



ECONOMIC RESEARCH
FEDERAL RESERVE BANK OF ST. LOUIS
WORKING PAPER SERIES

Optimal Ramsey Taxation in Heterogeneous Agent Economies with Quasi-Linear Preferences

Authors	YiLi Chien, and Yi Wen
Working Paper Number	2019-007F
Revision Date	August 2021
Citable Link	https://doi.org/10.20955/wp.2019.007
Suggested Citation	Chien, Y., Wen, Y., 2021; Optimal Ramsey Taxation in Heterogeneous Agent Economies with Quasi-Linear Preferences, Federal Reserve Bank of St. Louis Working Paper 2019-007. URL https://doi.org/10.20955/wp.2019.007

Federal Reserve Bank of St. Louis, Research Division, P.O. Box 442, St. Louis, MO 63166

The views expressed in this paper are those of the author(s) and do not necessarily reflect the views of the Federal Reserve System, the Board of Governors, or the regional Federal Reserve Banks. Federal Reserve Bank of St. Louis Working Papers are preliminary materials circulated to stimulate discussion and critical comment.

Optimal Ramsey Taxation in Heterogeneous Agent Economies with Quasi-Linear Preferences*

YiLi Chien Yi Wen

July 26, 2021

Abstract

We build a tractable heterogeneous-agent incomplete-markets model with quasi-linear preferences to address a set of long-standing issues in the optimal Ramsey taxation literature. The tractability of our model enables us to analytically prove the existence of the Ramsey steady state and establish several novel results within standard parameter spaces: (i) The failure of the modified golden rule (MGR) cannot by itself justify a positive steady-state capital tax—we prove that in the absence of wealth-redistribution effects the optimal capital tax is exclusively zero in the Ramsey steady state regardless of the validity of the MGR. (ii) The optimal capital tax is positive only along the transition path, and it depends positively on the elasticity of intertemporal substitution. (iii) The optimal debt-to-GDP ratio, however, is determined by a positive wedge times the MGR saving rate. The key insight behind our results is that in the absence of any wealth-redistribution effects, taxing capital in the steady state cannot eliminate the liquidity premium—the primal friction in the model—but instead permanently erodes individuals’ buffer-stock savings and self-insurance position; thus, the Ramsey planner opts to issue debt rather than impose a steady-state capital tax to correct the capital-overaccumulation problem whenever the interest rate lies below the time discount rate. Also, the MGR fails to hold in a Ramsey equilibrium whenever the government encounters a binding debt limit; but even in this case the optimal long-run capital tax is zero. Therefore, if there is a reason to tax capital in the Ramsey steady state, it may have something to do with the tax’s effect on wealth redistribution rather than on the failure of the MGR due to capital overaccumulation.

JEL Classification: E13; E62; H21; H30

Key Words: Optimal Capital Taxation, Ramsey Problem, Incomplete Markets

*This is a significantly revised version of our earlier working paper (Chien and Wen, 2017) titled “Optimal Ramsey Capital Income Taxation—A Reappraisal.” We thank Manuel Amador, Macro Bassetto, V. V. Chari, Wei Cui, Jonathan Heathcote, Sagiri Kitao, Dirk Krueger, Tomoyuki Nakajima, Juan Pablo Nicolini, Yena Park, and participants at various seminars and conferences for useful comments. The views expressed here are only those of the individual authors and do not necessarily reflect the official positions of the Federal Reserve Bank of St. Louis, the Federal Reserve System, or the Board of Governors. Yili Chien, Federal Reserve Bank of St. Louis, yilichien@gmail.com. Yi Wen, Federal Reserve Bank of St. Louis, yi.wen.08.08.2008@gmail.com.

1 Introduction

The seminal work of Aiyagari (1995) has inspired a large literature. However, despite several important revisits, such as Chamley (2001), Conesa, Kitao, and Krueger (2009), Dávila, Hong, Krusell, and Ríos-Rull (2012), and many others, many issues regarding optimal capital taxation in a heterogeneous-agent incomplete-markets (HAIM) economy (à la Aiyagari (1994)) remain unsettled.

For example, is a positive tax levied on capital in the Ramsey steady state motivated mainly by the failure of the modified golden rule (MGR) in light of capital overaccumulation, by wealth redistribution in light of inequality under borrowing constraints, or both? How does optimal capital taxation relate to the optimal level of public debt and the existence of an interior Ramsey steady state?

These issues are intertwined because government bonds not only serve as a self-insurance device for individuals, they also alleviate the overaccumulation of the aggregate capital stock and thus reduce the need for a distortionary capital tax.¹ In the meantime, capital taxation (in addition to a labor tax) may be necessary in order to finance the interest payments on public debt and to redistribute wealth from the rich to the poor.

Moreover, to the best of our knowledge, the existence of a Ramsey steady state in infinite-horizon HAIM models is often *assumed* instead of proven in the bulk of the existing literature. Without such an assumption, the Ramsey allocation is hard to analyze because of these models' intractability; but, optimal tax policies drawn from the analyses may hinge critically on the validity of such an assumption.

The issue at stake is a trade-off under capital taxation between (i) aggregate allocative efficiency (in terms of the MGR) and (ii) individual allocative efficiency (in terms of self-insurance). A positive capital tax improves aggregate allocative efficiency by equalizing the premium-adjusted private marginal product of capital (MPK) and socially optimal MPK but worsens individual allocative efficiency by increasing the liquidity premium (or deteriorating individuals' self-insurance positions). In addition, since the interest rate lies strictly below the time discount rate—a hallmark feature of infinite-horizon HAIM models—it would seem optimal for the Ramsey planner to keep increasing the bond supply until all individuals are no longer borrowing constrained and become fully self-insured against idiosyncratic risk, at which point the MGR would be automatically restored.

Specifically, the competitive market equilibrium in a HAIM model may appear to be dy-

¹Aiyagari (1995) (pp. 1160-1161) acknowledged the importance of government debts in alleviating capital overaccumulation.

namically inefficient due to overaccumulation of capital under precautionary saving motives, which results in (i) inequality across households, (ii) a liquidity premium on the rate of return to household savings, and (iii) an excessively low aggregate MPK. Consequently, the MGR fails to hold in a competitive equilibrium. This observation provides the key intuition that the Ramsey planner should tax capital to restore aggregate allocative efficiency. However, a capital tax has not only a strong redistributive effect on individual wealth, but also a strong adverse effect on individuals' self-insurance positions. So why is a capital tax more desirable than a labor tax?

In other words, the conventional wisdom for relying solely on the MGR to justify a positive capital tax in an infinite-horizon HAIM model is counterintuitive, at least from a micro viewpoint. By taxing capital income and thus reducing each individual's optimal buffer stock of savings, the government is hampering and effectively destroying individuals' ability to self-insure against idiosyncratic risks when lump-sum transfers are not available.² Since taxing capital *per se* does not directly address the lack-of-insurance problem for households (if anything, it intensifies the problem), why would taxing capital be always optimal for the Ramsey planner, especially when government bonds are less distortionary than capital taxes and thus more effective in addressing the problem of capital overaccumulation without hindering individuals' self-insurance positions?

These questions are intriguing because the lack of full self-insurance is the root cause of the capital overaccumulation problem in HAIM economies and should hence be the ultimate concern of a benevolent government. In other words, in the absence of any wealth-redistribution effect of a capital tax, eliminating the inefficiency of overaccumulation through capital taxation does not help to alleviate the primal friction in the model—the lack of full self-insurance under borrowing constraints; thus, why one would use only the MGR as the optimality criterion to justify a positive capital income tax regardless of its effect on individual allocative efficiency is not at all clear.

In short, intuition tells us that in the absence of any concerns for wealth redistribution, a Ramsey planner can improve welfare more likely through improving individuals' self-insurance positions (such as issuing enough bonds to substitute for capital) than through taxing individuals' buffer stock (capital) when labor taxes are available to finance the in-

²Presumably, a lump-sum tax can further reduce the need for capital taxation. A lump-sum transfer, however, has no effect on the Ramsey allocation (especially the optimal quantity of public debt) in our model. Using an entirely different HAIM model from this paper, Chen, Chien, Wen, and Yang (2021) also show that lump-sum taxes can improve welfare but lump-sum transfers have no effect on the Ramsey allocation—the reason is precisely that government bonds are a better tool than lump-sum transfers in providing self-insurance against idiosyncratic income risk.

terest payments on public debt and the interest rate (cost) is itself below the time discount rate.

The capital taxation trade-off between aggregate allocative efficiency and individual allocative efficiency thus pertains intrinsically to the determination of the optimal level of public debt, which in turn depends critically on the existence of an interior Ramsey steady state. That is, the greater the supply of government bonds, the greater the self-insurance position individuals can achieve, creating less of a need to impose a distortionary capital tax. Hence, optimality seems to require the bond interest rate to equal the time discount rate. Yet the demand for government bonds in a standard infinite-horizon HAIM model approaches infinity as the interest rate approaches the time discount rate, as pointed out by Aiyagari (1994, 1995). Hence, without any borrowing limits, the assumption of the existence of an interior Ramsey steady state necessarily implies that the optimal level of the bond supply is finite. But the question is why?

Therefore, daunting challenges in the determination of an optimal capital tax in the class of infinite-horizon HAIM models lie (critically) in analyzing the trade-off problem under capital taxation and in proving the existence of an interior Ramsey steady state, which in turn dictate the determination of the optimal level of public debt and optimal labor tax. In other words, since a capital tax can always be substituted by a labor tax and the capital tax rate can always be lowered further by increasing the bond supply, the policy mix must be simultaneously determined in a Ramsey equilibrium by investigating the full set of the Ramsey planner's first-order optimal conditions, such as those for the bond supply, capital tax, and labor tax.

The goal of this paper is to design a tractable HAIM model—where the redistribution effect and the saving effect of a capital tax can be separated—to analytically investigate the trade-off between aggregate allocative efficiency and individual allocative efficiency when an interior Ramsey steady state can be proven to exist—in the absence of any wealth-redistribution effects of capital taxation.

Our infinite-horizon model follows in the spirit of Aiyagari (1994), but with two key differences: Individuals in our model face idiosyncratic shocks to the marginal utility of consumption, and their preferences are quasi-linear. This preference structure completely eliminates any wealth-redistribution effects of a capital tax and enables us to solve both the competitive equilibrium and the Ramsey allocation in closed forms.

Our analytical approach shares a similar spirit to the recent work of Heathcote and Perri (2018). In their model, households can reshuffle asset holdings at the end of each period

such that the distribution of households' end-of-period wealth is degenerate. This feature allows their HAIM model to be analytically tractable despite idiosyncratic risk and precautionary saving motives. In our model, however, the degenerate distribution of wealth is an endogenous outcome of individuals' rational choices and we can characterize the distribution of consumption and savings by a single endogenous cutoff variable that is fully responsive to aggregate conditions. By eliminating the wealth redistribution effect, our contribution is to highlight what does and does not drive the results about positive long-run taxation in the Aiyagari model.

Our contributions are thus five-fold: First, we provide an analytically tractable model in which necessary and sufficient conditions for the existence of an interior Ramsey steady state can be explicitly proved and stated.³ Our model differs from the Aiyagari model in that the distribution of individual wealth is completely degenerate, thus ruling out any wealth-redistribution effects of a capital tax.

Second, we show analytically that within the standard parameter space and in the absence of wealth-redistribution effects, the Ramsey planner will never tax capital in the steady state, despite capital overaccumulation.

Third, our analysis provides clarification of the critical role of government debt in achieving the MGR and influencing the trade-offs of a capital tax between aggregate allocative efficiency and individual allocative efficiency. Specifically, we show that whether the MGR holds or not depends critically on the Ramsey planner's ability to issue bonds as an alternative store of value (aside from capital) for households to buffer idiosyncratic risks. In particular, the MGR would hold in our model if and only if the government can amass a sufficiently large stock of bonds to enable households to achieve full self-insurance. Once households are fully self-insured, the equilibrium interest rate in our model equals the time discount rate ($1/\beta$). In such a case, aggregate allocative efficiency and individual allocative efficiency are simultaneously achieved by the Ramsey planner. However, when it is impossible to equalize the interest rate and the time discount rate—either because of unbounded variance of the idiosyncratic shocks or due to an ad hoc debt-limit constraint on the government's capacity to issue bonds—the MGR *does not* hold. Despite the failure of the MGR, however, the optimal capital tax rate is *still* zero in the steady state (even in the case where the government cannot issue bonds).⁴ Hence, the MGR appears to have no bearing on the planner's long-run

³An interior Ramsey steady state refers to a Ramsey allocation where all aggregate variables and the moments of the distribution converge to constant and finite non-zero values.

⁴This result is striking yet very intuitive: If it is optimal to set the capital tax to zero without the tool of

capital tax scheme.

Fourth, we use numerical analysis to show that the Ramsey planner nonetheless opts to tax capital along the transition path—so as to front-load consumption by arbitraging between the relatively low interest rate and the time discount rate. In particular, the larger the elasticity of intertemporal substitution, the higher the short-run tax rate. In fact, the levels of aggregate consumption, the capital stock, and hours worked in a Ramsey steady state are lower than their counterparts in the laissez-faire competitive equilibrium. Hence, the welfare gains under optimal policies derive primarily from higher consumption in the short run and the improved distribution of household self-insurance positions in the long run. Notice that in our numerical analyses the Ramsey transition path is calculated based on a proven Ramsey steady state, in contrast to the existing numerical literature that calculates the Ramsey transition path based on an unproven Ramsey steady state that may or may not exist. Our numerical exercises also serve as independent verification of our theoretical analyses.

Last but not least, in the absence of a binding debt limit, the optimal debt-to-GDP ratio in our model is determined by a positive wedge times the aggregate saving rate implied by the MGR. This wedge is an increasing function of the extent of individual allocative inefficiency and would vanish only if idiosyncratic risk approaches zero or markets become complete. Namely, the optimal debt-to-GDP ratio in our model is zero if (and only if) households are fully self-insured and no longer borrowing constrained in a competitive equilibrium. This result suggests (again) that in the absence of wealth-redistribution effects, the single most important role of government debt is to improve individuals' self-insurance positions through which the MGR is achieved (if possible). In other words, in the absence of any wealth-redistribution effects, the MGR does not appear to be the primal concern of the Ramsey planner, because aggregate efficiency is the consequence of individual efficiency in Aiyagari-type models; so it is never optimal to tax capital in the steady state simply to achieve aggregate allocative efficiency even though it is feasible to do so (as long as other forms of distortionary taxes such as a labor income tax are available).

These results are intuitive. On the one hand, it is the lack of an insurance market that induces agents to overaccumulate capital to self-insure against idiosyncratic consumption risk. In the absence of wealth-redistribution effects, taxing capital in the steady state would permanently hamper individuals' self-insurance—the lack of which is the root cause of aggregate

government debt, then it should be optimal to maintain a zero capital tax when government bonds become available.

allocative inefficiency—and is thus not a desirable tool to restore the MGR.

On the other hand, government bonds meet individuals' demand for precautionary saving without creating pecuniary externalities on the marginal product of capital. Hence, by substituting for (or crowding out) capital, government debt can satisfy the household buffer-stock saving needs and, at the same time, correct aggregate inefficiency due to capital overaccumulation. Most importantly, since the interest rate lies below the time discount rate—a hallmark feature of Aiyagari-type models, the marginal benefit of improving individual allocative efficiency is always larger than the discounted future marginal costs of debt financing by distortionary labor taxes. This is why the Ramsey planner opts to flood the asset market with a sufficient amount of bonds to satisfy (as much as possible) the full self-insurance demand from all households across all states. However, when debt limits exist, the Ramsey planner is unable to issue enough bonds to achieve a full self-insurance allocation; but, in spite of this, the planner will not levy a permanent tax on capital simply to achieve the MGR.

Although our model is just a special case of standard infinite-horizon HAIM models, it serves to demonstrate that the classical result of zero capital taxation obtained in the representative-agent literature can still hold in infinite-horizon HAIM economies with over-accumulated capital stock—as long as the redistributational channel of capital taxation is shut down.

The remainder of the paper is organized as follows. Section 2 describes the model, derives the competitive equilibrium, and provides sufficient conditions for the Ramsey planner to support a competitive equilibrium. Section 3 shows how to solve for the Ramsey allocation analytically and how to prove the existence of a Ramsey steady state. Section 4 performs numerical exercises to study transition dynamics and also to confirm our theoretical analyses. Section 5 provides a brief literature review. The last section concludes the paper with remarks for future research.

2 The Model

This model is based on Bewley (1980), Lucas (1980), Huggett (1993), Aiyagari (1994), and especially Wen (2015). To fix notions for this paper (with some abuse of terminology), we define aggregate allocative efficiency as a competitive equilibrium allocation in which the (after-tax) premium-adjusted private MPK equals the socially optimal MPK, and we define individual allocative efficiency as a competitive equilibrium allocation in which all households

are fully self-insured and their borrowing constraints do not bind in all idiosyncratic states (for a given level of government bonds).

Obviously, in a competitive equilibrium without government intervention, individual allocative efficiency implies aggregate allocative efficiency but not necessarily vice versa. Also, the Ramsey planner may design policies that achieve individual allocative efficiency without achieving aggregate allocative efficiency, or vice versa.

We also define “efficient allocation” as a Ramsey allocation in which both aggregate allocative efficiency and individual allocative efficiency are achieved under optimal government policies. Accordingly, our discussions involve two different notions of steady state: the “competitive equilibrium steady state” for a given set of government policies, and the “Ramsey steady state” under optimal policies.

2.1 Environment

A representative firm produces output according to the constant-returns-to-scale Cobb-Douglas technology, $Y_t = F(K_t, N_t) = K_t^\alpha N_t^{1-\alpha}$, where Y , K , and N denote aggregate output, capital, and labor, respectively. The firm rents capital and hires labor from households by paying a competitive rental rate and real wage, denoted by q_t and w_t , respectively. The firm’s optimal conditions for profit maximization at time t satisfy

$$w_t = \frac{\partial F(K_t, N_t)}{\partial N_t} \equiv MP_{N,t}, \quad (1)$$

$$q_t = \frac{\partial F(K_t, N_t)}{\partial K_t} \equiv MP_{K,t}. \quad (2)$$

There is a unit measure of *ex ante* identical households that face idiosyncratic preference shocks, denoted by θ . The shocks are identically and independently distributed (iid) over time and across households, and have the mean $\bar{\theta}$ and the cumulative distribution $\mathbf{F}(\theta)$ with support $[\theta_L, \theta_H]$, where $\theta_H > \theta_L > 0$.

Time is discrete and indexed by $t = 0, 1, 2, \dots, \infty$. There are two subperiods in each period t . The idiosyncratic preference shock θ_t is realized only in the second subperiod, and the labor supply decision must be made in the first subperiod before observing θ_t . Namely, the idiosyncratic preference shock is uninsurable by wage income even when leisure enters the utility function linearly. Let $\theta^t \equiv (\theta_1, \dots, \theta_t)$ denote the history of idiosyncratic shocks. All households are endowed with the same asset holdings at the beginning of time 0.

Households are infinitely lived with a quasi-linear utility function and face borrowing

constraints. Their lifetime expected utility is given by

$$V = E_0 \sum_{t=0}^{\infty} \beta^t \left[\theta_t \frac{c_t(\theta^t)^{1-\sigma} - 1}{1-\sigma} - n_t(\theta^{t-1}) \right], \quad (3)$$

where $\beta \in (0, 1)$ is the discount factor; $\sigma \in (0, \infty)$ is a parameter that determines the elasticity of intertemporal substitution (EIS) and risk aversion of the household; $c_t(\theta^t)$ and $n_t(\theta^{t-1})$ denote consumption and the labor supply, respectively, for a household with history θ^t at time t . Note that the labor supply in period t is only measurable with respect to θ^{t-1} , reflecting the assumption that the labor-supply decision is made in the first subperiod before observing the preference shock θ_t .

The government needs to finance an exogenous stream of purchases, denoted by $G_t \geq 0$ for all t , and it can issue bonds and levy time-varying labor and capital taxes at flat rates $\tau_{n,t}$ and $\tau_{k,t}$, respectively. The flow government budget constraint in period t is

$$\tau_{n,t} w_t N_t + \tau_{k,t} q_t K_t + B_{t+1} \geq G_t + r_t B_t, \quad (4)$$

where B_{t+1} is the level of government bonds chosen in period t , and r_t is the gross risk-free rate.

2.2 Household Problem

We assume there is no aggregate uncertainty and that government bonds and capital are perfect substitutes for store of value for households. As a result, the after-tax gross rate of return to capital must equal the gross risk-free rate:

$$1 + (1 - \tau_{k,t}) q_t - \delta = r_t, \quad (5)$$

which constitutes a no-arbitrage condition for capital and bonds.

Given the sequence of interest rates, $\{r_t\}_{t=0}^{\infty}$, and after tax wage rates, $\{\bar{w}_t \equiv (1 - \tau_{n,t}) w_t\}_{t=0}^{\infty}$, a household maximizes (3) by choosing a plan of consumption, labor, and asset holdings, $\{c_t(\theta^t), n_t(\theta^{t-1}), a_{t+1}(\theta^t)\}_{t=0}^{\infty}$ subject to

$$c_t(\theta^t) + a_{t+1}(\theta^t) \leq \bar{w}_t n_t(\theta^{t-1}) + r_t a_t(\theta^{t-1}), \quad (6)$$

$$a_{t+1}(\theta^t) \geq 0, \quad (7)$$

with $a_0 > 0$ given and $n_t(\theta^{t-1}) \in [0, \bar{N}]$. The solution of the household problem can be characterized analytically by the following proposition.

Proposition 1. *Denoting household gross income (or total liquidity on hand) by $x_t(\theta^{t-1}) \equiv r_t a_t(\theta^{t-1}) + \bar{w}_t n_t(\theta^{t-1})$, the optimal decisions for $x_t(\theta^{t-1})$, consumption $c_t(\theta^t)$, savings $a_{t+1}(\theta^t)$, and the labor supply $n_t(\theta^{t-1})$ are given, respectively, by the following cutoff-policy rules⁵:*

$$x_t = [\bar{w}_t L(\theta_t^*) \theta_t^*]^{1/\sigma} \quad (8)$$

$$c_t(\theta_t) = \min \left\{ 1, \left(\frac{\theta_t}{\theta_t^*} \right)^{1/\sigma} \right\} x_t \quad (9)$$

$$a_{t+1}(\theta_t) = \max \left\{ 1 - \left(\frac{\theta_t}{\theta_t^*} \right)^{1/\sigma}, 0 \right\} x_t \quad (10)$$

$$n_t(\theta_{t-1}) = \frac{1}{\bar{w}_t} [x_t - r_t a_t(\theta_{t-1})], \quad (11)$$

where the cutoff θ_t^* is independent of individual history and determined by the Euler equation

$$\frac{1}{\bar{w}_t} = \beta \frac{r_{t+1}}{\bar{w}_{t+1}} L(\theta_t^*), \quad (12)$$

and the function $L(\theta_t^*) \geq 1$ captures the (gross) liquidity premium of savings and is given by

$$L(\theta_t^*) \equiv \int_{\theta \leq \theta_t^*} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta). \quad (13)$$

Proof. See Appendix A.1. □

Notice that the individual consumption function in equation (9) is reminiscent of that derived by Deaton (1991) under a numerical method, and the saving function in equation (10) exhibits a buffer-stock behavior: When the urge to consume is low ($\theta_t < \theta_t^*$), the individual opts to consume only a $\frac{\theta_t}{\theta_t^*} < 1$ fraction of total income and save the rest, anticipating that future consumption demand may be high. On the other hand, when the urge to consume is high ($\theta_t \geq \theta_t^*$), the agent opts to consume all gross income, up to the limit where the borrowing constraint binds, so the saving stock is reduced to zero. The function $L(\theta_t^*) - 1 \geq 0$ reflects the excess rate of return to savings due to the option value (liquidity premium) of

⁵The cutoff-policy rules hold if the individual labor decision is an interior one, namely, $n_t \in (0, \bar{N})$. In the proof of this proposition (Appendix A.1), we show that with reasonable parameter values and a sufficiently large chosen \bar{N} , individual hours worked are ensured to be interior.

the buffer stock.

