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WORKING PAPER SERIES

Insurance and Inequality with Persistent Private Information

Authors	Alex Bloedel, Vijay Krishna, and Oksana Leukhina
Working Paper Number	2018-020H
Revision Date	January 2025
Citable Link	https://doi.org/10.20955/wp.2018.020
Suggested Citation	Bloedel, A., Krishna, V., Leukhina, O., 2025; Insurance and Inequality with Persistent Private Information, Federal Reserve Bank of St. Louis Working Paper 2018-020. URL https://doi.org/10.20955/wp.2018.020

Federal Reserve Bank of St. Louis, Research Division, P.O. Box 442, St. Louis, MO 63166

ECONOMETRICA

JOURNAL OF THE ECONOMETRIC SOCIETY

An International Society for the Advancement of Economic Theory in its Relation to Statistics and Mathematics

https://www.econometricsociety.org/

Econometrica, Vol. 93, No. 3 (May, 2025), 821–857

INSURANCE AND INEQUALITY WITH PERSISTENT PRIVATE INFORMATION

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We study the implications of optimal insurance provision for long-run welfare and inequality in economies with persistent private information. A principal insures an agent whose private type follows an ergodic, finite-state Markov chain. The optimal contract always induces *immiseration*: the agent's consumption and utility decrease without bound. Under positive serial correlation, it also *backloads high-powered incentives*: the sensitivity of the agent's utility with respect to his reports increases without bound. These results extend—and help elucidate the limits of—the hallmark immiseration results for economies with i.i.d. private information. Numerically, we find that persistence yields faster immiseration, higher inequality, and novel short-run distortions. Our analysis uses recursive methods for contracting with persistent types and allows for binding global incentive constraints.

KEYWORDS: Immiseration, insurance, inequality, backloaded incentives, recursive contracts, persistent private information.

1. INTRODUCTION

> The idea that a society's income distribution arises, in large part, from the way it deals with individual risks is a very old and fundamental one, one that is at least implicit in all modern studies of distribution.
>
> Lucas (1992, p. 234)

MANY CONTEMPORARY debates over rising inequality concern institutions designed to facilitate risk-sharing, such as social insurance programs and redistributive tax systems. More generally, there is a fundamental connection between the way that a society provides insurance and its degree of inequality. In particular, because individuals typically have private information about their idiosyncratic risks—such as shocks to income, tastes, or productivity—and must be incentivized to reveal that information, the markets for insuring against these risks are inevitably incomplete. Consequently, the way that a society

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This paper subsumes the working paper Bloedel and Krishna (2015). We thank anonymous referees, Andy Atkeson, Simon Board, Gabriel Carroll, Hector Chade, V.V. Chari, Sebastian Di Tella, Ed Green, Christian Hellwig, Pablo Kurlat, Paul Milgrom, Andy Newman, Shunya Noda, Ilya Segal, Andy Skrzypacz, Bruno Strulovici, Bob Wilson, and various seminar and conference audiences for many helpful comments. Aristotle Magganas provided superb research assistance. Bloedel is grateful to the B.F. Haley and E.S. Shaw Fellowship for Economics for financial support and to Caltech for its hospitality while some of this work was completed. Krishna is grateful to the Economics Department at Duke University, where he was a visitor at the inception of this project.

 https://doi.org/10.3982/ECTA20404

resolves the tradeoff between providing insurance and incentives determines how these private shocks affect individuals' consumption, and hence how the population distributions of consumption and wealth evolve over time. What, then, is the *optimal* degree of inequality?

The classic answer is striking: in private information economies, the social optimum induced by a utilitarian planner with full commitment generates *immiseration*. That is, in the long run, almost all individuals become completely impoverished while a vanishing lucky few consume all of society's resources (Green (1987), Thomas and Worrall (1990)). Furthermore, cross-sectional consumption and wealth inequality increase without bound, so the economy does not converge to a well-defined steady state (Atkeson and Lucas (1992)). In other words, there is no meaningful long-run tradeoff between efficiency and equity: the optimal provision of incentives demands destitution and infinite inequality.

This extreme conclusion has sparked debates over immiseration's intuitive appeal and inspired the search for alternative models.[1] Nonetheless, immiseration is "often regarded as being the hallmark result of dynamic social contracting in the presence of private information" (Kocherlakota (2010, p. 70)) and constitutes an apparently fundamental feature of the workhorse normative models that economists use to study optimal risk-sharing. It is therefore important to understand its robustness within this canonical class of models.

A key gap in this understanding concerns the dynamic nature of individuals' private information. At one extreme, the classic literature universally assumes that private information is i.i.d. over time (i.e., shocks are "completely transient"), which is analytically convenient but unrealistic: shocks to individuals' incomes, tastes, and productivities are often highly persistent.[2] At the opposite extreme, in an influential paper, Williams (2011) shows that when individuals' private types follow a random walk (i.e., shocks are "permanent"), immiseration not only disappears, but the optimal contract generates long-run *bliss*: it sends individuals' consumption and utility to their *upper* bounds. This raises two questions. First, what is driving these diametrically opposing results? Second, what long-run outcomes emerge in the generic and empirically relevant intermediate case in which private information is imperfectly persistent?

In this paper, we study the long-run properties of optimal insurance contracts under general forms of persistent private information. We address the second question by showing that immiseration arises for a broad class of *ergodic* private information processes. While prior work (Zhang (2009)) has considered specific parametric examples, our generality permits us, in essence, to interpolate between the i.i.d. and permanent shock benchmarks. Consequently, we also elucidate the first question by identifying "mean-reversion" of the private information process as a key determinant of long-run outcomes. In particular, our analysis suggests that immiseration fails *only* in the knife-edge case of permanent shocks.

Model. We study a canonical principal-agent formulation of the planning problem.[3] A risk-neutral principal (she) interacts with a risk-averse agent (he) over an infinite horizon

[1]Various authors have critiqued immiseration on normative grounds (it seems perverse that ex ante efficiency should require ex post destitution), descriptive grounds (commitment to one's own impoverishment may not be enforceable), and practical grounds (without a well-defined steady state, one cannot meaningfully analyze the long-run tradeoff between equity and efficiency). Correspondingly, a number of papers provide foundations for bounded long-run inequality by appealing to alternative models with limited commitment, different normative criteria, and various other features (see Section 1.1 for references).

[2]See Storesletten, Telmer, and Yaron (2004) or Meghir and Pistaferri (2004) for evidence on income.

[3]Specifically, we consider the "open economy" planning problem of a utilitarian planner who maximizes the average welfare of a large population of agents subject to an *intertemporal* resource constraint (Green (1987)).

in discrete time. The agent's preferences in each period are determined by his privately observed *type*, which evolves stochastically over time. The principal provides insurance to the agent via an infinite-horizon insurance contract—which specifies the agent's allocation in each period conditional on the history of reported types—with the goal of achieving constrained Pareto optimality, that is, minimizing costs while delivering a prespecified lifetime utility to, and eliciting truthful reports from, the agent. Both parties commit to the contract at the initial date and discount the future at the same rate. For concreteness, we focus on a setting in which the agent's type corresponds to his privately observed endowment (as in Green (1987), Thomas and Worrall (1990)), but this is not essential.

We make two main assumptions about the environment. First, the agent's type evolves according to a fully connected, finite-state Markov chain. This implies that the agent's type process is ergodic and bounded but allows for otherwise arbitrary serial correlation, and thus permits forms of persistence studied in the recent theoretical literature and the kinds of asymmetric and skewed shocks documented in the recent empirical literature.[4] Second, the agent can neither covertly save outside of the contract nor over-report his type within the contract (though he can freely under-report his type). In our favored interpretation, this captures the idea that the agent cannot covertly engage in trade or production.[5]

Main Results. Our main result, Theorem 1, shows that the optimal contract always generates *immiseration*: the agent's consumption and utility converge in probability (and, under certain conditions, almost surely) to their lower bounds. This result encompasses prior immiseration results in i.i.d. settings (Green (1987), Thomas and Worrall (1990)) and specific settings with persistence (Zhang (2009)). Our second result, Theorem 2, shows that when the agent's type is positively serially correlated, the optimal contract additionally features *backloaded high-powered incentives*: in the long run, the sensitivity of the agent's continuation utility with respect to his reports increases without bound. This can be viewed as a kind of *relative immiseration* wherein the impact of an incrementally higher type realization is magnified over time: the welfare difference between two agents with identical histories before time T, but with different shocks at time T, grows unboundedly as $T \to \infty$.

By standard duality arguments (Golosov, Tsyvinski, and Werquin (2016)), such problems can be equivalently "decentralized" into a collection of one-on-one principal-agent problems in which the principal minimizes the expected cost of delivering a particular lifetime utility to each agent. Following Thomas and Worrall (1990) and others, we analyze this principal-agent version of the problem. We revisit the planning interpretation in Sections 5.1 and 6.3 and, in the latter, discuss how our analysis relates to the "closed economy" problem in which the planner faces a *per-period* resource constraint (Atkeson and Lucas (1992)).

[4]Recent theoretical studies of dynamic contracting primarily focus on either binary-state Markov chains or continuous-state ergodic Markov processes (see Footnote 6 and Section 1.1 for references). Our framework includes the former processes and distributional approximations of the latter ones. Empirically, Arellano, Blundell, and Bonhomme (2017) and Guvenen et al. (2021) find that the relative persistence of positive and negative income shocks depends sensitively on the individual's current income level.

[5]The restriction on savings is standard (cf. Allen (1985); Cole and Kocherlakota (2001b)). The restriction on over-reporting is common in the literature and often natural. When the agent's private type is his endowment, it amounts to the assumption that he cannot covertly borrow outside of the contract (Williams (2011); Bloedel, Krishna, and Strulovici (2023)). When his private type is his labor productivity, it amounts to the assumption that he cannot covertly engage in home production (Golosov and Tsyvinski (2007)). Under these interpretations, these restrictions permit us to isolate the incentive constraints corresponding to the agent's exogenous private information from any other constraints arising from imperfect enforcement of contractual terms. See Section 6.1 for further discussion.

A simple intuition for these results is as follows. Incentive compatibility requires that high and low endowment reports be, respectively, rewarded and punished with higher and lower average transfers from the principal in future periods. Thus, the agent's continuation utility must vary with his reported endowment, with larger variation corresponding to higher-powered incentives. Ceteris paribus, the *cost of incentive provision* is lower for the principal when the agent's continuation utility is *lower*: due to risk aversion, this is precisely when the agent's *marginal* utility of consumption is *higher*, so that a given variation in utility can be induced by smaller variations in consumption. Thus, by immiserating the agent, the principal drives her cost of incentive provision to zero and makes it affordable to provide arbitrarily high-powered incentives in later periods. Such *backloading* of high-powered incentives allows the principal to provide better insurance in early periods while maintaining incentive compatibility, and thus to optimally smooth her costs over time.

While similar intuitions have appeared in the classic i.i.d. setting (Thomas and Worrall (1990)), formalizing them is significantly more challenging in settings with persistence. The key subtlety is that, with persistence, the agent has two sources of information rents. First, there is the classic *i.i.d. information rent* arising from the agent's private information about his current type, and hence his *static* preferences over allocations in the current period. Second, there is the new *Markov information rent* arising from the agent's private information about the distribution of his future types, and hence his *intertemporal* preferences over the path of allocations in the "continuation contract" for future periods. Due to the latter, the principal can no longer screen the agent's static preferences by using continuation utility as a type-independent "numeraire" with which to deliver rewards and punishments. Instead, she must also screen his intertemporal preferences through continuation contracts that carefully spread those rewards and punishments over multiple periods. If the agent's type is sufficiently persistent, these new intertemporal screening considerations might dominate the classic motive to backload incentives and yield qualitatively different contractual dynamics (as in the permanent shock benchmark of Williams (2011)). In essence, our key lesson is that, given *any* mean-reversion in the agent's type, the classic i.i.d. forces remain dominant *in the long run* and render immiseration optimal.

Our formal proofs of Theorems 1 and 2 involve three main steps, the latter two of which constitute our primary methodological contribution. First, we formulate the principal's problem recursively, using as the main state variable a vector of *interim promised utilities* that encode the agent's continuation utility conditional on each possible current type. Second, we identify a specific directional derivative of the principal's value function with respect to this state variable—her *marginal cost martingale*—as the appropriate object with which to analyze long-run outcomes. This step builds on Thomas and Worrall's (1990) observations that, in the i.i.d. special case, (i) the principal's optimal cost-smoothing implies that her marginal cost of providing continuation utility to the agent follows a martingale process, and (ii) the optimal contract's long-run behavior is pinned down by convergence properties of this martingale. However, adapting these ideas to the general Markovian setting requires our analysis to overcome several nontrivial challenges. Third, in the final step, we identify a novel "renewal" property of the martingale's dynamics that lets us characterize long-run outcomes using probabilistic arguments.

Notably, this analysis neither (i) imposes any restrictions (beyond finiteness) on the number of agent types nor (ii) relies on the "first-order approach" in which one studies a relaxed problem that ignores the agent's global incentive constraints. In contrast, most work on contracting with persistent types either studies binary-type settings, which is restrictive, or studies continuous-type settings using the first-order approach, which can yield solutions that violate the agent's global incentive constraints.[6] Moving beyond these two approaches has been identified as an important goal for the contracting literature.[7]

Solved Examples. To obtain a more detailed description of the optimal contract, we specialize to the setting with CARA utility and two endowment levels (which facilitates simple graphical depictions and direct comparisons to prior work). We analytically characterize key structural properties of the optimal contract and use numerical analysis to fully solve for it. This exercise yields two main sets of insights.

First, persistence affects the *speed* of immiseration. To illustrate this, we simulate many sample paths of the optimal contract and study how the implied consumption distribution evolves over time. We interpret this as describing the dynamics of the cross-sectional distribution of consumption in an "open economy" that can borrow and lend in the world market, as in Green (1987) (cf. Footnote 3). As persistence increases, we find that both the mean and variance of consumption diverge *more quickly* (to minus and plus infinity, respectively). That is, the economy's aggregate consumption decreases without bound and its consumption inequality increases without bound, and both occur at a faster rate when individuals' endowment shocks are more persistent. Motivated by these findings, we discuss how our analysis speaks to "closed economy" settings in which aggregate consumption is required to be constant, as in Atkeson and Lucas (1992).

Second, persistence gives rise to novel *short-run* dynamics and distortions. For instance, the optimal contract (i) uses history-dependent rewards and punishments that condition on the *order* of the agent's shocks, in contrast to an order-invariance property from the i.i.d. case (Thomas and Worrall (1990)), and (ii) delivers these incentives using a fundamentally different mix of *intra*temporal and *inter*temporal distortions. Furthermore, the relative importance of these new features increases as the agent's type becomes more persistent; by several measures, higher persistence results in worse insurance.

Roadmap. Section 2 develops the model. Section 3 explains the recursive formulation. Section 4 presents our main long-run results. Section 5 presents our solved examples. Section 6 concludes with a discussion of modeling assumptions and various extensions. Appendices A and B present facts about the recursive formulation. Supplemental Appendices C–H in Bloedel, Krishna, and Leukhina (2025a) contain proofs of our main results. Proofs of auxiliary technical results are in Bloedel, Krishna, and Leukhina (2025b).

1.1. *Related Literature*

Insurance and Immiseration. Substantively, our work contributes to the literature on long-run properties of optimal insurance contracts initiated by Green (1987), Thomas and Worrall (1990), and Atkeson and Lucas (1992). We build on the connection between immiseration and the Martingale Convergence Theorem identified in Thomas and Worrall (1990), extending it to settings with general forms of persistent private information.

[6]See Section 1.1 for references to recent binary-type models. The first-order approach to dynamic mechanism design is developed in Pavan, Segal, and Toikka (2014) and applied to dynamic insurance problems in Williams (2011), Kapička (2013), Farhi and Werning (2013), Golosov, Troshkin, and Tsyvinski (2016), Hellwig (2021), and Brendon (2022). In a monopolistic screening setting, Battaglini and Lamba (2019) show that, if period lengths are short, the first-order approach fails for a generic class of finite-state Markov type processes; when it fails, the candidate contract it produces can differ qualitatively from the true optimal contract. It is natural to expect that these lessons also apply to dynamic insurance problems.

[7]See Pavan (2016), Garrett, Pavan, and Toikka (2018), and Battaglini and Lamba (2019).

Most directly, our work relates to several papers that study optimal insurance and immiseration under specific forms of persistent private information (and parametric assumptions on preferences).[8] Zhang (2009) studies the special case of our framework with two types and symmetric transition probabilities; using arguments that rely on this special structure, he shows that the optimal contract induces immiseration.[9] Williams (2011) studies a model in which the agent's type follows a Brownian motion (i.e., a continuous-time Gaussian random walk) and shows that the optimal contract yields long-run bliss, rather than immiseration.[10] Bloedel, Krishna, and Strulovici (2024) consider a more general class of Markovian type processes with "permanent shocks" and show that it is effectively impossible to elicit information from the agent, which helps explain why optimal contracts under permanent shocks (as in Williams (2011)) differ from those in settings with mean-reversion. In contrast to these works, we study a general class of Markovian type processes with mean-reversion (and make minimal assumptions on preferences).

Less directly, our work also relates to the literature that studies the robustness of immiseration under alternative assumptions on preferences, technologies, and institutions (while maintaining the i.i.d. assumption). Several papers show that bounded long-run inequality is optimal in settings with limited commitment, such as when the agent faces interim participation constraints (Atkeson and Lucas (1995), Phelan (1995)) or the principal faces credibility constraints (Sleet and Yeltekin (2006, 2008), Farhi, Sleet, Werning, and Yeltekin (2012)), and in settings with full commitment but alternative normative criteria that place Pareto weight directly on "future generations" of agents (Phelan (2006), Farhi and Werning (2007)). Other papers derive weaker forms of immiseration in economies with production (Khan and Ravikumar (2001), Khan, Popov, and Ravikumar (2020)) or endogenous fertility (Hosseini, Jones, and Shourideh (2013)), and under alternative assumptions on agents' risk preferences (Phelan (1998), Olszewski and Safronov (2021)). In each case, long-run outcomes can be viewed as deriving from a modified version of the principal's marginal cost martingale.

Recursive Contracts. Methodologically, our work contributes to the literature on recursive methods for dynamic contracting initiated by Green (1987), Thomas and Worrall (1990), Spear and Srivastava (1987). Fernandes and Phelan (2000) introduce the first recursive formulation for settings with persistence. Their main state variable comprises the agent's "ex ante" promised utility, which encodes his continuation payoff conditional on having reported truthfully in the previous period, and a vector of *off-path* "threat-point" utilities that encode his continuation payoffs conditional on having misreported in the previous period. Building on Cole and Kocherlakota (2001a) and Doepke and Townsend (2006), our main state variable instead comprises a vector of *on-path* "interim" promised utilities that encode the agent's continuation payoff conditional on his current type, which makes the agent's incentive constraints simpler to interpret and analyze.[11] For instance,

[8]Farinha Luz (2022) studies optimal insurance when the agent has a persistent private "risk type," but his realized income is contractible (as in Rothschild and Stiglitz (1976), Wilson (1977)). The latter feature yields qualitatively different optimal contracts than those in the social contracting literature on which we build.

[9]Formally, Zhang (2009) studies a model with private productivity shocks (an extension we discuss in Section 6.1) and works in continuous time, but these differences are inessential for the present comparison.

[10]Williams (2011) also permits the agent's type to follow a mean-reverting Gaussian AR(1) process, but does not characterize the optimal contract in that case (see Bloedel, Krishna, and Strulovici (2023)).

[11]Cole and Kocherlakota (2001a) develop their approach in the context of stochastic games with hidden states, building on Abreu, Pearce, and Stacchetti (1990); Athey and Bagwell (2008) apply their methods to study dynamic Bertrand competition. Doepke and Townsend (2006) develop their approach in the context of contracting problems with both hidden states and hidden actions, emphasizing settings with both frictions.

we are able to fully characterize the "recursive domain" of implementable promised utility vectors for any number of types (Theorem 3). See Appendix B.1 for further discussion.

Our main methodological contribution is not the recursive formulation itself, but our use of it to characterize optimal contracts in settings with persistence.[12] Several recent papers share this goal, but in different economic environments and under the restriction to binary-type settings. Halac and Yared (2014) use the Fernandes and Phelan (2000) formulation to study optimal discretion in fiscal policymaking. Concurrent to our working paper, Guo and Hörner (2020) and Fu and Krishna (2019) apply the same recursive formulation used here to study firm financing and allocation problems, respectively, with a risk-neutral agent and limited transfers. Fu and Krishna (2019) establish convergence to long-run efficiency using a special case of the martingale methods that we develop here to establish immiseration, while Guo and Hörner (2020) establish long-run "polarization" (cf. Phelan (1998)) through a detailed construction of the optimal contract. Krasikov and Lamba (2021) use methods similar to Fu and Krishna (2019) to study dynamic procurement problems with limited liability. As our martingale methods for characterizing long-run outcomes (Theorems 1 and 2) and construction of the recursive domain (Theorem 3) apply for a broad class of Markovian type processes, they may prove useful for extending the analysis of optimal contracts in these and other applications beyond the binary-type case.

2. MODEL

Time is discrete and runs over an infinite horizon. At $t = 0$, a principal (she), who is risk-neutral and has discount factor $\alpha \in (0, 1)$, offers an insurance contract to a risk-averse agent (he). Neither party can renege on the contract at a later date.

Agent's Preferences. The agent has the same discount factor $\alpha \in (0, 1)$. His Bernoulli utility of consumption is $U : (\underline{c}, \infty) \to \mathbb{R}$, where $\underline{c} \geq -\infty$ is a minimal consumption level.

ASSUMPTION 1—DARA: $U(\cdot)$ *satisfies the following properties*:

(a) *It is strictly increasing*, *strictly concave*, *continuously differentiable*, *and satisfies the Inada conditions* $\lim_{c\to\underline{c}} U'(c) = +\infty$ *and* $\lim_{c\to\infty} U'(c) = 0$.

(b) *It is bounded above and unbounded below*: $\lim_{c\to\infty} U(c) = 0$ *and* $\lim_{c\to\underline{c}} U(c) = -\infty$.

(c) *It has* decreasing absolute risk aversion: *the map* $c \mapsto -\log(U'(c))$ *is* (*weakly*) *concave*.

Assumption DARA is common in the literature and includes standard parametric classes of utilities, such as those in the CARA class and many in the HARA class.[13] We denote the range of U by $\mathcal{U} := U((\underline{c}, \infty)) \subset \mathbb{R}$. Note that part (b) implies $\mathcal{U} = (-\infty, 0)$.

[12] Fernandes and Phelan (2000) and Doepke and Townsend (2006) numerically solve binary-type examples (as in Section 5), but do not analytically study optimal contracts (see also Broer, Kapička, and Klein (2017)).

[13] We adopt DARA following Cole and Kocherlakota (2001b) and Thomas and Worrall (1990). CARA utility is the benchmark specification in the literature: see the models or main examples in Green (1987), Phelan (1995, 1998), Williams (2011), Bloedel, Krishna, and Strulovici (2023), Thomas and Worrall (1990), Wang (1995), Strulovici (2020), and Atkeson and Lucas (1992). DARA is satisfied by all HARA utilities $U(c) = (c + \eta)^{1-\gamma}/(1 - \gamma)$ with $\gamma > 1$ and $\eta \geq -\underline{c}$ (where CRRA utility corresponds to $\eta = \underline{c} = 0$).

Type Process. In each period t, the agent privately observes his endowment $\omega^{(t)} \in \mathbb{R}$, which can take any of the d values $\omega_d > \omega_{d-1} > \cdots > \omega_1$. If $\omega^{(t)} = \omega_i$, we say that the agent's time-t *type* is $i \in S := \{1, \ldots, d\}$. The *endowment (or type) process* $(\omega^{(t)})_{t=0}^{\infty}$ is Markovian and the measure over sample paths is given by $\mathsf{P} \in \Delta(S^{\infty})$.[14]

ASSUMPTION 2—Markov: *The agent's type follows a fully connected, time-homogeneous, first-order Markov chain with transition probabilities* $f_{ij} := \mathsf{P}(\omega^{(t+1)} = \omega_j | \omega^{(t)} = \omega_i) > 0$.

