \) in the case of high demand (\( \theta_t > \theta^*_t \)) with a binding borrowing constraint, which has probability \( \int_{\theta > \theta^*_t} dF(\theta) \). When the borrowing constraint binds, additional savings can yield a higher shadow marginal utility \( \frac{\theta_r}{\theta^*_t} \Lambda_{t+1} > \Lambda_t \). The optimal cutoff \( \theta^*_t \) is then determined at the point where the marginal cost of saving equals the expected marginal gains. Here, savings play the role of a buffer stock and the rate of return to savings is determined by the
real interest rate \( r_t \) compounded by a liquidity premium \( L(\theta^*_t) \). Notice that \( \frac{\partial L(\theta^*_t)}{\partial \theta^*_t} < 0 \) and \( L(\theta^*_t) > 1 \) for any \( \theta^*_t \neq \theta_H \).

Equation (12) also suggests that the cutoff \( \theta^*_t \) is independent of individual history. This property holds in this model because of the quasi-linear utility function and the assumption that the labor supply is predetermined in the first subperiod. In other words, the optimal level of liquidity on hand in period \( t \) is determined by a “target” income level given by 
\[
x_t = \left[ \theta^*_t \bar{w}_t L(\theta^*_t) \right]^{1/\sigma},
\]
which is also independent of the history of realized values of \( \theta_t \) but depends only on the cutoff \( \theta^*_t \). This target is essentially the optimal consumption level when the borrowing constraint binds. This target policy (uniform to all households) emerges because labor income \((\bar{w}_t n_t(\theta_{t-1}))\) can be adjusted elastically to meet the optimal target, given (and regardless of) the initial asset holdings \( a_t(\theta_{t-1}) \). Hence, in the beginning of each period, all households will choose the same level of gross income \( x_t \). Thus, the individual-history-independent cutoff variable \( \theta^*_t \) uniquely and fully characterizes the distributions of household decisions in the economy.

Since cash on hand \( x_t \) is independent of the idiosyncratic shock \( \theta_t \) and thus identical across households, capital taxation plays no redistributional role. Similarly, a lump-sum tax/transfer has no redistributional effect on households’ cash on hand either because any redistributional wealth effects of a lump-sum tax/transfer can be always offset by the perfectly elastic labor supply such that the target level of cash on hand \( x_t \) is degenerated and remains the same across households. This is why we do not need to consider lump-sum taxes/transfers in the model.

### 2.3 Competitive Equilibrium

Denote \( C_t, N_t, \) and \( K_{t+1} \) as the level of aggregate consumption, aggregate labor, and aggregate capital, respectively. A competitive equilibrium allocation can be defined as follows:

**Definition 1.** Given initial aggregate capital \( K_1 \) and bonds \( B_1 \), a sequence of taxes, and government spending and government bonds, \( \{\tau_{n,t}, \tau_{k,t}, G_t, B_{t+1}\}_{t=1}^\infty \), a competitive equilibrium is a sequence of prices \( \{w_t, q_t\}_{t=1}^\infty \), allocations \( \{c_t(\theta^t), n_t(\theta^{t-1}), a_{t+1}(\theta^t), K_{t+1}, N_t\}_{t=1}^\infty \), and the cutoff \( \{\theta^*_t\}_{t=1}^\infty \) such that

1. given the sequence \( \{w_t, q_t, \tau_{n,t}, \tau_{k,t}\}_{t=1}^\infty \), the sequences \( \{c_t(\theta^t), a_{t+1}(\theta^t), n_t(\theta^{t-1})\}_{t=1}^\infty \) solve the household problem;

2. given the sequence of \( \{w_t, q_t\}_{t=1}^\infty \), the sequences \( \{N_t, K_t\}_{t=1}^\infty \) solve the firm’s problem;
3. the no-arbitrage condition holds for each period: $r_t = 1 + (1 - \tau_{k,t})q_t - \delta$ for all $t \geq 1$;
4. the government budget constraint in equation (4) holds for each period; and
5. all markets clear for all $t \geq 1$:

$$K_{t+1} = \int a_{t+1}(\theta_t) dF(\theta_t) - B_{t+1} \quad (14)$$

$$N_t = \int n_t(\theta_{t-1}) dF(\theta_{t-1}) \quad (15)$$

$$\int c_t(\theta_t) dF(\theta_t) + G_t \leq F(K_t, N_t) + (1 - \delta)K_t - K_{t+1}.$$ (16)

**Proposition 2.** If the upper bound $\theta_H$ of the preference shocks is sufficiently large relative to the moment $\left[ \mathbb{E} \left( \theta^{1/\sigma} \right) \right]^\sigma$ such that the following condition holds:

$$\frac{\alpha \beta (\theta_H)^{1/\sigma}}{(\theta_H)^{1/\sigma} - \mathbb{E}(\theta^{1/\sigma})} + \beta(1 - \alpha)(1 - \delta) < 1,$$ (17)

then, in a laissez-faire competitive equilibrium, the steady-state risk-free rate is lower than the time discount rate, $r < 1/\beta$; and there exists overaccumulation of capital with a positive liquidity premium, $L(\theta^*) > 1$.

**Proof.** See Appendix A.2. □

Notice that when $\theta_H \to \infty$, as in the case of a Pareto distribution, the above condition is clearly satisfied. The intuition of Proposition 2 is straightforward. Since labor income is determined *(ex ante)* before the realization of the idiosyncratic preference shock $\theta_t$, a household’s total income may be insufficient to provide full insurance for large enough preference shocks under condition (17). In this case, precautionary saving motives lead to overaccumulation of capital, which reduces the equilibrium interest rate below the time discount rate. This outcome is clearly inefficient from a social point of view. It emerges because of the negative externalities of household savings on the aggregate interest rate (due to diminishing marginal product of capital), as noted by Aiyagari (1994).

However, unlike the Aiyagari (1994) model, a laissez-faire competitive equilibrium can achieve both AAE and IAE in our model if the idiosyncratic risk is sufficiently small (e.g., the upper bound $\theta_H$ is close enough to the mean $\mathbb{E}(\theta)$ such that condition (17) is violated). In this case, individual savings can become sufficiently large to fully buffer preference shocks.
and, as a result, household borrowing constraints will never bind. Clearly, with full self-insurance, it must be true that the optimal cutoff is at the upper corner, $\theta^* = \theta_H$; the liquidity premium vanishes, $L(\theta^*) = 1$; and the interest rate equals the time discount factor, $r = 1/\beta$.

A competitive equilibrium with full self-insurance is impossible in a typical HAIM model (such as in Aiyagari (1994, 1995)) because every household’s marginal utility of consumption follows a supermartingale when $r = 1/\beta$. This implies that household consumption and savings (or asset demand) diverge to infinity in the long run, which cannot constitute an equilibrium.\footnote{Please refer to Ljungqvist and Sargent (2012, Chapter 17) for details.}

In our model, however, because the household utility function is quasi-linear, the expected shadow price of consumption goods is thus the same across agents and given by $\frac{1}{\bar{w}_t}$ (as revealed by equations (43) and (45) in the proof of Proposition 1 (Appendix A.1)), which kills the supermartingale property of the household marginal utility of consumption. As a result, household savings (or asset demand) are bounded away from infinity even at the point $r = 1/\beta$. More specifically, equations (8) and (10) show that an individual’s asset demand is always bounded above by $(\theta_H - \theta_t) \bar{w}_t$ for any distribution $F$ of the shock $\theta_t \in [\theta_L, \theta_H]$ when $r = 1/\beta$. This upper bound $(\theta_H - \theta_t) \bar{w}_t$ is finite as long as the support $[\theta_L, \theta_H]$ of $\theta_t$ is bounded (a counter example is a Pareto distribution where $\theta_H = \infty$). This special property renders our model analytically tractable with closed-form solutions (provided that $\theta_t$ is iid), and it implies that the Ramsey planner has the potential to use government debt to achieve GRA in this economy when the competitive equilibrium is not IAE.

Nonetheless, the trade-off between AAE and IAE emphasized in this paper does not hinge on the special properties of our model. It is driven by the impact of distortional capital taxation upon precautionary saving motives—because a capital tax discourages households from savings, it mitigates the overaccumulation of capital but at the same time tightens individuals’ borrowing constraints (thus impeding the individual self-insurance position). Hence, such a trade-off channel should also exist in more general HAIM models. The only critical difference is that in a more general HAIM model a capital tax has redistributational effects but such effects are completely absent in our special model.
2.4 Conditions to Support a Competitive Equilibrium

Given that government policies are inside the aggregate state space of the competitive equilibrium and affect the endogenous distributions (including the average) of all endogenous economic variables, the Ramsey problem is to pick a competitive equilibrium (through policies) that attains the maximum of the expected household lifetime utility $V$ defined in (3). Since $V$ depends on the endogenous distributions (see below and Wen (2015)), the Ramsey planner needs to also pick a particular time path (sequence) of distributions to achieve the maximum.

This subsection expresses the necessary conditions, in terms of the aggregate variables and distributions characterized by the cutoff $\theta^*_t$, that the Ramsey planner must respect in order to construct a competitive equilibrium. We first show that since the cutoff $\theta^*_t$ is a sufficient statistic for describing the distributions of individual variables, all allocations and prices in the competitive equilibrium can be expressed as functions of the aggregate variables and the cutoff $\theta^*_t$. Hence, we also call the $\theta^*_t$ as the distribution statistic.

To facilitate the analysis, we first show the properties of aggregate consumption (or average consumption across households) by aggregating the individual consumption decision rules (9). By the law of large numbers, the aggregate consumption is determined by

$$C_t = D(\theta^*_t)x_t,$$

where the aggregate marginal propensity to consume (the function $D$) is given by

$$D(\theta^*_t) \equiv \int_{\theta \leq \theta^*_t} \left( \frac{\theta}{\theta^*_t} \right)^{1/\sigma} dF(\theta) + \int_{\theta > \theta^*_t} dF(\theta) \in (0, 1].$$

Then, we can express individual consumption and individual asset holding as functions of $C_t$ and $\theta^*_t$ by plugging equation (18) into equations (9) and (10). To fully describe the conditions necessary for constructing a competitive equilibrium, we rely on the following proposition:

**Proposition 3.** Given initial capital $K_1$, initial government bonds $B_1$, and the initial capital tax $\tau_{k,1}$, the sequences of aggregate allocations $\{C_t, N_t, K_{t+1}, B_{t+1}\}_{t=1}^\infty$ and distribution statistics $\{\theta^*_t\}_{t=1}^\infty$ can be supported as a competitive equilibrium if and only if the resource constraint (16), the asset market clearing condition

$$B_{t+1} = \left( \frac{1}{D(\theta^*_t)} - 1 \right) C_t - K_{t+1}, \text{ for all } t \geq 1,$$

for all $t \geq 1$.
and the following implementability conditions (for periods \( t = 1 \) and \( t \geq 2 \), respectively) are satisfied:

\[
C_1^{1-\sigma} D(\theta_1^*)^{\sigma-1} L(\theta_1^*) \theta_1^* \geq N_1 + r_1 C_1^{1-\sigma} D(\theta_1^*)^{\sigma} L(\theta_1^*) \theta_1^* (K_1 + B_1) \tag{21}
\]

and

\[
C_t^{1-\sigma} D(\theta_t^*)^{\sigma-1} L(\theta_t^*) \theta_t^* \geq N_t + \frac{1}{\beta} C_{t-1}^{1-\sigma} D(\theta_{t-1}^*)^{\sigma-1} \theta_{t-1}^* (1 - D(\theta_{t-1}^*)) , \tag{22}
\]

where \( r_1 = 1 + (1 - \tau_{k,1}) \frac{\partial F(K_1, N_1)}{\partial K_1} - \delta \).

**Proof.** See Appendix A.3

Note that the implementability conditions essentially enforce the flow government budget constraint and are comparable to those in the representative agent framework.

To derive the implementability conditions (21) and (22), we first replace the intertemporal prices and taxes with quantitative variables. As shown in Appendix A.3, the flow government budget constraint in a competitive equilibrium can be expressed as

\[
U_{C,t} D(\theta_t^*)^{\sigma} L(\theta_t^*) \theta_t^* C_t - N_t + U_{C,t} D(\theta_t^*)^{\sigma} L(\theta_t^*) \theta_t^* A_{t+1} \geq \frac{1}{\beta} U_{C,t-1} D(\theta_{t-1}^*)^{\sigma} \theta_{t-1}^* A_t,
\]

where \( U_{C,t} \) is defined as \( C_t^{-\sigma} \), the “marginal utility” of aggregate consumption in our setup. The above expression is analogous to that in a representative agent model, except with two additional terms: \( D(\theta_t^*)^{\sigma} \theta_t^* \) and \( L(\theta_t^*) \). These extra terms originate from the modification of the risk free rate, \( r_{t+1} \), which can be expressed as (see Appendix A.3):

\[
\frac{1}{r_{t+1}} = \beta \frac{U_{C,t+1} D(\theta_{t+1}^*)^{\sigma} \theta_{t+1}^*}{U_{C,t} D(\theta_t^*)^{\sigma} \theta_t^*} L(\theta_t^*). \tag{24}
\]

The function \( U_{C,t} \) captures the marginal utility of aggregate consumption; the function \( D(\theta_t^*)^{\sigma} \theta_t^* \) captures the marginal utility of the distribution of individual consumption; and the function \( L(\theta_{t+1}^*) \) captures the liquidity premium of bonds. In representative agent models, the last two terms are absent since there is no consumption distribution or liquidity premium to affect the risk free rate of bonds. Thus, as shown in the proof of Proposition 3, equation (23) together with the asset market clearing conditions imply the implementability conditions (21) and (22).

This proposition demonstrates that the Ramsey planner can construct a competitive equilibrium by simply choosing the sequences of aggregate allocations \( \{C_t, N_t, K_{t+1}, B_{t+1}\} \) and
distribution statistics \( \{ \theta^*_t \} \) to maximize expected welfare, subject to the aggregate resource constraint, asset market clearing condition, and the implementability condition.

Notice that in an interior steady state with \( \theta^* < \theta_H \), equations (24) and (5) imply

\[
1 = \beta \left[ 1 - \delta + (1 - \tau_k) \bar{q} \right] L(\theta^*),
\]

whereas the MGR is characterized by

\[
1 = \beta \left[ 1 - \delta + \tilde{q} \right];
\]

where \( q \) is private MPK and \( \tilde{q} \) denotes socially optimal MPK. Hence, even in the absence of any production externalities there exists a wedge between the socially optimal MPK and the laissez-faire competitive equilibrium MPK (where \( \tau_k = 0 \) and \( B = 0 \)—due to the positive liquidity premium: \( L(\theta^*) > 1 \)). Suppose a Ramsey steady state exists such that the MGR holds; then choosing \( \tau_k > 0 \) (for a given level of government bonds) such that

\[
[1 - \delta + (1 - \tau_k) q] L(\theta^*) = [1 - \delta + \bar{q}]
\]

is clearly a feasible policy. However, whether such a positive capital-tax policy is optimal depends also on the Ramsey planner’s first-order conditions for government debt and labor-income tax. For example, suppose the optimal level of government debt is such that \( L(\theta^*) = 1 \), then the implied optimal capital tax is zero \( (\tau_k = 0) \) instead of positive. In addition, if there exists a government borrowing limit such that the debt limit binds in the steady state, then equation (26) must be modified accordingly to include a Lagrangian multiplier for the government borrowing constraint; thus, the implications for optimal capital tax becomes less clear cut.

In what follows, we will solve the Ramsey allocation analytically to jointly determine the optimal level of government debt, optimal capital tax, and optimal labor tax. We also consider the effects of an ad hoc government borrowing limit on the Ramsey allocation when the debt limit binds in the Ramsey steady state.

## 3 Ramsey Allocations

Armed with Proposition 3, we are ready to write down the Ramsey planner’s problem and derive the first-order Ramsey conditions analytically.
3.1 Ramsey Problem

Using equations (9) and (18), the lifetime utility function $V$ can be rewritten as a function of the distribution statistic $\theta^*_t$ and aggregate variables:

$$V = \sum_{t=1}^{\infty} \beta^{t-1} \left[ W(\theta^*_t) \frac{C^1_{t-\sigma}}{1-\sigma} - \frac{\overline{\theta}}{1-\sigma} - N_t \right],$$

(27)

where $W(\theta^*_t)$ is defined as

$$W(\theta^*_t) \equiv \left( \int_{\theta \leq \theta^*_t} \theta \left( \frac{\theta}{\theta^*_t} \right)^{\frac{1-\sigma}{\sigma}} dF(\theta) + \int_{\theta > \theta^*_t} \theta dF(\theta) \right) D(\theta^*_t)^{\sigma-1}$$

(28)

Thus, the Ramsey problem can be represented alternatively as maximizing the welfare function (27) by choosing the sequences of $\{\theta^*_t, N_t, C_t, K_{t+1}, B_{t+1}\}$, subject to the resource constraint (16), the asset market clearing condition (20), and the implementability conditions (22) and (21).

In addition, an exogenous debt limit $B_{t+1} \leq \overline{B}$ is imposed on the Ramsey planner to facilitate our analysis on the role of government debt, which most of the existing literature has ignored or assumed away by implicitly setting $\overline{B} = \infty$—which is larger than any debt limit of the government.

