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Don't Tax Capital—

Optimal Ramsey Taxation in Heterogeneous Agent Economies with Quasi-Linear Preferences*

YiLi Chien   Yi Wen

February 20, 2019

Abstract

We build a tractable infinite-horizon Aiyagari-type model with quasi-linear preferences to address a set of long-standing issues in the optimal Ramsey taxation literature. The tractability of our model enables us to analytically prove the existence of Ramsey steady states and establish several strong and novel results: (i) Depending on the government's capacity to issue debts, there can exist different types of Ramsey steady state and their existence depends critically on model parameter values. (ii) The optimal capital tax is exclusively zero in a Ramsey steady state regardless of the modified golden rule and government debt limits. (iii) Along the transition path toward a Ramsey steady state, optimal capital tax depends positively on the elasticity of intertemporal substitution. (iv) When a Ramsey steady state (featuring a non-binding government debt limit) *does not* exist but is *erroneously assumed* to exist, the modified golden rule always "holds" and the implied "optimal" long-run capital tax is strictly positive, reminiscent of the result obtained by Aiyagari (1995). (v) Whether the modified golden rule holds depends critically on the government's capacity to issue debts, but has no bearing on the planner's long-run capital tax scheme. (vi) The optimal debt-to-GDP ratio in the absence of a binding debt limit, however, is determined by a positive wedge times the modified-golden-rule saving rate; the wedge is decreasing in the strength of the individual self-insurance position and approaches zero when the idiosyncratic risk vanishes or markets are complete. The key insight behind our results is the Ramsey planner's ultimate concern for self-insurance. Since taxing capital in the steady state permanently hinders individuals' self-insurance positions, the Ramsey planner prefers (i) issuing debt rather than imposing a steady-state capital tax to correct the capital-overaccumulation problem under precautionary saving motives, and

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(ii) taxing capital only in the short run regardless of its debt positions. Thus, in sharp contrast to Aiyagari's argument, permanent capital taxation is not the optimal tool to achieve aggregate allocative efficiency despite overaccumulation of capital, and the modified golden rule can fail to hold in a Ramsey equilibrium whenever the government encounters a debt-limit.

JEL Classification: E13; E62; H21; H30

Key Words: Optimal Capital Taxation, Ramsey Problem, Incomplete Markets

1 Introduction

The seminal work of Aiyagari (1995) has inspired a large literature. However, despite several important revisits, such as Chamley (2001), Conesa, Kitao, and Krueger (2009), Dávila, Hong, Krusell, and Ríos-Rull (2012), and many others, the issue regarding optimal capital income taxation in a heterogeneous-agent incomplete-markets economy remains unsettled. Key to the problem is the trade-off facing the Ramsey planner between aggregate allocative efficiency (in terms of the modified golden rule) and individual allocative efficiency (in terms of self-insurance). The first aspect in the trade-off involves production efficiency at the aggregate level, and the second aspect involves the (re)distributional effects of fiscal policies on the individual consumption risk and self-insurance position. Neither issue (aspect) is trivial.

For example, the competitive market equilibrium in the Aiyagari model (Aiyagari (1994)) may appear to be dynamically inefficient due to overaccumulation of capital under individual precautionary saving motives, which results in an excessively low marginal product of capital at the aggregate. Consequently, the modified golden rule (MGR) is violated in a competitive equilibrium, leading to an aggregate allocative inefficiency. This observation provides the key intuition of Aiyagari (1995) that the Ramsey planner should tax capital income to restore the aggregate allocative efficiency—the MGR, regardless of the distribution of individuals' self-insurance positions.

However, the above intuition for justifying a positive capital income tax is counterintuitive from a micro viewpoint. By taxing capital income and thus reducing each individual's optimal buffer stock of savings, the government is hampering and effectively destroying individuals' ability to self-insure themselves against idiosyncratic risks when lump-sum transfers are not available. Since taxing capital *per se* does not directly address the lack-of-insurance problem for households (if anything, it intensifies the problem), why would taxing capital always be optimal for the social planner? Or why would the MGR matter to individuals' welfare more than the need of self-insurance?

This question is particularly intriguing since the lack of full self-insurance is the root problem in the Aiyagari economy and should hence be the ultimate concern for the benevolent government. In other words, achieving the MGR through capital taxation does not help at all to alleviate the primal friction in the model—lack of full self-insurance under borrowing constraints—thus, using the MGR principle to justify a positive capital income tax regardless of its effect on individual allocative efficiency as well as specific model structures and parameter values is not at all clear and convincing.[1]

In short, two margins of allocative inefficiency exist in the Aiyagari model: aggregate allocative inefficiency in light of the MGR failure, and individual allocative inefficiency in light of insufficient self-insurance under borrowing constraints and incomplete credit markets. Clearly, the aggregate inefficiency is a consequence of the individual inefficiency (due to a pecuniary externality of precautionary savings on the rate of return to aggregate capital), suggesting that restoring individual allocative efficiency is the fundamental way to achieve aggregate allocative efficiency. Hence, intuition tells us that a social planner can improve welfare more likely through addressing the individual inefficiency problem by improving individuals' self-insurance positions (such as issuing bonds to substitute capital) than through restoring the MGR by taxing individuals' buffer stock.

This trade-off is not discussed clearly in the existing literature, including Aiyagari's original analysis. A daunting challenge in determining and unlocking the precise rationales behind the Ramsey planner's tax policies in confronting the trade-off problem lies in the existence of a Ramsey steady state, which is never properly proven in the existing literature in infinite-horizon Aiyagari-type models. But the proof of the existence of a Ramsey steady state hinges critically on the tractability of the models. When models are not tractable, not only does the Ramsey problem become difficult to solve, but the "proofs" also become highly non-transparent. Consequently, intuitions behind any policy inference often get lost and dubious claims may be made.

The goal of this paper is to design a specific model to analytically investigate the trade-off between the two margins of inefficiency and rebuke Aiyagari's arguments. Our model features heterogeneous agents and incomplete insurance markets—in the spirit of Aiyagari (1995)—but with two key differences: Individuals in our model face idiosyncratic shocks to the marginal utility of consumption and their preferences are quasi-linear. This preference structure enables us to solve our model analytically and derive both the competitive equilibrium and the Ramsey allocation in closed forms.

[1] In this paper, aggregate allocative efficiency refers to a situation in which the MGR holds, and individual allocative efficiency refers to a situation in which all households are fully self-insured and their borrowing constraints do not bind in all states. These terms are defined more precisely in Section 2.

More precisely, our contributions are five-fold. First, we construct a tractable heterogeneous-agent incomplete-markets model with closed-form solutions, in which the equilibrium distribution of market allocations and the Ramsey problem can be solved analytically. Our model has the desirable properties that a Ramsey steady state can be proven to exist only under certain parameter values and that a particular Ramsey steady state is one at which the interest rate equals the time discount rate ($1/\beta$). These properties allow us to use counterfactual analyses to address some important questions.

Second, we show analytically that (i) the Ramsey planner will never tax capital in a Ramsey steady state, regardless of the MGR; and (ii) when a Ramsey steady state (featuring a non-binding government debt limit) *does not* exist but is *erroneously* assumed to exist, the modified golden rule always *appears* to "hold" and the implied "optimal" long-run capital tax is strictly positive, reminiscent of the result obtained by Aiyagari (1995).[2]

Third, our analysis provides a clarification for the positive role of government debts and the distortionary effect of capital tax on the household self-insurance position. Specifically, we show that whether MGR should hold or not depends critically on the Ramsey planner's ability to issue bonds as an alternative store of value (substituting for capital) for households to buffer idiosyncratic risks. In particular, MGR would hold in our model if and only if the government can amass sufficiently large stock of bonds to enable households to achieve full self-insurance. Once households are fully self-insured, the equilibrium interest rate in our model equals the time discount rate ($1/\beta$). In such a case, aggregate allocative efficiency and individual allocative efficiency are simultaneously achieved by the Ramsey planner. However, when it is impossible to equalize the interest rate and the time discount rate or the government's capacity to issue debt is limited, such as in the case of a binding debt limit in a Ramsey steady state, then MGR *does not* hold. Despite the failure of the MGR, however, the optimal capital tax rate is *still* zero in the steady state (even in the case where the government cannot issue bonds at all).[3] Hence, MGR has no bearing on the planner's long-run capital tax scheme, in sharp contrast to the argument of Aiyagari (1995).

Fourth, the Ramsey planner nonetheless opts to tax capital along the transition path—

[2]A Ramsey steady state refers to a Ramsey allocation where all aggregate variables and the moments of the distribution converge to constant and finite non-zero values.

[3]This result is striking yet very intuitive: If it is optimal to set capital tax to zero without the tool of government debts, then it should be optimal to maintain a zero capital tax when government bonds become available. See proofs in the next section.

so as to front-load consumption and discourage capital accumulation. In particular, the larger the elasticity of intertemporal substitution, the higher the short-run tax rate. In fact, the levels of aggregate consumption, capital stock, and hours worked in a Ramsey steady state are lower than their counterparts in the competitive equilibrium. Hence, the steady state welfare gains under optimal policies derive primarily from the improved distribution of household self-insurance positions.

Last but not the least, in the absence of a binding debt limit the optimal debt-to-GDP ratio in our model is determined by a positive wedge times the aggregate saving rate implied by MGR. This wedge is an increasing function of the extent of individual allocative inefficiency and would vanish only when the idiosyncratic risk approaches zero or markets become complete. Namely, the optimal debt-to-GDP ratio in our model is zero if and only if households are fully self-insured and no longer borrowing-constrained in a competitive equilibrium. This result suggests (again) that the single most important role of government debts is to improve the individual self-insurance position through which the modified golden rule is achieved (if possible). In other words, MGR is *not* the primal concern of the social planner, and it is never optimal to tax capital in the steady state simply to achieve aggregate allocative efficiency even though it is feasible to do so.

The insight behind our results is intuitive. First of all, it is the lack of an insurance market that induces agents to overaccumulate capital to self-insure against idiosyncratic consumption risk. Taxing capital in the steady state would permanently hamper individuals' self-insurance—the lack of which is the root cause of the aggregate allocative inefficiency—and is thus not a desirable tool to restore the MGR. In other words, for the Ramsey planner, the long-term concern for individual allocative efficiency dominates that for aggregate allocative efficiency.

Secondly, in the absence of lump-sum transfers, government bonds meet individuals' demand for precautionary saving without creating pecuniary externalities on the marginal product of capital. Hence, by substituting for (or crowding out) capital, government debt can satisfy the household buffer-stock saving motives and, at the same time, correct the aggregate inefficiency due to overaccumulation of capital. This is why the Ramsey planner opts to flood the asset market with a sufficient amount of bonds to ensure (as much as possible) full self-insurance for all households across all states. However, when debt limits exist, the Ramsey planner is unable to issue enough bonds to implement a full self-insurance allocation; but, in spite of this, the planner will not levy a permanent tax on capital simply to achieve the MGR. Instead, the planner opts to tax capital in the short run to reduce as much as possible the steady-state capital stock.

In addition, with certain parameter values such as unbounded idiosyncratic shocks (as in the case of a Pareto distribution) and unbounded debt limits (which is implicitly assumed in Aiyagari (1995)), a Ramsey steady state does not exist in our model and in such a case the interest rate always lies below the time discount rate—because the Ramsey planner opts to issue an infinite amount of bonds to achieve a fully self-insured allocation, thus destroying the Ramsey steady state commonly assumed in the existing literature.

Our analysis suggests that the assumption of the existence of a Ramsey steady state in the original analysis of Aiyagari (1995) is dubious and may be incorrect. Since the original Aiyagari model is intractable, the existence of a Ramsey steady state is often assumed instead of proved in the existing literature. Our conjecture is supported independently by the recent work of Chen, Chien, and Yang (2019), which shows that the assumption of a Ramsey steady state in the typical Aiyagari model is inconsistent with some of the Ramsey planner's first-order conditions. It is also easy to show that if a debt limit exists and is binding in the original Aiyagari model, then MGR does not hold, as is also true in our model. Therefore, the difference between Aiyagari's (1995) analysis and ours may not be attributed to the difference in models, but instead to the validity of the assumption of a Ramsey steady state featuring non-binding government debt limit.

The remainder of the paper is organized as follows. Section 2 describes the model, derives the competitive equilibrium, and provides sufficient conditions for the Ramsey planner to support a competitive equilibrium. Section 3 shows how to solve for the Ramsey allocation analytically, to prove the existence of a Ramsey steady state, and to derive the optimal capital tax in a Ramsey steady state. Section 4 performs numerical exercises to confirm our theoretical analysis by studying transition dynamics. Section 5 studies the robustness of our results by extending our model to the case of endogenous government spending as in Aiyagari (1995); and we show that our results hold regardless of endogenous government spending. Section 6 provides a brief literature review. The last section concludes the paper with remarks for future research.

2 The Model

This model is based on Bewley (1980), Lucas (1980), Huggett (1993), Aiyagari (1994), and especially Wen (2009, 2015). To fix notions, for this paper (with some abuse of terminology) we define aggregate allocative efficiency (AAE) as a competitive equilibrium allocation in which the MGR holds, and we define individual allocative efficiency (IAE) as a competitive equilibrium allocation in which all households are fully self-insured and their borrowing constraints do not bind in all states.

Obviously, in a competitive equilibrium without government intervention, IAE implies AAE; but this may not be true in a Ramsey equilibrium. Namely, the Ramsey planner may design policies that achieve IAE without achieving AAE, or vice versa.

We also define the "optimal self-insured allocation" (OSIA) as a Ramsey steady state allocation in which both AAE and IAE are achieved under optimal government policies. Accordingly, our discussions involve two different notions of steady state: the "competitive equilibrium steady state" for a given set of government policies, and the "Ramsey steady state" under optimal policies.

2.1 Environment

A representative firm produces output according to the constant-returns-to-scale Cobb-Douglas technology, $Y_t = F(K_t, N_t) = K_t^{\alpha} N_t^{1-\alpha}$, where Y, K, and N denote aggregate output, capital, and labor, respectively. The firm rents capital and hires labor from households by paying a competitive rental rate and real wage, denoted by q_t and w_t, respectively. The firm's optimal conditions for profit maximization at time t satisfy

$$w_t = \frac{\partial F(K_t, N_t)}{\partial N_t}, \quad (1)$$

$$q_t = \frac{\partial F(K_t, N_t)}{\partial K_t}. \quad (2)$$

There is a unit measure of *ex ante* identical households that face idiosyncratic preference shocks, denoted by θ. The shocks are identically and independently distributed (iid) over time and across households, and have the mean $\overline{\theta}$ and the cumulative distribution $\mathbf{F}(\theta)$ with support $[\theta_L, \theta_H]$, where $\theta_H > \theta_L > 0$.

Time is discrete and indexed by $t = 1, 2, ..., \infty$. There are two subperiods in each period t. The idiosyncratic preference shock θ_t is realized only in the second subperiod, and the labor supply decision must be made in the first subperiod before observing θ_t. Namely, the idiosyncratic preference shock is uninsurable by wage income even when leisure enters the utility function linearly. Let $\theta^t \equiv (\theta_1, ..., \theta_t)$ denote the history of idiosyncratic shocks. All households are endowed with the same asset holdings at the beginning of time 1.

Households are infinitely lived with a quasi-linear utility function and face borrowing constraints. Their lifetime expected utility is given by

$$V = E_1 \sum_{t=1}^{\infty} \beta^{t-1} \left[\theta_t \frac{c_t(\theta^t)^{1-\sigma} - 1}{1-\sigma} - n_t\left(\theta^{t-1}\right) \right], \tag{3}$$

where $\beta \in (0, 1)$ is the discount factor; $\sigma \in (0, \infty)$ is a parameter that determines the elasticity of intertemporal substitution (EIS) and risk aversion of the household; $c_t(\theta^t)$ and $n_t(\theta^{t-1})$ denote consumption and the labor supply, respectively, for a household with history θ^t at time t. Note that the labor supply in period t is only measurable with respect to θ^{t-1}, reflecting the assumption that the labor-supply decision is made in the first subperiod before observing the preference shock θ_t.

The government needs to finance an exogenous stream of purchases, denoted by $G_t \geq 0$ for all t, and it can issue bonds and levy time-varying labor and capital taxes at flat rates $\tau_{n,t}$ and $\tau_{k,t}$, respectively. The flow government budget constraint in period t is

$$\tau_{n,t} w_t N_t + \tau_{k,t} q_t K_t + B_{t+1} \geq G_t + r_t B_t, \tag{4}$$

where B_{t+1} is the level of government bonds chosen in period t, and r_t is the gross risk-free rate.

2.2 Household Problem

We assume there is no aggregate uncertainty and that government bonds and capital are perfect substitutes for store of value for households. As a result, the after-tax gross rate of return to capital must equal the gross risk-free rate:

$$1 + (1 - \tau_{k,t}) q_t - \delta = r_t, \tag{5}$$

which constitutes a no-arbitrage condition for capital and bonds.

Given the sequence of interest rates, $\{r_t\}_{t=1}^{\infty}$, and after tax wage rates, $\{\overline{w}_t \equiv (1-\tau_{n,t})w_t\}_{t=1}^{\infty}$, a household maximizes (3) by choosing a plan of consumption, labor, and asset holdings, $\{c_t(\theta^t), n_t(\theta^{t-1}), a_{t+1}(\theta^t)\}$ subject to

$$c_t\left(\theta^t\right) + a_{t+1}\left(\theta^t\right) \leq \overline{w}_t n_t\left(\theta^{t-1}\right) + r_t a_t\left(\theta^{t-1}\right), \tag{6}$$

$$a_{t+1}\left(\theta^t\right) \geq 0, \tag{7}$$

with $a_1 > 0$ given and $n_t\left(\theta^{t-1}\right) \in \left[0, \overline{N}\right]$. The solution of the household problem can be characterized analytically by the following proposition.

Proposition 1. *Denoting household gross income (or total liquidity on hand) by* $x_t(\theta^{t-1}) \equiv r_t a_t(\theta^{t-1}) + \overline{w}_t n_t(\theta^{t-1})$*, the optimal decisions for* $x_t(\theta^{t-1})$*, consumption* $c_t\left(\theta^t\right)$*, savings* $a_{t+1}\left(\theta^t\right)$*, and the labor supply* $n_t\left(\theta^{t-1}\right)$ *are given, respectively, by the following cutoff-policy rules*[4] *:*

$$x_t = \left[\overline{w}_t L(\theta_t^*)\theta_t^*\right]^{1/\sigma} \tag{8}$$

$$c_t(\theta_t) = \min\left\{1, \left(\frac{\theta_t}{\theta_t^*}\right)^{1/\sigma}\right\} x_t \tag{9}$$

$$a_{t+1}(\theta_t) = \max\left\{1 - \left(\frac{\theta_t}{\theta_t^*}\right)^{1/\sigma}, 0\right\} x_t \tag{10}$$

$$n_t(\theta_{t-1}) = \frac{1}{\overline{w}_t}\left[x_t - r_t a_t(\theta_{t-1})\right], \tag{11}$$

where the cutoff θ_t^* *is independent of individual history and determined by the Euler equation*

$$\frac{1}{\overline{w}_t} = \beta \frac{r_{t+1}}{\overline{w}_{t+1}} L(\theta_t^*), \tag{12}$$

and the function $L\left(\theta_t^*\right) \geq 1$ *captures the (gross) liquidity premium of savings and is given by*

$$L(\theta_t^*) \equiv \int_{\theta \leq \theta_t^*} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta). \tag{13}$$

[4]The cutoff-policy rules hold if the individual labor decision is an interior one, namely, $n_t \in (0, \overline{N})$. In the proof of this proposition (Appendix A.1), we show that with reasonable parameter values and a sufficiently large chosen $\overline{N}$, individual hours worked are ensured to be interior.

Proof. See Appendix A.1. □

Notice that the individual consumption function in equation (9) is reminiscent of that derived by Deaton (1991) under a numerical method, and the saving function in equation (10) exhibits a buffer-stock behavior: When the urge to consume is low ($\theta_t < \theta_t^*$), the individual opts to consume only a $\frac{\theta_t}{\theta_t^*} < 1$ fraction of total income and save the rest, anticipating that future consumption demand may be high. On the other hand, when the urge to consume is high ($\theta_t \geq \theta_t^*$), the agent opts to consume all gross income, up to the limit where the borrowing constraint binds, so the saving stock is reduced to zero. The function $L\left(\theta_t^*\right) - 1 \geq 0$ reflects the extra rate of return to savings due to the option value (liquidity premium) of the buffer stock.

Denote $\Lambda_t \equiv \frac{1}{\overline{w}_t}$ as the expected marginal utility of income. Then the left-hand side of equation (12) is the average marginal cost of consumption in the current period, and the right-hand side is the discounted expected next-period return to savings (augmented by r_{t+1}), which takes two possible values in light of the two components for the liquidity premium in equation (13): The first is simply the discounted next-period marginal utility of consumption Λ_{t+1} in the case where the borrowing constraint does not bind, which has probability $\int_{\theta \leq \theta_t^*} d\mathbf{F}(\theta)$. The second is the discounted marginal utility of consumption $\Lambda_{t+1}\frac{\theta_t}{\theta_t^*}$ in the case of high demand ($\theta_t > \theta_t^*$) with a binding borrowing constraint, which has probability $\int_{\theta > \theta_t^*} d\mathbf{F}(\theta)$. When the borrowing constraint binds, additional savings can yield a higher shadow marginal utility $\frac{\theta_{t+1}}{\theta_{t+1}^*}\Lambda_{t+1} > \Lambda_{t+1}$. The optimal cutoff θ_t^* is then determined at the point where the marginal cost of saving equals the expected marginal gains. Here, savings play the role of a buffer stock and the rate of return to savings is determined by the real interest rate r_t compounded by a liquidity premium $L(\theta_t^*)$. Notice that $\frac{\partial L(\theta_t^*)}{\partial \theta_t^*} < 0$ and $L\left(\theta_t^*\right) > 1$ for any $\theta_t^* \neq \theta_H$.