Denote $\Lambda_t \equiv \frac{1}{\bar{w}_t}$ as the expected marginal utility of income. Then the left-hand side of equation (12) is the average marginal cost of consumption in the current period, and the right-hand side is the discounted expected next-period return to savings (augmented by r_{t+1}), which takes two possible values in light of the two components for the liquidity premium in equation (13): The first is simply the discounted next-period marginal utility of consumption Λ_{t+1} in the case where the borrowing constraint does not bind, which has probability $\int_{\theta \leq \theta_t^*} d\mathbf{F}(\theta)$. The second is the discounted marginal utility of consumption $\Lambda_{t+1} \frac{\theta_t}{\theta_t^*}$ in the case of high demand ($\theta_t > \theta_t^*$) with a binding borrowing constraint, which has probability $\int_{\theta > \theta_t^*} d\mathbf{F}(\theta)$. When the borrowing constraint binds, additional savings can yield a higher shadow marginal utility $\frac{\theta_{t+1}}{\theta_{t+1}^*} \Lambda_{t+1} > \Lambda_{t+1}$. The optimal cutoff θ_t^* is then determined at the point where the marginal cost of saving equals the expected marginal gains. Here, savings play the role of a buffer stock and the rate of return to savings is determined by the real interest rate r_t compounded by a liquidity premium $L(\theta_t^*)$. Notice that $\frac{\partial L(\theta_t^*)}{\partial \theta_t^*} < 0$ and $L(\theta_t^*) > 1$ for any $\theta_t^* \neq \theta_H$.

Equation (12) also suggests that the cutoff θ_t^* is independent of individual history. This property holds in this model because of the quasi-linear utility function and the assumption that the labor supply is predetermined in the first subperiod. In other words, the optimal level of liquidity on hand in period t is determined by a “target” income level given by $x_t = [\theta_t^* \bar{w}_t L(\theta_t^*)]^{1/\sigma}$, which is also independent of the history of realized values of θ_t but depends only on the cutoff θ_t^* . This target is essentially the optimal consumption level when the borrowing constraint binds. This target policy (uniform to all households) emerges because labor income ($\bar{w}_t n_t(\theta_{t-1})$) can be adjusted elastically to meet the optimal target, given (and regardless of) the initial asset holdings $a_t(\theta_{t-1})$. Hence, in the beginning of each period, all households will choose the same level of gross income x_t . Thus, the individual-history-independent cutoff variable θ_t^* uniquely and fully characterizes the distributions of household decisions in the economy.

Since cash on hand x_t is independent of the idiosyncratic shock θ_t and thus identical across households, capital taxation plays no redistributive role. Similarly, a lump-sum tax/transfer has no redistribution effect on households’ cash on hand either because any wealth-redistribution effects of a lump-sum tax/transfer can be always offset by the perfectly elastic labor supply such that the target level of cash on hand x_t is degenerate and remains the same across households. This is why we do not need to consider lump-sum taxes/transfers in the model.

2.3 Competitive Equilibrium

Denote C_t , N_t , and K_{t+1} as the level of aggregate consumption, aggregate labor, and aggregate capital, respectively. A competitive equilibrium allocation can be defined as follows:

Definition 1. *Given initial aggregate capital K_0 and bonds B_0 , a sequence of taxes, and government spending and government bonds, $\{\tau_{n,t}, \tau_{k,t}, G_t, B_{t+1}\}_{t=0}^\infty$, a competitive equilibrium is a sequence of prices $\{w_t, q_t\}_{t=0}^\infty$, allocations $\{c_t(\theta^t), n_t(\theta^{t-1}), a_{t+1}(\theta^t), K_{t+1}, N_t\}_{t=0}^\infty$, and the cutoff $\{\theta_t^*\}_{t=0}^\infty$ such that*

1. *given the sequence $\{w_t, q_t, \tau_{n,t}, \tau_{k,t}\}_{t=0}^\infty$, the sequences $\{c_t(\theta^t), a_{t+1}(\theta^t), n_t(\theta^{t-1})\}_{t=0}^\infty$ solve the household problem;*
2. *given the sequence of $\{w_t, q_t\}_{t=0}^\infty$, the sequences $\{N_t, K_t\}_{t=0}^\infty$ solve the firm's problem;*
3. *the no-arbitrage condition holds for each period: $r_t = 1 + (1 - \tau_{k,t})q_t - \delta$ for all $t \geq 0$;*
4. *the government budget constraint in equation (4) holds for each period; and*
5. *all markets clear for all $t \geq 0$:*

$$K_{t+1} = \int a_{t+1}(\theta_t) d\mathbf{F}(\theta_t) - B_{t+1} \quad (14)$$

$$N_t = \int n_t(\theta_{t-1}) d\mathbf{F}(\theta_{t-1}) \quad (15)$$

$$\int c_t(\theta_t) d\mathbf{F}(\theta_t) + G_t \leq F(K_t, N_t) + (1 - \delta)K_t - K_{t+1}. \quad (16)$$

Proposition 2. *If the upper bound θ_H of the preference shocks is sufficiently large relative to the moment $[\mathbb{E}(\theta^{1/\sigma})]^\sigma$ such that the following condition holds:*

$$\frac{\alpha\beta(\theta_H)^{1/\sigma}}{(\theta_H)^{1/\sigma} - \mathbb{E}(\theta^{1/\sigma})} + \beta(1 - \alpha)(1 - \delta) < 1, \quad (17)$$

then, in a laissez-faire competitive equilibrium, the steady-state risk-free rate is lower than the time discount rate, $r < 1/\beta$; and there exists overaccumulation of capital with a positive liquidity premium, $L(\theta^) > 1$.*

Proof. See Appendix A.2. □

Notice that when $\theta_H \rightarrow \infty$, as in the case of a Pareto distribution, the above condition is clearly satisfied. The intuition of Proposition 2 is straightforward. Since labor income is determined (*ex ante*) before the realization of the idiosyncratic preference shock θ_t , a household's total income may be insufficient to provide full insurance for large enough preference shocks under condition (17). In this case, precautionary saving motives lead to overaccumulation of capital, which reduces the equilibrium interest rate below the time discount rate. This outcome is clearly inefficient from a social point of view. It emerges because of the negative externalities of household savings on the aggregate interest rate (due to diminishing marginal product of capital), as noted by Aiyagari (1994).

However, unlike the Aiyagari (1994) model, a laissez-faire competitive equilibrium can achieve both aggregate allocative efficiency and individual allocative efficiency in our model if the idiosyncratic risk is sufficiently small (e.g., the upper bound θ_H is close enough to the mean $\mathbb{E}(\theta)$ such that condition (17) is violated). In this case, individual savings can become sufficiently large to fully buffer preference shocks and, as a result, household borrowing constraints will never bind. Clearly, with full self-insurance, it must be true that the optimal cutoff is at the upper corner, $\theta^* = \theta_H$; the liquidity premium vanishes, $L(\theta^*) = 1$; and the interest rate equals the time discount factor, $r = 1/\beta$.

A competitive equilibrium with full self-insurance is impossible in a typical HAIM model (such as in Aiyagari (1994, 1995)) because every household's marginal utility of consumption follows a supermartingale when $r = 1/\beta$. This implies that household consumption and savings (or asset demand) diverge to infinity in the long run, which cannot constitute an equilibrium.⁶

In our model, however, because the household utility function is quasi-linear, the expected shadow price of consumption goods is thus the same across agents and given by $\frac{1}{\bar{w}_t}$ (as revealed by equations (36) and (38) in the proof of Proposition 1 (Appendix A.1)), which kills the supermartingale property of the household marginal utility of consumption. As a result, household savings (or asset demand) are bounded away from infinity even at the point $r = 1/\beta$. More specifically, equations (8) and (10) show that an individual's asset demand is always bounded above by $(\theta_H - \theta_t) \bar{w}_t$ for any distribution \mathbf{F} of the shock $\theta_t \in [\theta_L, \theta_H]$ when $r = 1/\beta$. This upper bound $(\theta_H - \theta_t) \bar{w}_t$ is finite as long as the support $[\theta_L, \theta_H]$ of θ_t is bounded (a counter example is a Pareto distribution where $\theta_H = \infty$). This special property renders our model analytically tractable with closed-form solutions (provided that θ_t is iid), and it implies that the Ramsey planner has the potential to use government debt to achieve

⁶Please refer to Ljungqvist and Sargent (2012, Chapter 17) for details.

efficient allocation in this economy when the laissez-faire competitive equilibrium is not.

Nonetheless, the trade-off between aggregate allocative efficiency and individual allocative efficiency emphasized in this paper does not hinge on the special properties of our model. It is driven by the impact of distortionary capital taxation upon precautionary saving motives—because a capital tax discourages households from savings, it mitigates the overaccumulation of capital but at the same time tightens individuals’ borrowing constraints (thus impeding the individual self-insurance position). Hence, such a trade-off channel should also exist in more general HAIM models. The only critical difference is that in a more general HAIM model a capital tax has wealth-redistribution effects but such effects are completely absent in our special model.

2.4 Conditions to Support a Competitive Equilibrium

Given that government policies are inside the aggregate state space of the competitive equilibrium and affect the endogenous distributions (including the average) of all endogenous economic variables, the Ramsey problem is to pick a competitive equilibrium (through policies) that attains the maximum of the expected household lifetime utility V defined in (3). Since V depends on the endogenous distributions (see below and Wen (2015)), the Ramsey planner needs to also pick a particular time path (sequence) of distributions to achieve the maximum.

This subsection expresses the necessary conditions, in terms of the aggregate variables and distributions characterized by the cutoff θ_t^* , that the Ramsey planner must respect in order to construct a competitive equilibrium. We first show that since the cutoff θ_t^* is a sufficient statistic for describing the distributions of individual variables, all allocations and prices in the competitive equilibrium can be expressed as functions of the aggregate variables and the cutoff θ_t^* . Hence, we also call the θ_t^* as the distribution statistic.

To facilitate the analysis, we first show the properties of aggregate consumption (or average consumption across households) by aggregating the individual consumption decision rules (9). By the law of large numbers, the aggregate consumption is determined by

$$C_t = D(\theta_t^*)x_t, \tag{18}$$

where the aggregate marginal propensity to consume (the function D) is given by

$$D(\theta_t^*) \equiv \int_{\theta \leq \theta_t^*} \left(\frac{\theta}{\theta_t^*} \right)^{1/\sigma} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} d\mathbf{F}(\theta) \in (0, 1]. \quad (19)$$

Then, we can express individual consumption and individual asset holding as functions of C_t and θ_t^* by plugging equation (18) into equations (9) and (10). To fully describe the conditions necessary for constructing a competitive equilibrium, we rely on the following proposition:

Proposition 3. *Given initial capital K_0 , initial government bonds B_0 , and the initial capital tax $\tau_{k,0}$, the sequences of aggregate allocations $\{C_t, N_t, K_{t+1}, B_{t+1}\}_{t=0}^\infty$ and distribution statistics $\{\theta_t^*\}_{t=0}^\infty$ can be supported as a competitive equilibrium if and only if the resource constraint (16), the asset market clearing condition*

$$B_{t+1} = \left(\frac{1}{D(\theta_t^*)} - 1 \right) C_t - K_{t+1}, \text{ for all } t \geq 0, \quad (20)$$

and the following implementability conditions (for periods $t = 0$ and $t \geq 1$, respectively) are satisfied:

$$C_0^{1-\sigma} D(\theta_0^*)^{\sigma-1} L(\theta_0^*) \theta_0^* \geq N_0 + r_0 C_0^{-\sigma} D(\theta_0^*)^\sigma L(\theta_0^*) \theta_0^* (K_0 + B_0) \quad (21)$$

and

$$C_t^{1-\sigma} D(\theta_t^*)^{\sigma-1} L(\theta_t^*) \theta_t^* \geq N_t + \frac{1}{\beta} C_{t-1}^{1-\sigma} D(\theta_{t-1}^*)^{\sigma-1} \theta_{t-1}^* (1 - D(\theta_{t-1}^*)), \quad (22)$$

where $r_0 = 1 + (1 - \tau_{k,0})MP_{K,0} - \delta$.

Proof. See Appendix A.3 □

Note that the implementability conditions essentially enforce the flow government budget constraint and are comparable to those in the representative agent framework.

To derive the implementability conditions (21) and (22), we first replace the intertemporal prices and taxes with quantitative variables. As shown in Appendix A.3, the flow government budget constraint in a competitive equilibrium can be expressed as

$$\begin{aligned} & U_{C,t} D(\theta_t^*)^\sigma L(\theta_t^*) \theta_t^* C_t - N_t + U_{C,t} D(\theta_t^*)^\sigma L(\theta_t^*) \theta_t^* A_{t+1} \\ & \geq \frac{1}{\beta} U_{C,t-1} D(\theta_{t-1}^*)^\sigma \theta_{t-1}^* A_t, \end{aligned} \quad (23)$$

where $U_{C,t}$ is defined as $C_t^{-\sigma}$, the “marginal utility” of aggregate consumption in our setup. The above expression is analogous to that in a representative agent model, except with

two additional terms: $D(\theta_t^*)^\sigma \theta_t^*$ and $L(\theta_t^*)$. These extra terms capture the distribution of individual marginal utilities and originate from the risk free rate, r_{t+1} , which can be expressed as (see Appendix A.3):

$$\frac{1}{r_{t+1}} = \beta \frac{U_{C,t+1}}{U_{C,t}} \frac{D(\theta_{t+1}^*)^\sigma \theta_{t+1}^*}{D(\theta_t^*)^\sigma \theta_t^*} L(\theta_{t+1}^*). \quad (24)$$

The function $U_{C,t}$ captures the marginal utility of aggregate consumption; the function $D(\theta_t^*)^\sigma \theta_t^*$ captures the distribution of marginal utilities of the individual households⁷; and the function $L(\theta_{t+1}^*)$ captures the liquidity premium of bonds. In representative agent models, the last two terms are absent since there is no consumption distribution or liquidity premium to affect the risk free rate of bonds. Thus, as shown in the proof of Proposition 3, equation (23) together with the asset market clearing conditions imply the implementability conditions (21) and (22).

This proposition demonstrates that the Ramsey planner can construct a competitive equilibrium by simply choosing the sequences of aggregate allocations $\{C_t, N_t, K_{t+1}, B_{t+1}\}$ and distribution statistics $\{\theta_t^*\}$ to maximize expected welfare, subject to the aggregate resource constraint, asset market clearing condition, and the implementability condition.

Notice that in a steady state with $\theta^* < \theta_H$, equations (24) and (5) imply

$$1 = \beta [1 - \delta + (1 - \tau_k) q] L(\theta^*), \quad (25)$$

whereas the MGR is characterized by

$$1 = \beta [1 - \delta + \tilde{q}]; \quad (26)$$

where q is private MPK and \tilde{q} denotes socially optimal MPK. Hence, even in the absence of any production externalities there exists a wedge between the socially optimal MPK and the laissez-faire competitive equilibrium MPK (where $\tau_k = 0$ and $B = 0$)—due to the positive liquidity premium: $L(\theta^*) > 1$. Suppose a Ramsey steady state exists such that the MGR holds; then choosing $\tau_k > 0$ (for a given level of government bonds) such that

$$[1 - \delta + (1 - \tau_k) q] L(\theta^*) = [1 - \delta + q]$$

is clearly a feasible policy. However, whether such a positive capital-tax policy is optimal depends also on the Ramsey planner's first-order conditions for government debt and labor-

⁷If the distribution of the idiosyncratic shock θ is degenerate with $\theta = \bar{\theta}$, then $D(\theta_t^*)^\sigma \theta_t^* = \bar{\theta}$.

income tax. For example, suppose the optimal level of government debt is such that $L(\theta^*) = 1$, then the implied optimal capital tax is zero ($\tau_k = 0$) instead of positive. In addition, if there exists a government borrowing limit such that the debt limit binds in the steady state, then equation (26) must be modified accordingly to include a Lagrangian multiplier for the government borrowing constraint; thus, the implications for optimal capital tax becomes less clear cut.

In what follows, we will solve the Ramsey allocation analytically to jointly determine the optimal level of government debt, optimal capital tax, and optimal labor tax. We also consider the effects of an ad hoc government borrowing limit on the Ramsey allocation when the debt limit binds in the Ramsey steady state.

3 Ramsey Allocation

Armed with Proposition 3, we are ready to write down the Ramsey planner's problem and derive the first-order Ramsey conditions analytically.

3.1 Ramsey Problem

Using equations (9) and (18), the lifetime utility function V can be rewritten as a function of the distribution statistic θ_t^* and aggregate variables:

$$V = \sum_{t=0}^{\infty} \beta^t \left[W(\theta_t^*) \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{\bar{\theta}}{1-\sigma} - N_t \right], \quad (27)$$

where $W(\theta_t^*)$ captures the welfare effects of consumption *distribution* and is defined as

$$W(\theta_t^*) \equiv \left(\int_{\theta \leq \theta_t^*} \theta \left(\frac{\theta}{\theta_t^*} \right)^{\frac{1-\sigma}{\sigma}} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} \theta d\mathbf{F}(\theta) \right) D(\theta_t^*)^{\sigma-1}. \quad (28)$$

Thus, the Ramsey problem can be represented as maximizing the welfare function (27) by choosing the sequences of distributions and other aggregate variables $\{\theta_t^*, N_t, C_t, K_{t+1}, B_{t+1}\}$, subject to the resource constraint (16), the asset market clearing condition (20), and the implementability conditions (22) and (21).

In addition, an exogenous debt limit $B_{t+1} \leq \bar{B}$ is imposed on the Ramsey planner to facilitate our analysis on the role of government debt, which most of the existing literature has ignored or assumed away by implicitly setting $\bar{B} = \infty$ —which is larger than any debt

limit of the government.

Let μ_t , λ_t , and ϕ_t denote the multipliers for the aggregate resource constraint, the implementability condition, and the asset market clearing condition, respectively. Appendix A.4 provides the Lagrangian of the Ramsey problem and derives the associated first-order conditions. In what follows, any variable without subscript t denotes its steady-state value.

3.2 Characterization of Ramsey Allocations

Definition 2. *An interior Ramsey steady state is defined as a long-run Ramsey allocation where (i) the parameter restriction $\frac{\theta_L}{(1-\beta)^\sigma} \in (\theta_H, \infty)$ (to ensure positive labor $n > 0$ for all individuals in all states) is satisfied and (ii) all aggregate variables $\{K, N, C, B, \theta^*\}$ converge to positive finite values.*

Notice that the parameter restriction

$$\theta_H < \frac{\theta_L}{(1-\beta)^\sigma} \quad (29)$$

is required to ensure that all individuals' labor decisions are strictly positive, a necessary condition for Proposition 1. The intuition is that if the variance (support) of θ is too large (spread out), some agents may end up with too much savings in the end of last period and thus opt not to work this period. Our model becomes intractable in such a situation with possible binding zero labor supply, so it must be ruled out.

Proposition 4. *If an interior Ramsey steady state exists, then the optimal capital tax rate in such a steady state is determined by the following equation:*

$$1 - \tau_k = \frac{L(\theta^*)^{-1} - \beta(1-\delta)}{\frac{\mu_t - \phi_t}{\mu_{t+1}} - \beta(1-\delta)}, \quad (30)$$

where $\mu_t > \phi_t \geq 0$ are the Lagrangian multipliers of Ramsey problem (47) for the aggregate resource constraint and the debt limit constraint (implied by equation (53)), respectively.

Proof. See Appendix A.6 □

Notice that the definition of the interior Ramsey steady state does not require the multipliers $\{\mu_t, \phi_t\}$ to converge to finite constant values—but we will prove that they necessarily converge if an interior Ramsey steady state exists under conventional parameter specifications (i.e., $\sigma \geq 1$).

Proposition 4 immediately gives the following steady-state tax rate for capital,

$$\tau_k \begin{matrix} \geq \\ \leq \end{matrix} 0 \text{ if and only if } L(\theta^*) \begin{matrix} \geq \\ \leq \end{matrix} \frac{\mu_{t+1}}{\mu_t - \phi_t}, \quad (31)$$

which implies the following two points:

(i) If the government debt-limit constraint does not bind in the interior Ramsey steady state (i.e., $\lim_{t \rightarrow \infty} \phi_t = 0$) and if the multiplier μ_t converges such that $\frac{\mu_t - \phi_t}{\mu_{t+1}} = \frac{\mu_t}{\mu_{t+1}} = 1$; then, since the liquidity premium $L(\theta^*) \geq 1$, the right-hand side of equation (30) must be smaller than or equal to 1. Therefore, optimal capital tax must be non-negative: $\tau_k \geq 0$. Hence, subsidizing capital in this case is never optimal. In addition, if in this case the optimal cutoff is a corner solution at $\theta^* = \theta_H$, then $L = 1$, and it must be true that $\tau_k = 0$.

(ii) If the government debt-limit constraint is binding in the steady state, i.e., $\phi > 0$, and if the multiplier μ_t converges, then given that $\mu > \phi$, the right-hand side of equation (31) must be greater than 1. In this case, if the optimal cutoff is a corner solution at $\theta^* = \theta_H$, then $L = 1$, so it must be true that $\tau_k < 0$. On the other hand, if in this case the optimal capital tax $\tau_k = 0$, then it must be true that $\theta^* < \theta_H$ and $L(\theta^*) > 1$; namely, individual allocative efficiency is not achieved.

In what follows, we prove the existence of two types of interior Ramsey steady state—depending on the debt limit—and characterize their respective properties. Recall that the conditions (17) and (29) are assumed to hold throughout the paper; so in each of the two possible cases the competitive equilibrium without government intervention is inefficient by design. The interesting question is how (and by how much) the government can improve upon the allocation of laissez-faire competitive equilibrium.

3.2.1 Case 1: Efficient Ramsey Steady State

Definition 3. *An efficient Ramsey steady state is defined as an interior Ramsey steady state where both aggregate allocative efficiency and individual allocative efficiency are achieved; namely, in an efficient Ramsey steady state: (i) the socially optimal MPK equals the (after-tax) premium-adjusted private MPK and (ii) the individual borrowing constraint $a_{t+1} \geq 0$ does not bind for all households in all states.*

Proposition 5. *Suppose the debt limit \bar{B} is sufficiently large such that the constraint $B_{t+1} \leq \bar{B}$ never binds; then, under a sufficiently large $\beta \in (0, 1)$ and the parameter restriction $\sigma \geq 1$, an efficient Ramsey steady state exists and it has the following properties:*

1. The optimal choice of the cutoff θ^* is a corner solution at $\theta^* = \theta_H$ so that there is no liquidity premium ($L(\theta^*) = 1$) and no households are borrowing-constrained.
2. The socially optimal MPK equals the (after-tax) premium-adjusted private MPK.
3. The equilibrium interest rate $r = 1/\beta$, the capital tax $\tau_k = 0$, and the labor tax $\tau_n = \frac{\lambda}{1+\lambda}\sigma \in (0, 1)$. Consequently, government expenditures and bond interest payments are financed solely by labor income taxes.
4. The Lagrangian multipliers $\{\lambda, \mu\}$ always converge to finite and strictly positive values; in other words, there cannot exist a Ramsey steady state where the Lagrangian multipliers diverge and capital tax is nonzero if the parameter condition $\sigma \geq 1$ holds.

Proof. See Appendix A.7 □

This proposition states that if the debt-limit constraint $B \leq \bar{B}$ does not bind in the Ramsey steady state, then the Ramsey plan picks a long-run competitive equilibrium that achieves both aggregate allocative efficiency and individual allocative efficiency when the intertemporal elasticity of substitution parameter $\sigma \geq 1$.