Therefore, the Lagrangian of the Ramsey problem is given by

$$L = \max_{\{\theta^*_t, N_t, C_t, K_{t+1}, B_{t+1}\}} \sum_{t=1}^{\infty} \beta^{t-1} \left[ W(\theta^*_t) \frac{C^1_{t-\sigma}}{1-\sigma} - \frac{\overline{\theta}}{1-\sigma} - N_t \right]$$

(29)

$$+ \sum_{t=1}^{\infty} \beta^{t-1} \mu_t \left( F(K_{t}, N_t) + (1 - \delta)K_t - G_t - C_t - K_{t+1} \right)$$

$$+ \lambda_1 \left( C^1_{t-\sigma} D(\theta^*_t)^{\sigma-1} L(\theta^*_t) \theta^*_t - N_t - C^1_{t-\sigma} D(\theta^*_t)^{\sigma} L(\theta^*_t) \theta^*_t r_1(K_1 + B_1) \right)$$

$$+ \sum_{t=2}^{\infty} \beta^{t-1} \lambda_t \left( C^1_{t-\sigma} D(\theta^*_t)^{\sigma-1} L(\theta^*_t) \theta^*_t - N_t - \beta^{-1} C^1_{t-1-\sigma} D(\theta^*_t)^{\sigma-1} \theta^*_t (1 - D(\theta^*_t)) \right)$$

$$+ \sum_{t=1}^{\infty} \beta^{t-1} \phi_t \left( K_{t+1} + B_{t+1} - (D(\theta^*_t))^{-1} - 1 \right) C_t$$

$$+ \sum_{t=1}^{\infty} \beta^{t-1} \nu_t \left( \overline{B} - B_{t+1} \right),$$

where $\mu_t, \lambda_t, \text{and} \phi_t$ denote the multipliers for the resource constraints, the implementability conditions, and the asset market clearing conditions, respectively. In addition, the multiplier
of the debt limit constraint is denoted by \( \nu_t^B \). The \( K_1, B_1, \) and \( \tau_{k,1} \) are given, and hence \( r_1 \) depends only on \( N_1 \).

The first-order Ramsey conditions with respect to \( K_{t+1}, N_t, C_t, B_{t+1}, \) and \( \theta_t^* \) are given, respectively, by

\[
\mu_t - \phi_t = \beta \mu_{t+1} (MP_{K,t+1} + 1 - \delta) \tag{30}
\]

\[
1 + \lambda_t = \mu_t MP_{N,t} \text{ for } t \geq 2 \tag{31}
\]

\[
\mu_t = W(\theta_t^*)C_t^{1-\sigma} + (1 - \sigma)C_t^{1-\sigma}D(\theta_t^*)^{\sigma-1}\theta_t^* (\lambda_t L(\theta_t^*) - \lambda_{t+1} (1 - D(\theta_t^*))) - \phi_t (D(\theta_t^*)^{-1} - 1) \text{ for } t \geq 2 \tag{32}
\]

\[
\beta^t \phi_t - \beta^t \nu_t^B = 0 \tag{33}
\]

\[
\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} C_t^{1-\sigma} + \lambda_t C_t^{1-\sigma}H(\theta_t^*) - \lambda_{t+1}C_t^{1-\sigma}J(\theta_t^*) + \phi_t \frac{C_t}{D(\theta_t^*)^2} \frac{\partial D(\theta_t^*)}{\partial \theta_t^*} = 0 \text{ for } t \geq 2, \tag{34}
\]

where

\[
H(\theta_t^*) \equiv \frac{\partial (D(\theta_t^*)^{\sigma-1}L(\theta_t^*)\theta_t^*)}{\partial \theta_t^*}
\]

\[
J(\theta_t^*) \equiv \frac{\partial (D(\theta_t^*)^{\sigma-1}\theta_t^*(1 - D(\theta_t^*)))}{\partial \theta_t^*}
\]

To conserve space, the first-order Ramsey conditions with respect to \( N_1, C_1, \) and \( \theta_1^* \) in the initial period, as well as several useful lemmas for the upcoming proofs, are relegated in the Appendix A.4.

Note that the Lagrangian multiplier \( \phi_t \) for the asset market clearing condition and the multiplier \( \nu_t^B \) for the government debt-limit constraint are equal to each other according to equation (33), suggesting that the tightness of the asset market depends on the government debt limit.

### 3.2 Characterization of Ramsey Allocations

**Definition 2.** A Ramsey steady state is defined as a long-run Ramsey allocation where (i) the parameter restriction \( \theta_H < \frac{\theta}{(1-\beta)^{\sigma}} < \infty \) (to ensure positive labor \( n > 0 \) for all individuals in all states) is satisfied, and (ii) all aggregate variables \( \{K, N, C, B, \theta^*\} \) and the associated Lagrangian multipliers \( \{\lambda, \mu\} \) converge to finite and strictly positive values.
The condition
\[ \theta_H < \frac{\theta_L}{(1 - \beta)\sigma} \]  
(35)
is required to ensure that all individuals’ labor decisions are positive, a necessary condition for Proposition 1. The intuition is that if the variance (support) of \( \theta \) is too large (spread out), some agents may end up with too much savings in the end of last period and thus opt not to work this period. Our model becomes intractable in the situation with possible binding zero labor supply, so it must be ruled out. In addition, any variable without subscript \( t \) is referred as its steady-state value.

**Proposition 4.** If a Ramsey steady state exists, then the optimal capital tax rate in such a Ramsey steady state is determined by the following equation:

\[ 1 - \tau_k = \frac{L(\theta^*)^{-1} - \beta (1 - \delta)}{\mu - \phi} - \beta (1 - \delta), \]  
(36)

where \( \mu \geq 0 \) and \( \phi \geq 0 \) are the multipliers of Ramsey Lagrangian (29) for the aggregate resource constraint and the debt limit constraint (implied by equation (33)), respectively.

**Proof.** See Appendix A.5

Proposition 4 immediately gives the following steady-state tax rate for capital,

\[ \tau_k \geq 0 \] if and only if \( L(\theta^*) \geq \frac{\mu}{\mu - \phi} \),

(37)

which implies the following two points:

First, if the government debt-limit constraint does not bind in the Ramsey steady state—i.e., \( \phi = 0 \) and \( \frac{\mu - \phi}{\mu} = 1 \)—then, since the liquidity premium \( L(\theta^*) \geq 1 \), the right-hand side of equation (36) must be smaller than or equal to 1. Therefore, optimal capital tax must be non-negative: \( \tau_k \geq 0 \). Hence, subsidizing capital in this case is never optimal. In addition, if in this case the optimal cutoff is a corner solution at \( \theta^* = \theta_H \), then \( L = 1 \), and it must be true that \( \tau_k = 0 \).

Second, if the government debt-limit constraint is binding in the steady state, i.e., \( \phi > 0 \), then the left-hand side of equation (37) must be greater than 1. In this case, if the optimal cutoff is a corner solution at \( \theta^* = \theta_H \), then \( L = 1 \), so it must be true that \( \tau_k < 0 \). On the other hand, if in this case the optimal capital tax \( \tau_k = 0 \), then it must be true that \( \theta^* < \theta_H \) and \( L(\theta^*) > 1 \); namely, IAE is not achieved.
So in what follows, we prove the existence of two types of the Ramsey steady state and characterize their respective properties. Recall that the conditions (17) and (35) are assumed to hold throughout the paper; so in each of the two possible cases the competitive equilibrium without government intervention is inefficient by design. The interesting question is how (and by how much) can the government improve upon the allocations of laissez-faire competitive equilibrium.

3.2.1 Case 1: Ramsey (GRA) Steady State

The first case characterizes the GRA allocation with full self-insurance, which is defined formally as follows:

**Definition 3.** GRA is defined as a Ramsey steady state where both AAE and IAE are achieved; namely, in GRA (i) the socially optimal MPK equals the (after-tax) liquidity-premium-adjusted private MPK and (ii) the borrowing constraints \( a_{t+1} \geq 0 \) do not bind for all households in all states.

**Proposition 5.** Suppose \( \overline{B} \) is sufficiently large such that the constraint \( B_{t+1} \leq \overline{B} \) never binds. Then, under suitable parameter values there exists a unique Ramsey steady state and this steady state is an GRA with the following properties:

1. IAE is achieved—the optimal choice of \( \theta^* \) is a corner solution at \( \theta^* = \theta_H \) so that there is no liquidity premium \( L(\theta^*) = 1 \) and no households are borrowing constrained.

2. AAE is achieved—the socially optimal MPK equals the (after-tax) liquidity-premium-adjusted private MPK.

3. Consequently, the equilibrium interest rate equals \( 1/\beta \), the capital tax is zero, \( \tau_k = 0 \), and the labor tax is positive at the rate \( \tau_n = \frac{1}{1+\lambda}\sigma \). Government expenditures and bond interest payments are financed solely by revenues from the labor income tax.

**Proof.** See Appendix A.6

This proposition states that if the debt-limit constraint \( B \leq \overline{B} \) does not bind in the Ramsey steady state, then the Ramsey plan picks a long-run competitive equilibrium that achieves both AAE and IAE.

This proposition indicates that in the absence of any redistributional effects the Ramsey planner achieves the MGR without the need to tax capital in the steady state—as capital
taxation in the steady state would undermine IAE by decreasing the steady-state household saving rate and thus permanently hampering their self-insurance positions. Instead, the Ramsey planner opts to provide enough incentives for households to save through bonds by picking a sufficiently high interest rate \((= 1/\beta)\) on bonds, such that all households are fully self-insured in the long run with zero probability of encountering binding liquidity constraints.

It is this critical role of government debt in improving the individual self-insurance position that determines the optimal debt level in the model. This can be seen more clearly by inspecting the optimal debt-to-GDP ratio in a GRA in the special case where the EIS parameter \(\sigma = 1\), the rate of capital depreciation \(\delta = 1\), and government spending \(G = 0\).

**Corollary 1.** When \(\sigma = \delta = 1\) and \(G = 0\), the optimal debt-to-GDP ratio is determined by a wedge \(\tau_b\) times the MGR-saving rate \(\beta\alpha\):

\[
\frac{B}{Y} = \tau_b \beta\alpha,
\]

where the wedge

\[
\tau_b = \left(\frac{1 - D(\theta^*)}{D(\theta^*)}\right) \sqrt{\frac{\beta\alpha}{1 - \beta\alpha} - 1} \geq 0
\]

is essentially the gap between the competitive equilibrium saving ratio \((1 - D)/D\) and the conventional modified-golden-rule saving ratio \((\beta\alpha/(1 - \beta\alpha))\). This gap vanishes if and only if the competitive equilibrium under incomplete markets approaches the allocation of an economy with full self-insurance (or with complete markets).

**Proof.** See Appendix A.7

Recall that \(D(\theta^*)\) denotes the aggregate marginal propensity to consume and that \((1 - D(\theta^*))\) denotes the aggregate marginal propensity to save in a competitive equilibrium with incomplete markets. It can be shown easily that under complete markets (e.g., in a representative-agent model with \(\delta = 1\) and log utility), the optimal saving rate is \(\beta\alpha\). Hence, precautionary saving behavior implies that the saving rate under incomplete markets exceeds the saving rate under complete markets, i.e., \((1 - D) > \beta\alpha\) and \(D < (1 - \beta\alpha)\). Hence, the wedge is strictly positive: \(\tau_b > 0\). However, as the variance of \(\theta\) approaches zero, or as the insurance markets become complete, it must be true that \((1 - D) \to \beta\alpha\) and \(\tau_b \to 0\), regardless of the MGR saving rate \(\beta\alpha\).

Therefore, the wedge \(\tau_b\) is a measure of the degree of the individual allocative inefficiency in the competitive equilibrium. So Proposition 1 shows (again) that the single most impor-
tant role of government debt is to improve the individual self-insurance position, such that the optimal level of bond supply is proportional to the conventional MGR-saving rate by a factor that is determined solely by the wedge of inefficiency caused by incomplete insurance markets in a competitive equilibrium.

Obviously, GRA can be achieved only if the Ramsey planner is capable of supplying enough bonds to satisfy the liquidity demand of each household across all states. An important property of the GRA is that the equilibrium interest rate equals the time discount rate: \( r = 1/\beta \).

However, the zero-capital-tax policy does not necessarily depend on this property. To shed light on this issue further, we study what happens if the government’s ability to issue bonds is limited and the debt-limit constraint is binding at least in the long run.

### 3.2.2 Case 2: Ramsey Steady State with a Binding Debt Limit

To illustrate our point without loss of tractability, we impose an exogenous debt limit \( \bar{B} \) that lies strictly below the optimal debt level \( B^* \) determined in GRA.

**Proposition 6.** Suppose that the debt-limit constraint binds after a sufficiently large \( t > 1: B_{t+1} = \bar{B} \). Then, under suitable parameter values there exists a Ramsey steady state with the following properties:

1. **IAE fails to hold**—the optimal choice of the cutoff under Ramsey is interior, \( \theta^* \in (\theta_L, \theta_H) \); so there is always a non-zero fraction \( 1 - F(\theta^*) > 0 \) of households facing binding borrowing constraints in every period and there is a positive liquidity premium \( L(\theta^*) > 1 \) with equilibrium interest rate \( r < 1/\beta \).

2. **AAE fails to hold**—the socially optimal MPK does not equal the (after-tax) liquidity-premium-adjusted private MPK.

3. The capital tax is still zero, \( \tau_k = 0 \); namely, the Ramsey planner does not tax capital even if AAE fails to hold because of a binding debt-limit constraint. Notice that this is true even if \( \bar{B} = 0 \).

**Proof.** See Appendix A.8

Obviously, a special subcase of case 2 is when the government cannot issue bonds at all: \( \bar{B} = 0 \). This subcase is analogous to the situation discussed in Proposition 2 in the previous section, where the competitive-equilibrium interest rate is strictly less than the time discount rate.
rate. In such a situation, the Ramsey planner cannot use bonds to manipulate the market interest rate and divert household savings away from capital formation. In general, whenever the government is unable to supply enough bonds to meet household demand for full self-insurance, either because $\theta_H$ is sufficiently large or the debt limit $\overline{B}$ is sufficiently low, the pursuit of IAE by the Ramsey planner will necessarily lead to a binding debt-limit constraint on government bonds: $B = \overline{B}$. In this case, there exists a Ramsey steady state that features neither IAE nor AAE—albeit it is feasible for the planner to achieve the MGR (but not IAE) by taxing capital.

As shown in the proofs for Proposition 5 and Proposition 6, the zero capital tax is obtained in our model because of the strikingly simple and unique steady-state relationship:

$$L(\theta^*) = \frac{\mu}{\mu - \phi},$$

which by equation (36) implies that $\tau_k = 0$, regardless of the debt limit $\overline{B}$ and the other model parameter values (such as the elasticity of intertemporal substitution $\sigma$, the time discount factor $\beta$, the output elasticity of capital $\alpha$, and the rate of capital depreciation $\delta$, among others).

This simple analytical relationship (40) is very striking and surprising. The Ramsey planner opts to supply enough bonds to improve the individual self-insurance position until the debt limit $\overline{B}$ binds. But, regardless of the tightness of the binding debt limit ($\phi$), the Ramsey planner nonetheless opts to equalize the ratio of Lagrangian multipliers $\frac{\mu}{\mu - \phi}$ to the liquidity premium $L$, such that the steady-state capital tax is exactly zero. This result offers a strong case to support the view that steady-state capital tax is extremely distortionary and hence should not be used as a tool to achieve the MGR in the long run (provided that labor-income taxation is feasible).

Hence, the government borrowing limit does not matter for the optimal steady-state capital tax but does matter for the Ramsey planner’s ability to achieve the AAE and IAE. Yet such a critical role of government-debt limits is often ignored or has gone unnoticed in the existing literature.

The fundamental reason is that while capital taxation is effective in eliminating any wedge between (after-tax) liquidity-premium-adjusted private MPK and socially optimal MPK, but it is not an effective tool to eliminate the liquidity premium itself—because a positive capital tax enlarges the liquidity premium albeit lowering the (after-tax) private MPK.\(^8\)

\(^8\)Since we did not prove that the Ramsey steady state is unique in Proposition 6, we have not ruled out other types of Ramsey steady state where taxing or subsidizing capital may be optimal when the government
However, we will show in the next section that the MGR is not irrelevant to the Ramsey planner. Along a transition path toward the steady state, the Ramsey planner will tax capital to reduce the steady-state capital stock, albeit not to the degree of fully restoring the MGR. But before showing that, we consider two interesting situations where a Ramsey steady state with a non-binding debt limit may not exist.

