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The Real Term Premium in a Stationary Economy with Segmented Asset Markets

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Abstract

This paper proposes an equilibrium model to explain the positive and sizable term premia observed in the data. We introduce a slow mean-reverting process of consumption growth and a segmented asset market mechanism with heterogeneous trading technology to otherwise a standard heterogeneous agent general equilibrium model. First, a slow mean-reverting consumption growth process implies that the expected consumption growth rate is only slightly countercyclical and the process can exhibit a near zero first-order autocorrelation as seen in the data. The very small countercyclicality of the expected consumption growth rate suggests that the long term bonds are risky and hence the term premia are positive. Second, the segmented asset market mechanism amplifies the size and the magnitude of term premia since the aggregate risk is concentrated into a small fraction of marginal traders who demand high risk premia. For sensitivity analysis, the role of each assumption is further investigated by taking each factor out one by one.

JEL codes: G11, G12, E30.

Keywords: Limited Participation, Term Premia, Portfolio Heterogeneity, Household Finance.

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1 Introduction

The positive and sizable term premia have been observed in the data and they have been found hard to reconcile with a standard structural macroeconomic model. Backus, Gregory, and Zin (1989) demonstrates the failure of a standard model in accounting for the sign and the magnitude of real bond risk premia. Campbell (1986), Donaldson, Johnsen, and Mehra (1990), and den Haan (1995) also experience the same difficulty with standard macroeconomic models as well.[1] Even with the failures and difficulties in standard equiibrium models, understanding the fundamental mechanism of resulting positive and sizable term premia can not be over-emphasized. For macroeconomists, the disconnection between the observed term premia in data and the prediction from a standard structural macroeconomic model is often referred as "term premium puzzle". The issue is also important to central bankers since they often use term premia to evaluate the real and nominal economic activities and consequences of monetary policy. Moreover, it is the first order of importance for investors to understand the term premia in order to hedge against the interest rate risks.

The standard macroeconomic model where the pricing kernel or the stochastic discount factor is derived from utility maximization problem generally had great difficulty in matching the slope and the level of term structure. Campbell (1986) shows that the term premia depends on the nature of consumption processes. If the consumption growth process is positively auto-correlated, then the expected future growth rate falls and bond price rises in recession. The long-term bond then becomes a good hedge and hence the term premium is negative. On the other hand, if consumption growth rate is negatively auto-correlated, then the model predicts the positive term premium since the long-term bond becomes risky because of its procyclical pricing. This intuition together with a nearly zero auto-correlation of consumption growth, i.e. random walk in an empirical studies, implies that the term premium should be close to zero with the pricing kernel derived in a standard macroeconomic model.

In addition, it is also well known that the pricing kernel of standard model, which relies purely on the expected consumption growth rate, is not volatile enough to deliver a high market price of risk. Therefore, the standard model not only fails to match the sign of term premium but also falls short to generate the right magnitude of term premium.

In this paper, we assume that aggregate consumption process is a trend stationary with a long memory process, which shows a nearly zero but slightly negative auto-correlation of consumption process. This consumption process alone generates the positive term premia but with very small magnitude. This process is not easy to be statistically distinguished from the difference-stationary process such as the random walk process. This assumption is motivated by Christiano and Eichen-

[1]See also Grkaynak and Wright (2012) for the review on the issues of term premia

baum (1990) that argues there exits no clear statistical evidence to support either a trend stationary or difference-stationary process of the aggregate consumption. More specifically, we consider a slow trend reverting consumption process in our model economy, and hence the level of consumption can be well above or below its long run trend for an extended period of time. With this process, when a bad shock is realized, then the expected growth rate of consumption is slightly higher due to its slow meam-reverting property but the expected growth rate will not be very high because of the long memory property. Therefore, the expected growth rate of consumption is only sightly countercyclical and the autocorrelation of consumption growth between two consecutive periods could be very close to zero but slightly negative at the level (only -0.02 in our calibrated model), which is consistent with our observation from data.

We also have a segmented market mechanism in our model, following the work by Chien, Cole, and Lustig (2011). The segmented market mechanism features a large fraction of households who do not participate the equity market and hence they do not bear any aggregate risk. There is however a small fraction of households who participates the equity market and bears a great amount of aggregate risks. In this paper, high market price of risk results from the concentration of a significant amount of risks into the small fraction of total households. In equilibrium, those households who take a lot of aggregate risks, demand the risk compensation highly and the resulting high risk premia are obtained not only in equities but also reflected onto the long term bonds. Therefore, a segmented market mechanism results in the higher aggregate price of risk and hence amplifies the size and the magnitude of term premia.

Our calibrated model considers not only the segmented asset market mechanism but also the asymmetric bond positions of the U.S. households. The data on the U.S. households shows that there is a large fraction of population who carries long-term mortgage loans but saves in short term risk-free assets, such as checking or saving accounts. In other words, these households essentially borrow in the long term bond by using housing as a collateral and save in a short term bond. In our calibrated model, we also evaluate the extent to which this asymmetric bond positions of households matters to term premia quantitatively.

The assumptions in our model are built on a solid support from empirical evidences. The first assumption of mean-reverting consumption growth process comes naturally in macroeconomics literature. The growth of aggregate variables, such as output or consumption, are often decomposed into trend components and cyclical components (business cycles). The second assumption on the segmented market mechanism is firmly grounded in the empirical evidences on household finance studies. These evidences show that most households do not purchase most of the assets available to them.[2] In fact, the composition of household asset holdings varies greatly across households even in a developed country like the United States. Only 50% of U.S. households participate in the equity

[2]See Guiso and Sodini (2012)

market, according to the 2010 Survey of Consumer Finance (SCF hereafter) data. Moreover, even among the participants in the equity market, many investors still hold low risk portfolios and do not adjust their portfolios frequently.[3] On the other hand, there is a small fraction of households who actively adjust their portfolio and earns a high return by taking more aggregate risks. The SCF data also show that a large fraction of households carries mortgage loans and saves in short term safe assets. These households effectively have a long position in short term bond and short position in the long term bond. As also shown in the the data, there is a relatively small fraction of households who is wealthier tends to hold higher fraction of long term bons in their portfolios.