This proposition also indicates that in the absence of wealth-redistribution effects, the Ramsey planner achieves the MGR without the need to tax capital in the steady state—as capital taxation in the steady state would undermine individual allocative efficiency by decreasing the steady-state household saving rate and thus permanently hampering their self-insurance positions. Instead, the Ramsey planner opts to provide enough incentives for households to save through bonds by picking a sufficiently high interest rate ($= 1/\beta$) on bonds, such that all households are fully self-insured in the long run with zero probability of encountering binding liquidity constraints.

Furthermore, this Ramsey steady state is “unique” in the sense that there does not exist any interior Ramsey steady state with divergent multipliers nor any non-interior Ramsey steady state where the multipliers diverge and the aggregate consumption approaches zero.⁸

The intuition is that the Ramsey planner exploits an “arbitrage” opportunity that the marginal benefit of issuing debt is larger than the discounted present-value marginal costs

⁸However, as the Appendix A.7.3 shows, if $0 < \sigma < 1$ so that the consumption-utility function is sufficiently linear, not only can an interior steady state featuring FSI ($\theta^* = \theta_H$) and finite multipliers still exist, but there can also exist another type of steady state that features partial self-insurance ($\theta^* < \theta_H$), divergent multipliers (e.g., $\lim_{t \rightarrow \infty} \lambda_t = \infty$), and positive capital tax ($\tau_k > 0$). Since we are not able to determine which steady state yields the highest social welfare or qualifies as a Ramsey allocation and analyzing this is beyond the scope of this paper, we leave it to future research.

of distortionary labor taxes, because the risk-free rate of public debt lies below the time discount rate in a competitive equilibrium. Hence, the planner opts to issue a sufficiently large amount of public debt to fully meet the self-insurance demand of households such that the root cause of the capital overaccumulation problem is completely eliminated.⁹

It is this critical role of government debt along the intertemporal margin to improve the individual self-insurance position that determines the optimal debt level in the model. This can be seen more clearly by inspecting the optimal debt-to-GDP ratio in the special case below where the EIS parameter $\sigma = 1$, the rate of capital depreciation $\delta = 1$, and government spending $G = 0$:

Corollary 1. *When $\sigma = \delta = 1$ and $G = 0$, the optimal debt-to-GDP ratio is determined by a wedge τ_b times the MGR-saving rate $\beta\alpha$:*

$$\frac{B}{Y} = \tau_b \beta \alpha, \quad (32)$$

where the wedge

$$\tau_b = \left(\frac{1 - D(\theta^*)}{D(\theta^*)} \middle/ \frac{\beta\alpha}{1 - \beta\alpha} - 1 \right) \geq 0 \quad (33)$$

is essentially the gap between the competitive equilibrium saving ratio ($\frac{1-D(\theta^*)}{D(\theta^*)}$) and the conventional modified-golden-rule saving ratio ($\frac{\beta\alpha}{1-\beta\alpha}$). This gap vanishes if and only if the competitive equilibrium under incomplete markets approaches the allocation of an economy with full self-insurance (or with complete markets).

Proof. See Appendix A.8 □

Recall that $D(\theta^*)$ denotes the aggregate marginal propensity to consume and that $(1 - D(\theta^*))$ denotes the aggregate marginal propensity to save in a competitive equilibrium with incomplete markets. It can be shown easily that under complete markets (e.g., in a representative-agent model with $\delta = 1$ and log utility), the optimal saving rate is $\beta\alpha$. Hence, precautionary saving behavior implies that the saving rate under incomplete markets exceeds the saving rate under complete markets, i.e., $(1 - D) > \beta\alpha$ and $D < (1 - \beta\alpha)$. Hence, the wedge is strictly positive: $\tau_b > 0$. However, as the variance of θ approaches zero, or as the insurance markets become complete, it must be true that $(1 - D) \rightarrow \beta\alpha$ and $\tau_b \rightarrow 0$, regardless of the MGR saving rate $\beta\alpha$.

⁹In contrast, as shown recently by one of our companion papers in Chien and Wen (2020b), if the Ramsey planner maximizes only the steady-state welfare, then optimal quantity of public debt no longer achieves full self-insurance because the “arbitrage” opportunity is lost in a static Ramsey problem.

Therefore, the wedge τ_b is a measure of the degree of the individual allocative inefficiency in the competitive equilibrium. So Corollary 1 shows (again) that the single most important role of government debt is to improve the individual self-insurance position, such that the optimal level of bond supply is proportional to the conventional MGR-saving rate by a factor that is determined solely by the wedge of inefficiency caused by incomplete insurance markets in a competitive equilibrium.

Obviously, efficient allocation can be achieved only if the Ramsey planner is capable of supplying enough bonds to satisfy the liquidity demand of each household across all states. An important property of the efficient allocation is that the equilibrium interest rate equals the time discount rate: $r = 1/\beta$.

However, the zero-capital-tax policy does not necessarily depend on this property. To shed light on this issue further, we study what happens if the government's ability to issue bonds is limited and the debt-limit constraint is binding at least in the long run.

3.2.2 Case 2: Ramsey Steady State with a Binding Debt Limit

To illustrate our point without loss of tractability, we impose an exogenous debt limit \bar{B} that lies strictly below the optimal debt level B^* determined in the efficient Ramsey allocation.

Proposition 6. *Suppose the debt-limit constraint binds such that $B_{t+1} = \bar{B}$ beyond some time point $T > 1$. Then, under the parameter restriction $\sigma \geq 1$, an interior Ramsey steady state exists and has the following properties:*

1. *Individual allocative efficiency fails to hold—the optimal choice of the cutoff under Ramsey is interior: $\theta^* \in (\theta_L, \theta_H)$; so there is always a non-zero fraction $(1 - \mathbf{F}(\theta^*)) > 0$ of households facing binding borrowing constraints in every period and there is a positive liquidity premium ($L(\theta^*) > 1$) with equilibrium interest rate $r < 1/\beta$.*
2. *Aggregate allocative efficiency fails to hold—the socially optimal MPK does not equal the (after-tax) liquidity-premium-adjusted private MPK.*
3. *The optimal capital tax is still zero, $\tau_k = 0$; namely, the Ramsey planner does not tax capital even if aggregate allocative efficiency (MGR) fails to hold because of a binding debt-limit constraint. Notice that this is true even if $\bar{B} = 0$.*
4. *The Lagrangian multipliers $\{\lambda, \mu, \phi\}$ always converge to finite and strictly positive values; in other words, there cannot exist a Ramsey steady state where the Lagrangian multipliers diverge and capital tax is nonzero if the parameter condition $\sigma \geq 1$ holds.*

Proof. See Appendix A.9 □

Obviously, a special subcase of case 2 is when the government cannot issue bonds at all: $\bar{B} = 0$. This subcase is analogous to the situation discussed in Proposition 2 in the previous section, where the competitive-equilibrium interest rate is strictly less than the time discount rate. In such a situation, the Ramsey planner cannot use bonds to manipulate the market interest rate and divert household savings away from capital formation. In general, whenever the government is unable to supply enough bonds to meet household demand for full self-insurance, either because θ_H is sufficiently large or the debt limit \bar{B} is sufficiently low, the pursuit of individual allocative efficiency by the Ramsey planner will necessarily lead to a binding debt-limit constraint on government bonds. In this case, there exists an interior Ramsey steady state that features neither individual allocative efficiency nor aggregate allocative efficiency—albeit it is feasible for the planner to achieve the MGR by taxing capital, but it is not optimal to do so. Furthermore, if $\sigma \geq 1$ this Ramsey steady state is “unique” in the sense that there does not exist a non-interior Ramsey steady state where the multipliers diverge and the aggregate consumption approaches zero.¹⁰

As shown in the proofs for Proposition 5 and Proposition 6, the zero capital tax result is obtained in our model because of the strikingly simple and unique steady-state relationship:

$$L(\theta^*) = \frac{\mu}{\mu - \phi}, \quad (34)$$

which by equation (30) implies that $\tau_k = 0$, regardless of the debt limit \bar{B} .

This simple analytical relationship (34) is very striking and surprising. The Ramsey planner opts to supply enough bonds to improve the individual self-insurance position until the debt limit \bar{B} binds. But, regardless of the tightness of the binding debt limit (captured by the value of ϕ), the Ramsey planner nonetheless opts to equalize the ratio of Lagrangian multipliers $\frac{\mu}{\mu - \phi}$ to the liquidity premium L , such that the steady-state capital tax is exactly zero. This result offers a strong case to support the view that capital taxation should not be used as a policy tool simply to achieve the MGR.

Hence, the government borrowing limit does not matter for the optimal steady-state capital tax but does matter for the Ramsey planner’s ability to achieve the aggregate allocative efficiency and individual allocative efficiency. Yet such a critical role of government-debt limits is often ignored or has gone unnoticed in the existing literature.

¹⁰However, if $0 < \sigma < 1$, we cannot rule out the possibility that there exists an Ramsey steady state where the multipliers diverge to infinity and the optimal capital tax may be different from zero.

The fundamental reason is this: While capital taxation is effective in eliminating any wedge between (after-tax) premium-adjusted private MPK and socially optimal MPK, but it is not an effective tool to eliminate the liquidity premium itself—because a positive capital tax enlarges the liquidity premium albeit lowering the (after-tax) private MPK.

However, we will show in the next section that the optimal capital tax is positive along a transition path toward the steady state regardless of the bond supply, even though the optimal capital tax rate is zero in the steady state.

4 Numerical Analysis

This section performs numerical exercises not only to confirm our theoretical results but also to illustrate the optimal transition path and convergence of Ramsey allocation in comparison with laissez-faire equilibrium. Such numerical analyses are valid because the Ramsey steady state is proven to exist and is unique. Since our goal is not to simulate a realistic real-world economy, we do not intend to calibrate the model parameters to match real-world data. Instead, these numerical analyses are meant to demonstrate the transitional dynamics of the Ramsey allocation with/without a binding government debt limit and reveal the trade-off under capital taxation between aggregate allocative efficiency and individual allocative efficiency.

4.1 Parameter Values

Consider the case of log utility ($\sigma = 1$). The government spending G_t is set to zero for all periods. The initial government bond and capital tax are also set to zero ($B_0 = 0$ and $\tau_{k,0} = 0$). The initial capital stock K_0 is set to be 20% below the Ramsey steady-state level. The production function is assumed to be Cobb-Douglas with capital share $\alpha = 1/3$. Each model period corresponds to 3 years and hence the time discount rate β is set to 0.91 (implying an annual rate of 0.97) and the capital depreciation rate δ is 0.25 (implying an annual rate of roughly 9%). The distribution of preference shock θ follows a power function $\mathbf{F}(\theta) = \frac{\theta^\gamma - \theta_L^\gamma}{\theta_H^\gamma - \theta_L^\gamma}$, where $\theta_L = 1$, $\theta_H = 10$ and $\gamma = 0.1$. The results are qualitatively similar for other choices of parameter values.

These parameter values imply that the following conditions are satisfied: (i) In the steady state the condition for positive individual labor choice, $\theta_H < \frac{\theta_L}{(1-\beta)^\sigma} < \infty$, is satisfied. (ii) In the transition the condition for positive labor supply $n_t > 0$ is satisfied and verified

numerically in each time period. (iii) The condition (17) holds, such that the laissez-faire competitive equilibrium is neither individual allocative efficiency nor aggregate allocative efficiency.

Under these parameter values, the efficient allocation is feasible for the Ramsey planner with a non-binding debt limit and corresponds to Case 1 described in Proposition 5.

4.2 Ramsey Transition Paths

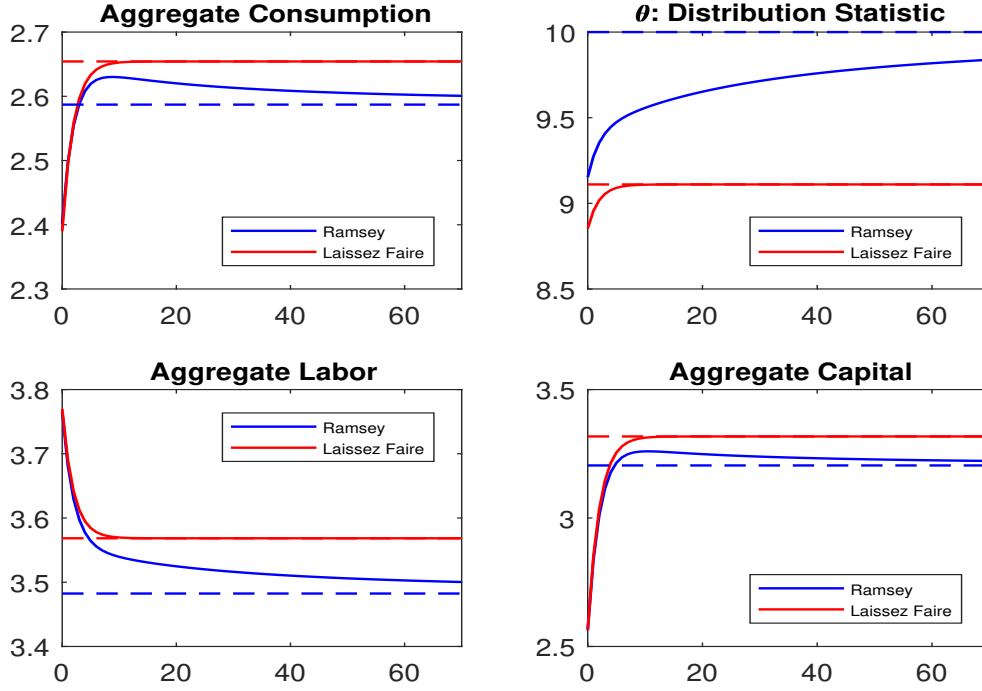
Figure 1 shows the transition paths of aggregate consumption C_t (top left panel), aggregate labor N_t (lower-left panel), the distribution statistic θ_t^* (top-right panel), and aggregate capital stock K_t (lower-right panel). In each panel, blue lines represent the Ramsey economy, red lines represent the laissez-faire economy, a solid line represents the transition, and a dashed line represents the corresponding steady state. The results can be summarized as follows.

First, the Ramsey transition paths are significantly slower than their counterparts in the laissez-faire economy, especially the transition of the distribution statistic θ_t^* . For example, consumption, labor, distribution statistic θ_t^* , and capital stock take about 10 periods to nearly approach their respective steady states under laissez faire, as opposed to more than 70 periods under Ramsey. Recall that the Ramsey steady state features both individual allocative efficiency and aggregate allocative efficiency, hence the steady state value of θ^* is $\theta_H (= 10)$ under Ramsey.

Second, the Ramsey allocation exhibits lower levels in aggregate labor and capital but a significantly higher cutoff value θ^* , compared with their counterparts in the laissez-faire economy. This suggests that the Ramsey planner opts to induce the households to work less and invest less in capital to improve aggregate allocative efficiency. Interestingly, aggregate consumption is also lower in the Ramsey allocation than in the laissez-faire competitive equilibrium (top-left panel). But this does not necessarily imply a lower welfare, because the distribution of consumption is significantly improved—a significantly higher cutoff θ^* implies a much lower probability of a binding borrowing constraint and hence a greatly improved individual self-insurance position (or individual allocative efficiency)—thanks to the availability of government bonds.

Third, under Ramsey, during the entire transition period (except the first few periods due to the initially low level of the capital stock), aggregate consumption is higher than its steady-state value and approaches the Ramsey steady state from above (top-left panel).

Figure 1: Ramsey Transition Paths of Aggregate Variables



Notes: Transition paths in the Ramsey economy (solid blue lines) and laissez-faire economy (solid red lines), as well as their corresponding steady states (dashed lines).

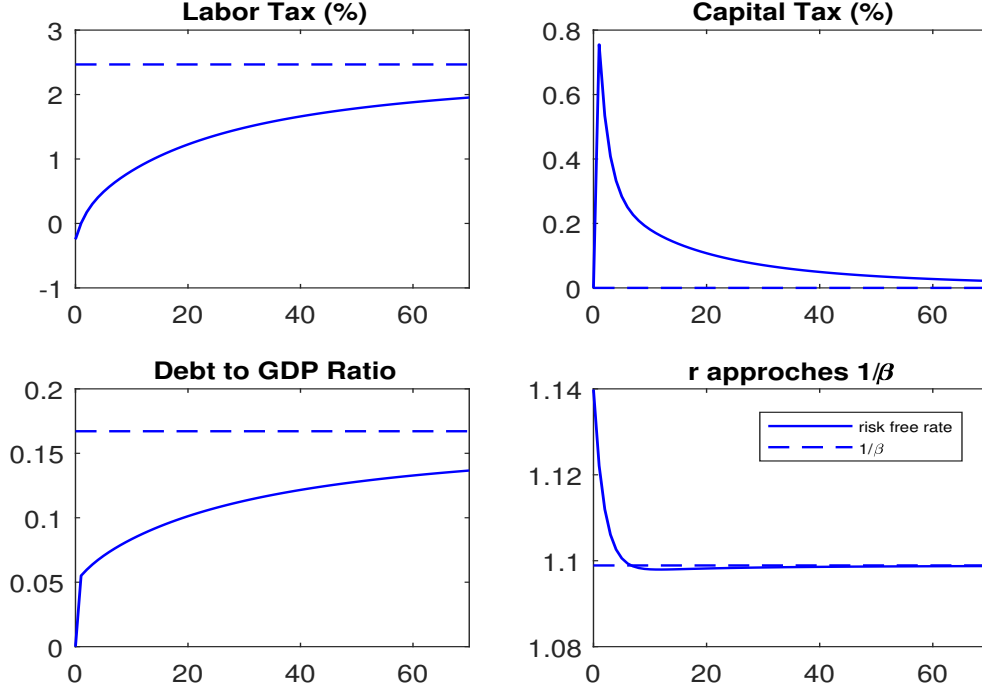
This suggests that the Ramsey planner intends to front-load consumption relative to its steady-state value.

In particular, Figure 2 shows that under the Ramsey planner, the debt-to-GDP ratio (bottom-left panel) increases rapidly from 0% to 6% in the short run to attract household savings and improve the individual self-insurance position. This crowding-out effect of government bonds on the capital stock slows down capital accumulation and lengthens the transition path, thus keeping the interest rate above its steady state for many initial periods (bottom-right panel). However, the interest rate subsequently falls below the time discount rate ($1/\beta$)—a hallmark feature of HAIM models—and converges only slowly back to the Ramsey steady state.¹¹

The rapidly increasing amount of debt clearly requires financing from tax revenues. The

¹¹Given that the initial capital stock is significantly below its steady-state, the MPK and the interest rate must be high during the initial periods. But the interest rate will eventually fall below the time discount rate due to capital overaccumulation under precautionary saving. In a model without capital, the risk-free interest rate will always lie below the time discount rate.

Figure 2: Ramsey Transition Paths of Policy Tools



Notes: Ramsey transition paths and their corresponding steady state values are shown as solid blue lines and dashed blue lines, respectively.

government can finance the debt through either a labor tax, a capital tax, or both. Interestingly, Figure 2 shows that the Ramsey planner opts to put the pressure of revenue collection on capital taxation in the short run and turn attention to labor taxation in the longer run—such that the capital tax rate is the highest initially (0.78% at $t = 1$) and gradually reduces to 0% in the long run (top-right panel); in the meanwhile, the labor tax rate is low initially (even slightly negative) and gradually approaches 2.5% in the long run (top-left panel). This suggests that the source of government revenues to finance public debt lies mainly in capital taxes in the very short run but exclusively in labor taxes in the long run.

The rapidly increasing government bonds and the positive capital tax rates in the transition periods significantly slowed down capital accumulation. This suggests that the Ramsey planner opts to tax capital *only* in the short run to subsidize labor so as to front-load consumption and work efforts. Consequently, we see the opposite transition paths of the capital tax and the labor tax in Figure 2.

Of course, it must be recognized that the key mechanism to enable the Ramsey planner

to achieve the MGR in the long run is its ability to issue plenty of debt. An important implication of this logic is that when the government cannot issue enough bonds or simply cannot issue debt at all, the Ramsey planner shall reduce short-run capital taxation (or simply do not tax capital at all in transition), even if the MGR fails to hold. In other words, in the absence of wealth-redistribution effects, the MGR appears to have no bearing on optimal capital taxation and consequently, as the debt limit \bar{B} reduces to zero, the Ramsey allocation approaches the laissez-faire allocation with zero capital tax in both the short run and the long run, as confirmed in the following subsection.

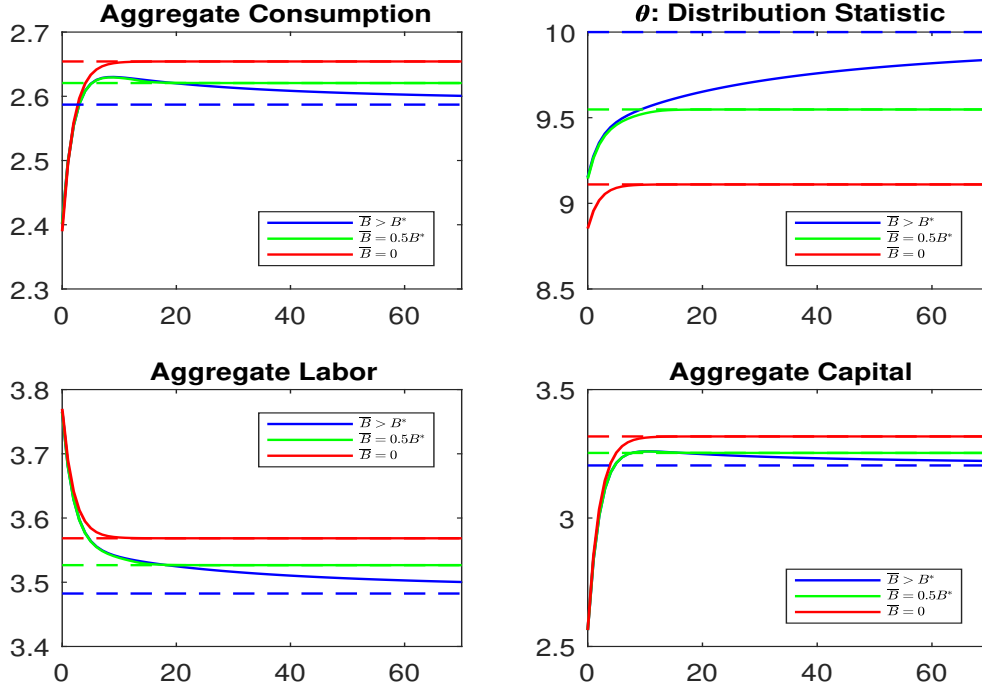
4.3 Ramsey Transitions under Binding Debt Limits

To study the dynamic and long-run effects of a binding debt limit on Ramsey allocation, we compare three scenarios in Figure 3 and Figure 4: (i) the scenario without any debt limit (blue lines), which is a reference point identical to the case shown in Figure 1 (blue lines); (ii) the scenario with a binding debt limit \bar{B} equal to 50% of the optimal debt level of efficient allocation (denoted by B^* , green lines); and (iii) the scenario with a zero debt limit $\bar{B} = 0$ (red lines).

Figure 3 shows that as the debt limit \bar{B} decreases step by step toward zero, the steady-state levels of aggregate consumption (top-left panel), labor (bottom-left panel), and capital stock (bottom-right panel) all increase and approach the corresponding laissez-faire level. Meanwhile, the steady-state cutoff decreases significantly toward the laissez-faire level (top-right panel), suggesting that the Ramsey planner becomes less and less capable of improving the individual allocative efficiency (or individual self-insurance position) when the government's capacity to issue debt is limited and reduced. Notice that as the debt limit decreases, the speed of transition also increases—because the Ramsey allocation behaves more and more like a laissez-faire competitive equilibrium without much crowding-out effect on capital from bonds, which speeds up capital accumulation and thus shortens the transition.