Assumption Markov implies that the type process is ergodic and bounded, but otherwise allows for essentially arbitrary serial correlation. Going forward, we represent the transition probabilities as the $d \times d$ transition matrix with rows $\mathbf{f}_i = (f_{i1}, \ldots, f_{id})$, where $\mathbf{f}_i$ denotes the conditional distribution of tomorrow's type if today's type is $i \in S$.[15]

Reporting Strategies. In each period, the agent submits a report about his current type. We assume, as is common in the literature, that the agent cannot over-report his type:

ASSUMPTION 3—NHB: *An agent of type i can only report to be of types $j \leq i$.*

Assumption NHB ("No Hidden Borrowing") captures two standard ideas: (i) endowments are partially verifiable, and (ii) the agent cannot covertly save, borrow, or produce the consumption good (e.g., via a market or private technology). For example, the principal might require the agent to deposit his reported endowment in a jointly monitored account; if he cannot covertly engage in such activities, he can deposit at most his true endowment.

Principal's Sequential Problem. A contract offered by the principal at $t = 0$ specifies transfers (of the consumption good) to the agent in each period as a function of the history of type reports. The principal aims to minimize her expected lifetime cost of financing the contract, subject to two constraints: (i) *promise keeping* and (ii) *incentive compatibility*. Formally, the contract must (i) deliver a given schedule of *promised utilities* $\mathbf{v}^{(0)} := (v_1, \ldots, v_d) \in \mathcal{U}^d$ to the agent, where v_i denotes the agent's lifetime utility if his initial endowment is $\omega^{(0)} = \omega_i$ and he reports truthfully, and (ii) ensure that truthful reporting in all periods maximizes the agent's lifetime utility.[16]

We call this the principal's *sequential problem*, as it describes her optimization over full sequences of history-dependent transfers. While this "sequential formulation" is standard, our main analysis uses a more tractable recursive formulation of contracts (see Section 3 below). Thus, for brevity, we defer the details of the sequential formulation to Appendix A.1, where the principal's sequential problem is stated as program (SP).

REMARK 2.1: Assumptions 1–3 hold throughout the paper. We discuss them further in Section 6.1. There, we also discuss model variants in which the agent's type represents his tastes or productivity, rather than his endowment.

[14]Throughout, we identify endowment realizations with their corresponding type realizations, and identify the space $\{\omega_1, \ldots, \omega_d\}^{\infty}$ of sample paths of endowments with the space S^{∞} of sample paths of types.

[15]We also implicitly assume that the initial distribution $\mathsf{P}(\omega^{(0)} = \cdot) = \mathbf{f}_i$ for some $i \in S$, which serves to simplify notation in the definition of the principal's problem below (see (RP) in Section 3.3 and (SP) in Appendix A.1). This assumption is not essential and can easily be relaxed.

[16]This initial $\mathbf{v}^{(0)}$ could be given exogenously or arise from the principal's cost-minimization given some prior belief over the agent's initial type (e.g., the "efficiency problem" (Eff_i) in Section 4.3 below).

3. RECURSIVE CONTRACTS

Section 3.1 develops the recursive formulation. Section 3.2 introduces regularity conditions used in our analysis. Section 3.3 presents the principal's Bellman equation.

3.1. *Principal's Recursive Problem*

State Variable. Recall from Green (1987), Thomas and Worrall (1990) that, when types are i.i.d., the principal's problem can be written recursively with the agent's *ex ante promised utility* as a state variable (i.e., his expected continuation utility under the contract *before* his type in the current period is realized, and *assuming* truthful reporting in the current and future periods). The fact that this one-dimensional state can encode all payoff-relevant aspects of the history relies on the *symmetry* of information *across* periods implied by the i.i.d. assumption. Intuitively, ex ante promised utility serves as a "numeraire" because the agent's preferences over "continuation contracts" for future periods are common knowledge, even at *off-path* histories where the agent has lied.[17]

Plainly, this logic fails in settings with persistence. Thus, as noted by Fernandes and Phelan (2000) and Doepke and Townsend (2006), we require a richer state variable. We use a state variable $(\mathbf{v}, s)$ with two components. First, $\mathbf{v} = (v_1, \dots, v_d) \in \mathcal{U}^d$ denotes a vector of *interim promised utilities*, where $v_i \in \mathcal{U}$ denotes the agent's continuation utility *contingent on* his current type being $i \in S$ and *assuming* truthful reporting in the future. As we will see below, $\mathbf{v}$ encodes all aspects of the history that are payoff-relevant to the *agent* (even off-path). Second, $s \in S$ denotes the agent's *report* in the *previous* period. This is payoff-relevant to the *principal*, as it determines her beliefs about the agent's current type. Thus, the overall state $(\mathbf{v}, s)$ encodes all payoff-relevant aspects of the history.[18]

Recursive Constraints. At each state $(\mathbf{v}, s)$, we view the contract as offering the agent report-contingent consumption transfers and interim continuation utility vectors. Formally, if the agent *reports* to be of type j, two things happen. First, he receives a consumption transfer c_j, so that his overall consumption is $c_j + \omega_i$, where i is his *true* type. Second, he is promised the continuation utilities $\mathbf{w}_j := (w_{j1}, \dots, w_{jd})$, where w_{jk} denotes his interim promised utility in the *next* period, contingent on k being his true type therein. Thus, the state transitions to $(\mathbf{w}_j, j)$ in the next period, and the process begins again.

It will be convenient to express the current period's allocation in terms of the agent's *flow utility*, rather than his consumption. To this end, let $C(u_j, j) := U^{-1}(u_j) - \omega_j$ denote the consumption transfer that delivers exactly u_j utiles to a *truthful* agent of type j. Suppose the principal intends to give u_j utiles to a type-j agent, and so transfers $c_j = C(u_j, j)$ units of consumption when the agent *reports* to be of type j. Then, if an agent of true type i reports to be type j, his flow utility is $\psi(u_j, i, j) := U(\omega_i + C(u_j, j))$.

Given this change of variables, at each state $(\mathbf{v}, s)$, we can view the contract as offering the agent a *menu* $(u_i, \mathbf{w}_i)_{i \in S} \in (\mathcal{U} \times \mathcal{U}^d)^d$ consisting of *flow utilities* $u_i \in \mathcal{U}$ and *interim continuation utilities* $\mathbf{w}_i \in \mathcal{U}^d$ for each type $i \in S$. Clearly, this menu should (i) deliver the correct promised utility to every type, and (ii) ensure that truthful reporting in the

[17] Formally, at each history, (i) the agent's beliefs about his future types do *not* depend on his current type, and (ii) the principal's beliefs about the agent's current type do *not* depend on his previous reports.

[18] In general, a contract may condition on aspects of the history beyond $(\mathbf{v}, s)$. However, it is well understood that *without loss of optimality*, we can restrict attention to contracts that are Markovian (or "recursive") with respect to $(\mathbf{v}, s)$. See, for example, Fernandes and Phelan (2000, Lemma 2.3) and Doepke and Townsend (2006, Proposition 4). As is standard in the literature, we implicitly restrict attention to such contracts throughout.

current period (assuming truthful reporting in future periods) is optimal for the agent. Formally, these *recursive constraints* consist of the (interim) *promise keeping* and *incentive compatibility* conditions

$$v_i = u_i + \alpha \mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_i], \tag{PK$_i$}$$

$$v_i \geq \psi(u_j, i, j) + \alpha \mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_j], \tag{IC$_{ij}$}$$

for all $i, j \in S$ with $i > j$ (per Assumption NHB), where $\mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_j] := \sum_{k=1}^{d} f_{ik} w_{jk}$ is the *expected* promised utility for an agent of type i who reports to be type j. We call $\mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_i]$ the ex ante promised utility for type i.[19]

The recursive constraints have two important properties. First, they are independent of s, the agent's previous report. Second, even if the agent lies today, his expectation over tomorrow's type in (IC$_{ij}$) is still determined by his true current type. Thus, the recursive constraints ensure one-step promise keeping and deter one-shot deviations from truthtelling at *every* history, regardless of the agent's previous true and reported types.[20]

Recursive Domain. Even if a menu $(u_i, \mathbf{w}_i)_{i\in S}$ satisfies the recursive constraints (PK$_i$)–(IC$_{ij}$) in state $(\mathbf{v}, s)$, there may not exist menus satisfying these constraints in all of the possible continuation states $(\mathbf{w}_i, i)$. No incentive compatible contract could ever lead to states at which such "dead ends" arise—there must always exist a well-defined continuation contract. Thus, an essential part of the recursive formulation is determining which promised utilities $\mathbf{v} \in \mathcal{U}^d$ are *implementable* via some underlying incentive compatible contract.

Formally, we call a set $D \subseteq \mathcal{U}^d$ *the (recursive) domain* if (i) for every $\mathbf{v} \in D$, there exists a menu $(u_i, \mathbf{w}_i)_{i\in S}$ satisfying the recursive constraints (PK$_i$)–(IC$_{ij}$) and $\mathbf{w}_i \in D$ for all $i \in S$, and (ii) any set $D' \subseteq \mathcal{U}^d$ with the preceding property satisfies $D' \subseteq D$.[21] The domain characterizes the implementable promised utilities: all (and only those) $\mathbf{v} \in D$ are viable. Theorem 3 in Appendix B shows that D exists and is a nonempty, open, convex cone. For type processes satisfying standard forms of positive serial correlation, Theorem 3 also yields the closed-form solution $D = V_d := \{\mathbf{v} \in \mathcal{U}^d : v_d > v_{d-1} > \cdots > v_1\}$.

To illustrate the logic of this solution, consider the $d = 2$ case with positive correlation depicted in Figure 1. Substituting the promise keeping constraint (PK$_i$) (for $i = 1$) into the incentive constraint (IC$_{ij}$) (for $i = 2$ and $j = 1$) yields an equivalent version of the latter:

$$v_2 - v_1 \geq \underbrace{\psi(u_1, 2, 1) - u_1}_{\text{i.i.d. info rent}} + \underbrace{\alpha\big[\mathsf{E}^{\mathbf{f}_2}(\mathbf{w}_1) - \mathsf{E}^{\mathbf{f}_1}(\mathbf{w}_1)\big]}_{\text{Markov info rent}}. \tag{IC$^*_{21}$}$$

[19]When types are i.i.d., so that $\mathbf{f}_i = \mathbf{f}_j$ for all $i, j \in S$, the quantity $w_i := \mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_i]$ is precisely the state variable from Green (1987) and Thomas and Worrall (1990).

[20]In principle, requiring that truthful reporting is optimal for the agent at *every* history might be stronger than requiring that it is optimal for him *conditional on truthful reporting in the past* (cf. incentive compatibility in the sequential problem, (SP)). However, since the type process is Markovian and payoffs are time-separable, these two requirements are, in fact, equivalent (e.g., Pavan, Segal, and Toikka (2014, Section 3.3)).

[21]In the terminology of Abreu, Pearce, and Stacchetti (1990), the domain D is (i) *self-generating* and (ii) the *largest* self-generating set. Note that D is clearly unique (if it exists).

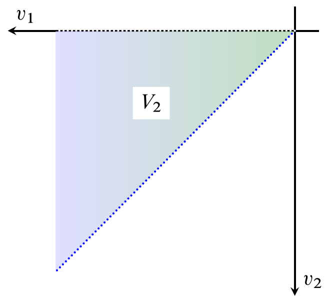

FIGURE 1.—The domain $D = V_2$ for $d = 2$ and positive correlation ($f_{11} \geq f_{21}$, $f_{22} \geq f_{12}$).

The *i.i.d. information rent* is clearly positive. If $\mathbf{w}_1 \in V_2$, then the *Markov information rent* is also positive, as $j \mapsto w_{1j}$ is increasing and $\mathbf{f}_2$ first-order stochastically dominates $\mathbf{f}_1$. Thus, implementability of $\mathbf{w}_1$ implies that $\mathbf{v} \in V_2$. Conversely, it can be shown that each $\mathbf{v} \in V_2$ can be induced by some menu $(u_i, \mathbf{w}_i)_{i=1,2}$ with $\mathbf{w}_1, \mathbf{w}_2 \in V_2$, which in turn implies that both $\mathbf{w}_1$ and $\mathbf{w}_2$ can be induced by menus with this property, and so on ad infinitum. In sum, all (and only those) $\mathbf{v} \in V_2 = \{\mathbf{v}' \in \mathcal{U}^d : v'_2 > v'_1\}$ are implementable in this example.

Recursive Contracts. With the domain D in hand, we can define contracts. A *(recursive) contract* is a map $\xi : D \times S \to (\mathcal{U} \times D)^d$ that assigns to each state $(\mathbf{v}, s) \in D \times S$ a menu $(u_i(\mathbf{v}, s), \mathbf{w}_i(\mathbf{v}, s))_{i\in S}$ of report-contingent flow utilities $u_i(\mathbf{v}, s) \in \mathcal{U}$ and continuation utilities $\mathbf{w}_i(\mathbf{v}, s) \in D$. We are interested only in those ξ that satisfy the recursive constraints at each step. To this end, define the *constraint correspondence* $\Gamma : D \rightrightarrows (\mathcal{U} \times D)^d$ as[22]

$$\Gamma(\mathbf{v}) := \big\{(u_i, \mathbf{w}_i)_{i\in S} \in (\mathcal{U} \times D)^d : \text{ (PK}_i\text{) and (IC}_{ij}\text{) hold } \forall i > j \in S \text{ at } \mathbf{v} \in D\big\}. \tag{3.1}$$

We say that a contract ξ is *feasible* if $\xi(\mathbf{v}, s) \in \Gamma(\mathbf{v})$ for all $(\mathbf{v}, s) \in D \times S$. Let Ξ denote the set of feasible contracts.

We can describe a contract's dynamics as follows. Each reporting strategy for the agent generates an S-valued stochastic process $(s^{(t+1)})_{t=0}^{\infty}$, where $s^{(t+1)}$ denotes his *current* report in period t (and thus his *previous* report from the perspective of period $t + 1$). Let $\mathcal{H} := S^\infty$ denote the set of *paths* of type reports. Given any $\xi \in \Xi$, initial $\mathbf{v} \in D$, and distribution over $\mathcal{H}$ (determined by the agent's reporting strategy), iterating on ξ in the natural way yields two stochastic processes: (i) the $\mathcal{U}$-valued process $\tilde{u}_\xi := (u_\xi^{(t)})_{t=0}^{\infty}$ of *induced allocations*, and (ii) the D-valued process $(\mathbf{v}_\xi^{(t)})_{t=0}^{\infty}$ of *induced promises* (where $\mathbf{v}_\xi^{(0)} := \mathbf{v}$).[23] In this notation, the recursive state variable at time t is $(\mathbf{v}_\xi^{(t)}, s^{(t)})$.

Optimality. Given an initial state $(\mathbf{v}, s) \in D \times S$, the principal minimizes her lifetime cost across all feasible recursive contracts $\xi \in \Xi$. Formally, her *recursive problem* is

$$P(\mathbf{v}, s) := \inf_{\xi\in\Xi} \mathbf{E}\left[\sum_{t=0}^{\infty} \alpha^t C\big(u_\xi^{(t)}, s^{(t+1)}\big) \big| \big(\mathbf{v}_\xi^{(0)}, s^{(0)}\big) = (\mathbf{v}, s)\right], \tag{RP}$$

[22]Note that $\Gamma(\cdot)$ depends only on the $\mathbf{v}$ component of the state $(\mathbf{v}, s)$ because, as discussed above, the recursive constraints are independent of s.

[23]Implicit in this description are the usual restrictions that (a) the agent's report process is adapted to his type process, and (b) the induced allocation and promised utility processes are adapted to the report process. See Appendix A for these details in the context of the sequential formulation of the contracting problem.

where the expectation is taken with respect to the distribution over paths induced by truthful reporting and the prior belief $\mathbf{f}_s$ over initial types $\omega^{(0)} = \omega_{s^{(1)}}$.[24] As this notation suggests, under truthful reporting, $s^{(t+1)}$ is the agent's true type in period t (i.e., $\omega^{(t)} = \omega_{s^{(t+1)}}$). For this reason, going forward, we will often refer to $(s^{(t+1)})_{t=0}^{\infty}$ itself as the *type process*.

A contract ξ^* is *optimal* if it attains the infimum in (RP) at every $(\mathbf{v}, s) \in D \times S$. We represent the infimal cost in (RP) by the principal's *value function* $P : D \times S \to \overline{\mathbb{R}}$.

Transversality. The recursive constraints only impose one-step promise keeping and deter one-step deviations. For a solution ξ^* in (RP) to be feasible in the sequential problem (SP), the induced *allocation* $\tilde{u}_{\xi^*}$ must actually (i) deliver the appropriate level of promised utility and (ii) deter infinite-length deviations from truthful reporting.[25] Intuitively, any contract violating (i) or (ii) corresponds to a Ponzi scheme in which the principal drives the induced *promises* to minus infinity too quickly. We wish to rule out such schemes.

To this end, we say that a contract $\xi \in \Xi$ is *(TVC)-implementable at* $\mathbf{v} \in D$ if, starting from the initial condition $\mathbf{v}_{\xi}^{(0)} := \mathbf{v}$, it satisfies

$$\lim_{t\to\infty} \inf_{h\in\mathcal{H}} \alpha^t \mathbf{v}_{\xi}^{(t)}(h) = \mathbf{0}, \tag{TVC}$$

where $(\mathbf{v}_{\xi}^{(t)}(h))_{t=0}^{\infty}$ is the sequence of promises along path $h \in \mathcal{H}$. All (TVC)-implementable contracts satisfy properties (i) and (ii) stated above (Lemma A.1 in Appendix A.2).

Full-Information Benchmark. For future reference, we briefly describe the *first-best* contract that is optimal when there is no private information, that is, there are no incentive constraints (see Supplemental Appendix F for details). In this setting, every promised utility vector $\mathbf{v} \in \mathcal{U}^d$ is implementable, and a *feasible full-information contract* is a map $\zeta : \mathcal{U}^d \times S \to (\mathcal{U} \times \mathcal{U}^d)^d$ such that $\zeta(\mathbf{v}, s)$ satisfies (PK$_i$) for all $i \in S$ at every $(\mathbf{v}, s) \in \mathcal{U}^d \times S$. We let Ξ^{FB} denote the set of all such contracts. The first-best contract ζ^* perfectly smooths the agent's consumption conditional on the initial type $s^{(1)}$: given any initial condition $(\mathbf{v}_{\zeta^*}^{(0)}, s^{(0)}) \in \mathcal{U}^d \times S$ and realization of $s^{(1)} \in S$, (i) the induced allocations $(u_{\zeta^*}^{(t)})_{t=0}^{\infty}$ are constant, and (ii) the induced promises $(\mathbf{v}_{\zeta^*}^{(t)})_{t=1}^{\infty}$ are constant and satisfy $v_{\zeta^*,1}^{(t)} = \cdots = v_{\zeta^*,d}^{(t)}$ for all $t \geq 1$. The principal's *first-best value function* is denoted by $Q^* : \mathcal{U}^d \times S \to \mathbb{R}$.[26] Naturally, the first-best contract is not incentive compatible: in every period $t \geq 1$, the agent's consumption is maximized by reporting to be of type $i = 1$ (cf. Lemma C.16 in Supplemental Appendix C.3).

[24]This distribution coincides with the measure $\mathbf{P}$ over paths of types defined in Section 2, except for one difference: the prior belief over the initial $\omega^{(0)}$ may differ. We formulate (RP) at all conceivable initial states, following the standard convention in dynamic programming.

[25]That is, (i) $v_j = \mathbf{E}[\sum_{t=0}^{\infty} \alpha^t u_{\xi^*}^{(t)} | s^{(1)} = j]$ for all $j \in S$, and (ii) the agent's reporting problem is suitably "continuous at infinity." Both (i) and (ii) would trivially hold if the agent's utility function were bounded.

[26]Formally, this characterization of the first-best holds under Regularity Condition R.2 (stated below in Definition 3.1), which ensures that the full-information problem (FB) in Supplemental Appendix F is well-behaved.

3.2. *Regularity Conditions*

To ensure that problem (RP) is mathematically well-behaved, we impose some regularity conditions on the environment. For expositional simplicity, we impose them directly on derived objects; sufficient conditions on model primitives can be obtained case-by-case.

DEFINITION 3.1: The environment is *Regular* if Conditions R.1–R.3 below hold, and is *(TVC)-Regular* if Conditions R.4–R.5 below also hold.

R.1 *(Finite Value)* The value function P for (RP) is well-defined and finite-valued on $D \times S$.

R.2 *(FB Regularity)* The first-best value function Q^* satisfies, for all $\zeta \in \Xi^{\mathrm{FB}}$ and $\mathbf{v}_\zeta^{(0)} \in \mathcal{U}^d$,

$$\liminf_{t\to\infty} \alpha^t \Big[\inf_{h\in\mathcal{H}} Q^*\big(\mathbf{v}_\zeta^{(t)}(h), s^{(t)}(h)\big)\Big] \geq 0.$$

R.3 *(Constraint Qualification)* For all $\mathbf{v} \in D$, we have $\Gamma_\circ(\mathbf{v}) \neq \emptyset$, where $\Gamma_\circ(\mathbf{v}) \subseteq \Gamma(\mathbf{v})$ is the subset of feasible menus at $\mathbf{v}$ that satisfy all of the incentive compatibility constraints (IC_{ij}) (for $i > j \in S$) as *strict* inequalities.

R.4 *((TVC) Existence)* There exists an optimal contract ξ^* that is (TVC)-implementable at every $\mathbf{v} \in D$.

R.5 *(Smooth Value)* The value function $P(\cdot, s)$ for (RP) is differentiable on D for each $s \in S$.

Regularity (R.1–R.3) is a mild technical assumption that facilitates analysis of (RP). Conditions R.1 and R.2 ensure that the principal's value function is well-defined and that an optimal contract exists. Condition R.3 ensures the existence of Lagrange multipliers, allowing us to use Lagrangian methods. One can obtain sufficient conditions for Regularity in terms of model primitives. For instance, Regularity holds for CARA utility and type processes with positive serial correlation (Lemma B.2 in Appendix B.2).

The assumption of (TVC)-Regularity comprises two additional technical conditions (R.4–R.5) that can be verified in special cases but are harder to check in general. First, Condition R.4 ensures that *some* optimal contract in the recursive problem (RP) is feasible in the sequential problem (SP). Were this conclusion to fail, (RP) would be a strict relaxation of (SP); we view this possibility as pathological. Second, Condition R.5 ensures that $P(\cdot, s)$ is continuously differentiable *everywhere*, rather than merely *almost* everywhere (see Proposition 3.2 below).[27] In Supplemental Appendix H, we outline an approach to verifying Condition R.5 based on the envelope theorem of Rincón-Zapatero and Santos (2009). Both of Conditions R.4 and R.5 (or suitable versions thereof) can be verified under CARA utility in the two limiting cases of i.i.d. and random walk type processes.[28]

3.3. *Bellman Equation*

To conclude the recursive formulation, we note that the principal's value function P satisfies a familiar Bellman equation. This equation is useful for generating optimal contracts and sketching the proofs of our main results in Section 4 below.

[27]Under Regularity, Proposition 3.2 shows that $P(\cdot, s)$ is convex, which implies that it is (i) differentiable almost everywhere and (ii) continuously differentiable at every point of differentiability.

[28]In the i.i.d. case, see Atkeson and Lucas (1992), Green (1987), and Thomas and Worrall (1990) for Condition R.4 and Supplemental Appendix H for Condition R.5. In the permanent shock case, see Bloedel, Krishna, and Strulovici (2023, 2024) for suitable versions of Conditions R.4 and R.5.

PROPOSITION 3.2: *Suppose the environment is Regular. Then the principal's value function* $P : D \times S \to \mathbb{R}$ *satisfies the functional equation*

$$P(\mathbf{v}, s) = \min_{(u_i, \mathbf{w}_i)_{i\in S} \in \Gamma(\mathbf{v})} \sum_{i\in S} f_{si}\big[C(u_i, i) + \alpha P(\mathbf{w}_i, i)\big], \tag{FE}$$

is the pointwise smallest solution to (FE) *that is pointwise greater than* Q^*, *and is convex in its first argument. Moreover*:

(a) *There exists an optimal contract* ξ^*, *and any contract* ξ *generated by* P *via a policy function from* (FE) *is optimal.*

(b) *If the environment is* (TVC)*-Regular, then* $P(\cdot, s)$ *is continuously differentiable and* strictly *convex for each* $s \in S$, *and there exists a* unique *optimal contract* ξ^*, *which is continuous on* $D \times S$.