There are only handful structural models in the literature that are able to deliver an average upward slopping nominal and/or real yield curve. Many of them modify the household preference to various forms in the standard macroeconomic model. Piazzesi and Schneider (2007) demonstrates that the nominal yield curve can be upward slopping even with flat or downward sloping real yield curve since a low frequency negative correlation between consumption growth and inflation rate causes an inflation risk. They assume the recursive preference and hence the agents are very willing to substitute consumptions over time even though they are risk averse enough. The recursive preference plays a critical role for the low frequency correlation to matter to the current price. Bansal and Shaliastovich (2013) also generates a positive nominal term premium with inflation risks and recursive preference. Rudebusch and Swanson (2012) further extends the endowment economy model to a production economy general equilibrium model. By introducing inflation ambiguity into a representative agent model, Ulrich (2013) explains the upward slopping nominal yield curve with log utility function. Our work is complimentary to the existing papers discussed above since we focus on the real term premia rather than the nominal one. Wachter (2006) utilizes a habit persistent model to explain both the positive real and nominal bond premia. In order to maintain the consumption level, the investors tend to sell long term bond during recession and vice versa during expansion. Namely, the demand of long bond is procyclical, which makes the bond price procyclical and hence the long term bond becomes a risky asset. Rudebusch and Swanson (2012) finds the habit formation mechanism in Wachter (2006) fails to generate a sizeable term premia without distorting the behavior of other macroeconomics variables.

Our benchmark model generates a high and volatile equity premium, 7.26% in mean with 15.63% in standard deviation as well as a low and stable risk-free return, 0.95% in mean with 1.45% in standard deviation; these values are quite close to the estimates in the asset-pricing literature. Most importantly, our quantitative result also predicts a high real term premium, 2.36% on average with 5.26% standard deviation. This paper delivers a nice term premium result with a reasonable calibration in the risk aversion rate at 4. For the sensitivity analysis, we further investigate the role of our assumptions by taking each factor out one by one. We find that the

[3] In this paper, the "household", "trader" and "investor" are used interchangeably

assumption on segmented market with heterogeneous trading technology increases the magnitudes of the price of aggregate risk and term premium as well and the assumption on trend-reverting process of the output growth rate generates the positive term premium.

Our main contribution to the literature is to provide a simple and intuitive story that can reconcile the puzzling disconnection between asset prices, equity and term premia in particular, and aggregate macroeconomic variables. The model in this paper integrates the empirical facts of heterogeneous portfolios across households found in the household finance literature and a mean reverting aggregate consumption process in the macroeconomic literature into an explanation for real term premia puzzle. Our model successfully matches both the sign and magnitude of real term premia. Specifically, we demonstrate the importance of household portfolio heterogeneity documented in macro-finance literature, while the majority of asset pricing models rely on a representative agent framework with modification of preferences.

2 The Model

We consider an endowment economy in which households sequentially trade assets and consume. There are two distinguishing features of our model from the standard one. First, our endowment (consumption) growth follows a slow mean-reverting process. After a realization of bad endowment shock, the expected consumption growth rate edges up a bit because of the trend reverting property and the impacts of shocks on the next period consumption growth growth is still roughly unchanged, which makes the auto-correlation of consumption growth process slightly negative while it is very close to zero. Hence, our shock process is consistent with the empirical fact that the consumption growth is well approximated by the random walk.

The second key feature of our model exhibits ex-ante heterogeneity in trading technologies. The trading technology is modelled on the menu of assets, specifically by imposing the restriction on the portfolios available that a household can trade and hold. These restrictions are imposed exogenously. The goal of these restrictions is to capture the observed portfolio behavior of most households. In our calibrated model, this form of ex-ante heterogeneity delivers a high market price of risk and hence helps to match the sizable level of bond risk premia.

2.1 Environment

There is a unit measure of households who are subject to both aggregate endowment growth risks and idiosyncratic income shocks. Households are ex ante identical, except for the trading technology that they are endowed with. Ex post, these households differ in terms of their idiosyncratic income shock realizations. All of the households face the same stochastic process for idiosyncratic income shocks, and all households start with the same present value of financial wealth.

In the model, time is discrete, infinite, and indexed by $t = 0, 1, 2, ...$ The first period, $t = 0$, is a planning period in which all trading takes place. We assume a constant average growth of endowment process while there is a transitory shock that makes the actual level of aggregate consumption deviated from its long term trend. More specifically, let m_t be the percentage deviation of aggregate endowment from the growth trend. Then, the total endowment in period t, denoted by Y_t, is

$$\ln Y_t = t \ln \overline{g} + m_t,$$

where $\overline{g}$ is the average growth rate of endowment. The output growth process is therefore affected by the evolution of m, which is assumed to follow an AR(1) process:

$$m_{t+1} = \rho m_t + \varepsilon_{t+1},\ \varepsilon_{t+1} \sim N(0, \sigma_\varepsilon^2).$$

With this specification of the endowment shock process, the growth rate of output, denoted by $g_{t+1} \equiv \frac{Y_{t+1}}{Y_t}$ is therefore given by

$$\begin{aligned} \ln \frac{Y_{t+1}}{Y_t} &\equiv \ln g_{t+1} \\ &= \ln \bar{g} + (\rho - 1)\, m_t + \varepsilon_{t+1}. \end{aligned} \tag{1}$$

If ρ is one, then the endowment process follows a random walk with draft, a difference-stationary process. If ρ is less than one but close to one, then the endowment process features trend stationary while with a slow rate of trend reverting (a long memory property). As mentioned in the Introduction, there is no clear evidence in favor of either trend stationary or difference stationary for macroeconomic variables, like consumption or output. Our model takes the view of a tend stationary endowment process with long memory. Hence the value of ρ is set to be 0.95 in the calibration.

Let z^t denote the history of aggregate states up to period t and hence the aggregate endowment is period t is denoted by $Y_t(z^t)$. In addition, aggregate endowment each period is divided into two parts: diversifiable income and nondiversifiable income. Claims to the diversifiable income can be traded in financial markets while claims to nondiversifiable income cannot. We assume a constant share of nondiversifiable income and the share is denoted by $\gamma \in (0, 1)$. The nondiversifiable component is subject to idiosyncratic stochastic shocks, which is denoted by η_t. The non-diversifiable incomes of a household are given by $\gamma Y_t(z^t)\eta_t$.