Equation (12) also suggests that the cutoff θ_t^* is independent of individual history. This property holds in this model because of the quasi-linear utility function and the assumption that the labor supply is predetermined in the first subperiod. In other words, the optimal level of liquidity on hand in period t is determined by a "target" income level given by $x_t = \left[\theta_t^* \overline{w}_t L\left(\theta_t^*\right)\right]^{1/\sigma}$, which is also independent of the history of realized values of θ_t but depends only on the distribution of θ_t. This target is essentially the optimal consumption level when the borrowing constraint binds. This target policy (uniform to all households) emerges because labor income ($\overline{w}_t n_t(\theta_{t-1})$) can be adjusted elastically to meet an optimal target, given (and regardless of) the initial asset holdings $a_t(\theta_{t-1})$. Hence, in the beginning of each period, all households will choose the same level of gross income x_t. Thus, the individual-history-independent cutoff variable θ_t^* uniquely and fully characterizes the distributions of household decisions in the economy.

2.3 Competitive Equilibrium

Denote C_t, N_t, and K_{t+1} as the level of aggregate consumption, aggregate labor, and aggregate capital, respectively. A competitive equilibrium allocation can be defined as follows:

Definition 1. *Given initial aggregate capital K_1 and bonds B_1, a sequence of taxes, and government spending and government bonds, $\{\tau_{n,t}, \tau_{k,t}, G_t, B_{t+1}\}_{t=1}^{\infty}$, a competitive equilibrium is a sequence of prices $\{w_t, q_t\}_{t=1}^{\infty}$, allocations $\{c_t(\theta^t), n_t(\theta^{t-1}), a_{t+1}(\theta^t), K_{t+1}, N_t\}_{t=1}^{\infty}$, and the cutoff $\{\theta_t^*\}_{t=1}^{\infty}$ such that*

1. *given the sequence $\{w_t, q_t, \tau_{n,t}, \tau_{k,t}\}_{t=1}^{\infty}$, the sequences $\{c_t(\theta^t), a_{t+1}(\theta^t), n_t(\theta^{t-1})\}_{t=1}^{\infty}$ solve the household problem;*

2. *given the sequence of $\{w_t, q_t\}_{t=1}^{\infty}$, the sequences $\{N_t, K_t\}_{t=1}^{\infty}$ solve the firm's problem;*

3. *the no-arbitrage condition holds for each period: $r_t = 1 + (1 - \tau_{k,t})q_t - \delta$ for all $t \geq 1$;*

4. *the government budget constraint in equation (4) holds for each period; and*

5. *all markets clear for all $t \geq 1$:*

$$K_{t+1} = \int a_{t+1}(\theta_t) d\mathbf{F}(\theta_t) - B_{t+1} \tag{14}$$

$$N_t = \int n_t(\theta_{t-1}) d\mathbf{F}(\theta_{t-1}) \tag{15}$$

$$\int c_t(\theta_t) d\mathbf{F}(\theta_t) + G_t \leq F(K_t, N_t) + (1 - \delta)K_t - K_{t+1}. \tag{16}$$

Proposition 2. *If the upper bound θ_H of the preference shocks is sufficiently large relative to the moment $\left[\mathbb{E}\left(\theta^{1/\sigma}\right)\right]^{\sigma}$ such that the following condition holds:*

$$\frac{\alpha\beta(\theta_H)^{1/\sigma}}{(\theta_H)^{1/\sigma} - \mathbb{E}(\theta^{1/\sigma})} + \beta(1-\alpha)(1-\delta) < 1, \tag{17}$$

then, in a laissez-faire competitive equilibrium, the steady-state risk-free rate is lower than the time discount rate, $r < 1/\beta$; and there exists overaccumulation of capital with a positive liquidity premium, $L(\theta^*) > 1$.

Proof. See Appendix A.2. □

Notice that when $\theta_H \to \infty$, as in the case of a Pareto distribution, the above condition is clearly satisfied. The intuition of Proposition 2 is straightforward. Since labor income is determined (*ex ante*) before the realization of the idiosyncratic preference shock θ_t, a household's total income may be insufficient to provide full insurance for large enough preference shocks under condition (17). In this case, precautionary saving motives lead to overaccumulation of capital, which reduces the equilibrium interest rate below the time discount rate. This outcome is clearly inefficient from a social point of view. It emerges because of the negative externalities of household savings on the aggregate interest rate (due to diminishing marginal product of capital), as noted by Aiyagari (1994).

However, unlike the Aiyagari (1994) model, a competitive equilibrium can achieve both AAE and IAE in our model if the idiosyncratic risk is sufficiently small (e.g., the upper bound θ_H is close enough to the mean $\mathbb{E}(\theta)$ when $\sigma = 1$ such that condition (17) is violated). In this case, individual savings can become sufficiently large to fully buffer preference shocks and, as a result, household borrowing constraints will never bind. Clearly, with full self-insurance, it must be true that the optimal cutoff is at corner, $\theta^* = \theta_H$; the liquidity premium vanishes, $L(\theta^*) = 1$; and the interest rate equals the time discount factor, $r = 1/\beta$.

A competitive equilibrium with full self-insurance is impossible in the Aiyagari model because every household's marginal utility of consumption follows a supermartingale when $r = 1/\beta$. This implies that household consumption and savings (or asset demand) diverge to infinity in the long run, which cannot constitute an equilibrium.[5]

[5] Please refer to Ljungqvist and Sargent (2012, Chapter 17) for details.

In our model, however, because the household utility function is quasi-linear, the expected shadow price of consumption goods is thus the same across agents and given by $\frac{1}{\overline{w}_t}$ (as revealed by equations (43) and (45) in the proof of Proposition 1 (Appendix A.1)), which kills the supermartingale property of the household marginal utility of consumption. As a result, household savings (or asset demand) are bounded away from infinity even at the point $r = 1/\beta$. More specifically, equations (8) and (10) show that an individual's asset demand is always bounded above by $(\theta_H - \theta_t)\,\overline{w}_t$ for any shock $\theta_t \in [\theta_L, \theta_H]$ when $r = 1/\beta$. This upper bound is finite as long as the support $[\theta_L, \theta_H]$ of θ_t is bounded (a counter example is a Pareto distribution where $\theta_H = \infty$). This special property renders our model analytically tractable with closed-form solutions (provided that θ_t is iid), and it implies that the Ramsey planner has the potential to use government debt to achieve OSIA in this economy when the competitive equilibrium is not IAE.

Nonetheless, the trade-off mechanism uncovered in this paper should be a universal property of any Aiyagari-type economies and does not hinge on the special property of our model. The trade-off between IAE (pertaining to self-insurance) and AAE (pertaining to the MGR) is driven entirely by distortionary capital taxation upon precautionary saving motives. Because a capital tax discourages households from savings, it mitigates the overaccumulation of capital but at the same time tightens individuals' borrowing constraints (thus impeding the individual self-insurance position). Hence, such a trade-off should exist in any heterogenous-agent incomplete-markets model with capital accumulation and an endogenously determined interest rate. It is therefore easy to see that government debt is an ideal tool to address this trade-off problem, as implicitly revealed in the model of Gottardi, Kajii, and Nakajima (2015) but made clear in this paper.

2.4 Conditions to Support a Competitive Equilibrium

Given that government policies are inside the aggregate state space of the competitive equilibrium and affect the endogenous distributions (including the average) of all endogenous economic variables, the Ramsey problem is to pick a competitive equilibrium (through policies) that attains the maximum of the expected household lifetime utility V defined in (3). Since V depends on the endogenous distributions (see below and Wen (2015)), the Ramsey planner needs to also pick a particular time path (sequence) of distributions to archive the maximum.

This subsection expresses the necessary conditions, in terms of the aggregate variables and distributions characterized by the cutoff θ_t^*, that the Ramsey planner must respect in order to construct a competitive equilibrium. We first show that since the cutoff θ_t^* is a sufficient statistic for describing the distributions of individual variables, all allocations and prices in the competitive equilibrium can be expressed as functions of the aggregate variables and the cutoff θ_t^*. Hence, we also call the θ_t^* as the distribution statistic.

To facilitate the analysis, we first show the properties of aggregate consumption (or average consumption across households) by aggregating the individual consumption decision rules (9). By the law of large numbers, the aggregate consumption is determined by

$$C_t = D(\theta_t^*)x_t, \tag{18}$$

where the aggregate marginal propensity to consume (the function D) is given by

$$D(\theta_t^*) \equiv \int_{\theta \leq \theta_t^*} \left(\frac{\theta}{\theta_t^*}\right)^{1/\sigma} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} d\mathbf{F}(\theta) \in (0, 1]. \tag{19}$$

Then, we can express individual consumption and individual asset holding as functions of C_t and θ_t^* by plugging equation (18) into equations (9) and (10). To fully describe the conditions necessary for constructing a competitive equilibrium, we rely on the following proposition:

Proposition 3. *Given initial capital K_1, initial government bonds B_1, and the initial capital tax $\tau_{k,1}$, the sequences of aggregate allocations $\{C_t, N_t, K_{t+1}, B_{t+1}\}_{t=1}^{\infty}$ and distribution statistics $\{\theta_t^*\}_{t=1}^{\infty}$ can be supported as a competitive equilibrium if and only if the resource constraint (16), the asset market clearing condition*

$$B_{t+1} = \left(\frac{1}{D(\theta_t^*)} - 1\right) C_t - K_{t+1}, \text{ for all } t \geq 1, \tag{20}$$

and the following implementability conditions (for periods $t = 1$ and $t \geq 2$, respectively) are satisfied:

$$C_1^{1-\sigma} D(\theta_1^*)^{\sigma-1} L(\theta_1^*)\theta_1^* \geq N_1 + r_1 C_1^{-\sigma} D(\theta_1^*)^{\sigma} L(\theta_1^*)\theta_1^*(K_1 + B_1) \tag{21}$$

and

$$C_t^{1-\sigma} D(\theta_t^*)^{\sigma-1} L(\theta_t^*)\theta_t^* \geq N_t + \frac{1}{\beta} C_{t-1}^{1-\sigma} D(\theta_{t-1}^*)^{\sigma-1} \theta_{t-1}^* \left(1 - D(\theta_{t-1}^*)\right), \tag{22}$$

where $r_1 = 1 + (1-\tau_{k,1})\frac{\partial F(K_1,N_1)}{\partial K_1} - \delta$.

Proof. See Appendix A.3 □

Note that the implementability conditions essentially enforce the flow government budget constraint and are comparable to those in the representative agent framework.

To derive the implementability conditions (21) and (22), we first replace the intertemporal prices and taxes with quantitative variables. As shown in Appendix A.3, the flow government budget constraint in a competitive equilibrium can be expressed as

$$\begin{aligned} & U_{C,t} D(\theta_t^*)^\sigma L(\theta_t^*)\theta_t^* C_t - N_t + U_{C,t} D(\theta_t^*)^\sigma L(\theta_t^*)\theta_t^* A_{t+1} \\ \geq\ & \frac{1}{\beta} U_{C,t-1} D(\theta_{t-1}^*)^\sigma \theta_{t-1}^* A_t, \end{aligned} \tag{23}$$

where $U_{C,t}$ is defined as $C_t^{-\sigma}$, the "marginal utility" of aggregate consumption in our setup. The above expression is analogous to that in a representative agent model, except with two additional terms: $D(\theta_t^*)^\sigma \theta_t^*$ and $L(\theta_t^*)$. These extra terms originate from the modification of the risk free rate, r_{t+1}, which can be expressed as (see Appendix A.3):

$$\frac{1}{r_{t+1}} = \beta \frac{U_{C,t+1}}{U_{C,t}} \frac{D(\theta_{t+1}^*)^\sigma \theta_{t+1}^*}{D(\theta_t^*)^\sigma \theta_t^*} L(\theta_{t+1}^*). \tag{24}$$

The function $U_{C,t}$ captures the marginal utility of aggregate consumption; the function $D(\theta_t^*)^\sigma \theta_t^*$ captures the marginal utility of the distribution of individual consumption; and the function $L\left(\theta_{t+1}^*\right)$ captures the liquidity premium of bonds. In representative agent models, the last two terms are absent since there is no consumption distribution or liquidity premium to affect the risk free rate of bonds. Thus, as shown in the proof of Proposition 3, equation (23) together with the asset market clearing conditions imply the implementability conditions (21) and (22).

This proposition demonstrates that the Ramsey planner can construct a competitive equilibrium by simply choosing the sequences of aggregate allocations $\{C_t, N_t, K_{t+1}, B_{t+1}\}$ and distribution statistics $\{\theta_t^*\}$ to maximize expected welfare, subject to the aggregate resource constraint, asset market clearing condition, and the implementability condition.

3 Ramsey Allocations

Armed with Proposition 3, we are ready to write down the Ramsey planner's problem and derive the first-order Ramsey conditions analytically.

3.1 Ramsey Problem

Using equations (9) and (18), the lifetime utility function V can be rewritten as a function of the distribution statistic θ_t^* and aggregate variables:

$$V = \sum_{t=1}^{\infty} \beta^{t-1} \left[W(\theta_t^*) \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{\overline{\theta}}{1-\sigma} - N_t \right], \tag{25}$$

where $W(\theta_t^*)$ is defined as

$$W(\theta_t^*) \equiv \left(\int_{\theta \leq \theta_t^*} \theta \left(\frac{\theta}{\theta_t^*} \right)^{\frac{1-\sigma}{\sigma}} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} \theta d\mathbf{F}(\theta) \right) D \left(\theta_t^* \right)^{\sigma-1} \tag{26}$$

Thus, the Ramsey problem can be represented alternatively as maximizing the welfare function (25) by choosing the sequences of $\{\theta_t^*, N_t, C_t, K_{t+1}, B_{t+1}\}$, subject to the resource constraint (16), the asset market clearing condition (20), and the implementability conditions (22) and (21).

In addition, an exogenous debt limit $B_{t+1} \leq \overline{B}$ is imposed on the Ramsey planner to facilitate our analysis of the role of government debt, which most of the existing literature has ignored or assumed away by implicitly setting $\overline{B} = \infty$—which is larger than the natural debt limit of the government.

Therefore, the Lagrangian of the Ramsey problem is given by

$$
\begin{aligned}
L \;=\; & \max_{\{\theta_t^*, N_t, C_t, K_{t+1}, B_{t+1}\}} \sum_{t=1}^{\infty} \beta^{t-1} \left[W(\theta_t^*) \frac{C_t^{1-\sigma}}{1-\sigma} - \frac{\overline{\theta}}{1-\sigma} - N_t \right] \qquad (27) \\
& + \sum_{t=1}^{\infty} \beta^{t-1} \mu_t \left(F(K_t, N_t) + (1-\delta) K_t - G_t - C_t - K_{t+1} \right) \\
& + \lambda_1 \left(C_1^{1-\sigma} D(\theta_1^*)^{\sigma-1} L(\theta_1^*) \theta_1^* - N_1 - C_1^{-\sigma} D(\theta_1^*)^{\sigma} L(\theta_1^*) \theta_1^* r_1 (K_1 + B_1) \right) \\
& + \sum_{t=2}^{\infty} \beta^{t-1} \lambda_t \left(C_t^{1-\sigma} D(\theta_t^*)^{\sigma-1} L(\theta_t^*) \theta_t^* - N_t - \beta^{-1} C_{t-1}^{1-\sigma} D(\theta_{t-1}^*)^{\sigma-1} \theta_{t-1}^* \left(1 - D(\theta_{t-1}^*) \right) \right) \\
& + \sum_{t=1}^{\infty} \beta^{t-1} \phi_t \left(K_{t+1} + B_{t+1} - \left(D(\theta_t^*)^{-1} - 1 \right) C_t \right) \\
& + \sum_{t=1}^{\infty} \beta^{t-1} \nu_t^B (\overline{B} - B_{t+1}),
\end{aligned}
$$

where μ_t, λ_t, and ϕ_t denote the multipliers for the resource constraints, the implementability conditions, and the asset market clearing conditions, respectively. In addition, the multiplier of the debt limit constraint is denoted by ν_t^B. The K_1, B_1, and $\tau_{k,1}$ are given. and hence r_1 depends only on N_1.

The first-order Ramsey conditions with respect to K_{t+1}, N_t, C_t, B_{t+1}, and θ_t^* are given, respectively, by

$$\mu_t - \phi_t = \beta \mu_{t+1} \left(MP_{K,t+1} + 1 - \delta \right) \qquad (28)$$

$$1 + \lambda_t = \mu_t MP_{N,t} \text{ for } t \geq 2 \qquad (29)$$

$$
\begin{aligned}
\mu_t \;=\; & W(\theta_t^*) C_t^{-\sigma} + (1-\sigma) C_t^{-\sigma} D(\theta_t^*)^{\sigma-1} \theta_t^* \left(\lambda_t L(\theta_t^*) - \lambda_{t+1} \left(1 - D(\theta_t^*) \right) \right) \qquad (30) \\
& - \phi_t \left(D(\theta_t^*)^{-1} - 1 \right) \text{ for } t \geq 2
\end{aligned}
$$

$$\beta^t \phi_t - \beta^t \nu_t^B = 0 \qquad (31)$$

$$\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{C_t^{1-\sigma}}{1-\sigma} + \lambda_t C_t^{1-\sigma} H(\theta_t^*) - \lambda_{t+1} C_t^{1-\sigma} J(\theta_t^*) + \phi_t \frac{C_t}{D(\theta_t^*)^2} \frac{\partial D(\theta_t^*)}{\partial \theta_t^*} = 0 \text{ for } t \geq 2, \qquad (32)$$

where

$$
\begin{aligned}
H\left(\theta_t^*\right) &\equiv \frac{\partial\left(D(\theta_t^*)^{\sigma-1}L(\theta_t^*)\theta_t^*\right)}{\partial\theta_t^*} \\
J\left(\theta_t^*\right) &\equiv \frac{\partial\left(D(\theta_t^*)^{\sigma-1}\theta_t^*\left(1-D(\theta_t^*)\right)\right)}{\partial\theta_t^*}.
\end{aligned}
$$

To conserve space, the first-order Ramsey conditions with respect to N_1, C_1, and θ_1^* in the initial period, as well as several useful lemmas for the upcoming proofs, are relegated in the Appendix A.4.

Note that the Lagrangian multiplier ϕ_t for the asset market clearing condition and the multiplier v_t^B for the government debt-limit constraint are equal to each other according to equation (31), suggesting that the tightness of the asset market depends on the government debt limit.

3.2 Characterization of Ramsey Allocations

Definition 2. *A Ramsey steady state is defined as a long-run Ramsey allocation where (i) the parameter restriction $\theta_H < \frac{\theta_L}{(1-\beta)^\sigma} < \infty$ (to ensure positive labor $n > 0$ for all individuals in all states) is satisfied, and (ii) all aggregate variables $\{K, N, C, B, \theta^*\}$ and the associated Lagrangian multipliers $\{\lambda, \mu\}$ converge to finite and strictly positive values.*

The condition

$$
\theta_H < \frac{\theta_L}{(1-\beta)^\sigma} \tag{33}
$$

is required to ensure that all individuals' labor decisions are positive, a necessary condition for Proposition 1. The intuition is that if the variance (support) of θ is too large (spread out), some agents may end up with too much savings in the last period and thus opt not to work this period. Our model becomes intractable in the situation with possible binding zero labor supply, so it must be ruled out. In addition, any variable without subscript t is referred as its steady-state value.

Proposition 4. *If a Ramsey steady state exists, then the optimal capital tax rate in such a Ramsey steady state is determined by the following equation:*

$$
1-\tau_k = \frac{L\left(\theta^*\right)^{-1} - \beta\left(1-\delta\right)}{\frac{\mu-\phi}{\mu} - \beta\left(1-\delta\right)}, \tag{34}
$$

where $\mu \geq 0$ and $\phi \geq 0$ are the multipliers of Ramsey Lagrangian (27) for the aggregate resource constraint and the debt limit constraint (implied by equation (31)), respectively.

Proof. See Appendix A.5 □

Proposition 4 immediately gives the following steady-state tax rate for capital,

$$\tau_k \gtreqless 0 \text{ if and only if } L(\theta^*) \gtreqless \frac{\mu}{\mu - \phi}, \tag{35}$$

which implies the following two points:

First, if the government debt-limit constraint does not bind in the Ramsey steady state—i.e., $\phi = 0$ and $\frac{\mu-\phi}{\mu} = 1$—then, since the liquidity premium $L(\theta^*) \geq 1$, the right-hand side of equation (34) must be smaller than or equal to 1. Therefore, optimal capital tax must be non-negative: $\tau_k \geq 0$. Hence, subsidizing capital in this case is never optimal. In addition, if in this case the optimal cutoff is a corner solution at $\theta^* = \theta_H$, then $L = 1$, and it must be true that $\tau_k = 0$.

Second, if the government debt-limit constraint is binding in the steady state, i.e., $\phi > 0$, then the left-hand side of equation (35) must be greater than 1. In this case, if the optimal cutoff is a corner solution at $\theta^* = \theta_H$, then $L = 1$, so it must be true that $\tau_k < 0$. On the other hand, if in this case the optimal capital tax $\tau_k = 0$, then it must be true that $\theta^* < \theta_H$ and $L(\theta^*) > 1$; namely, IAE is not achieved.

So in what follows, we prove the existence of two types of the Ramsey steady state and characterize their respective properties. Recall that the conditions (17) and (33) are assumed to hold throughout the paper; so in each of the two possible cases the competitive equilibrium without government intervention is inefficient by design. The interesting question is how and by how much can the government improve upon the allocations of laissez-faire competitive equilibrium.