The routine but lengthy proof of Proposition 3.2 is in Section J of Bloedel, Krishna, and Leukhina (2025b), where we also establish some further properties of P and ξ^*.[29]

4. LONG-RUN OUTCOMES

Sections 4.1 and 4.2 present our main results on long-run properties of optimal contracts. Section 4.3 provides a sketch of the proof.

4.1. *Immiseration*

Our first main result establishes that the optimal contract always induces *immiseration*: the agent's utility and consumption tend to their lower bounds, so he becomes impoverished in the long run. To state it, let $c^{(t)} := C(u^{(t)}, s^{(t+1)})$ denote the process of consumption transfers under the optimal contract. (For convenience, we henceforth omit ξ^* subscripts on all processes induced by the optimal contract.)

THEOREM 1—Immiseration: *Suppose the environment is* (TVC)*-Regular. Under the optimal contract, as* $t \to \infty$:

(a) *Promised utilities decrease without bound*: $v_i^{(t)} \to -\infty$ *in probability for all* $i \in S$.

(b) *Flow utilities decrease without bound*: $u^{(t)} \to -\infty$ *in probability.*

(c) *Consumption converges to its lower bound*: $c^{(t)} + \omega^{(t)} \to \underline{c}$ *in probability.*

Consequently, none of the promised utility, flow utility, or consumption processes have limiting distributions, and the state process $(\mathbf{v}^{(t)}, s^{(t)})$ *does not have a stationary distribution.*

The proof of Theorem 1 is in Supplemental Appendix C. We sketch the argument in Section 4.3 and discuss the importance of Assumption Markov for this result in Section 6.2. As a corollary of *the proof of* Theorem 1, we also obtain a stronger mode of convergence in the two special cases considered in prior work.[30]

COROLLARY 4.1: *The convergence in Theorem 1 can be strengthened from "in probability" to "almost surely" if types are either* (*i*) *i.i.d.* ($\mathbf{f}_i = \mathbf{f}_j$ *for all* $i, j \in S$) *or* (*ii*) *binary* ($d = 2$).

The proof of Corollary 4.1 is in Supplemental Appendix D. The i.i.d. case essentially restates the main result of Thomas and Worrall (1990). The binary-state case can be viewed

[29]The value function P is unbounded and diverges near the boundaries of the domain D, so we cannot use standard contraction-mapping results. Instead, we appeal to first-principles and order-theoretic arguments.

[30]We conjecture that, in general, the convergence in parts (a)–(c) of Theorem 1 occurs almost surely.

as a generalization of the immiseration result in Zhang (2009), which further assumes positive correlation and symmetric transition probabilities (i.e., $f_{11} = f_{22} \geq f_{12} = f_{21}$).

The basic intuition for Theorem 1 is that the agent's risk aversion implies that providing incentives is cheaper for the principal when the level of the agent's (promised) utility is lower (Thomas and Worrall (1990)). In particular, while effective insurance provision requires the principal to give larger transfers to the agent when his reported endowment is lower, incentive compatibility requires these larger transfers in the current period to be accompanied by lower continuation utility (i.e., lower transfers, on average, in future periods). Thus, the agent's flow and continuation utilities must vary with his reported endowment, with larger variation corresponding to higher-powered incentives. All else equal, the cost of incentive provision is lower for the principal when the level of the agent's utility is lower, as this is when the agent's *marginal* utility of consumption is higher. For a simple illustration, suppose that $d = 2$ and the principal wants to induce a spread of $\varepsilon > 0$ between the two types' flow utilities (i.e., $u_2 - u_1 = \varepsilon$). If the previous report was $s \in S$ and $\varepsilon > 0$ is small, this costs the principal approximately

$$\underbrace{f_{s1}C(u_1, 1) + f_{s2}C(u_1, 2)}_{\text{level}} + \underbrace{\varepsilon \cdot f_{s2}C'(u_1, 2)}_{\text{variability}}.$$

Thus, because $C(\cdot, 2)$ is convex (due to risk aversion), the cost of inducing utility variability increases with the utility level.[31] In the long run, it is optimal for the principal to drive this cost of incentive provision to zero by immiserating the agent. The next two sections explain why this is so: Section 4.2 provides economic intuition in terms of the power of the optimal contract's incentives, and Section 4.3 sketches the formal proof of Theorem 1.

4.2. *Backloaded Incentives and Relative Immiseration*

Intuitively, immiserating the agent allows the principal to optimally smooth her costs over time. As the level of (promised) utility decreases, it becomes affordable for the principal to provide *high-powered incentives*, that is, make the sensitivity of the agent's continuation utility with respect to his report increase without bound. By *backloading* high-powered but cheap incentives in later periods, the principal can make utility and consumption less variable in earlier periods while maintaining incentive compatibility, thereby reducing the cost of incentive provision in those early periods. This pattern of incentive provision allows her to intertemporally smooth, and thus minimize, her costs.

Our second main result formalizes this idea. To state it, we require two definitions. First, note that at the beginning of period t, when the state $(\mathbf{v}^{(t)}, s^{(t)})$ is given but before the current type $s^{(t+1)}$ is realized, the agent's type-contingent continuation utility $v^{(t)}_{s^{(t+1)}}$ is a random variable with support $\{v^{(t)}_1, \dots, v^{(t)}_d\}$. We denote by

$$\mathsf{V}\big(v^{(t)}_{s^{(t+1)}} \big| \mathbf{v}^{(t)}, s^{(t)}\big) := \sum_{i=1}^{d} f_{s^{(t)},i} \left(v^{(t)}_i - \sum_{i=1}^{d} f_{s^{(t)},i} v^{(t)}_i \right)^2 \tag{4.1}$$

the variance of this random variable, conditional on information available at the beginning of period t. Second, we focus on type processes that exhibit positive serial correlation:

[31] See Banerjee and Newman (1991) or Newman (2007) for a clear articulation of this relation between wealth (or utility) levels and risk-bearing in a related class of principal-agent problems with production.

DEFINITION 4.2: The type process is *FOSD-ordered* (or simply *FOSD*) if $\mathbf{f}_i \in \Delta(S)$ first-order stochastically dominates $\mathbf{f}_j \in \Delta(S)$ whenever $i > j$.

FOSD is the standard notion of positive serial correlation in the literature. Every i.i.d. process is FOSD. When $d = 2$, FOSD is equivalent to $f_{22} \geq f_{12}, f_{11} \geq f_{21}$.

THEOREM 2—Backloaded Incentives: *Suppose the environment is* (TVC)*-Regular and the type process is FOSD. Under the optimal contract, as* $t \to \infty$:
(a) $v_i^{(t)} - v_{i-1}^{(t)} \to +\infty$ *in probability for all* $i = 2, \dots, d$.
(b) $\mathsf{V}(v^{(t)}_{s^{(t+1)}} | \mathbf{v}^{(t)}, s^{(t)}) \to +\infty$ *in probability.*

The proof of Theorem 2 is in Supplemental Appendix E. Again, we obtain a stronger mode of convergence in the two special cases from Corollary 4.1.

COROLLARY 4.3: *The convergence in Theorem 2 can be strengthened from "in probability" to "almost surely" if types are either* (*i*) *i.i.d.* ($\mathbf{f}_i = \mathbf{f}_j$ *for all* $i, j \in S$) *or* (*ii*) *binary* ($d = 2$).

While we have argued that Theorem 2 in some sense explains why immiseration arises in the first place, we derive it as a simple consequence of Theorem 1. We sketch the argument here. As in Section 3.1, substituting the promise keeping constraint (PK_i) for type j into the incentive constraint (IC_{ij}) for type $i > j$ gives an alternative way to write the latter:

$$v_i - v_j \geq \underbrace{\psi(u_j, i, j) - u_j}_{\text{i.i.d. info rent}} + \underbrace{\alpha\big(\mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_j] - \mathsf{E}^{\mathbf{f}_j}[\mathbf{w}_j]\big)}_{\text{Markov info rent}}. \qquad (\text{IC}^*_{ij})$$

For FOSD type processes, Theorem 3 in Appendix B guarantees that the Markov information rent term is non-negative, so that both the i.i.d. and Markov information rents work in the same direction (recall Section 3.1 for the $d = 2$ case). Theorem 1(b) implies that the i.i.d. information rent term, which is always non-negative, grows without bound. It then follows from (IC^*_{ij}) (with $j = i - 1$) that the difference $v_i - v_{i-1}$ must also explode.

Consequently, the agent's continuation utility becomes increasingly uncertain over time: Theorem 2(b) shows that the conditional variance (4.1) diverges. That is, the agent's uncertainty at the start of period t (before observing his period-t type) about his future prospects (his continuation utility after reporting his period-t type) increases without bound as $t \to \infty$. In this sense, the quality of risk sharing degrades in the long run.

Another interpretation of Theorem 2 is that the optimal contract induces *relative immiseration*: the difference in promised utilities across different types of agents increases without bound, so that low-type agents become impoverished (in utility terms) *relative to* high-type agents (who also become impoverished). More concretely, imagine two agents, Alice and Bob, who have received the same sequence of realized endowments up through period $t - 1$. In period t, Alice receives a higher endowment than Bob. Theorem 2(a) states that this *transient* difference in Alice and Bob's endowments translates to an arbitrarily large difference in *lifetime* continuation utility as $t \to \infty$. Viewed in this light, Theorem 2 expresses the idea that "pathwise welfare inequality" increases without bound in the long run.[32]

[32]This complements Atkeson and Lucas's (1992) main finding that the *unconditional* (i.e., from the period-0 perspective) variance of period-t ex ante promised utility grows without bound as $t \to \infty$. In their model,

4.3. *Proof Sketch for Theorem 1*

The proof consists of several steps, which we sketch in sequence below. Building on Thomas and Worrall (1990), the basic idea is to identify a martingale that embodies the principal's optimal cost-smoothing and to show that convergence of this martingale corresponds to the contract inducing immiseration. However, relative to the i.i.d. benchmark, persistence gives rise to both substantively new economic features of the optimal contract and technical complications that must be overcome in the proof.

Step 1: Marginal Cost Martingale. We first formalize the principal's optimal cost-smoothing in terms of a martingale process (cf. an "Euler equation"). In the special case of i.i.d. types, Thomas and Worrall (1990) show that the principal's marginal cost of increasing the agent's ex ante promised utility is such a martingale. In particular, when types are i.i.d., we can write the principal's value function as $\hat{P}(v)$, where $v = \mathsf{E}^{\mathbf{f}}[\mathbf{v}]$ is the agent's ex ante promised utility under the (type-independent) transition probability $\mathbf{f} \in \Delta(S)$. The derivative $\hat{P}'(v)$ represents the principal's marginal cost of promised utility.

In our general Markovian setting, the appropriate martingale is a subtle but natural generalization of this object. Denote the derivative of P at $(\mathbf{v}, s)$ by $\mathsf{D}P(\mathbf{v}, s) = (P_1(\mathbf{v}, s), \dots, P_d(\mathbf{v}, s))$, where $P_i(\mathbf{v}, s)$ is the partial derivative with respect to the component v_i. Let $\mathsf{D}_{\mathbf{1}}P(\mathbf{v}, s) := \lim_{\varepsilon \to 0}[P(\mathbf{v} + \varepsilon\mathbf{1}, s) - P(\mathbf{v}, s)]/\varepsilon = \sum_{i \in S} P_i(\mathbf{v}, s)$ denote the directional derivative of $P(\cdot, s)$ in direction $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^d$.

PROPOSITION 4.4: *If the environment is* (TVC)*-Regular, then the process* $(\mathsf{D}_{\mathbf{1}}P(\mathbf{v}^{(t)}, s^{(t)}))_{t=0}^{\infty}$ *induced by the optimal contract is a strictly positive martingale.*

The proof of Proposition 4.4 is in Supplemental Appendix C.2. We henceforth refer to this process as the principal's *marginal cost martingale*. Proposition 4.4 has three parts:

(a) *Martingale Property*: The marginal cost process defines a martingale because the principal *optimally smooths costs* over time and across states. Fix some $\mathbf{v} \in D$ and consider the cost to the principal of increasing this promise to $\mathbf{v}' := \mathbf{v} + \varepsilon\mathbf{1}$ for some $\varepsilon > 0$. When $\varepsilon > 0$ is small, this cost is approximated by the marginal cost $\mathsf{D}_{\mathbf{1}}P(\mathbf{v}, s)$. One way to deliver the additional utility in an incentive compatible manner is to increase the continuation promises from $\mathbf{w}_i$ to $\mathbf{w}_i' := \mathbf{w}_i + (\varepsilon/\alpha)\mathbf{1}$ for all $i \in S$. Using the Bellman equation (FE), the cost of this perturbation is approximately $\sum_{i=1}^{d} f_{si}\mathsf{D}_{\mathbf{1}}P(\mathbf{w}_i, i)$. By an envelope argument, this perturbation is locally optimal, yielding equality of these two expressions and thus the martingale property $\mathsf{D}_{\mathbf{1}}P(\mathbf{v}, s) = \sum_{i=1}^{d} f_{si}\mathsf{D}_{\mathbf{1}}P(\mathbf{w}_i, i)$.

(b) *Differentiation in Direction* $\mathbf{1}$: This is the *unique* direction of change for $\mathbf{v}$ that increases the agent's ex ante continuation utility *while leaving his information rent unchanged*. This can be seen from the incentive constraint (IC_{ij}^*) in two ways. First, a perturbation of $\mathbf{v}$ leaves the left-hand side of (IC_{ij}^*) unchanged if and only if it is taken in the direction $\mathbf{1}$. Second, suppose the principal wants to perturb $\mathbf{w}_i$ to some $\hat{\mathbf{w}}_i$ such that type i's ex ante continuation utility increases by $\varepsilon > 0$, that is,

which considers a society with a continuum of agents and a per-period aggregate resource constraint, this implies that the society's *cross-sectional* inequality explodes due to the *cumulative* effect of all past shocks (in that some agents will have received unboundedly more total endowment in periods $\tau \leq t$ than other agents as $t \to \infty$). Theorem 2 identifies the complementary property that a *single* shock leads to unboundedly large welfare differences between two agents with *otherwise identical histories* as $t \to \infty$.

$\mathsf{E}^{\mathbf{f}_i}[\hat{\mathbf{w}}_i] - \mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_i] = \varepsilon$. Other types $j \neq i$ may value this perturbation differently if $\mathbf{f}_j \neq \mathbf{f}_i$. By setting $\hat{\mathbf{w}}_i - \mathbf{w}_i = \varepsilon\mathbf{1}$, the principal guarantees that *all* types value this perturbation in the same way, that is, that $\mathsf{E}^{\mathbf{f}_j}[\hat{\mathbf{w}}_i - \mathbf{w}_i] = \varepsilon$ for *all* $j \in S$, ensuring that the perturbation preserves (IC^*_{ij}).[33]

(c) *Strict Positivity*: Somewhat subtly, the partial derivatives $P_i(\mathbf{v}, s)$ may be *negative*, as increasing a single component v_i of $\mathbf{v}$ has two countervailing effects.[34] First, it increases the promise to type i, which mechanically increases costs. Second, it tightens incentive constraints for higher types $j > i$ that can misreport as type i, but also adds slack to type i's incentive constraints. Whenever it adds "enough" slack to the latter incentive constraints, increasing v_i can lead to an overall decrease in costs. Nevertheless, the marginal cost martingale is always positive because, as described above, differentiation in the direction $\mathbf{1}$ does not affect any incentive constraints. This will be important for applying the Martingale Convergence Theorem below.

To preview how Proposition 4.4 leads to Theorem 1, we recall two benchmarks. First, when the agent does not have private information, the first-best contract fully stabilizes the agent's consumption and utility (Section 3.1), yielding a *constant* marginal cost martingale.

Second, when the agent's type is private but i.i.d. over time (Thomas and Worrall (1990)), the principal provides incentives by making the agent's ex ante continuation utility $w_i = \mathsf{E}^{\mathbf{f}}[\mathbf{w}_i]$ vary with his reported type i (where $\mathbf{f}$ denotes the type-independent transition probability). The optimal contract sets $w_d > v > w_1$ in order to (a) reward the highest-type agents (who get the lowest current transfers) for not under-reporting and (b) punish the lowest-type agents, so as to deter other types from under-reporting. Consequently, the Thomas and Worrall (1990) martingale always *splits*: $\hat{P}'(w_1) < \hat{P}'(v) < \hat{P}'(w_d)$. This is costly to the principal because her value function is convex, where the "size" of martingale splitting quantifies the *cost of incentives*, for example, the difference $\hat{P}'(w_d) - \hat{P}'(v) > 0$ is determined by the magnitude of the Lagrange multipliers on the agent's incentive constraints. Thomas and Worrall (1990) show that $\hat{P}'(v^{(t)}) \to 0$ almost surely, that is, the cost of incentives *vanishes* in the long run. We sketch their argument as follows. First, since their value function $\hat{P}$ is increasing, the Martingale Convergence Theorem guarantees that $\hat{P}'(v^{(t)})$ converges to *some* non-negative random variable. Second, because their state variable v is one-dimensional and their value function $\hat{P}$ is strictly convex and satisfies Inada conditions, the derivative $\hat{P}' : \mathcal{U} \to \mathbb{R}_{++}$ is a *bijection*. These two facts imply that $\hat{P}'(v^{(t)})$ must converge to zero along almost every path. To see why, note that for any path along which $\hat{P}'(v^{(t)})$ converges to a strictly positive number, bijectivity implies that $v^{(t)}$ must converge to some $v^* \in \mathcal{U}$. But then, letting $(w^*_i)_{i\in S}$ denote the optimal (ex ante) continuation utilities at v^*, martingale convergence requires that $\hat{P}'(w^*_1) = \hat{P}'(v^*) = \hat{P}'(w^*_d)$, which contradicts martingale splitting at v^*. Thus, such paths must arise with zero probability. It follows that $\hat{P}'(v^{(t)}) \to 0$ and therefore (by bijectivity) that $v^{(t)} \to -\infty$ almost surely.

[33] In particular, for generic transition matrices $[\mathbf{f}_i]_{i=1}^d$, the direction $\mathbf{1}$ is the *unique* direction with this property.

[34] See Theorem 6 in Section J of Bloedel, Krishna, and Leukhina (2025b) for a formal statement. Economically, the "interim" Pareto frontier that traces the principal's costs as a function of $\mathbf{v}$ is generally *not* monotonic. This is closely related to the fact that the optimal contract is generally not renegotiation-proof (see Step 2 below).

Step 2: Martingale Splitting and Convergence. Persistence greatly complicates the above logic. Economically, it implies that the agent's current type determines his preferences over continuation contracts, giving the agent new Markov information rents and the principal new opportunities to *screen the agent through continuation contracts* that manipulate those rents (recall (IC^*_{ij})). For example, with persistence, it may be possible for the principal to incentivize truthtelling while also making the agent's ex ante continuation utility $\mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_i]$ constant across all types $i \in S$ by suitably choosing the interim continuation promises $(\mathbf{w}_i)_{i\in S}$ to distort the continuation payoffs $\mathsf{E}^{\mathbf{f}_j}[\mathbf{w}_i]$ for non-truthful types $j \neq i$. Given this new channel for incentive provision, it is no longer clear whether the marginal cost martingale splits (i.e., $\mathsf{D}_1 P(\mathbf{w}_1, 1) < \mathsf{D}_1 P(\mathbf{v}, s) < \mathsf{D}_1 P(\mathbf{w}_d, d)$) and, if it does, what relation this has to the cost of incentives. Furthermore, because $\mathsf{D}_1 P(\cdot, s) : D \to \mathbb{R}_{++}$ is *not* injective in our setting due to the multidimensionality of $\mathbf{v} \in D$, it is unclear how to connect convergence of the martingale to limiting properties of the $\mathbf{v}^{(t)}$ process.

To overcome these challenges, we first identify a special class of histories at which the principal does not need to screen through continuation contracts. Suppose that the agent has revealed himself to be of type i in period t, and the principal wants to give him ex ante promised utility $w \in \mathcal{U}$ starting in period $t+1$ (before his period-$(t+1)$ type is realized). The cost-minimizing way to do this is to solve the *efficiency problem*

$$K(w, i) := \min_{\mathbf{w}_i \in D} P(\mathbf{w}_i, i) \quad \text{s.t. } \mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_i] \geq w. \tag{Eff_i}$$

We call solutions to (Eff_i) *efficient*. Efficiency corresponds to a kind of *renegotiation-proofness*: even *after* the agent reveals himself to be of type i, so that incentive compatibility in the current period is no longer a concern, there is no way for the principal to reduce her continuation costs while also improving the agent's (expected) continuation payoff. The i.i.d. case is simple precisely because the optimal contract is always renegotiation-proof in this sense (see Remark C.7).[35] But with persistence, the optimal contract is typically *not* renegotiation-proof because the principal's choice of $\mathbf{w}_j$ affects the agent's Markov information rent in each (IC^*_{ij}) with $i > j$. However, Assumption NHB implies that the highest type's promised utility $\mathbf{w}_d$ does not enter any of these incentive constraints. Thus, the optimal contract *is* efficient at those histories where the agent had reported the highest endowment ω_d in the previous period (Lemma C.8). Moreover, at such histories, the marginal cost martingale splits and the size of its splitting pins down the cost of incentives, in analogy to the i.i.d. case (Lemmas C.9 and C.10). In particular, at such histories, the martingale does *not* split if and only if the contract perfectly stabilizes the agent's consumption, which is inconsistent with incentive compatibility (Lemma C.18).

Next, we use these observations to show that the marginal cost martingale converges to 0 almost surely. By Proposition 4.4, the Martingale Convergence Theorem implies that the marginal cost martingale converges to a non-negative, integrable random variable. To show that this limit random variable equals 0, we argue as in Figure 2, which illustrates the special case of binary types ($d = 2$). The *efficiency rays* $\tilde{E}_i$ characterize the solutions of

[35]This is the same notion of renegotiation-proofness that is informally discussed in Thomas and Worrall (1990, p. 369), and which corresponds to having a monotonic Pareto frontier. We do not delve into the subtleties of formally defining renegotiation-proof contracts (cf. Strulovici (2017, 2020)). Also note that the efficiency problem (Eff_i) is analogous to what Fernandes and Phelan (2000) call the "planner's problem," while our recursive problem (RP) is analogous to what they call the "auxiliary planner's problem."

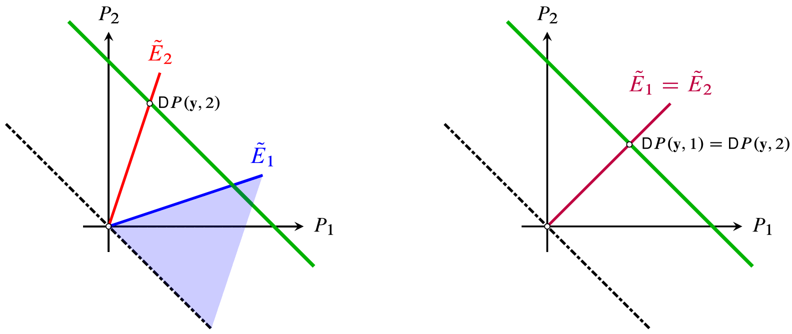

FIGURE 2.—Martingale convergence argument for $d = 2$ and FOSD (left) and i.i.d. (right) types.

the efficiency problem (Eff$_i$) in terms of dual variables, namely, the derivative $\mathsf{D}P(\mathbf{w}_i, i)$, which is pinned down by the first-order conditions for (Eff$_i$). When types are i.i.d., there is a single efficiency ray (corresponding to the one-dimensional state space in Thomas and Worrall (1990)), and the derivative process traverses it because the optimal contract is renegotiation-proof. Consequently, a similar argument to that in Thomas and Worrall (1990) shows that if the marginal cost martingale were to converge to a positive number along some path, then the agent's promised utility vector would also converge: $\mathbf{v}^{(t)} \to \mathbf{y} \in D$ almost surely.[36] This is impossible by the martingale splitting results described above, so the martingale must converge to zero along almost every path. When types are persistent, we show that a similar argument applies *along the subsequence of histories* at which $s^{(t)} = d$ (i.e., the agent truthfully reported to have the highest endowment ω_d in the previous period), where the optimal contract is efficient. Finally, we show that this convergence *extends to (almost) all histories*, establishing that $\mathsf{D}_1 P(\mathbf{v}^{(t)}, s^{(t)}) \to 0$ almost surely (Lemma C.19).