Similarly, let η^t denote the history of idiosyncratic shocks up to period t. In addition, we use $\pi(z^t, \eta^t)$ to denote the unconditional probability of that state (z^t, η^t) being realized. The idiosyncratic shock events are governed by first order Markov process and their probabilities are assumed to be independent between z shocks and η shocks:

$$\pi(z^{t+1}, \eta^{t+1} | z^t, \eta^t) = \pi(z_{t+1} | z^t) \pi(\eta_{t+1} | \eta^t)$$

Since we can appeal to a law of large number, $\pi(\eta^t)$ also denotes the fraction of agents in state z^t that have drawn a history η^t. We introduce some additional notation: $z^{t+1} \succ z^t$ or $\eta^{t+1} \succ \eta^t$ means that the left hand side node is a successor node to the right hand side node. We denote by $\{z^\tau \succ z^t\}$ the set of successor aggregate histories for z^t including those many periods in the future; ditto for $\{\eta^\tau \succ \eta^t\}$. When we use $\succeq$, we include the current nodes z^t or η^t in the set.

All households live for infinite periods and rank a stream of consumptions according to the following criterion

$$U(\{c\}) = \sum_{t \geq 1, (z^t, \eta^t)}^{\infty} \beta^t \frac{1}{1-\alpha} c_t(z^t, \eta^t)^{1-\alpha} \pi(z^t, \eta^t) \tag{2}$$

where α denotes the coefficient of relative risk aversion, β is the time discount factor, and $c_t(z^t, \eta^t)$ denotes the household's consumption in state (z^t, η^t).

In this economy, there are four type of assets are available: state-contingent claims on aggregate shock, a long term bond (consol) with a constant stream of payments, risky equities and one period risk-free bonds. Note that the market is incomplete in our environment since there is no state-contingent claims available for idiosyncratic shocks. Equity is assumed to be leveraged aggregate output process, with its dividend growth determined by the following equation

$$\Delta \ln D_{t+1} = E_t(\Delta \ln Y_{t+1}) + \phi \left[\Delta \ln Y_{t+1} - E_t(\Delta \ln Y_{t+1}) \right],$$

where ϕ is the leverage ratio, which is assumed constant over time. Finally, we denote the value of total equity by $V_t(z^t)$. The gross returns of leveraged equity, or $R^e_{t,t-1}(z^t)$, are given by

$$R^e_{t,t-1}(z^t) = \frac{D_t(z^t) + V_t(z^t)}{V_{t-1}(z^{t-1})} \tag{3}$$

2.2 Heterogeneity in Trading Technologies

To match the size of term premium, we introduce the mechanism of segmented market, in particular the portfolio heterogeneity in household levels. As mentioned in the introduction, the heterogeneity in portfolio choices is widely supported by the data. As we shall demonstrate latter, the concentration of a large portion of the aggregate risks into a relatively small fraction of households amplifies the price of risk in the calibrated model. Without such a mechanism, the model fails short to match the size of term premium quantitatively. To capture such a portfolio heterogeneity, we adopt the approach by Chien, Cole, and Lustig (2011), which exogenously imposes different restrictions on investors' portfolio choices. These restrictions apply to the menu of assets that these households can trade as well as the composition of households' portfolios.

There are two classes of investors in terms of their asset trading technologies. The first class of investors faces no restrictions on portfolio choices and hence the menu of tradable assets. Specifically, these investors are capable to trade a complete set of contingent claims on the aggregate endowment. We call these investors Mertonian traders. They optimally adjust their portfolio choices in response to changes in the investment opportunity set. Therefore, they act as market arbitrageurs and price the aggregate risk in our model.

The second class of investors faces restrictions on their portfolios and are called non-Mertonian traders. Specifically, their portfolio composition is restricted to be constant over time. There are two types of non-Mertonian traders as follows: The first type are non-Mertonian equity investors, who can trade equities, risk-free bonds and long term bonds but the state-contingent claims on aggregate shocks. The other type are called non-participants, who do not hold equity but only invest in risk-free bonds and console bonds. Even though the portfolio composition of non-Mertonian traders is exogenously given, they can still choose how much to save and consume.

Non-Mertonian investors deviate from the optimal portfolio choices in two dimensions. First, they cannot change the share of equities, long term bonds and short term bonds in their portfolios in response to changes in the market price of risk, which indicates missed market timing. Second, their portfolio share in each asset might deviate from the optimal one on average, implying that their average exposure to aggregate risk might not be optimal. We denote the measure of different types of households by μ_j, where $j \in \{me, et, np\}$ represents Mertonian investors, non-Mertonian equity investors, and nonparticipants, respectively.

2.3 The Household's Problem

Budget Constraints of Mertonian Traders Consider a Mertonian trader entering the period with a net financial wealth $a_t(z^t, \eta^{t-1})$ given the event history (z^t, η^{t-1}). Note that the net financial wealth is not spanned by the realization of idiosyncratic shocks, η_t, since there are no contingent claims on idiosyncratic shocks. At the end of the period, Mertonian traders buy shares of equities $s_t(z^t, \eta^t)$, one period risk free bonds $b_t(z^t, \eta^t)$, long term consol bond $b_t^c(z^t, \eta^t)$ and state contingent claims, $\widehat{a}_t(z^t, \eta^{t-1})$ in financial markets and consumption $c_t(z^t, \eta^t)$ in the goods markets subject to this one-period budget constraint:

$$\begin{aligned} & s_t(z^t, \eta^t)V_t(z^t) + b_t(z^t, \eta^t) + b_t^c(z^t, \eta^t) + \sum_{z_{t+1}} Q(z^{t+1}|z^t)\widehat{a}_{t+1}(z^{t+1}, \eta^t) + c_t(z^t, \eta^t) \\ \leq \;& a_t(z^t, \eta^{t-1}) + \gamma Y_t(z^t)\eta_t, \text{ for all } z^t, \eta^t. \end{aligned} \tag{4}$$