### 3.2.3 Non-existence of Ramsey Steady State

As argued by Aiyagari (1994, 1995), the long-run equilibrium interest rate in a standard HAIM model must be lower than $1/\beta$; otherwise, individual asset demand goes to infinity—which cannot be a competitive equilibrium. This property led to Aiyagari (1995) arguing that the Ramsey plan should tax capital to achieve the MGR regardless of the individual allocative inefficiency (which is intensified by a positive capital tax). Aiyagari may be right in general, especially in models with redistributive effects of a capital tax. However, under special circumstances (in the absence of any redistributional effects) we show above that such an argument is incorrect. We will also show below by counterfactual analyses that the assumption of the existence of a Ramsey steady state (without a binding debt limit) can lead to erroneous results when such a steady state does not exist. We show this through conducting two counterfactual analyses:

(i) Suppose under Case 1 (with a non-binding debt limit) we assume *incorrectly* that there exists another Ramsey steady state where $\phi = 0$ and that the interest rate $r^* < \frac{1}{\beta}$, where $r^* \equiv 1 + q_t - \delta$. Since the interest rate lies below the time discount rate (as in the model of Aiyagari), according to equations (12) and (13), it must be true that the gross liquidity premium $L(\theta^*) > 1$ and the cutoff is interior, $\theta^* < \theta_H$; namely, individual borrowing constraints strictly bind for some households. Given that $\phi = 0$, equation (30) implies that the MGR holds. Given that the MGR holds and $L(\theta^*) > 1$, equation (12) implies that the Ramsey planner should tax capital so that the two equations are mutually consistent—because $(q + 1 - \delta) = L(\theta^*) [(1 - \tau_k) q + 1 - \delta]$ implies $\tau_k > 0$. In other words, the Ramsey planner should set $\tau_k > 0$ to achieve a competitive equilibrium consistent with the MGR and should ignore the individual allocative inefficiency, exactly as Aiyagari (1995) has argued. However, the assumption of the existence of such a Ramsey steady state with a non-binding debt limit and an interior cutoff $\theta^* < \theta_H$ is inconsistent with the other Ramsey first-order conditions in our model—because under such assumptions it can be shown (see the proof of Proposition 5) that the Lagrangian multiplier $\mu = 0$, which contradicts the definition of a
Ramsey steady state where the aggregate resource constraint strictly binds.

(ii) Let $\theta_H \to \infty$ (as in the case of a Pareto distribution) and assume again that there is no debt limit (i.e., $\overline{B} = \infty$). Note that in this case a competitive-equilibrium steady state without government intervention still exists. Also suppose (for the sake of argument) that the assumption of non-negative individual labor supply ($\theta_H < \theta_L/(1 - \beta)^{\sigma}$) can be relaxed so that labor supply can be negative for some individuals but remain positive at the aggregate level (—otherwise, a binding individual labor supply at zero would invalidate Proposition 1). Since by the arguments in Propositions 5 and 6, the Ramsey planner would always opt to keep increasing the interest rate by issuing enough government bonds to achieve IAE; consequently, the optimal bond demand diverges to infinity as $\theta_H$ approaches infinity. This point can be seen from the asset market clearing condition (20), which pins down the optimal level of aggregate bond demand $B_{t+1}$ as a function of the distribution parameter $\theta_H$; namely, under IAE, the debt-to-GDP ratio becomes

$$\frac{B}{Y} = \left( \frac{\theta_H^{1/\sigma}}{E(\theta^{1/\sigma})} - 1 \right) \left( 1 - \frac{\delta \beta \alpha}{1 - \beta (1 - \delta)} - \frac{G}{Y} \right) - \frac{\beta \alpha}{1 - \beta (1 - \delta)},$$

(41)

which implies that under finite values of aggregate consumption-to-output ratio $\left(1 - \frac{\delta \beta \alpha}{1 - \beta (1 - \delta)} - \frac{G}{Y}\right)$ and capital-to-output ratio $\frac{\beta \alpha}{1 - \beta (1 - \delta)}$, both the optimal debt-to-output ratio $\frac{B}{Y}$ and the Lagrangian multiplier $\mu$ approach infinity as $\theta_H$ goes to infinity. An infinite amount of government bonds clearly contradicts the existence of a Ramsey steady state featuring a non-binding debt limit. In addition, when $\mu_t \to \infty$ (because of an increasingly tight aggregate resource constraint under unlimited bond demand), the Ramsey first-order condition (30) cannot pin down the steady-state marginal product of capital or support the validity of the MGR. Therefore, assuming the existence of a Ramsey steady state in this case can lead to the erroneous result that the MGR holds and the optimal capital tax is positive.

Note that in both situations considered above, a Ramsey steady state would exist if the debt-limit constraint binds: $\lim_{t \to \infty} B_{t+1} = \overline{B}$ and $\lim_{t \to \infty} \phi_t = \phi > 0$. Then it can be shown that the Lagrangian multiplier $\mu \in (0, \infty)$ and that this value is consistent with all of the Ramsey first-order conditions. But in such a case, equation (30) suggests that the MGR fails to hold; even in this case, Proposition 6 shows that the steady-state capital tax $\tau_k = 0$.

In other words, when a Ramsey steady state featuring a non-binding debt limit does not exist but is erroneously assumed to exist in our model, we would conclude incorrectly that it is optimal to pursue the MGR by levying a positive tax on capital and ignore the individual
allocative inefficiency (or ignore the adverse effect of capital tax on household self-insurance position).

4 Numerical Analysis

This section performs numerical exercises not only to confirm our theoretical results but also to illustrate the optimal transition path of Ramsey allocation in comparison with laissez-faire equilibrium. Such numerical analyses are valid because the Ramsey steady state is proven to exist. Since our goal is not to simulate a realistic real-world economy, we do not intend to calibrate the model parameters to match real-world data. Instead, these numerical analyses are meant to demonstrate the transitional dynamics of the Ramsey allocation with/without a binding government debt limit and reveal the trade-off under capital taxation between AAE and IAE.

4.1 Parameter Values

The government spending $G_t$ is set to zero for all periods. The initial government bond and capital tax are also set to zero ($B_1 = 0$ and $\tau_{k,1} = 0$). The initial capital stock $K_1$ is set to be 70% of the Ramsey steady-state level. The production function is assumed to be Cobb-Douglas with capital share $\alpha = 1/3$, the time discount rate $\beta = 0.95$, and the capital depreciation rate $\delta = 0.75$. The distribution of preference shock $\theta$ follows a power function $F(\theta) = \frac{\theta - \theta_L}{\theta_H - \theta_L}$, where $\theta_L = 1$, $\theta_H = 10$ and $\gamma = 0.1$. The results are qualitatively similar for other choices of parameter values.

These parameter values imply that the following conditions are satisfied: (i) In the steady state the condition for positive individual labor choice, $\theta_H < \frac{\theta_L}{(1-\beta)^{\gamma}} < \infty$, is satisfied. (ii) In the transition the condition for positive labor supply $n_t > 0$ is satisfied and verified numerically in each time period. (iii) The condition (17) holds, such that the laissez-faire competitive equilibrium is neither IAE nor AAE.

Under these parameter values, the GRA is feasible for the Ramsey planner with a non-binding debt limit and corresponds to Case 1 described in Proposition 5.

The proof of Proposition 6 shows that the existence of a Ramsey steady state requires $\beta$ to be sufficiently large $\delta$ to be sufficiently small but without specifying feasible numerical ranges, we therefore choose a small value for $\beta$ and a large value for $\delta$ to demonstrate that the permissible intervals for their values are reasonably wide.
4.2 Ramsey Transition Paths

Consider the case of log utility ($\sigma = 1$) first. Figure 1 shows the transition paths of aggregate consumption $C_t$ (top left panel), aggregate labor $N_t$ (lower-left panel), the distribution statistic $\theta^*_t$ (top-right panel), and aggregate capital stock $K_t$ (lower-right panel). In each panel, blue lines represent the Ramsey economy, red lines represent the laissez-faire economy, a solid line represents the transition, and a dashed line represents the corresponding steady state. The results can be summarized as follows.

First, the Ramsey transition paths are significantly slower than their counterparts in the laissez-faire economy, especially the transition of the distribution statistic $\theta^*_t$. For example, consumption, labor, and capital stock take about 5 periods to nearly approach their respective steady states under laissez faire, as opposed to more than 50 periods under Ramsey. In particular, it takes only about 2 periods for the distribution statistic $\theta^*_t$ to nearly approach its steady state under laissez faire, as opposed to more than 50 periods under Ramsey (top-right panel). Recall that the Ramsey steady state is both IAE and AAE, hence the steady state of $\theta^*$ is $\theta_H$ ($= 10$) under Ramsey.

Second, the Ramsey allocation exhibits lower steady-state levels in aggregate labor and capital but a significantly higher cutoff value $\theta^*$, compared with those in the laissez-faire economy. This suggests that the Ramsey planner opts to induce the households to work less and invest less in capital to improve AAE. Interestingly, aggregate consumption is also lower in the Ramsey steady state than in the laissez-faire steady state (top-left panel). But this does not necessarily imply a lower welfare, because the distribution of consumption is significantly improved—a significantly higher cutoff $\theta^*$ implies a much lower probability of a binding borrowing constraint and hence a greatly improved individual self-insurance position (or IAE)—thanks to the availability of government bonds.

Third, however, in the initial several periods aggregate consumption is slightly higher in the Ramsey economy than in the laissez-faire economy and approaches the Ramsey steady state from above (top-left panel). This suggests that the Ramsey planner intends to front-load consumption during the transition path to increase welfare under time discounting, which the laissez-faire economy is unable to do because of strong precautionary saving motives. Unlike the competitive equilibrium, the Ramsey planner is able to front-load consumption by over-shooting aggregate consumption above its Ramsey steady state even in the intermediate run—because the planner can reduce the interest rate to a level below the time discount rate along its transition path in the intermediate run, as can be seen in Figure 2 where the interest rate over-shoots its Ramsey steady state from above and then converges.
Notes: Transition paths in the Ramsey economy (solid blue lines) and laissez-faire economy (solid red lines), as well as their corresponding steady states (dashed lines).

slowly back to the steady state from below (bottom-right panel).

In particular, Figure 2 shows that under the Ramsey planner, the debt-to-GDP ratio (bottom-left panel) increases rapidly from 0% to 40% in the short run to attract household savings and improve the individual self-insurance position, resulting in higher-than-steady-state short-run interest rates in the initial several periods (bottom-right panel). However, the interest rate subsequently falls below the time discount rate ($1/\beta$) and converges only slowly back to the Ramsey steady state.

The rapidly increasing amount of debt clearly requires financing from tax revenues. The government can finance the debt through either labor tax, capital tax, or both. Interestingly, Figure 2 shows that the Ramsey planner opts to put the pressure of revenue collection on capital tax in the short run and turn attention to labor tax in the longer run—such that capital tax is the highest initially (0.9% at $t = 2$) and gradually reduces to 0% in the long run (top-right panel); in the meanwhile, labor tax is low initially (even slightly negative) and gradually approaches 4.7% in the long run (top-left panel). This suggests that the source of
government revenues to finance public debt lies mainly in capital tax in the very short run but exclusively in labor tax in the long run.

The rapidly increasing government bonds and the positive capital tax rates in the transition periods significantly slowed down capital accumulation so that the steady-state capital is consistent with the MGR. In contrast, the capital stock under laissez faire is 20% above the MGR-capital stock (bottom-right panel in Figure 1). This suggests that, instead of taxing capital permanently as argued by Aiyagari (1995), the Ramsey planner opts to tax capital only in the short run to improve aggregate allocative efficiency. Consequently, we see the opposite transition paths of capital tax and labor tax in Figure 2.

Of course, it must be recognized that the key mechanism to enable the Ramsey planner to achieve the MGR in the long run is its ability to issue plenty of debt. As explained earlier, the supply of government bonds helps to improve the household self-insurance position and at the same time to reduce the overaccumulation of capital.

An important implication of this logic is that when the government cannot issue enough
bonds or simply cannot issue debt at all, the Ramsey planner shall reduce short-run capital taxation (or simply do not tax capital at all), even if the MGR fails to hold. In other words, in the absence of any redistributio nal effects, the MGR appears to have no bearing on optimal capital taxation and consequently, as the debt limit $\bar{B}$ reduces to zero, the Ramsey allocation approaches the laissez-faire allocation with zero capital tax in both the short run and the long run, as confirmed in the following subsection. These findings are in sharp contrast to the conventional wisdom embraced by the main literature.

4.3 Ramsey Transitions under Binding Debt Limits

To study the dynamic and long-run effects of a binding debt limit on Ramsey allocation, we compare three scenarios in Figure 3 and Figure 4: (i) the scenario without any debt limit (blue lines), which is identical to the case shown in Figure 1 (blue lines); (ii) the scenario with a binding debt limit $\bar{B}$ equal to 50% of the optimal debt level of GRA (denoted by $B^*$, green lines); and (iii) the scenario with a zero debt limit $\bar{B} = 0$ (red lines).

Figure 3 shows that as the debt limit $\bar{B}$ decreases step by step toward zero, the steady-state levels of aggregate consumption (top-left panel), labor (bottom-left panel), and capital stock (bottom-right panel) all increase and approach the corresponding laissez-faire level. Meanwhile, the steady-state cutoff decreases significantly toward the laissez-faire level (top-right panel), suggesting that the Ramsey planner becomes less and less capable of improving the individual self-insurance position when the government capacity to issue debt is reduced. Notice that as the debt limit decreases, the speed of transition also increases—because the Ramsey allocation behaves more and more like a laissez-faire competitive equilibrium.

These results are anticipated by Proposition 6, according to which (under the assumption of zero government spending) the Ramsey steady state in scenario (iii) coincides with the laissez-faire steady state where $G = B = \tau_k = \tau_n = 0$. But here we show that when the government cannot issue bonds at all (equivalent to $\bar{B} = 0$), the entire transition path is also identical to the laissez-faire case (red lines) as shown in Figure 1. Therefore, scenario (ii) with $\bar{B} = 50$ percent of the optimal debt level of GRA lies in between scenario (i) and scenario (iii).

Figure 4 shows the effects of debt limits on the transition paths (as well as the steady state) of policy variables. It offers explanations for the transition patterns of aggregate variables shown in Figure 3. First, as the debt limit reduces, the optimal debt-to-GDP ratio and its transition time needed to approach the steady state also decline (bottom-right
Notes: Scenario (i), (ii) and (iii) are shown as blue, green, and red lines, respectively. Their corresponding steady state levels are indicated by dashed lines.

Panel). Since a lower debt-to-GDP ratio implies that the government has a smaller burden of financing interest payments, the average tax rates for both capital and labor along the transition are reduced as well (top row panels). Interestingly, although in the initial transition period capital tax under scenario (ii) is higher than that under scenario (i), the average rate is lower because capital tax converges to the zero-steady state much faster under scenario (ii) than under scenario (i), suggesting a feature of non-linearity.

As anticipated, as the debt limit reduces, the equilibrium interest rate $r_t$ also declines both during transition and in the steady state (bottom-right panel). Keep in mind that the steady-state interest rate under scenario (i) equals the time discount rate $1/\beta = 1.0526$, so the steady-state interest rates under scenarios (ii) and (iii) are strictly lower than that under scenario (i).

The main lesson taken away from this subsection is that in the absence of any redistributional concerns, tax policies in an HAIM economy are shaped by the government’s ability to issue debt and that the single most important function of public debt is to improve the
individual self-insurance position. Since a binding debt limit handicaps the government’s ability to improve individual allocative efficiency, as a result, the associated burden of interest payments and the average tax rate during transition are also reduced. However, given any level of required tax revenues to finance interest payments, the composition of the tax revenue in terms of capital tax and labor tax is dictated by the trade-off between IAE and AAE under capital taxation; so optimal capital tax is high in the short run and zero in the long run, while optimal labor tax is low in the short run but high in the long run. Hence, as the debt limit $B$ approaches zero, the Ramsey allocation approaches the competitive equilibrium under laissez faire both along transition and in steady state. In other words, in the absence of government spending, steady-state tax policies have no independent role to play without the tool of government debt, and they are used solely to finance interest payments on government debt. For this very reason, since labor tax is less distortionary, a permanent capital tax in the steady state is never optimal regardless of government debt limits and the MGR (unless a capital tax can redistribute wealth from the rich to the poor, which is
ruled out in our model). To compromise, the planner opts to improve aggregate allocative efficiency (or reduce the problem of capital overaccumulation) by taxing capital in the short run and taxing labor in the long run.\footnote{Overaccumulation of capital at the aggregate is merely a competitive-equilibrium outcome of individuals' precautionary saving behaviors. Hence, it does not appear to be a genuine "externality" that the Ramsey planner should aim to correct in the steady state by using distortionary capital tax.}

Nonetheless, even though capital taxation is not the right tool to restore the MGR, it is not optimal either to subsidize capital in spite of the government debt limit—because doing so will further reduce the equilibrium interest rate below the time discount rate and intensify the overaccumulation problem, which means that the Ramsey planner has to increase capital taxation during the transition period and completely offset the steady-state capital subsidization rate.

### 4.4 Effects of Elasticity of Intertemporal Substitution

This subsection investigates the short- and long-run effects of elasticity of intertemporal substitution (EIS) on the Ramsey allocation when debt-limit constraints do not bind. Keep in mind that if EIS is too small or $\sigma$ is too large (such as in the limiting case $\sigma = \infty$), then the laissez-faire competitive equilibrium features both IAE and AAE. The reason is that extremely risk-averse households would opt to save a lot (enough) to provide full self-insurance when $\sigma$ is large.

So we consider cases with $\sigma$ not too far away from unity: the case of $\sigma = 0.8$ and the case of $\sigma = 1.5$, respectively. In each case, the initial capital stock is set to be 70% of the corresponding Ramsey steady-state level, as in the log utility case. As before, all parameter values imply that (i) the condition for positive individual labor choice, $\theta_H < \frac{\theta_L}{(1-\beta)^{\infty}} < \infty$, is satisfied in the steady state; (ii) the condition for positive labor supply $n_t > 0$ is satisfied and verified numerically in each time period along the entire transition path; (iii) the condition (17) holds such that the laissez-faire competitive equilibrium is neither IAE nor AAE; and (iv) the GRA allocation is feasible for the Ramsey planner with a non-binding debt limit, and the model’s Ramsey steady state corresponds to Case 1 described in Proposition 5.