3.2.1 Case 1: OSIA Ramsey Steady State

The first case characterizes the optimal self-insured allocation (OSIA), which is defined formally as follows:

Definition 3. *OSIA is defined as a Ramsey steady state where both AAE and IAE are achieved; namely, in OSIA the MGR holds and the borrowing constraints $a_{t+1} \geq 0$ do not* *bind for all households in all states.*

Proposition 5. *Suppose $\overline{B}$ is sufficiently large (but smaller than the natural debt limit) such that the constraint $B_{t+1} \leq \overline{B}$ never binds. Then there exists a unique Ramsey steady state and this steady state is an OSIA with the following properties:*

1. *IAE is achieved—the optimal choice of θ^* is a corner solution at $\theta^* = \theta_H$ so that no households face a positive probability of strictly binding borrowing constraints.*
2. *AAE is achieved—the MGR holds, there is no liquidity premium ($L(\theta^*) = 1$), and the equilibrium interest rate equals $1/\beta$.*
3. *The capital tax is zero, $\tau_k = 0$; and the labor tax is positive at the rate $\tau_n = \frac{\lambda}{1+\lambda}\sigma$. Government expenditures and bond interest payments are financed solely by revenues from the labor income tax.*

Proof. See Appendix A.6 □

The above proposition states that if the debt-limit constraint $B \leq \overline{B}$ does not bind in the steady state, then the Ramsey planner will pick a long-run competitive equilibrium that achieves OSIA—the allocation with both AAE and IAE satisfied.

This proposition indicates that the Ramsey planner achieves the MGR without the need to tax capital in the steady state—as capital taxation in the steady state would undermine IAE by decreasing the steady-state household saving rate and thus permanently hampering their self-insurance positions. Instead, the Ramsey planner opts to provide enough incentives for households to save through bonds by picking a sufficiently high interest rate ($= 1/\beta$) on bonds, such that all households are fully self-insured in the long run with zero probability of encountering binding liquidity constraints.

It is this critical role of government debts in improving the individual self-insurance position that determines the optimal debt level in the model. This can be seen more clearly by inspecting the optimal debt-to-GDP ratio in an OSIA in the special case where the EIS parameter $\sigma = 1$, the rate of capital depreciation $\delta = 1$, and government spending $G = 0$.

Corollary 1. *When $\sigma = \delta = 1$ and $G = 0$, the optimal debt-to-GDP ratio is determined by a wedge τ_b times the MGR-saving rate $\beta\alpha$:*

$$\frac{B}{Y} = \tau_b \beta \alpha, \tag{36}$$

where the wedge

$$\tau_b = \left(\frac{1 - D(\theta^*)}{D(\theta^*)} \Big/ \frac{\beta\alpha}{1 - \beta\alpha} - 1 \right) \geq 0 \tag{37}$$

is essentially the gap between the competitive equilibrium saving ratio ($\frac{1-D}{D}$) and the modified-golden-rule saving ratio ($\frac{\beta\alpha}{1-\beta\alpha}$). This gap vanishes if and only if the competitive equilibrium under incomplete markets approaches the allocation of an economy with full self-insurance (or with complete markets).

Proof. See Appendix A.7 □

Recall that $D(\theta^*)$ denotes the aggregate marginal propensity to consume and that $(1 - D(\theta^*))$ denotes the aggregate marginal propensity to save in a competitive equilibrium with incomplete markets. It can be shown easily that under complete markets (e.g., in a representative-agent model with $\delta = 1$ and log utility), the optimal saving rate is $\beta\alpha$. Hence, precautionary saving behavior implies that the saving rate under incomplete markets exceeds the saving rate under complete markets, i.e., $(1 - D) > \beta\alpha$ and $D < (1 - \beta\alpha)$. Hence, the wedge is strictly positive: $\tau_b > 0$. However, as the variance of θ approaches zero, or as the insurance markets become complete, it must be true that $(1 - D) \to \beta\alpha$ and $\tau_b \to 0$, regardless of the MGR saving rate $\beta\alpha$.

Therefore, the wedge τ_b is a measure of the degree of the individual allocative inefficiency in the competitive equilibrium. So Proposition 1 shows (again) that the single most important role of government debts is to improve the individual self-insurance position, such that the optimal level of bond supply is proportional to the MGR-saving rate by a factor that is determined solely by the wedge of inefficiency caused by incomplete insurance markets in a competitive equilibrium.

Obviously, OSIA can be archived only if the Ramsey planner is capable of supplying enough bonds to satisfy the liquidity demanded of each household across all states. An important property of the OSIA is that the equilibrium interest rate equals the time discount rate: $r = 1/\beta$. However, the zero-capital-tax policy does not depend on this property. To shed light on this issue further, we study what happens if the government's ability to issue bonds is limited.

3.2.2 Case 2: Ramsey Steady State with a Binding Debt Limit

To illustrate our point without loss of tractability, we impose an exogenous debt limit $\overline{B}$ that lies strictly below both the natural debt limit and the optimal debt level B^* determined in OSIA.

Proposition 6. *Suppose that the debt-limit constraint binds after a sufficiently large $t > 1$: $B_{t+1} = \overline{B}$. Then, there exists a Ramsey steady state with the following properties:*

1. *IAE fails to hold—the optimal choice of the cutoff is interior, $\theta^* \in (\theta_L, \theta_H)$; so there is always a non-zero fraction $(1 - \mathbf{F}(\theta^*) > 0)$ of households facing binding borrowing constraints in every period.*

2. *AAE fails to hold—the MGR does not hold and there is a positive liquidity premium $(L > 1)$ with equilibrium interest rate $r < 1/\beta$.*

3. *The capital tax is still zero, $\tau_k = 0$; namely, the Ramsey planner does not tax capital even if MGR fails to hold because of a binding debt-limit constraint. Notice that this is true even if $\overline{B} = 0$.*

Proof. See Appendix A.8 □

Obviously, a special subcase of case 2 is when the government cannot issue bonds at all: $\overline{B} = 0$. This subcase is analogous to the situation discussed in Proposition 2 in the previous section, where the competitive-equilibrium interest rate is strictly less than the time discount rate. In such a situation, the Ramsey planner cannot use bonds to manipulate the market interest rate and divert household savings away from capital formation. In general, whenever the government is unable to supply enough bonds to meet household demand for buffer-stock savings, either because θ_H is sufficiently large or the debt limit $\overline{B}$ is sufficiently low, the pursuit of individual efficiency by the Ramsey planner will necessarily lead to a binding debt-limit constraint on government bonds: $B = \overline{B}$. In this case, a Ramsey steady state exists, but in such a steady state, neither IAE nor AAE is achieved by the planner—albeit it is feasible for the planner to achieve the AAE by taxing capital.

As shown in the proofs for Proposition 5 and Proposition 6, the zero capital tax is obtained in our model because of the strikingly simple steady-state relationship:

$$L(\theta^*) = \frac{\mu}{\mu - \phi}, \tag{38}$$

which by equation (34) implies that $\tau_k = 0$, regardless of the debt limit $\overline{B}$ and the other model parameter values (such as the elasticity of intertemporal substitution σ, the time discount factor β, the output elasticity of capital α, and the rate of capital depreciation δ, among others).

This simple analytical relationship (38) is very striking and surprising. The Ramsey planner opts to supply enough bonds to improve the individual self-insurance position until the debt limit $\overline{B}$ is binding. But regardless of the tightness of the binding debt limit (ϕ), the Ramsey planner nonetheless opts to equalize the ratio of Lagrangian multipliers, $\frac{\mu}{\mu-\phi}$, to the liquidity premium, such that the steady-state capital tax is exactly zero. This result offers a strong case to support the view that steady-state capital tax is extremely distortionary and hence should not be used as a tool to achieve AAE (or MGR) in the long run.

Hence, the government borrowing limit does not matter for the optimal steady-state capital tax but does matter for the Ramsey planner's ability to achieve MGR and IAE. Yet such a critical role of government-debt limits is often ignored or has gone unnoticed in the existing literature.

We will show in the next section, however, that the MGR is not entirely irrelevant to the social planner. Along the transition path toward the steady state, the social planner will tax capital to reduce the steady-state capital stock, albeit not to the degree of fully restoring the MGR. But before showing that, we consider two interesting situations where a Ramsey steady state with a non-binding debt limit may not exist.

3.2.3 Non-existence of Ramsey Steady State

As argued by Aiyagari (1994, 1995), the long-run equilibrium interest rate in a standard Aiyagari model must be lower than $1/\beta$; otherwise, individual asset demand goes to infinity—which cannot be a competitive equilibrium. This property led to Aiyagari (1995) arguing that the Ramsey plan should tax capital to achieve the MGR regardless of the individual allocative inefficiency (which is intensified by a positive capital tax). We assert below that this line of argument is based on the dubious assumption that a Ramsey steady state (without a binding debt limit) always exists in the Aiyagari model.

To see our point, we can conduct two counterfactual analyses:

(i) Suppose under Case 1 (with a non-binding debt limit) we assume *incorrectly* that there exists another Ramsey steady state where $\phi = 0$ and that the interest rate $r^* < \frac{1}{\beta}$, where $r^* \equiv 1 + q_t - \delta$. Since the interest rate lies below the time discount rate (as in the model of Aiyagari), according to equations (12) and (13), it must be true that the gross liquidity premium $L(\theta^*) > 1$ and the cutoff is interior, $\theta^* < \theta_H$; namely, individual borrowing constraints strictly bind for some households. Given that $\phi = 0$, equation (28) implies that the MGR holds. Given that the MGR holds and $L(\theta^*) > 1$, equation (12) implies that the Ramsey planner should tax capital so that the two equations are mutually consistent—because $(q + 1 - \delta) = L(\theta^*) [(1 - \tau_k) q + 1 - \delta]$ implies $\tau_k > 0$. In other words, the Ramsey planner should set $\tau_k > 0$ to achieve a competitive equilibrium consistent with MGR and should ignore the individual allocative inefficiency, exactly as Aiyagari (1995) has argued. However, the assumption of the existence of *such* a Ramsey steady state with a non-binding debt limit and an interior cutoff $\theta^* < \theta_H$ is inconsistent with the other Ramsey first-order conditions—because under such assumptions it can be shown (see the proof of Proposition 5) that the Lagrangian multiplier $\mu = 0$, which contradicts the definition of a Ramsey steady state where the aggregate resource constraint strictly binds. Unfortunately, Aiyagari's (1995) original analysis failed to check if the assumption of the existence of a Ramsey steady state is fully consistent with the other first-order conditions of the Ramsey problem.

(ii) Let $\theta_H \to \infty$ (as in the case of a Pareto distribution) and assume again that there is no debt limit (i.e., $\overline{B} = \infty$). Note that in this case a competitive equilibrium without government intervention still exists. Also suppose (for the sake of argument) that the assumption of non-negative individual labor supply ($\theta_H < \theta_L/(1-\beta)^\sigma$) can be relaxed so that labor supply can be negative for some individuals but remain positive at the aggregate level—otherwise, a binding individual labor supply at zero would invalidate Proposition 1. Since by the arguments in Propositions 5 and 6, the Ramsey planner would always opt to keep increasing the interest rate by issuing enough government bonds to achieve IAE; consequently, the optimal bond demand diverges to infinity as θ_H approaches infinity. This point can be seen from the asset market clearing condition (20), which pins down the optimal level of aggregate bond demand B_{t+1} as a function of the distribution parameter θ_H; namely, under IAE, the debt-to-GDP ratio becomes

$$\frac{B}{Y}=\left(\frac{\theta_H^{1/\sigma}}{E\left(\theta^{1/\sigma}\right)}-1\right)\left(1-\frac{\delta\beta\alpha}{1-\beta\left(1-\delta\right)}-\frac{G}{Y}\right)-\frac{\beta\alpha}{1-\beta\left(1-\delta\right)}, \tag{39}$$

which implies that under finite values of aggregate consumption-to-output ratio $\left(1-\frac{\delta\beta\alpha}{1-\beta(1-\delta)}-\frac{G}{Y}\right)$ and capital-to-output ratio $\frac{\beta\alpha}{1-\beta(1-\delta)}$, both the optimal debt-to-output ratio $\frac{B}{Y}$ and the Lagrangian multiplier μ approach infinity as θ_H goes to infinity. An infinite amount of government bonds clearly violates the natural debt limit and hence leads to a contradiction of the existence of a Ramsey steady state featuring a non-binding debt limit. In addition, when $\mu_t\to\infty$ (because of an increasingly tight aggregate resource constraint under unlimited bond demand), the Ramsey first-order condition (28) cannot pin down the steady-state marginal product of capital or support the validity of MGR.

Note that in both situations considered above, a Ramsey steady state would exist if the debt-limit constraint is binding: $\lim_{t\to\infty}B_{t+1}=\overline{B}$ and $\lim_{t\to\infty}\phi_t=\phi>0$. Then it can be shown that the Lagrangian multiplier $\mu\in(0,\infty)$ and that this value is consistent with all of the Ramsey first-order conditions. But in such a case, equation (28) suggests that the MGR fails to hold; even in this case, Proposition 6 shows that the steady-state capital tax $\tau_k=0$.

In other words, when a Ramsey steady state featuring a non-binding debt limit does not exist but is erroneously assumed to exist in our model, we would conclude incorrectly that it is optimal to pursue MGR by levying a positive tax on capital and ignore the individual allocative inefficiency (or the adverse effect of capital tax on household self-insurance position).

These counterfactual analyses suggest that the different results for optimal capital tax between our analysis and Aiyagari's (1995) analysis may be caused not by the difference in models, but rather by the validity of the assumption of the existence of a Ramsey steady state (with a non-binding natural government debt limit) made by Aiyagari in his analysis. This argument is supported by the independent analysis of Chen, Chien, and Yang (2019), which employs a standard Aiyagari-type heterogeneous-agent incomplete-markets model to show that the assumption of a Ramsey steady state with a non-binding debt limit is indeed inconsistent with some of the first-order necessary conditions of the Ramsey planner.[6] There-

[6]It can be shown that the MGR would fail to hold in the original Aiyagari model whenever a debt-limit constraint binds in a Ramsey steady state (if it exists).

fore, our work provides a strong warning for the existing literature that relies on numerical methods to compute Ramsey taxation, which often assumes (without proof) the existence of a Ramsey steady state and a non-binding debt limit.

4 Numerical Analysis

This section performs numerical exercises to confirm our theoretical results and illustrate the optimal transition path of Ramsey allocation in comparison with laissez-faire equilibrium. These numerical analyses further reveal the trade-off between AAE and IAE and why IAE matters to the Ramsey planner.

4.1 Parameter Values

The government spending G_t is set to zero for all periods. The initial government bond and capital tax are also set to zero ($B_1 = 0$ and $\tau_{k,1} = 0$). The initial capital stock K_1 is set to be 70% of the Ramsey steady-state level. The production function is assumed to be Cobb-Douglas with capital share $\alpha = 1/3$, the time discount rate $\beta = 0.95$, and the capital deprecation rate $\delta = 0.75$. The distribution of preference shock θ follows a power function $\mathbf{F}(\theta) = \frac{\theta^\gamma - \theta_L^\gamma}{\theta_H^\gamma - \theta_L^\gamma}$, where $\theta_L = 1$, $\theta_H = 10$ and $\gamma = 0.1$. The results are qualitatively similar for other choices of parameter values.

These parameter values imply that the following conditions are satisfied: (i) In the steady state the condition for positive individual labor choice, $\theta_H < \frac{\theta_L}{(1-\beta)^\sigma} < \infty$, is satisfied. (ii) In the transition the condition for positive labor supply $n_t > 0$ is satisfied and verified numerically in each time period. (iii) The condition (17) holds, such that the laissez-faire competitive equilibrium is neither IAE nor AAE.

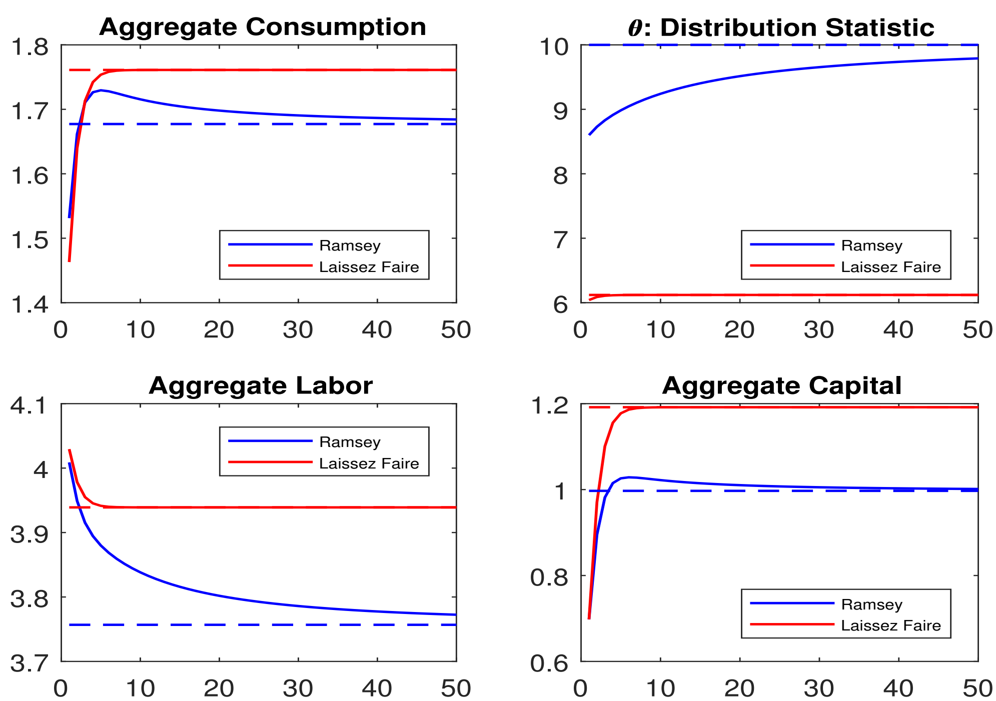
Under these parameter values, the OSIA allocation is feasible for the Ramsey planner with a non-binding debt limit and corresponds to Case 1 described in Proposition 5.

4.2 Ramsey Transition Paths

Consider the case of log utility ($\sigma = 1$) first. Figure 1 shows the transition paths of aggregate consumption C_t (top left panel), aggregate labor N_t (lower-left panel), the distribution statistic θ_t^* (top-right panel), and aggregate capital stock K_t (lower-right panel). In each panel, blue lines represent the Ramsey economy, red lines represent the laissez-faire economy, a solid line represents the transition, and a dashed line represents the corresponding steady state. The results can be summarized as follows.

First, the Ramsey transition paths are significantly slower than their counterparts in the laissez-faire economy, especially the transition of the distribution statistic θ_t^*. For example, consumption, labor, and capital stock take about 5 periods to nearly approach their respective steady states under laissez faire, as opposed to more than 50 periods under Ramsey. In particular, it takes only about 2 periods for the distribution statistic θ_t^* to nearly approach its steady state under laissez faire, as opposed to more than 50 periods under Ramsey (top-right panel). Recall that the Ramsey steady state is both IAE and AAE, hence the steady state of θ^* is θ_H $(= 10)$ under Ramsey.

Figure 1: Ramsey Transition Paths of Aggregate Variables



Notes: Transition paths in the Ramsey economy (solid blue lines) and laissez-faire economy (solid red lines), as well as their corresponding steady states (dashed lines).

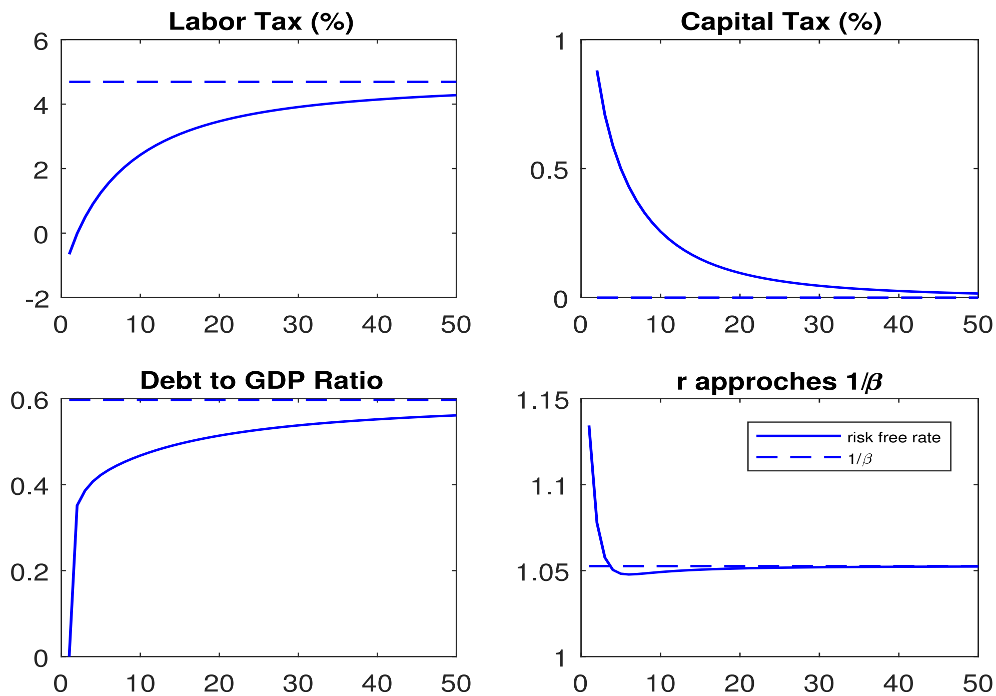
Second, the Ramsey allocation exhibits lower steady-state levels in aggregate labor and capital but a significantly higher cutoff value θ^*, compared with those in the laissez-faire economy. This suggests that the Ramsey planner opts to induce the households to work less and invest less in capital to improve AAE. Interestingly, aggregate consumption is also lower in the Ramsey steady state than in the laissez-faire steady state (top-left panel). But this does not necessarily imply a lower welfare, because the distribution of consumption is significantly improved—a significantly higher cutoff θ^* implies a much lower probability of a binding borrowing constraint and hence a greatly improved individual self-insurance position (or IAE)—thanks to the availability of government bonds.

Third, however, in the initial several periods aggregate consumption is slightly higher in the Ramsey economy than in the laissez-faire economy and approaches the Ramsey steady state from above (top-left panel). This suggests that the Ramsey planner intends to front-load consumption during the transition path to increase welfare under time discounting, which the laissez-faire economy is unable to do because of strong precautionary saving motives. Unlike the competitive equilibrium, the Ramsey planner is able to front-load consumption by over-shooting aggregate consumption above its Ramsey steady state even in the intermediate run—because the planner can reduce the interest rate to a level below the time discount rate along its transition path in the intermediate run, as can be seen in Figure 2 where the interest rate over-shoots its Ramsey steady state from above and then converges slowly back to the steady state from below (bottom-right panel).