where $Q(z^{t+1}|z^t)$ denotes the state contingent price of a unit contingent claim to consumption good in aggregate state z^{t+1} acquired in aggregate state z^t. The agent's net financial wealth, $a_t(z^t, \eta^{t-1})$, in state (z^t, η^t), is given by the payoffs from her return of portfolio formed in the last period:

$$\begin{aligned} a_t(z^t, \eta^{t-1}) \;=\; & s_{t-1}(z^{t-1}, \eta^{t-1})\left[D_t(z^t) + V_t(z^t)\right] + R^f_{t,t-1}(z^{t-1})b_{t-1}(z^{t-1}, \eta^{t-1}) \\ & + R^c_{t,t-1}(z^t)b^c_{t-1}(z^{t-1}, \eta^{t-1}) + \widehat{a}_t(z^t, \eta^{t-1}). \end{aligned} \tag{5}$$

where $R^c_{t,t-1}(z^t)$ and $R^f_{t,t-1}(z^{t-1})$ denote the return of long term consol bond and one period risk free bond at period t, respectively. Note that the total equity share of this economy, $s_t(z^t, \eta^t)$, is normalized to be 1.

Budget Constraints of Non-Mertonian Traders The non-Mertonian investors have no access to state-contingent claims on aggregate shock and are restricted to fixed portfolio weights among the equity, short term risk free bond and long term consol bond. At the end of period t, the household buys equity shares, one period risk-free bond and long term consol bond, subject to a fixed target portfolio equity share and long bond share, denoted by $\overline{\omega}^e$ and $\overline{\omega}^c$, respectively. As a result, in addition to the equations (4) and (5), their constraints also include a portfolio restriction:

$$\begin{aligned} \overline{\omega}^e &= \frac{s_t(z^t, \eta^t)V_t(z^t)}{s_t(z^t, \eta^t)V_t(z^t) + b_t(z^t, \eta^t) + b^c_t(z^t, \eta^t)} \\ \overline{\omega}^c &= \frac{b^c_t(z^t, \eta^t)}{s_t(z^t, \eta^t)V_t(z^t) + b_t(z^t, \eta^t) + b^c_t(z^t, \eta^t)} \end{aligned}$$

and no access to state contingent claims

$$\widehat{a}_t(z^t, \eta^{t-1}) = 0 \text{ for all } z^t \text{ and } \eta^{t-1}.$$

The portfolio share of short term bond is therefore $1 - \overline{\omega}^e - \overline{\omega}^c$. Alternatively, we can simplify the budget constraint of non-mertonian traders as follows:

$$\widehat{s}_t(z^t, \eta^t) + c(z^t, \eta^t) \leq R^p_{t,t-1}(z^t)\widehat{s}_{t-1}(z^{t-1}, \eta^{t-1}) + \gamma Y_t(z^t)\eta_t,$$

where $\widehat{s}_t$ denotes for the asset holdings in the end of period t. $R^p_{t,t-1}(z^t)$ represent for the gross return on the fixed portfolio imposed into the non-Mertonian traders and is given by

$$R^p_{t,t-1}(z^t, \eta^t) = \overline{\omega}^e R^e_{t,t-1}(z^t) + \overline{\omega}^c R^c_{t,t-1}(z^t) + (1 - \overline{\omega}^e - \overline{\omega}^c) R^f_{t,t-1}(z^t).$$

In the case of nonparticipants, $\overline{\omega}^e$ is zero.

Finally, all households are subject to nonnegative net wealth constraints, given by $a_t(z^t, \eta^{t-1}) \geq 0$ for Mertonian and $\widehat{s}_t(z^t, \eta^t) \geq 0$ for non-Mertonian. The details of the household problem and its associated optimal conditions are provided in the Appendix A.1.

2.4 Competitive Equilibrium

A competitive equilibrium for this economy is defined in the standard way. It consists of a consumption allocation, allocations of state contingent claims, one period risk-free bonds, long term consol bonds, and equity choices, and a list of prices such that (i) given these prices, households' asset and consumption choices maximize their expected utility subject to the budget constraints, the solvency constraints, and the constraints on portfolio choices, and (ii) all asset markets clear.

3 Quantitative Results

This section evaluates the extent to which our model can account for average real term premia in the United State data in three steps. The following subsection explains how we calibrate idiosyncratic shocks and aggregate shocks. Next we describe the trader's pool in the benchmark case, which is chosen to match several key features of data in asset pricing as well as household portfolio behaviors. Finally, we report our benchmark asset-pricing quantitative results, especially the size of positive real term premia, in subsection 3.4.

3.1 Calibration

The calibration of aggregate shocks is critical to our results. There are two key observations of consumption growth data that guide our calibration. First, empirical studies has shown that the growth rate of consumption is hard to predict and is well approximated by a random walk in the short run. Second, the level of aggregate consumption exhibit a trend reverting property, which suggest that the expected long term growth rate should be higher (lower) than average of the current level of consumption when it is below (above) the trend. Based on these two observation, the model setup of the aggregate process has a constant growth trend and an iid innovation shock that makes the realization of output deviate from its trend.

$$\ln Y_t = t \ln \overline{g} + m_t,$$

where $\overline{g}$ is the average growth rate of endowment and the deviation form the trend is captured by a variable m, which assumed to follow an AR(1) process:

$$m_{t+1} = \rho m_t + \varepsilon_{t+1},\ \varepsilon_{t+1} \sim N(0, \sigma_\varepsilon^2).$$

We use a two state of Markov process to approximate the iid innovation ε_t. More specifically, since expansions occur more often than recessions, the probability of good innovation shock is set to is set to 27.4% as in Alvarez and Jermann (2001). However, the expected endowment growth rate in each period will depends on how far the current consumption level is deviated from its trend, which depends on the whole past history of innovation shocks. In computation, we therefore has to keep track of one extra state variable, m, in order to compute the conditional expected growth rate.

Our model operates at annual frequency. The average aggregate consumption growth rate $\overline{g}$ is set to be 1.8 percent with standard deviation 3.15 percent. Given the consumption growth is well approximation by random walk, the persistency of m has to be high. We set $\rho = 0.95$, which makes the consumption growth autocorrelation sufficiently close to zero at level of -0.02.