These results are anticipated by Proposition 6, according to which (under the assumption of zero government spending) the Ramsey steady state in scenario (iii) coincides with the laissez-faire steady state where $G = B = \tau_k = \tau_n = 0$. But here we show that when the government cannot issue bonds at all (equivalent to $\bar{B} = 0$), the entire transition path is also identical to the laissez-faire case (red lines) as shown in Figure 1. Therefore, scenario (ii) with $\bar{B} = 50$ percent of the optimal debt level of efficient allocation lies in between scenario (i) and scenario (iii).

Figure 3: Ramsey Transition Paths of Aggregate Variables with Binding Debt Limits



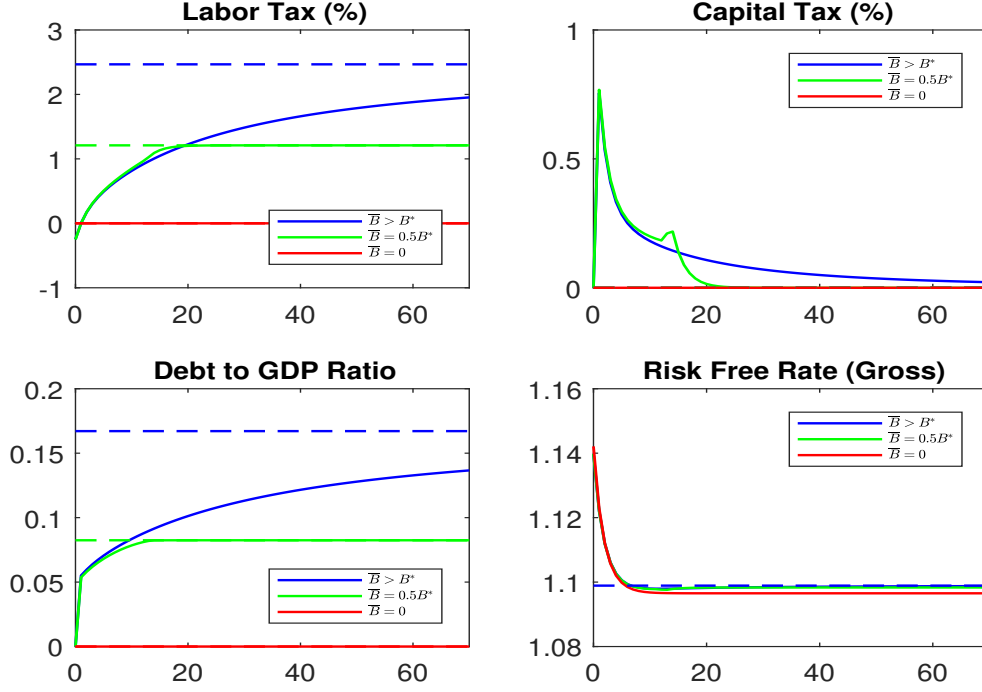
Notes: Scenario (i), (ii) and (iii) are shown as blue, green, and red lines, respectively. Their corresponding steady state levels are indicated by dashed lines.

Figure 4 shows the effects of debt limits on tax policies during the transition paths as well as in the steady state. It also offers explanations for the transition patterns of aggregate variables shown in Figure 3. First, as the debt limit reduces, the debt-to-GDP ratio and its transition time needed to approach the steady state also decline (bottom-right panel). Since a lower debt-to-GDP ratio implies that the government has a smaller burden of financing interest payments, the average tax rates for both capital and labor along the transition are reduced as well (top row panels). Interestingly, although in the initial transition period capital tax under scenario (ii) is higher than that under scenario (i), the average rate is lower because capital tax converges to the zero-steady state much faster under scenario (ii) than under scenario (i), suggesting a feature of non-linearity.

As anticipated, as the debt limit reduces, the equilibrium interest rate r_t also declines both during transition and in the steady state (bottom-right panel). Keep in mind that the steady-state interest rate under scenario (i) equals the time discount rate $1/\beta = 1.1$, so the steady-state interest rates under scenarios (ii) and (iii) are strictly lower than that under

scenario (i).

Figure 4: Ramsey Transition Paths of Policies with Binding Debt Limits



Notes: Scenario (i), (ii) and (iii) are shown as blue, green, and red lines, respectively. Their corresponding steady-state levels are indicated by dashed lines.

The main lesson taken away from this numerical subsection is that in the absence of concerns for wealth redistribution, tax policies in a heterogeneous-agents economy are shaped by the government's ability to issue debt and that the single most important function of public debt is to improve the individual self-insurance position. Since a binding debt limit handicaps the government's ability to improve individual allocative efficiency, as a result, the associated burden of interest payments and the average tax rate during transition are also reduced. However, given any level of required tax revenues to finance interest payments, the composition of the tax revenue in terms of capital tax and labor tax is dictated by the trade-off between individual allocative efficiency and aggregate allocative efficiency; so optimal capital tax is high in the short run and zero in the long run, while optimal labor tax is low in the short run but high in the long run. Hence, as the debt limit \bar{B} approaches zero, the Ramsey allocation approaches the competitive equilibrium under laissez faire both along transition and in steady state.

In other words, in the absence of government spending, steady-state tax policies have no independent role to play without the tool of government debt, and they are used solely to finance interest payments on government debt. For this very reason, since labor tax is less distortionary, a permanent capital tax in the steady state is never optimal regardless of government debt limits and the MGR (unless a capital tax can redistribute wealth from the rich to the poor, which is ruled out in our model). Since the interest rate lies below the time discount rate, the planner opts to front-load consumption by subsidizing labor and taxing capital in the short run but using labor taxes to finance public debt in the long run.

Nonetheless, even though steady-state capital taxation is not the right tool to restore the MGR, it is not optimal either to subsidize capital in spite of the government debt limit—because doing so will further reduce the equilibrium interest rate below the time discount rate and intensify the overaccumulation problem.

Elasticity of Intertemporal Substitution (EIS). The effects of EIS (σ) on Ramsey allocation are discussed in detail in our working paper, interested readers are referred to Chien and Wen (2020a). To conserve space, here we provide only a brief summary of our findings:

(i) When $0 < \sigma < 1$, an interior steady state with convergent multipliers still exists but we can no longer rule out the possibility of other types of steady states where the multipliers $\{\lambda, \mu, \phi\}$ diverge to infinity and aggregate consumption approaches zero or some finite positive values in the limit. In addition, in the interior steady state with convergent multipliers the policy implications are identical (such as a zero capital tax) and share qualitatively similar transitional dynamics to those discussed above under the parameter value $\sigma \geq 1$. The following discussions focus only on such interior Ramsey steady state.

(ii) The relative speed of convergence under Ramsey (benchmarked by the corresponding laissez-faire economy) depends negatively on the EIS, or positively on the value of σ . In particular, the transition speed is fastest in the case of low EIS (or large value of σ) and slowest in the case of high EIS (or small value of σ). This result suggests that the Ramsey planner has less room to engage in intertemporal “arbitrage” to alter the competitive equilibrium through the use of policies when the market participants’ EIS is low; consequently, the economy converges faster to its Ramsey steady state when σ is larger. The implication is that in the limit when $\sigma \rightarrow \infty$, it must be true that the Ramsey allocation approaches that of the laissez-faire competitive equilibrium—because when agents’ EIS is close to zero, the laissez-faire competitive equilibrium can achieve full self-insurance, and hence the Ramsey planner has no room (or desire) to improve the welfare of the laissez-faire economy through

fiscal policies.

(iii) The implication is that the welfare gains under the Ramsey allocation from improving laissez-faire competitive equilibrium increase with EIS. This also implies that in the limit as $\sigma \rightarrow \infty$, welfare gains approach zero both in transition and in steady state.

(iv) The Ramsey planner opts to issue a far larger amount of debt relative to GDP under a high EIS than under a low EIS. Although the long-run capital tax rate is exclusively zero regardless of σ , the short-run capital tax rate is higher under a larger EIS. The insight is that the Ramsey planner is able to front-load consumption more aggressively by improving the individual self-insurance position when the EIS is high, but doing so requires the government to reduce labor tax to stimulate work efforts and amass a larger stock of claims on the private economy, which calls for a higher capital tax to cover the interest payments.

5 A Brief Literature Review

The literature that relates to optimal capital taxation is vast. Here we review only the most relevant works in the infinite-horizon HAIM tradition à la Aiyagari (1994).

The seminal work of Aiyagari (1995) shows that precautionary saving behaviors under incomplete markets can cause overaccumulation of capital, thus calling for a positive capital tax in a Ramsey steady state (if it exists), despite the availability of a labor tax and government bonds. Our zero-capital-tax result should not be seen as a contradiction to the result of Aiyagari (1995), given the difference in models. Our contribution, instead, is to help interpret Aiyagari’s result by showing analytically that his positive-capital-tax result is not driven purely by the MGR but more likely by the wealth-redistribution effect of a capital tax. By the same reason, our result does not necessarily contradict the recent works by Acikgoz, Hagedorn, Holter, and Wang (2018), Dyrda and Pedroni (2018) and Ragot and Grand (2017), who numerically solve optimal fiscal policies along the transitional path in an infinite-horizon heterogeneous-agents economy.

Straub and Werning (2020) revisit the classical zero-capital-taxation result by Chamley (1986) and Judd (1985), and show that the common assumption that the endogenous multipliers associated with the Ramsey problem will always converge in the limit is not necessarily correct. In our model, however, we can prove the existence and uniqueness of the interior Ramsey steady state—in the sense that there cannot exist a non-interior Ramsey steady state as long as the elasticity of intertemporal substitution parameter $\sigma \geq 1$. In addition, we are able to numerically solve our model for the entire transitional path of the Ramsey problem

and confirm that the Ramsey allocation does converge to the interior Ramsey steady state. In the case of $0 < \sigma < 1$, although we are unable to rule out the existence of a non-interior Ramsey steady state, we can prove analytically that an interior Ramsey steady state could exist and show numerically that the Ramsey allocation can converge to this interior steady state.

A recent work by Bassetto and Cui (2020) studies the optimal fiscal policy in an environment where the capital stock tends to be under-accumulated due to frictions in firms' financing constraints. The capital stock in our model, however, tends to be overaccumulated due to households' borrowing constraints that lead to precautionary (excessive) savings. Despite such a difference, their analysis also indicates that the optimal debt policy is to provide a sufficient amount of public liquidity to completely alleviate firms' financial frictions whenever the fiscal capacity is not binding. As a result, the steady-state capital tax is also zero in their model, consistent with our findings. They also show an alternative type of Ramsey steady state where the Lagrangian multiplier associated with the government budget constraint diverges to infinity and the tax revenue converges to the top of the Laffer curve. This alternative steady state could occur if the consumption EIS is sufficiently high. In the extreme case of linear consumption utility (infinite EIS), the optimal capital tax is shown to be positive in this alternative steady state. Similarly, we prove analytically in the Appendix A.7 that when $0 < \sigma < 1$, there may exist multiple types of steady state: one type features FSI with zero capital tax and convergent multipliers, another type features partial self-insurance with divergent multipliers and strictly positive capital tax. Unfortunately, we are unable to determine which type yields the highest social welfare and thus constitutes a Ramsey allocation. Nonetheless, our results do not contradict theirs and, to the best of our knowledge, they are the first to study this alternative type of Ramsey steady state (with divergent multipliers and nonzero capital tax) and bring the Laffer curve argument into the picture.

Since our framework is in the class of models pioneered by Aiyagari (1995) in which infinitely lived agents are ex ante identical but ex post heterogeneous, our result thus may not apply to models with ex ante heterogeneity, such as the two-class model of Judd (1985) or the overlapping generation models. For example, in a two-period ex-ante heterogeneous-agents model, Azzimonti and Yared (2017) show that the Ramsey planner optimally limits the supply of bonds such that not all households are slack in their borrowing constraints. In an economy where agents are heterogeneous in their initial wealth, Bassetto and Benhabib (2006) demonstrates that the capital tax could stay high at its limit forever if the redis-

tribution concerns are sufficiently strong. Krueger and Ludwig (2018) use an overlapping generation model to show that optimal capital tax may be positive. Since in their model all tax revenues are lump-sum transferred back to households (to serve as an insurance device) and there are no government bonds, their result is expected to be different from ours.

Gottardi, Kajii, and Nakajima (2015) revisits optimal Ramsey taxation in an environment with uninsurable human-capital risk. As in our model, model tractability enables them to provide transparent analysis on the Ramsey problem. In a special case where government spending and the bond supply are both set to zero, they find that in the steady state the Ramsey planner should tax human capital and subsidize physical capital, despite the overaccumulation of physical capital.

Aiyagari and McGrattan (1998) and Flodén (2001) study optimal supply of government debt in the Aiyagari (1994) model. Similar to our finding, government bonds are shown to play an important role in providing self-insurance for households and to help in relaxing their borrowing constraints. However, their analysis focus on maximizing the households' steady-state welfare instead of their time-zero dynamic welfare. In addition, they do not separate the distinctive roles that a capital tax plays in the MGR and wealth redistribution. However, Domeij and Heathcote (2004) show that welfare along the transitional path is an important concern for the Ramsey planner and their finding indicates that steady-state welfare maximization could be misleading when designing optimal policies. More recently, Chien and Wen (2020b) show that, because Ramsey plans in HAIM models are time inconsistent, the optimal tax and debt policies are doomed to be different between steady-state welfare analysis and dynamic welfare analysis.

6 Conclusion with Remarks for Future Research

This paper designs a special and tractable infinite-horizon HAIM model that can isolate the long-run impact of a capital tax and public debt on the MPK without any wealth-redistribution effects, so as to address a set of long-standing issues in the optimal Ramsey capital taxation literature. Especially, the separation of government bonds' effects on wealth redistribution and liquidity provision is achieved by assuming quasi-linear preferences, which gives rise to the tractability of our model. Analytical tractability enables us to provide necessary and sufficient conditions for the existence of an interior Ramsey steady state and establish several novel results under conventional parameter values: (i) In line with the classical result of zero capital taxation based on representative-agent models—where redis-

tribution is never a concern—the optimal steady-state capital tax is also exclusively zero in our infinite-horizon HAIM economy (regardless of government debt limits). (ii) The Ramsey planner opts to levy a capital tax only in the transition path before the Ramsey steady state is reached. (iii) Whether the MGR holds or not in a Ramsey steady state depends critically on the government’s capacity to issue debt, but it has no bearing on the planner’s long-run capital tax scheme. (iv) The optimal debt-to-GDP ratio in the absence of a binding debt limit, however, is determined by a positive wedge times the MGR saving rate; the wedge is decreasing in the strength of individual self-insurance and approaches zero when idiosyncratic risk vanishes or markets are complete.

The key insight behind our results is that under incomplete markets there exist both an intertemporal wedge and an intratemporal wedge in the failure of the MGR—the former wedge pertaining to a positive liquidity premium (or a gap between the interest rate and the time discount rate) and the latter wedge to a difference between the socially optimal MPK and the premium-adjusted private MPK. The intertemporal wedge indicates individual allocative inefficiency in terms of insufficient self-insurance and the intratemporal wedge indicates aggregate allocative inefficiency in terms of overaccumulation of capital. However, the second wedge is the consequence of the first wedge. So in the absence of any wealth-redistribution effects of government policies, the Ramsey planner’s ultimate concern for self-insurance—the root cause of aggregate inefficiency—implies that it is optimal to issue a sufficient amount of bonds to achieve full self-insurance until a debt limit binds. Since a capital tax has little effect on the liquidity premium or the first wedge, therefore, taxing capital in the steady state is never optimal because it permanently erodes individuals’ self insurance positions without much effect on eliminating the intertemporal wedge when public debt is available. Consequently, the Ramsey planner prefers taxing capital only in the short run and using a labor tax to finance public debt in the long run, thus correcting the capital-overaccumulation problem through the provision of public liquidity. Consequently, the optimal steady-state capital tax is exclusively zero even if the MGR fails to hold, such as in the case of a binding debt limit.¹²

We thus reveal in a transparent manner that government debt plays an important role in determining the optimal Ramsey tax scheme when facing the trade-off between aggregate

¹²Subsidizing capital in the steady state is not optimal either, because it permanently worsens the capital-overaccumulation problem and thus would force the Ramsey planner to dramatically increase transitory capital taxation, offsetting the original intention (and potential welfare gains) of boosting household savings. Therefore, when the government is unable to issue debt at all, it is optimal to do nothing by setting both the capital tax and labor tax to zero, instead of imposing a steady-state labor tax and using the receipts to subsidize capital while taxing capital heavily in the transition period.

allocative efficiency and individual allocative efficiency in HAIM models. Hence, the issues of the optimal quantity of public debt and optimal capital tax cannot be studied in isolation. The fundamental reason behind our result is that under precautionary saving with borrowing constraints, the risk-free rate lies below the time discount rate—a hallmark feature of Aiyagari-type models—which suggests the existence of an “arbitrage” opportunity for the Ramsey planner to exploit along the transition path since the planner can finance the debt cheaply by discounting distortionary future taxes at a higher rate than the inverse of the interest rate. In other words, because the interest rate lies below the time discount rate in a competitive equilibrium, the planner’s motive to improve individuals’ self-insurance positions and relax their borrowing constraints dominates the planner’s concerns for distortionary labor taxes, dictating the planner to amass an ever-increasing amount of public debt to relax households’ borrowing constraints. This motive, however, may also result in a dynamic path featuring no Ramsey steady state, such as in the case of a fat-tailed distribution of the idiosyncratic shocks (i.e., $\theta_H \rightarrow \infty$ as in a Pareto distribution of θ_t).¹³

Nonetheless, we still ask to what extent that our zero-capital-tax result in this paper is driven by the quasi-linear preference structure. The assumption of linear disutility from labor explicitly implies infinite Frisch elasticity of the labor supply; consequently, the model implies that for given financial wealth, a decrease in the after tax wage (for example, due to increase in the labor-tax rate) leads to an increase in the labor supply and that issuing more debt financed with labor income tax crowds in the labor supply and hence increases the MPK, which counters the crowding out of capital via the interest rate channel. Since a non-linear preference structure would not only render the model analytically intractable as in the Aiyagari (1994) model but also brings back the wealth redistribution effect of government policies, fully answering this important question is beyond the scope of the current paper. However, in a separate companion paper (Chen, Chien, Wen, and Yang (2020)), we use an infinite-horizon HAIM model (in which preferences are non-linear and the Frisch elasticity of labor supply is finite) to demonstrate analytically that in the absence of any wealth-redistribution effects of government policies, the optimal capital tax is zero in the Ramsey steady state despite capital overaccumulation under precautionary saving motives and borrowing constraints. More specifically, that paper relaxes the assumption of

¹³In a different paper with an entirely different model setup—i.e., without capital but with non-linear preferences and a partially degenerate wealth distribution (Chien and Wen (2020b)), we show that a full self-insurance allocation is still the goal of the Ramsey planner precisely for the same reason that the interest rate lies below the time discount rate. In that paper we also show that a non-interior Ramsey steady state exists under a sufficiently persistent idiosyncratic shock process.

quasi-linear preferences and reintroduces idiosyncratic labor income shocks as the source of idiosyncratic risk as in the Aiyagari (1994) model, but assumes instead that households can reshuffle asset holdings at the end of each period such that the distribution of households' end-of-period wealth is degenerate, as in Heathcote and Perri (2018). Our result in that paper indicates that in the long run, public debt is a better tool than capital taxation to restore aggregate productive efficiency. Therefore, we can conclude once again that if there is a reason to tax capital in the Ramsey steady state at all (given that such a steady state does exist), it must have something to do with the tax's effect on wealth redistribution rather than on capital overaccumulation or the failure of the MGR.

Our simple and tractable model can be extended in many directions. For example, there is a strong tradition and renewed interest in studying the optimal responses of fiscal policies to aggregate government spending shocks. The works by Barro (1979) and Lucas and Stokey (1983) both identify the importance of tax smoothing but give different predictions on the optimal behavior of government bonds. Aiyagari, Marcet, Sargent, and Seppala (2002) show that the (in)complete-market assumption explains the different findings of these two classical papers. Farhi (2010) studies the optimal capital taxation by adding capital into the model of Aiyagari, Marcet, Sargent, and Seppala (2002). A more recent work by Bhandari, Evans, Golosov, and Sargent (2017) contributes to this literature by investigating the optimal portfolio decision in an incomplete-markets economy with aggregate risks. Also more recently, Bassetto (2014) and Bhandari, Evans, Golosov, and Sargent (2018) extend this literature to a heterogeneous-agent framework; but the numerical approach taken by Bhandari, Evans, Golosov, and Sargent (2018) sidesteps the issue of possible nonexistence of a Ramsey steady state. Our model can be extended to an environment with both idiosyncratic and aggregate risks and complement this literature by offering a more tractable and transparent analysis.¹⁴

¹⁴This task is currently being undertaken by the authors.

References

- ACIKGOZ, O., M. HAGEDORN, H. HOLTER, AND Y. WANG (2018): “The Optimum Quantity of Capital and Debt,” CEPR Discussion Papers 12952, C.E.P.R. Discussion Papers.
- AIYAGARI, S. R. (1994): “Uninsured Idiosyncratic Risk and Aggregate Saving,” *Quarterly Journal of Economics*, 109(3), 659–684.
- AIYAGARI, S. R. (1995): “Optimal Capital Income Taxation with Incomplete Markets, Borrowing Constraints, and Constant Discounting,” *Journal of Political Economy*, 103(6), 1158–75.
- AIYAGARI, S. R., A. MARCET, T. J. SARGENT, AND J. SEPPALA (2002): “Optimal Taxation without State-Contingent Debt,” *Journal of Political Economy*, 110(6), 1220–1254.
- AIYAGARI, S. R., AND E. R. MCGRATTAN (1998): “The optimum quantity of debt,” *Journal of Monetary Economics*, 42(3), 447–469.
- AZZIMONTI, M., AND P. YARED (2017): “A note on optimal fiscal policy in an economy with private borrowing limits,” *Economics Letters*, 151(C), 62–65.
- BARRO, R. J. (1979): “On the Determination of the Public Debt,” *Journal of Political Economy*, 87(5), 940–71.
- BASSETTO, M. (2014): “Optimal fiscal policy with heterogeneous agents,” *Quantitative Economics*, 5(3), 675–704.
- BASSETTO, M., AND J. BENHABIB (2006): “Redistribution, Taxes and the Median Voter,” *Review of Economic Dynamics*, 9(2), 211–223.
- BASSETTO, M., AND W. CUI (2020): “A Ramsey Theory of Financial Distortions,” Working Papers 775, Federal Reserve Bank of Minneapolis.
- BEWLEY, T. (1980): “The Optimum Quantity of Money,” in *Models of Monetary Economics*, ed. by J. H. Kareken, and N. Wallace, pp. 169–210. Federal Reserve Bank of Minneapolis, Minneapolis.
- BHANDARI, A., D. EVANS, M. GOLOSOV, AND T. J. SARGENT (2017): “Fiscal Policy and Debt Management with Incomplete Markets,” *The Quarterly Journal of Economics*, 132(2), 617–663.