Step 3: Convergence of Multipliers and Allocations. As noted above and depicted in Figure 2, the principal's marginal cost $\mathsf{D}_1 P(\cdot, s)$ is *not* injective when types are persistent. Thus, even though the marginal cost martingale converges to 0, the agent's promised utility $\mathbf{v}^{(t)}$ may remain transient along histories where the optimal contract is not efficient.[37]

To rule this out, we show that martingale convergence implies that the cost of incentives (i.e., the vector of Lagrange multipliers on the agent's incentive constraints) also converges to zero. Given this, it can be deduced that the contract either (a) converges to the first-best or (b) immiserates the agent, only the latter of which is consistent with incentive compatibility. These arguments are involved, but the main ideas are as follows. When types are i.i.d. or binary (as in Corollary 4.1), we can show that, in essence, the marginal cost martingale provides a uniform bound for the multipliers, so that almost sure convergence of the former implies almost sure convergence of the latter. For general type processes, while we cannot rule out that the vector of multipliers take infinitely-many excursions away from zero, Assumption Markov implies that the agent's endowment almost surely returns to the highest level ω_d infinitely often. Using the fact that the contract is efficient at such histories, we show that the multiplier process exhibits a "renewal property"

[36]The facts that (i) $\mathbf{v}^{(t)}$ converges and (ii) its limit point is in D (rather than being a boundary point) rely on the Inada conditions in Assumption DARA(a). See Section 6.1 for further discussion.

[37]For instance, in the left panel of Figure 2, the derivative $\mathsf{D}P(\mathbf{v}^{(t)}, s^{(t)})$ may traverse the blue region below $\tilde{E}_1$ at such histories even as $\mathsf{D}P_1(\mathbf{v}^{(t)}, s^{(t)})$ (depicted by the green line) converges to zero.

whereby it returns to a neighborhood of zero infinitely often; moreover, it mixes quickly enough to ensure that long excursions away from zero are very rare. This lets us show that the multipliers converge to zero in probability (Lemma C.22). Given convergence of the multipliers, convergence of consumption and utility follows quickly (Lemma C.23).

5. SOLVED EXAMPLES

We now turn to the optimal contract's short- and medium-run properties. Section 5.1 investigates the *speed* of immiseration. Section 5.2 studies short-run dynamics and distortions.

For these purposes, we focus on the special case of CARA utility and $d = 2$ types with FOSD transitions (and assume (TVC)-Regularity). While restrictive, these assumptions yield two key benefits. First, they permit a direct comparison to the i.i.d. benchmark, in which full solutions are known only under CARA utility (Thomas and Worrall (1990), Green (1987)). Second, they allow for a simple graphical depiction of the optimal contract's dynamics. We can analytically derive structural properties of the optimal contract in this setting (Properties (a)–(e) in Section 5.2), but a full solution requires numerical simulations.

Parameters. For the numerical simulations, we use the following parameters. The two endowment levels are $\omega_1 = -\log(5)$ and $\omega_2 = \log(10)$. The agent's utility function is $U(c + \omega_i) = -e^{-(\omega_i + c)}$ and his discount factor is $\alpha = 0.5$. For simplicity, the type process has *symmetric* transitions: $q := f_{11} = f_{22}$ and $q \geq 1/2$.[38] We focus on three persistence levels: $q = 0.5$ (i.i.d.), $q = 0.65$ (low persistence), and $q = 0.8$ (high persistence). In Section 5.1, we also consider $q = 0.95$ (very high persistence).

5.1. *How Fast Is the Ride on the Highway to Hell?*

To begin, we investigate how persistence affects the *speed* of immiseration. Our main finding is that, in the medium- and long-run, greater persistence yields *faster* immiseration.

For this exercise, it is useful to slightly reinterpret our model. Consider, as in Green (1987), the problem of a utilitarian planner presiding over an economy populated by a continuum of homogeneous agents who face idiosyncratic endowment shocks. The economy is *open*: the planner can finance consumption in each period by borrowing and saving in the "world market" at the risk-free rate $R = 1/\alpha$, and therefore faces a single intertemporal resource constraint. It is well known that our principal-agent problem is equivalent to the *dual* of the open economy planning problem (by viewing the open-economy resource constraint as our principal's objective function).[39] Given this correspondence, we find it most insightful to focus on the time evolution of two statistics: (i) *mean consumption* $\mu_{C,t} := \mathbf{E}[c^{(t)} + \omega^{(t)}]$ at time t and (ii) the *variance of consumption* $\sigma^2_{C,t} := \mathbf{V}[c^{(t)} + \omega^{(t)}]$ at time t, where the expectations are over paths of the type process from the perspective of time 0. We can then interpret (i) $\mu_{C,t}$ as describing the open economy's *aggregate consumption* at time t, and (ii) $\sigma^2_{C,t}$ as describing its *cross-sectional consumption inequality* at time t. We explore how persistence shapes their evolution.

[38]Neither our analytical results nor the qualitative findings from our numerical simulations rely on this symmetry assumption. It merely simplifies the exposition by (i) letting us parameterize persistence with a *single* number and (ii) ensuring that the stationary distribution of types is *fixed* (at (0.5, 0.5)) as we vary persistence.

[39]See, for example, Golosov, Tsyvinski, and Werquin (2016) or Green (1987).

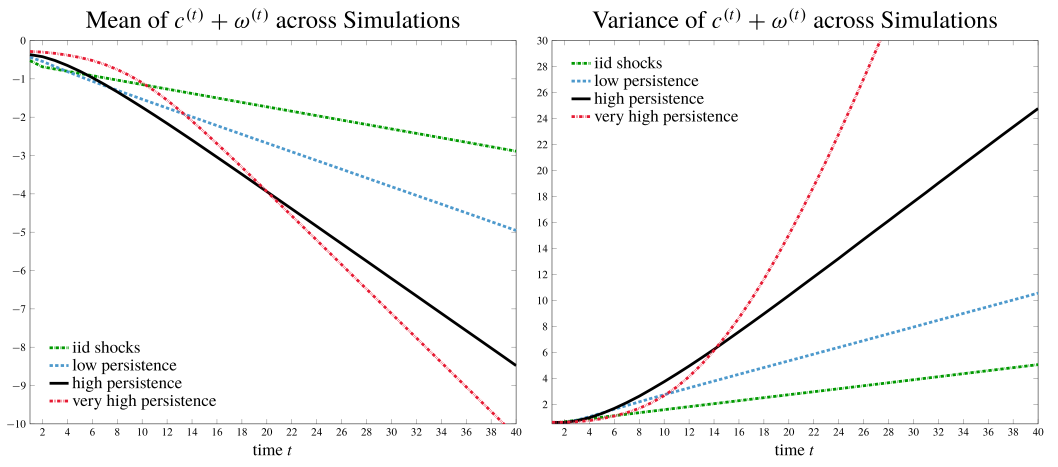

FIGURE 3.—Mean consumption (left) and variance of consumption (right) across simulations.

To implement this numerically, for each of the persistence levels $q \in \{0.5, 0.65, 0.8, 0.95\}$, we simulate 420,000 sample paths of the optimal contract for 40 time periods.[40] Figure 3 plots the corresponding sample means and variances of consumption (which serve as proxies for $\mu_{C,t}$ and $\sigma^2_{C,t}$, respectively). The following key patterns emerge:

1. *The economy tends towards aggregate misery and infinite inequality*: at all persistence levels, $\mu_{C,t} \to -\infty$ and $\sigma^2_{C,t} \to +\infty$. These features are consistent with Theorem 1 and the fact that, in the i.i.d. case, consumption follows a random walk with negative drift (Green (1987), Thomas and Worrall (1990)).[41] As in Green (1987), the fact that aggregate consumption $\mu_{C,t} \to -\infty$ effectively means that the planner finances consumption in early periods by going into unbounded debt in later periods.
2. *In the medium-run, greater persistence yields faster immiseration and higher inequality*: after the first few periods, greater persistence causes $\mu_{C,t}$ to decrease more quickly and $\sigma^2_{C,t}$ to increase more quickly.
3. *In the short-run, the patterns in Point 2 are reversed*: for the first few periods, greater persistence causes $\mu_{C,t}$ to decrease more slowly and $\sigma^2_{C,t}$ to increase more slowly. Furthermore, the initial aggregate consumption level $\mu_{C,0}$ increases with persistence.

We view Point 2 as being the primary lesson of this exercise, and interpret it as suggesting that private information has an even more pernicious effect on welfare and equity outside the classic i.i.d. benchmark. In Section 5.2 below, we attribute this comparative static result to novel intertemporal distortions that arise from the agent's Markov information rents, and which become more severe as persistence increases.

To reconcile Points 2 and 3, note that the $\mu_{C,t}$ and $\sigma^2_{C,t}$ curves in Figure 3 are *eventually* linear, but (except in the i.i.d. case) are *initially* concave and convex, respectively.[42]

[40] We draw 21 initial $\mathbf{v}^{(0)}$ points uniformly from a grid in the domain $D = V_2$ (recall Figure 1), with the $v_1^{(0)}$ and $v_2^{(0)}$ values evenly spaced between -10 and -1. For each such $\mathbf{v}^{(0)}$, we run 20,000 simulations, half of which start from $\omega^{(0)} = \omega_1$. Thus, we effectively initialize the type process at its stationary distribution.

[41] Two clarifications: (i) as consumption is unbounded, Theorem 1 does not *directly* imply that $\mu_{C,t} \to -\infty$, and (ii) beyond the i.i.d. case, consumption does *not* follow a random walk (see Section 5.2 below).

[42] In the i.i.d. case, these curves are linear after an initial kink at $t = 1$. This kink arises because, in the i.i.d. case, the $\mathbf{v}^{(t)}$ process "immediately jumps" at $t = 1$ from its exogenous initial condition $\mathbf{v}^{(0)}$ to a specific subset of the domain that it remains in thereafter (see properties (c) and (d) in Section 5.2 for details).

Moreover, increasing persistence makes these curves "flatter" in the initial periods. Intuitively, this arises from the mechanical fact that greater persistence increases the "mixing time" that it takes for the promised utility process to become "stationary" in an appropriate sense (see Section 5.2 for a more precise discussion). For reference, it can be shown that the optimal contract under *perfect* persistence ($q = 1$) is "static," that is, it specifies a constant path of consumption conditional on the initial endowment report (which it imperfectly insures against). Thus, as we approach the perfect persistence limit ($q \to 1$), it is natural for the optimal contract to specify "almost constant" consumption in early periods (see the $q = 0.95$ case in Figure 3).[43] Finally, note that because we are holding the initial promised utility fixed, $\mu_{C,0}$ *must* increase with persistence to compensate for the faster negative trend in later periods.

Next, in Section 5.2, we study the optimal contract's short-run properties in detail. Later, in Section 6.3, we revisit Point 1 when discussing a *closed economy* variant of the planning problem described here, as in Atkeson and Lucas (1992).

5.2. *Short-Run Properties*

We begin by describing the optimal contract's basic structural properties. We then turn to the qualitative and quantitative implications of persistence.

Basic Structure. The joint assumptions of CARA utility and binary types ($d = 2$) yield two simplifications. First, CARA implies that optimal contracts are *homogeneous of degree* 1 *(HD1)* with respect to interim promises: $\xi^*(a\mathbf{v}, s) = a \cdot \xi^*(\mathbf{v}, s)$ for all states $(\mathbf{v}, s) \in D \times S$ and positive scalars $a > 0$.[44] Second, $d = 2$ implies that the domain $D = V_2 := \{\mathbf{v} \in \mathcal{U} : v_2 > v_1\}$ (by Theorem 3 in Appendix B), as illustrated in Figure 1. Thus, on the equilibrium path, the optimal contract has a simple structure determined by a countable infinity of rays in V_2. Figure 4 illustrates the following properties:

(a) There are a countable infinity of *rays* in V_2, denoted E_1, E_2, and B_k for $k \in \mathbb{N}$. Each E_i ray traces out solutions to the corresponding efficiency problem (Eff_i).
(b) Starting from *any* $\mathbf{v} \in V_2$, promised utility jumps to E_2 after a high shock ($i = 2$).
(c) Starting from $\mathbf{v}^{(0)} \in E_2$, a high shock sends promised utility to the "northeast" along E_2, while k *consecutive* low shocks send it to B_k. If there is a high shock after k consecutive low shocks, promised utility returns to E_2, and the cycle starts again. Due to HD1, the proportional scaling $\|\mathbf{v}^{(t+1)}\|/\|\mathbf{v}^{(t)}\|$ between pre- and post-shock promised utilities at each step does not depend on $\|\mathbf{v}^{(t)}\|$, only on the ray that contains $\mathbf{v}^{(t)}$.
(d) If types are i.i.d., all the rays coincide ($E_1 = E_2 = B_k$ for all k). But if types are persistent (FOSD but not i.i.d.), E_2 and the B_k rays lie strictly "below" E_1.
(e) The cyclicality and HD1 in Property (c) imply that the optimal contract induces a stationary distribution over the E_2 and B_k rays.[45] The right panel of Figure 4 plots

[43]Specifically, it is natural for the optimal contract to converge *pointwise* in the $q \to 1$ limit to the optimal "static" contract at $q = 1$ *along the constant histories that arise with positive probability at* $q = 1$. Point 2 indicates that this convergence is not uniform or does not occur along non-constant histories, so that even when $q \approx 1$, the compound effect of multiple type transitions eventually becomes dominant. See Battaglini and Lamba (2019, Section 4.3) and Krasikov and Lamba (2021, Proposition 2) for related points.

[44]This fact and Properties (a)–(e) below follow directly from the CARA, $d = 2$, and FOSD assumptions and the optimality conditions stated in Supplemental Appendices C.1 and C.3.3. For completeness, derivations are in Section K of Bloedel, Krishna, and Leukhina (2025b).

[45]Formally, the "slope" process $v_2^{(t)}/v_1^{(t)}$ has a stationary distribution on $(0, 1)$. The support characterizes $\{E_2, B_1, \dots\}$, while the probabilities are determined by the stationary distribution of the "counting process"

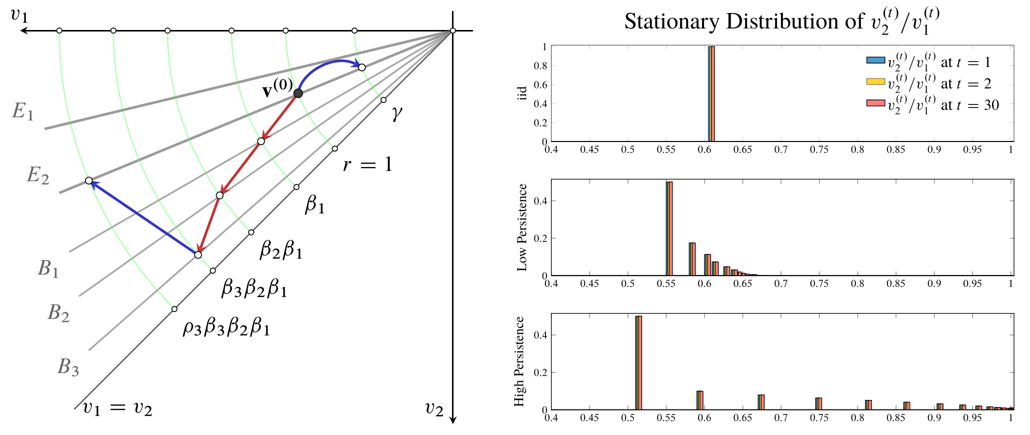

FIGURE 4.—*Left: Qualitative depiction of optimal contract.* Initial $\mathbf{v}^{(0)} \in E_2$ normalized so that $\|\mathbf{v}^{(0)}\| = 1$. Proportional scalings are $\gamma \in (0, 1)$ if first shock is *good*, $\beta_k > 1$ if kth consecutive shock is *bad*, and $\rho_k > 1$ when *returning* to E_2 with the first good shock after k consecutive bad shocks. *Right: Stationary distribution over E_2 and B_k rays.* Simulated distribution of $v_2^{(t)}/v_1^{(t)}$ given initial $\mathbf{v}^{(0)}$ drawn from the stationary distribution over rays, which is $\mathsf{P}(\mathbf{v}^{(0)} \in E_2) = 0.5$ and $\mathsf{P}(\mathbf{v}^{(0)} \in B_k) = 0.5(1 - q)q^{k-1}$ if all rays are distinct.

the simulated stationary distributions in our examples, indicating both (i) the slopes of E_2 and the B_k rays and (ii) the fraction of time spent on each. (Both panels of Figure 4 depict that B_1 lies strictly below E_2 and that, for all k, B_k lies strictly below B_{k-1}. This is true in all of our numerical examples, but we did not prove these properties.)

Next, we unpack the economic meaning and implications of these Properties (a)–(e).

Managing Markov Information Rents. As noted in Section 4.3, a fundamental implication of persistence is that the optimal contract is *not* renegotiaton-proof: it need *not* solve the efficiency problem (Eff_i), except after the highest type realizations ($i = d$). Intuitively, these inefficiencies *optimally reduce the agent's Markov information rents*.

Property (d) and our simulations help clarify this intuition. The fact that the B_k rays lie below E_1 means that, after a low shock, the agent's continuation utility $\mathbf{w}_1$ is pushed *inefficiently close* to the lower boundary of V_2. For any fixed expected continuation utility $\mathsf{E}^{\mathbf{f}_1}[\mathbf{w}_1]$ for a low-type agent, this distortion *compresses* the difference $w_{12} - w_{11} > 0$ and thus reduces a high-type agent's Markov information rent $\alpha(2q - 1)(w_{12} - w_{11}) > 0$, adding slack to ($\text{IC}^*_{21}$). However, it also yields a *tighter* incentive constraint (i.e., left-hand side of (IC^*_{21})) in the *next* period, when the new promised utility state is $\mathbf{w}_1$. This makes further compression *even more* attractive after a *second* consecutive low shock, and so on.

This is borne out in Figure 4. As depicted in the left panel, except in the i.i.d. case, B_k lies below B_{k-1} for all k, indicating greater compression as the number of consecutive low shocks increases. Furthermore, as depicted in the right panel, the B_k rays become steeper and more "spread out" as persistence increases, indicating greater compression for each fixed number of consecutive low shocks.

over the number of consecutive low shocks, that is, the time elapsed since the last high shock. This process is defined as $n^{(t)} := \inf\{k \in \mathbb{N} \cup \{0\} : s^{(t-k)} = 2\}$ and its stationary distribution is given by $\lim_{t\to\infty} \mathsf{P}(n^{(t)} = 0) = 0.5$ and $\lim_{t\to\infty} \mathsf{P}(n^{(t)} = k) = 0.5(1 - q)q^{k-1}$ for all $k \in \mathbb{N}$. If all the rays are distinct, $n^{(t)} = 0$ corresponds to $\mathbf{v}^{(t)} \in E_2$ and $n^{(t)} = k \geq 1$ corresponds to $\mathbf{v}^{(t)} \in B_k$, as described in Figure 4.

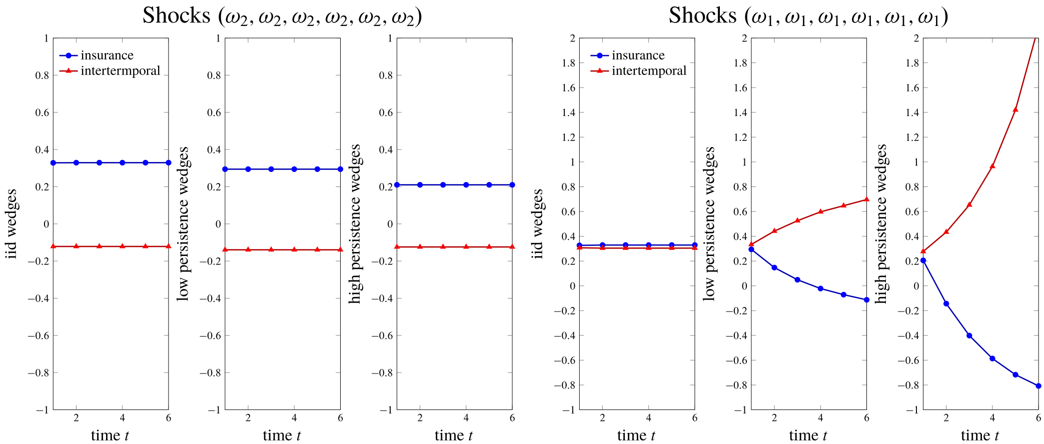

FIGURE 5.—Wedges along strings of six consecutive high (left) and low (right) shocks in periods $t \in \{1, \dots, 6\}$, starting from $\mathbf{v}^{(0)} \in E_2$. *Insurance wedge* $\equiv U'(c_1^{(t)} + \omega_1)/U'(c_2^{(t)} + \omega_2) - 1$ measures departures from within-period perfect insurance. *Intertemporal wedge* $\equiv \mathbf{E}[U'(c^{(t+1)} + \omega^{(t+1)})|\omega^{(t)}]/U'(c^{(t)} + \omega^{(t)}) - 1$ measures departures from Euler equation. Both are 0 in the first-best.

Implications. These forces shape economically important features of optimal contracts:

1. *Speed of Immiseration*: As noted in Section 5.1, greater persistence yields (i) *faster* immiseration in the medium-run, but (ii) *slower* immiseration in the short-run. We can now explain these findings as follows: greater persistence simultaneously (i) yields steeper B_k rays, so that $\mathbf{v}^{(t)}$ decreases (to the "southwest" in Figure 4) *more quickly* after long strings of consecutive low shocks (which also become more likely), and (ii) makes initial strings of high shocks more likely, increasing the "mixing time" for $\mathbf{v}^{(t)}$ to settle into its cyclical dynamics on the E_2 and B_k rays (Properties (c) and (e)).
2. *Mix of Incentives*: Persistence causes the optimal contract to use a different mix of *intra-* and *inter*-temporal incentives. To illustrate, Figure 5 plots the standard *insurance* and *intertemporal wedges* along sequences of consecutive high shocks (left panel) and consecutive low shocks (right panel). In the i.i.d. case, (i) the insurance wedge is *always* positive because the contract *under*-insures the agent (Thomas and Worrall (1990)) and (ii) the intertemporal wedge is positive after low shocks (which cause consumption to drift down). By contrast, with persistence, *after consecutive low shocks*, (i) the insurance wedge *becomes negative* (*over*-insurance), while (ii) the intertemporal wedge remains positive but *becomes an order of magnitude larger*. Both features arise from the fact that persistence yields steeper B_k rays: (i) over-insurance ($u_1 > u_2$) after low shocks serves to "compress" promised utilities (by increasing w_{11} relative to w_{12}), while (ii) promised (and hence flow) utilities decrease more quickly after low shocks.
3. *Order-Dependence*: With i.i.d. types and CARA utility, the optimal contract is order-independent: the agent's consumption and continuation utility depend only on the *number* of high and low shocks to date, but *not* the order in which they arrived. Thomas and Worrall (1990) attribute this to the absence of wealth effects under CARA. We find that it also depends critically on the i.i.d. assumption. In our examples with persistence, for a fixed number of high and low shocks, the agent is *worse*

off if the low shocks occurred *earlier*—initial bad luck is punished more than later bad luck.[46] Intuitively, this is driven by the following property (depicted in Figure 4): following a high shock, the low-type continuation utility $v_1^{(t)}$ *increases* if the preceding shock was also high ($\mathbf{v}^{(t-1)} \in E_2$) but may *decrease* if the preceding shock was low ($\mathbf{v}^{(t-1)} \in B_k$). The latter feature, which does not arise in the i.i.d. case, serves to *re-introduce slack* into the (IC_{21}^*) constraint after it was successively tightened along a string of low shocks.

6. DISCUSSION

Section 6.1 revisits our baseline model assumptions. Section 6.2 discusses the role of mean-reversion. Section 6.3 considers an extension to a closed-economy planning problem.

6.1. *The Role of Baseline Assumptions*

We discuss (i) Assumptions DARA and NHB and (ii) other sources of private information.