We also consider a two-state first-order Markov chain for idiosyncratic shocks. The first state is low and the second state is high. Following Alvarez and Jermann (2001) and Storesletten, Telmer, and Yaron (2004), we calibrate this shock process by two moments: the standard deviation of idiosyncratic shocks and the first-order autocorrelation of the shocks, except we eliminate the countercyclical variation in idiosyncratic risks. The Markov process for the log of the non-diversified income share, $\ln \eta$, has a standard deviation of 0.71, and its autocorrelation is 0.89. The transition probability is denoted by

$$\pi(\eta'|\eta) = \begin{bmatrix} 0.945 & 0.055 \\ 0.055 & 0.945 \end{bmatrix}$$

The two states of idiosyncratic shocks, of which the mean is normalized to 1, are $\eta_L = 0.3894$ and $\eta_H = 1.6106$.

All households have the same CRRA preference. In our calibration, there are strong incentive for household to save because of the idiosyncratic shock in an incomplete market environment, which causes the risk free rate lower than the reciprocal of preference discount factor β despite in a growth economy. As a result, we set the time discount factor $\beta = 0.95$ to match the low risk-free rate in our benchmark model. The risk-aversion rate α is set to 4 to produce a high risk premium in our benchmark calibration.

Following Mendoza, Quadrini, and Rios-Rull (2009), the fraction of nondiversifiable output is set to 88.75%. As shown in Section 2, equity in our model is simply a leveraged claim to diversifiable income process. Following Abel (1999) and Bansal and Yaron (2004), the leverage ratio parameter is set to 3.

3.2 The Composition of Traders

The 50% of US households are set as stock market non-participants, as shown in the 2010 SCF data. The remaining 50% of populations do hold equities and we need to divided them into non-Mertonian equity traders and Mertonian traders according to our model. To match the high risk premium, a small fraction of Mertonian traders must absorb a large amount of aggregate risks. We therefore set the fraction of Mertonian traders to 5% for our benchmark economy. The remaining fraction, 45%, of households are classified as non-Mertonian equity traders who can own fix portfolio shares in short risk free bonds, long term risky bonds and equities.

In addition to the equity market participation rate, the portfolio shares of non-Mertonian traders are also important parameters. Again, we use the 2010 SCF data to calibrate the portfolio share of non-Mertonian equity traders and non-participants in our model, which account for 45% and 50% of the population, respectively.

For the portfolio choice of non-Mertonian equity traders, we first need to distinguish them form the Mertonian traders in the data pool. We first sort the 50% of households holding equities in the data by their equity position and compute the average equity share excluding the top 5% of equity holders. The average computed equity share is 21.1%, which we use as the equity share of non-Mertonian equity traders in the benchmark case. This calibration reflects the observations both from the data and from our model that more sophisticated households tend to hold larger amounts of equities.

3.3 Real Yield Curve in the Data

The average real treasury yield for maturity 5 year, 7 year and 10 years are listed in table I for different sample period. From 2003 to 2007, the average 5 year, 7 year and 10 year real treasury yields are 1.646%, 1.871% and 2.061%. Our model is not far from that.

The real yield data after 2009 is descending as sample period prolonged. This is because the short term real rate drops to negative. Obviously, our results miss that. We think that is maybe due to a permanent change in the average consumption growth process.

[Table 1 about here.]

3.4 Benchmark Results

The benchmark asset-pricing results are shown in the column labeled as "Benchmark" case in Table II. In the table, we report the market price of risk, $\sigma(Q)/E(Q)$, the conditional standard deviation of market price of risk, $std(\sigma(Q)/E(Q))$, equity premium $E(R^e - R^f)$, the standard deviation of equity excess return $\sigma(R^e - R^f)$, the Sharpe ratio on equity return, the long term consol bond premium $E(R^c - R^f)$, the standard deviation of long bond excess return $\sigma(R^c - R^f)$, the Sharpe ratio on the long bond, the the average risk-free rate $E(R^f)$, and the standard deviation of the risk-free rate $\sigma(R^f)$.

Our benchmark economy produces a high and volatile market price of risk as well as a low and stable risk-free rate. In the benchmark case of Table II, the market price of risk is high, 0.475, and volatile, with standard deviation 9.766%. The equity premium reaches 7.262% and the Sharpe ratio on equity is 0.465. The average risk-free rate is low at 0.949% and its volatility is only 1.449%. Hence, our calibrated model is capable of producing reasonable asset-pricing results. In our model, the success of matching high risk premiums and low risk-free rates relies on two key frictions. The first friction is the incomplete market with respect to idiosyncratic risk. It is well known that incomplete market models can produce reasonable risk-free rates in a growing economy. The second friction, which is limited participation combined with a relatively small fraction of Mertonian traders, produces a high equity premium by concentrating the aggregate risk on Mertonian traders.

As explained earlier, the long bond is risky because its price tends to fall during recession, which is simply resulted form a higher expected growth after recessions. In our benchmark case, the average excess return for long term consol bond reaches 2.361% with a Sharpe ratio 0.449. To illustrate the upward slopping real yield curve, Figure 1 plots the yield curve and real term premia of zero coupon bonds in the benchmark case, respectively.

[Figure 1 about here.]

Our results show the mechanism of our model is able to match the sign as well as the size of real term premium observed in the data by imposing the trend-reverting consumption process and heterogenous portfolio choices into an otherwise standard macroeconomic model. In the next section, we explore the relative importance of these assumptions.

[Table 2 about here.]

4 Trend Reverting versus Random Walk

The trend reverting endowment process is important to our results. If the endowment growth process is truly follows a random walk process, then the return of long bond is not necessary fall during the recession and hence the standard model might fail to generate even a positive bond premium. In this subsection, we demonstrate this point analytically in the representative agent economy.

Given the assumption of our shock process, the following lemma describes the expression for the term premia as well as its property in a representative agent economy.

Lemma 1. *The unconditional expectation of term premium for a k period zero coupon bond is*

$$E\left[r_t^k - r_t^1\right] = \left[1 - \frac{1}{k}\frac{1-\rho^{2k}}{1-\rho^2}\right]\frac{\alpha^2\sigma_\varepsilon^2}{2}. \tag{6}$$

In addition, the term premium, $E\left[r_t^k - r_t^1\right]$, is increasing in k given $0 < \rho < 1$.