- (2018): “Inequality, Business Cycles and Monetary-Fiscal Policy,” Working papers, University of Chicago.
- CHAMLEY, C. (1986): “Optimal Taxation of Capital Income in General Equilibrium with Infinite Lives,” *Econometrica*, 54(3), 607–22.
- (2001): “Capital income taxation, wealth distribution and borrowing constraints,” *Journal of Public Economics*, 79(1), 55–69.
- CHEN, Y., Y. CHIEN, Y. WEN, AND C. YANG (2020): “Should Capital Be Taxed?,” Working Papers 2020-033, Federal Reserve Bank of St. Louis.
- (2021): “Why Might Lump-sum Transfers Not Be a Good Idea?,” Working Papers 2021-002, Federal Reserve Bank of St. Louis.
- CHIEN, Y., AND Y. WEN (2020a): “Optimal Ramsey Taxation in Heterogeneous Agent Economies with Quasi-Linear Preferences,” Working Papers 2019-007D, Federal Reserve Bank of St. Louis.
- (2020b): “Time-Inconsistent Optimal Quantity of Debt,” Working Papers 2020-037B, Federal Reserve Bank of St. Louis.
- CONESA, J. C., S. KITAO, AND D. KRUEGER (2009): “Taxing Capital? Not a Bad Idea after All!,” *American Economic Review*, 99(1), 25–48.
- DÁVILA, J., J. H. HONG, P. KRUSELL, AND J.-V. RÍOS-RULL (2012): “Constrained Efficiency in the Neoclassical Growth Model with Uninsurable Idiosyncratic Shocks,” *Econometrica*, 80(6), 2431–2467.
- DEATON, A. (1991): “Saving and Liquidity Constraints,” *Econometrica*, 59(5), 1221–1248.
- DOMELIJ, D., AND J. HEATHCOTE (2004): “On the Distributional Effects of Reducing Capital Taxes,” *International Economic Review*, 45(2), 523–554.
- DYRDA, S., AND M. PEDRONI (2018): “Optimal Fiscal Policy in a Model with Uninsurable Idiosyncratic Shocks,” Working papers, University of Toronto, Department of Economics.
- FARHI, E. (2010): “Capital Taxation and Ownership When Markets Are Incomplete,” *Journal of Political Economy*, 118(5), 908–948.

- FLODÉN, M. (2001): “The effectiveness of government debt and transfers as insurance,” *Journal of Monetary Economics*, 48(1), 81–108.
- GOTTARDI, P., A. KAJII, AND T. NAKAJIMA (2015): “Optimal Taxation and Debt with Uninsurable Risks to Human Capital Accumulation,” *American Economic Review*, 105(11), 3443–3470.
- HEATHCOTE, J., AND F. PERRI (2018): “Wealth and Volatility,” *Review of Economic Studies*, 85(4), 2173–2213.
- HUGGETT, M. (1993): “The Risk-Free Rate in Heterogeneous-Agent Incomplete-Insurance Economies,” *Journal of Economic Dynamics and Control*, 17(5-6), 953–969.
- JUDD, K. L. (1985): “Redistributive Taxation in a Simple Perfect Foresight Model,” *Journal of Public Economics*, 28(1), 59–83.
- KRUEGER, D., AND A. LUDWIG (2018): “Optimal Taxes on Capital in the OLG Model with Uninsurable Idiosyncratic Income Risk,” NBER Working Papers 24335, National Bureau of Economic Research, Inc.
- LJUNGQVIST, L., AND T. J. SARGENT (2012): *Recursive Macroeconomic Theory, Third Edition*, MIT Press Books. The MIT Press.
- LUCAS, ROBERT E, J. (1980): “Equilibrium in a Pure Currency Economy,” *Economic Inquiry*, 18(2), 203–220.
- LUCAS, R. J., AND N. L. STOKEY (1983): “Optimal Fiscal and Monetary Policy in an Economy without Capital,” *Journal of Monetary Economics*, 12(1), 55–93.
- RAGOT, X., AND F. L. GRAND (2017): “Optimal Fiscal Policy with Heterogeneous Agents and Aggregate Shocks,” 2017 Meeting Papers 969, Society for Economic Dynamics.
- STRAUB, L., AND I. WERNING (2020): “Positive Long-Run Capital Taxation: Chamley-Judd Revisited,” *American Economic Review*, 110(1), 86–119.
- WEN, Y. (2015): “Money, liquidity and welfare,” *European Economic Review*, 76(C), 1–24.

A Appendix

A.1 Proof of Proposition 1

Denoting $\{\beta^t \lambda_t^h(\theta^t), \beta^t \mu_t^h(\theta^t)\}$ as the Lagrangian multipliers for constraints (6) and (7), respectively, the first-order conditions for $\{c_t(\theta^t), n_t(\theta^{t-1}), a_{t+1}(\theta^t)\}$ are given, respectively, by

$$\frac{\theta_t}{c_t(\theta^t)^\sigma} = \lambda_t^h(\theta^t) \quad (35)$$

$$1 = \bar{w}_t \int \lambda_t^h(\theta^t) d\mathbf{F}(\theta_t) \quad (36)$$

$$\lambda_t^h(\theta^t) = \beta r_{t+1} E_t [\lambda_{t+1}^h(\theta^{t+1})] + \mu_t^h(\theta^t), \quad (37)$$

where equation (36) reflects that the labor supply $n_t(\theta^{t-1})$ must be chosen before the idiosyncratic taste shocks (and hence before the value of $\lambda_t^h(\theta^t)$) are realized. By the law of iterated expectations and the *iid* assumption of idiosyncratic shocks, equation (37) can be written (using equation (36)) as

$$\lambda_t^h(\theta^t) = \beta \frac{r_{t+1}}{\bar{w}_{t+1}} + \mu_t^h(\theta^t), \quad (38)$$

where $\frac{1}{\bar{w}}$ is the marginal utility of consumption in terms of labor income.

We adopt a guess-and-verify strategy to derive the decision rules. The decision rules for an individual's consumption and savings are characterized by a cutoff strategy, taking as given the aggregate states (such as the interest rate and real wage). Anticipating that the optimal cutoff θ_t^* is independent of an individual's history of shocks, consider two possible cases:

Case A. $\theta_t \leq \theta_t^*$. In this case the urge to consume is low. It is hence optimal to save so as to prevent possible liquidity constraints in the future. So $a_{t+1}(\theta^t) \geq 0$, $\mu_t^h(\theta^t) = 0$, and the shadow value is

$$\lambda_t^h(\theta^t) = \beta \frac{r_{t+1}}{\bar{w}_{t+1}} \equiv \Lambda_t,$$

where Λ_t depends only on aggregate states. Notice that $\lambda_t^h(\theta^t) = \lambda_t^h$ is independent of the history of idiosyncratic shocks. Equation (35) implies that consumption is given by $c_t(\theta^t)^\sigma = \theta_t \Lambda_t^{-1}$. Defining $x_t(\theta^{t-1}) \equiv r_t a_t(\theta^{t-1}) + \bar{w}_t n_t(\theta^{t-1})$ as the gross income of the household, the budget identity (6) then implies $a_{t+1}(\theta^t) = x_t(\theta^{t-1}) - (\theta_t \Lambda_t^{-1})^{1/\sigma}$. The requirement

$a_{t+1}(\theta^t) \geq 0$ then implies

$$\theta_t \leq \Lambda_t x_t^\sigma \equiv \theta_t^*, \quad (39)$$

which defines the cutoff θ_t^* .

We conjecture that the cutoff is independent of the idiosyncratic state. Then the optimal gross income x_t is also independent of the idiosyncratic state. The intuition is that x_t is determined before the realization of θ_t and that all households face the same distribution of idiosyncratic shocks. Since the utility function is quasi-linear, the household is able to adjust labor income to meet any target level of liquidity in hand. As a result, the distribution of x_t is degenerate. This property simplifies the model tremendously.

Case B. $\theta_t > \theta_t^*$. In this case the urge to consume is high. It is then optimal not to save, so $a_{t+1}(\theta^t) = 0$ and $\mu_t^h(\theta^t) > 0$. By the resource constraint (6), we have $c_t(\theta^t) = x_t$, which by equation (39) implies $c_t(\theta^t)^\sigma = \theta_t^* \Lambda_t^{-1}$. Equation (35) then implies that the shadow value is given by $\lambda_t^h(\theta^t) = \frac{\theta_t}{\theta_t^*} \Lambda_t$. Since $\theta_t > \theta_t^*$, equation (38) implies $\mu_t^h(\theta^t) = \Lambda_t \left[\frac{\theta_t}{\theta_t^*} - 1 \right] > 0$. Notice that the shadow value of goods (the marginal utility of income), $\lambda_t^h(\theta^t)$, is higher under case B than under case A because of binding borrowing constraints.

The above analyses imply that the expected shadow value of income, $\int \lambda_t^h(\theta) d\mathbf{F}(\theta)$, and hence the optimal cutoff value θ^* , is determined by equation (36) by plugging in the expressions for $\lambda_t^h(\theta^t)$ under cases A and B, which immediately gives equation (12). Specifically, combining case A and case B, we have

$$\begin{aligned} \lambda_t^h(\theta^t) &= \beta \frac{r_{t+1}}{\bar{w}_{t+1}} \text{ for } \theta \leq \theta_t^* \\ \lambda_t^h(\theta^t) &= \frac{\theta_t}{\theta_t^*} \beta \frac{r_{t+1}}{\bar{w}_{t+1}} \text{ for } \theta_t \geq \theta_t^*. \end{aligned}$$

The aggregate Euler equation is therefore given by

$$\frac{1}{\bar{w}_t} = \int \lambda_t^h(\theta) d\mathbf{F}(\theta) = \beta \frac{r_{t+1}}{\bar{w}_{t+1}} \left[\int_{\theta \leq \theta_t^*} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) \right] = \beta \frac{r_{t+1}}{\bar{w}_{t+1}} L(\theta_t^*),$$

which is equation (12). This equation reveals that the optimal cutoff depends only on aggregate states and is independent of individual history.

We also immediately obtain

$$x_t = \left[\theta_t^* \left(\beta \frac{r_{t+1}}{\bar{w}_{t+1}} \right)^{-1} \right]^{1/\sigma} = [\theta_t^* L(\theta_t^*) \bar{w}_t]^{1/\sigma},$$

which leads to equation (8). By the discussion of cases A and B, as well as the use of equation (8), the decision rules of household consumption and saving can then be summarized by equations (9) and (10), respectively. Finally, the decision rule of the household labor supply, equation (11), is decided residually to satisfy the household budget constraint.

Finally, to ensure that the above proof and hence the associated cutoff-policy rules are consistent with the assumption of interior choices of labor, namely, $n_t \in (0, \overline{N})$, we need to consider the following two cases:

First, to ensure that $n_t(\theta^{t-1}) > 0$, consider the worst situation where $n_t(\theta^{t-1})$ takes its minimum value. Given $x_t = r_t a_t(\theta^{t-1}) + \overline{w}_t n_t(\theta^{t-1})$, $n_t(\theta^{t-1})$ is at its minimum if $\mu_t^h = 0$ and $a_t(\theta^{t-1})$ takes the maximum possible value, $a_t(\theta^{t-1}) = \left[1 - \left(\frac{\theta_L}{\theta_{t-1}^*}\right)^{1/\sigma}\right] x_{t-1}$. So $n_t(\theta^{t-1}) > 0$ if

$$x_t - r_t \left[1 - \left(\frac{\theta_L}{\theta_{t-1}^*}\right)^{1/\sigma}\right] x_{t-1} > 0, \quad (40)$$

which is independent of the shock θ_t . This condition in the steady state becomes $1 - r \left[1 - \left(\frac{\theta_L}{\theta^*}\right)^{1/\sigma}\right] > 0$, or equivalently (by using equation (12)),

$$\beta L(\theta^*) > 1 - \left(\frac{\theta_L}{\theta^*}\right)^{1/\sigma}. \quad (41)$$

Given that $L(\theta^*)$ is a monotonic decreasing function in θ^* with a lower bound of 1, the necessary condition to satisfy (41) in the steady state is $\beta > 1 - \left(\frac{\theta_L}{\theta^*}\right)^{1/\sigma}$, which is easy to satisfy when $\theta_H < \infty$. Therefore, as long as the condition $\beta > 1 - \left(\frac{\theta_L}{\theta^*}\right)^{1/\sigma}$ is met, the condition (40) is assumed to hold throughout the paper.

Second, to ensure that $n_t < \overline{N}$, consider those agents who encounter the borrowing constraint last period such that $a_t(\theta^{t-1}) = 0$. Their labor supply reaches the maximum value at $n_t(\theta^{t-1}) = \frac{x_t}{\overline{w}_t} = \theta_t^* L(\theta_t^*)$. Given a finite steady state value of θ^* , the value of \overline{N} can be chosen such that

$$\overline{N} = \theta_H \geq \theta^* L(\theta^*). \quad (42)$$

A.2 Proof of Proposition 2

In the laissez-faire economy, the capital tax, the labor tax, government spending and government bond are all equal to zero. In this laissez-faire competitive equilibrium, the capital-to-labor ratio $\frac{K_t}{N_t}$ satisfies two conditions. The first condition is derived from the resource

constraint (16), which can be expressed as

$$F(K_t, N_t) + (1 - \delta)K_t = C_t + K_{t+1} = x_t,$$

where the last equality uses the definition of x_t . Dividing both sides of the equation by K_t gives

$$\left(\frac{K_t}{N_t}\right)^{\alpha-1} + (1 - \delta) = \frac{1}{1 - D(\theta_t^*)}, \quad (43)$$

where x_t/K_t is substituted out by $\frac{1}{1-D(\theta_t^*)}$.

The second condition is derived by combining equation (12) and the no-arbitrage condition, $r_t = 1 + q_t - \delta$, which gives

$$1 = \beta \left(1 + \alpha \left(\frac{K_t}{N_t} \right)^{\alpha-1} - \delta \right) L(\theta_t^*), \quad (44)$$

where the marginal product of capital q_t is replaced by $\alpha \left(\frac{K_t}{N_t} \right)^{\alpha-1}$. Since the capital-to-labor ratio must be the same in both equations, conditions (43) and (44) imply the following equation in the steady state:

$$\frac{\alpha\beta}{(1 - D(\theta^*))} + \beta(1 - \alpha)(1 - \delta) = \frac{1}{L(\theta^*)}, \quad (45)$$

which solves for the steady-state value of θ^* .

It can be shown easily that both $L(\theta^*)$ and $D(\theta^*)$ are monotonically decreasing in θ^* , thus the right-hand side (RHS) of equation (45) increases monotonically in θ^* and the left-hand side (LHS) of equation (45) decreases monotonically in θ^* .

It remains to be seen if the RHS and the LHS cross each other at an interior value of $\theta^* \in [\theta_L, \theta_H]$. The RHS of equation (45) reaches its minimum value of 1 when $\theta^* = \theta_H$ and its maximum value of $\bar{\theta}/\theta_L > 1$ when $\theta^* = \theta_L$. The LHS of equation (45) takes the maximum value of infinity when $\theta^* = \theta_L$ and the minimum value of $\frac{\alpha\beta(\theta_H)^{1/\sigma}}{(\theta_H)^{1/\sigma} - \mathbb{E}(\theta^{1/\sigma})} + \beta(1 - \alpha)(1 - \delta)$ when $\theta^* = \theta_H$. Thus, an interior solution exists if and only if

$$\frac{\alpha\beta(\theta_H)^{1/\sigma}}{(\theta_H)^{1/\sigma} - \mathbb{E}(\theta^{1/\sigma})} + \beta(1 - \alpha)(1 - \delta) < 1.$$

Clearly, $\theta^* = \theta_L$ cannot constitute a solution for any positive value when $\theta_L > 0$. On the other hand, $\theta^* = \theta_H$ may constitute a solution if the above condition is violated. For

example, if θ_H is small and close enough to the $E(\theta^{\frac{1}{\sigma}})^\sigma$, then the above condition does not hold since its LHS approaches infinity when $\theta_H \rightarrow E(\theta^{\frac{1}{\sigma}})^\sigma$. Therefore, an interior solution for θ^* exists if the upper bound of the idiosyncratic shock is large enough. Otherwise, we have the corner solution $\theta^* = \theta_H$. Finally, if θ^* is an interior solution, then $L(\theta^*) > 1$ and $r < 1/\beta$ by equation (12).

A.3 Proof of Proposition 3

A.3.1 The “only if” Part

Assume that we have the allocation $\{\theta_t^*, C_t, N_t, K_{t+1}, B_{t+1}\}_{t=0}^\infty$ and the initial risk-free rate r_0 . We then can directly construct the prices, taxes, and individual allocations in the competitive equilibrium in the following 7 steps:

1. w_t and q_t are set by (1) and (2), which are $w_t = MP_{N,t}$ and $q_t = MP_{K,t}$, respectively.
2. Given C_t and θ_t^* , the total liquidity in hand can be set by equation (18), $x_t = \frac{C_t}{D(\theta_t^*)}$.
3. The individual consumption and asset holdings, $c_t(\theta_t)$ and $a_{t+1}(\theta_t)$, are pinned down by equations (9) and (10).
4. $\tau_{n,t}$ is determined by equation (8), which implies $\tau_{n,t} = 1 - \frac{x_t^\sigma}{L(\theta_t^*)\theta_t^* MP_{N,t}}$. Hence, \bar{w}_t can be expressed as

$$\bar{w}_t = \frac{x_t^\sigma}{L(\theta_t^*)\theta_t^*} = \frac{C_t^\sigma}{D(\theta_t^*)^\sigma L(\theta_t^*)\theta_t^*}.$$

Given \bar{w}_t , the interest rate r_{t+1} can be backed out by the Euler equation (12):

$$\frac{1}{r_{t+1}} = \beta L(\theta_t^*) \frac{\bar{w}_t}{\bar{w}_{t+1}} = \beta \frac{U_{C,t+1}}{U_{C,t}} \frac{D(\theta_{t+1}^*)^\sigma \theta_{t+1}^*}{D(\theta_t^*)^\sigma \theta_t^*} L(\theta_{t+1}^*) \text{ for all } t \geq 0$$

where $U_{C,t}$ is defined as $C_t^{-\sigma}$, the “marginal utility of aggregate consumption” given our preference assumption.

Given r_0 and the expression of $\{r_{t+1}\}_{t=0}^\infty$, the capital tax $\{\tau_{k,t+1}\}_{t=0}^\infty$ is chosen to satisfy the no-arbitrage condition: $r_t = 1 + (1 - \tau_{k,t})MP_{K,t} - \delta$ for all $t \geq 1$.

5. Finally, set $n_t(\theta_{t-1})$ to satisfy equation (11), which is implied by the individual household budget constraint.

6. Define A_{t+1} as the aggregate asset holding in period t . Integrating (10) gives

$$A_{t+1} \equiv \int a_{t+1}(\theta_t) dF(\theta_t) = \int \max \left\{ 1 - \left(\frac{\theta_t}{\theta_t^*} \right)^{1/\sigma}, 0 \right\} x_t dF(\theta_t) = \left(\frac{1}{D(\theta_t^*)} - 1 \right) C_t,$$

where the last equality utilizes equation (18). The above equation together with the condition defined in equation (20) gives the competitive equilibrium asset market clearing condition (14).

7. The implementability conditions are

$$C_0^{1-\sigma} D(\theta_0^*)^{\sigma-1} L(\theta_0^*) \theta_0^* \geq N_0 + r_0 C_0^{-\sigma} D(\theta_0^*)^{\sigma} L(\theta_0^*) \theta_0^* (K_0 + B_0)$$

and for $t \geq 1$,

$$C_t^{1-\sigma} D(\theta_t^*)^{\sigma-1} L(\theta_t^*) \theta_t^* \geq N_t + \frac{1}{\beta} C_{t-1}^{1-\sigma} D(\theta_{t-1}^*)^{\sigma-1} \theta_{t-1}^* \left(\frac{1}{D(\theta_{t-1}^*)} - 1 \right)$$

Multiplying both side of the above equation with $\frac{C_t^\sigma}{D(\theta_t^*)^\sigma L(\theta_t^*) \theta_t^*}$ leads to

$$\begin{aligned} \frac{C_0}{D(\theta_0^*)} &\geq \frac{C_0^\sigma}{D(\theta_0^*)^\sigma L(\theta_0^*) \theta_0^*} N_0 + r_0 (K_0 + B_0) \\ \frac{C_t}{D(\theta_t^*)} &\geq \frac{C_t^\sigma}{D(\theta_t^*)^\sigma L(\theta_t^*) \theta_t^*} N_t + \frac{1}{\beta} \frac{C_{t-1}^{-\sigma} D(\theta_{t-1}^*)^\sigma \theta_{t-1}^*}{C_t^{-\sigma} D(\theta_t^*)^\sigma \theta_t^*} \frac{1}{L(\theta_t^*)} (1 - D(\theta_{t-1}^*)) C_{t-1}. \end{aligned}$$

Using the relationship constructed in steps 2, 4, and 6 for x_t , C_t , A_{t+1} , \bar{w}_t , and r_t in the above equation gives

$$C_t + A_{t+1} \geq \bar{w}_t N_t + r_t A_t \text{ for all } t \geq 0,$$

which together with the resource constraint and asset market clearing condition enforce the government budget constraint.

Step 1 ensures that the representative firm's problem is solved. Steps 2 to 5 guarantee that the individual household problem is solved. Steps 6 and 7 ensure that the asset market clearing condition and government budget constraint are satisfied, respectively. The labor market clearing condition is satisfied by Walras law.

A.3.2 The “if” Part

Note that the resource constraint and asset market clearing condition are trivially implied by a competitive equilibrium since they are part of the definition. The implementability condition is constructed as follows. First, we rewrite the government budget constraint as

$$G_t \leq F(K_t, N_t) - (1 - \tau_{k,t})q_t K_t - (1 - \tau_{n,t})w_t N_t + B_{t+1} - r_t B_t.$$

Combining this equation with the resource constraint (16), no-arbitrage condition, and the asset market clearing condition (14) implies

$$(1 - \tau_{n,t})w_t N_t + r_t A_t \leq C_t + A_{t+1},$$

and hence leading to the following equation:

$$C_t + A_{t+1} \geq \bar{w}_t N_t + r_t A_t. \quad (46)$$

For $t \geq 1$, the equilibrium conditions (8), (18), and (12) suggest that \bar{w}_t and r_t can be expressed as

$$\bar{w}_t = \frac{1}{C_t^{-\sigma} D(\theta_t^*)^\sigma L(\theta_t^*) \theta_t^*}$$

and

$$r_t = \frac{1}{\beta} \frac{C_{t-1}^{-\sigma} D(\theta_{t-1}^*)^\sigma L(\theta_{t-1}^*) \theta_{t-1}^*}{C_t^{-\sigma} D(\theta_t^*)^\sigma L(\theta_t^*) \theta_t^*} \frac{1}{L(\theta_t^*)}.$$

Substituting the above two equations into (46) and rearranging terms, we get

$$C_t^{-\sigma} D(\theta_t^*)^\sigma L(\theta_t^*) \theta_t^* C_t + C_t^{-\sigma} D(\theta_t^*)^\sigma L(\theta_t^*) \theta_t^* A_{t+1} \geq N_t + \frac{1}{\beta} C_{t-1}^{-\sigma} D(\theta_{t-1}^*)^\sigma L(\theta_{t-1}^*) \theta_{t-1}^* A_t,$$

The implementability condition (22) follows by plugging $A_{t+1} = \left(\frac{1}{D(\theta_t^*)} - 1\right) C_t$ and $A_t = \left(\frac{1}{D(\theta_{t-1}^*)} - 1\right) C_{t-1}$ to the equation above.