In particular, Figure 2 shows that under the Ramsey planner, the debt-to-GDP ratio (bottom-left panel) increases rapidly from 0% to 40% in the short run to attract household savings and improve the individual self-insurance position, resulting in higher-than-steady-state short-run interest rates in the initial several periods (bottom-right panel). However, the interest rate subsequently falls below the time discount rate ($1/\beta$) and converges only slowly back to the Ramsey steady state.

The rapidly increasing amount of debts clearly requires financing from tax revenues. The government can finance the debts through either labor tax, capital tax, or both. Interestingly, Figure 2 shows that the Ramsey planner opts to put the pressure of revenue collection on capital tax in the short run and turn attention to labor tax in the longer run—such that capital tax is the highest initially (0.9% at $t = 2$) and gradually reduces to 0% in the long run (top-right panel); in the meanwhile, labor tax is low initially (even slightly negative) and gradually approaches 4.7% in the long run (top-left panel). This suggests that the source of government revenues to finance public debts lies mainly in capital tax in the very short run

Figure 2: Ramsey Transition Paths of Policy Tools



Notes: Ramsey transition paths and their corresponding steady state values are shown as solid blue lines and dashed blue lines, respectively. Since the initial capital tax, $\tau_{k,1}$, is zero, the plot of capital tax starts at $t = 2$.

but exclusively in labor tax in the long run.

The rapidly increasing government bonds and the positive capital tax rates in the transition periods significantly slowed down capital accumulation so that the steady-state capital is consistent with the MGR. In contrast, the capital stock under laissez faire is 20% above the MGR-capital stock (bottom-right panel in Figure 1). This suggests that, instead of taxing capital permanently as argued by Aiyagari (1995), the Ramsey planner opts to tax capital *only* in the short run to improve aggregate allocative efficiency. Consequently, we see the opposite transition paths of capital tax and labor tax in Figure 2.

Of course, it must be recognized that the key mechanism to enable the Ramsey planner to achieve MGR in the long run is its ability to issue plenty of debts. As explained earlier, the supply of government bonds helps to improve the household self-insurance position and at the same time to reduce the overaccumulation of capital.

An important implication of this logic is that when the government cannot issue enough bonds or simply cannot issue debts at all, the Ramsey planner shall reduce short-run capital taxation (or simply do not tax capital at all), even if MGR fails to hold. In other words, MGR has no bearing on optimal capital taxation and consequently, as the debt limit $\overline{B}$ reduces to zero, the Ramsey allocation approaches the laissez-faire allocation with zero capital tax in both the short run and the long run, as confirmed in the following subsection. These findings are in sharp contrast to the conventional Aiyagarian wisdom embraced by the main literature.

4.3 Ramsey Transitions under Binding Debt Limits

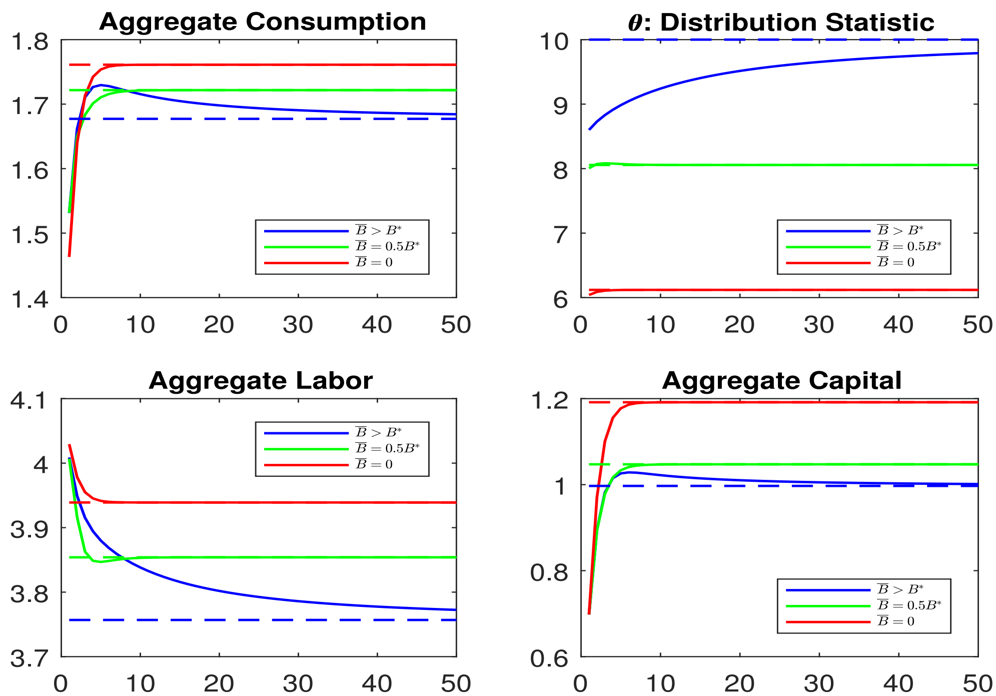
To study the dynamic and long-run effects of a binding debt limit on Ramsey allocation, we compare three scenarios in Figure 3 and Figure 4: (i) the scenario without any debt limit (blue lines), which is identical to the case shown in Figure 1 (blue lines); (ii) the scenario with a binding debt limit $\overline{B}$ equal to 50% of the optimal debt level of OSIA (denoted by B^*, green lines); and (iii) the scenario with a zero debt limit $\overline{B} = 0$ (red lines).

Figure 3 shows that as the debt limit $\overline{B}$ decreases step by step toward zero, the steady-state levels of aggregate consumption (top-left panel), labor (bottom-left panel), and capital stock (bottom-right panel) all increase and approach the corresponding laissez-faire level. Meanwhile, the steady-state cutoff decreases significantly toward the laissez-faire level (top-right panel), suggesting that the Ramsey planner becomes less and less capable of improving the individual self-insurance position when the government capacity to issue debts is reduced. Notice that as the debt limit decreases, the speed of transition also increases—because the Ramsey allocation behaves more and more like a laissez-faire competitive equilibrium.

These results are anticipated by Proposition 6, according to which (under the assumption of zero government spending) the Ramsey steady state in scenario (iii) coincides with the laissez-faire steady state where $G = B = \tau_k = \tau_n = 0$. But here we show that when the government cannot issue bonds at all (equivalent to $\overline{B} = 0$), the entire transition path is also identical to the laissez-faire case (red lines) as shown in Figure 1. Therefore, scenario (ii) with $\overline{B} = 50\%$ of the optimal debt level of OSIA lies in between scenario (i) and scenario (iii).

Figure 4 shows the effects of debt limits on the transition paths (as well as the steady state) of policy variables. It offers explanations for the transition patterns of aggregate

Figure 3: Ramsey Transition Paths of Aggregate Variables with Binding Debt Limits

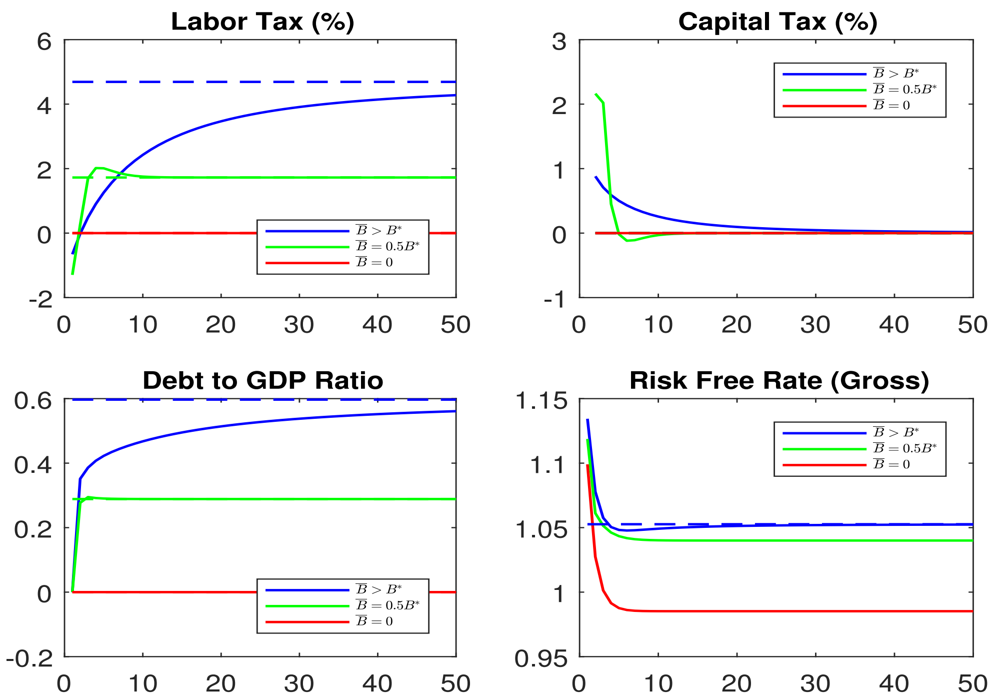


Notes: Scenario (i), (ii) and (iii) are shown as blue, green, and red lines, respectively. Their corresponding steady state levels are indicated by dashed lines.

variables shown in Figure 3. First, as the debt limit reduces, the optimal debt-to-GDP ratio and its transition time needed to approach the steady state also decline (bottom-right panel). Since a lower debt-to-GDP ratio implies that the government has a smaller burden of financing interest payments, the average tax rates for both capital and labor along the transition are reduced as well (top row panels). Interestingly, although in the initial transition period capital tax under scenario (ii) is higher than that under scenario (i), the average rate is lower because capital tax converges to the zero-steady state much faster under scenario (ii) than under scenario (i), suggesting a feature of non-linearity.

As anticipated, as the debt limit reduces, the equilibrium interest rate r_t also declines both during transition and in the steady state (bottom-right panel). Keep in mind that the steady-state interest rate under scenario (i) equals the time discount rate $1/\beta = 1.0526$, so the steady-state interest rates under scenarios (ii) and (iii) are strictly lower than that under scenario (i).

Figure 4: Ramsey Transition Paths of Policies with Binding Debt Limits



Notes: Scenario (i), (ii) and (iii) are shown as blue, green, and red lines, respectively. Their corresponding steady-state levels are indicated by dashed lines. Since the initial capital tax, $\tau_{k,1}$, is given at zero, the plot of capital tax starts at $t = 2$.

The main lesson taken away from this subsection is that tax policies are shaped by the government's ability to issue debts and that the single most important function of public debts is to improve the individual self-insurance position. Since a binding debt limit handicaps the government's ability to improve individual allocative efficiency, as a result, the associated burden of interest payments and the average tax rate during transition are also reduced. However, given any level of required tax revenues to finance interest payments, the composition of the tax revenue in terms of capital tax and labor tax is dictated by the trade-off between IAE and AAE; so optimal capital tax is high in the short run and zero in the long run, while optimal labor tax is low in the short run but high in the long run. Hence, as the debt limit $\overline{B}$ approaches zero, the Ramsey allocation approaches the competitive equilibrium under laissez faire both along transition and in steady state. In other words, in the absence of government spending, steady-state tax policies have no independent role to play without the tool of government debts, and they are used solely to finance interest payments on government debts. For this very reason, since labor tax is less distortionary, a permanent capital tax in the steady state is never optimal regardless of government debt limits and the MGR. To compromise, the planner opts to improve aggregate allocative efficiency (or reduce the problem of capital overaccumulation) by taxing capital in the short run and taxing labor in the long run.[7]

4.4 Effects of Elasticity of Intertemporal Substitution

This subsection investigates the short- and long-run effects of elasticity of intertemporal substitution (EIS) on the Ramsey allocation when debt-limit constraints do not bind. Keep in mind that if EIS is too small or σ is too large (such as in the limiting case $\sigma = \infty$), then the laissez-faire competitive equilibrium features both IAE and AAE. The reason is that extremely risk-averse households would opt to save a lot (enough) to provide full self-insurance when σ is large.

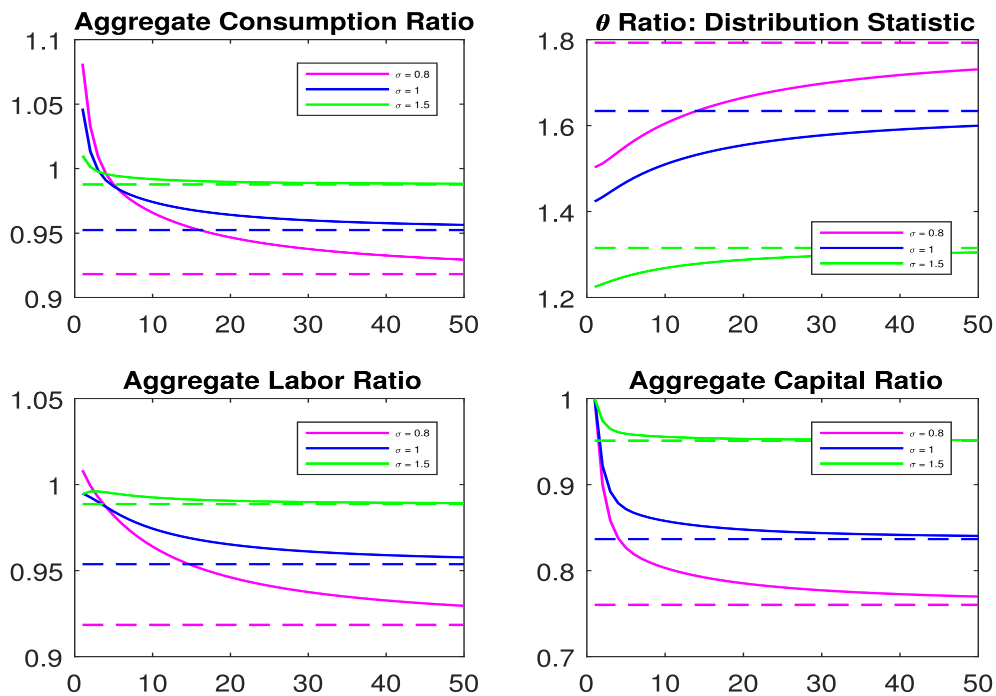
So we consider cases with σ not too far away from unity: the case of $\sigma = 0.8$ and the case of $\sigma = 1.5$, respectively. In each case, the initial capital stock is set to be 70% of the corresponding Ramsey steady-state level, as in the log utility case. As before, all parameter values imply that (i) the condition for positive individual labor choice, $\theta_H < \frac{\theta_L}{(1-\beta)^\sigma} < \infty$, is satisfied in the steady state; (ii) the condition for positive labor supply $n_t > 0$ is satisfied and verified numerically in each time period along the entire transition path; (iii) the condition (17) holds such that the laissez-faire competitive equilibrium is neither IAE nor AAE; and (iv) the OSIA allocation is feasible for the Ramsey planner with a non-binding debt limit, and the model's Ramsey steady state corresponds to Case 1 described in Proposition 5.

To make the different cases more comparable to each other from the perspective of their respective laissez-faire economy, we report the ratios of Ramsey allocations to the laissez-faire allocations for different values of σ. The transition paths for aggregate consumption, labor, capital, and the cutoff are reported in Figure 5, where green lines represent the low EIS case with $\sigma = 1.5$, blue lines the benchmark case with $\sigma = 1$, and pink lines the high EIS case with $\sigma = 0.8$.

The figure shows that the key mechanism driving the optimal Ramsey allocation dis-

[7] Overaccumulation of capital at the aggregate is merely a competitive-equilibrium outcome of individuals' precautionary saving behaviors. Hence, it does not appear to be a genuine "externality" that the Ramsey planner should aim to correct in the steady state by using distortionary capital tax.

Figure 5: Ramsey Transition Paths of Aggregate Variable under Different σ



Notes: Blue lines represent $\sigma = 1$*, green lines represent* $\sigma = 1.5$*, and pink lines represent* $\sigma = 0.8$.

cussed in the previous subsections remains unchanged, except with the following important differences:

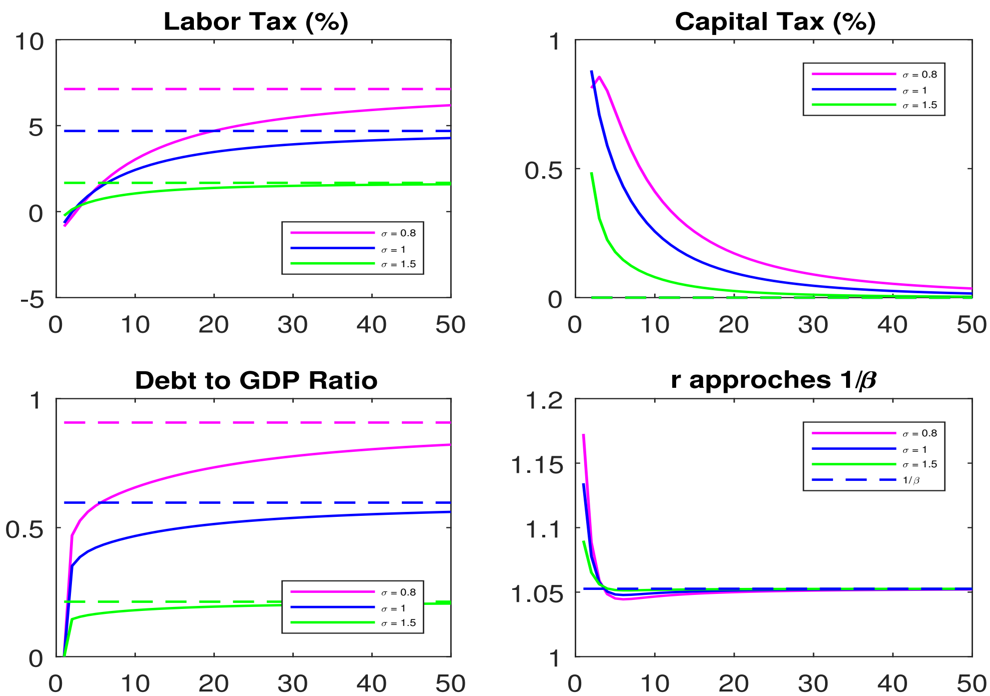
(i) The relative speed of convergence under Ramsey (benchmarked by the corresponding laissez-faire economy) depends negatively on the EIS, or positively on the value of σ. In particular, the transition speed is fastest in the case of low EIS ($\sigma = 1.5$) and slowest in the case of high EIS ($\sigma = 0.8$). Because we have shown previously under $\sigma = 1$ that the laissez-faire economy converges faster than the Ramsey economy, this result suggests that the Ramsey planner has less room to engage in intertemporal "arbitrage" to alter the competitive equilibrium through the use of policies when the market participants' EIS is low; consequently, the economy converges faster when σ is larger. The implication is that in the limit when $\sigma \to \infty$, it must be true that the Ramsey allocation approaches that of the laissez-faire competitive equilibrium and vise versa—because when agents' EIS is close to zero, the laissez-faire competitive equilibrium can achieve full self-insurance, and hence the Ramsey planner has no room (or desire) to improve the welfare of the laissez-faire economy through fiscal policies.

(ii) When the EIS is lower or σ is larger, the steady-state relative levels of aggregate consumption, labor, and capital (relative to their laissez-faire counterparts) are higher and closer to 1; however, the optimal cutoff θ_t^* (relative to its corresponding laissez-faire level) converges to 1 from above (top-right panel in Figure 5). The reason is the same as pointed out before—namely, a lower EIS implies a smaller amount of room for the Ramsey planner to improve upon the associated laissez-faire competitive equilibrium because the competitive equilibrium is closer to the optimum under a larger σ; hence, the ratio of the Ramsey allocation and the Laissez-faire allocation approaches 1 as σ increases. In other words, in the limiting case when $\sigma \to \infty$, the transition path must becomes a straight line of 1 for all variables shown in the figure—because the Ramsey allocation is identical to the laissez-faire competitive-equilibrium allocation both in transition and in steady state when EIS is close enough to zero.

(iii) The implication is that the welfare gains under the Ramsey allocation from improving laissez-faire competitive equilibrium increase with EIS. Figure 7 reports welfare gains of the Ramsey allocation from the laissez-faire competitive equilibrium over time. A welfare gain is measured by the compensation variation in terms of aggregate consumption. The figure shows that welfare gains from the Ramsey allocation are larger under the case of $\sigma = 0.8$ than under the case of $\sigma = 1.5$, especially during the transition. The figure also implies that in the limit as $\sigma \to \infty$, welfare gains approach zero both in transition and in steady state.

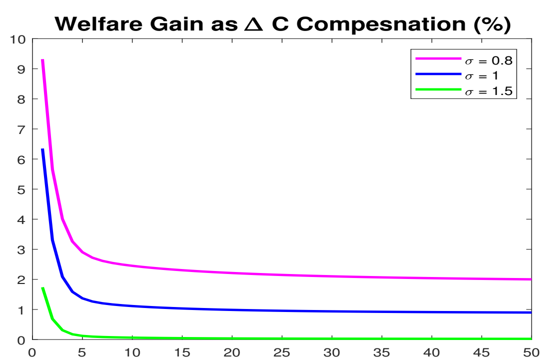
(iv) Figure 6 shows that the Ramsey planner opts to issue a far larger amount of debts relative to GDP under a higher EIS than under a lower EIS (bottom-left panel). As a result, the market interest rate is significantly higher when $\sigma = 0.8$ than when $\sigma = 1.5$ (bottom-right panel)—it is more than 800 basis points higher in the former case than in the latter case in the initial period. The labor tax rate approaches 7.13% in the long run when $\sigma = 0.8$, as compared with only 1.67% when $\sigma = 1.5$. Although the long-run capital tax rate is exclusively zero regardless of σ, the short-run capital tax rate is higher under a larger EIS. The insight is that the Ramsey planner is able to front-load consumption more aggressively by improving the individual self-insurance position when the EIS is high, but doing so requires the government to amass a larger stock of claims on the private economy, which calls for higher tax revenues to cover the interest payments.

Figure 6: Ramsey Transitions of Policy Tools under Different EIS



Notes: The plot of capital tax starts at $t = 2$.