Proof. Please refer to Appendix A.2 □

With $0 < \rho < 1$ in the representative agent economy, Lemma 1 not only shows a positive term premia but also show that the term premium is increasing in k, an indication of an upward slopping real yield curve. However, if $\rho = 1$, the random walk case, the average term premium shown in Equation (6) becomes zero and independent of k. The independence implies a flat yield curve.

This result is not surprising in the sense that the auto-correlation of consumption growth rate is negative when $0 < \rho < 1$ and becomes zero when $\rho = 1$. This can be seen clearly from the fact that $cov\left(\ln g_{t+1}, \ln g_t\right) = -\left(\frac{1-\rho}{1+\rho}\right)\sigma^2 < 0$ if $\rho < 1$. However, if ρ is sufficiently close to 1, then the consumption growth correlation is close to zero, which is not far from what we observe from the data.

The quantitative result of a representative agent economy with trend-reverting process are reported in the fifth column of Table II, labeled as "RA" case. The term premium is positive as shown in this subsection. However, the term premium is not sizable, only 0.487%. This is because of the absent of high market price of risk, which drops significantly to only 0.133 from 0.475 of our benchmark economy.

5 Inspection the Mechanism of Quantitative Results

In our benchmark economy, there are two features that contribute to the results of sizable real term premia quantitatively: the heterogenous trading technologies and the mortgage effects. In this section, we decompose the contribution for each feature by removing each of them from our benchmark economy.

5.1 No Mortgage Effects

Part of the real bond risk premia could be come from the asymmetric bond portfolio holdings across households, which is motivated by the heterogeneous amounts of mortgage held across households. Here, we simply shout down this channel by assuming that both non-participants and non-Mertonian equity holders do not have a position on long term bond. The third column of Table II, labeled as "No Mortgage" case, reports the results. We find that this asymmetric bond portfolio channel has minor effects on the market price of risk and equity risk premia. The market price of risk drops slightly from 0.475 to 0.464 as well as standard deviation of market price of risk from 9.766% to 9.494%. The equity premium decreases 32.5 basis points to 6.937%. The risk free rate is up to 1.206% from 0.949% and the standard deviation of risk free rate falls to 1.296% from 1.449%.

As for the impact on real term premia, the long term bond premia drops by 24.8 basis points and sharpe ratio falls slightly to 0.440. This exercise revels that the asymmetric portfolios in terms of bond maturity only plays a minor role on lifting up bond risk premia.

5.2 No Heterogenous Trading Technologies

In this subsection, we remove the heterogenous trading technologies assumed in the model but keep the trend-reverting process of the endowment growth rates. All households are now Mertonian traders and face no restriction on their portfolio choices. The results are reported in the second column of Table II, labeled as "No HTT" case. Once the force of heterogeneous risk loading is absent, the aggregate risk is no longer being concentrated into a small fraction of households. The model acts similar to a Bewley type of model and exhibits low risk premia. Quantitatively, the market price of risk drops significantly to 0.135 from 0.475. The equity premium is only 1.699% and the term premium for 10-year bond is down significantly to 0.575%.

This sensitive analysis suggests that a high market price of risk is essential to our results. The large fraction of none and low equity market participants not only helps to match the high and volatile equity risk premia but also goes a long way to increase the real bond risk premia.

6 Conclusion

We find that a slow mean-reverting shock process of consumption growth and a segmented asset market mechanism with heterogeneous trading technology can quantitatively account for the positive and sizable term premium of the bond, which we observe in the data.

In this paper, we show that the slow mean-reverting consumption process at low frequency alone explains the positive term premia while the size of the premia is still small quantitatively . Our quantitative exercise shows that with this slight modification of the aggregate shock process, the long term bond is risky since the risk free rate is slightly countercyclical even in the representative agent model. We also show that the segmented market mechanism with heterogeneous trading technology and asymmetric bond positions across households can amplify the size and magnitude of the term premium with raising the market price of aggregate risks.

We think that our model is the first step to resolve the inconsistency between the theoretical macroeconomic model and the empirical asset pricing findings of the yield curve without a modification of preferences. There are two obvious directions that the model can be improved further for the future research. For asset pricing literature, one can enrich the model by additionally introducing the long term consumption growth shock in order to match more properties of real bond premia found in the empirical literature. For macroeconomics, our mechanism does not reply on the modification of preference and the behavior of asset pricing is pinned down by a relative small set of marginal investors in a segmented market mechanism. Therefore, the consumption and saving behavior of most households stay close to those in a standard macroeconomic model. It is more likely to extend our results into a general equilibrium production economy without comprising dynamics of other macroeconomic variables.

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A Appendix

A.1 Time-Zero Trading Household Problem

This Appendix describes an equivalent version of this economy in which all households trade at time zero. The time-zero price of a claim that pays one unit of consumption in node z^t can be constructed recursively from the one-period-ahead Arrow prices:

$$P(z^t)\pi(z^t) = Q(z_t|z^{t-1})Q(z_{t-1}|z^{t-2})...Q(z_1|z^0)Q(z_0),$$

The net financial wealth position of any trader in the home country given the history can be stated as

$$-a_t(z^t,\eta^t) = \sum_{s\geq t}\sum_{(z^s,\eta^s)\succeq(z^t,\eta^t)} \widetilde{P}(z^s,\eta^s)\left[\gamma Y(z^s)\eta_s - c(z^s,\eta^s)\right],$$

where $\widetilde{P}(z^t,\eta^t) = \pi(z^t,\eta^t)P(z^t)$. From the above equation, we are able to write the household problem in the form of time-zero trading fashion as shown in the next subsection.

A.1.1 Household Optimization Problem

Following Chien, Cole, and Lustig (2011), we state the household problem in this Arrow-Debreu economy.