For the initial period, $\{B_0, K_0, \tau_{k,0}\}$ are given, which implies that $r_0 = 1 + (1 - \tau_{k,0})MP_{K,0} - \delta$ is pinned down by N_0 . Therefore, the first period implementability condition could be rewritten as

$$C_0^{1-\sigma} D(\theta_0^*)^{\sigma-1} L(\theta_0^*) \theta_0^* \geq N_0 + r_0 C_0^{-\sigma} D(\theta_0^*)^\sigma L(\theta_0^*) \theta_0^* (K_0 + B_0).$$

A.4 Ramsey Problem and Optimality Conditions

The Lagrangian of the Ramsey problem is given by

$$\begin{aligned}
L = & \max_{\{\theta_t^*, N_t, C_t, K_{t+1}, B_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \left[W(\theta_t^*) \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{\bar{\theta}}{1-\sigma} - N_t \right] \\
& + \sum_{t=0}^{\infty} \beta^t \mu_t (F(K_t, N_t) + (1-\delta)K_t - G_t - C_t - K_{t+1}) \\
& + \lambda_0 (C_0^{1-\sigma} D(\theta_0^*)^{\sigma-1} L(\theta_0^*) \theta_0^* - N_0 - C_0^{-\sigma} D(\theta_0^*)^{\sigma} L(\theta_0^*) \theta_0^* r_0 (K_0 + B_0)) \\
& + \sum_{t=1}^{\infty} \beta^t \lambda_t (C_t^{1-\sigma} D(\theta_t^*)^{\sigma-1} L(\theta_t^*) \theta_t^* - N_t - \beta^{-1} C_{t-1}^{1-\sigma} D(\theta_{t-1}^*)^{\sigma-1} \theta_{t-1}^* (1 - D(\theta_{t-1}^*))) \\
& + \sum_{t=0}^{\infty} \beta^t \phi_t (K_{t+1} + B_{t+1} - (D(\theta_t^*)^{-1} - 1) C_t) \\
& + \sum_{t=0}^{\infty} \beta^t \nu_t^B (\bar{B} - B_{t+1}),
\end{aligned} \tag{47}$$

where μ_t , λ_t , and ϕ_t denote the multipliers for the resource constraints, the implementability conditions, and the asset market clearing conditions, respectively. In addition, the multiplier of the debt limit constraint is denoted by ν_t^B . The K_0 , B_0 , and $\tau_{k,0}$ are given and hence r_0 depends only on N_0 . The first-order Ramsey conditions with respect to K_{t+1} , N_0 , N_t , C_0 , C_t , B_{t+1} , θ_0 and θ_t^* are given, respectively, by

$$\mu_t - \phi_t = \beta \mu_{t+1} (MP_{K,t+1} + 1 - \delta) \tag{48}$$

$$1 + \lambda_0 + \lambda_0 C_0^{-\sigma} D(\theta_0^*)^{\sigma} L(\theta_0^*) \theta_0^* \frac{\partial r_0}{\partial N_0} (K_0 + B_0) = \mu_0 MP_{N,0} \tag{49}$$

$$1 + \lambda_t = \mu_t MP_{N,t} \text{ for } t \geq 1 \tag{50}$$

$$\begin{aligned}
\mu_0 = & W(\theta_0^*) C_0^{-\sigma} + \lambda_0 (1 - \sigma) C_0^{-\sigma} D(\theta_0^*)^{\sigma-1} L(\theta_0^*) \theta_0^* \\
& + \lambda_0 \sigma C_0^{-\sigma-1} D(\theta_0^*)^{\sigma} L(\theta_0^*) \theta_0^* r_0 (K_0 + B_0) \\
& - \lambda_1 (1 - \sigma) C_0^{-\sigma} D(\theta_0^*)^{\sigma} \theta_0^* \left(\frac{1}{D(\theta_0^*)} - 1 \right) - \phi_0 \left(\frac{1}{D(\theta_0^*)} - 1 \right)
\end{aligned} \tag{51}$$

$$\begin{aligned}
\mu_t = & W(\theta_t^*) C_t^{-\sigma} + (1 - \sigma) C_t^{-\sigma} D(\theta_t^*)^{\sigma-1} \theta_t^* (\lambda_t L(\theta_t^*) - \lambda_{t+1} (1 - D(\theta_t^*))) \\
& - \phi_t (D(\theta_t^*)^{-1} - 1) \text{ for } t \geq 1
\end{aligned} \tag{52}$$

$$\beta^t \phi_t - \beta^t \nu_t^B = 0 \quad (53)$$

$$\begin{aligned} & \frac{\partial W(\theta_0^*)}{\partial \theta_0^*} \frac{C_0^{1-\sigma}}{1-\sigma} + \lambda_0 C_0^{1-\sigma} H(\theta_0^*) - \lambda_1 C_0^{1-\sigma} J(\theta_0^*) + \phi_0 \frac{C_0}{D(\theta_0^*)^2} \frac{\partial D(\theta_0^*)}{\partial \theta_0^*} \\ &= \lambda_0 C_0^{-\sigma} \frac{\partial (D(\theta_0^*)^\sigma L(\theta_0^*) \theta_0^*)}{\partial \theta_0^*} r_0 (K_0 + B_0). \end{aligned} \quad (54)$$

$$\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{C_t^{1-\sigma}}{1-\sigma} + \lambda_t C_t^{1-\sigma} H(\theta_t^*) - \lambda_{t+1} C_t^{1-\sigma} J(\theta_t^*) + \phi_t \frac{C_t}{D(\theta_t^*)^2} \frac{\partial D(\theta_t^*)}{\partial \theta_t^*} = 0 \text{ for } t \geq 1, \quad (55)$$

where

$$\begin{aligned} H(\theta_t^*) &\equiv \frac{\partial (D(\theta_t^*)^{\sigma-1} L(\theta_t^*) \theta_t^*)}{\partial \theta_t^*} \\ J(\theta_t^*) &\equiv \frac{\partial (D(\theta_t^*)^{\sigma-1} \theta_t^* (1 - D(\theta_t^*)))}{\partial \theta_t^*}. \end{aligned}$$

Note that the Lagrangian multiplier ϕ_t for the asset market clearing condition and the multiplier ν_t^B for the government debt-limit constraint are equal to each other according to equation (53), suggesting that the tightness of the asset market depends on the government debt limit.

A.5 Proofs of Lemmas

In addition, since the relative magnitude of $H(\theta^*)$ and $J(\theta^*)$ affects the dynamics of λ_t in equation (55), we provide the following four lemmas to help characterize the optimal Ramsey allocation.

Lemma 1. *The derivative of $W(\theta_t^*)$ is given by*

$$\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} = \frac{(1-\sigma)}{\sigma} D(\theta_t^*)^{\sigma-1} X(\theta_t^*) \left[\frac{M(\theta_t^*)}{D(\theta_t^*) \theta_t^*} - 1 \right],$$

where $X(\theta_t^*)$ and $M(\theta_t^*)$ are defined as

$$X(\theta_t^*) \equiv \int_{\theta \leq \theta_t^*} \left(\frac{\theta}{\theta_t^*} \right)^{1/\sigma} d\mathbf{F}(\theta), \quad (56)$$

$$M(\theta_t^*) \equiv \int_{\theta \leq \theta_t^*} \theta_t^* \left(\frac{\theta}{\theta_t^*} \right)^{\frac{1}{\sigma}} d\mathbf{F} + \int_{\theta > \theta_t^*} \theta d\mathbf{F}, \quad (57)$$

respectively. In addition, $M(\theta_t^*) \geq \theta_t^* D(\theta_t^*)$ with equality if $\theta_t^* = \theta_H$. Hence, the sign of $\frac{\partial W(\theta^*)}{\partial \theta^*}$ is determined by the sign of the elasticity of intertemporal substitution coefficient, $1/\sigma$.

Proof. By the definition of $M(\theta_t^*)$, the $W(\theta_t^*)$ can be rewritten as $W(\theta_t^*) = M(\theta_t^*) D(\theta_t^*)^{\sigma-1}$. The derivative of $M(\theta_t^*)$ and hence $W(\theta_t^*)$ are given by

$$\frac{\partial M(\theta_t^*)}{\partial \theta_t^*} = -\frac{1-\sigma}{\sigma} \int_{\theta \leq \theta_t^*} \left(\frac{\theta}{\theta_t^*} \right)^{\frac{1}{\sigma}} d\mathbf{F} = -\frac{1-\sigma}{\sigma} X(\theta_t^*)$$

and

$$\begin{aligned} \frac{\partial W(\theta_t^*)}{\partial \theta_t^*} &= (\sigma-1) D(\theta_t^*)^{\sigma-2} \frac{\partial D(\theta_t^*)}{\partial \theta_t^*} M(\theta_t^*) - D(\theta_t^*)^{\sigma-1} \frac{1-\sigma}{\sigma} X(\theta_t^*) \\ &= \frac{(1-\sigma)}{\sigma} D(\theta_t^*)^{\sigma-1} X(\theta_t^*) \left[\frac{M(\theta_t^*)}{D(\theta_t^*) \theta_t^*} - 1 \right] \end{aligned}$$

In addition,

$$M(\theta_t^*) - \theta_t^* D(\theta_t^*) = \int_{\theta > \theta_t^*} (\theta - \theta_t^*) d\mathbf{F}(\theta) \geq 0$$

Hence, $M(\theta_t^*) \geq \theta_t^* D(\theta_t^*)$ with equality holds at $\theta_t^* = \theta_H$ □

Lemma 2. $J(\theta_t^*)$ and $H(\theta_t^*)$ can be expressed as

$$H(\theta_t^*) = D(\theta_t^*)^{\sigma-1} X(\theta_t^*) \left[\frac{1-\sigma}{\sigma} \frac{L(\theta_t^*)}{D(\theta_t^*)} + \frac{\mathbf{F}(\theta_t^*)}{X(\theta_t^*)} \right], \quad (58)$$

$$J(\theta_t^*) = D(\theta_t^*)^{\sigma-1} X(\theta_t^*) \left[\frac{1-\sigma}{\sigma} \frac{1}{D(\theta_t^*)} + \frac{\mathbf{F}(\theta_t^*)}{X(\theta_t^*)} \right], \quad (59)$$

which have the following properties: (1) if $\sigma < 1$, then $H(\theta_t^*) > J(\theta_t^*) > 0$ for all $\theta_t^* \in (\theta_L, \theta_H)$; (2) if $\sigma = 1$, then $H(\theta_t^*) = J(\theta_t^*) > 0$; and (3) if $\sigma > 1$, then $J(\theta_t^*) > 0$ and $J(\theta_t^*) > H(\theta_t^*)$ for all $\theta_t^* \in (\theta_L, \theta_H)$.

Proof. $H(\theta_t^*)$ can be expressed as

$$\begin{aligned} H(\theta_t^*) &\equiv \frac{\partial (D(\theta_t^*)^{\sigma-1} \theta_t^* L(\theta_t^*))}{\partial \theta_t^*} = (\sigma-1) D(\theta_t^*)^{\sigma-2} \frac{\partial D(\theta_t^*)}{\partial \theta_t^*} \theta_t^* L(\theta_t^*) + D(\theta_t^*)^{\sigma-1} \mathbf{F}(\theta_t^*) \\ &= D(\theta_t^*)^{\sigma-1} \left[(\sigma-1) D(\theta_t^*)^{-1} \frac{\partial D(\theta_t^*)}{\partial \theta_t^*} \theta_t^* L(\theta_t^*) + \mathbf{F}(\theta_t^*) \right] \\ &= D(\theta_t^*)^{\sigma-1} X(\theta_t^*) \left[\frac{1-\sigma}{\sigma} \frac{L(\theta_t^*)}{D(\theta_t^*)} + \frac{\mathbf{F}(\theta_t^*)}{X(\theta_t^*)} \right]. \end{aligned}$$

$J(\theta_t^*)$ can be rewritten as

$$\begin{aligned}
J(\theta_t^*) &\equiv \frac{\partial(D(\theta_t^*)^{\sigma-1}\theta_t^*(1-D(\theta_t^*)))}{\partial\theta_t^*} \\
&= (\sigma-1)D(\theta_t^*)^{\sigma-2}\frac{\partial D(\theta_t^*)}{\partial\theta_t^*}\theta_t^*(1-D(\theta_t^*)) + D(\theta_t^*)^{\sigma-1}\left(1-D(\theta_t^*) - \theta_t^*\frac{\partial D(\theta_t^*)}{\partial\theta_t^*}\right) \\
&= \frac{1-\sigma}{\sigma}D(\theta_t^*)^{\sigma-2}X(\theta_t^*)(1-D(\theta_t^*)) + D(\theta_t^*)^{\sigma-1}\left(\mathbf{F}(\theta_t^*) + \left(\frac{1}{\sigma}-1\right)X(\theta_t^*)\right) \\
&= D(\theta_t^*)^{\sigma-1}X(\theta_t^*)\left[\frac{1-\sigma}{\sigma}\frac{1}{D(\theta_t^*)} + \frac{\mathbf{F}(\theta_t^*)}{X(\theta_t^*)}\right]
\end{aligned}$$

By $H(\theta_t^*)$ and $J(\theta_t^*)$ expressed above, we reach that

1. if $\sigma < 1$, $0 < J(\theta_t^*) \leq H(\theta_t^*)$ and $J(\theta_t^*) = H(\theta_t^*)$ if $\theta_t^* = \theta_H$
2. if $\sigma = 1$, $0 \leq J(\theta_t^*) = H(\theta_t^*)$.
3. if $\sigma > 1$, $J(\theta_t^*) \geq H(\theta_t^*)$ and $J(\theta_t^*) = H(\theta_t^*)$ only if $\theta_t^* = \theta_H$. For $J(\theta_t^*) > 0$, we need to show that $\frac{1-\sigma}{\sigma}\frac{1}{D(\theta_t^*)} + \frac{\mathbf{F}(\theta_t^*)}{X(\theta_t^*)} > 0$. By the definition of $\mathbf{F}(\theta_t^*)$ and $X(\theta_t^*)$, we know that

$$X(\theta_t^*) \leq \mathbf{F}(\theta_t^*),$$

which implies

$$X(\theta_t^*) \leq \mathbf{F}(\theta_t^*)(1 - \mathbf{F}(\theta_t^*)) + \mathbf{F}(\theta_t^*)X(\theta_t^*) = \mathbf{F}(\theta_t^*)D(\theta_t^*)$$

where the last equality utilities the definition of $D(\theta_t^*)$ and $X(\theta_t^*)$. The equation above can further be rewritten as

$$\frac{\mathbf{F}(\theta_t^*)}{X(\theta_t^*)} \geq \frac{1}{D(\theta_t^*)}$$

Under $\sigma > 1$, the equation above implies $\frac{1-\sigma}{\sigma}\frac{1}{D(\theta_t^*)} + \frac{\mathbf{F}(\theta_t^*)}{X(\theta_t^*)} > 0$.

□

Lemma 3. $\frac{\partial D(\theta_t^*)}{\partial\theta_t^*} = -\frac{X(\theta_t^*)}{\sigma\theta_t^*} < 0$ for all $\theta_t^* \in (\theta_L, \theta_H]$.

Proof. $D(\theta_t^*)$ is therefore given by

$$D(\theta_t^*) = X(\theta_t^*) + \int_{\theta > \theta_t^*} d\mathbf{F}(\theta).$$

The derivative of $D(\theta_t^*)$ is

$$\frac{\partial D(\theta_t^*)}{\partial \theta_t^*} = -\frac{1}{\sigma} \frac{1}{\theta_t^*} \int_{\theta \leq \theta_t^*} \left(\frac{\theta}{\theta_t^*} \right)^{\frac{1}{\sigma}} d\mathbf{F}(\theta) = -\frac{X(\theta_t^*)}{\sigma \theta_t^*} < 0$$

□

Lemma 4. $L(\theta_t^*) - 1 + D(\theta_t^*) = \frac{M(\theta_t^*)}{\theta_t^*}$.

Proof. By the definition of $L(\theta_t^*)$ and $D(\theta_t^*)$, we get

$$\begin{aligned} L(\theta_t^*) - 1 + D(\theta_t^*) &= \int_{\theta \leq \theta_t^*} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) + X(\theta_t^*) + \int_{\theta > \theta_t^*} d\mathbf{F}(\theta) - 1 \\ &= \int_{\theta > \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) + X(\theta_t^*) = \frac{M(\theta_t^*)}{\theta_t^*}, \end{aligned}$$

where the last equality utilize the definition of $M(\theta_t^*)$.

□

A.6 Proof of Proposition 4

The optimal capital tax is chosen such that the Euler equation in the competitive equilibrium (12) is consistent with the one chosen by the Ramsey planner shown in (48). Hence, $\tau_{k,t+1}$ is pinned down by

$$1 - \tau_{k,t+1} = \frac{\frac{\bar{w}_{t+1}}{\bar{w}_t} \frac{1}{L(\theta_t^*)} - \beta(1 - \delta)}{\frac{\mu_t - \phi_t}{\mu_{t+1}} - \beta(1 - \delta)},$$

which is equation (30) in an interior Ramsey steady state.

A.7 Proof of Proposition 5

From equation (53), $\nu_t^B = 0$ implies that $\phi_t = 0$. The first-order condition with respect to θ_t^* is then reduced to

$$\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{1}{1 - \sigma} + \lambda_t H(\theta_t^*) - \lambda_{t+1} J(\theta_t^*) = 0,$$

which can be further simplified according to Lemma 1 and Lemma 2 as

$$\widehat{W}(\theta_t^*) + \lambda_t \widehat{H}(\theta_t^*) = \lambda_{t+1} \widehat{J}(\theta_t^*), \quad (60)$$

where $\widehat{H}(\theta_t^*)$, $\widehat{J}(\theta_t^*)$, and $\widehat{W}(\theta_t^*)$ are defined, respectively, as

$$\widehat{H}(\theta_t^*) \equiv \frac{1-\sigma}{\sigma} \frac{L(\theta_t^*)}{D(\theta_t^*)} + \frac{\mathbf{F}(\theta_t^*)}{X(\theta_t^*)}, \quad (61)$$

$$\widehat{J}(\theta_t^*) \equiv \frac{1-\sigma}{\sigma} \frac{1}{D(\theta_t^*)} + \frac{\mathbf{F}(\theta_t^*)}{X(\theta_t^*)}, \quad (62)$$

$$\widehat{W}(\theta_t^*) \equiv \frac{1}{\sigma} \frac{M(\theta_t^*)}{D(\theta_t^*)\theta_t^*} - \frac{1}{\sigma} \geq 0. \quad (63)$$

Note that $\widehat{W}(\theta_t^*) > 0$ for all $\theta_t^* \in [\theta_L, \theta_H)$ and $\sigma \in (0, \infty)$. Also, for $\theta_t^* = \theta_H$, $\widehat{W}(\theta_t^*) = 0$ and $\widehat{H}(\theta_t^*) = \widehat{J}(\theta_t^*)$.

A.7.1 Existence of FSI Interior Ramsey Steady State

In what follows, we first sketch the proof that an interior Ramsey steady state featuring $\theta^* = \theta_H$ exists if β is sufficiently large. We proceed by the following steps, which show that the conjecture $\theta^* = \theta_H$ satisfies all of the Ramsey FOCs and these FOC-implied steady-state values of the aggregate allocation $\{C, N, K, B\} > 0$ as well as Lagrangian multipliers $\{\lambda, \mu\}$ are mutually consistent, strictly positive, and finitely-valued:

1. The FOC with respect to θ_t^* in equation (55) is satisfied at $\theta_t^* = \theta_H$. Specifically, plugging $\theta^* = \theta_H$ into the definitions of $\widehat{J}(\theta_t^*)$, $\widehat{H}(\theta_t^*)$, and $\widehat{W}(\theta_t^*)$ leads to $\widehat{J}(\theta_H) = \widehat{H}(\theta_H) \neq 0$ and $\widehat{W}(\theta_H) = 0$. Therefore, the FOC (55) implies that $\lambda_t = \lambda_{t+1} = \lambda$ in the steady state, which together with the FOC (50) ensure the convergence of multiplier μ_t .
2. Then, the steady-state version of FOC with respect to K in equation (48) gives

$$1 = \beta (MP_K + 1 - \delta),$$

which implies $MP_K = \alpha \left(\frac{K}{N}\right)^{\alpha-1} = \frac{1-\beta(1-\delta)}{\beta} \in (0, \infty)$ (i.e., the capital-to-labor ratio is unique, strictly positive, and bounded). Given the assumption of the production function, it must be true that the following ratios are unique, strictly positive, and finite: $\{K/N, Y/K, MP_N, Y/N\} \in (0, \infty)$. More specifically, the Y/N and Y/K ratios can be expressed, respectively, as

$$\frac{Y}{N} = \left(\frac{K}{N}\right)^\alpha = \left(\frac{1-\beta(1-\delta)}{\beta\alpha}\right)^{\frac{\alpha}{\alpha-1}} \quad (64)$$

and

$$\frac{Y}{K} = \left(\frac{K}{N}\right)^{\alpha-1} = \frac{1 - \beta(1 - \delta)}{\alpha\beta}. \quad (65)$$

3. The resource constraint,

$$F(N, K) = K^\alpha N^{1-\alpha} = C + \delta K + G,$$

together with a finite level of government spending G implies a unique ratio $C/K \in (0, \infty)$ as

$$\frac{C}{Y} = \left(1 - \delta \frac{K}{Y} - \frac{G}{Y}\right) = \left(1 - \frac{\alpha\beta\delta}{1 - \beta(1 - \delta)} - \frac{G}{Y}\right), \quad (66)$$

where the last equality uses (65).

4. We know that under our parameter restrictions the level of labor is interior, $N \in (0, \overline{N})$; hence, it must be true that the aggregate allocation is also unique and interior: $\{C, K, Y\} \in (0, \infty)$.

5. Next, we show that $\{\mu, \lambda\} \in (0, \infty)$. Given $\theta^* = \theta_H$ in the steady state, the FOCs (50) and (52) become

$$1 + \lambda = \mu MP_N \quad (67)$$

and

$$\mu C^\sigma D(\theta_H)^{-\sigma} \theta_H^{-1} = 1 + \lambda(1 - \sigma), \quad (68)$$

respectively. These two equations uniquely solve for $\{\lambda, \mu\}$. Note that $\mu \in (0, \infty)$ if $\lambda \in (0, \infty)$ according to (67). These two equations above imply

$$1 + \lambda = Q(1 + (1 - \sigma)\lambda),$$

or equivalently,

$$\lambda = \frac{Q - 1}{1 + Q(\sigma - 1)},$$

where $Q = \frac{MP_N D(\theta_H)^\sigma \theta_H}{C^\sigma} \in (0, \infty)$. For $\sigma \geq 1$, it is straightforward to verify that $\lambda \in (0, \infty)$ if and only if $Q > 1$. For $0 < \sigma < 1$, we can easily show that $\lambda \in (0, \infty)$ if and only if $1 < Q < \frac{1}{1-\sigma}$.

6. What is Q ? Note that the steady-state version of the implementability condition (22)

with $\theta_t^* = \theta_H$ becomes

$$N = C^{1-\sigma} D(\theta_H)^{\sigma-1} \theta_H \left(1 - \frac{1 - D(\theta_H)}{\beta} \right),$$

which together with equations (64) and (66) imply

$$C^\sigma = \left(1 - \frac{\beta\alpha\delta}{1 - \beta(1-\delta)} - \frac{G}{Y} \right) \left(\frac{1 - \beta(1-\delta)}{\beta\alpha} \right)^{\frac{\alpha}{\alpha-1}} D(\theta_H)^{\sigma-1} \theta_H \left(1 - \frac{1 - D(\theta_H)}{\beta} \right). \quad (69)$$

Thus, by equations (69) and (64), Q can be rewritten as

$$\begin{aligned} Q &= (1 - \alpha) \frac{Y}{N} \frac{D(\theta_H)^\sigma \theta_H}{C^\sigma} \\ &= \frac{1 - \alpha}{\left(1 - \frac{\beta\alpha\delta}{1 - \beta(1-\delta)} - \frac{G}{Y} \right)} \frac{D(\theta_H)}{\left(1 - \frac{1 - D(\theta_H)}{\beta} \right)}. \end{aligned} \quad (70)$$

Now we are ready to find parameter conditions to ensure first $Q > 1$ and $\lambda \in (0, \infty)$ when $\sigma \geq 1$. If $\beta > 1 - D(\theta_H)$, then $\left(1 - \frac{1 - D(\theta_H)}{\beta} \right) > 0$ and hence the second term $\frac{D(\theta_H)}{\left(1 - \frac{1 - D(\theta_H)}{\beta} \right)} > 1$. As a result, a sufficient condition for $Q > 1$ is that for a sufficiently large β the first term $\frac{1 - \alpha}{\left(1 - \frac{\beta\alpha\delta}{1 - \beta(1-\delta)} - \frac{G}{Y} \right)}$ is no less than 1, which is clearly true because $\lim_{\beta \rightarrow 1} \frac{1 - \alpha}{\left(1 - \frac{\beta\alpha\delta}{1 - \beta(1-\delta)} - \frac{G}{Y} \right)} = \frac{1 - \alpha}{1 - \alpha - \frac{G}{Y}} \geq 1$. In addition, notice that the second term is independent of α , and the first term increases as α decreases and approaches the limit $\lim_{\alpha \rightarrow 0} \frac{1 - \alpha}{\left(1 - \frac{\beta\alpha\delta}{1 - \beta(1-\delta)} - \frac{G}{Y} \right)} = \frac{1}{1 - \frac{G}{Y}} \geq 1$. So the set of sufficient conditions for $Q > 1$ is easy to satisfy as long as $\beta > 1 - D(\theta_H)$.