To make the different cases more comparable to each other from the perspective of their respective laissez-faire economy, we report the ratios of Ramsey allocations to the laissez-faire allocations for different values of $\sigma$. The transition paths for aggregate consumption, labor, capital, and the cutoff are reported in Figure 5, where green lines represent the low EIS case with $\sigma = 1.5$, blue lines the benchmark case with $\sigma = 1$, and pink lines the high...
EIS case with $\sigma = 0.8$.

The figure shows that the key mechanism driving the optimal Ramsey allocation discussed in the previous subsections remains unchanged, except with the following important differences:

(i) The relative speed of convergence under Ramsey (benchmarking by the corresponding laissez-faire economy) depends negatively on the EIS, or positively on the value of $\sigma$. In particular, the transition speed is fastest in the case of low EIS ($\sigma = 1.5$) and slowest in the case of high EIS ($\sigma = 0.8$). Because we have shown previously under $\sigma = 1$ that the laissez-faire economy converges faster than the Ramsey economy, this result suggests that the Ramsey planner has less room to engage in intertemporal “arbitrage” to alter the competitive equilibrium through the use of policies when the market participants’ EIS is low; consequently, the economy converges faster when $\sigma$ is larger. The implication is that in the limit when $\sigma \to \infty$, it must be true that the Ramsey allocation approaches that of the laissez-faire competitive equilibrium and vice versa—because when agents’ EIS is close to zero, the laissez-faire competitive equilibrium can achieve full self-insurance, and hence the
Ramsey planner has no room (or desire) to improve the welfare of the laissez-faire economy through fiscal policies.

(ii) When the EIS is lower or $\sigma$ is larger, the steady-state relative levels of aggregate consumption, labor, and capital (relative to their laissez-faire counterparts) are higher and closer to 1; however, the optimal cutoff $\theta^+_t$ (relative to its corresponding laissez-faire level) converges to 1 from above (top-right panel in Figure 5). The reason is the same as pointed out before—namely, a lower EIS implies a smaller amount of room for the Ramsey planner to improve upon the associated laissez-faire competitive equilibrium because the competitive equilibrium is closer to the optimum under a larger $\sigma$; hence, the ratio of the Ramsey allocation and the Laissez-faire allocation approaches 1 as $\sigma$ increases. In other words, in the limiting case when $\sigma \to \infty$, the transition path must becomes a straight line of 1 for all variables shown in the figure—because the Ramsey allocation is identical to the laissez-faire competitive-equilibrium allocation both in transition and in steady state when EIS is close enough to zero.

(iii) The implication is that the welfare gains under the Ramsey allocation from improving laissez-faire competitive equilibrium increase with EIS. Figure 7 reports welfare gains of the Ramsey allocation from the laissez-faire competitive equilibrium over time. A welfare gain is measured by the compensation variation in terms of aggregate consumption. The figure shows that welfare gains from the Ramsey allocation are larger under the case of $\sigma = 0.8$ than under the case of $\sigma = 1.5$, especially during the transition. The figure also implies that in the limit as $\sigma \to \infty$, welfare gains approach zero both in transition and in steady state.

(iv) Figure 6 shows that the Ramsey planner opts to issue a far larger amount of debt relative to GDP under a higher EIS than under a lower EIS (bottom-left panel). As a result, the market interest rate is significantly higher when $\sigma = 0.8$ than when $\sigma = 1.5$ (bottom-right panel)—it is more than 800 basis points higher in the former case than in the latter case in the initial period. The labor tax rate approaches 7.13% in the long run when $\sigma = 0.8$, as compared with only 1.67% when $\sigma = 1.5$. Although the long-run capital tax rate is exclusively zero regardless of $\sigma$, the short-run capital tax rate is higher under a larger EIS. The insight is that the Ramsey planner is able to front-load consumption more aggressively by improving the individual self-insurance position when the EIS is high, but doing so requires the government to amass a larger stock of claims on the private economy, which calls for higher tax revenues to cover the interest payments.
5 A Brief Literature Review

The literature related to optimal capital taxation is vast. Here we review only the most relevant papers in the incomplete-markets literature.
Gottardi, Kajii, and Nakajima (2015) revisits optimal Ramsey taxation in an incomplete-markets model with uninsurable human-capital risk. As in our model, tractability in their model enables them to provide transparent analysis on Ramsey taxation and facilitates intuitive interpretations for their results. When government spending and the bond supply are both set to zero, they find that in the steady state the Ramsey planner should tax human capital and subsidize physical capital, despite the overaccumulation of physical capital. The purpose or the benefit of taxing human capital is to reduce uninsurable risk from human-capital returns; and the rationale for subsidizing physical capital despite overaccumulation is to satisfy household demand for a buffer stock, similar in spirit to our finding. However, the authors solve the Ramsey problem indirectly, and they can characterize analytically the properties of optimal taxes only in a neighborhood of zero government bonds and zero government spending. In contrast, we can solve the Ramsey problem analytically and directly along the entire dynamic path of the model, which permits transparent examinations of how the Ramsey planner takes into account the impact of its policies on the dynamic distributions of household decisions and aggregate productive efficiency. Our model also enables us to show analytically the exact roles played by government debt and how such roles are hindered by debt limits.

In an environment of uninsurable idiosyncratic risk, Krueger and Ludwig (2018) use an overlapping generation model to study Ramsey capital tax policy. Similar to our analytical approach, they are able to fully characterize the Ramsey solution and justify the existence of a Ramsey steady state. Since in their model all tax revenues are lump-sum transferred back to households (to serve as an insurance device) and there is no government bond, their result of a positive capital tax by the Ramsey planner does not contradict our results. The insight behind their result is that the planner opts to use both capital tax to achieve production efficiency and lump-sum transfers to improve IAE.

Aiyagari and McGrattan (1998) study optimal government debt in the Aiyagari (1994) model. Similar to our finding, government bonds are shown to play an important role in providing self-insurance for households and to help in relaxing their borrowing constraints. However, the authors restrict their analysis to the special case of the same tax rates across capital and labor incomes and analyze welfare only in the steady state. In addition, they are unable to separate the distinctive roles that the MGR and income/wealth redistribution play in the determination of optimal capital taxation.

In an overlapping generations model with uninsured individual risk, Conea, Kitao, and Krueger (2009) conduct a numerical exercise to derive optimal capital tax and non-linear
labor tax. As in Aiyagari and McGrattan (1998), they mainly consider welfare in the steady state, and the transitional path is therefore ignored. However, Domeij and Heathcote (2004) show that welfare along the transitional path is an important concern for the Ramsey planner. Their findings indicate that steady-state welfare maximization could be misleading when designing optimal policies. But, instead of solving optimal tax policies, they numerically evaluate the welfare consequences of tax changes.

Three recent works by Acikgoz, Hagedorn, Holter, and Wang (2018), Dyrda and Pedroni (2018) and Ragot and Grand (2017) numerically solve optimal fiscal policies along the transitional path in an infinite horizon HAIM economy. In contrast to our findings, their results are consistent with Aiyagari’s analysis. The sources of difference between their results and ours could be the difference between our models.

Instead of using the Ramsey approach, Dávila, Hong, Krusell, and Ríos-Rull (2012) characterize and decentralize constrained efficient allocations in an Aiyagari-type economy where the government can levy an individual specific labor tax, which is not allowed in the traditional Ramsey framework. In addition, there is no government bond in their analysis. They found that in a competitive equilibrium, the capital stock could be either too high or too low compared with the constraint-efficient allocation, thus the optimal capital income tax rate can be either positive or negative.

Finally, Park (2014) considers Ramsey taxation in a complete-markets environment featuring enforcement constraints, à la Kehoe and Levine (1993). She shows that capital accumulation improves the outside option of default, which is not internalized by household decisions. Therefore, capital income should be taxed in order to internalize such an adverse externality.

6 Conclusion

This paper designs a special and tractable infinite-horizon HAIM model without redistribut- tional effects to address a set of long-standing issues in the optimal Ramsey capital taxation literature. The tractability of our model enables us to provide necessary and sufficient conditions for the existence of the Ramsey steady state and establish several novel results in the absence of any redistributional effects of capital taxation or lump-sum transfers: (i) In line with the classical result of zero capital taxation based on representative-agent models where redistribution is never a concern, the optimal steady-state capital tax is also exclusively zero in our infinite-horizon HAIM economy (regardless of government debt limits). (ii) The
Ramsey planner opts to levy a capital tax only during transition periods; and the optimal tax rate depends positively on the elasticity of intertemporal substitution. (iii) It is not innocuous to assume the existence of a Ramsey steady state without rigorous proof—when a Ramsey steady state (featuring a non-binding government debt limit) does not exist but is erroneously assumed to exist, we are led to the erroneous conclusion that the MGR “holds” and the implied “optimal” long-run capital tax is strictly positive. (iv) Whether the MGR holds depends critically on the government’s capacity to issue debt, but it has no bearing on the planner’s long-run capital tax scheme. (v) The optimal debt-to-GDP ratio in the absence of a binding debt limit, however, is determined by a positive wedge times the MGR saving rate; the wedge is decreasing in the strength of individual self-insurance and approaches zero when idiosyncratic risk vanishes or markets are complete.

The key insight behind our results is that under incomplete markets there exist both an intertemporal wedge and an intratemporal wedge in the failure of the MGR—the former wedge pertaining to a positive liquidity premium and the latter wedge to a difference between the socially optimal MPK and the liquidity-premium-adjusted private MPK. The intertemporal wedge indicates individual allocative inefficiency in terms of insufficient self-insurance and the intratemporal wedge indicates aggregate allocative inefficiency in terms of overaccumulation of capital. However, the second wedge is the consequence of the first wedge. So in the absence of any redistributional effects of income taxation, the Ramsey planner’s ultimate concern for self-insurance implies that it is optimal to issue a sufficient amount of bonds to achieve full self-insurance unless a debt limit binds. Since taxing capital in the steady state permanently hinders individuals’ self insurance positions, it is not an effective tool for eliminating the intertemporal wedge. Consequently, the Ramsey planner prefers taxing capital only in the short run rather than imposing a capital tax in the steady state to correct the capital-overaccumulation problem under precautionary saving motives. Thus, permanent capital taxation always remains zero even when the MGR fails to hold in a Ramsey steady state because of a binding debt limit. In addition, subsidizing capital in the steady state is not optimal either, because it permanently worsens the capital-overaccumulation problem and thus would force the Ramsey planner to dramatically increase transitory capital taxation to address the problem, offsetting the original intention (and potential welfare gains) of boosting household savings. Therefore, when the government is unable to issue debt, it is optimal to do nothing by setting both the capital tax and labor tax to zero, instead of imposing a steady-state labor tax and using the receipts to subsidize capital while taxing capital heavily in the transition period.
We reveal in a transparent manner that government debt plays an important role in determining the optimal Ramsey tax scheme when facing the trade-off between AAE and IAE. In particular, the planner’s motive to improve individuals’ self-insurance positions and relax their borrowing constraints may induce the planner to amass an ever-increasing amount of public debt, which may result in a dynamic path featuring no Ramsey steady state.

Our model can be extended in many directions. For example, there is a strong tradition and renewed interest in studying the optimal responses of fiscal policies to aggregate shocks. The works by Barro (1979) and Lucas and Stokey (1983) both identify the importance of tax smoothing but give different predictions on the optimal behavior of government bonds. Aiyagari, Marcet, Sargent, and Seppala (2002) show that the (in)complete-market assumption explains the different findings of these two classical papers. Farhi (2010) and Bhandari, Evans, Golosov, and Sargent (2017) contribute to this literature by considering incomplete markets with aggregate risks in a representative-agent framework. More recently, Bassetto (2014) and Bhandari, Evans, Golosov, and Sargent (2018) extend this literature to a heterogeneous-agent framework; but the numerical approach taken by Bhandari, Evans, Golosov, and Sargent (2018) sidesteps the issue of possible nonexistence of a Ramsey steady state. Our model can be extended to an environment with aggregate risks and complement this literature by offering a more tractable and transparent analysis.11

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11 This task is currently undertaken by the authors.
References


A Appendix

A.1 Proof of Proposition 1

Denoting \(\{\beta^t \lambda^h_t(\theta^t), \beta^t \mu^h_t(\theta^t)\}\) as the Lagrangian multipliers for constraints (6) and (7), respectively, the first-order conditions for \(\{c_t(\theta^t), n_t(\theta^{t-1}), a_{t+1}(\theta^t)\}\) are given, respectively, by

\[
\frac{\theta_t}{c_t(\theta^t)^\sigma} = \lambda^h_t(\theta^t) \tag{42}
\]

\[
1 = \overline{w}_t \int \lambda^h_t(\theta^t) dF(\theta_t) \tag{43}
\]

\[
\lambda^h_t(\theta^t) = \beta r_{t+1} E_t \left[ \lambda^h_{t+1}(\theta^{t+1}) \right] + \mu^h_t(\theta^t), \tag{44}
\]

where equation (43) reflects that the labor supply \(n_t(\theta^{t-1})\) must be chosen before the idiosyncratic taste shocks (and hence before the value of \(\lambda^h_t(\theta^t)\)) are realized. By the law of iterated expectations and the iid assumption of idiosyncratic shocks, equation (44) can be written (using equation (43)) as

\[
\lambda^h_t(\theta^t) = \beta \frac{r_{t+1}}{\overline{w}_{t+1}} + \mu^h_t(\theta^t), \tag{45}
\]

where \(\frac{1}{\overline{w}}\) is the marginal utility of consumption in terms of labor income.

We adopt a guess-and-verify strategy to derive the decision rules. The decision rules for an individual’s consumption and savings are characterized by a cutoff strategy, taking as given the aggregate states (such as the interest rate and real wage). Anticipating that the optimal cutoff \(\theta_t^*\) is independent of an individual’s history of shocks, consider two possible cases:

Case A. \(\theta_t \leq \theta_t^*\). In this case the urge to consume is low. It is hence optimal to save so as to prevent possible liquidity constraints in the future. So \(a_{t+1}(\theta^t) \geq 0\), \(\mu^h_t(\theta^t) = 0\), and the shadow value is

\[
\lambda^h_t(\theta^t) = \beta \frac{r_{t+1}}{\overline{w}_{t+1}} \equiv \Lambda_t,
\]

where \(\Lambda_t\) depends only on aggregate states. Notice that \(\lambda^h_t(\theta^t) = \lambda^h_t\) is independent of the history of idiosyncratic shocks. Equation (42) implies that consumption is given by \(c_t(\theta^t)^\sigma = \theta_t \Lambda_t^{-1}\). Defining \(x_t(\theta^{t-1}) \equiv r_t a_t(\theta^{t-1}) + \overline{w}_t n_t(\theta^{t-1})\) as the gross income of the household, the budget identity (6) then implies \(a_{t+1}(\theta^t) = x_t(\theta^{t-1}) - \left(\theta_t \Lambda_t^{-1}\right)^{1/\sigma}\). The requirement
\( a_{t+1}(\theta^t) \geq 0 \) then implies
\[
\theta_t \leq \Lambda_t x_t^\sigma \equiv \theta_t^*, \tag{46}
\]
which defines the cutoff \( \theta_t^* \).

We conjecture that the cutoff is independent of the idiosyncratic state. Then the optimal gross income \( x_t \) is also independent of the idiosyncratic state. The intuition is that \( x_t \) is determined before the realization of \( \theta_t \) and that all households face the same distribution of idiosyncratic shocks. Since the utility function is quasi-linear, the household is able to adjust labor income to meet any target level of liquidity in hand. As a result, the distribution of \( x_t \) is degenerate. This property simplifies the model tremendously.

Case B. \( \theta_t > \theta_t^* \). In this case the urge to consume is high. It is then optimal not to save, so \( a_{t+1}(\theta^t) = 0 \) and \( \mu_t^h(\theta^t) > 0 \). By the resource constraint (6), we have \( c_t(\theta^t) = x_t \), which by equation (46) implies \( c_t(\theta^t)^\sigma = \theta_t^* \Lambda_t^{-1} \). Equation (42) then implies that the shadow value is given by \( \lambda_t^h(\theta^t) = \frac{\theta_t}{\theta_t^*} \Lambda_t \). Since \( \theta_t > \theta^* \), equation (45) implies \( \mu_t^h(\theta^t) = \Lambda_t \left[ \frac{\theta_t}{\theta_t^*} - 1 \right] > 0 \). Notice that the shadow value of goods (the marginal utility of income), \( \lambda_t^h(\theta^t) \), is higher under case B than under case A because of binding borrowing constraints.

The above analyses imply that the expected shadow value of income, \( \int \lambda_t^h(\theta)dF(\theta) \), and hence the optimal cutoff value \( \theta^* \), is determined by equation (43) by plugging in the expressions for \( \lambda_t^h(\theta^t) \) under cases A and B, which immediately gives equation (12). Specifically, combining case A and case B, we have
\[
\lambda_t^h(\theta^t) = \begin{cases} 
\beta \frac{r_{t+1}}{w_{t+1}} & \text{for } \theta \leq \theta_t^* \\
\frac{\theta_t}{\theta_t^*} \beta \frac{r_{t+1}}{w_{t+1}} & \text{for } \theta \geq \theta_t^* 
\end{cases}
\]

The aggregate Euler equation is therefore given by
\[
\frac{1}{w_t} = \int \lambda_t^h(\theta)dF(\theta) = \beta \frac{r_{t+1}}{w_{t+1}} \left[ \int_{\theta \leq \theta_t^*} dF(\theta) + \int_{\theta > \theta_t^*} \frac{\theta}{\theta^*} dF(\theta) \right] = \beta \frac{r_{t+1}}{w_{t+1}} L(\theta_t^*),
\]
which is equation (12). This equation reveals that the optimal cutoff depends only on aggregate states and is independent of individual history.