Shape of Utility Function. Assumption DARA plays both substantive and technical roles in our analysis. Part (a) embeds the economically substantive conditions needed for Theorem 1: strict monotonicity and concavity of U ensure that first-best insurance is not implementable (i.e., the incentive problem is nontrivial), while the upper Inada condition $\lim_{c\to\infty} U'(c) = 0$ ensures that there is no upper bound for the principal's cost of incentive provision (i.e., maintaining truthtelling requires unboundedly large variability in consumption as the level of utility increases).[47] The remaining pieces of Assumption DARA play technical roles: smoothness of U in part (a) lets us take well-behaved first-order conditions and is related to smoothness of the value function P, unboundedness below of U in part (b) implies that the optimal contract is interior, and DARA in part (c) implies that the principal's problem is convex, so that report-contingent randomization is not needed. It is an open question whether these technical assumptions can be relaxed.[48]

Feasible Reporting Strategies. Assumption NHB prevents the agent from over-reporting his type. We view this as a relatively mild assumption. First, under-reporting constraints appear to be the empirically relevant concern in hidden endowment settings (see Feldman and Slemrod (2007) for evidence from U.S. tax data). Second, it is often argued that NHB is without loss of generality if (i) the agent's private type represents a

[46]For example, consider period $t = 6$ after the agent has had three high shocks and three low shocks. In the high persistence ($q = 0.8$) case, the agent's flow and continuation utilities contingent on a low shock at $t = 6$ ($u_1^{(6)}$ and $v_1^{(6)}$) are approximately *two times lower* if the three prior low shocks occurred in periods $\{0, 1, 2\}$ than if they occurred in periods $\{3, 4, 5\}$.

[47]In the i.i.d. setting, Phelan (1998) observes that if $M := \lim_{c\to\infty} U'(c) > 0$, then the principal's marginal cost martingale is bounded above and thus converges to either 0 *or its upper bound*, both with positive probability. Consequently, the optimal contract generates *polarization*: both immiseration and long-run bliss arise with positive probability. The probability of immiseration nonetheless tends to 1 as $M \to 0$.

[48]We note that Theorem 2 *does* rely on the agent's utility function being unbounded below, for otherwise it is impossible to simultaneously immiserate the agent and provide unboundedly high-powered incentives. For instance, if $\inf_{c\in\mathcal{C}} U(c) = 0$, then immiseration corresponds to $v_i^{(t)} \to 0$ for all $i \in S$. However, the basic intuition—that risk-aversion makes incentive provision cheaper at lower utility levels—is robust.

partially verifiable variable, like his endowment or labor productivity, and (ii) he cannot covertly trade or produce outside of the contract.[49] While covert trade and production are surely important in some settings (Allen (1985), Cole and Kocherlakota (2001b)), they are conceptually distinct from the pure reporting problems studied herein and in much of the prior literature.

Alternatively, one can interpret NHB as a *relaxation* of the "full" problem in which over-reporting is feasible for the agent, which is natural when his type is purely subjective (e.g., a taste shock). Even then, NHB corresponds to a significantly *less* relaxed problem than that considered in the popular first-order approach (FOA), which in our context would include only the "local downward" incentive constraints (IC_{ij}) with $j = i - 1$. This additional robustness is notable because the FOA may be "invalid" (i.e., yield solutions that violate global incentive constraints) without strong assumptions on the type process.[50]

Source of Private Information. For concreteness, we have assumed that the agent's private information concerns his endowment, but this is not essential. Our main long-run analysis (Sections 3 and 4) can be adapted to other canonical insurance settings, including:

- *Taste shocks*: The agent has utility $U(\omega, c) = \omega u(c)$ over consumption c, where $\omega \in \mathbb{R}_+$ is privately observed taste shock, and the principal minimizes the lifetime cost of providing consumption to the agent. This model has been a workhorse specification since Atkeson and Lucas (1992), and coincides with our baseline hidden endowment model under CARA utility. Our analysis extends to this model almost verbatim, provided that $u(\cdot)$ satisfies parts (a) and (b) of Assumption DARA.
- *Productivity shocks*: The agent has utility $U(\omega, c, \ell) = u(c) - v(\omega, \ell)$ over consumption c and labor effort ℓ, where $\omega \in \mathbb{R}$ is a privately observed productivity shock. In each period, the principal offers a menu of report-contingent (c, ℓ) bundles, transfers consumption c to the agent, and collects his labor output $\omega\ell$, aiming to minimize the lifetime cost of the contract. This is the workhorse specification in the dynamic Mirrleesian taxation literature (e.g., Zhang (2009), Farhi and Werning (2013)). Many aspects of our analysis apply to this model with minor modification; we leave a full analysis to future work.[51]

6.2. *The Role of Mean-Reversion*

To clarify the role of Assumption Markov, it is useful to recall the analysis of Williams (2011) (henceforth W11), which provides an important counterpoint to our Theorems 1

[49]See Phelan (1998), Williams (2011), and Golosov and Tsyvinski (2007) for versions of this interpretation. For hidden endowment settings, Fernandes and Phelan (2000, fn. 4) also argue that NHB is without loss of generality if the agent's minimal consumption $\underline{c} > -\infty$, as the principal can then use stochastic contracts that punish an over-report with infinite disutility (with vanishing probability and cost). NHB also appears in the related family of "cash-flow diversion" models (e.g., Fu and Krishna (2019) and references therein).

[50]In the i.i.d. case, Thomas and Worrall (1990) argue that only local downward constraints bind. Battaglini and Lamba (2019), Pavan (2016) discuss potential issues with the FOA when types are highly persistent.

[51]With separable preferences $U(\omega, c, \ell) = u(c) - v(\omega, \ell)$, consumption utility $u(c)$ serves as a "numeraire" for delivering value to the agent in a type-independent manner. Thus, the principal's marginal cost martingale reduces to $\mathsf{D_1}P(\mathbf{v}, s) = 1/u'(c(\mathbf{v}, s))$, yielding the classic *Inverse Euler Equation* (IEE) which states that the inverse marginal utility of consumption is a martingale (e.g., Golosov and Tsyvinski (2007)). When preferences are not separable between consumption and type—as in hidden endowment, taste shock, and nonseparable Mirrleesian models—this correspondence and the classic IEE break down. In such cases, the martingale $\mathsf{D_1}P(\mathbf{v}, s)$ remains the relevant object for characterizing long-run outcomes, while short-run distortions can be characterized with augmented versions of the IEE that include additional Lagrange multiplier terms (e.g., Farhi and Werning (2013), Hellwig (2021), Golosov, Troshkin, and Tsyvinski (2016)).

and 2. W11 studies the special case of our baseline model in which the agent has CARA utility, but with one key difference: the agent's endowment evolves as a Gaussian random walk, rather than as an ergodic finite-state Markov chain.[52] This seemingly minor difference yields markedly different results: the optimal contract generates long-run *bliss* ($c^{(t)} + \omega^{(t)} \to +\infty$ and $u^{(t)} \to 0$ almost surely) and *vanishing incentives* ($v_i^{(t)} - v_{i-1}^{(t)} \to 0$ almost surely).

Taken together, our and W11's results indicate that ergodicity (or "mean-reversion") of the type process is a key determinant of the optimal contract's long-run properties. In particular, we interpret our results as suggesting that the failure of immiseration in W11 hinges on the knife-edge assumption of *zero* mean-reversion ("permanent shocks").[53]

Intuitively, with *any* nonzero mean-reversion, the agent's current type (in period t) is approximately independent of his distant-future types (in period $t + T$ as $T \to \infty$), implying that the principal and agent have approximately symmetric information about the distant future. This suggests that the classic forces underlying immiseration in the i.i.d. benchmark "should" kick in over long time horizons. Indeed, this logic is borne out in parts of our proof of Theorem 1 (e.g., the "renewal" dynamics in Step 3 of Section 4.3).

Formally, Assumption Markov ensures that the type process has *impulse response functions* (Pavan, Segal, and Toikka (2014)) that vanish over long time horizons, and thus that the agent's Markov information rents are "not too large."[54] In contrast, when the agent's type follows a random walk as in W11, the impulse response functions are *constant* over time because a change in the current type has a permanent additive effect on all future types. In a generalization of W11's model, Bloedel, Krishna, and Strulovici (2024) show that this implies that the agent's Markov information rents are "so large" that (i) he is globally indifferent among *all* reporting strategies under *any* incentive compatible contract and (ii) the principal optimally elicits *no* information from the agent and provides *no* insurance. In other words, permanent shocks effectively leave the principal no latitude over the power of incentives, shutting off the classic rationale for immiseration by construction.

This discussion invites two conjectures: (i) immiseration arises for all utility functions satisfying Assumption DARA if and only if the agent's type process has asymptotically vanishing impulse response functions, and (ii) immiseration occurs "more slowly" as we approach the permanent shock limit.[55] While our results offer a significant step towards (i), definitive answers require extending our analysis to unbounded, continuous, and non-Markovian type processes. We leave this important and challenging task to future work.

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[52]More precisely, W11 studies a continuous-time model in which the agent's type follows a Brownian motion. Bloedel, Krishna, and Strulovici (2023, 2024) clarify that W11's results carry over to the discrete-time version of the model, permitting a more direct comparison to our analysis. W11 and Bloedel, Krishna, and Strulovici (2023) also allow for the agent's type to follow a mean-reverting AR(1) process (which is closer to the Markov chains studied here) but do not characterize the optimal contract in that case.

[53]Note that this notion of "permanent shocks" is different from the "perfect persistence" limit in which types become absorbing (cf. Section 5.1). Immiseration also fails when there are absorbing states for the mechanical reason that the optimal contract becomes "static" when such a state is reached. See Ravikumar and Zhang (2012), Fuller, Ravikumar, and Zhang (2015), and Golosov and Tsyvinski (2006) for models of dynamic insurance with absorbing states.

[54]See Pavan, Segal, and Toikka (2014, Theorem 1) for the definition of impulse response functions and their relation to information rents (see also Battaglini and Lamba (2019, Lemma 1)).

[55]In our model, the permanent shock limit corresponds to increasing the number of types d without bound while suitably adjusting the transition probabilities $[\mathbf{f}_i]_{i=1}^d$ so that the type process approximates a random walk as $d \to \infty$. Theorem 1 suggests that the long-run outcome "discontinuously" switches from immiseration to bliss at the permanent shock limit (i.e., the order in which we send $d \to \infty$ and $t \to \infty$ matters), which is consistent with *pointwise* but *non-uniform* convergence of optimal contracts in this limit.

6.3. *Closing the Economy*

Our analysis has focused on a principal-agent version of the insurance problem (Thomas and Worrall (1990)). As noted in Section 5.1, this framework is dual to the utilitarian planning problem for a continuum-agent *open economy* in which the planner can borrow and save at the risk-free rate $R = 1/\alpha$ on the world market (Green (1987)). Both of these formulations are "partial equilibrium" because they endow the principal/planner with deep pockets. As such, we view them as applying most readily to "small" economies (e.g., individual communities, private insurance exchanges, or social insurance programs at the state or national level) and over "short" horizons (e.g., the lifespan of a single individual).

Equally important is the "general equilibrium" perspective pioneered by Atkeson and Lucas (1992), which considers the more constrained *closed economy* problem in which the utilitarian planner faces a sequence of date-by-date resource constraints. Specifically, in the notation of Section 5.1, aggregate consumption $\mu_{C,t} := \mathsf{E}[c^{(t)} + \omega^{(t)}]$ must satisfy $\mu_{C,t} \leq e$ at all dates t, where $e \in \mathbb{R}$ is a per-period aggregate endowment. (The corresponding open economy problem is subject to the weaker intertemporal constraint $\sum_{t=0}^{\infty} \alpha^t \mu_{C,t} \leq \frac{e}{1-\alpha}$ on the present values of consumption and the endowment.) This approach is well suited for modeling "large" economies over "long" horizons, such as when each period corresponds to a new generation in a dynasty (Farhi and Werning (2007), Phelan (2006)).

It is important to understand the extent to which Theorems 1 and 2 translate to such settings. Notably, our optimal contracts may be infeasible in the closed economy problem: if the open economy constraint binds and aggregate consumption is declining over time (as in Section 5.1), then the closed economy constraints must be violated in the initial periods. While it is beyond the scope of this paper to solve the closed economy problem, our analysis does have some notable implications for it. We outline these implications and associated challenges below, with the hope of stimulating further work on this problem.

To begin, note that the most tractable way to tackle the closed economy problem is to study the *decentralized (dual) problem* from Atkeson and Lucas (1992), in which the planner minimizes the expected lifetime cost of consumption transfers $\mathsf{E}[\sum_{t=0}^{\infty} q_t c^{(t)}]$ under a given sequence $\tilde{q} := \{q_t\}_{t=0}^{\infty}$ of intertemporal prices, subject to promise keeping and incentive compatibility for each agent.[56] The analysis then consists of two steps:

- *Step 1*: For each fixed $\tilde{q}$, the decentralized problem separates agent-by-agent into a collection of principal-agent problems of the sort we have studied, *except that the principal's discount factor between periods t and* $(t+1)$ *is now* q_{t+1}/q_t, *which may differ from the agent's discount factor* α. Solve the principal-agent problem for each $\tilde{q}$.
- *Step 2*: Find a $\tilde{q}^*$ at which the Step 1 solution satisfies the closed economy constraints with equality ("market clearing"). This yields a solution to the closed economy problem.

A General Immiseration Result?. Our recursive approach extends, with minor modifications and technical qualifications, to the Step 1 subproblem. Let $P_{\tilde{q}}(\cdot, \cdot, t) : D \times S \to \mathbb{R}$ denote the principal's value function in the time-t continuation of this subproblem, which is analogous to (RP) but may now depend on calendar time if $t \mapsto q_{t+1}/q_t$ is non-constant. Adapting Proposition 4.4, we see that the principal's marginal cost process now satisfies

$$\mathsf{D}_1 P_{\tilde{q}}\big(\mathbf{v}^{(t)}, s^{(t)}, t\big) = \frac{q_{t+1}/q_t}{\alpha} \cdot \mathsf{E}\big[\mathsf{D}_1 P_{\tilde{q}}\big(\mathbf{v}^{(t+1)}, s^{(t+1)}, t+1\big) \big| \mathbf{v}^{(t)}, s^{(t)}\big] \tag{6.1}$$

[56]See, for example, Atkeson and Lucas (1995), Farhi and Werning (2007), and Phelan (1998, 1994).

and thus is a martingale only if $q_{t+1}/q_t = \alpha$ for all t. More generally, however, it is a non-negative *super*martingale and thus guaranteed to converge whenever $q_{t+1}/q_t \geq \alpha$ for all t. Consequently, Theorem 1 and its proof can be extended to $\tilde{q}$ satisfying this inequality. If such $\tilde{q}$ could be shown to satisfy market clearing in Step 2, this would yield a general proof of immiseration for the closed economy problem.

Unfortunately, market clearing (at least) sometimes requires the *opposite* inequality $q_{t+1}/q_t < \alpha$, which makes it *cheaper* than in our baseline model to backload consumption. Intuitively, this is needed to counteract the downward drift in aggregate consumption that arises under "equal discounting" (Section 5.1).[57] For such $\tilde{q}$, the principal's marginal cost (6.1) is a non-negative (and unbounded) *sub*martingale, and hence not guaranteed to converge in general. Thus, our proof method for Theorem 1 does not *directly* extend to the closed economy model, at least without further structure on market clearing prices.

Towards a Special Case. As it is hard to imagine a general immiseration proof that does *not* use martingale convergence, further progress apparently requires parametric assumptions. This is not surprising: even in the i.i.d. case, immiseration-type results have only been obtained for CARA utility (or CRRA utility in taste-shock settings).[58]

In particular, Atkeson and Lucas (1992) and most of the subsequent literature rely on the wealth (or scale) invariance of these preferences to ensure that optimal contracts are "scale invariant," that is, characterized by d numbers that (i) determine how promised utility is (proportionally or additively) rescaled following each of the d shocks and (ii) crucially, are *independent* of the current promised utility state. Without such scale-invariance, the market clearing prices in Step 2 will generally depend on the (entire path of the) economy's cross-sectional promised utility distribution, which in turn depends on the path of prices; this requires one to solve an infinite-dimensional fixed-point problem before determining whether or not (6.1) is a supermartingale.[59]

With this in mind, the analysis in Section 5 suggests a roadmap for how to generalize the CARA (or CRRA) example from Atkeson and Lucas (1992) to persistent types. Even with such preferences, the scale-invariance from the i.i.d. case *fails* with persistence due to the distortions described in Section 5.2. Nonetheless, because optimal contracts are HD1 and induce a stationary distribution over the E_2 and B_k rays, we expect that analogous scale-invariance holds *at the population level*. Thus, we conjecture that the long-run evolution of the *cross-sectional promised utility distribution* can be characterized through an eigendistribution/eigenvalue problem. This is an exciting direction for future research.

APPENDIX

These appendices contain additional details concerning the recursive formulation. Appendix A presents the sequential formulation of the contracting problem described informally in Section 2. Appendix B then presents a characterization of the recursive domain (Theorem 3) and sufficient conditions for Regularity from Section 3.

[57] Atkeson and Lucas (1992) and Phelan (1998) formalize this for the i.i.d. case with CARA utility.

[58] See Atkeson and Lucas (1992) and Phelan (1998, 1994). Atkeson and Lucas (1995), Farhi and Werning (2007), and Phelan (2006) study long-run outcomes for more general preferences in alternative models where non-degenerate stationary distributions exist, using arguments that rely on this different structure.

[59] See, for example, Atkeson and Lucas (1992, p. 446) for further discussion of the role of scale-invariance.

APPENDIX A: SEQUENTIAL CONTRACTS, RECURSIVE CONTRACTS, AND THEIR EQUIVALENCE

A.1. *Sequential Formulation*

By the Revelation Principle, we may restrict attention to truthful direct revelation mechanisms.[60] Let $\mathscr{H} := S^\infty$ denote the set of paths of type *reports*, and let $\mathscr{G} := S^\infty$ denote the set of paths of type *realizations*. For notational consistency with our formulation of the recursive problem (RP) in Section 3.1, we adopt the following timing convention. Denote a generic path of reports by $h = (s^{t+1})_{t=0}^\infty$, where $s^{t+1} \in S$ represents the agent's reported type in period t. Analogously, denote a generic path of type realizations by $g = (\mathscr{s}^{t+1})_{t=0}^\infty$, where $\mathscr{s}^{t+1} \in S$ represents the agent's realized type in period t. That is, along path $g = (\mathscr{s}^{t+1})_{t=0}^\infty$, the agent's realized endowment in period t is $\omega^t = \omega_{\mathscr{s}^{t+1}}$.

Let G^t and H^t denote the sets of length-t private and public histories, respectively. That is, $g^t := (\mathscr{s}^1, \dots, \mathscr{s}^t) \in G^t$ and $h^t := (s^1, \dots, s^t) \in H^t$ encode the sequences of realized and reported types, respectively, in periods $\{0, \dots, t-1\}$. Together, g^t and h^t specify the full history at the start of period t, before the agent's period-t type or report are realized.

A *(pure) reporting strategy* $\sigma := (\sigma_t)_{t=0}^\infty$ for the agent is a sequence of functions $\sigma_t : G^{t+1} \times H^t \to S$. A strategy σ is *admissible* if it never specifies over-reporting the current type, that is, $\sigma_t((g^t, \mathscr{s}^{t+1}), h^t) \le \mathscr{s}^{t+1}$ for all $g^t \in G^t$, $h^t \in H^t$, and $\mathscr{s}^{t+1} \in S$. Under Assumption NHB, the agent can only use admissible strategies. The set of admissible strategies is denoted by Σ. Every $\sigma \in \Sigma$, together with the measure $\mathsf{P} \in \Delta(\mathscr{G})$ over type paths, generates a stochastic process $(\mathscr{s}^{(t+1)}, s^{(t+1)})_{t=0}^\infty$ on $S \times S$. A strategy $\sigma^* \in \Sigma$ is *on-path truthful* if $\sigma^*((g^t, \mathscr{s}^{t+1}), h^t) = \mathscr{s}^{t+1}$ for all $\mathscr{s}^{t+1} \in S$ and $g^t \in G^t$, $h^t \in H^t$ such that $g^t = h^t$ (where we define $g^0 \equiv h^0$). All on-path truthful strategies generate the same joint distribution on $\mathscr{G} \times \mathscr{H}$, under which $\mathscr{s}^{(t+1)} = s^{(t+1)}$ for all $t \ge 0$ almost surely and the marginal distribution on $\mathscr{H}$ equals P. The *globally truthful* strategy $\sigma^\star$ is defined as $\sigma^\star((g^t, \mathscr{s}^{t+1}), h^t) = \mathscr{s}^{t+1}$ for all $\mathscr{s}^{t+1} \in S$ and $g^t \in G^t$, $h^t \in H^t$ (i.e., including $g^t \ne h^t$).

Following Section 3.1, we define contracts in terms of flow utilities, rather than consumption transfers. A *sequential contract* is a controlled stochastic process $\tilde{u} := (u^{(t)})_{t=0}^\infty$, where (i) $u^{(t)} : H^{t+1} \to \mathcal{U}$ maps histories of type reports in periods $\{0, \dots, t\}$ to flow utilities for period t, and (ii) the law of $\tilde{u}$ is controlled by the agent's reporting strategy σ. In particular, given a type history $(g^t, \mathscr{s}^{t+1}) \in G^{t+1}$ and report history $(h^t, s^{t+1}) \in H^{t+1}$, the agent's true flow utility in period t is $\psi(u^{(t)}(h^t, s^{t+1}), \mathscr{s}^{t+1}, s^{t+1}) := U(C(u^{(t)}(h^t, s^{t+1}), s^{t+1}) + \omega_{\mathscr{s}^{t+1}})$. We denote the set of sequential contracts by Π.

The lifetime utility function $V : \Pi \times \Sigma \times S \to \mathcal{U} \cup \{-\infty\}$, defined as

$$V(\tilde{u}, \sigma, i) := \mathsf{E}^\sigma\left[\sum_{t=0}^\infty \alpha^t \psi\big(u^{(t)}, \mathscr{s}^{(t+1)}, s^{(t+1)}\big) \,\big|\, \mathscr{s}^{(1)} = i\right],$$

represents the agent's time-0 preferences over sequential contracts and reporting strategies, conditional on observing the realization of his time-0 type $\mathscr{s}^{(1)}$. A sequential contract $\tilde{u} \in \Pi$ *implements* $\mathbf{v} \in \mathcal{U}^d$ if it satisfies promise keeping and incentive compatibility:

$$v_i = V\big(\tilde{u}, \sigma^*, i\big) \quad \forall i \in S, \tag{S-PK}$$

[60]Because the agent can only under-report his type (Assumption NHB), we technically require a version of the Revelation Principle suited for environments with partial verification. The reporting constraints in each period satisfy the "nested range condition" of Green and Laffont (1986), so straightforward adaptations of their arguments to our dynamic setting establish the appropriate version.

$$V(\tilde{u}, \sigma^*, i) \geq V(\tilde{u}, \sigma, i) \quad \forall i \in S, \sigma \in \Sigma, \tag{S-IC}$$

where $\sigma^* \in \Sigma$ denotes any on-path truthful strategy. For each $\mathbf{v} \in \mathcal{U}^d$, we denote by $\Pi(\mathbf{v}) \subseteq \Pi$ the subset of sequential contracts that implement $\mathbf{v}$. Let $D_S := \{\mathbf{v} \in \mathcal{U}^d : \Pi(\mathbf{v}) \neq \emptyset\}$ denote the set of *sequentially implementable* promised utility vectors.

Given any promised utility vector $\mathbf{v} \in D_S$ and type $s \in S$, the principal's *sequential problem* is to implement $\mathbf{v}$ at minimal cost:

$$P_S(\mathbf{v}, s) := \inf_{\tilde{u} \in \Pi(\mathbf{v})} \mathbf{E}\left[\sum_{t=0}^{\infty} \alpha^t C(u^{(t)}, s^{(t+1)}) | s^{(0)} = s\right], \tag{SP}$$

where the expectation is taken with respect to the measure $\mathbf{P} \in \Delta(\mathcal{H})$ over paths of reports induced by (on-path) truthful reporting and the prior belief $\mathbf{f}_s$ over initial types $\omega^{(0)} = \omega_{s^{(1)}}$. A sequential contract is *sequentially optimal* at initial state $(\mathbf{v}, s)$ if it attains the infimum in (SP) at $(\mathbf{v}, s)$. For every $\mathbf{v} \in D_S$ and $s \in S$, we assume that (a) $P_S(\mathbf{v}, s)$ is well-defined and finite, and (b) a sequentially optimal contract $\tilde{u}^* \in \Pi(\mathbf{v})$ exists.[61] We refer to the map $P_S : D_S \times S \to \mathbb{R}$ as the principal's *sequential value function*.