Figure 7: Ramsey Welfare Gain under Different EIS



5 A Robustness Analysis

It is well acknowledged that the theoretical proof in Aiyagari (1995) relies not only on the implicit assumption of a Ramsey steady state, but also on a non-standard environment with endogenous government spending in household utilities. In this section, we show that the endogenous government spending assumed by Aiyagari (1995) is innocuous and does not contribute to the different results found in this paper. More specifically, the reason the Ramsey planner appears to care only about MGR (or AAE) but not IAE in the analysis of Aiyagari (1995) is not because of the peculiar assumption of the endogenous government spending that equalizes the marginal utility of government spending across heterogeneous households, but rather because of the dubious assumption of the existence of a Ramsey steady state.

To make the point, we show here that even with endogenous government spending introduced into our model, our previous results remain robust and unchanged. In particular, the introduction of endogenous government spending into household utilities does not eliminate the trade-off problem between IAE and AAE facing the Ramsey planner, nor does it ensure the existence of a Ramsey steady state in our model. Namely, the existence of a Ramsey steady state does not depend on endogenous government spending, and the Ramsey planner still cares about IAE and the distribution of individual self-insurance positions despite the fact that government spending equalizes all individuals' marginal utilities of public expenditure, as shown below.

Following Aiyagari (1995), the household preference in our model is modified to

$$E_1 \sum_{t=1}^{\infty} \beta^{t-1} \left[\theta_t \frac{c_t(\theta^t)^{1-\sigma} - 1}{1-\sigma} - n_t \left(\theta^{t-1}\right) + U(G_t) \right], \tag{40}$$

where $U(G)$ is the utility function of government spending. Note that the introduction of G_t into the household utility does not alter the household decision rules since households take the sequence of G_t as given. Hence, the definition of competitive equilibrium remains the same.

The Ramsey problem does change slightly since G_t is now an endogenous choice variable for the Ramsey planner. However, the construction of the Ramsey problem follows the proof of Proposition 3 in exactly the same way except for one additional first-order condition with respect to $G_t \geq 0$, which is chosen by respecting the aggregate resource constraint. A non-negative G_t could be easily ensured by the assumption of $U'(0) = \infty$, which is a common practice in the literature. Therefore, the first-order conditions of the Ramsey problem with respect to K_{t+1}, N_t, C_t, B_{t+1}, and θ_t^* are identical to those listed in Section 3. The additional first-order condition with respect to G_t is given by

$$U'(G_t) = \mu_t, \tag{41}$$

which together with equation (28) gives exactly the same Ramsey-Euler equation as in Aiyagari (1995, eq.(20) p. 1170) if $\phi_t = 0$ (i.e., no debt-limit constraints). This additional first-order condition has no important role to play in determining the Ramsey allocation $\{K_{t+1}, N_t, C_t, B_{t+1}, \theta_t^*\}$ except that it helps pin down the endogenous government spending G_t once μ_t is known. Therefore, regardless of the value of G_t, all propositions and arguments stated previously remain valid. Thus, the long-run optimal level of government spending G depends on the existence of a Ramsey steady state and, in particular, on the Lagrangian multiplier μ—if μ is bounded and strictly positive, then G is also bounded and strictly positive.

But, when a Ramsey steady state does not exist, such as when $\mu = 0$ or $\mu \to \infty$, yet if we assume (incorrectly) that a Ramsey steady state exists and the debt limit is not binding ($\phi = 0$), then equation (28) implies that the MGR holds: $\beta(MP_k + 1 - \delta) = 1$. Given this MGR condition, if we also pretend (incorrectly assume) that the equilibrium interest rate in our model is below the time discount rate (which would be true only if $\theta^* < \theta_H$ or $\theta_H = \infty$), we would immediately conclude that the"optimal" capital tax should be positive in the Ramsey steady state.

But such conclusion is based on the erroneous assumption of the existence of a Ramsey steady state, yet such an assumption is inconsistent with the other Ramsey first-order conditions and a non-binding government debt limit. In particular, after taking all necessary first-order conditions into account, we have shown previously that a Ramsey steady state consistent with the MGR exists if and only if θ_H is sufficiently small and the debt limit $\overline{B}$ is sufficiently large so that $\theta^* = \theta_H$ and the constraint $B_t \leq \overline{B}$ does not bind. And even in this case we have shown that the Ramsey planner achieves the MGR only through achieving the IAE instead of by taxing capital. It is straightforward to see that such arguments continue to hold here with endogenous government spending by following our analysis in Section 3 and the first-order conditions provided therein.

Also take note that if $\mu = 0$ or $\mu \to \infty$ in a Ramsey equilibrium, then equation (41) implies that G either becomes unbounded or converges to zero. Neither case is consistent with the definition of a Ramsey steady state. Hence, introducing endogenous government spending does not change our results, nor does it help ensure the existence of a Ramsey steady state if it does not exist in the first place without endogenous government spending.

6 A Brief Literature Review

The literature related to optimal capital taxation is vast. Here we review only the most relevant papers in the incomplete-markets literature.

The work of Aiyagari (1995) is the first attempt at investigating optimal Ramsey taxation in heterogeneous-agent incomplete-markets economies. Under the assumptions of the existence of a Ramsey state steady and no government debt limit, Aiyagari (1995) shows that the MGR holds and argues that since the risk-free rate is below the time preference rate under precautionary saving motives, a positive capital tax should be levied by the Ramsey planner to correct the problem of overaccumulation of capital.

Aiyagari's analysis suggests that the planner should care more about the AAE but not about the IAE, regardless of model structures, parameter values, and the government's (natural) debt limits. Our results clearly indicate otherwise. In our view, the different results are caused not by the difference between our models, but rather by the validity of the implicit assumption on the existence of a Ramsey steady state made in Aiyagari's analysis. Such an assumption is problematic since it implicitly implies that the optimal level of government bond supply is always finite and that the debt limit (if exists) does not bind, so the optimal debt level is always feasible and sustainable (and does not violate the natural borrowing limit of the government). Our analysis shows instead that the Ramsey planner intends to increase the supply of government bond until both AAE and IAE are achieved. This intention of the Ramsey planner does not fade away until the interest rate equals the time discount rate. Therefore, if such an intention cannot be realized with a finite and feasible level of government debt, a Ramsey steady state cannot exist. Given that the real interest rate lies strictly below the time discount rate in the standard Aiyagari model with any finite demand for government bonds, IAE can never be archived. Hence, the implicit assumption of the existence of a Ramsey steady state and non-binding natural government debt limit in Aiyagari's analysis appears to be unjustified.

Indeed, the recent work by Chen, Chien, and Yang (2019) employs a standard Aiyagari-type heterogeneous-agent incomplete-markets model to show that the assumption of a Ramsey steady state is inconsistent with some of the first-order necessary conditions of the Ramsey problem. Therefore, the optimal Ramsey allocation may feature no steady state when assuming a non-binding government debt limit. Their finding supports our analysis in this paper. However, standard heterogeneous-agent models such as that adopted by Chen, Chien, and Yang (2019) have no closed-form solutions. Therefore, they are not able to determine the optimal Ramsey capital tax rate (either in transition or in steady state) and to study the Ramsey planner's trade-off problem in terms of IAE and AAE.

An important recent paper by Gottardi, Kajii, and Nakajima (2015) revisits optimal Ramsey taxation in an incomplete-markets model with uninsurable human-capital risk. As in our model, tractability in their model enables them to provide transparent analysis on Ramsey taxation and facilitates intuitive interpretations for their results. When government spending and the bond supply are both set to zero, they find that in the steady state the Ramsey planner should tax human capital and subsidize physical capital, despite the overaccumulation of physical capital. The purpose or the benefit of taxing human capital is to reduce uninsurable risk from human-capital returns; and the rationale for subsidizing physical capital despite overaccumulation is to satisfy household demand for a buffer stock, similar in spirit to our finding but in contrast to Aiyagari's results. However, the authors solve the Ramsey problem indirectly, and they can characterize analytically the properties of optimal taxes only in a neighborhood of zero government bonds and zero government spending. In contrast, we can solve the Ramsey problem analytically and directly along the entire dynamic path of the model, which permits transparent examinations of how the Ramsey planner takes into account the impact of its policies on the dynamic distributions of household decisions and aggregate productive efficiency. Our model also enables us to show analytically the exact roles played by government debt and how such roles are hindered by debt limits.

In an environment of uninsurable idiosyncratic risk, Krueger and Ludwig (2018) use an overlapping generation model to study Ramsey capital tax policy. Similar to our analytical approach, they are able to fully characterize the Ramsey solution and justify the existence of a Ramsey steady state. Since in their model all tax revenues are lump-sum transferred back to households (to serve as an insurance device) and there is no government bond, their result of a positive capital tax by the Ramsey planner does not contradict our results. The insight behind their result is that the planner opts to use both capital tax to archive production efficiency and lump-sum transfers to improve IAE.

Aiyagari and McGrattan (1998) study optimal government debt in the Aiyagari model. Similar to our finding, government bonds are shown to play an important role in providing self-insurance for households and to help in relaxing their borrowing constraints. However, the authors restrict their analysis to the special case of the same tax rates across capital and labor incomes and analyze welfare only in the steady state.

In an overlapping generations model with uninsured individual risk, Conesa, Kitao, and Krueger (2009) conduct a numerical exercise to derive optimal capital tax and non-linear labor tax. As in Aiyagari and McGrattan (1998), they mainly consider welfare in the *assumed* steady state, and the transitional path is therefore ignored. However, Domeij and Heathcote (2004) show that welfare along the transitional path is an important concern for the Ramsey planner. Their findings indicate that steady-state welfare maximization could be misleading when designing optimal policies. But, instead of solving optimal tax policies, they numerically evaluate the welfare consequences of tax changes.

Three recent works by Acikgoz, Hagedorn, Holter, and Wang (2018), Dyrda and Pedroni (2018) and Ragot and Grand (2017) numerically solve optimal fiscal policies along the transitional path in an Aiyagari-type economy. In contrast to our findings, their results are consistent with Aiyagari's analysis. The sources of difference between their results and ours could be their implicit assumptions about unbounded debt limits and the existence of a Ramsey steady state in their numerical analyses. Such assumptions are not well justified in light of our studies. In fact, Straub and Werning (2014) also argue that the assumption of the existence of a Ramsey steady state commonly made in the Ramsey literature may be incorrect and could be misleading even in a representative agent framework.

Instead of using the Ramsey approach, Dávila, Hong, Krusell, and Ríos-Rull (2012) characterize and decentralize constrained efficient allocations in an Aiyagari-type economy where the government can levy an individual specific labor tax, which is not allowed in the traditional Ramsey framework. They found that in a competitive equilibrium, the capital stock could be either too high or too low compared with the constraint-efficient allocation, thus the optimal capital income tax rate can be either positive or negative.

Finally, Park (2014) considers Ramsey taxation in a complete-markets environment featuring enforcement constraints, á la Kehoe and Levine (1993). She shows that capital accumulation improves the outside option of default, which is not internalized by household decisions. Therefore, capital income should be taxed in order to internalize such an adverse externality.

7 Conclusion

This paper designs a tractable infinite-horizon model of heterogeneous agents and incomplete markets to address a set of long-standing issues in the optimal Ramsey capital taxation literature. The tractability of our model enables us to prove the existence of Ramsey steady states and establish several novel results: (i) The optimal capital tax is exclusively zero in a Ramsey steady state regardless of the modified golden rule and government debt limits. (ii) The Ramsey planner opts to levy capital tax only during transition periods; and the optimal tax rate depends positively on the elasticity of intertemporal substitution. (iii) When a Ramsey steady state (featuring a non-binding government debt limit) *does not* exist but is *erroneously* assumed to exist, the modified golden rule always *appears* to "hold" and the implied "optimal" long-run capital tax is strictly positive, reminiscent of the result obtained by Aiyagari (1995). (iv) Whether the modified golden rule holds depends critically on the government's capacity to issue debts, but it has no bearing on the planner's long-run capital tax scheme. (v) The optimal debt-to-GDP ratio in the absence of a binding debt limit, however, is determined by a positive wedge times the modified-golden-rule saving rate; the wedge is decreasing in the strength of the individual self-insurance position and approaches zero when idiosyncratic risk vanishes or markets are complete.

The key insight behind our results is the Ramsey planner's ultimate concern for self-insurance. Since taxing capital in the steady state permanently hinders individuals' self insurance positions, the Ramsey planner prefers taxing capital only in the short run and issuing debt rather than imposing a steady-state capital tax to correct the capital-overaccumulation problem under precautionary saving motives. Thus, in sharp contrast to Aiyagari's argument, permanent capital taxation is not the optimal tool to achieve AAE despite overaccumulation of capital, and the MGR can fail to hold in a Ramsey steady state whenever the government encounters a debt-limit.

We reveal in a transparent manner that the government debt plays an important role in determining the optimal Ramsey outcome when facing the trade-off between aggregate allocative efficiency and individual allocative efficiency. In particular, the planner's motive to improve individuals' self-insurance positions and relax their borrowing constraints may induce it to amass an ever-increasing amount of public debts, which may result in a dynamic path featuring no Ramsey steady state. Yet the existence of a Ramsey steady state is simply assumed in Aiyagari's analysis without proof, which leads to Aiyagari's conclusion that the Ramsey planner should ignore the trade-off and focus only on the MGR. Our analysis thus provides a strong caution to all numerical approaches that solve the Ramsey taxation problem by assuming (without proof) the existence of a Ramsey steady state featuring non-binding government debt limit.

Our model can be extended in many possible dimensions. For example, there is a strong tradition and renewed interest in studying the optimal responses of fiscal policies to aggregate shocks. The works by Barro (1979) and Lucas and Stokey (1983) both identify the importance of tax smoothing but give different predictions on the optimal behavior of government bonds. Aiyagari, Marcet, Sargent, and Seppala (2002) show that the (in)complete-market assumption explains the different findings of these two classical papers. Farhi (2010) and Bhandari, Evans, Golosov, and Sargent (2017) contribute to this literature by considering incomplete markets with aggregate risks in a representative-agent framework. More recently, Bhandari, Evans, Golosov, and Sargent (2018) extend this literature to a heterogeneous-agent framework. However, their numerical approach sidesteps the issue of possible nonexistence of a Ramsey steady state. Our model could be extended to an environment with aggregate risks and complement this literature by offering a more tractable and transparent analysis.[8]

[8]This task is currently undertaken by the authors.

References

Acikgoz, O., M. Hagedorn, H. Holter, and Y. Wang (2018): "The Optimum Quantity of Capital and Debt," CEPR Discussion Papers 12952, C.E.P.R. Discussion Papers.

Aiyagari, S. R. (1994): "Uninsured Idiosyncratic Risk and Aggregate Saving," *Quarterly Journal of Economics*, 109(3), 659–684.

Aiyagari, S. R. (1995): "Optimal Capital Income Taxation with Incomplete Markets, Borrowing Constraints, and Constant Discounting," *Journal of Political Economy*, 103(6), 1158–75.

Aiyagari, S. R., A. Marcet, T. J. Sargent, and J. Seppala (2002): "Optimal Taxation without State-Contingent Debt," *Journal of Political Economy*, 110(6), 1220–1254.

Aiyagari, S. R., and E. R. McGrattan (1998): "The optimum quantity of debt," *Journal of Monetary Economics*, 42(3), 447–469.

Barro, R. J. (1979): "On the Determination of the Public Debt," *Journal of Political Economy*, 87(5), 940–71.

Bewley, T. (1980): "The Optimum Quantity of Money," in *Models of Monetary Economies*, ed. by J. H. Kareken, and N. Wallace, pp. 169–210. Federal Reserve Bank of Minneapolis, Minneapolis.

Bhandari, A., D. Evans, M. Golosov, and T. J. Sargent (2017): "Fiscal Policy and Debt Management with Incomplete Markets," *The Quarterly Journal of Economics*, 132(2), 617–663.

——— (2018): "Inequality, Business Cycles and Monetary-Fiscal Policy," Working papers, University of Chicago.

Chamley, C. (2001): "Capital income taxation, wealth distribution and borrowing constraints," *Journal of Public Economics*, 79(1), 55–69.

Chen, Y., Y. Chien, and C. Yang (2019): "Implementing the Modified Golden Rule? Optimal Ramsey Capital Taxation with Incomplete Markets Revisited," Working Papers 2017-003D, Federal Reserve Bank of St. Louis.

Conesa, J. C., S. Kitao, and D. Krueger (2009): "Taxing Capital? Not a Bad Idea after All!," *American Economic Review*, 99(1), 25–48.

Dávila, J., J. H. Hong, P. Krusell, and J.-V. Ríos-Rull (2012): "Constrained Efficiency in the Neoclassical Growth Model with Uninsurable Idiosyncratic Shocks," *Econometrica*, 80(6), 2431–2467.

Deaton, A. (1991): "Saving and Liquidity Constraints," *Econometrica*, 59(5), 1221–1248.

Domeij, D., and J. Heathcote (2004): "On the Distributional Effects of Reducing Capital Taxes," *International Economic Review*, 45(2), 523–554.

Dyrda, S., and M. Pedroni (2018): "Optimal Fiscal Policy in a Model with Uninsurable Idiosyncratic Shocks," Working papers, University of Toronto, Department of Economics.

Farhi, E. (2010): "Capital Taxation and Ownership When Markets Are Incomplete," *Journal of Political Economy*, 118(5), 908–948.

Gottardi, P., A. Kajii, and T. Nakajima (2015): "Optimal Taxation and Debt with Uninsurable Risks to Human Capital Accumulation," *American Economic Review*, 105(11), 3443–3470.

Huggett, M. (1993): "The Risk-Free Rate in Heterogeneous-Agent Incomplete-Insurance Economies," *Journal of Economic Dynamics and Control*, 17(5-6), 953–969.

Kehoe, T. J., and D. K. Levine (1993): "Debt-Constrained Asset Markets," *Review of Economic Studies*, 60(4), 865–888.

Ljungqvist, L., and T. J. Sargent (2012): *Recursive Macroeconomic Theory, Third Edition*, MIT Press Books. The MIT Press.

Lucas, Robert E, J. (1980): "Equilibrium in a Pure Currency Economy," *Economic Inquiry*, 18(2), 203–220.

Lucas, R. J., and N. L. Stokey (1983): "Optimal Fiscal and Monetary Policy in an Economy without Capital," *Journal of Monetary Economics*, 12(1), 55–93.

Park, Y. (2014): "Optimal Taxation in a Limited Commitment Economy," *Review of Economic Studies*, 81(2), 884–918.

Ragot, X., and F. L. Grand (2017): "Optimal Fiscal Policy with Heterogeneous Agents and Aggregate Shocks," 2017 Meeting Papers 969, Society for Economic Dynamics.

Straub, L., and I. Werning (2014): "Positive Long-Run Capital Taxation: Chamley-Judd Revisited," NBER Working Papers 20441, National Bureau of Economic Research, Inc.

Wen, Y. (2009): "An analytical approach to buffer-stock saving," Working Papers 2009-026, Federal Reserve Bank of St. Louis.

——— (2015): "Money, liquidity and welfare," *European Economic Review*, 76(C), 1–24.

A Appendix

A.1 Proof of Proposition 1

Denoting $\left\{\beta^t\lambda_t^h(\theta^t), \beta^t\mu_t^h(\theta^t)\right\}$ as the Lagrangian multipliers for constraints (6) and (7), respectively, the first-order conditions for $\{c_t(\theta^t), n_t(\theta^{t-1}), a_{t+1}(\theta^t)\}$ are given, respectively, by

$$\frac{\theta_t}{c_t(\theta^t)^\sigma} = \lambda_t^h(\theta^t) \tag{42}$$

$$1 = \overline{w}_t \int \lambda_t^h(\theta^t)\, d\mathbf{F}(\theta_t) \tag{43}$$

$$\lambda_t^h(\theta^t) = \beta r_{t+1} E_t\left[\lambda_{t+1}^h(\theta^{t+1})\right] + \mu_t^h(\theta^t), \tag{44}$$

where equation (43) reflects that the labor supply $n_t(\theta^{t-1})$ must be chosen before the idiosyncratic taste shocks (and hence before the value of $\lambda_t^h(\theta^t)$) are realized. By the law of iterated expectations and the *iid* assumption of idiosyncratic shocks, equation (44) can be written (using equation (43)) as

$$\lambda_t^h(\theta^t) = \beta\frac{r_{t+1}}{\overline{w}_{t+1}} + \mu_t^h(\theta^t), \tag{45}$$

where $\frac{1}{\overline{w}}$ is the marginal utility of consumption in terms of labor income.

We adopt a guess-and-verify strategy to derive the decision rules. The decision rules for an individual's consumption and savings are characterized by a cutoff strategy, taking as given the aggregate states (such as the interest rate and real wage). Anticipating that the optimal cutoff θ_t^* is independent of an individual's history of shocks, consider two possible cases:

Case A. $\theta_t \le \theta_t^*$. In this case the urge to consume is low. It is hence optimal to save so as to prevent possible liquidity constraints in the future. So $a_{t+1}(\theta^t) \ge 0$, $\mu_t^h(\theta^t) = 0$, and the shadow value is

$$\lambda_t^h(\theta^t) = \beta\frac{r_{t+1}}{\overline{w}_{t+1}} \equiv \Lambda_t,$$

where Λ_t depends only on aggregate states. Notice that $\lambda_t^h(\theta^t) = \lambda_t^h$ is independent of the history of idiosyncratic shocks. Equation (42) implies that consumption is given by $c_t(\theta^t)^\sigma = \theta_t\Lambda_t^{-1}$. Defining $x_t(\theta^{t-1}) \equiv r_t a_t(\theta^{t-1}) + \overline{w}_t n_t(\theta^{t-1})$ as the gross income of the household, the budget identity (6) then implies $a_{t+1}(\theta^t) = x_t(\theta^{t-1}) - \left(\theta_t \Lambda_t^{-1}\right)^{1/\sigma}$. The requirement $a_{t+1}(\theta^t) \geq 0$ then implies

$$\theta_t \leq \Lambda_t x_t^\sigma \equiv \theta_t^*, \tag{46}$$

which defines the cutoff θ_t^*.