We also immediately obtain
\[
x_t = \left[ \theta_t^* \left( \beta \frac{r_{t+1}}{w_{t+1}} \right)^{-1} \right]^{1/\sigma} = \left[ \theta_t^* L(\theta_t^*) \frac{1}{w_t} \right]^{1/\sigma},
\]
45
which leads to equation (8). By the discussion of cases A and B, as well as the use of equation (8), the decision rules of household consumption and saving can then be summarized by equations (9) and (10), respectively. Finally, the decision rule of the household labor supply, equation (11), is decided residually to satisfy the household budget constraint.

Finally, to ensure that the above proof and hence the associated cutoff-policy rules are consistent with the assumption of interior choices of labor, namely, \( n_t \in (0, N) \), we need to consider the following two cases:

First, to ensure that \( n_t(\theta^{t-1}) > 0 \), consider the worst situation where \( n_t(\theta^{t-1}) \) takes its minimum value. Given \( x_t = r_t a_t(\theta^{t-1}) + \theta^t n_t(\theta^{t-1}) \), \( n_t(\theta^{t-1}) \) is at its minimum if \( \mu_t^h = 0 \) and \( a_t(\theta^{t-1}) \) takes the maximum possible value, \( a_t(\theta^{t-1}) = \left[ 1 - \left( \frac{\theta_t}{\theta_{t-1}^*} \right)^{1/\sigma} \right] x_{t-1} \). So \( n_t(\theta^{t-1}) > 0 \) if

\[
x_t - r_t \left[ 1 - \left( \frac{\theta_t}{\theta_{t-1}^*} \right)^{1/\sigma} \right] x_{t-1} > 0,
\]

which is independent of the shock \( \theta_t \). This condition in the steady state becomes \( 1 - r \left[ 1 - \left( \frac{\theta_t}{\theta_{t-1}^*} \right)^{1/\sigma} \right] > 0 \), or equivalently (by using equation (12)),

\[
\beta L (\theta^*) > 1 - \left( \frac{\theta_t}{\theta^*} \right)^{1/\sigma}.
\]

Given that \( L (\theta^*) \) is a monotonic decreasing function in \( \theta^* \) with a lower bound of 1, the necessary condition to satisfy (48) in the steady state is \( \beta > 1 - \left( \frac{\theta_t}{\theta^*} \right)^{1/\sigma} \), which is easy to satisfy when \( \theta_H < \infty \). Therefore, as long as the condition \( \beta > 1 - \left( \frac{\theta_t}{\theta^*} \right)^{1/\sigma} \) is met, the condition (47) is assumed to hold throughout the paper.

Second, to ensure that \( n_t < N \), consider those agents who encounter the borrowing constraint last period such that \( a_t(\theta^{t-1}) = 0 \). Their labor supply reaches the maximum value at \( n_t(\theta^{t-1}) = \frac{\theta_t}{\theta^*} = \theta^*_t L(\theta^*_t) \). Given a finite steady state value of \( \theta^* \), the value of \( N \) can be chosen such that

\[
N = \theta_H \geq \theta^*_t L(\theta^*_t).
\]

A.2 Proof of Proposition 2

In the laissez-faire economy, the capital tax, the labor tax, government spending and government bond are all equal to zero. In this laissez-faire competitive equilibrium, the capital-to-labor ratio \( \frac{K^t}{N_t} \) satisfies two conditions. The first condition is derived from the resource
constraint (16), which can be expressed as

\[ F(K_t, N_t) + (1 - \delta)K_t = C_t + K_{t+1} = x_t, \]

where the last equality uses the definition of \( x_t \). Dividing both sides of the equation by \( K_t \) gives

\[ \left( \frac{K_t}{N_t} \right)^{\alpha - 1} + (1 - \delta) = \frac{1}{1 - D(\theta_t^*)}, \tag{50} \]

where \( x_t/K_t \) is substituted out by \( \frac{1}{1 - D(\theta_t^*)} \).

The second condition is derived by combining equation (12) and the no-arbitrage condition, \( r_t = 1 + q_t - \delta \), which gives

\[ 1 = \beta \left( 1 + \alpha \left( \frac{K_t}{N_t} \right)^{\alpha - 1} - \delta \right) L(\theta_t^*), \tag{51} \]

where the marginal product of capital \( q_t \) is replaced by \( \alpha \left( \frac{K_t}{N_t} \right)^{\alpha - 1} \). Since the capital-to-labor ratio must be the same in both equations, conditions (50) and (51) imply the following equation in the steady state:

\[ \frac{\alpha \beta}{(1 - D(\theta^*))} + \beta(1 - \alpha)(1 - \delta) = \frac{1}{L(\theta^*)}, \tag{52} \]

which solves for the steady-state value of \( \theta^* \).

It can be shown easily that both \( L(\theta^*) \) and \( D(\theta^*) \) are monotonically decreasing in \( \theta^* \), thus the right-hand side (RHS) of equation (52) increases monotonically in \( \theta^* \) and the left-hand side (LHS) of equation (52) decreases monotonically in \( \theta^* \).

It remains to be seen if the RHS and the LHS cross each other at an interior value of \( \theta^* \in [\theta_L, \theta_H] \). The RHS of equation (52) reaches its minimum value of 1 when \( \theta^* = \theta_H \) and its maximum value of \( \theta^*/\theta_L > 1 \) when \( \theta^* = \theta_L \). The LHS of equation (52) takes the maximum value of infinity when \( \theta^* = \theta_L \) and the minimum value of \( \frac{\alpha \beta (\theta_H)^{1/\sigma}}{(\theta_H)^{1/\sigma} - E(\theta^*/\sigma)} + \beta(1 - \alpha)(1 - \delta) \) when \( \theta^* = \theta_H \). Thus, an interior solution exists if and only if

\[ \frac{\alpha \beta (\theta_H)^{1/\sigma}}{(\theta_H)^{1/\sigma} - E(\theta^*/\sigma)} + \beta(1 - \alpha)(1 - \delta) < 1. \]

Clearly, \( \theta^* = \theta_L \) cannot constitute a solution for any positive value when \( \theta_L > 0 \). On the other hand, \( \theta^* = \theta_H \) may constitute a solution if the above condition is violated. For
example, if $\theta_H$ is small and close enough to the $E(\theta^\frac{1}{2})^\sigma$, then the above condition does not hold since its LHS approaches infinity when $\theta_H \to E(\theta^\frac{1}{2})^\sigma$. Therefore, an interior solution for $\theta^*$ exists if the upper bound of the idiosyncratic shock is large enough. Otherwise, we have the corner solution $\theta^* = \theta_H$. Finally, if $\theta^*$ is an interior solution, then $L(\theta^*) > 1$ and $r < 1/\beta$ by equation (12).

A.3 Proof of Proposition 3

A.3.1 The “only if” Part

Assume that we have the allocation $\{\theta^*_t, C_t, N_t, K_{t+1}, B_{t+1}\}_{t=1}^\infty$ and the initial risk-free rate $r_1$. We then can directly construct the prices, taxes, and individual allocations in the competitive equilibrium in the following 7 steps:

1. $w_t$ and $q_t$ are set by (1) and (2), which are $w_t = MP_{N,t}$ and $q_t = MP_{K,t}$, respectively.

2. Given $C_t$ and $\theta^*_t$, the total liquidity in hand can be set by equation (18), $x_t = \frac{C_t}{D(\theta^*_t)}$.

3. The individual consumption and asset holdings, $c_t(\theta_t)$ and $a_{t+1}(\theta_t)$, are pinned down by equations (9) and (10).

4. $\tau_{n,t}$ is determined by equation (8), which implies $\tau_{n,t} = 1 - \frac{x_t^\sigma}{L(\theta^*_t)\theta^*_t MP_{N,t}}$. Hence, $\overline{w}_t$ can be expressed as

$$\overline{w}_t = \frac{x_t^\sigma}{L(\theta^*_t)\theta^*_t MP_{N,t}} = \frac{C_t^\sigma}{D(\theta^*_t)^\sigma L(\theta^*_t)\theta^*_t}.$$ 

Given $\overline{w}_t$, the interest rate $r_{t+1}$ can be backed out by the Euler equation (12):

$$\frac{1}{r_{t+1}} = \beta L(\theta^*_t)\overline{w}_t = \beta \frac{U_{C,t+1}}{U_{C,t}} \frac{D(\theta^*_t)^\sigma \theta^*_t}{D(\theta^*_t)^\sigma} L(\theta^*_t)$$ for all $t \geq 1$

where $U_{C,t}$ is defined as $C_{t}^{-\sigma}$, the "marginal utility of aggregate consumption" given our preference assumption.

Given $r_1$ and the expression of $\{r_{t+1}\}_{t=1}^\infty$, the capital tax $\{\tau_{k,t+1}\}_{t=0}^\infty$ is chosen to satisfy the no-arbitrage condition: $r_t = 1 + (1 - \tau_{k,t})MP_{K,t} - \delta$ for all $t \geq 1$.

5. Finally, set $n_t(\theta_{t-1})$ to satisfy equation (11), which is implied by the individual household budget constraint.
6. Define $A_{t+1}$ as the aggregate asset holding in period $t$. Integrating (10) gives
\[ A_{t+1} \equiv \int a_{t+1}(\theta_t)dF(\theta_t) = \int \max \left\{ 1 - \left( \frac{\theta_t}{\theta^*_t} \right)^{1/\sigma}, 0 \right\} x_t dF(\theta_t) = \left( \frac{1}{D(\theta^*_t)} - 1 \right) C_t, \]
where the last equality utilizes equation (18). The above equation together with the condition defined in equation (20) gives the competitive equilibrium asset market clearing condition (14).

7. The implementability conditions are
\[ C_1^{1-\sigma} D(\theta^*_t)^{\sigma - 1} L(\theta^*_t)\theta^*_t \geq N_1 + r_1 C_1^{1-\sigma} D(\theta^*_t)^{\sigma} L(\theta^*_t)\theta^*_t (K_1 + B_1) \]
and for $t \geq 2$,
\[ C_t^{1-\sigma} D(\theta^*_t)^{\sigma - 1} L(\theta^*_t)\theta^*_t \geq N_t + \frac{1}{\beta} C_{t-1}^{1-\sigma} D(\theta^*_t-1)^{\sigma} \theta^*_t \left( \frac{1}{D(\theta^*_t-1)} - 1 \right) \]
Multiplying both side of the above equation with $\frac{C_t^\sigma}{D(\theta^*_t)^\sigma L(\theta^*_t)\theta^*_t}$ leads to
\[ \frac{C_1}{D(\theta^*_1)} \geq \frac{C_1^\sigma}{D(\theta^*_1)^\sigma L(\theta^*_1)\theta^*_1} N_1 + r_1 (K_1 + B_1) \]
\[ \frac{C_t}{D(\theta^*_t)} \geq \frac{C_t^\sigma}{D(\theta^*_t)^\sigma L(\theta^*_t)\theta^*_t} N_t + \frac{1}{\beta} \frac{C_{t-1}^\sigma}{D(\theta^*_t-1)^\sigma \theta^*_t} \frac{1}{L(\theta^*_t)} \left( 1 - D(\theta^*_t-1) \right) C_{t-1}. \]
Using the relationship constructed in steps 2, 4, and 6 for $x_t$, $C_t$, $A_{t+1}$, $w_t$, and $r_t$ in the above equation gives
\[ C_t + A_{t+1} \geq w_t N_t + r_t A_t \text{ for all } t \geq 1, \]
which together with the resource constraint and asset market clearing condition enforce the government budget constraint.

Step 1 ensures that the representative firm’s problem is solved. Steps 2 to 5 guarantee that the individual household problem is solved. Steps 6 and 7 ensure that the asset market clearing condition and government budget constraint are satisfied, respectively. The labor market clearing condition is satisfied by Walras law.
A.3.2 The “if” Part

Note that the resource constraint and asset market clearing condition are trivially implied by a competitive equilibrium since they are part of the definition. The implementability condition is constructed as follows. First, we rewrite the government budget constraint as

\[ G_t \leq F(K_t, N_t) - (1 - \tau_{k,t})q_tK_t - (1 - \tau_{n,t})w_tN_t + B_{t+1} - r_tB_t. \]

Combining this equation with the resource constraint (16), no-arbitrage condition, and the asset market clearing condition (14) implies

\[ (1 - \tau_{n,t})w_tN_t + r_tA_t \leq C_t + A_{t+1}, \]

and hence leading to the following equation:

\[ C_t + A_{t+1} \geq \bar{w}_tN_t + r_tA_t. \]  

(53)

For \( t \geq 2 \), the equilibrium conditions (8), (18), and (12) suggest that \( \bar{w}_t \) and \( r_t \) can be expressed as

\[ \bar{w}_t = \frac{1}{C_t^{-\sigma}D(\theta_t^*)L(\theta_t^*)\theta_t^*} \]

and

\[ r_t = \frac{1}{\beta} \frac{C_{t-1}^{-\sigma}D(\theta_{t-1}^*)L(\theta_{t-1}^*)\theta_{t-1}^*}{C_t^{-\sigma}D(\theta_t^*)L(\theta_t^*)\theta_t^*}. \]

Substituting the above two equations into (53) and rearranging terms, we get

\[ C_t^{-\sigma}D(\theta_t^*)L(\theta_t^*)\theta_t^*C_t + C_t^{-\sigma}D(\theta_t^*)L(\theta_t^*)\theta_t^*A_{t+1} \geq N_t + \frac{1}{\beta}C_{t-1}^{-\sigma}D(\theta_{t-1}^*)\theta_{t-1}^*A_t, \]

The implementability condition (22) follows by plugging \( A_{t+1} = \left(\frac{1}{D(\theta_t^*)} - 1\right)C_t \) and \( A_t = \left(\frac{1}{D(\theta_{t-1}^*)} - 1\right)C_{t-1} \) to the equation above.

For the first period, \( \{B_1, K_1, \tau_{k,1}\} \) are given, which implies that \( r_1 = 1 + (1 - \tau_{k,1})MP_{K,1} - \delta \) is pinned down by \( N_1 \). Therefore, the first period implementability condition could be rewritten as

\[ C_1^{-\sigma}D(\theta_1^*)L(\theta_1^*)\theta_1^* \geq N_1 + r_1C_1^{-\sigma}D(\theta_1^*)L(\theta_1^*)\theta_1^*(K_1 + B_1). \]
### A.4 Appendix A.4

The first-order conditions with respect to $N_1$, $C_1$, and $\theta_1^*$ in the Ramsey problem (29) are given, respectively, by

\[
1 + \lambda_1 + \lambda_1 C_1^{-\sigma} D(\theta_1^*)^{\sigma} L(\theta_1^*) \theta_1^* \frac{\partial r_1}{\partial N_1} (K_1 + B_1) = \mu_1 M P_{N,1} \tag{54}
\]

\[
\mu_1 = W(\theta_1^*) C_1^{-\sigma} + \lambda_1 (1 - \sigma) C_1^{-\sigma} D(\theta_1^*)^{\sigma-1} L(\theta_1^*) \theta_1^*
\]

\[
+ \lambda_1 \sigma C_1^{-\sigma-1} D(\theta_1^*)^{\sigma} L(\theta_1^*) \theta_1^* r_1 (K_1 + B_1)
\]

\[
- \lambda_2 (1 - \sigma) C_1^{-\sigma} D(\theta_1^*)^{\sigma} \theta_1^* \left( \frac{1}{D(\theta_1^*)} - 1 \right) - \phi_1 \left( \frac{1}{D(\theta_1^*)} - 1 \right)
\]

and

\[
\frac{\partial W(\theta_1^*)}{\partial \theta_1^*} C_1^{1-\sigma} \frac{1}{1 - \sigma} + \lambda_1 C_1^{1-\sigma} H(\theta_1^*) - \lambda_2 C_1^{1-\sigma} J(\theta_1^*) + \phi_1 \frac{C_1}{D(\theta_1^*)} \frac{\partial D(\theta_1^*)}{\partial \theta_1^*} = \lambda_1 C_1^{-\sigma} \frac{\partial (D(\theta_1^*) \sigma L(\theta_1^*) \theta_1^*)}{\partial \theta_1^*} r_1 (K_1 + B_1). \tag{56}
\]

In addition, since the relative magnitude of $H(\theta_1^*)$ and $J(\theta_1^*)$ affects the dynamics of $\lambda_t$ in equation (34), we provide the following four lemmas to help characterize the optimal Ramsey allocation.