A.2. *Transversality and the Equivalence of Formulations*

For every recursive contract $\xi \in \Xi$ and initial $\mathbf{v}_\xi^{(0)} := \mathbf{v} \in D$, the induced allocation $\tilde{u}_\xi$ defines a sequential contract, although it is possible that $\tilde{u}_\xi \notin \Pi(\mathbf{v})$ (recall the discussion of (TVC) in Section 3.1). We say that $\tilde{u} \in \Pi(\mathbf{v})$ is *recursively generated* if $\tilde{u} = \tilde{u}_\xi$ for some $\xi \in \Xi$ initialized at $\mathbf{v}_\xi^{(0)} = \mathbf{v}$. As is well understood, for every $(\mathbf{v}, s) \in D_S \times S$, there exists a sequentially optimal contract $\tilde{u} \in \Pi(\mathbf{v})$ that is recursively generated.[62] It follows that $D_S \subseteq D$ and $P_S \geq P$. Furthermore, as claimed in Sections 3.1 and 3.2, the sequential problem (SP) and the recursive problem (RP) are equivalent under (TVC)-Regularity:

LEMMA A.1: *If a recursive contract $\xi \in \Xi$ is* (TVC)*-implementable at $\mathbf{v} \in D$, then the induced allocation $\tilde{u}_\xi$ is feasible in* (SP), *that is, $\tilde{u}_\xi \in \Pi(\mathbf{v})$. In particular*:

(a) *$\tilde{u}_\xi$ delivers promises at $\mathbf{v}$, that is, for all $i \in S$,*

$$v_i = \mathbf{E}\left[\sum_{t=0}^{\infty} \alpha^t \tilde{u}_\xi^{(t)} | s^{(1)} = i\right]. \tag{DP}$$

(b) *Under $\tilde{u}_\xi$, the globally truthful reporting strategy $\sigma^\star$ is optimal for the agent.*

(c) *Moreover, if ξ is optimal in* (RP) *at initial state $(\mathbf{v}, s)$, then $\tilde{u}_\xi$ is optimal in* (SP) *at $(\mathbf{v}, s)$.*

Consequently, if the environment is (TVC)*-Regular, then $D = D_S$ and $P = P_S$.*

PROOF: For part (a), let ξ be (TVC)-implementable at $\mathbf{v}$. Iterating (PK_i) forward T times gives $v_i = \mathbf{E}[\sum_{t=0}^{T} \alpha^t \tilde{u}_\xi^{(t)} | s^{(1)} = i] + \mathbf{E}[\alpha^{T+1} \mathbf{E}^{\mathbf{f}_{s^{(T+1)}}}[\mathbf{v}_\xi^{(T+1)}] | s^{(1)} = i]$ for all $i \in S$. Sending $T \to \infty$, using the Monotone Convergence Theorem on the first term, and using the Bounded Convergence Theorem on the second term (it applies under (TVC)) yields

[61] These two assumptions mirror Conditions R.1 and R.4, respectively, from the recursive problem (RP).

[62] See, for example, Fernandes and Phelan (2000), Atkeson and Lucas (1992), or Doepke and Townsend (2006).

(DP). The proof of part (b) is analogous (see, e.g., (Green, 1987, Lemma 2)). Part (c) follows from parts (a) and (b) and the fact that $P_S \geq P$. Under (TVC)-Regularity, parts (a)–(c) then imply that $D_S \supseteq D$ and $P_S = P$ on $D \times S$; the fact that $D_S \subseteq D$ completes the proof. *Q.E.D.*

APPENDIX B: STRUCTURAL RESULTS OMITTED FROM SECTION 3

B.1. *Recursive Domain*

Herein, we characterize the recursive domain D. We need two definitions. First, for every $\mathbf{v} \in D$, we denote by $\Xi^*(\mathbf{v}) \subseteq \Xi$ the subset of feasible contracts that are (TVC)-implementable at $\mathbf{v}$. Thus, the set of initial $\mathbf{v} \in D$ at which (TVC) can be satisfied is

$$D^* := \{\mathbf{v} \in D : \Xi^*(\mathbf{v}) \neq \emptyset\}. \tag{B.1}$$

Second, consider the following classes of type processes:

DEFINITION B.1: The type process is:
- *MLRP-ordered* (or simply *MLRP*) if f_{ki}/f_{kj} is non-decreasing in k whenever $i > j$.
- *Persistent pseudo-renewal* (or simply *PPR*) if it is FOSD and, moreover, there exists a vector $\boldsymbol{\pi} \in \mathbb{R}^S_+$ such that $f_{ij} = \pi_j$ whenever $i \neq j$.[63]

When $d = 2$, the three properties FOSD, MLRP, and PPR are equivalent. When $d \geq 3$, (i) MLRP is stronger than FOSD but satisfied by many type processes considered in applications, such as discretized AR(1) processes, and (ii) PPR processes are distinguished within the FOSD class by the property that, conditional on a transition occurring, the probability over new types does not depend on the previous type. The MLRP and PPR classes always include all i.i.d. processes and, when $d \geq 3$, have nontrivial intersection.

THEOREM 3: *Fix $d \geq 2$ and define the set $V_d := \{\mathbf{v} \in \mathcal{U}^d : v_d > v_{d-1} > \cdots > v_1\}$.*
(a) *The domain D exists. It is an open, convex cone satisfying $V_d \subseteq D$. It is also independent of the discount factor $\alpha \in (0, 1)$ and utility function U (within the DARA class).*
(b) *$\Gamma : D \rightrightarrows (\mathcal{U} \times D)^d$ is nonempty-valued and has a convex graph.*
(c) *$D^* \subseteq D$ is nonempty, convex, unbounded below, and has decreasing returns.*[64]
(d) *If FOSD holds, then $D^* \subseteq V_d$.*
(e) *If either MLRP or PPR holds, then $D = V_d$.*
(f) *If either MLRP or PPR holds and the agent has CARA utility, then $D = D^* = V_d$.*

We prove Theorem 3(d) in Supplemental Appendix E during the proof of Theorem 2. The proofs of all other parts of Theorem 3 are in Section I of Bloedel, Krishna, and Leukhina (2025b); while the details are nontrivial and lengthy, the core logic builds on standard iterative arguments (Abreu, Pearce, and Stacchetti (1990)). We prove parts (b)–(c) and most of part (a) by characterizing fixed points of a set-valued operator via Tarski's theorem. We prove parts (e)–(f) and the cone property in part (a) by characterizing solutions to systems of linear and conic programs.

Notably, alternative recursive formulations would yield different domains. For instance, in the well-known Fernandes and Phelan (2000) formulation based on ex ante promised

[63]The latter property is from Renault, Solan, and Vielle (2013) and Hörner, Mu, and Vielle (2017).

[64]We say that D^* is *unbounded below* if, for every $r < 0$, there exists some $\mathbf{v} \in D^*$ such that $\mathbf{v} \leq r\mathbf{1}$. We say that D^* has *decreasing returns* if $\mathbf{v} \in D^*$ implies that $a\mathbf{v} \in D^*$ for all $a \in (0, 1]$.

and threat-point utilities, the domain is a set-valued *function* of past reports, the shape of which *changes* with the type process.[65] In contrast, our domain is independent of the agent's reports (by construction) and type process (at least within the MLRP and PPR classes). These features are important for the tractability of our analysis.

B.2. *Regularity*

Regularity Conditions R.1–R.3 can be verified in some cases of special interest.

LEMMA B.2: *The following hold*:

(a) *If either* (*i*) *the type process is FOSD or* (*ii*) *the transition probabilities* $\{\mathbf{f}_i\}_{i\in S}$ *are affinely independent*, *then Condition R*.3 *holds*.

(b) *If the agent has CARA utility and the type process is MLRP or PPR*, *then the environment is Regular* (*i.e*., *Conditions R*.1–*R*.3 *hold*).

The proof of Lemma B.2 is in Section J of Bloedel, Krishna, and Leukhina (2025b). In part (a), the FOSD case covers all instances of (weakly) positive correlation and the affine independence case covers many instances of negative correlation. Part (b) implies that the example in Section 5 is Regular. Since AR(1) processes satisfy MLRP, it also implies that a suitably discretized version of the setup from Williams (2011) and Bloedel, Krishna, and Strulovici (2023) is Regular.

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[65]To illustrate, consider the case in which $d = 2$ and types are MLRP. The Fernandes and Phelan (2000) recursive state variable consists of the previous report $s \in S$, the ex ante promised utility $v^p(s) := \mathsf{E}^{\mathbf{f}_s}[\mathbf{v}]$ for an agent whose report of s was truthful, and the *threat-point* utility $v^\dagger(s) := \mathsf{E}^{\mathbf{f}_{3-s}}[\mathbf{v}]$ for an agent whose report of s was a lie. The domain is then a correspondence $W : S \rightrightarrows \mathcal{U}^2$, where $W(s)$ consists of implementable $(v^p(s), v^\dagger(s))$ pairs. The sets $W(1)$ and $W(2)$ must be solved for jointly. Theorem 3(e) implies that $W(1) = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} V_2$ and $W(2) = \begin{bmatrix} f_{21} & f_{22} \\ f_{11} & f_{12} \end{bmatrix} V_2$. Thus, $W(1)$ and $W(2)$ are reflections of each other about the diagonal in $\mathcal{U}^2 = \mathbb{R}^2_{--}$, and each $W(s)$ is a convex cone whose shape depends on the transition probabilities. If types are i.i.d., both sets collapse to the diagonal, that is, $W(1) = W(2) = \{(t, t) : t < 0\}$.

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Co-editor Charles I. Jones handled this manuscript.

Manuscript received 13 December, 2021; final version accepted 24 January, 2025; available online 30 January, 2025.

The replication package for this paper is available at https://doi.org/10.5281/zenodo.14720557. The Journal checked the data and codes included in the package for their ability to reproduce the results in the paper and approved online appendices.

Submitted to *Econometrica*

SUPPLEMENT TO "INSURANCE AND INEQUALITY WITH PERSISTENT PRIVATE INFORMATION"

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In this Supplemental Appendix (henceforth SA), we prove Theorems 1 and 2 and associated results, and discuss Condition R.5. SA-C proves Theorem 1 and Proposition 4.4. SA-D proves Corollary 4.1. SA-E presents the proof of Theorem 2, an important step of which is proving Theorem 3(d). SA-F and SA-G collect facts about the first-best and pathwise properties of Markov chains, respectively. SA-H discusses Condition R.5. While this SA is mostly self-contained, some auxiliary results are proved in Bloedel, Krishna, and Leukhina (2025).

APPENDIX C: PROOF OF THEOREM 1

Herein, we assume that the environment is (TVC)-Regular (as in the statement of Theorem 1). SA-C.1 presents the Lagrangian and first-order optimality conditions for the Bellman equation (FE). SA-C.2 proves Proposition 4.4 (Step 1 in the sketch from Section 4.3). SA-C.3 presents intermediate steps towards the proof of Theorem 1 (most of Step 2 in the sketch). SA-C.4 presents the main convergence proofs (most of Step 3 in the sketch).

C.1. *Optimality Conditions*

Recall that the set of *recursive constraints* consists of the *promise keeping* constraints

$$v_i = u_i + \alpha\, \mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_i] \tag{PK$_i$}$$

for all $i \in S$, and the *incentive compatibility* constraints

$$u_i + \alpha\, \mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_i] \geq \psi(u_j, i, j) + \alpha\, \mathsf{E}^{\mathbf{f}_i}\left[\mathbf{w}_j\right] \tag{IC$_{ij}$}$$

for all $i, j \in S$ with $i > j$ (per Assumption NHB).[1]

Under (TVC)-Regularity, Proposition 3.2 reduces the principal's problem to a family of smooth, strictly convex, finite-dimensional minimization problems. Thus, under Condition R.3, standard results imply that optimal menus in (FE) can be characterized via saddle points of a Lagrangian function (see, e.g., Exercise 7 on p. 236 and Theorem 2 on p. 221 of Luenberger (1969)). Letting $\lambda_i \in \mathbb{R}$ denote a multiplier on the promise keeping constraint (PK$_i$) and $\mu_{ij} \geq 0$

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[1]The incentive constraints are written here in a slightly different, but equivalent, form than in Section 3.

denote a multiplier on the incentive constraint (IC_{ij}), the Lagrangian for this problem is

$$\mathscr{L}(\mathbf{v}, s, \mathbf{u}, \mathbf{w}, \boldsymbol{\lambda}, \boldsymbol{\mu}) = \sum_{i=1}^{d} f_{si}\big(C(u_i, i) + \alpha P(\mathbf{w}_i, i)\big) + \sum_{i=1}^{d} \lambda_i\big(v_i - u_i - \alpha\, \mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_i]\big)$$
$$- \sum_{i=2}^{d} \sum_{j=1}^{i-1} \mu_{ij}\big(u_i + \alpha\, \mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_i] - \psi(u_j, i, j) - \alpha\, \mathsf{E}^{\mathbf{f}_i}\big[\mathbf{w}_j\big]\big).$$

For notational ease, we henceforth extend μ_{ij} to all pairs $i, j \in \mathbb{N}$, with the convention that $\mu_{ij} = 0$ if $j \geq i$, $i \notin S$, or $j \notin S$.

The necessary and sufficient optimality equations consist of the envelope conditions

$$P_i(\mathbf{v}, s) = \lambda_i(\mathbf{v}, s) \tag{Env_i}$$

for all $i \in S$, the first-order conditions for flow utilities

$$f_{si} C'(u_i(\mathbf{v}, s), i) = \lambda_i(\mathbf{v}, s) + \sum_{k=1}^{i-1} \mu_{ik}(\mathbf{v}, s) - \sum_{k=i+1}^{d} \psi'(u_i(\mathbf{v}, s), k, i)\mu_{ki}(\mathbf{v}, s) \tag{$\mathrm{FOC}u_i$}$$

for all $i \in S$, and the first-order conditions for contingent continuation utilities

$$f_{si} P_j(\mathbf{w}_i(\mathbf{v}, s), i) = f_{ij}\left(\lambda_i(\mathbf{v}, s) + \sum_{k=1}^{i-1} \mu_{ik}(\mathbf{v}, s)\right) - \sum_{k=i+1}^{d} f_{kj}\mu_{ki}(\mathbf{v}, s) \tag{$\mathrm{FOC}\mathbf{w}_{ij}$}$$

for all $i, j \in S$ with $i > j$ (per Assumption NHB), where $(u_i(\mathbf{v}, s), \mathbf{w}_i(\mathbf{v}, s))_{i\in S} \in \Gamma(\mathbf{v})$ is the optimal menu at state $(\mathbf{v}, s)$ and $\lambda_i(\mathbf{v}, s)$ and $\mu_{ij}(\mathbf{v}, s)$ are the corresponding multipliers.[2]

By Proposition 3.2, the policy functions characterized by the above optimality conditions induce the (unique) optimal contract. Accordingly, we henceforth let ξ denote the optimal contract, let $\xi^f(\mathbf{v}, s, i) := u_i(\mathbf{v}, s)$ denote the *flow* utility policy functions, and let $\xi^c(\mathbf{v}, s, i) := \mathbf{w}_i(\mathbf{v}, s)$ denote the *continuation* utility policy functions. Going forward, we will utilize both of these notational conventions for the policy functions, depending on which one is more convenient for the task at hand.

C.2. *Proof of Proposition 4.4*

By Proposition 3.2(b), the value function $P(\cdot, s) \in \mathbf{C}^1(D)$. Hence, the directional derivative $\mathsf{D}_\mathbf{1} P(\cdot, s) = \sum_{i\in S} P_i(\cdot, s)$ and is real-valued on D. For each $t \in \mathbb{N}$, integrability of the random variable $\mathsf{D}_\mathbf{1} P(\mathbf{v}^{(t)}, s^{(t)})$ then follows from finiteness of S and positivity of the directional derivative (established in Lemma C.1 below). For the martingale property, let $(\mathbf{v}, s) \in D \times S$ be given. Summing the (Env_i) over $i \in S$ delivers

$$\mathsf{D}_\mathbf{1} P(\mathbf{v}, s) = \sum_{i=1}^{d} \lambda_i(\mathbf{v}, s). \tag{C.1}$$

[2]We omit the usual complementary slackness conditions.

For each fixed $i \in S$, summing the (FOC$\mathbf{w}_{ij}$) over $j \in S$ yields

$$f_{si} \cdot \mathsf{D}_{\mathbf{1}} P(\mathbf{w}_i(\mathbf{v}, s), i) = \lambda_i(\mathbf{v}, s) + \sum_{k=1}^{i-1} \mu_{ik}(\mathbf{v}, s) - \sum_{k=i+1}^{d} \mu_{ki}(\mathbf{v}, s). \tag{C.2}$$

Now, summing (C.2) over $i \in S$ and noting that $\sum_{i=1}^{d}\sum_{k=1}^{i-1} \mu_{ik}(\mathbf{v}, s) = \sum_{i>k\in S} \mu_{ik}(\mathbf{v}, s) = \sum_{i=1}^{d}\sum_{k=i+1}^{d} \mu_{ki}(\mathbf{v}, s)$ delivers $\sum_{i\in S} f_{si}\mathsf{D}_{\mathbf{1}} P(\mathbf{w}_i(\mathbf{v}, s), i) = \sum_{i=1}^{d} \lambda_i(\mathbf{v}, s)$. Combined with (C.1), this delivers the martingale property $\mathsf{D}_{\mathbf{1}} P(\mathbf{v}, s) = \sum_{i=1}^{d} f_{si}\mathsf{D}_{\mathbf{1}} P(\mathbf{w}_i(\mathbf{v}, s), i)$.

The following lemma then completes the proof:

LEMMA C.1: *For every $s \in S$, the directional derivative $\mathsf{D}_{\mathbf{1}} P(\cdot, s)$ is strictly positive.*

PROOF: We first show that each $\mathsf{D}_{\mathbf{1}} P(\cdot, s)$ is non-negative on D, i.e., that $P(\cdot, s)$ is non-decreasing in direction $\mathbf{1}$. Lemma F.1 shows that the first-best value function Q^* is non-decreasing in this direction. We will show that P inherits this property from Q^*. The proof is order-theoretic. Let $[Q^*, P]$ denote the order interval (in the pointwise order) of functions $Q : D \times S \to \mathbb{R}$ that lie weakly above Q^* and weakly below P. Let $\Phi := \{Q \in [Q^*, P] : Q(\mathbf{v} + \varepsilon\mathbf{1}, s) \geqslant Q(\mathbf{v}, s)\ \forall\, \mathbf{v} \in D,\ \varepsilon > 0 \text{ s.t. } \mathbf{v} + \varepsilon\mathbf{1} \in D\}$ denote the subset of functions in $[Q^*, P]$ that are non-decreasing in direction $\mathbf{1}$. By Lemma F.1, $Q^* \in \Phi$.

CLAIM C.2: *Φ is a lattice in the pointwise order.*

PROOF OF CLAIM C.2: Let $f, g \in \Phi$ be given. Clearly, $f \vee g, f \wedge g \in [Q^*, P]$. Now, fix $(\mathbf{v}, s) \in D \times S$ and $\varepsilon > 0$ such that $\mathbf{v}' := \mathbf{v} + \varepsilon\mathbf{1} \in D$. If f and g are ordered the same way at $(\mathbf{v}, s)$ and $(\mathbf{v}', s)$, we are done. So suppose, without loss of generality, that $f(\mathbf{v}, s) \geq g(\mathbf{v}, s)$ and $g(\mathbf{v}', s) \geq f(\mathbf{v}', s)$. Then, $(f \wedge g)(\mathbf{v}', s) = f(\mathbf{v}', s) \geqslant f(\mathbf{v}, s) \geq (f \wedge g)(\mathbf{v}, s)$. Similarly, $(f \vee g)(\mathbf{v}', s) \geq f(\mathbf{v}', s) \geqslant f(\mathbf{v}, s) = (f \vee g)(\mathbf{v}, s)$. Thus, $f \vee g$, $f \wedge g \in \Phi$ as desired. *Q.E.D.*

CLAIM C.3: *The lattice Φ is complete.*

PROOF OF CLAIM C.3: Let $F \subseteq \Phi$ be nonempty and define $\overline{f}(\mathbf{v}, s) := \sup_{f\in F} f(\mathbf{v}, s)$ and $\underline{f}(\mathbf{v}, s) := \inf_{f\in F} f(\mathbf{v}, s)$ for each $(\mathbf{v}, s) \in D \times S$. Clearly, we have $\overline{f}, \underline{f} \in [Q^*, P]$. We show that $\overline{f}$ is non-decreasing in direction $\mathbf{1}$, and hence $\overline{f} \in \Phi$ (the proof that $\underline{f} \in \Phi$ is symmetric). Suppose towards a contradiction that there exists $(\mathbf{v}, s) \in D \times S$ and some $\varepsilon > 0$ such that $(\mathbf{v}', s) \in D \times S$, where $\mathbf{v}' = \mathbf{v} + \varepsilon\mathbf{1}$, and $\overline{f}(\mathbf{v}, s) > \overline{f}(\mathbf{v}', s)$. Then $\overline{f}(\mathbf{v}, s) - \delta \geq \overline{f}(\mathbf{v}', s)$ for some $\delta > 0$. By definition of $\overline{f}$, there exists an $f \in F$ such that $f(\mathbf{v}, s) > \overline{f}(\mathbf{v}, s) - \delta$. Combining these inequalities and the definition of $\overline{f}$ yields $f(\mathbf{v}, s) > \overline{f}(\mathbf{v}', s) \geq f(\mathbf{v}', s)$, which contradicts that $f \in F \subseteq \Phi$. We conclude that $\overline{f} \in \Phi$, as desired. *Q.E.D.*

Let $\overline{\mathbb{R}}$ denote the extended reals, and let $\overline{\mathbb{R}}^{D\times S}$ denote the space of functions $f : D \times S \to \overline{\mathbb{R}}$. Define the Bellman operator $T : \overline{\mathbb{R}}^{D\times S} \to \overline{\mathbb{R}}^{D\times S}$ by

$$TQ(\mathbf{v}, s) := \inf_{(u_i, \mathbf{w}_i)_{i\in S} \in \Gamma(\mathbf{v})} \sum_{i=1}^{d} f_{si} \left[C(u_i, i) + \alpha Q(\mathbf{w}_i, i)\right]. \tag{T}$$

CLAIM C.4: *$T : \Phi \to \Phi$ is well-defined and monotone.*[3]

[3] That is, the image $T(\Phi) \subseteq \Phi$ and, if $Q, \hat{Q} \in \Phi$ satisfy $Q \leq \hat{Q}$, then $TQ \leq T\hat{Q}$.

PROOF OF CLAIM C.4: It is easy to see that the map $T:\Phi\to\overline{\mathbb{R}}^{D\times S}$ is monotone. It remains to show that the image $T(\Phi)\subseteq\Phi$. Let $Q\in\Phi$ be given. Since $Q\leq P$ by definition and $P=TP$ by Proposition 3.2, the monotonicity of T implies that $TQ\leq P$. Since $Q^*\leq Q$ by definition and $Q^*\leq TQ^*$ by Lemma F.1,[4] the monotonicity of T implies that $Q^*\leq TQ$. Thus, $TQ\in[Q^*,P]$. To show that $TQ\in\Phi$, let $(\mathbf{v},s)\in D\times S$ and $\varepsilon>0$ such that $\mathbf{v}-\varepsilon\mathbf{1}\in D$ be given. For every $\delta>0$, there exists $(u_i^\delta,\mathbf{w}_i^\delta)_{i\in S}\in\Gamma(\mathbf{v})$ attaining within δ of the infimal payoff in (T). Then $\delta+TQ(\mathbf{v},s)\geq\sum_{i\in S}f_{si}\big[C(u_i^\delta,i)+\alpha Q(\mathbf{w}_i^\delta,i)\big]\geq\sum_{i\in S}f_{si}\big[C(u_i^\delta,i)+\alpha Q(\mathbf{w}_i^\delta-\frac{\varepsilon}{\alpha}\mathbf{1},i)\big]\geq TQ(\mathbf{v}-\varepsilon\mathbf{1},s)$ where the first inequality is by δ-optimality of the menu, the second inequality is by $Q\in\Phi$, and the third inequality is because $(u_i^\delta,\mathbf{w}_i^\delta-\frac{\varepsilon}{\alpha}\mathbf{1})_{i\in S}\in\Gamma(\mathbf{v}-\varepsilon\mathbf{1})$. Sending $\delta\to 0$ yields $TQ(\mathbf{v},s)\geq TQ(\mathbf{v}-\varepsilon\mathbf{1},s)$. We conclude that $TQ\in\Phi$. Thus, the map $T:\Phi\to\Phi$ is well-defined. *Q.E.D.*

Given Claims C.2, C.3, and C.4, Tarski's Fixed Point Theorem implies that T has a smallest fixed point in Φ. This smallest fixed point must be P, since P is the smallest fixed point of $T:\overline{\mathbb{R}}^{D\times S}\to\overline{\mathbb{R}}^{D\times S}$ that lies pointwise above Q^* (Proposition 3.2) and $Q^*\in\Phi$. Hence, $P\in\Phi$. We conclude that $\mathsf{D}_{\mathbf{1}}P(\cdot,s)\geq 0$ on D for each $s\in S$, as desired.