We start with the Mertonian traders' problem in the home country. There are two constraints. Let χ denote the multiplier on the present value budget constraint and $\varphi(z^t,\eta^t)$ denote the multiplier on debt constraints. The saddle-point problem of a Mertonian trader can be stated as follows:

$$\begin{aligned} L \quad = \quad & \min_{\{\chi,\nu,\varphi\}}\max_{\{c\}}\sum_{t=1}^{\infty}\beta^t\sum_{(z^t,\eta^t)}\frac{1}{1-\alpha}c(z^t,\eta^t)^{1-\alpha}\pi(z^t,\eta^t) \\ & +\chi\left\{\sum_{t=1}^{\infty}\sum_{(z^t,\eta^t)}\widetilde{P}(z^t,\eta^t)\left[\gamma Y(z^t)\eta_t - c(z^t,\eta^t)\right] + a_0(z^0)\right\} \\ & -\sum_{t=1}^{\infty}\sum_{(z^t,\eta^t)}\varphi_t(z^t,\eta^t)\left\{\sum_{s\geq t}\sum_{(z^s,\eta^s)\succeq(z^t,\eta^t)}\widetilde{P}(z^s,\eta^s)\left[\gamma Y(z^t)\eta_s - c(z^s,\eta^s)\right]\right\}. \end{aligned}$$

The first-order condition with respect to consumption is given by

$$\beta^t c(z^t,\eta^t)^{-\alpha} = \zeta(z^t,\eta^t) P(z^t) \;\; for \;\; all \;\; (z^t,\eta^t), \tag{7}$$

where $\zeta(z^t,\eta^t)$ is defined recursively as

$$\zeta_t(z^t,\eta^t) = \zeta_{t-1}(z^{t-1},\eta^{t-1}) - \varphi_t(z^t,\eta^t),$$

with initial $\zeta_0 = \chi$. It is easy to show that this is a standard convex constraint maximization problem. Therefore, the first-order conditions are necessary and sufficient.

Non-Mertonian traders face additional restrictions on their portfolio choices. Let $\nu_t(z^t,\eta^t)$ denote the multiplier on portfolio restrictions. Given the same definition of other multipliers as in the active trader problem, the saddle-point problem of a nonparticipant trader whose asset in the end of the period is $\widehat{a}_{t-1}(z^{t-1},\eta^{t-1})$ in the home country can be stated as

$$\begin{aligned} L \;\; = \;\; & \min_{\{\chi,\nu,\varphi\}} \max_{\{c,\widehat{s}\}} \sum_{t=1}^{\infty} \beta^t \sum_{(z^t,\eta^t)} \frac{1}{1-\alpha} c_t(z^t,\eta^t)^{1-\alpha} \pi(z^t,\eta^t) \\ & \chi \left\{ \sum_{t=1}^{\infty} \sum_{(z^t,\eta^t)} \widetilde{P}(z^t,\eta^t) \left[\gamma Y(z^t)\eta_t - c(z^t,\eta^t) \right] + a_0(z^0) \right\} \\ & + \sum_{t=1}^{\infty} \sum_{(z^t,\eta^t)} \nu_t(z^t,\eta^t) \left\{ \begin{array}{c} \sum_{s\geq t} \sum_{(z^s,\eta^s)\succeq(z^t,\eta^t)} \widetilde{P}(z^s,\eta^s) \left[\gamma Y(z^s)\eta_s - c(z^s,\eta^s) \right] \\ + \widetilde{P}(z^t,\eta^t) R^p_{t,t-1}(z^t) \widehat{s}_{t-1}(z^{t-1},\eta^{t-1}) \end{array} \right\} \\ & - \sum_{t=1}^{\infty} \sum_{(z^t,\eta^t)} \varphi_t(z^t,\eta^t) \left\{ \sum_{s\geq t} \sum_{(z^s,\eta^s)\succeq(z^t,\eta^t)} \widetilde{P}(z^s,\eta^s) \left[\gamma Y(z^s)\eta_s - c(z^s,\eta^s) \right] \right\}. \end{aligned}$$

The first-order condition with respect to consumption is given by

$$\beta^t c(z^t,\eta^t)^{-\alpha} = \zeta(z^t,\eta^t) P(z^t) \;\; for \;\; all \;\; (z^t,\eta^t),$$

where $\zeta(z^t,\eta^t)$ is defined as

$$\zeta_t(z^t,\eta^t) = \zeta_{t-1}(z^{t-1},\eta^{t-1}) + \nu_t(z^t,\eta^t) - \varphi_t(z^t,\eta^t).$$

Therefore, the first-order condition with respect to consumption is independent of trading restrictions. The first-order condition with respect to total asset holdings at the end of period $t-1$, $\widehat{s}_{t-1}(z^{t-1}, \eta^{t-1})$, is

$$\sum_{(z^t, \eta^t)} R^p_{t,t-1}(z^t)\nu_t(z^t, \eta^t)P(z^t)\pi(z^t, \eta^t) = 0 \ \ for \ \ all \ \ z^t, \eta^t.$$

This condition varies according to different trading restrictions.

A.1.2 Stochastic Discount Factor

By summing the first-order conditions with respect to consumption, equation (7), across all domestic households at period t, we can obtain the consumption sharing rule as follow:

$$\frac{c(z^t, \eta^t)}{C(z^t)} = \frac{\zeta(z^t, \eta^t)^{-\frac{1}{\alpha}}}{h_t(z^t)},$$

where $h_t(z^t)$ is defined as $h_t(z^t) \equiv \sum_{\eta^t} \zeta(z^t, \eta^t)^{-\frac{1}{\alpha}}\pi(\eta^t)$. In addition, by plugging back the consumption sharing rule back to the first order condition with respect to consumption, equation (7), we can obtain the price of home consumption basket at state z^t:

$$P(z^t) = \beta^t C(z^t)^{-\alpha} h_t(z^t)^{\alpha}$$

Therefore, the stochastic discount factor is given by the Breeden-Lucas stochastic discount factor (SDF) with a multiplicative adjustment:

$$Q_{t+1}(z^{t+1}|z^t) \equiv \frac{P(z^{t+1})}{P(z^t)} = \beta \left(\frac{C(z^{t+1})}{C(z^t)}\right)^{-\alpha} \left(\frac{h_{t+1}(z^{t+1})}{h_t(z^t)}\right)^{\alpha}.$$