**Lemma 1.** The derivative of $W(\theta_1^*)$ is given by

\[
\frac{\partial W(\theta_1^*)}{\partial \theta_1^*} = \frac{(1 - \sigma)}{\sigma} D(\theta_1^*)^{\sigma-1} X(\theta_1^*) \left[ \frac{M(\theta_1^*)}{D(\theta_1^*)} \theta_1^* - 1 \right],
\]

where $X(\theta_1^*)$ and $M(\theta_1^*)$ are defined as

\[
X(\theta_1^*) \equiv \int_{\theta \leq \theta_1^*} \left( \frac{\theta}{\theta_1^*} \right)^{1/\sigma} dF(\theta), \tag{57}
\]

\[
M(\theta_1^*) \equiv \int_{\theta \leq \theta_1^*} \theta \left( \frac{\theta}{\theta_1^*} \right)^{\frac{1}{\sigma}} dF + \int_{\theta > \theta_1^*} \theta dF, \tag{58}
\]

respectively. In addition, $M(\theta_1^*) \geq \theta_1^* D(\theta_1^*)$ with equality if $\theta_1^* = \theta_H$. Hence, the sign of $\frac{\partial W(\theta_1^*)}{\partial \theta_1^*}$ is determined by the sign of the elasticity of intertemporal substitution coefficient, $1/\sigma$. 51
Proof. By the definition of $M(\theta^*_t)$, the $W(\theta^*_t)$ can be rewritten as $W(\theta^*_t) = M(\theta^*_t)D(\theta^*_t)^{\sigma-1}$. The derivative of $M(\theta^*_t)$ and hence $W(\theta^*_t)$ are given by

$$\frac{\partial M(\theta^*_t)}{\partial \theta^*_t} = -\frac{1-\sigma}{\sigma} \int_{\theta \leq \theta^*_t} \left(\frac{\theta}{\theta^*_t}\right)^{\frac{1}{\sigma}} dF = -\frac{1-\sigma}{\sigma} X(\theta^*_t)$$

and

$$\frac{\partial W(\theta^*_t)}{\partial \theta^*_t} = (\sigma - 1)D(\theta^*_t)^{\sigma-2}\frac{\partial D(\theta^*_t)}{\partial \theta^*_t}M(\theta^*_t) - D(\theta^*_t)^{\sigma-1}\frac{1-\sigma}{\sigma}X(\theta^*_t)$$

$$= \left(\frac{1-\sigma}{\sigma}\right)D(\theta^*_t)^{\sigma-1}X(\theta^*_t) \left[\frac{M(\theta^*_t)}{D(\theta^*_t)} \theta^*_t - 1\right]$$

In addition,

$$M(\theta^*_t) - \theta^*_t D(\theta^*_t) = \int_{\theta > \theta^*_t} (\theta - \theta^*_t)dF(\theta) \geq 0$$

Hence, $M(\theta^*_t) \geq \theta^*_t D(\theta^*_t)$ with equality holds at $\theta^*_t = \theta_H$.

**Lemma 2.** $J(\theta^*_t)$ and $H(\theta^*_t)$ can be expressed as

$$H(\theta^*_t) = D(\theta^*_t)^{\sigma-1}X(\theta^*_t) \left[\frac{1-\sigma}{\sigma} \frac{L(\theta^*_t)}{D(\theta^*_t)} + \frac{F(\theta^*_t)}{X(\theta^*_t)}\right], \quad (59)$$

$$J(\theta^*_t) = D(\theta^*_t)^{\sigma-1}X(\theta^*_t) \left[\frac{1-\sigma}{\sigma} \frac{1}{D(\theta^*_t)} + \frac{F(\theta^*_t)}{X(\theta^*_t)}\right], \quad (60)$$

which have the following properties: (1) if $\sigma < 1$, then $H(\theta^*_t) > J(\theta^*_t) > 0$; (2) if $\sigma = 1$, then $H(\theta^*_t) = J(\theta^*_t) > 0$; and (3) if $\sigma > 1$, then $H(\theta^*_t) < J(\theta^*_t)$.

Proof. $H(\theta^*_t)$ can be expressed as

$$\begin{align*}
H(\theta^*_t) &= \frac{\partial(D(\theta^*_t)^{\sigma-1} \theta^*_t L(\theta^*_t))}{\partial \theta^*_t} = (\sigma - 1)D(\theta^*_t)^{\sigma-2}\frac{\partial D(\theta^*_t)}{\partial \theta^*_t} \theta^*_t L(\theta^*_t) + D(\theta^*_t)^{\sigma-1} F(\theta^*_t) \\
&= D(\theta^*_t)^{\sigma-1} \left[ (\sigma - 1)D(\theta^*_t)^{-1} \frac{\partial D(\theta^*_t)}{\partial \theta^*_t} \theta^*_t L(\theta^*_t) + F(\theta^*_t) \right] \\
&= D(\theta^*_t)^{\sigma-1}X(\theta^*_t) \left[\frac{1-\sigma}{\sigma} \frac{L(\theta^*_t)}{D(\theta^*_t)} + \frac{F(\theta^*_t)}{X(\theta^*_t)}\right].
\end{align*}$$
\( J(\theta^*_t) \) can be rewritten as

\[
J(\theta^*_t) = \frac{\partial}{\partial \theta^*_t} (D(\theta^*_t)^{\sigma-1} \theta^*_t (1 - D(\theta^*_t))) \\
= (\sigma - 1)D(\theta^*_t)^{\sigma-2} \frac{\partial D(\theta^*_t)}{\partial \theta^*_t} \theta^*_t (1 - D(\theta^*_t)) + D(\theta^*_t)^{\sigma-1} \left( 1 - D(\theta^*_t) - \theta^*_t \frac{\partial D(\theta^*_t)}{\partial \theta^*_t} \right) \\
= \frac{1 - \sigma}{\sigma} D(\theta^*_t)^{\sigma-2} X(\theta^*_t) (1 - D(\theta^*_t)) + D(\theta^*_t)^{\sigma-1} \left( F(\theta^*_t) + \left( \frac{1}{\sigma} - 1 \right) X(\theta^*_t) \right) \\
= D(\theta^*_t)^{\sigma-1} X(\theta^*_t) \left[ \frac{1 - \sigma}{\sigma} \frac{1}{D(\theta^*_t)} + \frac{F(\theta^*_t)}{X(\theta^*_t)} \right]
\]

By \( H(\theta^*_t) \) and \( J(\theta^*_t) \) expressed above, we reach that

1. if \( \sigma < 1, 0 < J(\theta^*_t) \leq H(\theta^*_t) \) and \( J(\theta^*_t) = H(\theta^*_t) \) if \( \theta^*_t = \theta_H \)

2. if \( \sigma = 1, 0 \leq J(\theta^*_t) = H(\theta^*_t) \).

3. if \( \sigma > 1, J(\theta^*_t) \geq H(\theta^*_t) \) and \( J(\theta^*_t) = H(\theta^*_t) \) if \( \theta^*_t = \theta_H \) or \( \theta_L \)

**Lemma 3.** \( \frac{\partial D(\theta^*_t)}{\partial \theta^*_t} = -\frac{X(\theta^*_t)}{\sigma \theta^*_t} < 0 \) for all \( \theta^*_t \in (\theta_L, \theta_H] \).

**Proof.** \( D(\theta^*_t) \) is therefore given by

\[
D(\theta^*_t) = X(\theta^*_t) + \int_{\theta > \theta^*_t} dF(\theta).
\]

The derivative of \( D(\theta^*_t) \) is

\[
\frac{\partial D(\theta^*_t)}{\partial \theta^*_t} = -\frac{1}{\sigma} \int_{\theta \leq \theta^*_t} \left( \frac{\theta}{\theta^*_t} \right)^{\frac{1}{\sigma}} dF(\theta) = -\frac{X(\theta^*_t)}{\sigma \theta^*_t} < 0
\]

**Lemma 4.** \( L(\theta^*_t) - 1 + D(\theta^*_t) = \frac{M(\theta^*_t)}{\theta^*_t} \).

**Proof.** By the definition of \( L(\theta^*_t) \) and \( D(\theta^*_t) \), we get

\[
L(\theta^*_t) - 1 + D(\theta^*_t) = \int_{\theta \leq \theta^*_t} dF(\theta) + \int_{\theta > \theta^*_t} \frac{\theta}{\theta^*_t} dF(\theta) + X(\theta^*_t) + \int_{\theta > \theta^*_t} dF(\theta) - 1
= \int_{\theta > \theta^*_t} \frac{\theta}{\theta^*_t} dF(\theta) + X(\theta^*_t) = \frac{M(\theta^*_t)}{\theta^*_t},
\]

where the last equality utilize the definition of \( M(\theta^*_t) \).
A.5 Proof of Proposition 4

The optimal capital tax is chosen such that the Euler equation in the competitive equilibrium (12) is consistent with the one chosen by the Ramsey planner shown in (30). Hence, \( \tau_{k,t+1} \) is pinned down by

\[
1 - \tau_{k,t+1} = \frac{\frac{w_{t+1}}{w_t} \frac{1}{L(\theta_t^*)} - \beta (1 - \delta)}{\frac{\mu - \phi_t}{\mu_{t+1}} - \beta (1 - \delta)},
\]

which is equation (36) in the steady state.

A.6 Proof of Proposition 5

From equation (33), \( \nu_t^B = 0 \) implies that \( \phi_t = 0 \). The first-order condition with respect to \( \theta_t^* \) is then reduced to

\[
\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{1}{1 - \sigma} + \lambda_t \hat{H}(\theta_t^*) - \lambda_{t+1} \hat{J}(\theta_t^*) = 0,
\]

which can be further simplified according to Lemma 1 and Lemma 2 as

\[
\hat{W}(\theta_t^*) + \lambda_t \hat{H}(\theta_t^*) = \lambda_{t+1} \hat{J}(\theta_t^*),
\]

(61)

where \( \hat{H}(\theta_t^*), \hat{J}(\theta_t^*), \) and \( \hat{W}(\theta_t^*) \) are defined, respectively, as

\[
\hat{H}(\theta_t^*) \equiv \frac{1 - \sigma}{\sigma} \frac{L(\theta_t^*)}{D(\theta_t^*)} + \frac{F(\theta_t^*)}{X(\theta_t^*)},
\]

(62)

\[
\hat{J}(\theta_t^*) \equiv \frac{1 - \sigma}{\sigma} \frac{1}{D(\theta_t^*)} + \frac{F(\theta_t^*)}{X(\theta_t^*)},
\]

(63)

\[
\hat{W}(\theta_t^*) \equiv \frac{1}{\sigma} \frac{M(\theta_t^*)}{D(\theta_t^*) \theta_t^*} - \frac{1}{\sigma} \geq 0.
\]

(64)

Note that \( \hat{W}(\theta_t^*) > 0 \) for all \( \theta_t^* \in [\theta_L, \theta_H] \) and \( \sigma \in (0, \infty) \). Also, for \( \theta_t^* = \theta_H, \hat{W}(\theta_t^*) = 0 \) and \( \hat{H}(\theta_t^*) = \hat{J}(\theta_t^*) \).

In what follows, we first sketch the proof that a Ramsey steady state featuring \( \theta^* = \theta_H \) exists if (1) \( \beta \) is sufficiently large for \( \sigma \geq 1 \) and (2) \( \beta \) is sufficiently large and government spending is sufficiently small if \( \sigma < 1 \). We proceed by the following steps, which show that the conjecture \( \theta^* = \theta_H \) satisfies all of the Ramsey FOCs and these FOC-implied steady-state values of the aggregate allocation \( \{C, N, K, B\} \) as well as Lagrangian multipliers \( \{\lambda, \mu\} \) are unique, mutually consistent, strictly positive, and finitely-valued:

1. The FOC with respect to \( \theta_t^* \) in equation (34) is satisfied at \( \theta_t^* = \theta_H \). Specifically,
plugging $\theta^* = \theta_H$ into the definitions of $\widehat{J}(\theta_H^*)$, $\widehat{H}(\theta_H^*)$, and $\widehat{W}(\theta_H^*)$ leads to $\widehat{J}(\theta_H) = \widehat{H}(\theta_H) \neq 0$ and $\widehat{W}(\theta_H) = 0$. Therefore the FOC (34) is satisfied.

2. The FOC with respect to $K$ in equation (30) gives

$$1 = \beta (MP_K + 1 - \delta),$$

which implies $MP_K = \alpha \left( \frac{K}{N} \right)^{\alpha-1} = \frac{1 - \beta (1 - \delta)}{\beta \alpha} \in (0, \infty)$ (i.e., the capital-to-labor ratio is unique, strictly positive, and bounded). Given the assumption of the production function, it must be true that the following ratios are unique, strictly positive, and finite: \( \{ K/N, Y/K, MP_N, Y/N \} \in (0, \infty) \). More specifically, the $Y/N$ and $Y/K$ ratios can be expressed, respectively, as

$$\frac{Y}{N} = \left( \frac{K}{N} \right)^{\alpha} = \left( \frac{1 - \beta (1 - \delta)}{\beta \alpha} \right)^{\frac{\alpha}{\alpha-1}} \quad (65)$$

and

$$\frac{Y}{K} = \left( \frac{K}{N} \right)^{\alpha-1} = \frac{1 - \beta (1 - \delta)}{\alpha \beta}. \quad (66)$$

3. The resource constraint,

$$F(N, K) = K^\alpha N^{1-\alpha} = C + \delta K + G,$$

together with a finite level of government spending $G$ implies a unique ratio $C/K \in (0, \infty)$ as

$$\frac{C}{Y} = \left( \frac{1 - \delta K}{Y} - \frac{G}{Y} \right) = \left( 1 - \frac{\alpha \beta \delta}{1 - \beta (1 - \delta)} \right) - \frac{G}{Y}, \quad (67)$$

where the last equality uses (66).

4. We know that under our parameter restrictions the level of labor is interior, $N \in (0, \overline{N})$; hence, it must be true that the aggregate allocation is also unique and interior: \( \{ C, K, Y \} \in (0, \infty) \).

5. Next, we show that \( \{ \mu, \lambda \} \in (0, \infty) \) and that these steady-state values are unique. Given $\theta^* = \theta_H$ in the steady state, the FOCs (31) and (32) become

$$1 + \lambda = \mu MP_N \quad (68)$$
and
\[ \mu C^\sigma D(\theta_H)^{-\sigma} \theta_H^{-1} = 1 + \lambda (1 - \sigma), \quad (69) \]
respectively. These two equations uniquely solve for \{\lambda, \mu\}. Note that \( \mu \in (0, \infty) \) if \( \lambda \in (0, \infty) \) according to (68). These two equations above imply
\[ 1 + \lambda = Q \left( 1 + (1 - \sigma) \lambda \right), \]
or equivalently,
\[ \lambda = \frac{Q - 1}{1 + Q(\sigma - 1)}, \]
where \( Q = \frac{M P_N D(\theta_H)\sigma \theta_H}{C^\sigma} \in (0, \infty) \) is unique. To show that \( \lambda > 0 \), we will discuss two subcases:

(a) For \( \sigma \geq 1 \), it is straightforward to verify that \( \lambda > 0 \) if and only if \( Q > 1 \).
(b) For \( 1 > \sigma > 0 \), the condition \( \lambda > 0 \) can be ensured if \( Q \in (1, \frac{1}{1 - \sigma}) \).

Therefore, for \( \{\mu, \lambda\} \in (0, \infty) \), we only need to find parameter conditions to ensure (i) \( Q > 1 \) when \( \sigma \geq 1 \) and (ii) \( Q \in (1, \frac{1}{1 - \sigma}) \) when \( 1 > \sigma > 0 \).

6. What is \( Q \)? Note that the steady-state version of the implementability condition (22) with \( \theta_t^* = \theta_H \) becomes
\[ N = C^{1 - \sigma} D(\theta_H)^{\sigma - 1} \theta_H \left( 1 - \frac{1 - D(\theta_H)}{\beta} \right), \]
which together with equations (65) and (67) imply
\[ C^\sigma = \left( 1 - \frac{\beta \alpha \delta}{1 - \beta (1 - \delta)} - \frac{G}{Y} \right) \left( 1 - \frac{\beta (1 - \delta)}{\beta \alpha} \right)^{\frac{\alpha}{\beta}} D(\theta_H)^{\sigma - 1} \theta_H \left( 1 - \frac{1 - D(\theta_H)}{\beta} \right). \quad (70) \]

Thus, by equations (70) and (65), \( Q \) can be rewritten as
\[ Q = \frac{(1 - \alpha) \frac{Y}{N} D(\theta_H)\sigma \theta_H}{C^\sigma} \]
\[ = \frac{1 - \alpha}{\left( 1 - \frac{\beta \alpha \delta}{1 - \beta (1 - \delta)} - \frac{G}{Y} \right) \left( 1 - \frac{1 - D(\theta_H)}{\beta} \right)}. \quad (71) \]
Now we are ready to find parameter conditions to ensure (i) $Q > 1$ when $\sigma \geq 1$ and (ii) $Q \in (1, \frac{1}{1-\sigma})$ when $0 < \sigma < 1$.

(a) Consider the case with $\sigma \geq 1$ first. Since the second term \( \frac{D(\theta_H)}{1 - D(\theta_H)} > 1 \) if $\beta > 1 - D(\theta_H)$, a sufficient condition for $Q > 1$ is that for a sufficiently large $\beta$ the first term \( \frac{1 - \alpha}{1 - \beta(1 - \beta(1 - \delta))} \) is no less than 1, which is clearly true because 
\[
\lim_{\beta \to 1} \left( \frac{1 - \alpha}{1 - \beta(1 - \beta(1 - \delta))} \right) = \frac{1 - \alpha}{1 - \delta} \geq 1.
\]
In addition, notice that the second term is independent of $\alpha$, and the first term increases as $\alpha$ decreases and approaches the limit 
\[
\lim_{\alpha \to 0} \left( \frac{1 - \alpha}{1 - \beta(1 - \beta(1 - \delta))} \right) = \frac{1}{1 - \delta} \geq 1.
\]
So the set of sufficient conditions for $Q > 1$ is easy to satisfy as long as $\beta > 1 - D(\theta_H)$.