To complete the proof, we now show that the directional derivative is strictly positive. Suppose, towards a contradiction, that there exists $(\mathbf{v},s)\in D\times S$ such that $\mathsf{D}_{\mathbf{1}}P(\mathbf{v},s)=0$. For this fixed $(\mathbf{v},s)$, define the function $g:[-\bar{\varepsilon},\bar{\varepsilon}]\to\mathbb{R}$ by $g(y):=P(\mathbf{v}+y\mathbf{1},s)$, where $\bar{\varepsilon}>0$ is chosen sufficiently small that $\mathbf{v}+y\mathbf{1}\in D$ for all for $y\in[-\bar{\varepsilon},\bar{\varepsilon}]$. (Such $\bar{\varepsilon}>0$ exist by Theorem 3(a).) By construction, $g'(y)=\mathsf{D}_{\mathbf{1}}P(\mathbf{v}+y\mathbf{1},s)\geq 0$ for all such y. At the same time, $g'(\cdot)$ is strictly increasing because P is strictly convex (Proposition 3.2(b)). Therefore, $g'(0)=\mathsf{D}_{\mathbf{1}}P(\mathbf{v},s)=0$ requires that $g'(y)<0$ for $y<0$, delivering the desired contradiction. We conclude that $\mathsf{D}_{\mathbf{1}}P(\cdot,s)>0$ on D, as desired. *Q.E.D.*

C.3. *Intermediate Steps Towards the Proof of Theorem 1*

This SA consists of several parts and culminates in Lemma C.18, which shows that the marginal cost martingale necessarily "splits" after consecutive realizations of the highest type d. To this end, SA-C.3.1 presents preliminary facts about the efficiency problem (Eff$_i$) from Section 4.3. SA-C.3.2 shows that the optimal contract is efficient (i.e., solves (Eff$_i$)) after consecutive d-type realizations, and records important properties of the marginal cost martingale at such histories. SA-C.3.3 studies an "interim" reformulation of the Bellman equation (FE), which lets us relate policy functions and optimal Lagrange multipliers across different values of the previous report $s\in S$. Finally, SA-C.3.4 uses the preceding results and facts about the first-best solution (recorded in SA-F) to prove Lemma C.18.

In what follows, we will make repeated use of the following fact:

LEMMA C.5: *Let $Y_s:=\mathsf{D}P(D,s)\subseteq\mathbb{R}^d$ denote the image of D under the derivative map $\mathsf{D}P(\cdot,s):D\to\mathbb{R}^d$. For every $s\in S$, the map $\mathsf{D}P(\cdot,s):D\to Y_s$ is a homeomorphism.*

PROOF: Since $P(\cdot,s)$ is strictly convex, $\mathsf{D}P(\cdot,s):D\to\mathbb{R}^d$ is injective.[5] Then, since $D\subseteq\mathbb{R}^d$ is open (Theorem 3(a)) and $\mathsf{D}P(\cdot,s)$ is continuous (Proposition 3.2(b)), Brouwer's Invariance of Domain Theorem (e.g., Hatcher, 2001, Theorem 2B.3) implies that $\mathsf{D}P(\cdot,s):D\to\mathbb{R}^d$ is an open map. Hence, the bijection $\mathsf{D}P(\cdot,s):D\to Y_s$ is a homeomorphism. *Q.E.D.*

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[4]In particular, Lemma F.1 shows that Q^* satisfies the Bellman equation (F.1), in which the feasible set $\Gamma(\mathbf{v})$ from (T) is replaced by the larger feasible set $\Gamma^{\text{FB}}(\mathbf{v})$ (which omits incentive constraints). This implies that $Q^*\leq TQ^*$, because infimizing over a smaller feasible set can only increase the principal's costs.

[5]Strict convexity implies that the derivative is *strictly monotone*: for all distinct $\mathbf{v},\mathbf{v}'\in D$, $\langle\mathbf{v}'-\mathbf{v},\mathsf{D}P(\mathbf{v}',s)-\mathsf{D}P(\mathbf{v},s)\rangle>\mathbf{0}$. Clearly, if $\mathsf{D}P(\cdot,s)$ were not injective, strict monotonicity would fail.

C.3.1. *The Efficiency Problem*

Recall the *efficiency problem* (Eff_i) from Section 4.3, re-stated here for convenience:

$$
\begin{aligned}
K(w,i) := \min_{\mathbf{w}_i \in D} & \; P(\mathbf{w}_i, i) \\
\text{s.t.} \qquad & \mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_i] \geq w.
\end{aligned}
\tag{Eff_i}
$$

LEMMA C.6: *For every $i \in S$, the following properties hold:*

(a) *For each $w \in \mathcal{U}$, (Eff_i) has a unique solution $\mathbf{w}^\dagger(w,i)$ and $w = \mathsf{E}^{\mathbf{f}_i}[\mathbf{w}^\dagger(w,i)]$.*

(b) *The policy function $\mathbf{w}^\dagger(\cdot,i) : \mathcal{U} \to D$ is continuous and injective, and thus defines a bijection between $\mathcal{U}$ and its image $E_i := \{\mathbf{v} \in D : \exists w \in \mathcal{U} \text{ s.t. } \mathbf{v} = \mathbf{w}^\dagger(w,i)\}$.*

(c) *The value function $K(\cdot,i) : \mathcal{U} \to \mathbb{R}$ is well-defined, strictly increasing, strictly convex, continuously differentiable, unbounded above, and satisfies the Inada conditions $\lim_{w\to-\infty} K'(w,i) = 0$ and $\lim_{w\to 0} K'(w,i) = +\infty$.*

PROOF: For part (a), existence follows from routine arguments analogous to those used to establish existence of optimal contracts in Proposition 3.2 (cf. Lemma J.9 in Section J of Bloedel, Krishna, and Leukhina (2025)), uniqueness follows from the strict convexity of $P(\cdot,i)$, and the binding constraint follows from Lemma C.1. For part (b), continuity follows from uniqueness and an application of Berge's Theorem (cf. Lemma J.9 in Section J of Bloedel, Krishna, and Leukhina (2025)). Since the constraint in (Eff_i) binds, $\mathbf{w}^\dagger(\cdot,i)$ is clearly injective. Thus, $\mathbf{w}^\dagger(\cdot,i) : \mathcal{U} \to E_i$ is a bijection. For part (c), $K(\cdot,i)$ is finite-valued (hence, well-defined) and strictly convex because P satisfies these properties (Proposition 3.2), strictly increasing by Lemma C.1, and inherits continuously differentiability from $P(\cdot,i)$ via a standard envelope argument for smooth convex problems.[6] It remains to show that $K(\cdot,i)$ is unbounded above and satisfies the claimed Inada conditions. These follow from analogous properties of the value function $K^*(w,i)$ for the first-best analogue of (Eff_i), in which P is replaced by Q^* (see (Eff_i^{FB}) in SA-F for the formal definition). Since $Q^* \leq P$ on $D \times S$, clearly $K^* \leq K$ on $\mathcal{U} \times S$. Lemma F.2 shows that $\lim_{w\to 0} K^*(w,i) = +\infty$. If $K(\cdot,i)$ were bounded above, i.e., $\lim_{w\to 0} K(w,i) < +\infty$, then there would exist some $v \in \mathcal{U}$ such that $K^*(v,i) > K(v,i)$, a contradiction. Thus, $\lim_{w\to 0} K(w,i) = +\infty$ and therefore $\lim_{w\to 0} K'(w,i) = +\infty$. Next, Lemma F.2 shows that $\lim_{w\to-\infty}(K^*)'(w,i) = 0$. If $K(\cdot,i)$ were to satisfy $\lim_{w\to-\infty} K'(w,i) > 0$, then there would exist some $v \in \mathcal{U}$ such that $K^*(v,i) > K(v,i)$, again a contradiction. Thus, $\lim_{w\to-\infty} K'(w,i) = 0$. *Q.E.D.*

The efficiency problem (Eff_i) admits a Lagrangian $\mathcal{L}^E(w,i,\mathbf{w},\zeta) = P(\mathbf{w},i) - \zeta \cdot (\mathsf{E}^{\mathbf{f}_i}[\mathbf{w}] - w)$ where $\zeta \geq 0$. By Lemma C.6, the unique solution to (Eff_i) at $w \in \mathcal{U}$ is characterized by the first-order and envelope conditions

$$P_j(\mathbf{w}^\dagger(w,i),i) = \zeta(w,i) f_{ij} \tag{$\text{FOC}_j\text{-Eff}_i$}$$

$$K'(w,i) = \zeta(w,i) \tag{$\text{Env}_j\text{-Eff}_i$}$$

where $\zeta(w,i) > 0$ denotes the optimal multiplier. Since Lemma C.6(c) implies that the image $K'(\mathcal{U},i) = \mathbb{R}_{++}$, these optimality conditions imply that the image of the *efficient set* $E_i =$

[6]For instance, part (a) and $P(\cdot,i) \in \mathbf{C}^1(D)$ imply, via the necessary and sufficient first-order condition stated below as ($\text{FOC}_j\text{-Eff}_i$), that there is a unique Lagrange multiplier $\zeta(w,i) \in \mathbb{R}_+$ for (Eff_i) at every $w \in \mathcal{U}$. A simple adaptation of Milgrom and Segal (2002, Corollary 5) then delivers $K(\cdot,i) \in \mathbf{C}^1(\mathcal{U})$.

$\{\mathbf{v} \in D : \exists w \in \mathcal{U} \text{ s.t. } \mathbf{v} = \mathbf{w}^\dagger(w,i)\}$ (as defined in Lemma C.6(b)) under the derivative map $\mathsf{D}P(\cdot,i)$ is given by the *efficiency ray*[7]

$$\mathsf{D}P(E_i,i) = \tilde{E}_i := \left\{(P_1,\dots,P_d) \in \mathbb{R}^d_{++} : \frac{P_1}{f_{i1}} = \cdots = \frac{P_d}{f_{id}}\right\}. \tag{$\tilde{\mathrm{E}}_i$}$$

Moreover, by summing the first-order conditions (FOC$_j$-Eff$_i$) over $j \in S$ and combining with the envelope condition (Env$_j$-Eff$_i$), we obtain

$$K'(w,i) = \mathsf{D}_\mathbf{1} P(\mathbf{w}^\dagger(w,i),i). \tag{C.3}$$

REMARK C.7—Efficiency in the i.i.d. Case: When types are i.i.d., the optimal contract satisfies $\xi^c(\cdot,\cdot,i) \in E_i$ for all $i \in S$. This follows mechanically from the absence of Markov information rents (i.e., $\mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_i] = \mathsf{E}^{\mathbf{f}_j}[\mathbf{w}_i]$ for all $i,j \in S$), the Bellman equation (FE) in Proposition 3.2, and the definition of (Eff$_i$). (It can also be seen by comparing ($\tilde{\mathrm{E}}_i$) to the optimality conditions in SA-C.1.) Moreover, $P(\cdot,i) = P(\cdot,j)$, $K(\cdot,i) = K(\cdot,j)$, and $E_i = E_j$ for all $i,j \in S$, so we can dispense with these type indices. It follows that the $\mathbf{v}^{(t)}$ process evolves in a subset $E \subset D$ that is in bijection to $\mathcal{U} \subset \mathbb{R}$ (Lemma C.6(b)). In effect, E is the image in D of the one-dimensional state space $\mathcal{U}$ from Thomas and Worrall (1990), $K(\cdot)$ is their value function, and $K'(\cdot)$ is their martingale (under NHB).

C.3.2. *Special Properties after d-Type Reports*

LEMMA C.8: *The optimal contract is efficient after type d reports. That is, for every $(\mathbf{v},s) \in D \times S$, we have $\xi^c(\mathbf{v},s,d) \in E_d$.*

PROOF: This follows directly from the Bellman equation (FE) in Proposition 3.2, the fact that the variable $\mathbf{w}_d \in D$ does not enter into any of the (IC$_{ij}$) constraints for $i \neq d$, and the definition of (Eff$_i$) (for $i = d$). (Alternatively, the optimality conditions (FOC$\mathbf{w}_{ij}$) for $i = d$ imply that $\mathsf{D}P(\xi^c(\mathbf{v},s,d),d) \in \tilde{E}_d$, so that ($\tilde{\mathrm{E}}_i$) and Lemma C.5 deliver $\xi^c(\mathbf{v},s,d) \in E_d$.) *Q.E.D.*

LEMMA C.9: *Given any $(\mathbf{v},s) \in D \times S$ and $i \in S$, define $\mathbf{w}_d := \xi^c(\mathbf{v},s,d)$ and $\tilde{\mathbf{w}}_i := \xi^c(\mathbf{w}_d,d,i)$. The following property holds:*

$$\mathsf{D}_\mathbf{1} P(\tilde{\mathbf{w}}_i,i) = \mathsf{D}_\mathbf{1} P(\mathbf{w}_d,d) + \frac{1}{f_{di}}\sum_{k=1}^{i-1}\mu_{ik}(\mathbf{w}_d,d) - \frac{1}{f_{di}}\sum_{k=i+1}^{d}\mu_{ki}(\mathbf{w}_d,d). \tag{MS$_i$}$$

PROOF: Let $(\mathbf{v},s) \in D \times S$ be given. We begin with the $i = d$ case. The optimality condition (FOC$\mathbf{w}_{ij}$) (for $i = j = d$) at state $(\mathbf{v},s)$ and the envelope condition (Env$_i$) (for $i = d$) at state $(\mathbf{w}_d,d)$ deliver $f_{dd}\big(\lambda_d(\mathbf{v},s) + \sum_{k=1}^{d-1}\mu_{dk}(\mathbf{v},s)\big) = f_{sd}P_d(\mathbf{w}_d,d) = f_{sd}\lambda_d(\mathbf{w}_d,d)$. Adding the common term $f_{sd}\sum_{k=1}^{d-1}\mu_{dk}(\mathbf{w}_d,d)$ to each side of the preceding equality then yields

$$\begin{aligned} f_{sd}\left[\lambda_d(\mathbf{w}_d,d) + \sum_{k=1}^{d-1}\mu_{dk}(\mathbf{w}_d,d)\right] &= f_{dd}\left[\lambda_d(\mathbf{v},s) + \sum_{k=1}^{d-1}\mu_{dk}(\mathbf{v},s)\right] \\ &\quad + f_{sd}\sum_{k=1}^{d-1}\mu_{dk}(\mathbf{w}_d,d). \end{aligned} \tag{C.4}$$

[7]The efficiency rays $\tilde{E}_i \subset \mathbb{R}^d_{++}$ are as described in Section 4.3, and the efficient sets $E_i \subset D$ are as described in Section 5.2 (where the assumption of CARA utility implies that each $E_i \subset D$ is a ray).

At state $(\mathbf{v}, s)$, summing the (FOCw$_{ij}$) for $i = d$ over $j \in S$ delivers

$$f_{sd}\mathsf{D}_1 P(\mathbf{w}_d, d) = \lambda_d(\mathbf{v}, s) + \sum_{k=1}^{d-1} \mu_{dk}(\mathbf{v}, s). \tag{C.5}$$

At state $(\mathbf{w}_d, d)$, performing the analogous sum delivers

$$f_{dd}\mathsf{D}_1 P(\tilde{\mathbf{w}}_d, d) = \lambda_d(\mathbf{w}_d, d) + \sum_{k=1}^{d-1} \mu_{dk}(\mathbf{w}_d, d). \tag{C.6}$$

Plugging (C.5) and (C.6) into (C.4) and dividing through by $f_{sd} \cdot f_{dd}$ yields (MS$_i$) for $i = d$.

Next, let $i < d$ be given. At state $(\mathbf{w}_d, d)$, summing the (FOCw$_{ij}$) over $j \in S$ yields

$$f_{di}\mathsf{D}_1 P(\tilde{\mathbf{w}}_i, i) = \lambda_i(\mathbf{w}_d, d) + \sum_{k=1}^{i-1} \mu_{ik}(\mathbf{w}_d, d) - \sum_{k=i+1}^{d} \mu_{ki}(\mathbf{w}_d, d). \tag{C.7}$$

Combining (C.6) and (C.7) delivers

$$\begin{aligned}\mathsf{D}_1 P(\tilde{\mathbf{w}}_i, i) = \mathsf{D}_1 P(\tilde{\mathbf{w}}_d, d) &- \frac{1}{f_{dd}}\sum_{k=1}^{d-1} \mu_{dk}(\mathbf{w}_d, d) - \frac{1}{f_{di}}\sum_{k=i+1}^{d} \mu_{ki}(\mathbf{w}_d, d) \\ &+ \frac{1}{f_{di}}\sum_{k=1}^{i-1} \mu_{ik}(\mathbf{w}_d, d) - \left[\frac{\lambda_d(\mathbf{w}_d, d)}{f_{dd}} - \frac{\lambda_i(\mathbf{w}_d, d)}{f_{di}}\right].\end{aligned} \tag{C.8}$$

The envelope conditions (Env$_i$) at state $(\mathbf{w}_d, d)$ imply that the final term in brackets equals $\frac{P_d(\mathbf{w}_d, d)}{f_{dd}} - \frac{P_i(\mathbf{w}_d, d)}{f_{di}}$, which vanishes by Lemma C.8 and ($\tilde{\text{E}}_i$). Then, plugging the rest of (C.8) into the $i = d$ case of (MS$_i$) (established above) delivers (MS$_i$) for the given $i < d$. *Q.E.D.*

LEMMA C.10: *Given any* $(\mathbf{v}, s) \in D \times S$ *and* $i \in S$, *define* $\mathbf{w}_d := \xi^c(\mathbf{v}, s, d)$ *and* $\tilde{\mathbf{w}}_i := \xi^c(\mathbf{w}_d, d, i)$. *The following two properties are equivalent:*

(a) $\mathsf{D}_1 P(\tilde{\mathbf{w}}_i, i) = \mathsf{D}_1 P(\mathbf{w}_d, d)$ *for all* $i \in S$.

(b) $\mu_{ij}(\mathbf{w}_d, d) = 0$ *for all* $i, j \in S$.

PROOF: By (MS$_i$) in Lemma C.9, (b) implies (a). To show the converse, suppose that (a) holds. We proceed by induction through the type space. For the base step, let $i = d$. From (MS$_i$) for $i = d$ and dual feasibility (i.e., $\mu_{dk}(\cdot, \cdot) \geq 0$ on $D \times S$ for every $k \in S$), we see that $\mathsf{D}_1 P(\tilde{\mathbf{w}}_d, d) = \mathsf{D}_1 P(\mathbf{w}_d, d)$ implies that $\mu_{dk}(\mathbf{w}_d, d) = 0$ for all $k \in S$. For the inductive step, let $i < d$ be given and suppose that $\mu_{jk}(\mathbf{w}_d, d) = 0$ for all $j \geq i + 1$ and $k < j$. Then (MS$_i$) reduces to $\mathsf{D}_1 P(\tilde{\mathbf{w}}_i, i) = \mathsf{D}_1 P(\mathbf{w}_d, d) + (1/f_{di}) \sum_{k=1}^{i-1} \mu_{ik}(\mathbf{w}_d, d)$. As in the base step, it then follows from dual feasibility that $\mathsf{D}_1 P(\tilde{\mathbf{w}}_i, i) = \mathsf{D}_1 P(\mathbf{w}_d, d)$ implies that $\mu_{ik}(\mathbf{w}_d, d) = 0$ for all $k < i$. This completes the induction. We conclude that (a) implies (b), as desired. *Q.E.D.*

C.3.3. *An Equivalent "Interim" Formulation*

Herein, we introduce an equivalent *interim* formulation of the principal's problem, in which she optimizes over contractual variables contingent on the *current* period's report (rather than the previous period's report). This shift in timing convention is merely cosmetic, but useful for relating the Lagrange multipliers at a given $\mathbf{v} \in D$ across different $s \in S$.

Formally, given any $\mathbf{v} \in D$ and $i \in S$, we consider principal's i^{th} *interim problem*:

$$Q^i(\mathbf{v}) := \inf_{(u_i,\mathbf{w}_i)\in\mathcal{U}\times D} [C(u_i,i) + \alpha P(\mathbf{w}_i,i)] \tag{FE-Q^i}$$

$$\text{s.t.} \qquad v_i = u_i + \alpha\, \mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_i] \tag{PK$_i$}$$

$$v_j - v_i \geq \psi(u_i,j,i) - u_i + \alpha\left(\mathsf{E}^{\mathbf{f}_j}[\mathbf{w}_i] - \mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_i]\right) \tag{IC$^*_{ji}$}$$

for all $j \in S$ with $j > i$.[8] That is, if the agent reports to be of type $i \in S$ in the current period, the principal optimizes over flow and continuation utility pairs $(u_i, \mathbf{w}_i) \in \mathcal{U} \times D$, subject to promise keeping (PK$_i$) for type i and incentive compatibility (IC$^*_{ji}$) for all *higher* types $j > i$. Notably, the function $Q^i : D \to \mathbb{R}$ depends on $\mathbf{v}$ only through the components $(v_i, v_{i+1}, \ldots, v_d)$, as these are the only components that enter the constraints. For each $i \in S$, we define the i^{th} *constraint correspondence* $\Gamma_i : D \to \mathcal{U} \times D$ as

$$\Gamma_i(\mathbf{v}) := \left\{(u_i,\mathbf{w}_i) \in \mathcal{U} \times D : (u_i,\mathbf{w}_i) \text{ satisfies (PK}_i) \text{ and (IC}^*_{ji}) \ \forall j \in S,\ j > i\right\}. \tag{C.9}$$

It is easy to see that, for every $\mathbf{v} \in D$, the constraint set $\Gamma(\mathbf{v})$ defined in (3.1) is given by the Cartesian product $\Gamma(\mathbf{v}) = \Gamma_1(\mathbf{v}) \times \cdots \times \Gamma_d(\mathbf{v})$.

LEMMA C.11: *For every* $(\mathbf{v}, s) \in D \times S$, *the following properties hold:*
(a) *The value functions* P *and* $\{Q^i\}_{i\in S}$ *satisfy* $P(\mathbf{v}, s) = \sum_{i=1}^d f_{si} Q^i(\mathbf{v})$.
(b) *A menu* $(u_i, \mathbf{w}_i)_{i\in S} \in \Gamma(\mathbf{v})$ *is a minimizer in* (FE) *at* $(\mathbf{v}, s)$ *if and only if, for every* $i \in S$, $(u_i, \mathbf{w}_i) \in \Gamma_i(\mathbf{v})$ *is a minimizer in* (FE-Q^i) *at* $\mathbf{v}$.
Consequently, under the optimal contract, for every $(\mathbf{v}, s) \in D \times S$ *and* $i \in S$, *it holds that* $\xi(\mathbf{v}, s, i) = (\xi^f(\mathbf{v}, s, i), \xi^c(\mathbf{v}, s, i)) \in \Gamma_i(\mathbf{v})$ *is a minimizer in* (FE-Q^i) *at* $\mathbf{v}$.

PROOF: Because $\Gamma(\mathbf{v}) = \Gamma_1(\mathbf{v}) \times \cdots \times \Gamma_d(\mathbf{v})$, we can equivalently write the Bellman equation (FE) as $P(\mathbf{v}, s) = \sum_{i=1}^d f_{si} \min_{(u_i,\mathbf{w}_i)\in\Gamma_i(\mathbf{v})} [C(u_i, i) + \alpha P(\mathbf{w}_i, i)] = \sum_{i=1}^d f_{si} Q^i(\mathbf{v})$. The lemma follows immediately from this observation and Proposition 3.2. *Q.E.D.*

COROLLARY C.12: *Under the optimal contract, for every* $i \in S$, *the function* $\xi(\cdot,\cdot,i) : D \times S \to \mathcal{U} \times D$ *depends on the argument* $(\mathbf{v}, s)$ *only through the components* $(v_i, \ldots, v_d)$.