On the other hand, if $0 < \sigma < 1$, we need the following additional condition to ensure $\lambda \in (0, \infty)$:

$$Q < \frac{1}{1 - \sigma},$$

which is equivalent to

$$\frac{1 - \alpha}{\left(1 - \frac{\beta\alpha\delta}{1 - \beta(1-\delta)} - \frac{G}{Y} \right)} \frac{D(\theta_H)}{\left(1 - \frac{1 - D(\theta_H)}{\beta} \right)} < \frac{1}{1 - \sigma}.$$

Given that $\frac{D(\theta_H)}{\left(1 - \frac{1 - D(\theta_H)}{\beta} \right)} > 1$ is required to ensure $Q > 1$, a necessary condition for

$Q < \frac{1}{1-\sigma}$ is

$$\frac{1-\alpha}{\left(1 - \frac{\beta\alpha\delta}{1-\beta(1-\delta)} - \frac{G}{Y}\right)} < \frac{1}{1-\sigma}, \quad (71)$$

which implies that the government spending-to-GDP ratio must be sufficiently small:

$$\frac{G}{Y} < 1 - \frac{\beta\alpha\delta}{1-\beta(1-\delta)} - (1-\alpha)(1-\sigma).$$

Since the right-hand side of the above inequality is between 0 and 1, this necessary condition makes economic sense. For example, suppose $\frac{G}{Y} = 0$, then condition (71) becomes $\frac{1-\alpha}{\left(1 - \frac{\beta\alpha\delta}{1-\beta(1-\delta)}\right)} < \frac{1}{1-\sigma}$, which is clearly true since $\frac{1-\alpha}{\left(1 - \frac{\beta\alpha\delta}{1-\beta(1-\delta)}\right)} < 1$. Therefore, a FSI Ramsey steady state may also exist when $0 < \sigma < 1$.

Given $\lambda \in (0, \infty)$, it is then easy to see that $\mu \in (0, \infty)$ from condition (67).

7. Notice that the required values for β discussed above do not contradict the competitive-equilibrium condition (17) for an interior cutoff and the condition for an interior $N \in (0, \overline{N})$ because these interior conditions hold true for all $\{\beta, \delta, \alpha\} \in (0, 1)$ as long as θ_H is sufficiently large relative to $E(\theta)$ (such that a competitive equilibrium does not feature full self-insurance).
8. It remains to be shown that the optimal quantity of government debt B is strictly positive. Given that aggregate output $Y \in (0, \infty)$, it suffices to show that the debt-to-GDP ratio $\frac{B}{Y} \in (0, \infty)$ and is unique. Since $K + B = \left(\frac{1}{D(\theta^*)} - 1\right) C$, we have

$$\begin{aligned} \frac{B}{Y} &= \left(\frac{1}{D(\theta_H)} - 1\right) \frac{C}{Y} - \frac{K}{Y} \\ &= \left(\frac{1}{D(\theta_H)} - 1\right) \left(1 - \frac{\delta\beta\alpha}{1-\beta(1-\delta)} - \frac{G}{Y}\right) - \left(\frac{\beta\alpha}{1-\beta(1-\delta)}\right), \end{aligned} \quad (72)$$

which is finite because each term in the parentheses is finite. The debt-to-GDP ratio is also strictly positive provided that the output elasticity of capital α is small enough. To see this, noting that the first term $\left(\frac{1}{D(\theta_H)} - 1\right) > 0$, the second term $\left(1 - \frac{\delta\beta\alpha}{1-\beta(1-\delta)} - \frac{G}{Y}\right) > 0$ because it is the consumption-to-output ratio $\frac{C}{Y}$, and the last term vanishes as α approaches zero. Notice that such a restriction on the value for α does not contradict the competitive-equilibrium condition (17) for interior cutoff and the conditions for interior $N \in (0, \overline{N})$.

9. Finally, by Proposition 4, since $\phi = 0$ and $L(\theta^*) = \frac{\mu}{\mu-\phi} = 1$, it must be true that $\theta^* = \theta_H$ and $\tau_k = 1 - \frac{L(\theta^*)^{-1} - \beta(1-\delta)}{\frac{\mu-\phi}{\mu} - \beta(1-\delta)} = 0$ in a Ramsey steady state. The steady-state equilibrium interest rate is therefore $1/\beta$ by the Euler equation (12). In addition, the labor tax rate is determined by equation (8), which implies $\tau_n = \frac{\lambda}{1+\lambda}\sigma$ in the steady state. Namely,

$$\tau_n = 1 - \frac{1}{L(\theta_H^*)\theta_H^*D(\theta_H^*)^\sigma MP_N} \frac{C^\sigma}{MP_N} = \frac{\lambda}{1+\lambda}\sigma,$$

where the last equality uses equations (67) and (68). Therefore, government expenditures and bond interest payments are financed solely by labor tax income in the Ramsey steady state.

This above steps finish the proof for the existence of an interior Ramsey steady state with convergent multipliers at the corner $\theta^* = \theta_H$ for all possible values of $\sigma \in (0, \infty)$.

A.7.2 Uniqueness of the FSI Interior Ramsey Steady State When $\sigma \geq 1$

By “uniqueness” we mean that if $\sigma \geq 1$ there do not exist other types of Ramsey steady states with divergent multipliers and a different cutoff $\theta^* \in [\theta_L, \theta_H)$ that implies only partial self-insurance and a non-zero capital tax.

Given the FSI allocation, it is trivial to see that the steady-state values of all other endogenous variables are strictly positive and finitely valued by checking the equations in the previous steps.

In what follows we prove by contradiction the uniqueness of the FSI allocation when $\sigma \geq 1$. First, consider a steady state where $\theta^* \in (\theta_L, \theta_H)$. Rewrite (60) as

$$\lambda_{t+1} = \rho(\theta_t^*)\lambda_t + \varepsilon(\theta_t^*),$$

where $\rho(\theta_t^*) \equiv \widehat{H}(\theta_t^*)/\widehat{J}(\theta_t^*)$ and $\varepsilon(\theta_t^*) \equiv \widehat{W}(\theta_t^*)/\widehat{J}(\theta_t^*)$. The sign and value of $\rho(\theta_t^*)$ and $\varepsilon(\theta_t^*)$ depend on the value of σ ; hence, we can discuss the possible steady states according to the values of $\sigma > 1$ or $\sigma = 1$.

1. $\sigma = 1$. Then $0 < \widehat{J}(\theta_t^*) = \widehat{H}(\theta_t^*)$ according to Lemma 2. Hence, $\rho(\theta_t^*) = 1$ and $\varepsilon(\theta_t^*) > 0$. In this case λ_t does not converge and will go to infinity in the long run. A divergent λ_t implies that μ_t must also diverge to infinity according to FOC (50). On the other hand, when $\sigma = 1$, the FOC (52) implies that μ_t must be finite and positive in the steady state. This leads to a contradiction and hence cannot be a Ramsey steady state.

2. $\sigma > 1$. Then Lemma 2 implies that $\widehat{J}(\theta_t^*) > 0$ and $\widehat{J}(\theta_t^*) > \widehat{H}(\theta_t^*)$. The value and sign of $\rho(\theta_t^*)$ then depend on the sign of $\widehat{H}(\theta_t^*)$; so we discuss two subcases below.

(a) $\widehat{H}(\theta_t^*) > 0$, which implies $0 < \rho(\theta_t^*) < 1$ and $\varepsilon(\theta_t^*) > 0$. The steady-state λ is then equal to $\varepsilon(\theta_t^*)/(1 - \rho(\theta_t^*))$. We next check if this λ satisfies the FOC (52), which in the steady state can be expressed as

$$\begin{aligned} \mu C^\sigma D(\theta^*)^{1-\sigma} &= M(\theta^*) + \lambda(1 - \sigma)\theta^* (L(\theta^*) - 1 + D(\theta^*)) \\ &= M(\theta^*)(1 + \lambda(1 - \sigma)), \end{aligned} \quad (73)$$

where the last equality holds according to Lemma 4. Plugging the steady-state λ into equation (73) and using the definitions of $\widehat{J}(\theta_t^*)$, $\widehat{H}(\theta_t^*)$, and $\widehat{W}_\theta(\theta_t^*)$ give

$$\mu C^\sigma D(\theta^*)^{1-\sigma} = M(\theta^*) \left(1 + \frac{\frac{M(\theta^*)}{\theta^*} - D(\theta^*)}{1 - L(\theta^*)} \right) = M(\theta^*) \left(1 + \frac{L(\theta^*) - 1}{1 - L(\theta^*)} \right) = 0,$$

where the second equality uses the Lemma 4 again. This equation holds if and only if $\mu = 0$. This cannot be a steady state.

(b) $\widehat{H}(\theta_t^*) < 0$. In this subcase, $\rho(\theta_t^*) < 0$. There are two possibilities regarding $\rho(\theta_t^*)$: $|\rho(\theta_t^*)| < 1$ or $|\rho(\theta_t^*)| > 1$. But in both cases, the steady-state λ either converges to a negative constant or diverges to negative infinity. Therefore, this subcase cannot be a Ramsey steady-state equilibrium.

Next, we show that the case of $\theta_t^* = \theta_L$ cannot be a Ramsey equilibrium although the necessary FOC with respect to θ_t^* is satisfied. The reason is that the first term of the Ramsey objective function (27), $W(\theta_t^*)C_t^{1-\sigma}/(1 - \sigma)$, is monotonically increasing in $\theta_t^* \in (\theta_L, \theta_H)$. Hence, for a global maximum, a cutoff θ_t^* at its lower corner cannot be a Ramsey equilibrium.

To ensure that $n \in (0, \overline{N})$ (see Proposition 1), note that we have assumed $\theta_H < \frac{\theta_L}{(1-\beta)^\sigma}$, which ensures that the minimum individual labor input remains positive, as shown in Appendix A.1. Moreover, by equation (42), the maximum value of n is less than \overline{N} if $\overline{N} > \theta_H$ in this case.

In addition, we can show that the maximum individual asset demand remains finite in the Ramsey steady state even if the risk-free rate is equal to the time discount rate, $r = 1/\beta$.

Since $\theta_H < \frac{\theta_L}{(1-\beta)^\sigma}$, we have

$$x_t = \frac{C_t}{D(\theta_H)} = \frac{C_t}{E(\theta^{1/\sigma})} \theta_H^{1/\sigma} < \infty.$$

Given the finite value of cash-on-hand x_t , the individual asset holding a_{t+1} is determined by the size of the idiosyncratic shock θ_t , and the agents with the largest asset holdings are those who receive the smallest shock $\theta_t = \theta_L$, i.e.,

$$a_{t+1}(\theta_L) = \left[1 - \left(\frac{\theta_L}{\theta_H} \right)^{1/\sigma} \right] x_t,$$

which is strictly positive and finite.

A.7.3 If $\sigma < 1$, other steady states with $\lambda_t \rightarrow \infty$ and $\theta^* \in [\theta_L, \theta_H)$ are Possible

We have shown that in a FSI Ramsey steady state ($\theta^* = \theta_H$), λ and μ must be finitely positive. Furthermore, we have shown that if $\sigma \geq 1$, this FSI allocation is the only possible Ramsey steady state. Now we study the possibility of multiple types of possible Ramsey steady state when $0 < \sigma < 1$; in particular, we show that besides the FSI interior steady state there can also exist steady states with divergent multipliers and partial self-insurance: $\lambda_t \rightarrow \infty$ and $\theta^* \in [\theta_L, \theta_H)$.

We first check if a steady state with divergent multipliers satisfies the constraint set of the Ramsey problem.

1. The resource constraint (16) can be satisfied.
2. The asset market clearing condition,

$$K + B = (D(\theta^*)^{-1} - 1) C,$$

can be satisfied.

3. The implementability condition is given by

$$C^{1-\sigma} D(\theta^*)^{\sigma-1} L(\theta^*) \theta^* = N + \beta^{-1} C^{1-\sigma} D(\theta^*)^{\sigma-1} \theta^* (1 - D(\theta^*))$$

which implies

$$C^{1-\sigma} D(\theta^*)^{\sigma-1} L(\theta^*) \theta^* - \beta^{-1} C^{1-\sigma} D(\theta^*)^{\sigma-1} \theta^* (1 - D(\theta^*)) = N$$

or

$$C^{1-\sigma} D(\theta^*)^{\sigma-1} \theta^* [\beta L(\theta^*) - (1 - D(\theta^*))] = \beta N.$$

The above equation is feasible if β is large enough such that $[\beta L(\theta^*) - (1 - D(\theta^*))] > 0$.

Next, we check if all Ramsey FOCs can be satisfied:

1. Rewrite (60) as

$$\lambda_{t+1} = \rho(\theta_t^*) \lambda_t + \varepsilon(\theta_t^*),$$

where $\rho(\theta_t^*) \equiv \widehat{H}(\theta_t^*)/\widehat{J}(\theta_t^*)$ and $\varepsilon(\theta_t^*) \equiv \widehat{W}(\theta_t^*)/\widehat{J}(\theta_t^*)$. Dividing both sides by λ_t gives

$$\frac{\lambda_{t+1}}{\lambda_t} = \rho(\theta_t^*) + \frac{\varepsilon(\theta_t^*)}{\lambda_t}.$$

For the value of λ to diverge to infinity, the growth rate $\frac{\lambda_{t+1}}{\lambda_t}$ must be greater than 1, or it must be the case that

$$\lim_{t \rightarrow \infty} \frac{\lambda_{t+1}}{\lambda_t} = \rho(\theta^*) \equiv \frac{\frac{1-\sigma}{\sigma} \frac{L(\theta^*)}{D(\theta^*)} + \frac{\mathbf{F}(\theta^*)}{X(\theta^*)}}{\frac{1-\sigma}{\sigma} \frac{1}{D(\theta^*)} + \frac{\mathbf{F}(\theta^*)}{X(\theta^*)}} > 1,$$

which is true since $L(\theta^*) > 1$ whenever $\theta^* \in [\theta_L, \theta_H)$. But notice that if $\theta^* = \theta_H$, then $\lim_{t \rightarrow \infty} \frac{\lambda_{t+1}}{\lambda_t} = 1$ since $L(\theta^*) = L(\theta_H) = 1$; hence, a FSI steady state must feature finite multipliers and vice versa.

2. According to the FOC (50),

$$1 + \lambda_t = \mu_t M P_{N,t},$$

so $\lambda_t \rightarrow \infty$ implies $\mu_t \rightarrow \infty$ and $\frac{\lambda_t}{\mu_t} \rightarrow M P_N$, so the growth rates of the two multipliers must be equal: $\lim_{t \rightarrow \infty} \frac{\lambda_{t+1}}{\lambda_t} = \lim_{t \rightarrow \infty} \frac{\mu_{t+1}}{\mu_t}$. These are all possible.

3. The Ramsey FOC with respect to consumption (52) implies

$$\mu_t = W(\theta_t^*) C_t^{-\sigma} + (1 - \sigma) C_t^{-\sigma} D(\theta_t^*)^{\sigma-1} \theta_t^* (\lambda_t L(\theta_t^*) - \lambda_{t+1} (1 - D(\theta_t^*))) \quad (74)$$

or

$$\frac{\mu_t}{\lambda_t} = \frac{W(\theta_t^*)C_t^{-\sigma}}{\lambda_t} + (1 - \sigma)C_t^{-\sigma}D(\theta_t^*)^{\sigma-1}\theta_t^* \left(L(\theta_t^*) - \frac{\lambda_{t+1}}{\lambda_t}(1 - D(\theta_t^*)) \right). \quad (75)$$

Taking limit as $t \rightarrow \infty$ and $\{\lambda, \mu\} \rightarrow \infty$ gives

$$\lim_{t \rightarrow \infty} \frac{\mu_t}{\lambda_t} = (1 - \sigma)C^{-\sigma}D(\theta^*)^{\sigma-1}\theta^* \left(L(\theta^*) - (1 - D(\theta^*)) \lim_{t \rightarrow \infty} \frac{\lambda_{t+1}}{\lambda_t} \right).$$

This limiting ratio is positive and finite under parameter value $\sigma \in (0, 1)$ as long as the limiting growth rate $\lim_{t \rightarrow \infty} \frac{\lambda_{t+1}}{\lambda_t}$ is not too large such that $\left(L(\theta^*) - (1 - D(\theta^*)) \lim_{t \rightarrow \infty} \frac{\lambda_{t+1}}{\lambda_t} \right) > 0$.

4. The FOCs with respect to K_{t+1} implies

$$1 = \beta \frac{\mu_{t+1}}{\mu_t} (MP_{K,t+1} + 1 - \delta).$$

Taking limit as $t \rightarrow \infty$ gives

$$\frac{1}{\beta(MP_K + 1 - \delta)} = \lim_{t \rightarrow \infty} \frac{\mu_{t+1}}{\mu_t} = \lim_{t \rightarrow \infty} \frac{\lambda_{t+1}}{\lambda_t} > 1,$$

which is true if and only if $\theta^* < \theta_H$ because when $\theta^* < \theta_H$ we must have capital overaccumulation, so $MP_K < MP_K^*$, where MP_K^* is the MGR at which $\frac{1}{\beta(MP_K^* + 1 - \delta)} = 1$.

5. Therefore, a steady state with divergent multipliers and partial self-insurance is possible. What does this imply for capital tax? Appendix A.6 indicates that regardless of the value of $\sigma \in (0, \infty)$ the implied capital tax rate is given by

$$1 - \tau_{k,t+1} = \frac{\frac{1}{L(\theta_t^*)} - \beta(1 - \delta)}{\frac{\mu_t}{\mu_{t+1}} - \beta(1 - \delta)}.$$

We have shown before that if $\sigma \geq 1$, then it must be true that $L(\theta_t^*) \rightarrow 1$, $\frac{\mu_{t+1}}{\mu_t} \rightarrow 1$, and $\tau_{k,t} \rightarrow 0$. We can also show easily that if $\sigma \in (0, 1)$ and the multipliers converge, then $\theta_t^* \rightarrow \theta_H$ and $\tau_{k,t} \rightarrow 0$ is a possible Ramsey steady state. In addition, we have just showed above that there may also exist other types of Ramsey steady state where the multipliers diverge if $0 < \sigma < 1$. To see the implications for optimal capital tax in

such a steady state with divergent multipliers, taking the limit of the equation above gives

$$\tau_k = 1 - \frac{\frac{1}{L(\theta^*)} - \beta(1 - \delta)}{\lim_{t \rightarrow \infty} \frac{\mu_t}{\mu_{t+1}} - \beta(1 - \delta)} = \frac{\lim_{t \rightarrow \infty} \frac{\mu_t}{\mu_{t+1}} - \frac{1}{L(\theta^*)}}{\lim_{t \rightarrow \infty} \frac{\mu_t}{\mu_{t+1}} - \beta(1 - \delta)}. \quad (76)$$

Note that the denominator is positive:

$$\lim_{t \rightarrow \infty} \frac{\mu_t}{\mu_{t+1}} = \beta(MP_K + 1 - \delta) > \beta(1 - \delta);$$

hence, the sign of capital tax depends on the sign of the numerator. Notice that

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{\mu_{t+1}}{\mu_t} - L(\theta^*) &= \frac{\frac{1-\sigma}{\sigma} \frac{L(\theta^*)}{D(\theta^*)} + \frac{\mathbf{F}(\theta^*)}{X(\theta^*)}}{\frac{1-\sigma}{\sigma} \frac{1}{D(\theta^*)} + \frac{\mathbf{F}(\theta^*)}{X(\theta^*)}} - L(\theta^*) \\ &= \frac{\frac{1-\sigma}{\sigma} \frac{L(\theta^*)}{D(\theta^*)} + \frac{\mathbf{F}(\theta^*)}{X(\theta^*)} - \frac{1-\sigma}{\sigma} \frac{L(\theta^*)}{D(\theta^*)} - \frac{\mathbf{F}(\theta^*)}{X(\theta^*)} L(\theta^*)}{\frac{1-\sigma}{\sigma} \frac{1}{D(\theta^*)} + \frac{\mathbf{F}(\theta^*)}{X(\theta^*)}} \\ &= \frac{\frac{\mathbf{F}(\theta^*)}{X(\theta^*)} (1 - L(\theta^*))}{\frac{1-\sigma}{\sigma} \frac{1}{D(\theta^*)} + \frac{\mathbf{F}(\theta^*)}{X(\theta^*)}} < 0. \end{aligned}$$

Therefore, we have

$$\lim_{t \rightarrow \infty} \frac{\mu_t}{\mu_{t+1}} > \frac{1}{L(\theta^*)},$$

and equation (76) implies a positive steady-state capital tax:

$$\tau_k = \frac{\lim_{t \rightarrow \infty} \frac{\mu_t}{\mu_{t+1}} - \frac{1}{L(\theta^*)}}{\lim_{t \rightarrow \infty} \frac{\mu_t}{\mu_{t+1}} - \beta(1 - \delta)} > 0.$$

However, since when $0 < \sigma < 1$ an FSI interior steady state with zero capital tax is also possible, it is not clear which steady state yields the highest welfare and constitutes the Ramsey allocation. Since analyzing this issue is beyond the scope of this paper, we leave it as a future research to compare whether the steady state with divergent multipliers and positive capital tax yields higher social welfare than the FSI interior steady state with convergent multipliers and zero capital tax.

A.8 Proof of Corollary 1

Assume that $\delta = 1$ and $G = 0$, then the steady-state resource constraint is $Y = C + K$. In the modified-golden-rule Ramsey steady state, the MGR holds and hence $1 = \beta MP_K = \beta \alpha \frac{Y}{K}$,

which together with the resource constraint imply $\frac{K}{Y} = \alpha\beta$ and $\frac{C}{Y} = 1 - \alpha\beta$. As a result, the optimal steady-state debt-to-GDP ratio can be inferred by the asset-market clearing condition:

$$\begin{aligned}\frac{B}{Y} &= \left(\frac{1}{D} - 1\right)\frac{C}{Y} - \frac{K}{Y} = \left(\frac{1}{D} - 1\right)(1 - \alpha\beta) - \alpha\beta \\ &= \alpha\beta \left[\frac{1 - D}{D} \frac{1 - \alpha\beta}{\alpha\beta} - 1 \right],\end{aligned}$$

which is equation (32) by the definition of τ_b .

A.9 Proof of Proposition 6

In light of the discussions above, it suffices to consider below only the situations with convergent multipliers under the parameter values $\sigma \geq 1$, and we leave the case of divergent multipliers under the parameter values $\sigma < 1$ to future research.