We conjecture that the cutoff is independent of the idiosyncratic state. Then the optimal gross income x_t is also independent of the idiosyncratic state. The intuition is that x_t is determined before the realization of θ_t and that all households face the same distribution of idiosyncratic shocks. Since the utility function is quasi-linear, the household is able to adjust labor income to meet any target level of liquidity in hand. As a result, the distribution of x_t is degenerate. This property simplifies the model tremendously.

Case B. $\theta_t > \theta_t^*$. In this case the urge to consume is high. It is then optimal not to save, so $a_{t+1}(\theta^t) = 0$ and $\mu_t^h(\theta^t) > 0$. By the resource constraint (6), we have $c_t(\theta^t) = x_t$, which by equation (46) implies $c_t(\theta^t)^\sigma = \theta_t^* \Lambda_t^{-1}$. Equation (42) then implies that the shadow value is given by $\lambda_t^h(\theta^t) = \frac{\theta_t}{\theta_t^*}\Lambda_t$. Since $\theta_t > \theta^*$, equation (45) implies $\mu_t^h(\theta^t) = \Lambda_t \left[\frac{\theta_t}{\theta_t^*} - 1\right] > 0$. Notice that the shadow value of goods (the marginal utility of income), $\lambda_t^h(\theta^t)$, is higher under case B than under case A because of binding borrowing constraints.

The above analyses imply that the expected shadow value of income, $\int \lambda_t^h(\theta) d\mathbf{F}(\theta)$, and hence the optimal cutoff value θ^*, is determined by equation (43) by plugging in the expressions for $\lambda_t^h(\theta^t)$ under cases A and B, which immediately gives equation (12). Specifically, combining case A and case B, we have

$$\begin{aligned} \lambda_t^h(\theta^t) &= \beta \frac{r_{t+1}}{\overline{w}_{t+1}} \text{ for } \theta \leq \theta_t^* \\ \lambda_t^h(\theta^t) &= \frac{\theta_t}{\theta_t^*} \beta \frac{r_{t+1}}{\overline{w}_{t+1}} \text{ for } \theta_t \geq \theta_t^*. \end{aligned}$$

The aggregate Euler equation is therefore given by

$$\frac{1}{\overline{w}_t} = \int \lambda_t^h(\theta) d\mathbf{F}(\theta) = \beta \frac{r_{t+1}}{\overline{w}_{t+1}} \left[\int_{\theta \leq \theta_t^*} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} \frac{\theta}{\theta^*} d\mathbf{F}(\theta)\right] = \beta \frac{r_{t+1}}{\overline{w}_{t+1}} L(\theta_t^*),$$

which is equation (12). This equation reveals that the optimal cutoff depends only on aggregate states and is independent of individual history.

We also immediately obtain

$$x_t = \left[\theta_t^* \left(\beta \frac{r_{t+1}}{\overline{w}_{t+1}} \right)^{-1} \right]^{1/\sigma} = \left[\theta_t^* L(\theta_t^*) \overline{w}_t \right]^{1/\sigma},$$

which leads to equation (8). By the discussion of cases A and B, as well as the use of equation (8), the decision rules of household consumption and saving can then be summarized by equations (9) and (10), respectively. Finally, the decision rule of the household labor supply, equation (11), is decided residually to satisfy the household budget constraint.

Finally, to ensure that the above proof and hence the associated cutoff-policy rules are consistent with the assumption of interior choices of labor, namely, $n_t \in (0, \overline{N})$, we need to consider the following two cases:

First, to ensure that $n_t(\theta^{t-1}) > 0$, consider the worst situation where $n_t(\theta^{t-1})$ takes its minimum value. Given $x_t = r_t a_t(\theta^{t-1}) + \overline{w}_t n_t(\theta^{t-1})$, $n_t(\theta^{t-1})$ is at its minimum if $\mu_t^h = 0$ and $a_t(\theta^{t-1})$ takes the maximum possible value, $a_t(\theta^{t-1}) = \left[1 - \left(\frac{\theta_L}{\theta_{t-1}^*} \right)^{1/\sigma} \right] x_{t-1}$. So $n_t(\theta^{t-1}) > 0$ if

$$x_t - r_t \left[1 - \left(\frac{\theta_L}{\theta_{t-1}^*} \right)^{1/\sigma} \right] x_{t-1} > 0, \tag{47}$$

which is independent of the shock θ_t. This condition in the steady state becomes $1 - r \left[1 - \left(\frac{\theta_L}{\theta_{t-1}^*} \right)^{1/\sigma} \right] > 0$, or equivalently (by using equation (12)),

$$\beta L(\theta^*) > 1 - \left(\frac{\theta_L}{\theta^*} \right)^{1/\sigma}. \tag{48}$$

Given that $L(\theta^*)$ is a monotonic decreasing function in θ^* with a lower bound of 1, the necessary condition to satisfy (48) in the steady state is $\beta > 1 - \left(\frac{\theta_L}{\theta^*} \right)^{1/\sigma}$, which is easy to satisfy when $\theta_H < \infty$. Therefore, as long as the condition $\beta > 1 - \left(\frac{\theta_L}{\theta^*} \right)^{1/\sigma}$ is met, the condition (47) is assumed to hold throughout the paper.

Second, to ensure that $n_t < \overline{N}$, consider those agents who encounter the borrowing constraint last period such that $a_t(\theta^{t-1}) = 0$. Their labor supply reaches the maximum value at $n_t(\theta^{t-1}) = \frac{x_t}{\overline{w}_t} = \theta_t^* L(\theta_t^*)$. Given a finite steady state value of θ^*, the value of $\overline{N}$ can be chosen such that

$$\overline{N} > \theta^* L(\theta^*). \tag{49}$$

A.2 Proof of Proposition 2

In the laissez-faire economy, the capital tax, the labor tax, government spending and government bond are all equal to zero. In this laissez-faire competitive equilibrium, the capital-to-labor ratio $\frac{K_t}{N_t}$ satisfies two conditions. The first condition is derived from the resource constraint (16), which can be expressed as

$$F(K_t, N_t) + (1-\delta)K_t = C_t + K_{t+1} = x_t,$$

where the last equality uses the definition of x_t. Dividing both sides of the equation by K_t gives

$$\left(\frac{K_t}{N_t}\right)^{\alpha-1} + (1-\delta) = \frac{1}{1-D(\theta_t^*)}, \tag{50}$$

where x_t/K_t is substituted out by $\frac{1}{1-D(\theta_t^*)}$.

The second condition is derived by combining equation (12) and the no-arbitrage condition, $r_t = 1 + q_t - \delta$, which gives

$$1 = \beta\left(1 + \alpha\left(\frac{K_t}{N_t}\right)^{\alpha-1} - \delta\right) L(\theta_t^*), \tag{51}$$

where the marginal product of capital q_t is replaced by $\alpha\left(\frac{K_t}{N_t}\right)^{\alpha-1}$. Since the capital-to-labor ratio must be the same in both equations, conditions (50) and (51) imply the following equation in the steady state:

$$\frac{\alpha\beta}{(1-D(\theta^*))} + \beta(1-\alpha)(1-\delta) = \frac{1}{L(\theta^*)}, \tag{52}$$

which solves for the steady-state value of θ^*.

It can be shown easily that both $L(\theta^*)$ and $D(\theta^*)$ are monotonically decreasing in θ^*, thus the right-hand side (RHS) of equation (52) increases monotonically in θ^* and the left-hand side (LHS) of equation (52) decreases monotonically in θ^*.

It remains to be seen if the RHS and the LHS cross each other at an interior value of $\theta^* \in [\theta_L, \theta_H]$. The RHS of equation (52) reaches its minimum value of 1 when $\theta^* = \theta_H$ and its maximum value of $\bar{\theta}/\theta_L > 1$ when $\theta^* = \theta_L$. The LHS of equation (52) takes the maximum value of infinity when $\theta^* = \theta_L$ and the minimum value of $\frac{\alpha\beta(\theta_H)^{1/\sigma}}{(\theta_H)^{1/\sigma} - E(\theta^{1/\sigma})} + \beta(1-\alpha)(1-\delta)$

when $\theta^* = \theta_H$. Thus, an interior solution exists if and only if

$$\frac{\alpha\beta(\theta_H)^{1/\sigma}}{(\theta_H)^{1/\sigma} - E(\theta^{1/\sigma})} + \beta(1-\alpha)(1-\delta) < 1.$$

Clearly, $\theta^* = \theta_L$ cannot constitute a solution for any positive value when $\theta_L > 0$. On the other hand, $\theta^* = \theta_H$ may constitute a solution if the above condition is violated. For example, if θ_H is small and close enough to the $E(\theta^{\frac{1}{\sigma}})^\sigma$, then the above condition does not hold since its LHS approaches infinity when $\theta_H \to E(\theta^{\frac{1}{\sigma}})^\sigma$. Therefore, an interior solution for θ^* exists if the upper bound of the idiosyncratic shock is large enough. Otherwise, we have the corner solution $\theta^* = \theta_H$. Finally, if θ^* is an interior solution, then $L(\theta^*) > 1$ and $r < 1/\beta$ by equation (12).

A.3 Proof of Proposition 3

A.3.1 The "only if" Part

Assume that we have the allocation $\{\theta_t^*, C_t, N_t, K_{t+1}, B_{t+1}\}_{t=1}^\infty$ and the initial risk-free rate r_1. We then can directly construct the prices, taxes, and individual allocations in the competitive equilibrium in the following 7 steps:

1. w_t and q_t are set by (1) and (2), which are $w_t = MP_{N,t}$ and $q_t = MP_{K,t}$, respectively.

2. Given C_t and θ_t^*, the total liquidity in hand can be set by equation (18), $x_t = \frac{C_t}{D(\theta_t^*)}$.

3. The individual consumption and asset holdings, $c_t(\theta_t)$ and $a_{t+1}(\theta_t)$, are pinned down by equations (9) and (10).

4. $\tau_{n,t}$ is determined by equation (8), which implies $\tau_{n,t} = 1 - \frac{x_t^\sigma}{L(\theta_t^*)\theta_t^* MP_{N,t}}$. Hence, $\overline{w}_t$ can be expressed as
$$\overline{w}_t = \frac{x_t^\sigma}{L(\theta_t^*)\theta_t^*} = \frac{C_t^\sigma}{D(\theta_t^*)^\sigma L(\theta_t^*)\theta_t^*}.$$
Given $\overline{w}_t$, the interest rate r_{t+1} can be backed out by the Euler equation (12):
$$\frac{1}{r_{t+1}} = \beta L(\theta_t^*)\frac{\overline{w}_t}{\overline{w}_{t+1}} = \beta\frac{U_{C,t+1}}{U_{C,t}}\frac{D(\theta_{t+1}^*)^\sigma\theta_{t+1}^*}{D(\theta_t^*)^\sigma\theta_t^*}L(\theta_{t+1}^*) \text{ for all } t \geq 1$$

where $U_{C,t}$ is defined as $C_t^{-\sigma}$, the "marginal utility of aggregate consumption" given our preference assumption.

Given r_1 and the expression of $\{r_{t+1}\}_{t=1}^{\infty}$, the capital tax $\{\tau_{k,t+1}\}_{t=0}^{\infty}$ is chosen to satisfy the no-arbitrage condition: $r_t = 1 + (1-\tau_{k,t})MP_{K,t} - \delta$ for all $t \geq 1$.

5. Finally, set $n_t(\theta_{t-1})$ to satisfy equation (11), which is implied by the individual household budget constraint.

6. Define A_{t+1} as the aggregate asset holding in period t. Integrating (10) gives

$$A_{t+1} \equiv \int a_{t+1}(\theta_t)dF(\theta_t) = \int \max\left\{1 - \left(\frac{\theta_t}{\theta_t^*}\right)^{1/\sigma}, 0\right\} x_t dF(\theta_t) = \left(\frac{1}{D(\theta_t^*)} - 1\right) C_t,$$

where the last equality utilizes equation (18). The above equation together with the condition defined in equation (20) gives the competitive equilibrium asset market clearing condition (14).

7. The implementability conditions are

$$C_1^{1-\sigma} D(\theta_1^*)^{\sigma-1} L(\theta_1^*)\theta_1^* \geq N_1 + r_1 C_1^{-\sigma} D(\theta_1^*)^{\sigma} L(\theta_1^*)\theta_1^*(K_1 + B_1)$$

and for $t \geq 2$,

$$C_t^{1-\sigma} D(\theta_t^*)^{\sigma-1} L(\theta_t^*)\theta_t^* \geq N_t + \frac{1}{\beta} C_{t-1}^{1-\sigma} D(\theta_{t-1}^*)^{\sigma-1}\theta_{t-1}^* \left(\frac{1}{D(\theta_{t-1}^*)} - 1\right)$$

Multiplying both side of the above equation with $\frac{C_t^{\sigma}}{D(\theta_t^*)^{\sigma} L(\theta_t^*)\theta_t^*}$ leads to

$$\frac{C_1}{D(\theta_1^*)} \geq \frac{C_1^{\sigma}}{D(\theta_1^*)^{\sigma} L(\theta_1^*)\theta_1^*} N_1 + r_1(K_1 + B_1)$$

$$\frac{C_t}{D(\theta_t^*)} \geq \frac{C_t^{\sigma}}{D(\theta_t^*)^{\sigma} L(\theta_t^*)\theta_t^*} N_t + \frac{1}{\beta} \frac{C_{t-1}^{-\sigma} D(\theta_{t-1}^*)^{\sigma}\theta_{t-1}^*}{C_t^{-\sigma} D(\theta_t^*)^{\sigma}\theta_t^*} \frac{1}{L(\theta_t^*)} \left(1 - D(\theta_{t-1}^*)\right) C_{t-1}.$$

Using the relationship constructed in steps 2, 4, and 6 for x_t, C_t, A_{t+1}, $\overline{w}_t$, and r_t in the above equation gives

$$C_t + A_{t+1} \geq \overline{w}_t N_t + r_t A_t \text{ for all } t \geq 1,$$

which together with the resource constraint and asset market clearing condition enforce the government budget constraint.

Step 1 ensures that the representative firm's problem is solved. Steps 2 to 5 guarantee that the individual household problem is solved. Steps 6 and 7 ensure that the asset market clearing condition and government budget constraint are satisfied, respectively. The labor market clearing condition is satisfied by Walras law.

A.3.2 The "if" Part

Note that the resource constraint and asset market clearing condition are trivially implied by a competitive equilibrium since they are part of the definition. The implementability condition is constructed as follows. First, we rewrite the government budget constraint as

$$G_t \leq F(K_t, N_t) - (1-\tau_{k,t})q_t K_t - (1-\tau_{n,t})w_t N_t + B_{t+1} - r_t B_t.$$

Combining this equation with the resource constraint (16), no-arbitrage condition, and the asset market clearing condition (14) implies

$$(1-\tau_{n,t})w_t N_t + r_t A_t \leq C_t + A_{t+1},$$

and hence leading to the following equation:

$$C_t + A_{t+1} \geq \overline{w}_t N_t + r_t A_t. \tag{53}$$

For $t \geq 2$, the equilibrium conditions (8), (18), and (12) suggest that $\overline{w}_t$ and r_t can be expressed as

$$\overline{w}_t = \frac{1}{C_t^{-\sigma} D(\theta_t^*)^\sigma L(\theta_t^*)\theta_t^*}$$

and

$$r_t = \frac{1}{\beta}\frac{C_{t-1}^{-\sigma} D(\theta_{t-1}^*)^\sigma \theta_{t-1}^*}{C_t^{-\sigma} D(\theta_t^*)^\sigma \theta_t^*}\frac{1}{L(\theta_t^*)}.$$

Substituting the above two equations into (53) and rearranging terms, we get

$$C_t^{-\sigma} D(\theta_t^*)^\sigma L(\theta_t^*)\theta_t^* C_t + C_t^{-\sigma} D(\theta_t^*)^\sigma L(\theta_t^*)\theta_t^* A_{t+1} \geq N_t + \frac{1}{\beta} C_{t-1}^{-\sigma} D(\theta_{t-1}^*)^\sigma \theta_{t-1}^* A_t,$$

The implementability condition (22) follows by plugging $A_{t+1} = \left(\frac{1}{D(\theta_t^*)} - 1\right) C_t$ and $A_t = \left(\frac{1}{D(\theta_{t-1}^*)} - 1\right) C_{t-1}$ to the equation above.

For the first period, $\{B_1, K_1, \tau_{k,1}\}$ are given, which implies that $r_1 = 1+(1-\tau_{k,1})MP_{K,1}-\delta$ is pinned down by N_1. Therefore, the first period implementability condition could be rewritten as

$$C_1^{1-\sigma} D(\theta_1^*)^{\sigma-1} L(\theta_1^*)\theta_1^* \geq N_1 + r_1 C_1^{-\sigma} D(\theta_1^*)^{\sigma} L(\theta_1^*)\theta_1^*(K_1 + B_1).$$

A.4 Appendix A.4

The first-order conditions with respect to N_1, C_1, and θ_1^* in the Ramsey problem (27) are given, respectively, by

$$1 + \lambda_1 + \lambda_1 C_1^{-\sigma} D(\theta_1^*)^{\sigma} L(\theta_1^*)\theta_1^* \frac{\partial r_1}{\partial N_1}(K_1 + B_1) = \mu_1 MP_{N,1} \tag{54}$$

$$\begin{aligned} \mu_1 \;=\;& W(\theta_1^*)C_1^{-\sigma} + \lambda_1(1-\sigma)C_1^{-\sigma} D(\theta_1^*)^{\sigma-1} L(\theta_1^*)\theta_1^* \\ &+\lambda_1 \sigma C_1^{-\sigma-1} D(\theta_1^*)^{\sigma} L(\theta_1^*)\theta_1^* r_1(K_1 + B_1) \\ &-\lambda_2(1-\sigma)C_1^{-\sigma} D(\theta_1^*)^{\sigma}\theta_1^* \left(\frac{1}{D(\theta_1^*)} - 1\right) - \phi_1 \left(\frac{1}{D(\theta_1^*)} - 1\right) \end{aligned} \tag{55}$$

and

$$\begin{aligned} &\frac{\partial W(\theta_1^*)}{\partial \theta_1^*}\frac{C_1^{1-\sigma}}{1-\sigma} + \lambda_1 C_1^{1-\sigma} H(\theta_1^*) - \lambda_2 C_1^{1-\sigma} J(\theta_1^*) + \phi_1 \frac{C_1}{D(\theta_1^*)^2}\frac{\partial D(\theta_1^*)}{\partial \theta_1^*} \\ =\;& \lambda_1 C_1^{-\sigma}\frac{\partial \left(D(\theta_1^*)^{\sigma} L(\theta_1^*)\theta_1^*\right)}{\partial \theta_1^*} r_1(K_1 + B_1). \end{aligned} \tag{56}$$

In addition, since the relative magnitude of $H(\theta^*)$ and $J(\theta^*)$ affects the dynamics of λ_t in equation (32), we provide the following four lemmas to help characterize the optimal Ramsey allocation.

Lemma 1. *The derivative of $W(\theta_t^*)$ is given by*

$$\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} = \frac{(1-\sigma)}{\sigma} D(\theta_t^*)^{\sigma-1} X(\theta_t^*) \left[\frac{M(\theta_t^*)}{D(\theta_t^*)\theta_t^*} - 1\right],$$

where $X(\theta_t^*)$ and $M(\theta_t^*)$ are defined as

$$X(\theta_t^*) \equiv \int_{\theta \leq \theta_t^*} \left(\frac{\theta}{\theta_t^*}\right)^{1/\sigma} d\mathbf{F}(\theta), \tag{57}$$

$$M(\theta_t^*) \equiv \int_{\theta \leq \theta_t^*} \theta_t^* \left(\frac{\theta}{\theta_t^*}\right)^{\frac{1}{\sigma}} d\mathbf{F} + \int_{\theta > \theta_t^*} \theta d\mathbf{F}, \tag{58}$$

respectively. In addition, $M(\theta_t^*) \geq \theta_t^* D(\theta_t^*)$ with equality if $\theta_t^* = \theta_H$. Hence, the sign of $\frac{\partial W(\theta^*)}{\partial \theta^*}$ is determined by the sign of the elasticity of intertemporal substitution coefficient, $1/\sigma$.

Proof. By the definition of $M(\theta_t^*)$, the $W(\theta_t^*)$ can be rewritten as $W(\theta_t^*) = M(\theta_t^*) D(\theta_t^*)^{\sigma-1}$. The derivative of $M(\theta_t^*)$ and hence $W(\theta_t^*)$ are given by

$$\frac{\partial M(\theta_t^*)}{\partial \theta_t^*} = -\frac{1-\sigma}{\sigma} \int_{\theta \leq \theta_t^*} \left(\frac{\theta}{\theta_t^*}\right)^{\frac{1}{\sigma}} d\mathbf{F} = -\frac{1-\sigma}{\sigma} X(\theta_t^*)$$

and

$$\begin{aligned} \frac{\partial W(\theta_t^*)}{\partial \theta_t^*} &= (\sigma-1) D(\theta_t^*)^{\sigma-2} \frac{\partial D(\theta_t^*)}{\partial \theta_t^*} M(\theta_t^*) - D(\theta_t^*)^{\sigma-1} \frac{1-\sigma}{\sigma} X(\theta_t^*) \\ &= \frac{(1-\sigma)}{\sigma} D(\theta_t^*)^{\sigma-1} X(\theta_t^*) \left[\frac{M(\theta_t^*)}{D(\theta_t^*)\theta_t^*} - 1\right] \end{aligned}$$

In addition,

$$M(\theta_t^*) - \theta_t^* D(\theta_t^*) = \int_{\theta > \theta_t^*} (\theta - \theta_t^*) d\mathbf{F}(\theta) \geq 0$$

Hence, $M(\theta_t^*) \geq \theta_t^* D(\theta_t^*)$ with equality holds at $\theta_t^* = \theta_H$ □

Lemma 2. *$J(\theta_t^*)$ and $H(\theta_t^*)$ can be expressed as*

$$\begin{aligned} H(\theta_t^*) &= D(\theta_t^*)^{\sigma-1} X(\theta_t^*) \left[\frac{1-\sigma}{\sigma} \frac{L(\theta_t^*)}{D(\theta_t^*)} + \frac{\mathbf{F}(\theta_t^*)}{X(\theta_t^*)}\right], & (59) \\ J(\theta_t^*) &= D(\theta_t^*)^{\sigma-1} X(\theta_t^*) \left[\frac{1-\sigma}{\sigma} \frac{1}{D(\theta_t^*)} + \frac{\mathbf{F}(\theta_t^*)}{X(\theta_t^*)}\right], & (60) \end{aligned}$$

which have the following properties: (1) if $\sigma < 1$, then $H(\theta_t^*) > J(\theta_t^*) > 0$; (2) if $\sigma = 1$, then $H(\theta_t^*) = J(\theta_t^*) > 0$; and (3) if $\sigma > 1$, then $H(\theta_t^*) < J(\theta_t^*)$.