A.2 Proof of Lemma 1

Given that our assumed growth rate of output is $g_{t+1} = \bar{g}e^{(\rho-1)m_t+\varepsilon_{t+1}}$, the one period ahead pricing kernel is $M_{t,t+1} = \beta\bar{g}^{-\alpha}e^{-\alpha(\rho-1)m_t}e^{-\alpha\varepsilon_{t+1}}$. The price of one period bond is therefore:

$$P^1_t = E_t M_{t,t+1} = \beta\bar{g}^{-\alpha}e^{-\alpha(\rho-1)m_t}e^{\alpha^2\frac{\sigma^2_\varepsilon}{2}},$$

which is a function of the current deviation from trend m_t. The one-period yield is then

$$r_t^1 = -\ln P_t^1 = -\ln \beta + \alpha \ln \bar{g} + \alpha(\rho - 1)m_t - \frac{\alpha^2\sigma_\varepsilon^2}{2}.$$

We can also derive the price of k period zero coupon bond and its yields as follows;

$$\begin{aligned}
P_t^k &= E_t\left[M_{t,t+k}\right] = E_t\left[\prod_{\tau=0}^{k-1} M_{t+\tau,t+\tau+1}\right] \\
&= \beta^k \bar{g}^{-k\alpha} e^{-\alpha(\rho-1)\left[\sum_{\tau=0}^{k-1}\rho^\tau\right]m_t} e^{\left[\sum_{\tau=0}^{k-1}\rho^{2\tau}\right]\frac{\alpha^2\sigma_\varepsilon^2}{2}}, \\
r_t^k &= -\frac{1}{k}\ln P_t^k \\
&= -\ln\beta + \alpha\ln g + \alpha(\rho-1)\frac{1}{k}\frac{1-\rho^k}{1-\rho}m_t - \frac{1-\rho^{2k}}{1-\rho^2}\frac{\alpha^2\sigma_\varepsilon^2}{2}
\end{aligned}$$

The term premium at period t for a k-period zero counpon bond is

$$r_t^k - r_t^1 = \alpha\left(1-\rho\right)\left[1 - \frac{1}{k}\frac{1-\rho^k}{1-\rho}\right]m_t + \left[1 - \frac{1}{k}\frac{1-\rho^{2k}}{1-\rho^2}\right]\frac{\alpha^2\sigma_\varepsilon^2}{2},$$

which is again a function of m_t. Taking the unconditional expectation of $r_t^k - r_t^1$ gives equation (6) because of $E(m_t) = 0$.

Moreover, we want to show that the term premium is increasing with k, i.e. $\frac{\partial E\left[r_t^k - r_t^1\right]}{\partial k} > 0$ for $k > 1$. First, notice that $\rho^{2k}(1 - \ln\rho^{2k}) < 1$ because (1) $\lim_{\rho^{2k}\to 1}\rho^{2k}(1-\ln\rho^{2k}) = 1$ and (2)

$$\frac{\partial\left[\rho^{2k}(1-\ln\rho^{2k})\right]}{\partial\rho^{2k}} = -\ln\rho^{2k} > 0 \text{ if } \rho^{2k} < 1$$

Second, by utilizing the fact that $\sum_{\tau=0}^{k-1}\rho^\tau = \frac{1-\rho^{2k}}{1-\rho^2}$, the average term premium can be re-written as

$$E\left[r_t^k - r_t^1\right] = \frac{\alpha^2\sigma_\varepsilon^2}{2} - \frac{\alpha^2\sigma_\varepsilon^2}{2\left(1-\rho^2\right)} \times \frac{1-\rho^{2k}}{k}$$

and hence the derivative of $E\left[r_t^k - r_t^1\right]$ with respect to k is

$$\begin{aligned}
\frac{\partial E\left[r_t^k - r_t^1\right]}{\partial k} &= \frac{\alpha^2\sigma_\varepsilon^2}{2\left(1-\rho^2\right)} \times \frac{1-\rho^{2k}(1-\ln\rho^{2k})}{k^2} \\
&> 0 \text{ if } \rho < 1 \text{ and } k > 1.
\end{aligned}$$

The last inequality utilizes the fatc that $\rho^{2k}(1 - \ln \rho^{2k}) < 1$ if $\rho^{2k} < 1$.

Table I: Real Treasury Yield

Period/Maturity	5 year	7 year	10 year
2003 to 2007	1.646	1.871	2.061
2003 to 2009	1.513	1.757	1.962
2003 to 2011	1.160	1.452	1.715
2003 to 2013	0.772	1.082	1.367
2003 to 2015	0.658	0.968	1.226

Data source: U.S. Department of Treasury. https://www.treasury.gov/resource-center/data-chart-center/interest-rates/Pages/TextView.aspx?data=realyield

Table II: Quantitative Results

Case	Benchmark	No Mortgage	No HTT	RA Economy
me	5%	5%	100%	NA
et	45%	45%	0%	NA
np	50%	50%	0%	NA
ω_{et}	$(0.211, -0.195)$	$(0.211, 0)$	NA	NA
ω_{np}	$(0, -.537)$	$(0, 0)$	NA	NA
$E(Q)$	0.982	0.980	0.960	0.894
$\sigma(Q)$	0.467	0.455	0.130	0.119
$\frac{\sigma(Q)}{E(Q)}$	0.475	0.464	0.135	0.133
$Std(\frac{\sigma_t(Q)}{E_t(Q)})$	9.766	9.494	0.033	0.000
$E(R^e - R^f)$	7.262	6.937	1.699	1.741
$\sigma(R^e - R^f)$	15.625	15.249	12.607	13.155
$\frac{E(R^e - R^f)}{\sigma(R^e - R^f)}$	0.465	0.455	0.135	0.132
$E(R^c - R^f)$	2.361	2.113	0.575	0.487
$\sigma(R^c - R^f)$	5.255	4.798	4.268	3.680
$\frac{E(R^c - R^f)}{\sigma(R^e - R^f)}$	0.449	0.440	0.135	0.132
$E(R^f)$	0.949	1.206	3.922	11.714
$\sigma(R^f)$	1.449	1.296	1.366	2.049

Notes: Parameters setting: risk aversion rate, $\gamma = 4$, discount factor, $\beta = 0.95$, non-diversifiable share of income: $\gamma = 0.885$, leverage ratio: $\phi = 3$. The simulation results are generated by an economy with 3000 agents and 10000 periods.

Figure 1: Average Real Yield Curve in Benchmark Case, Zero Coupon Bond