A.9.1 Existence of a Ramsey Steady State

Here we first prove that a Ramsey steady state featuring a binding debt limit exists if β is sufficiently large and δ is sufficiently small. We proceed by the following steps, which show that the constructed steady state—characterized by $\phi > 0$ and $\theta^* \in (\theta_L, \theta_H)$ —satisfies all of the Ramsey FOCs and these FOC-implied values of the steady-state aggregate allocation $\{C, N, K, \theta^*, B\}$ and Lagrangian multipliers $\{\lambda, \mu\}$ are mutually consistent, strictly positive, and finitely-valued.

1. We first show that any possible Ramsey steady state satisfying (i) the Ramsey FOC with respect to θ_t^* and (ii) the convergence of $\mu \in (0, \infty)$ must have the following property:

$$\frac{\mu}{\mu - \phi} = L(\theta^*), \quad (77)$$

where the liquidity premium $L(\theta^*)$ equals 1 if and only if $\phi = 0$, and is greater than 1 only if $\phi > 0$.

2. Starting from equation (53), $\nu_t^B > 0$ implies that $\phi_t > 0$. The FOC with respect to θ_t^* can be rewritten as

$$\widehat{W}(\theta_t^*) - \phi_t C_t^\sigma \widehat{Z}(\theta_t^*) + \lambda_t \widehat{H}(\theta_t^*) = \lambda_{t+1} \widehat{J}(\theta_t^*), \quad (78)$$

where $\widehat{H}(\theta_t^*)$, $\widehat{J}(\theta_t^*)$, and $\widehat{W}(\theta_t^*)$ are defined as before, and $\widehat{Z}(\theta_t^*)$ is defined as

$$\widehat{Z}(\theta_t^*) \equiv \frac{D(\theta_t^*)^{-1-\sigma}}{\sigma \theta_t^*}, \quad (79)$$

which is positive for any $\theta_t^* \in [\theta_L, \theta_H]$. Rewrite (78) as

$$\lambda_{t+1} = \rho(\theta_t^*)\lambda_t + \widehat{\varepsilon}(\theta_t^*),$$

where $\rho(\theta_t^*) \equiv \widehat{H}(\theta_t^*)/\widehat{J}(\theta_t^*)$ and

$$\widehat{\varepsilon}(\theta_t^*) \equiv \frac{\widehat{W}(\theta_t^*) - \phi_t C_t^\sigma \widehat{Z}(\theta_t^*)}{\widehat{J}(\theta_t^*)}. \quad (80)$$

The existence of a finitely positive steady-state value of λ depends on the sign and value of $\rho(\theta_t^*)$ and $\widehat{\varepsilon}(\theta_t^*)$, which in turn depend on the value of σ . Hence we discuss the following three cases of $\sigma \leq 1$ with a focus only on convergent multipliers.

- (a) $\sigma < 1$. Then $0 < \widehat{J}(\theta_t^*) < \widehat{H}(\theta_t^*)$ according to Lemma 2, and $\rho(\theta_t^*) > 1$. In this case λ_t converges to a positive steady-state value if and only if $\widehat{\varepsilon}(\theta_t^*) < 0$. Namely,

$$\lambda = \widehat{\varepsilon}(\theta^*)/(1 - \rho(\theta^*)) > 0, \quad (81)$$

if and only if $\widehat{\varepsilon}(\theta^*) < 0$. We will check later on that $\widehat{\varepsilon}(\theta^*) < 0$ indeed constitutes a Ramsey steady state with $\sigma \in (0, 1)$, $\theta^* < \theta_H$, and $\{\lambda, \mu, \phi\} > 0$. According to FOC (52) and Lemma 4, the steady-state value of μ is given by

$$\mu = C^{-\sigma} D(\theta^*)^{\sigma-1} M(\theta^*) (1 + \lambda(1 - \sigma)) - \phi \frac{1 - D(\theta^*)}{D(\theta^*)}, \quad (82)$$

or equivalently,

$$\frac{C^\sigma}{D(\theta^*)^\sigma} = \frac{D(\theta^*)^{-1} M(\theta^*) (1 + \lambda(1 - \sigma))}{\mu + \phi \frac{1 - D(\theta^*)}{D(\theta^*)}}. \quad (83)$$

Plugging the steady-state λ into equation (82) and using the definitions of $\widehat{J}(\theta_t^*)$, $\widehat{H}(\theta_t^*)$, $\widehat{W}(\theta_t^*)$, and $\widehat{Z}(\theta_t^*)$ as well as Lemma 4, gives

$$\mu = \phi \frac{M(\theta^*)}{\theta^*} \frac{D(\theta^*)^{-1}}{L(\theta^*) - 1} - \phi \frac{1 - D(\theta^*)}{D(\theta^*)} = \phi \frac{L(\theta^*)}{L(\theta^*) - 1}, \quad (84)$$

where the last equality uses the Lemma 4 again. This equation is exactly identical to (77) and the mapping is unique because $L(\theta^*)$ is strictly monotone.

- (b) $\sigma = 1$. Then $0 < \widehat{J}(\theta_t^*) = \widehat{H}(\theta_t^*)$ according to Lemma 2, and $\rho(\theta_t^*) = 1$. The steady-state λ is a finitely positive value (and unique) if and only if $\widehat{\varepsilon}(\theta^*) = 0$. We will show later on that $\widehat{\varepsilon}(\theta^*) = 0$ is indeed consistent with an interior Ramsey steady state with $\sigma = 1$, $\theta^* \in (\theta_L, \theta_H)$, and $\{\lambda, \mu, \phi\} > 0$. First, given $\widehat{\varepsilon}(\theta^*) = 0$, the steady-state ϕ can be solved by equation (78), which is

$$\phi = C^{-1} \frac{\widehat{W}(\theta^*)}{\widehat{Z}(\theta^*)} = C^{-1} D(\theta^*) \theta^* (L(\theta^*) - 1).$$

Note that $\phi > 0$ if and only if $L(\theta^*) > 1$, or equivalently, $\theta^* < \theta_H$. According to FOC (52) and lemma 4, the steady-state value of μ at $\sigma = 1$ is given by

$$\begin{aligned} \mu &= C^{-1} M(\theta^*) - \phi D(\theta^*)^{-1} (1 - D(\theta^*)) \\ &= C^{-1} \theta^* D(\theta^*) L(\theta^*) \\ &> 0, \end{aligned}$$

where the second equality holds according to Lemma 4 and the equation $\phi = C^{-1} D(\theta^*) \theta^* (L(\theta^*) - 1)$ solved above. As a result, the value of $\mu/(\mu - \phi)$ is uniquely given by

$$\frac{\mu}{\mu - \phi} = \frac{C^{-1} \theta^* D(\theta^*) L(\theta^*)}{C^{-1} \theta^* D(\theta^*) L(\theta^*) - C^{-1} D(\theta^*) \theta^* (L(\theta^*) - 1)} = L(\theta^*),$$

which is identical to (77).

- (c) $\sigma > 1$. Then Lemma 2 implies that $\widehat{J}(\theta_t^*) > \widehat{H}(\theta_t^*)$ and $\widehat{J}(\theta_t^*) > 0$. The value and sign of $\rho(\theta_t^*)$ and $\widehat{\varepsilon}(\theta_t^*)$ depend on the sign of $\widehat{H}(\theta_t^*)$; there are two possible subcases, which are discussed below:

- i. $\widehat{H}(\theta_t^*) > 0$ and $\widehat{J}(\theta_t^*) > 0$. Then $0 < \rho(\theta_t^*) < 1$. The steady-state $\lambda = \widehat{\varepsilon}(\theta^*)/(1 - \rho(\theta^*)) > 0$ if and only if $\widehat{\varepsilon}(\theta^*) > 0$.
- ii. $\widehat{H}(\theta_t^*) < 0$ and $\widehat{J}(\theta_t^*) > 0$. In this case $\rho(\theta^*) < 0$. Then there are two possibilities regarding to $\rho(\theta_t^*)$: $|\rho(\theta_t^*)| < 1$ or $|\rho(\theta_t^*)| > 1$. In either case we can show that $\lambda = \widehat{\varepsilon}(\theta^*)/(1 - \rho(\theta^*)) > 0$ if and only if $\widehat{\varepsilon}(\theta^*) > 0$.

Since all of the possible values of λ in the case of $\sigma > 1$ are analogous to those

in the case of $\sigma < 1$, following the proof in the $\sigma < 1$ case, we can show that equation (77), $\mu/(\mu - \phi) = L(\theta^*)$, also holds.

Therefore, any possible steady state satisfying the FOC with respect to θ_t^* and with finite positive μ must imply $\mu/(\mu - \phi) = L(\theta^*)$. We will verify at the end that the required conditions for (the sign of) $\hat{\varepsilon}(\theta^*)$ stated above are all automatically met once the condition for $\lambda > 0$ is satisfied.

3. The FOC with respect to K in (48) together with equation (77) give

$$1 - \frac{\phi}{\mu} = \beta (MP_K + 1 - \delta) = \frac{1}{L(\theta^*)} < 1, \quad (85)$$

which suggests that MP_K is bounded above. In addition, if δ is sufficiently small such that $MP_K > \delta$, then $MP_K \in (\delta, \frac{1-\beta(1-\delta)}{\beta})$. Given the assumption of the production function, it must be true that the following ratios are strictly positive and finite: $\{K/N, Y/K, MP_N, Y/N\} \in (0, \infty)$. First, the Y/K ratio is bounded above and below:

$$\frac{\delta}{\alpha} < \frac{Y}{K} = \left(\frac{K}{N}\right)^{\alpha-1} < \frac{1 - \beta(1 - \delta)}{\alpha\beta}. \quad (86)$$

4. Then the resource constraint,

$$Y = K^\alpha N^{1-\alpha} = C + \delta K + G,$$

together with a finite level of government spending $G \geq 0$ implies a bounded C/Y ratio as

$$\left(1 - \alpha - \frac{G}{Y}\right) < \frac{C}{Y} = \left(1 - \delta \frac{K}{Y} - \frac{G}{Y}\right) < \left(1 - \frac{\alpha\beta\delta}{1 - \beta(1 - \delta)} - \frac{G}{Y}\right), \quad (87)$$

where the last inequality uses equation (86).

5. We know that under our parameter restrictions the level of labor is interior, $N \in (0, \bar{N})$; hence, it must be true that the aggregate allocation is also interior: $\{C, K, Y\} \in (0, \infty)$.

6. Next, we show that $\{\mu, \lambda\} \in (0, \infty)$. Plugging equation (77) into the steady-state FOC (52) gives

$$\mu = \frac{M(\theta^*)D(\theta^*)^{\sigma-1}C^{-\sigma}[1 + \lambda(1 - \sigma)]}{1 + (1 - L(\theta^*)^{-1})(D(\theta^*)^{-1} - 1)}, \quad (88)$$

where we have used Lemma 4 and the definition of $W(\theta^*)$, which implies $W(\theta^*) = M(\theta^*)D(\theta^*)^{\sigma-1}$. The above equation together with the steady-state FOC (50) imply the mapping,

$$\lambda = \frac{Q - 1}{1 + Q(\sigma - 1)}, \quad (89)$$

where

$$Q = \frac{MP_N M(\theta^*) D(\theta^*)^{\sigma-1} L(\theta^*) D(\theta^*)}{C^\sigma [L(\theta^*) - 1 + D(\theta^*)]} \in (0, \infty).$$

Depending on the value of σ , we discuss two subcases below:

- (a) When $\sigma \geq 1$, it is straightforward to verify that $\lambda > 0$ if and only if $Q > 1$.
- (b) When $1 > \sigma > 0$, then $\lambda > 0$ can be ensured if $Q \in (1, \frac{1}{1-\sigma})$.

Therefore, for $\{\mu, \lambda\} \in (0, \infty)$, we need (i) $Q > 1$ if $\sigma \geq 1$ and (ii) $Q \in (1, \frac{1}{1-\sigma})$ if $1 > \sigma > 0$.

7. To derive these conditions, note that in the steady state the implementability condition (22) becomes

$$\frac{N}{Y} = \frac{C}{Y} C^{-\sigma} D(\theta^*)^{\sigma-1} \theta^* \left(L(\theta^*) - \frac{1 - D(\theta^*)}{\beta} \right),$$

and hence C^σ can be expressed as

$$C^\sigma = \left(\frac{C}{Y} \right) \left(\frac{Y}{N} \right) D(\theta^*)^{\sigma-1} \theta^* \left(L(\theta^*) - \frac{1 - D(\theta^*)}{\beta} \right). \quad (90)$$

By using equation (90), Lemma 4, and $MP_N = (1 - \alpha) \left(\frac{Y}{N} \right)$, we can express Q as:

$$Q = \frac{(1 - \alpha) L(\theta^*) D(\theta^*)}{\left(\frac{C}{Y} \right) \left(L(\theta^*) - \frac{1 - D(\theta^*)}{\beta} \right)},$$

which together with equation (87) imply an upper and lower bound for Q :

$$\begin{aligned} & \frac{1 - \alpha}{\left(1 - \frac{\alpha\beta\delta}{1-\beta(1-\delta)} - \frac{G}{Y} \right)} \frac{L(\theta^*) D(\theta^*)}{\left(L(\theta^*) - \frac{1 - D(\theta^*)}{\beta} \right)} \\ & < Q < \frac{1 - \alpha}{\left(1 - \alpha - \frac{G}{Y} \right)} \frac{L(\theta^*) D(\theta^*)}{\left(L(\theta^*) - \frac{1 - D(\theta^*)}{\beta} \right)}. \end{aligned} \quad (91)$$

Now consider the following subcases:

- (a) When $\sigma \geq 1$, we need $Q > 1$ to ensure $\lambda > 0$. Since $L(\theta^*) > 1$, the term $\frac{L(\theta^*)D(\theta^*)}{(L(\theta^*) - \frac{1-D(\theta^*)}{\beta})} > 1$ if $\beta L(\theta^*) < 1$. Notice that $\beta L(\theta^*) < 1$ can be ensured by a sufficiently small δ according to equation (85). Consequently, $Q > 1$ would be ensured if the first term $\frac{1-\alpha}{(1 - \frac{\alpha\beta\delta}{1-\beta(1-\delta)} - \frac{G}{Y})} \geq 1$, which is clearly true if β is sufficiently large because in the limit we have

$$\lim_{\beta \rightarrow 1} \frac{1-\alpha}{\left(1 - \frac{\alpha\beta\delta}{1-\beta(1-\delta)} - \frac{G}{Y}\right)} = \frac{1-\alpha}{1-\alpha-\frac{G}{Y}} \geq 1.$$

- (b) When $0 < \sigma < 1$, $\lambda > 0$ requires an additional condition that $Q < \frac{1}{1-\sigma}$. From equation (91), the following condition ensures $Q < \frac{1}{1-\sigma}$:

$$\left(1 - \alpha - \frac{G}{Y}\right) \frac{1}{L(\theta^*)} > \frac{(1-\sigma)(1-\alpha)D(\theta^*)}{\left(L(\theta^*) - \frac{1-D(\theta^*)}{\beta}\right)},$$

which together with equation (85) implies

$$\frac{1}{(1-\sigma)} \frac{(1-\alpha-\frac{G}{Y})}{(1-\alpha)} \beta (MP_K + 1 - \delta) > \frac{D(\theta^*)}{\left(L(\theta^*) - \frac{1-D(\theta^*)}{\beta}\right)}. \quad (92)$$

Notice that the RHS of equation (92) is close to but less than 1 if β is sufficiently close to 1, as shown by the following inequality:

$$\lim_{\beta \rightarrow 1} \frac{D(\theta^*)}{\left(L(\theta^*) - \frac{1-D(\theta^*)}{\beta}\right)} = \frac{D(\theta^*)}{L(\theta^*) - 1 + D(\theta^*)} < 1.$$

In addition, if G is sufficiently small, then the LHS of equation (92) is larger than 1 when β is sufficiently close to 1, because $MP_K > \delta$ and $\frac{1}{(1-\sigma)} > 1$. Therefore, even when $\sigma \in (0, 1)$, there can still exist a Ramsey steady state if β is sufficiently large, δ is sufficiently small, and G is sufficiently small. These parameter restrictions suggest that when the utility function of consumption is sufficiently flat (or risk neutral), the Ramsey planner can still improve social welfare by issuing a limited amount of bonds so long as capital is sufficiently durable and households are sufficiently patient.

8. Notice that these requirements for $\{\beta, \delta, \alpha\}$ do not contradict the competitive-equilibrium condition (17) for an interior cutoff and the condition for an interior $N \in (0, \bar{N})$ be-

cause these interior conditions hold true for all $\{\beta, \delta, \alpha\} \in (0, 1)$ as long as θ_H is sufficiently large relative to $\mathbb{E}(\theta)$ (such that a competitive equilibrium does not feature full self-insurance). In fact, the existence of a Ramsey steady state under a binding debt limit is also confirmed by our numerical analysis in Section 4.

9. We verify that the required conditions for (the sign of) $\widehat{\varepsilon}(\theta^*)$ stated above are all automatically met once the condition for $\lambda > 0$ is satisfied. Using equation (79) and (63) to substitute out $\widehat{Z}(\theta^*)$ and $\widehat{W}(\theta^*)$ in the definition of $\widehat{\varepsilon}(\theta^*)$, which is listed in equation (80), gives

$$\begin{aligned}\widehat{\varepsilon}(\theta^*) &= \frac{\widehat{W}(\theta^*) - \phi C^\sigma \widehat{Z}(\theta^*)}{\widehat{J}(\theta_t^*)} = \frac{1}{\widehat{J}(\theta_t^*)} \left(\frac{1}{\sigma} \frac{M(\theta_t^*)}{D(\theta_t^*) \theta_t^*} - \frac{1}{\sigma} - \phi \frac{C^\sigma}{\sigma \theta^* D^{1+\sigma}} \right) \\ &= \frac{1}{\widehat{J}(\theta_t^*) \sigma} \left(\frac{M(\theta_t^*)}{D(\theta_t^*) \theta_t^*} - 1 - \frac{M(\theta^*)}{\theta^* D(\theta^*)} \frac{D(\theta^*)^{-1} (1 + \lambda(1 - \sigma))}{\frac{\mu}{\phi} + \frac{1 - D(\theta^*)}{D(\theta^*)}} \right),\end{aligned}$$

where the last equality uses equation (83) to substitute out $\frac{C^\sigma}{D(\theta^*)^\sigma}$. Simplifying under Lemma 4 and equation (84), the above equation becomes

$$\begin{aligned}\widehat{\varepsilon}(\theta^*) &= \frac{1}{\widehat{J}(\theta_t^*) \sigma} \left(\frac{(L(\theta_t^*) - 1 + D(\theta_t^*))}{D(\theta_t^*)} - 1 - \frac{(L(\theta_t^*) - 1 + D(\theta_t^*))}{D(\theta^*)} \frac{D(\theta^*)^{-1} (1 + \lambda(1 - \sigma))}{\frac{L(\theta^*)}{L(\theta^*) - 1} + \frac{1 - D(\theta^*)}{D(\theta^*)}} \right) \\ &= \frac{1}{\widehat{J}(\theta_t^*) \sigma} \left(\frac{L(\theta_t^*) - 1}{D(\theta_t^*)} - \frac{L(\theta_t^*) - 1 + D(\theta_t^*)}{D(\theta^*)} \frac{(L(\theta^*) - 1) (1 + \lambda(1 - \sigma))}{L(\theta^*) - 1 + D(\theta^*)} \right) \\ &= \frac{1}{\widehat{J}(\theta_t^*)} \frac{1}{\sigma} \frac{L(\theta_t^*) - 1}{D(\theta_t^*)} \lambda (\sigma - 1),\end{aligned}$$

where $\lambda(\sigma - 1) \gtrless 0$ if and only if $\sigma \gtrless 1$ (given that $\lambda > 0$). Also note that $\widehat{J}(\theta^*)$, $D(\theta_t^*)$ and $L(\theta_t^*) - 1$ are all strictly positive. Hence, under the parameter conditions for $\lambda > 0$, it must be true that (i) $\widehat{\varepsilon}(\theta^*) \leq 0$ if and only if $\sigma \leq 1$, which are consistent with the stated conditions for $\widehat{\varepsilon}(\theta^*)$ under cases (2a) and (2b) discussed above; (ii) $\widehat{\varepsilon}(\theta^*) > 0$ if and only if $\sigma > 1$, which are consistent with the stated conditions for $\widehat{\varepsilon}(\theta^*)$ under case (2c) discussed above.

10. Finally, we show that if $\sigma \geq 1$, a Ramsey steady state with divergent multipliers is impossible. Namely, there is no interior Ramsey steady state where μ_t diverges to infinity if $\sigma \geq 1$. From the Ramsey FOCs, we know that in the multipliers of Ramsey problem, μ_t , λ_t and ϕ_t have to grow at the same rate in steady state and their ratios

in the long run have to be finite, positive, and constant. Consider the following two cases.

- (a) Suppose $\sigma = 1$, the Ramsey FOC with respect to C_t , (52), in steady state becomes

$$\mu_t - \phi_t = W(\theta^*)C^{-\sigma} - \phi_t D(\theta^*)^{-1},$$

which implies

$$\lim_{t \rightarrow \infty} \frac{\mu_t - \phi_t}{\mu_{t+1}} = - \lim_{t \rightarrow \infty} \frac{\phi_t}{\mu_{t+1}} D(\theta^*)^{-1} < 0.$$

The above inequality leads to a contradiction to the steady state version of Ramsey FOC with respect to K_{t+1} , (48).

- (b) Consider the case where $\sigma > 1$. Notice that by Lemma 2, we know that $\frac{\hat{H}(\theta^*)}{\hat{J}(\theta^*)} < 1$ and $\hat{J}(\theta^*) > 0$ for any $\theta^* \in (\theta_L, \theta_H)$. However, for $\lambda_t \rightarrow \infty$, it has to be the case that the growth rate of λ_t is larger than 1, which by equation (81) gives that

$$\lim_{t \rightarrow \infty} \frac{\lambda_{t+1}}{\lambda_t} = \rho(\theta^*) + \lim_{t \rightarrow \infty} \frac{\hat{\varepsilon}(\theta^*)}{\lambda_t} = \frac{\hat{H}(\theta^*)}{\hat{J}(\theta^*)} - \frac{C^\sigma \hat{Z}(\theta^*)}{\hat{J}(\theta^*)} \lim_{t \rightarrow \infty} \frac{\phi_t}{\lambda_t} > 1.$$

The above inequality leads to a contradiction since (1) $\frac{\hat{H}(\theta^*)}{\hat{J}(\theta^*)} < 1$ (by Lemma 2) (2) $\frac{\phi_t}{\lambda_t}$ converges to a finite positive constant and (3) $C^\sigma \hat{Z}(\theta^*)/\hat{J}(\theta^*) > 0$.

11. In short, we show the following properties of the Ramsey steady state with a binding government debt limit:

- (a) Individual allocative efficiency fails to hold. Equation (85) implies that $L(\theta^*) > 1$ and hence θ^* must be interior.
- (b) Aggregate allocative efficiency fails to hold. Equation (48) suggests that the MGR does not hold in the steady state.
- (c) Despite the failure of the MGR, the steady-state capital tax must be zero by Proposition 4, since $\mu/(\mu - \phi) = L(\theta^*)$ if the multipliers converge. However, if $\sigma < 1$, we still cannot rule out the possibility that there exists an Ramsey steady state where the multipliers diverge to infinity and the optimal capital tax may be different from zero. The steady-state labor tax τ_n is given by

$$\tau_n = 1 - \frac{1}{L(\theta^*)\theta^* D(\theta^*)^\sigma} \frac{C^\sigma}{MP_N}.$$

- (d) The condition (41) ensures $n > 0$ and the condition (42) ensures $n < \overline{N}$ if $\overline{N} > \theta^* L(\theta^*)$. Hence, $n \in (0, \overline{N})$ is guaranteed.