Proof. $H(\theta_t^*)$ can be expressed as

$$\begin{aligned} H(\theta_t^*) &\equiv \frac{\partial \left(D(\theta_t^*)^{\sigma-1}\theta_t^* L(\theta_t^*)\right)}{\partial \theta_t^*} = (\sigma-1)D(\theta_t^*)^{\sigma-2}\frac{\partial D(\theta_t^*)}{\partial \theta_t^*}\theta_t^* L(\theta_t^*) + D(\theta_t^*)^{\sigma-1}\mathbf{F}(\theta_t^*) \\ &= D(\theta_t^*)^{\sigma-1}\left[(\sigma-1)D(\theta_t^*)^{-1}\frac{\partial D(\theta_t^*)}{\partial \theta_t^*}\theta_t^* L(\theta_t^*) + \mathbf{F}(\theta_t^*)\right] \\ &= D(\theta_t^*)^{\sigma-1}X(\theta_t^*)\left[\frac{1-\sigma}{\sigma}\frac{L(\theta_t^*)}{D(\theta_t^*)} + \frac{\mathbf{F}(\theta_t^*)}{X(\theta_t^*)}\right]. \end{aligned}$$

$J(\theta_t^*)$ can be rewritten as

$$\begin{aligned} J(\theta_t^*) &\equiv \frac{\partial \left(D(\theta_t^*)^{\sigma-1}\theta_t^*\left(1-D(\theta_t^*)\right)\right)}{\partial \theta_t^*} \\ &= (\sigma-1)D(\theta_t^*)^{\sigma-2}\frac{\partial D(\theta_t^*)}{\partial \theta_t^*}\theta_t^*\left(1-D(\theta_t^*)\right) + D(\theta_t^*)^{\sigma-1}\left(1-D(\theta_t^*)-\theta_t^*\frac{\partial D(\theta_t^*)}{\partial \theta_t^*}\right) \\ &= \frac{1-\sigma}{\sigma}D(\theta_t^*)^{\sigma-2}X(\theta_t^*)\left(1-D(\theta_t^*)\right) + D(\theta_t^*)^{\sigma-1}\left(\mathbf{F}(\theta_t^*) + \left(\frac{1}{\sigma}-1\right)X(\theta_t^*)\right) \\ &= D(\theta_t^*)^{\sigma-1}X(\theta_t^*)\left[\frac{1-\sigma}{\sigma}\frac{1}{D(\theta_t^*)} + \frac{\mathbf{F}(\theta_t^*)}{X(\theta_t^*)}\right] \end{aligned}$$

By $H(\theta_t^*)$ and $J(\theta_t^*)$ expressed above, we reach that

1. if $\sigma < 1$, $0 < J(\theta_t^*) \le H(\theta_t^*)$ and $J(\theta_t^*) = H(\theta_t^*)$ if $\theta_t^* = \theta_H$
2. if $\sigma = 1$, $0 \le J(\theta_t^*) = H(\theta_t^*)$.
3. if $\sigma > 1$, $J(\theta_t^*) \ge H(\theta_t^*)$ and $J(\theta_t^*) = H(\theta_t^*)$ if $\theta_t^* = \theta_H$ or θ_L

□

Lemma 3. $\frac{\partial D(\theta_t^*)}{\partial \theta_t^*} = -\frac{X(\theta_t^*)}{\sigma\theta_t^*} < 0$ *for all* $\theta_t^* \in (\theta_L, \theta_H]$.

Proof. $D(\theta_t^*)$ is therefore given by

$$D(\theta_t^*) = X(\theta_t^*) + \int_{\theta > \theta_t^*} d\mathbf{F}(\theta).$$

The derivative of $D(\theta_t^*)$ is

$$\frac{\partial D(\theta_t^*)}{\partial \theta_t^*} = -\frac{1}{\sigma}\frac{1}{\theta_t^*}\int_{\theta \le \theta_t^*}\left(\frac{\theta}{\theta_t^*}\right)^{\frac{1}{\sigma}} d\mathbf{F}(\theta) = -\frac{X(\theta_t^*)}{\sigma\theta_t^*} < 0$$

□

Lemma 4. $L(\theta_t^*) - 1 + D(\theta_t^*) = \frac{M(\theta_t^*)}{\theta_t^*}$.

Proof. By the definition of $L(\theta_t^*)$ and $D(\theta_t^*)$, we get

$$\begin{aligned} L(\theta_t^*) - 1 + D(\theta_t^*) &= \int_{\theta > \theta_t^*} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) + X(\theta_t^*) + \int_{\theta > \theta_t^*} d\mathbf{F}(\theta) - 1 \\ &= \int_{\theta > \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) + X(\theta_t^*) = \frac{M(\theta_t^*)}{\theta_t^*}, \end{aligned}$$

where the last equality utilize the definition of $M(\theta_t^*)$. □

A.5 Proof of Proposition 4

The optimal capital tax is chosen such that the Euler equation in the competitive equilibrium (12) is consistent with the one chosen by the Ramsey planner shown in (28). Hence, $\tau_{k,t+1}$ is pinned down by

$$1 - \tau_{k,t+1} = \frac{\frac{\overline{w}_{t+1}}{\overline{w}_t} \frac{1}{L(\theta_t^*)} - \beta(1-\delta)}{\frac{\mu_t - \phi_t}{\mu_{t+1}} - \beta(1-\delta)},$$

which is equation (34) in the steady state.

A.6 Proof of Proposition 5

From equation (31), $\nu_t^B = 0$ implies that $\phi_t = 0$. The first-order condition with respect to θ_t^* is then reduced to

$$\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} \frac{1}{1-\sigma} + \lambda_t H(\theta_t^*) - \lambda_{t+1} J(\theta_t^*) = 0,$$

which can be further simplified according to Lemma 1 and Lemma 2 as

$$\widehat{W}(\theta_t^*) + \lambda_t \widehat{H}(\theta_t^*) = \lambda_{t+1} \widehat{J}(\theta_t^*), \tag{61}$$

where $\widehat{H}(\theta_t^*)$, $\widehat{J}(\theta_t^*)$, and $\widehat{W}(\theta_t^*)$ are defined, respectively, as

$$\widehat{H}(\theta_t^*) \equiv \frac{1-\sigma}{\sigma}\frac{L(\theta_t^*)}{D(\theta_t^*)} + \frac{\mathbf{F}(\theta_t^*)}{X(\theta_t^*)}, \tag{62}$$

$$\widehat{J}(\theta_t^*) \equiv \frac{1-\sigma}{\sigma}\frac{1}{D(\theta_t^*)} + \frac{\mathbf{F}(\theta_t^*)}{X(\theta_t^*)}, \tag{63}$$

$$\widehat{W}(\theta_t^*) \equiv \frac{1}{\sigma}\frac{M(\theta_t^*)}{D(\theta_t^*)\,\theta_t^*} - \frac{1}{\sigma} \geq 0. \tag{64}$$

Note that $\widehat{W}(\theta_t^*) > 0$ for all $\theta_t^* \in [\theta_L, \theta_H)$ and $\sigma \in (0,\infty)$, and $\widehat{W}(\theta_t^*) = 0$ for $\theta_t^* = \theta_H$.

In what follows, we first sketch the proof that a Ramsey steady state featuring $\theta^* = \theta_H$ exists if β is sufficiently large. We proceed by the following steps, which show that the conjecture $\theta^* = \theta_H$ satisfies all of the Ramsey FOCs and these FOC-implied steady-state values of the aggregate allocation $\{C, N, K, B\}$ and Lagrangian multipliers $\{\lambda, \mu\}$ are unique, mutually consistent, strictly positive, and finitely-valued:

1. The FOC with respect to θ_t^* in equation (32) is satisfied at $\theta_t^* = \theta_H$. Specifically, plugging $\theta^* = \theta_H$ into the definitions of $\widehat{J}(\theta_t^*)$, $\widehat{H}(\theta_t^*)$, and $\widehat{W}(\theta_t^*)$ leads to $\widehat{J}(\theta_H) = \widehat{H}(\theta_H) \neq 0$ and $\widehat{W}(\theta_H) = 0$. Therefore the FOC (32) is satisfied.

2. The FOC with respect to K in equation (28) gives

$$1 = \beta\left(MP_K + 1 - \delta\right),$$

which implies $MP_K = \alpha\left(\frac{K}{N}\right)^{\alpha-1} = \frac{1-\beta(1-\delta)}{\beta} \in (0,\infty)$ (i.e., the capital-to-labor ratio is unique, strictly positive, and bounded). Given the assumption of the production function, it must be true that the following ratios are unique, strictly positive, and finite: $\{K/N, Y/K, MP_N, Y/N\} \in (0,\infty)$. More specifically, the Y/N and Y/K ratios can be expressed, respectively, as

$$\frac{Y}{N} = \left(\frac{K}{N}\right)^{\alpha} = \left(\frac{1-\beta(1-\delta)}{\beta\alpha}\right)^{\frac{\alpha}{\alpha-1}} \tag{65}$$

and

$$\frac{Y}{K} = \left(\frac{K}{N}\right)^{\alpha-1} = \frac{1-\beta(1-\delta)}{\alpha\beta}. \tag{66}$$

3. The resource constraint,

$$F(N,K) = K^{\alpha}N^{1-\alpha} = C + \delta K + G,$$

together with a finite level of government spending G implies a unique ratio $C/K \in (0,\infty)$ as

$$\frac{C}{Y} = \left(1 - \delta\frac{K}{Y} - \frac{G}{Y}\right) = \left(1 - \frac{\alpha\beta\delta}{1-\beta(1-\delta)} - \frac{G}{Y}\right), \tag{67}$$

where the last equality uses (66).

4. We know that under our parameter restrictions the level of labor is interior, $N \in \left(0,\overline{N}\right)$; hence, it must be true that the aggregate allocation is also unique and interior: $\{C,K,Y\} \in (0,\infty)$.

5. Next, we show that $\{\mu,\lambda\} \in (0,\infty)$ and that these steady-state values are unique. Given $\theta^{*} = \theta_H$ in the steady state, the FOCs (29) and (30) become

$$1 + \lambda = \mu MP_N \tag{68}$$

and

$$\mu C^{\sigma} D(\theta_H)^{-\sigma}\theta_H^{-1} = 1 + \lambda(1-\sigma), \tag{69}$$

respectively. These two equations uniquely solve for $\{\lambda,\mu\}$. Note that $\mu \in (0,\infty)$ if $\lambda \in (0,\infty)$ according to (68). These two equations above imply

$$1 + \lambda = Q\left(1 + (1-\sigma)\lambda\right),$$

or equivalently,

$$\lambda = \frac{Q-1}{1+Q(\sigma-1)},$$

where $Q = \frac{MP_N D(\theta_H)^{\sigma}\theta_H}{C^{\sigma}} \in (0,\infty)$ is unique. So $\lambda > 0$ if and only if $Q > 1$ because the denominator is strictly positive for $Q > 1$ and $\sigma > 0$. To see this, suppose the denominator is negative, then $\sigma < 1 - 1/Q$, and then $\lambda > 0$ implies $Q < 1$; but $Q \in (0,1)$ implies $\sigma = 1 - 1/Q < 0$, a contradiction. Therefore, for $\{\mu,\lambda\} \in (0,\infty)$, we only need to ensure $Q > 1$.

Note that the steady-state version of the implementability condition (22) with $\theta_t^* = \theta_H$ becomes

$$N = C^{1-\sigma} D(\theta_H)^{\sigma-1} \theta_H \left(1 - \frac{1 - D(\theta_H)}{\beta}\right),$$

which together with equations (65) and (67) imply

$$C^{\sigma} = \left(1 - \frac{\beta\alpha\delta}{1-\beta(1-\delta)} - \frac{G}{Y}\right)\left(\frac{1-\beta(1-\delta)}{\beta\alpha}\right)^{\frac{\alpha}{\alpha-1}} D(\theta_H)^{\sigma-1}\theta_H \left(1 - \frac{1-D(\theta_H)}{\beta}\right). \tag{70}$$

Thus, by equations (70) and (65), Q can be rewritten as

$$\begin{aligned} Q &= (1-\alpha)\frac{Y}{N}\frac{D(\theta_H)^{\sigma}\theta_H}{C^{\sigma}} \\ &= \frac{1-\alpha}{\left(1 - \frac{\beta\alpha\delta}{1-\beta(1-\delta)} - \frac{G}{Y}\right)} \frac{D(\theta_H)}{\left(1 - \frac{1-D(\theta_H)}{\beta}\right)}, \end{aligned}$$

where it is easy to show that $\frac{D(\theta_H)}{\left(1-\frac{1-D(\theta_H)}{\beta}\right)} > 1$ if β is sufficiently large. Hence, for $Q > 1$, we only need to ensure that the first term $\frac{1-\alpha}{\left(1-\frac{\beta\alpha\delta}{1-\beta(1-\delta)}-\frac{G}{Y}\right)}$ is no less than 1, which is true if β is sufficiently close to 1 because $\lim_{\beta\to 1} \frac{1-\alpha}{\left(1-\frac{\beta\alpha\delta}{1-\beta(1-\delta)}-\frac{G}{Y}\right)} = \frac{1-\alpha}{1-\alpha-\frac{G}{Y}} \geq 1$.

6. Hence, $\lambda > 0$ *only if* β is sufficiently large such that $Q > 1$. Notice that such a required value for β does not contradict the competitive-equilibrium condition (17) for interior cutoff and the conditions for interior $N \in \left(0, \overline{N}\right)$. Given $\lambda > 0$, it is then easy to see that $\mu > 0$.

7. It remains to show that the steady-state level of government debt $B \in (0, \infty)$. Given that aggregate output $Y \in (0, \infty)$, it suffices to show that the debt-to-GDP ratio $\frac{B}{Y} \in (0, \infty)$. Since $K + B = \left(\frac{1}{D(\theta^*)} - 1\right) C$, we have

$$\begin{aligned} \frac{B}{Y} &= \left(\frac{1}{D(\theta_H)} - 1\right)\frac{C}{Y} - \frac{K}{Y} \\ &= \left(\frac{1}{D(\theta_H)} - 1\right)\left(1 - \frac{\delta\beta\alpha}{1-\beta(1-\delta)} - \frac{G}{Y}\right) - \left(\frac{\beta\alpha}{1-\beta(1-\delta)}\right), \end{aligned} \tag{71}$$

which is finite because each term in the parentheses is finite. The debt-to-GDP ratio is also strictly positive provided that the output elasticity of capital α is small enough. To see this, noting that the first term $\left(\frac{1}{D(\theta_H)} - 1\right) > 0$, the second term

$\left(1-\frac{\delta\beta\alpha}{1-\beta(1-\delta)}-\frac{G}{Y}\right)>0$ because it is the consumption-to-output ratio $\frac{C}{Y}$, and the last term vanishes as α approaches zero. Notice that such a restriction on the value for α does not contradict the competitive-equilibrium condition (17) for interior cutoff and the conditions for interior $N \in \left(0, \overline{N}\right)$.

8. Finally, by Proposition 4, since $\phi = 0$ and $L(\theta^*) = \frac{\mu}{\mu-\phi} = 1$, it must be true that $\theta^* = \theta_H$ and $\tau_k = 1 - \frac{L(\theta^*)^{-1}-\beta(1-\delta)}{\frac{\mu-\phi}{\mu}-\beta(1-\delta)} = 0$ in a Ramsey steady state. The steady-state equilibrium interest rate is therefore $1/\beta$ by the Euler equation (12). In addition, the labor tax rate is determined by equation (8), which implies $\tau_n = \frac{\lambda}{1+\lambda}\sigma$ in the steady state. Namely,
$$\tau_n = 1 - \frac{1}{L(\theta_H^*)\theta_H^* D(\theta_H^*)^\sigma}\frac{C^\sigma}{MP_N} = \frac{\lambda}{1+\lambda}\sigma,$$
where the last equality uses equations (68) and (69). Therefore, government expenditures and bond interest payments are financed solely by labor tax income in the Ramsey steady state.

This finishes the proof for the existence of the Ramsey steady state. To show uniqueness, we first show that when $\phi = 0$, there is no Ramsey steady state for $\theta^* \in (\theta_L, \theta_H)$. Rewrite (61) as
$$\lambda_{t+1} = \rho(\theta_t^*)\lambda_t + \varepsilon(\theta_t^*),$$
where $\rho(\theta_t^*) \equiv \widehat{H}(\theta_t^*)/\widehat{J}(\theta_t^*)$ and $\varepsilon(\theta_t^*) \equiv \widehat{W}(\theta_t^*)/\widehat{J}(\theta_t^*)$. The sign and value of $\rho(\theta_t^*)$ and $\varepsilon(\theta_t^*)$ depend on the value of σ; hence, we can discuss the possible steady states according to the values of $\sigma \lesseqgtr 1$.

1. $\sigma < 1$. Then $0 < \widehat{J}(\theta_t^*) < \widehat{H}(\theta_t^*)$ according to Lemma 2. Hence, $\rho(\theta_t^*) > 1$ and $\varepsilon(\theta_t^*) > 0$. In this case, if $\theta^* \in (\theta_L, \theta_H)$, then the steady-state $\lambda = \frac{\varepsilon(\theta^*)}{1-\rho(\theta^*)}$ must be negative, which violates the FOC (29) in the steady state. Hence, an interior solution for the cutoff $\theta^* \in (\theta_L, \theta_H)$ cannot constitute a Ramsey steady state.

2. $\sigma = 1$. Then $0 < \widehat{J}(\theta_t^*) = \widehat{H}(\theta_t^*)$ according to Lemma 2. Hence, $\rho(\theta_t^*) = 1$ and $\varepsilon(\theta_t^*) > 0$. In this case λ_t does not converge and goes to infinity in the long run. A divergent λ_t implies that μ_t must also diverge to infinity according to FOC (29). On the other hand, when $\sigma = 1$, the FOC (30) implies that μ_t must be finite and positive

in the steady state. This leads to a contradiction and hence cannot be a Ramsey steady state.

3. $\sigma > 1$. Then Lemma 2 implies that $\widehat{J}(\theta_t^*) > \widehat{H}(\theta_t^*)$. The value and sign of $\rho(\theta_t^*)$ and $\varepsilon(\theta_t^*)$ then depend on the sign of $\widehat{H}(\theta_t^*)$ and $\widehat{J}(\theta_t^*)$; so we discuss three subcases below.

   (a) $\widehat{H}(\theta_t^*) > 0$. Then $\widehat{J}(\theta_t^*) > 0$; which implies $0 < \rho(\theta_t^*) < 1$ and $\varepsilon(\theta_t^*) > 0$. The steady-state λ is then equal to $\varepsilon(\theta^*)/(1-\rho(\theta^*))$. We next check if this λ satisfies the FOC (30), which in the steady state can be expressed as

$$\begin{aligned} \mu C^{\sigma} D(\theta^*)^{1-\sigma} &= M(\theta^*) + \lambda(1-\sigma)\theta^* \left(L(\theta^*) - 1 + D(\theta^*)\right) \\ &= M(\theta^*)(1+\lambda(1-\sigma)), \end{aligned} \tag{72}$$

   where the last equality holds according to Lemma 4. Plugging the steady-state λ into equation (72) and using the definitions of $\widehat{J}(\theta_t^*)$, $\widehat{H}(\theta_t^*)$, and $\widehat{W_\theta}(\theta_t^*)$ give

$$\mu C^{\sigma} D(\theta^*)^{1-\sigma} = M(\theta^*) \left(1 + \frac{\frac{M(\theta^*)}{\theta^*} - D(\theta^*)}{1 - L(\theta^*)}\right) = M(\theta^*)(1 + \frac{L(\theta^*) - 1}{1 - L(\theta^*)}) = 0,$$

   where the second equality uses the Lemma 4 again. This equation holds if and only if $\mu = 0$. This cannot be a steady state. Note that $\mu = 0$ as long as $\lambda = \varepsilon(\theta^*)/(1-\rho(\theta^*)) > 0$, regardless of the signs of $\rho(\theta^*)$ and $\varepsilon(\theta_t^*)$.

   (b) $\widehat{H}(\theta_t^*) < 0$ and $\widehat{J}(\theta_t^*) > 0$. Then $\varepsilon(\theta_t^*) > 0$. There are two possibilities regarding $\rho(\theta_t^*) : |\rho(\theta_t^*)| < 1$ or $|\rho(\theta_t^*)| > 1$. But in both cases, the steady-state $\lambda = \varepsilon(\theta^*)/(1-\rho(\theta^*)) > 0$. Such a steady-state λ implies $\mu = 0$ according to (72) regardless of the sign of $\rho(\theta^*)$, as shown in the proof of the previous subcase. Therefore, this subcase cannot be a Ramsey steady-state equilibrium.

   (c) $\widehat{H}(\theta_t^*) < 0$ and $\widehat{J}(\theta_t^*) < 0$. Then $\rho(\theta_t^*) > 1$ and $\varepsilon(\theta_t^*) < 0$, and the steady-state $\lambda = \varepsilon(\theta^*)/(1-\rho(\theta^*))$, which again leads to $\mu = 0$ by equation (72). Hence, this subcase cannot be a Ramsey steady state equilibrium.