(b) When $0 < \sigma < 1$, $\lambda > 0$ requires an additional condition that $Q < \frac{1}{1-\sigma}$, which can be rewritten as 
\[
\frac{1 - \alpha}{1 - \beta(1 - \beta(1 - \delta))} - \frac{G}{Y} < \frac{1}{1 - \sigma} - \frac{\beta - (1 - D(\theta_H))}{\beta D(\theta_H)}.
\] (72)

The above condition is satisfied if (i) $G$ is sufficiently small and (ii) $\beta$ is sufficiently close to 1. To see this, the LHS of condition (72) converges to 1 as $G \to 0$ and $\beta \to 1$. The RHS of (72) is larger than 1 if $\beta$ is sufficiently close to 1. Therefore, even when $\sigma \in (0, 1)$, there can still exist a unique Ramsey steady state if $\beta$ is sufficiently large and $G$ is sufficiently small. These parameter restrictions suggest that when the utility function of consumption is sufficiently flat (or risk neutral), the Ramsey planner can still improve social welfare by issuing plenty of bonds so long as capital is sufficiently durable and households are sufficiently patient.

7. Notice that such required values for $\beta$ (and $\delta$) discussed above do not contradict the competitive-equilibrium condition (17) for an interior cutoff and the condition for an interior $N \in (0, \bar{N})$ because these interior conditions hold true for all $\{\beta, \delta, \alpha\} \in (0, 1)$ as long as $\theta_H$ is sufficiently large relative to $E(\theta)$ (such that a competitive equilibrium does not feature full self-insurance).

8. Given $\lambda > 0$, it is then easy to see that $\mu > 0$ from condition (68).

9. It remains to show that the optimal quantity of government debt $B$ is strictly positive and unique. Given that aggregate output $Y \in (0, \infty)$, it suffices to show that the
The debt-to-GDP ratio \(\frac{B}{Y}\) is \((0, \infty)\) and is unique. Since \(K + B = \left(\frac{1}{D(\theta^*)} - 1\right)C\), we have

\[
\frac{B}{Y} = \left(\frac{1}{D(\theta_H)} - 1\right)\frac{C}{Y} - \frac{K}{Y}
= \left(\frac{1}{D(\theta_H)} - 1\right)\left(1 - \frac{\delta\beta\alpha}{1 - \beta(1 - \delta)} - \frac{G}{Y}\right) - \left(\frac{\beta\alpha}{1 - \beta(1 - \delta)}\right),
\]

which is finite and unique because each term in the parentheses is finite and unique. The debt-to-GDP ratio is also strictly positive provided that the output elasticity of capital \(\alpha\) is small enough. To see this, noting that the first term \(\left(\frac{1}{D(\theta_H)} - 1\right) > 0\), the second term \(\left(1 - \frac{\delta\beta\alpha}{1 - \beta(1 - \delta)} - \frac{G}{Y}\right) > 0\) because it is the consumption-to-output ratio \(\frac{C}{Y}\), and the last term vanishes as \(\alpha\) approaches zero. Notice that such a restriction on the value for \(\alpha\) does not contradict the competitive-equilibrium condition (17) for interior cutoff and the conditions for interior \(N \in (0, \infty)\).

10. Finally, by Proposition 4, since \(\phi = 0\) and \(L(\theta^*) = \frac{\mu}{\mu - \phi} = 1\), it must be true that \(\theta^* = \theta_H\) and \(\tau_k = 1 - \frac{L(\theta^*)^{-1} - \beta(1 - \delta)}{L(\theta^*)^{-1} - \beta(1 - \delta)} = 0\) in a Ramsey steady state. The steady-state equilibrium interest rate is therefore \(1/\beta\) by the Euler equation (12). In addition, the labor tax rate is determined by equation (8), which implies \(\tau_n = \frac{\lambda}{1 + \lambda}\sigma\) in the steady state. Namely,

\[
\tau_n = 1 - \frac{1}{L(\theta^*)\theta_H D(\theta^*)^{\sigma}} \frac{C^{\sigma}}{MP_N} = \frac{\lambda}{1 + \lambda}\sigma,
\]

where the last equality uses equations (68) and (69). Therefore, government expenditures and bond interest payments are financed solely by labor tax income in the Ramsey steady state.

This above steps finish the proof for the existence of a Ramsey steady state at the corner \(\theta^* = \theta_H\). This steady state is also shown to have a unique allocation \(\{C, N, Y, K, B\} \in (0, \infty)\) and a unique mix of tax policies \(\{\tau_k, \tau_n\} = \{0, \frac{\lambda}{1 + \lambda}\sigma\} \in [0, 1]\).

However, to show that this is the only Ramsey steady state, or there do not exist other Ramsey steady states with the cutoff \(\theta^* \in [\theta_L, \theta_H]\), we need the following steps.

We first show that when \(\phi = 0\), there is no Ramsey steady state for \(\theta^* \in (\theta_L, \theta_H)\). Rewrite (61) as

\[
\lambda_{t+1} = \rho(\theta^*_t)\lambda_t + \varepsilon(\theta^*_t),
\]

where \(\rho(\theta^*_t) \equiv \tilde{H}(\theta^*_t)/\tilde{J}(\theta^*_t)\) and \(\varepsilon(\theta^*_t) \equiv \tilde{W}(\theta^*_t)/\tilde{J}(\theta^*_t)\). The sign and value of \(\rho(\theta^*_t)\) and \(\varepsilon(\theta^*_t)\) depend on the value of \(\sigma\); hence, we can discuss the possible steady states according to the
values of $\sigma \lessgtr 1$.

1. $\sigma < 1$. Then $0 < \tilde{J}(\theta^*_t) < \tilde{H}(\theta^*_t)$ according to Lemma 2. Hence, $\rho(\theta^*_t) > 1$ and $\varepsilon(\theta^*_t) > 0$. In this case, if $\theta^* \in (\theta_L, \theta_H)$, then the steady-state $\lambda = \frac{\varepsilon(\theta^*)}{1-\rho(\theta^*)}$ must be negative, which violates the FOC (31) in the steady state. Hence, an interior solution for the cutoff $\theta^* \in (\theta_L, \theta_H)$ cannot constitute a Ramsey steady state.

2. $\sigma = 1$. Then $0 < \tilde{J}(\theta^*_t) = \tilde{H}(\theta^*_t)$ according to Lemma 2. Hence, $\rho(\theta^*_t) = 1$ and $\varepsilon(\theta^*_t) > 0$. In this case $\lambda_t$ does not converge and goes to infinity in the long run. A divergent $\lambda_t$ implies that $\mu_t$ must also diverge to infinity according to FOC (31). On the other hand, when $\sigma = 1$, the FOC (32) implies that $\mu_t$ must be finite and positive in the steady state. This leads to a contradiction and hence cannot be a Ramsey steady state.

3. $\sigma > 1$. Then Lemma 2 implies that $\tilde{J}(\theta^*_t) > \tilde{H}(\theta^*_t)$. The value and sign of $\rho(\theta^*_t)$ and $\varepsilon(\theta^*_t)$ then depend on the sign of $\tilde{H}(\theta^*_t)$ and $\tilde{J}(\theta^*_t)$; so we discuss three subcases below.

   (a) $\tilde{H}(\theta^*_t) > 0$. Then $\tilde{J}(\theta^*_t) > 0$; which implies $0 < \rho(\theta^*_t) < 1$ and $\varepsilon(\theta^*_t) > 0$. The steady-state $\lambda$ is then equal to $\varepsilon(\theta^*)/(1 - \rho(\theta^*))$. We next check if this $\lambda$ satisfies the FOC (32), which in the steady state can be expressed as

   \[
   \mu C^\sigma D(\theta^*)^{1-\sigma} = M(\theta^*) + \lambda (1 - \sigma)\theta^* (L(\theta^*) - 1 + D(\theta^*)) = M(\theta^*)(1 + \lambda(1 - \sigma)),
   \]

   where the last equality holds according to Lemma 4. Plugging the steady-state $\lambda$ into equation (74) and using the definitions of $\tilde{J}(\theta^*_t)$, $\tilde{H}(\theta^*_t)$, and $\tilde{W}_\theta(\theta^*_t)$ give

   \[
   \mu C^\sigma D(\theta^*)^{1-\sigma} = M(\theta^*) \left(1 + \frac{M(\theta^*) - D(\theta^*)}{1 - L(\theta^*)}\right) = M(\theta^*)(1 + \frac{L(\theta^*) - 1}{1 - L(\theta^*)}) = 0,
   \]

   where the second equality uses the Lemma 4 again. This equation holds if and only if $\mu = 0$. This cannot be a steady state. Note that $\mu = 0$ as long as $\lambda = \varepsilon(\theta^*)/(1 - \rho(\theta^*)) > 0$, regardless of the signs of $\rho(\theta^*)$ and $\varepsilon(\theta^*_t)$.

   (b) $\tilde{H}(\theta^*_t) < 0$ and $\tilde{J}(\theta^*_t) > 0$. Then $\varepsilon(\theta^*_t) > 0$. There are two possibilities regarding $\rho(\theta^*_t)$: $|\rho(\theta^*_t)| < 1$ or $|\rho(\theta^*_t)| > 1$. But in both cases, the steady-state $\lambda = \varepsilon(\theta^*)/(1 - \rho(\theta^*)) > 0$. Such a steady-state $\lambda$ implies $\mu = 0$ according to (74) regardless of
the sign of \( \rho(\theta^*) \), as shown in the proof of the previous subcase. Therefore, this subcase cannot be a Ramsey steady-state equilibrium.

(c) \( \hat{H}(\theta^*) < 0 \) and \( \hat{J}(\theta^*) < 0 \). Then \( \rho(\theta^*) > 1 \) and \( \varepsilon(\theta^*) < 0 \), and the steady-state \( \lambda = \varepsilon(\theta^*)/(1 - \rho(\theta^*)) \), which again leads to \( \mu = 0 \) by equation (74). Hence, this subcase cannot be a Ramsey steady state equilibrium.

Finally, we show that the case of \( \theta^* \) cannot be a Ramsey equilibrium although the necessary FOC with respect to \( \theta^* \) is satisfied. The reason is that the first term of the Ramsey objective function (27), \( W(\theta^*)C_1^{1-\sigma}C_1/(1-\sigma) \), is monotonically increasing in \( \theta^* \in (\theta_L, \theta_H) \). Hence, for a global maximum, a cutoff \( \theta^*_t \) at its lower corner cannot be a Ramsey equilibrium.

To ensure that \( n \in (0, N) \) (see Proposition 1), note that we have assumed \( \theta_H < \frac{\theta_L}{(1-\beta)^\sigma} \), which ensures that the minimum individual labor input remains positive, as shown in Appendix A.1. Moreover, by equation (49), the maximum value of \( n \) is less than \( N \) if \( N > \theta_H \) in this case.

In addition, we can show that the maximum individual asset demand remains finite in the Ramsey steady state even if the risk-free rate is equal to the time discount rate, \( r = 1/\beta \). Since \( \theta_H < \frac{\theta_L}{(1-\beta)^\sigma} \), we have

\[
x_t = \frac{C_t}{D(\theta_H)} = \frac{C_t}{E(\theta_1^{1/\sigma}) \theta_H^{1/\sigma}} < \infty.
\]

Given the finite value of \( x_t \), the individual asset holding \( a_{t+1} \) is determined by the size of the idiosyncratic shock \( \theta_t \), and the agents with the largest asset holdings are those who receive the smallest shock \( \theta_t = \theta_L \), i.e.,

\[
a_{t+1}(\theta_L) = \left[ 1 - \left( \frac{\theta_L}{\theta_H} \right)^{1/\sigma} \right] x_t,
\]

which is strictly positive and finite.

A.7 Proof of Corollary 1

Assume that \( \delta = 1 \) and \( G = 0 \), then the steady-state resource constraint is \( Y = C + K \). In the GRA Ramsey steady state, the MGR holds and hence \( 1 = \beta MPK = \beta \frac{Y}{K} \), which together with the resource constraint imply \( \frac{K}{\theta} = \alpha \beta \) and \( \frac{C}{\theta} = 1 - \alpha \beta \). As a result, the optimal
steady-state debt-to-GDP ratio can be inferred by the asset-market clearing condition:

\[
\frac{B}{Y} = (\frac{1}{D} - 1)\frac{C}{Y} - K = (\frac{1}{D} - 1)(1 - \alpha\beta) - \alpha\beta
\]

\[
= \alpha\beta \left[ \frac{1 - D(1 - \alpha\beta)}{D} - 1 \right],
\]

which is equation (38) by the definition of \( \tau_b \).

A.8 Proof of Proposition 6

Here we prove that a Ramsey steady state featuring a binding debt limit exists if \( \beta \) is sufficiently large and \( \delta \) is sufficiently small. We proceed by the following steps, which show that the constructed steady state—characterized by \( \phi > 0 \) and \( \theta^* \in (\theta_L, \theta_H) \)—satisfies all of the Ramsey FOCs and these FOC-implied values of the steady-state aggregate allocation \( \{C, N, K, \theta^*, B\} \) and Lagrangian multipliers \( \{\lambda, \mu\} \) are mutually consistent, strictly positive, and finitely-valued.

1. We first show that any possible Ramsey steady state satisfying the Ramsey FOC with respect to \( \theta^*_t \) must have the following property:

\[
\frac{\mu}{\mu - \phi} = L(\theta^*_t),
\]

(75)

where the liquidity premium \( L(\theta^*_t) \) equals 1 if and only if \( \phi = 0 \), and is greater than 1 only if \( \phi > 0 \).

Starting from equation (33), \( \nu^B_t > 0 \) implies that \( \phi_t > 0 \). The FOC with respect to \( \theta^*_t \) can be rewritten as

\[
\hat{W}(\theta^*_t) - \phi_tC_t^s \hat{Z}(\theta^*_t) + \lambda_t\hat{H}(\theta^*_t) = \lambda_{t+1}\hat{J}(\theta^*_t),
\]

(76)

where \( \hat{H}(\theta^*_t), \hat{J}(\theta^*_t), \) and \( \hat{W}(\theta^*_t) \) are defined as before, and \( \hat{Z}(\theta^*_t) \) is defined as

\[
\hat{Z}(\theta^*_t) \equiv \frac{D(\theta^*_t)^{-1-\sigma}}{\sigma \theta^*_t},
\]

(77)

which is positive for any \( \theta^*_t \in [\theta_L, \theta_H] \). Rewrite (76) as

\[
\lambda_{t+1} = \rho(\theta^*_t)\lambda_t + \hat{\varepsilon}(\theta^*_t),
\]

(61)
where \( \rho(\theta^*_t) \equiv \hat{H}(\theta^*_t)/\hat{J}(\theta^*_t) \) and
\[
\hat{\varepsilon}(\theta^*_t) \equiv \frac{\hat{W}(\theta^*_t) - \phi_t C^\sigma_t \hat{Z}(\theta^*_t)}{\hat{J}(\theta^*_t)}.
\]
(78)
The existence of a finitely positive steady-state value of \( \lambda \) depends on the sign and value of \( \rho(\theta^*_t) \) and \( \hat{\varepsilon}(\theta^*_t) \), which in turn depend on the value of \( \sigma \). Hence we discuss the following three cases of \( \sigma \leq 1 \):

(a) \( \sigma < 1 \). Then \( 0 < \hat{J}(\theta^*_t) < \hat{H}(\theta^*_t) \) according to Lemma 2, and \( \rho(\theta^*_t) > 1 \). In this case \( \lambda_t \) converges to a positive steady-state value if and only if \( \hat{\varepsilon}(\theta^*_t) < 0 \); namely,
\[
\lambda = \frac{\hat{\varepsilon}(\theta^*)}{1 - \rho(\theta^*)} > 0
\]
if and only if \( \hat{\varepsilon}(\theta^*) < 0 \). We will check later on that \( \hat{\varepsilon}(\theta^*) < 0 \) indeed constitutes a Ramsey steady state with \( \sigma \in (0,1) \), \( \theta^* < \theta_H \), and \( \{\lambda, \mu, \phi\} > 0 \). According to FOC (32) and Lemma 4, the steady-state value of \( \mu \) is given by
\[
\mu = C^{-\sigma} D(\theta^*)^{-1} M(\theta^*) (1 + \lambda (1 - \sigma)) - \phi \frac{1 - D(\theta^*)}{D(\theta^*)},
\]
or equivalently,
\[
\frac{C^\sigma}{D(\theta^*)^\sigma} = \frac{D(\theta^*)^{-1} M(\theta^*) (1 + \lambda (1 - \sigma))}{\mu + \phi \frac{1 - D(\theta^*)}{D(\theta^*)}}.
\]
(81)
Plugging the steady-state \( \lambda \) into equation (80) and using the definitions of \( \hat{J}(\theta^*_t) \), \( \hat{H}(\theta^*_t) \), \( \hat{W}(\theta^*_t) \), and \( \hat{Z}(\theta^*_t) \) as well as Lemma 4, gives
\[
\mu = \phi \frac{M(\theta^*)}{\theta^*} \frac{D(\theta^*)^{-1}}{L(\theta^*)} - \phi \frac{1 - D(\theta^*)}{D(\theta^*)} = \phi \frac{L(\theta^*)}{L(\theta^*) - 1},
\]
where the last equality uses the Lemma 4 again. This equation is exactly identical to (75) and the mapping is unique because \( L(\theta^*) \) is strictly monotone.