Finally, we show that the case of $\theta_t^* = \theta_L$ cannot be a Ramsey equilibrium although the necessary FOC with respect to θ_t^* is satisfied. The reason is that the first term of the Ramsey objective function (25), $W(\theta_t^*)C_t^{1-\sigma}/(1-\sigma)$, is monotonically increasing in $\theta_t^* \in (\theta_L, \theta_H)$. Hence, for a global maximum, a cutoff θ_t^* at its lower corner cannot be a Ramsey equilibrium.

To ensure that $n \in \left(0, \overline{N}\right)$ (see Proposition 1), note that we have assumed $\theta_H < \frac{\theta_L}{(1-\beta)^\sigma}$, which ensures that the minimum individual labor input remains positive, as shown in Appendix A.1. Moreover, by equation (49), the maximum value of n is less than $\overline{N}$ if $\overline{N} > \theta_H$ in this case.

In addition, we can show that the maximum individual asset demand remains finite in the steady state even if the risk-free rate is equal to the time discount rate, $r = 1/\beta$. Since $\theta_H < \frac{\theta_L}{(1-\beta)^\sigma}$, we have

$$x_t = \frac{C_t}{D(\theta_H)} = \frac{C_t}{E(\theta^{1/\sigma})}\theta_H^{1/\sigma} < \infty.$$

Given the finite value of x_t, the individual asset holding a_{t+1} is determined by the size of the idiosyncratic shock θ_t, and the agents with the largest asset holdings are those who receive the smallest shock $\theta_t = \theta_L$, i.e.,

$$a_{t+1}(\theta_L) = \left[1 - \left(\frac{\theta_L}{\theta_H}\right)^{1/\sigma}\right] x_t,$$

which is strictly positive and finite.

A.7 Proof of Corollary 1

Assume that $\delta = 1$ and $G = 0$, then the steady-state resource constraint is $Y = C + K$. In the OSIA Ramsey steady state, MGR holds and hence $1 = \beta MP_K = \beta\alpha\frac{Y}{K}$, which together with the resource constraint imply $\frac{K}{Y} = \alpha\beta$ and $\frac{C}{Y} = 1 - \alpha\beta$. As a result, the optimal steady-state debt-to-GDP ratio can be inferred by the asset-market clearing condition:

$$\begin{aligned}\frac{B}{Y} &= (\frac{1}{D} - 1)\frac{C}{Y} - \frac{K}{Y} = (\frac{1}{D} - 1)(1-\alpha\beta) - \alpha\beta \\ &= \alpha\beta\left[\frac{1-D}{D}\frac{1-\alpha\beta}{\alpha\beta} - 1\right],\end{aligned}$$

which is equation (36) by the definition of τ_b.

A.8 Proof of Proposition 6

Here we prove that a Ramsey steady state featuring a binding debt limit exists if β is sufficiently large and δ is sufficiently small. We proceed by the following steps, which show that the constructed steady state—characterized by $\phi > 0$ and $\theta^* \in (\theta_L, \theta_H)$—satisfies all of the Ramsey FOCs and these FOC-implied values of the steady-state aggregate allocation $\{C, N, K\}$ and Lagrangian multipliers $\{\lambda, \mu\}$ are mutually consistent, strictly positive, and finitely-valued.

1. We first show that any possible Ramsey steady state satisfying the Ramsey FOC with respect to θ_t^* must have the following property:

$$\frac{\mu}{\mu - \phi} = L(\theta^*), \tag{73}$$

which is strictly positive and equals 1 if $\phi = 0$, and is greater than 1 if $\phi > 0$.

Starting from equation (31), $\nu_t^B > 0$ implies that $\phi_t > 0$. The FOC with respect to θ_t^* can be rewritten as

$$\widehat{W}(\theta_t^*) - \phi_t C_t^\sigma \widehat{Z}(\theta_t^*) + \lambda_t \widehat{H}(\theta_t^*) = \lambda_{t+1} \widehat{J}(\theta_t^*), \tag{74}$$

where $\widehat{H}(\theta_t^*)$, $\widehat{J}(\theta_t^*)$, and $\widehat{W}(\theta_t^*)$ are defined as before, and $\widehat{Z}(\theta_t^*)$ is defined as

$$\widehat{Z}(\theta_t^*) \equiv \frac{D(\theta_t^*)^{-1-\sigma}}{\sigma \theta_t^*}, \tag{75}$$

which is positive for any $\theta_t^* \in [\theta_L, \theta_H]$. Rewrite (74) as

$$\lambda_{t+1} = \rho(\theta_t^*)\lambda_t + \widehat{\varepsilon}(\theta_t^*),$$

where $\rho(\theta_t^*) \equiv \widehat{H}(\theta_t^*)/\widehat{J}(\theta_t^*)$ and

$$\widehat{\varepsilon}(\theta_t^*) \equiv \frac{\widehat{W}(\theta_t^*) - \phi_t C_t^\sigma \widehat{Z}(\theta_t^*)}{\widehat{J}(\theta_t^*)}. \tag{76}$$

The existence of a finitely positive steady-state value of λ depends on the sign and value of $\rho(\theta_t^*)$ and $\widehat{\varepsilon}(\theta_t^*)$, which in turn depend on the value of σ. Hence we discuss the following three cases of $\sigma \lesseqgtr 1$:

(a) $\sigma < 1$. Then $0 < \widehat{J}(\theta_t^*) < \widehat{H}(\theta_t^*)$ according to Lemma 2, and $\rho(\theta_t^*) > 1$. In this case λ_t converges to a positive steady-state value if and only if $\widehat{\varepsilon}(\theta_t^*) < 0$; namely,

$$\lambda = \widehat{\varepsilon}(\theta^*)/(1-\rho(\theta^*)) > 0 \tag{77}$$

if and only if $\widehat{\varepsilon}(\theta^*) < 0$. We will check later on that $\widehat{\varepsilon}(\theta^*) < 0$ indeed constitutes a Ramsey steady state with $\sigma \in (0,1)$, $\theta^* < \theta_H$, and $\{\lambda, \mu, \phi\} > 0$. According to FOC (30) and Lemma 4, the steady-state value of μ is given by

$$\mu = C^{-\sigma} D(\theta^*)^{\sigma-1} M(\theta^*)(1+\lambda(1-\sigma)) - \phi \frac{1-D(\theta^*)}{D(\theta^*)}, \tag{78}$$

or equivalently,

$$\frac{C^{\sigma}}{D(\theta^*)^{\sigma}} = \frac{D(\theta^*)^{-1} M(\theta^*)(1+\lambda(1-\sigma))}{\mu + \phi \frac{1-D(\theta^*)}{D(\theta^*)}}. \tag{79}$$

Plugging the steady-state λ into equation (78) and using the definitions of $\widehat{J}(\theta_t^*)$, $\widehat{H}(\theta_t^*)$, $\widehat{W}(\theta_t^*)$, and $\widehat{Z}(\theta_t^*)$ as well as Lemma 4, gives

$$\mu = \phi \frac{M(\theta^*)}{\theta^*} \frac{D(\theta^*)^{-1}}{L(\theta^*)-1} - \phi \frac{1-D(\theta^*)}{D(\theta^*)} = \phi \frac{L(\theta^*)}{L(\theta^*)-1}, \tag{80}$$

where the last equality uses the Lemma 4 again. This equation is exactly identical to (73).

(b) $\sigma = 1$. Then $0 < \widehat{J}(\theta_t^*) = \widehat{H}(\theta_t^*)$ according to Lemma 2, and $\rho(\theta_t^*) = 1$. The steady-state λ is a finitely positive value if and only if $\widehat{\varepsilon}(\theta^*) = 0$. We will show later on that $\widehat{\varepsilon}(\theta^*) = 0$ is indeed consistent with an interior Ramsey steady state with $\sigma = 1$, $\theta^* \in (\theta_L, \theta_H)$, and $\{\lambda, \mu, \phi\} > 0$. First, given $\widehat{\varepsilon}(\theta^*) = 0$, the steady-state ϕ can be solve by equation (74), which is

$$\phi = C^{-1} \frac{\widehat{W}(\theta^*)}{\widehat{Z}(\theta^*)} = C^{-1} D(\theta^*) \theta^* \left(L(\theta^*) - 1\right).$$

Note that $\phi > 0$ if and only if $L(\theta^*) > 1$, or equivalently, $\theta^* < \theta_H$. According to

FOC (30) and lemma 4, the steady-state value of μ at $\sigma = 1$ is given by

$$\begin{aligned}
\mu &= C^{-1}M(\theta^*) - \phi D(\theta^*)^{-1}(1 - D(\theta^*)) \\
&= C^{-1}\theta^* D(\theta^*) L(\theta^*) \\
&> 0,
\end{aligned}$$

where the second equality holds according to Lemma 4 and the equation $\phi = C^{-1}D(\theta^*)\theta^*(L(\theta^*) - 1)$ solved above. As a result, the value of $\mu/(\mu - \phi)$ is

$$\frac{\mu}{\mu - \phi} = \frac{C^{-1}\theta^* D(\theta^*) L(\theta^*)}{C^{-1}\theta^* D(\theta^*) L(\theta^*) - C^{-1}D(\theta^*)\theta^*(L(\theta^*) - 1)} = L(\theta^*),$$

which is identical to (73).

(c) $\sigma > 1$. Then Lemma 2 implies that $\widehat{J}(\theta_t^*) > \widehat{H}(\theta_t^*)$. The value and sign of $\rho(\theta_t^*)$ and $\widehat{\varepsilon}(\theta_t^*)$ depend on the sign of $\widehat{H}(\theta_t^*)$ and $\widehat{J}(\theta_t^*)$; there are three possible subcases, which are discussed below:

   i. $\widehat{H}(\theta_t^*) > 0$ and $\widehat{J}(\theta_t^*) > 0$. Then $0 < \rho(\theta_t^*) < 1$. The steady-state $\lambda = \widehat{\varepsilon}(\theta^*)/(1 - \rho(\theta^*)) > 0$ if and only if $\widehat{\varepsilon}(\theta^*) > 0$.

   ii. $\widehat{H}(\theta_t^*) < 0$ and $\widehat{J}(\theta_t^*) > 0$. In this case $\rho(\theta^*) < 0$. Then there are two possibilities regarding to $\rho(\theta_t^*)$: $|\rho(\theta_t^*)| < 1$ or $|\rho(\theta_t^*)| > 1$. In either case we can show that $\lambda = \widehat{\varepsilon}(\theta^*)/(1 - \rho(\theta^*)) > 0$ if and only if $\widehat{\varepsilon}(\theta^*) > 0$.

   iii. $\widehat{H}(\theta_t^*) < 0$ and $\widehat{J}(\theta_t^*) < 0$. Then $\rho(\theta_t^*) > 1$. For a Ramsey steady state with $\lambda = \widehat{\varepsilon}(\theta^*)/(1 - \rho(\theta^*)) > 0$ to exist, it requires $\widehat{\varepsilon}(\theta^*) < 0$.

   Since all of the possible values of λ in the case of $\sigma > 1$ are analogous to those in the case of $\sigma < 1$, following the proof in the $\sigma < 1$ case, we can show that $\mu/(\mu - \phi) = L(\theta^*)$ also holds.

Therefore, any possible steady state satisfying the FOC with respect to θ_t^* must imply $\mu/(\mu - \phi) = L(\theta^*)$. We will verify at the end that the required conditions for (the sign of) $\widehat{\varepsilon}(\theta^*)$ stated above are all automatically met once the condition for $\lambda > 0$ is satisfied.

2. The FOC with respect to K in (28) together with equation (73) give

$$1-\frac{\phi}{\mu}=\beta\left(MP_K+1-\delta\right)=\frac{1}{L(\theta^*)}<1, \tag{81}$$

which suggests MP_K is bounded above. In addition, if δ is sufficiently small such that $MP_K>\delta$, then $MP_K\in(\delta,\frac{1-\beta(1-\delta)}{\beta})$. Given the assumption of the production function, it must be true that the following ratios are strictly positive and finite: $\{K/N,Y/K,MP_N,Y/N\}\in(0,\infty)$. More specifically, Y/K ratio is bounded above:

$$\frac{Y}{K}=\left(\frac{K}{N}\right)^{\alpha-1}<\frac{1-\beta\left(1-\delta\right)}{\alpha\beta}. \tag{82}$$

3. The resource constraint,

$$Y=K^{\alpha}N^{1-\alpha}=C+\delta K+G,$$

together with a finite level of government spending G implies an upper bounded of the C/Y ratio as

$$\frac{C}{Y}=\left(1-\delta\frac{K}{Y}-\frac{G}{Y}\right)<\left(1-\frac{\alpha\beta\delta}{1-\beta\left(1-\delta\right)}-\frac{G}{Y}\right), \tag{83}$$

where the last inequality uses equation (82).

4. We know that under our parameter restrictions the level of labor is interior, $N\in\left(0,\overline{N}\right)$; hence, it must be true that the aggregate allocation is also interior: $\{C,K,Y\}\in(0,\infty)$.

5. Next, we show that $\{\mu,\lambda\}\in(0,\infty)$. Plugging equation (73) into the steady-state FOC (30) gives

$$\mu=\frac{M(\theta^*)D(\theta^*)^{\sigma-1}C^{-\sigma}[1+\lambda(1-\sigma)]}{1+(1-L(\theta^*)^{-1})(D(\theta^*)^{-1}-1)}, \tag{84}$$

where we have used Lemma 4 and the definition of $W(\theta^*)$, which implies $W(\theta^*)=M(\theta^*)D(\theta^*)^{\sigma-1}$. The above equation together with the steady-state FOC (29) imply

$$\lambda=\frac{Q-1}{1+Q(\sigma-1)}, \tag{85}$$

where

$$Q = \frac{MP_N M(\theta^*)D(\theta^*)^{\sigma-1}L(\theta^*)D(\theta^*)}{C^\sigma\left[L(\theta^*) - 1 + D(\theta^*)\right]} \in (0, \infty) .$$

So $\lambda > 0$ if and only if $Q > 1$ because the denominator, $1 + Q(\sigma - 1)$, is strictly positive for $Q > 1$ and $\sigma > 0$. To see this, suppose the denominator is negative, then $\sigma < 1 - 1/Q$, and then $\lambda > 0$ implies $Q < 1$; but $Q \in (0, 1)$ implies $\sigma = 1 - 1/Q < 0$, a contradiction. Therefore, we only need to ensure $Q > 1$.

Note that in the steady state the implementability condition (22) becomes

$$\frac{N}{Y} = \frac{C}{Y}C^{-\sigma}D(\theta^*)^{\sigma-1}\theta^*\left(L(\theta^*) - \frac{1 - D(\theta^*)}{\beta}\right),$$

and hence C^σ can be expressed as

$$C^\sigma = \left(\frac{C}{Y}\right)\left(\frac{Y}{N}\right)D(\theta^*)^{\sigma-1}\theta^*\left(L(\theta^*) - \frac{1 - D(\theta^*)}{\beta}\right). \tag{86}$$

By using equation (86), Lemma 4, and $MP_N = (1 - \alpha)\left(\frac{Y}{N}\right)$, we can express Q as:

$$\begin{aligned} Q &= \frac{(1-\alpha)L(\theta^*)D(\theta^*)}{\left(\frac{C}{Y}\right)\left(L(\theta^*) - \frac{1-D(\theta^*)}{\beta}\right)} \\ &> \frac{1-\alpha}{\left(1 - \frac{\alpha\beta\delta}{1-\beta(1-\delta)} - \frac{G}{Y}\right)}\frac{L(\theta^*)D(\theta^*)}{\left(L(\theta^*) - \frac{1-D(\theta^*)}{\beta}\right)}, \end{aligned}$$

where the last inequality uses equation (83). Hence, to ensure $Q > 1$, it suffices to show that

$$\frac{1-\alpha}{\left(1 - \frac{\alpha\beta\delta}{1-\beta(1-\delta)} - \frac{G}{Y}\right)}\frac{L(\theta^*)D(\theta^*)}{\left(L(\theta^*) - \frac{1-D(\theta^*)}{\beta}\right)} > 1.$$

It is easy to show that the first term on the left hand side is no less than 1 if β is sufficiently large— because

$$\lim_{\beta\to1}\frac{1-\alpha}{\left(1 - \frac{\alpha\beta\delta}{1-\beta(1-\delta)} - \frac{G}{Y}\right)} = \frac{1-\alpha}{1-\alpha-\frac{G}{Y}} \geq 1.$$

Hence, given a sufficiently large β, we need to ensure that the second term, $\frac{L(\theta^*)D(\theta^*)}{\left(L(\theta^*) - \frac{1-D(\theta^*)}{\beta}\right)} \geq$

1, which requires the following conditions:

$$\frac{1}{\beta} \geq L(\theta^*) > \frac{1 - D(\theta^*)}{\beta},$$

where the second inequality can be ensured if β is sufficiently close to 1, since $L(\theta^*) \geq 1$ and $D(\theta^*) \in (0,1)$. In addition, if δ is sufficiently small such that $MP_K \geq \delta$, the first inequality can be ensured according to equation (81), which implies

$$\frac{1}{\beta} = L(\theta^*)\left(1 + MP_K - \delta\right) \geq L(\theta^*)$$

if $MP_k - \delta \geq 0$. As a result, $Q \in (0, \infty)$, which implies that $\lambda \in (0, \infty)$ by (85) and hence $\mu \in (0, \infty)$ by (84).

6. Finally, we verify that the required conditions for (the sign of) $\widehat{\varepsilon}(\theta^*)$ stated above are all automatically met once the condition for $\lambda > 0$ is satisfied. Using equation (75) and (64) to substitute out $\widehat{Z}(\theta^*)$ and $\widehat{W}(\theta^*)$ in the definition of $\widehat{\varepsilon}(\theta^*)$, which is listed in equation (76), gives

$$\begin{aligned}
\widehat{\varepsilon}(\theta^*) &= \frac{\widehat{W}(\theta^*) - \phi C^{\sigma} \widehat{Z}(\theta^*)}{\widehat{J}(\theta_t^*)} = \frac{1}{\widehat{J}(\theta_t^*)}\left(\frac{1}{\sigma}\frac{M(\theta_t^*)}{D(\theta_t^*)\,\theta_t^*} - \frac{1}{\sigma} - \phi\frac{C^{\sigma}}{\sigma\theta^* D^{1+\sigma}}\right) \\
&= \frac{1}{\widehat{J}(\theta_t^*)\sigma}\left(\frac{M(\theta_t^*)}{D(\theta_t^*)\,\theta_t^*} - 1 - \frac{M(\theta^*)}{\theta^* D(\theta^*)}\frac{D(\theta^*)^{-1}\left(1 + \lambda(1-\sigma)\right)}{\frac{\mu}{\phi} + \frac{1 - D(\theta^*)}{D(\theta^*)}}\right),
\end{aligned}$$

where the last equality uses equation (79) to substitute out $\frac{C^{\sigma}}{D(\theta^*)^{\sigma}}$. Simplifying under Lemma 4 and equation (80), the above equation becomes

$$\begin{aligned}
\widehat{\varepsilon}(\theta^*) &= \frac{1}{\widehat{J}(\theta_t^*)\sigma}\left(\frac{\left(L(\theta_t^*) - 1 + D(\theta_t^*)\right)}{D(\theta_t^*)} - 1 - \frac{\left(L(\theta_t^*) - 1 + D(\theta_t^*)\right)}{D(\theta^*)}\frac{D(\theta^*)^{-1}\left(1 + \lambda(1-\sigma)\right)}{\frac{L(\theta^*)}{L(\theta^*)-1} + \frac{1 - D(\theta^*)}{D(\theta^*)}}\right) \\
&= \frac{1}{\widehat{J}(\theta_t^*)\sigma}\left(\frac{L(\theta_t^*) - 1}{D(\theta_t^*)} - \frac{L(\theta_t^*) - 1 + D(\theta_t^*)}{D(\theta^*)}\frac{\left(L(\theta^*) - 1\right)\left(1 + \lambda(1-\sigma)\right)}{L(\theta^*) - 1 + D(\theta^*)}\right) \\
&= \frac{1}{\widehat{J}(\theta_t^*)}\frac{1}{\sigma}\frac{L(\theta_t^*) - 1}{D(\theta_t^*)}\lambda(\sigma - 1),
\end{aligned}$$

where $\lambda(\sigma - 1) \gtreqless 0$ if and only if $\sigma \gtreqless 1$ (given that $\lambda > 0$). Also note that $\widehat{J}(\theta^*) > 0$ when $\sigma \leq 1$, and $\widehat{J}(\theta^*) < 0$ only if $\sigma > 1$. Hence, under the parameter conditions for

$\lambda > 0$, it must be true that (i) $\widehat{\varepsilon}(\theta^*) \leq 0$ if and only if $\sigma \leq 1$, which are consistent with the stated conditions for $\widehat{\varepsilon}(\theta^*)$ under cases (1a) and (1b) discussed above; (ii) $\widehat{\varepsilon}(\theta^*) > 0$ if and only if $\sigma > 1$ and $J(\theta^*) > 0$, which are consistent with the stated conditions for $\widehat{\varepsilon}(\theta^*)$ under the first two subcases in case (1c) discussed above; and (iii) $\widehat{\varepsilon}(\theta^*) < 0$ if and only if $\sigma > 1$ and $J(\theta^*) < 0$, which are also consistent with the stated conditions for $\widehat{\varepsilon}(\theta^*)$ under the third subcase in case (1c) discussed above.

We further show that the following properties of the Ramsey steady state with a binding government debt limit hold true regardless of σ:

1. IAE fails. Equation (81) implies that $L(\theta^*) > 1$ and hence θ^* must be interior.

2. AAE fails. Equation (28) suggests that the MGR does not hold in the steady state.

3. Despite the failure of the MGR, the steady-state capital tax must be zero by Proposition 4, since $\mu/(\mu - \phi) = L(\theta^*)$ regardless of the value of $\sigma \in (0, \infty)$. The steady-state labor tax τ_n is given by

$$\tau_n = 1 - \frac{1}{L(\theta^*)\theta^* D(\theta^*)^\sigma} \frac{C^\sigma}{MP_N}.$$

Finally, the condition (48) ensures $n > 0$ and the condition (49) ensures $n < \overline{N}$ if $\overline{N} > \theta^* L(\theta^*)$. Hence, $n \in \left(0, \overline{N}\right)$ is guaranteed.