(b) \( \sigma = 1 \). Then \( 0 < \hat{J}(\theta^*_t) = \hat{H}(\theta^*_t) \) according to Lemma 2, and \( \rho(\theta^*_t) = 1 \). The steady-state \( \lambda \) is a finitely positive value (and unique) if and only if \( \hat{\varepsilon}(\theta^*) = 0 \). We will show later on that \( \hat{\varepsilon}(\theta^*) = 0 \) is indeed consistent with an interior Ramsey steady state with \( \sigma = 1 \), \( \theta^* \in (\theta_L, \theta_H) \), and \( \{\lambda, \mu, \phi\} > 0 \). First, given \( \hat{\varepsilon}(\theta^*) = 0 \),
Therefore, any possible steady state satisfying the FOC with respect to \( \frac{\mu}{\sigma} \) can be solved by equation (76), which is

\[
\phi = C^{-1} \frac{\bar{W}(\theta^*)}{\bar{Z}(\theta^*)} = C^{-1} D(\theta^*) \theta^* (L(\theta^*) - 1).
\]

Note that \( \phi > 0 \) if and only if \( L(\theta^*) > 1 \), or equivalently, \( \theta^* < \theta_H \). According to FOC (32) and lemma 4, the steady-state value of \( \mu \) at \( \sigma = 1 \) is given by

\[
\mu = C^{-1} M(\theta^*) - \phi D(\theta^*)^{-1} (1 - D(\theta^*)) = C^{-1} \theta^* D(\theta^*) L(\theta^*) > 0,
\]

where the second equality holds according to Lemma 4 and the equation \( \phi = C^{-1} D(\theta^*) \theta^* (L(\theta^*) - 1) \) solved above. As a result, the value of \( \mu/(\mu - \phi) \) is uniquely given by

\[
\frac{\mu}{\mu - \phi} = \frac{C^{-1} \theta^* D(\theta^*) L(\theta^*)}{C^{-1} \theta^* D(\theta^*) L(\theta^*) - C^{-1} D(\theta^*) \theta^* (L(\theta^*) - 1)} = L(\theta^*),
\]

which is identical to (75).

(c) \( \sigma > 1 \). Then Lemma 2 implies that \( \tilde{J}(\theta^*_1) > \tilde{H}(\theta^*_1) \). The value and sign of \( \rho(\theta^*_t) \) and \( \tilde{\varepsilon}(\theta^*_t) \) depend on the sign of \( \tilde{H}(\theta^*_t) \) and \( \tilde{J}(\theta^*_t) \); there are three possible subcases, which are discussed below:

i. \( \tilde{H}(\theta^*_t) > 0 \) and \( \tilde{J}(\theta^*_t) > 0 \). Then \( 0 < \rho(\theta^*_t) < 1 \). The steady-state \( \lambda = \tilde{\varepsilon}(\theta^*)/(1 - \rho(\theta^*)) > 0 \) if and only if \( \tilde{\varepsilon}(\theta^*) > 0 \).

ii. \( \tilde{H}(\theta^*_t) < 0 \) and \( \tilde{J}(\theta^*_t) > 0 \). In this case \( \rho(\theta^*) < 0 \). Then there are two possibilities regarding to \( \rho(\theta^*_t) : |\rho(\theta^*_t)| < 1 \) or \( |\rho(\theta^*_t)| > 1 \). In either case we can show that \( \lambda = \tilde{\varepsilon}(\theta^*)/(1 - \rho(\theta^*)) > 0 \) if and only if \( \tilde{\varepsilon}(\theta^*) > 0 \).

iii. \( \tilde{H}(\theta^*_t) < 0 \) and \( \tilde{J}(\theta^*_t) < 0 \). Then \( \rho(\theta^*_t) > 1 \). For a Ramsey steady state with \( \lambda = \tilde{\varepsilon}(\theta^*)/(1 - \rho(\theta^*)) > 0 \) to exist, it requires \( \tilde{\varepsilon}(\theta^*) < 0 \).

Since all of the possible values of \( \lambda \) in the case of \( \sigma > 1 \) are analogous to those in the case of \( \sigma < 1 \), following the proof in the \( \sigma < 1 \) case, we can show that \( \mu/(\mu - \phi) = L(\theta^*) \) also holds.

Therefore, any possible steady state satisfying the FOC with respect to \( \theta^*_t \) must imply \( \mu/(\mu - \phi) = L(\theta^*) \). We will verify at the end that the required conditions for (the
sign of) \( \hat{\epsilon}(\theta^*) \) stated above are all automatically met once the condition for \( \lambda > 0 \) is satisfied.

2. The FOC with respect to \( K \) in (30) together with equation (75) give

\[
1 - \frac{\phi}{\mu} = \beta (MP_K + 1 - \delta) = \frac{1}{L(\theta^*)} < 1,
\]

which suggests that \( MP_K \) is bounded above. In addition, if \( \delta \) is sufficiently small such that \( MP_K > \delta \), then \( MP_K \in (\delta, \frac{1-\beta(1-\delta)}{\beta}) \). Given the assumption of the production function, it must be true that the following ratios are strictly positive and finite: \( \{K/N, Y/K, MP_N, Y/N\} \in (0, \infty) \). First, the \( Y/K \) ratio is bounded above and below:

\[
\frac{\delta}{\alpha} < \frac{Y}{K} < \left( \frac{K}{N} \right)^{\alpha-1} < \frac{1 - \beta (1 - \delta)}{\alpha \beta}.
\] (84)

3. Then the resource constraint,

\[
Y = K^\alpha N^{1-\alpha} = C + \delta K + G,
\]

together with a finite level of government spending \( G \geq 0 \) implies a bounded \( C/Y \) ratio as

\[
\left(1 - \alpha - \frac{G}{Y}\right) < \frac{C}{Y} = \left(1 - \delta \frac{K}{Y} - \frac{G}{Y}\right) < \left(1 - \frac{\alpha \beta \delta}{1 - \beta (1 - \delta)} - \frac{G}{Y}\right),
\] (85)

where the last inequality uses equation (84).

4. We know that under our parameter restrictions the level of labor is interior, \( N \in (0, \bar{N}) \); hence, it must be true that the aggregate allocation is also interior: \( \{C, K, Y\} \in (0, \infty) \).

5. Next, we show that \( \{\mu, \lambda\} \in (0, \infty) \). Plugging equation (75) into the steady-state FOC (32) gives

\[
\mu = \frac{M(\theta^*)D(\theta^*)^{\sigma-1}C^{-\sigma}[1 + \lambda(1 - \sigma)]}{1 + (1 - L(\theta^*)^{-1})(D(\theta^*)^{-1} - 1)},
\] (86)

where we have used Lemma 4 and the definition of \( W(\theta^*) \), which implies \( W(\theta^*) = M(\theta^*)D(\theta^*)^{\sigma-1} \). The above equation together with the steady-state FOC (31) imply the mapping,

\[
\lambda = \frac{Q - 1}{1 + Q(\sigma - 1)},
\] (87)
where
\[
Q = \frac{MP_NM(\theta^*)D(\theta^*)^{\sigma-1}L(\theta^*)D(\theta^*)}{C^\sigma [L(\theta^*) - 1 + D(\theta^*)]} \in (0, \infty).
\]

Depending on the value of \(\sigma\), we discuss two subcases below:

(a) When \(\sigma \geq 1\), it is straightforward to verify that \(\lambda > 0\) if and only if \(Q > 1\).

(b) When \(1 > \sigma > 0\), then \(\lambda > 0\) can be ensured if \(Q \in (1, \frac{1}{1-\sigma})\).

Therefore, for \(\{\mu, \lambda\} \in (0, \infty)\), we need (i) \(Q > 1\) if \(\sigma \geq 1\) and (ii) \(Q \in (1, \frac{1}{1-\sigma})\) if \(1 > \sigma > 0\).

6. To derive these conditions, note that in the steady state the implementability condition (22) becomes
\[
\frac{N}{Y} = \frac{C}{Y} C^{-\sigma} D(\theta^*)^{\sigma-1} \theta^* \left( L(\theta^*) - \frac{1 - D(\theta^*)}{\beta} \right),
\]
and hence \(C^\sigma\) can be expressed as
\[
C^\sigma = \left( \frac{C}{Y} \right) \left( \frac{Y}{N} \right) D(\theta^*)^{\sigma-1} \theta^* \left( L(\theta^*) - \frac{1 - D(\theta^*)}{\beta} \right).
\]

By using equation (88), Lemma 4, and \(MP_N = (1 - \alpha) \left( \frac{Y}{N} \right)\), we can express \(Q\) as:
\[
Q = \frac{(1 - \alpha)L(\theta^*)D(\theta^*)}{\left( \frac{C}{Y} \right) \left( L(\theta^*) - \frac{1 - D(\theta^*)}{\beta} \right)},
\]
which together with equation (85) imply an upper and lower bound for \(Q\):
\[
\frac{1 - \alpha}{\left( 1 - \frac{\alpha \beta \delta}{1 - \beta (1 - \delta) - \frac{C}{Y}} \right) \left( L(\theta^*) - \frac{1 - D(\theta^*)}{\beta} \right)} < Q < \frac{L(\theta^*)D(\theta^*)}{\left( 1 - \alpha - \frac{Q}{N} \right) \left( L(\theta^*) - \frac{1 - D(\theta^*)}{\beta} \right)}.
\]

Now consider the following subcases:

(a) When \(\sigma \geq 1\), we need \(Q > 1\) to ensure \(\lambda > 0\). Since \(L(\theta^*) > 1\), the term \(\frac{L(\theta^*)D(\theta^*)}{L(\theta^*) - \frac{1 - D(\theta^*)}{\beta}} > 1\) if \(\beta L(\theta^*) < 1\). Notice that \(\beta L(\theta^*) < 1\) can be ensured by a sufficiently small \(\delta\) according to equation (83). Consequently, \(Q > 1\) would be
ensured if the first term \( \frac{1-\alpha}{1-\beta(1-\alpha)} \geq 1 \) is clearly true if \( \beta \) is sufficiently large because in the limit we have

\[
\lim_{\beta \to 1} \frac{1-\alpha}{1-\beta(1-\alpha)} = \frac{1-\alpha}{1-\alpha - \frac{G}{Y}} \geq 1.
\]

(b) When \( 0 < \sigma < 1, \lambda > 0 \) requires an additional condition that \( Q < \frac{1}{1-\sigma} \). From equation (89), the following condition ensures \( Q < \frac{1}{1-\sigma} \):

\[
\left(1 - \alpha - \frac{G}{Y}\right) \frac{1}{L(\theta^*)} > \frac{(1-\sigma)(1-\alpha)D(\theta^*)}{\left(L(\theta^*) - \frac{1-D(\theta^*)}{\beta}\right)},
\]

which together with equation (83) implies

\[
\frac{1}{(1-\sigma)} \frac{1-\alpha - \frac{G}{Y}}{(1-\alpha)} \beta (MP_K + 1 - \delta) > \frac{D(\theta^*)}{\left(L(\theta^*) - \frac{1-D(\theta^*)}{\beta}\right)}.
\]

Notice that the RHS of equation (90) is close to but less than 1 if \( \beta \) is sufficiently close to 1, as shown by the following inequality:

\[
\lim_{\beta \to 1} \frac{D(\theta^*)}{L(\theta^*) - \frac{1-D(\theta^*)}{\beta}} = \frac{D(\theta^*)}{L(\theta^*) - 1 + D(\theta^*)} < 1.
\]

In addition, if \( G \) is sufficiently small, then the LHS of equation (90) is larger than 1 when \( \beta \) is sufficiently close to 1, because \( MP_K > \delta \) and \( \frac{1}{1-\sigma} > 1 \). Therefore, even when \( \sigma \in (0,1) \), there can still exist a Ramsey steady state if \( \beta \) is sufficiently large, \( \delta \) is sufficiently small, and \( G \) is sufficiently small. These parameter restrictions suggest that when the utility function of consumption is sufficiently flat (or risk neutral), the Ramsey planner can still improve social welfare by issuing a limited amount of bonds so long as capital is sufficiently durable and households are sufficiently patient.

7. Notice that these requirements for \( \{\beta, \delta, \alpha\} \) do not contradict the competitive-equilibrium condition (17) for an interior cutoff and the condition for an interior \( N \in (0, \bar{N}) \) because these interior conditions hold true for all \( \{\beta, \delta, \alpha\} \in (0,1) \) as long as \( \theta_H \) is sufficiently large relative to \( E(\theta) \) (such that a competitive equilibrium does not feature full self-insurance). In fact, the existence of a Ramsey steady state under a binding debt limit
is also confirmed by our numerical analysis in Section 4.3.

8. Finally, we verify that the required conditions for (the sign of) \( \hat{\varepsilon} (\theta^*) \) stated above are all automatically met once the condition for \( \lambda > 0 \) is satisfied. Using equation (77) and (64) to substitute out \( \hat{Z} (\theta^*) \) and \( \hat{W} (\theta^*) \) in the definition of \( \hat{\varepsilon} (\theta^*) \), which is listed in equation (78), gives

\[
\hat{\varepsilon} (\theta^*) = \frac{\hat{W} (\theta^*) - \phi C^* \hat{Z} (\theta^*)}{\tilde{J} (\theta^*)} = \frac{1}{\tilde{J} (\theta^*)} \left( \frac{1}{\sigma} \frac{M (\theta^*)}{D (\theta^*)} \theta^* - \frac{1}{\sigma} - \hat{\phi} \frac{C^*}{\sigma \theta^* D^{1+\sigma}} \right)
\]

where the last equality uses equation (81) to substitute out \( \frac{C^*}{D(\theta^*)} \). Simplifying under Lemma 4 and equation (82), the above equation becomes

\[
\hat{\varepsilon} (\theta^*) = \frac{1}{\tilde{J} (\theta^*) \sigma} \left( \frac{L (\theta^*) - 1 + D (\theta^*)}{D (\theta^*)} - 1 - \frac{(L (\theta^*) - 1 + D (\theta^*)) D (\theta^*)^{-1} (1 + \lambda (1 - \sigma))}{L (\theta^*)^{-1} \frac{1 - D (\theta^*)}{D (\theta^*)}} \right)
\]

\[
\hat{\varepsilon} (\theta^*) = \frac{1}{\tilde{J} (\theta^*) \sigma} \left( \frac{L (\theta^*) - 1}{D (\theta^*)} - \frac{L (\theta^*) - 1 + D (\theta^*) (L (\theta^*) - 1) (1 + \lambda (1 - \sigma))}{L (\theta^*) - 1 + D (\theta^*)} \right)
\]

\[
\hat{\varepsilon} (\theta^*) = \frac{1}{\tilde{J} (\theta^*) \sigma} \frac{1}{D (\theta^*)} L (\theta^*) - 1 \lambda (\sigma - 1),
\]

where \( \lambda (\sigma - 1) \geq 0 \) if and only if \( \sigma \geq 1 \) (given that \( \lambda > 0 \)). Also note that \( \tilde{J} (\theta^*) > 0 \) when \( \sigma \leq 1 \), and \( \tilde{J} (\theta^*) < 0 \) only if \( \sigma > 1 \). Hence, under the parameter conditions for \( \lambda > 0 \), it must be true that (i) \( \hat{\varepsilon} (\theta^*) \leq 0 \) if and only if \( \sigma \leq 1 \), which are consistent with the stated conditions for \( \hat{\varepsilon} (\theta^*) \) under cases (1a) and (1b) discussed above; (ii) \( \hat{\varepsilon} (\theta^*) > 0 \) if and only if \( \sigma > 1 \) and \( \tilde{J} (\theta^*) > 0 \), which are consistent with the stated conditions for \( \hat{\varepsilon} (\theta^*) \) under the first two subcases in case (1c) discussed above; and (iii) \( \hat{\varepsilon} (\theta^*) < 0 \) if and only if \( \sigma > 1 \) and \( \tilde{J} (\theta^*) < 0 \), which are also consistent with the stated conditions for \( \hat{\varepsilon} (\theta^*) \) under the third subcase in case (1c) discussed above.

We further show that the following properties of the Ramsey steady state with a binding government debt limit hold true regardless of \( \sigma \):

1. IAE fails. Equation (83) implies that \( L(\theta^*) > 1 \) and hence \( \theta^* \) must be interior.

2. AAE fails. Equation (30) suggests that the MGR does not hold in the steady state.
3. Despite the failure of the MGR, the steady-state capital tax must be zero by Proposition 4, since 
\( \mu/(\mu - \phi) = L(\theta^*) \) regardless of the value of \( \sigma \in (0, \infty) \). The steady-state labor tax \( \tau_n \) is given by

\[
\tau_n = 1 - \frac{1}{L(\theta^*)\theta^*D(\theta^*)^\sigma} \frac{C^\sigma}{MP_N}.
\]

Finally, the condition (48) ensures \( n > 0 \) and the condition (49) ensures \( n < \overline{N} \) if \( \overline{N} > \theta^*L(\theta^*) \). Hence, \( n \in (0, \overline{N}) \) is guaranteed.