PROOF: Immediate from Lemma C.11 and the above observation that the constraint set $\Gamma_i(\mathbf{v})$ depends on $\mathbf{v}$ only through the components $(v_i, \ldots, v_d)$. *Q.E.D.*

LEMMA C.13: *For every* $i \in S$, *the following properties hold:*
(a) *The value function* $Q^i : D \to \mathbb{R}$ *is convex and continuously differentiable.*
(b) *For every* $\mathbf{v} \in D$ *and* $i \in S$, *there exists some* $(u_i, \mathbf{w}_i) \in \Gamma_i(\mathbf{v})$ *such that all of the* (IC$^*_{ji}$) *(for* $j > i$*) hold as strict inequalities.*

PROOF: For part (a), convexity follows from the definition of Q^i in (FE-Q^i) and the convexity of $P(\cdot, i)$ (Proposition 3.2). Since $P(\cdot, s) \in \mathbf{C}^1(D)$ (Proposition 3.2), Lemma C.11(a) and the sum rule for subdifferentials of convex functions imply that $\{\mathsf{D}P(\mathbf{v}, s)\} = \partial P(\mathbf{v}, s) =$

[8] Note that (IC$^*_{ji}$) is the same as (IC$^*_{ij}$) from the main text, except that the $i, j \in S$ indices are flipped.

$\sum_{i=1}^{d} f_{si} \partial Q^i(\mathbf{v})$ for every $(\mathbf{v}, s) \in D \times S$.[9] Thus, for every $\mathbf{v} \in D$ and $i \in S$, $|\partial Q^i(\mathbf{v})| = 1$ and therefore Q^i is differentiable at $\mathbf{v}$. It follows that each $Q^i \in \mathbf{C}^1(D)$, as every differentiable convex function on the open set D (Theorem 3) is, in fact, continuously differentiable.

Part (b) follows from the fact that $\Gamma(\mathbf{v}) = \Gamma_1(\mathbf{v}) \times \cdots \times \Gamma_d(\mathbf{v})$ and Condition R.3. *Q.E.D.*

By Lemmas C.11 and C.13, the optimal $(u_i, \mathbf{w}_i) \in \Gamma_i(\mathbf{v})$ in (FE-Q^i) is characterized by saddle points of the Lagrangian (see, e.g., Exercise 7 on p. 236 and Theorem 2 on p. 221 of Luenberger (1969))

$$\begin{aligned}\mathscr{L}^i(\mathbf{v}, \mathbf{u}, \mathbf{w}, \boldsymbol{\eta}, \boldsymbol{\sigma}) = C(u_i, i) + \alpha \sum_{k=1}^{d} f_{ik} Q^k(\mathbf{w}_i) + \eta_i \Big[v_i - u_i - \alpha \, \mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_i] \Big] \\ - \sum_{k=i+1}^{d} \sigma_{ki} \Big[v_k - v_i - \psi(u_i, k, i) + u_i - \alpha \big(\mathsf{E}^{\mathbf{f}_k}[\mathbf{w}_i] - \mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_i] \big) \Big],\end{aligned} \tag{L$_i$}$$

where $\eta_i \in \mathbb{R}$ is the multiplier on (PK$_i$) and $\sigma_{ji} \geq 0$ is the multiplier on (IC$^*_{ji}$). We let $\eta_i(\mathbf{v}) \in \mathbb{R}$ denote the optimal multiplier on (PK$_i$) at $\mathbf{v} \in D$, and let $\sigma_{ji}(\mathbf{v}) \in \mathbb{R}_+$ denote the optimal multiplier on (IC$^*_{ji}$) at $\mathbf{v} \in D$. Lemma C.11 implies that, for every $(\mathbf{v}, s) \in D \times S$ and $i \in S$, the interim problem (FE-Q^i) is (uniquely) solved by the pair $(u_i(\mathbf{v}, s), \mathbf{w}_i(\mathbf{v}, s))$ derived from the (unique) optimal menu $(u_i(\mathbf{v}, s), \mathbf{w}_i(\mathbf{v}, s))_{i \in S}$ characterized in SA–C.1 by the optimality conditions (Env$_i$), (FOCu_i), and (FOC$\mathbf{w}_{ij}$). We use this fact repeatedly below.

LEMMA C.14: *The following properties hold:*

(a) *For every* $(\mathbf{v}, s) \in D \times S$ *and* $i \in S$, *the multipliers satisfy*

$$\frac{\lambda_i(\mathbf{v}, s)}{f_{si}} = \eta_i(\mathbf{v}) + \sum_{k=i+1}^{d} \sigma_{ki}(\mathbf{v}) - \sum_{k=1}^{i-1} \frac{f_{sk}}{f_{si}} \sigma_{ik}(\mathbf{v}). \tag{C.10}$$

(b) *For every* $(\mathbf{v}, s) \in D \times S$ *and* $i \in S$, *the multipliers satisfy*

$$\sum_{k=1}^{i-1} \Big[f_{sk} \sigma_{ik}(\mathbf{v}) - \mu_{ik}(\mathbf{v}, s) \Big] = \sum_{k=i+1}^{d} \psi'(u_i(\mathbf{v}, s), k, i) \Big[f_{si} \sigma_{ki}(\mathbf{v}) - \mu_{ki}(\mathbf{v}, s) \Big]. \tag{C.11}$$

(c) *For every* $\mathbf{v} \in D$, *the following are equivalent: (i)* $\sigma_{ij}(\mathbf{v}) = 0$ *for all* $i, j \in S$, *(ii) for* some $s \in S$, $\mu_{ij}(\mathbf{v}, s) = 0$ *for all* $i, j \in S$, *(iii) for* every $s \in S$, $\mu_{ij}(\mathbf{v}, s) = 0$ *for all* $i, j \in S$.

PROOF: The proof proceeds by comparing the optimality conditions from SA–C.1 to those derived from the interim Lagrangians (L$_i$). For part (a), let $(\mathbf{v}, s) \in D \times S$ be given. For each $i \in S$, the envelope conditions for (L$_i$) are

$$Q^i_j(\mathbf{v}) = \mathbf{1}(j = i) \cdot \Big[\eta_i(\mathbf{v}) + \sum_{k=i+1}^{d} \sigma_{ki}(\mathbf{v}) \Big] - \mathbf{1}(j > i) \cdot \sigma_{ji}(\mathbf{v}) \tag{C.12}$$

[9]For any convex function $f : D \to \mathbb{R}$, $\partial f(\mathbf{v}) \subseteq \mathbb{R}^d$ denotes its subdifferential at $\mathbf{v} \in D$. Since D is open (Theorem 3), we have $\partial f(\mathbf{v}) \neq \varnothing$ for all $\mathbf{v} \in D$. See Chapters 23 and 25 of Rockafellar (1970) for the relevant facts about subdifferentials (in particular, Theorem 23.8, Theorem 25.1, and Corollary 25.5.1).

for all $j \in S$. Meanwhile, Lemma C.11(a) implies that $P_j(\mathbf{v}, s) = \sum_{i=1}^{d} f_{si} Q_j^i(\mathbf{v})$ for every $j \in S$. Thus, fixing $j \in S$ and summing over (C.12) over $i \in S$ delivers

$$P_j(\mathbf{v}, s) = f_{sj}\left[\eta_j(\mathbf{v}) + \sum_{k=j+1}^{d} \sigma_{kj}(\mathbf{v})\right] - \sum_{k=1}^{j-1} f_{sk}\sigma_{jk}(\mathbf{v}). \tag{C.13}$$

The envelope condition (Env_i) (for $i = j$) at $(\mathbf{v}, s)$ yields $\lambda_j(\mathbf{v}, s) = P_j(\mathbf{v}, s)$. Plugging this into (C.13) yields (C.10) (for $i = j$). Since the fixed $j \in S$ was arbitrary, this proves part (a).

For part (b), let $(\mathbf{v}, s) \in D \times S$ and $i \in S$ be given. The FOC for $u_i(\mathbf{v}, s)$ in (L_i) is

$$C'(u_i(\mathbf{v}, s), i) = \eta_i(\mathbf{v}) + \sum_{k=i+1}^{d} \sigma_{ki}(\mathbf{v})\left(1 - \psi'(u_i(\mathbf{v}, s), k, i)\right). \tag{C.14}$$

Meanwhile, ($\text{FOC}u_i$) at $(\mathbf{v}, s)$ can be written as

$$C'(u_i(\mathbf{v}, s), i) = \frac{\lambda_i(\mathbf{v}, s)}{f_{si}} + \sum_{k=1}^{i-1} \frac{\mu_{ik}(\mathbf{v}, s)}{f_{si}} - \sum_{k=i+1}^{d} \psi'(u_i(\mathbf{v}, s), k, i)\frac{\mu_{ki}(\mathbf{v}, s)}{f_{si}}. \tag{C.15}$$

Equating (C.14) and (C.15) and substituting in (C.10) delivers

$$\begin{aligned}
&\frac{\lambda_i(\mathbf{v}, s)}{f_{si}} + \sum_{k=1}^{i-1} \frac{\mu_{ik}(\mathbf{v}, s)}{f_{si}} - \sum_{k=i+1}^{d} \psi'(u_i(\mathbf{v}, s), k, i)\frac{\mu_{ki}(\mathbf{v}, s)}{f_{si}} \\
&\quad = \sum_{k=i+1}^{d} \sigma_{ki}(\mathbf{v})\left(1 - \psi'(u_i(\mathbf{v}, s), k, i)\right) + \underbrace{\left[\frac{\lambda_i(\mathbf{v}, s)}{f_{si}} - \sum_{k=i+1}^{d} \sigma_{ki}(\mathbf{v}) + \sum_{k=1}^{i-1} \frac{f_{sk}}{f_{si}}\sigma_{ik}(\mathbf{v})\right]}_{= \eta_i(\mathbf{v})}.
\end{aligned}$$

Simplifying the above display yields (C.11), as desired.

For part (c), let $\mathbf{v} \in D$ be given. We show that (i) implies (iii) by induction. So suppose that (i) holds, and let $s \in S$ be given. For the base step, note that (C.11) with $i = d$ becomes $0 = \sum_{k=1}^{d-1}\left[\mu_{dk}(\mathbf{v}, s) - f_{sk}\sigma_{dk}(\mathbf{v})\right] = \sum_{k=1}^{d-1} \mu_{dk}(\mathbf{v}, s)$ where the second equality is because (i) holds at $\mathbf{v}$. Because $\mu_{dk}(\cdot, \cdot) \geq 0$ on $D \times S$ for all $k < d$, it follows that $\mu_{dk}(\mathbf{v}, s) = 0$ for all $k < d$. For the inductive step, let $\ell < d$ be given and suppose we have shown, for all $k > \ell$, that $\mu_{kj}(\mathbf{v}, s) = 0$ for all $j < k$. Then we have

$$\begin{aligned}
0 &= \sum_{k=1}^{\ell-1}\left[\mu_{\ell k}(\mathbf{v}, s) - f_{sk}\sigma_{\ell k}(\mathbf{v})\right] + \sum_{k=\ell+1}^{d} \psi'(u_\ell(\mathbf{v}, s), k, \ell)\left[f_{s\ell}\sigma_{k\ell}(\mathbf{v}) - \mu_{k\ell}(\mathbf{v}, s)\right] \\
&= \sum_{k=1}^{\ell-1} \mu_{\ell k}(\mathbf{v}, s) - \sum_{k=\ell+1}^{d} \psi'(u_\ell(\mathbf{v}, s), k, \ell)\mu_{k\ell}(\mathbf{v}, s) = \sum_{k=1}^{\ell-1} \mu_{\ell k}(\mathbf{v}, s),
\end{aligned}$$

where the first equality is (C.11) for $i = \ell$, the second equality is because (i) holds at $\mathbf{v}$, and the third equality is by the induction hypothesis. As before, it follows that $\mu_{\ell k}(\mathbf{v}, s) = 0$ for all $k < \ell$. This completes the induction. Since the given $s \in S$ was arbitrary, we conclude that (i) implies (iii). By full connectedness (Assumption Markov), the proof that (ii) implies (i) is completely analogous. Obviously, (iii) implies (ii). This proves part (c). *Q.E.D.*

LEMMA C.15: *For every* $\mathbf{v} \in D$*, the following hold:*
(a) *If there exists an* $s \in S$ *such that* $\mu_{ij}(\mathbf{v}, s) = 0$ *for all* $i, j \in S$*, then*

$$\frac{\lambda_i(\mathbf{v}, s')}{f_{s'i}} = \eta_i(\mathbf{v}) \qquad \textit{for all } s', i \in S. \tag{C.16}$$

(b) *If there exists an* $s \in S$ *such that* $\mathbf{v} \in E_s$ *and* $\mu_{ij}(\mathbf{v}, s) = 0$ *for all* $i, j \in S$,[10] *then (i)* $\mathbf{v} \in E_{s'}$ *for all* $s' \in S$ *and (ii) there exists an* $\hat{\eta}(\mathbf{v}) \in \mathbb{R}$ *such that*

$$\frac{\lambda_i(\mathbf{v}, s')}{f_{s'i}} = \hat{\eta}(\mathbf{v}) \qquad \textit{for all } s', i \in S. \tag{C.17}$$

PROOF: Let $\mathbf{v} \in D$ and such an $s \in S$ be given. For part (a), the hypothesis and Lemma C.14(c) imply that $\sigma_{ij}(\mathbf{v}) = 0$ for all $i, j \in S$. Plugging this into (C.10) (for the given $\mathbf{v}$ and across all $s', i \in S$) delivers (C.16). For part (b), the same logic delivers (C.16). At the same time, the hypothesis that $\mathbf{v} \in E_s$ and ($\tilde{\text{E}}_i$) (for $i = s$) imply that $\frac{P_1(\mathbf{v},s)}{f_{s1}} = \cdots = \frac{P_d(\mathbf{v},s)}{f_{sd}}$, which by the envelope condition (Env_i) at $(\mathbf{v}, s)$ is equivalent to $\frac{\lambda_1(\mathbf{v},s)}{f_{s1}} = \cdots = \frac{\lambda_d(\mathbf{v},s)}{f_{sd}}$. Plugging the latter into (C.16) delivers $\eta_1(\mathbf{v}) = \cdots = \eta_d(\mathbf{v})$. Denoting the common value of these multipliers by $\hat{\eta}(\mathbf{v}) \in \mathbb{R}$ yields (C.17). Now, let $s' \in S$ be given. We obtain from (C.17) that $\frac{\lambda_1(\mathbf{v},s')}{f_{s'1}} = \cdots = \frac{\lambda_d(\mathbf{v},s')}{f_{s'd}}$, which by the envelope condition (Env_i) at $(\mathbf{v}, s')$ is equivalent to $\frac{P_1(\mathbf{v},s')}{f_{s'1}} = \cdots = \frac{P_d(\mathbf{v},s')}{f_{s'd}}$. Then, ($\tilde{\text{E}}_i$) (for $i = s'$) and Lemma C.5 imply that $\mathbf{v} \in E_{s'}$, as desired. Since the given $s' \in S$ was arbitrary, this completes the proof of part (b). *Q.E.D.*

C.3.4. *First-Best Efficiency and Martingale Splitting*

The optimal contract ξ *self-generates at* $\mathbf{v} \in D$ if $\xi^c(\mathbf{v}, s, i) = \mathbf{v}$ for all $s, i \in S$. The optimal contract ξ is *first-best efficient at* $\mathbf{v} \in D$ if (i) ξ self-generates at $\mathbf{v}$, (ii) there is some $v \in \mathcal{U}$ such that $\mathbf{v} = v\mathbf{1}$, and (iii) $\xi^f(\mathbf{v}, s, i) = (1-\alpha)v$ for all $s, i \in S$.[11] First-best efficient contracts arise from solutions to the "first-best efficiency problem" (Eff_i^{FB}) defined in SA–F (see Lemma F.1(a) and Lemma F.2 therein), which is the analogue of the efficiency problem (Eff_i) without IC constraints.

LEMMA C.16: *The optimal contract* ξ *is not first-best efficient at any* $\mathbf{v} \in D$.

PROOF: If the domain D does not intersect the diagonal $\{\mathbf{v} \in \mathcal{U}^d : \exists v \in \mathcal{U} \text{ s.t. } \mathbf{v} = v\mathbf{1}\}$, then it is impossible for any $\mathbf{v} \in D$ to satisfy property (ii) in the definition of first-best efficiency, so we are done. Suppose, towards a contradiction, that D intersects the diagonal and that ξ is first-best efficient at some $\mathbf{v} \in D$, where $\mathbf{v} = v\mathbf{1}$ and $v \in \mathcal{U}$. Let $i, j \in S$ with $i > j$ be given. Then, for every $s \in S$, the (IC_{ij}^*) constraint at state $(\mathbf{v}, s)$ reads $v - v \geq \psi((1-\alpha)v, i, j) - (1-\alpha)v + \alpha\left(\mathbf{E}^{\mathbf{f}_i}[v\mathbf{1}] - \mathbf{E}^{\mathbf{f}_j}[v\mathbf{1}]\right)$, which reduces to $0 \geq \psi((1-\alpha)v, i, j) - (1-\alpha)v$. But $\psi(u, i, j) - u > 0$ for all $u \in \mathcal{U}$, a contradiction. *Q.E.D.*

LEMMA C.17: *Given any* $s \in S$ *and* $\mathbf{v} \in E_s$*, there exist some* $i, j \in S$ *(with* $i > j$*) such that* $\mu_{ij}(\mathbf{v}, s) > 0$.

[10]Recall the efficient sets $E_{s'} \subset D$ defined in Appendix C.3.1 (see, e.g., Lemma C.6(b)).
[11]Given condition (i), conditions (ii) and (iii) are equivalent (see, e.g., the proof of Lemma C.17 below).

PROOF: Let $s \in S$ and $\mathbf{v} \in E_s$ be given. Suppose, towards a contradiction, that $\mu_{ij}(\mathbf{v}, s) = 0$ for all $i, j \in S$. Then Lemma C.14(c) implies that $\mu_{ij}(\mathbf{v}, s') = 0$ for all $s', i, j \in S$ and Lemma C.15(b) implies that (C.17) holds. Plugging the former into the the optimality conditions (FOCu_i) and (FOC$\mathbf{w}_{ij}$) at $(\mathbf{v}, s')$ for every $s' \in S$ delivers

$$f_{s'i} C'(u_i(\mathbf{v}, s'), i) = \lambda_i(\mathbf{v}, s') \qquad \forall s', i \in S$$
$$f_{s'i} P_j(\mathbf{w}_i(\mathbf{v}, s'), i) = f_{ij} \lambda_i(\mathbf{v}, s') \qquad \forall s', i, j \in S.$$

Plugging (C.17) into the above display delivers

$$\frac{P_j(\mathbf{w}_i(\mathbf{v}, s'), i)}{f_{ij}} = \hat{\eta}(\mathbf{v}) = C'(u_i(\mathbf{v}, s'), i) \qquad \forall s', i, j \in S. \tag{C.18}$$

Plugging (C.17) into the envelope conditions (Env$_i$) at $(\mathbf{v}, i)$ for every $i \in S$ delivers

$$\frac{P_j(\mathbf{v}, i)}{f_{ij}} = \hat{\eta}(\mathbf{v}) \qquad \forall i, j \in S. \tag{C.19}$$

We claim that (C.18) and (C.19) together imply that the optimal contract ξ is first-best efficient at $\mathbf{v}$. Recall that, by definition, $\xi^f(\cdot, \cdot, i) = u_i(\cdot, \cdot)$ and $\xi^c(\cdot, \cdot, i) = \mathbf{w}_i(\cdot, \cdot)$ for all $i \in S$. To show that ξ self-generates at $\mathbf{v}$ (part (i) in the definition), note that combining (C.19) and the first equality in (C.18) delivers

$$\mathsf{D}P(\mathbf{v}, i) = \mathsf{D}P(\mathbf{w}_i(\mathbf{v}, s'), i) \qquad \forall s', i \in S.$$

It then follows from Lemma C.5 that $\mathbf{v} = \mathbf{w}_i(\mathbf{v}, s')$ for every $s', i \in S$, as desired. To establish parts (ii) and (iii) in the definition, first recall that $C(\cdot, i) := U^{-1}(\cdot) - \omega_i$ for every $i \in S$. Thus, the Inverse Function Theorem yields $C'(\cdot, i) = 1/U'(U^{-1}(\cdot))$ for every $i \in S$. Plugging this into the second equality in (C.18) delivers

$$U'(U^{-1}(u_i(\mathbf{v}, s'))) = \frac{1}{\hat{\eta}(\mathbf{v})} \qquad \forall s', i \in S.$$

Thus, since $U'(\cdot)$ and $U^{-1}(\cdot)$ are both injective (Assumption DARA), there exists some $z(\mathbf{v}) \in \mathcal{U}$ such that $u_i(\mathbf{v}, s') = z(\mathbf{v})$ for all $s', i \in S$. Given this property and the fact (shown above) that ξ self-generates at $\mathbf{v}$, the (PK$_i$) constraints (for all $i \in S$) at state $(\mathbf{v}, s')$ (for any $s' \in S$) yield $\mathbf{v} = z(\mathbf{v})\mathbf{1} + \alpha \mathbf{F} \mathbf{v}$, where $\mathbf{F} = [\mathbf{f}_i]_{i=1}^d$ is the matrix of transition probabilities. Thus, $\mathbf{v} = z(\mathbf{v})(\mathbf{I} - \alpha\mathbf{F})^{-1}\mathbf{1} = \frac{z(\mathbf{v})}{1-\alpha}\mathbf{1}$.[12] That is, $\mathbf{v} = v\mathbf{1}$ for $v := \frac{z(\mathbf{v})}{1-\alpha} \in \mathcal{U}$, and hence $u_i(\mathbf{v}, s') = (1-\alpha)v$ for all $s', i \in S$. This establishes that ξ is first-best efficient at $\mathbf{v}$.

Since Lemma C.16 establishes that ξ cannot be first-best efficient at $\mathbf{v}$, this delivers the desired contradiction. We conclude that $\mu_{ij}(\mathbf{v}, s) > 0$ for some $i, j \in S$ (with $i > j$). *Q.E.D.*

LEMMA C.18: *Given any $(\mathbf{v}, s) \in D \times S$, define $\mathbf{w}_d := \xi^c(\mathbf{v}, s, d)$ and $\tilde{\mathbf{w}}_i := \xi^c(\mathbf{w}_d, d, i)$ for every $i \in S$. There exists an $i \in S$ such that $\mathsf{D}_1 P(\tilde{\mathbf{w}}_i, i) \neq \mathsf{D}_1 P(\mathbf{w}_d, d)$.*

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[12] In particular, note that $\mathbf{v} = z(\mathbf{v})\mathbf{1} + \alpha\mathbf{F}\mathbf{v}$ can be rewritten as $(\mathbf{I} - \alpha\mathbf{F})\mathbf{v} = z(\mathbf{v})\mathbf{1}$, where $\mathbf{I} \in \mathbb{R}^{d \times d}$ is the identity matrix. The matrix $\mathbf{I} - \alpha\mathbf{F}$ is strictly diagonally dominant: for every $i \in S$, $|[\mathbf{I} - \alpha\mathbf{F}]_{ii}| = 1 - \alpha f_{ii} > \alpha(1 - f_{ii}) = \sum_{j \neq i} |[\mathbf{I} - \alpha\mathbf{F}]_{ij}|$. Thus, $\mathbf{I} - \alpha\mathbf{F}$ is invertible. Since $(\mathbf{I} - \alpha\mathbf{F})\mathbf{1} = (1-\alpha)\mathbf{1}$, we have $(\mathbf{I} - \alpha\mathbf{F})^{-1}\mathbf{1} = \frac{1}{1-\alpha}\mathbf{1}$. It follows that $\mathbf{v} = z(\mathbf{v})(\mathbf{I} - \alpha\mathbf{F})^{-1}\mathbf{1} = \frac{z(\mathbf{v})}{1-\alpha}\mathbf{1}$, as claimed.

PROOF: By Lemma C.8, we have $\mathbf{w}_d \in E_d$. Thus, Lemma C.17 implies that there exist some $i, j \in S$ (with $i > j$) such that $\mu_{ij}(\mathbf{w}_d, d) > 0$. Lemma C.10 then implies that there exists an $i \in S$ such that $\mathsf{D}_1 P(\tilde{\mathbf{w}}_i, i) \neq \mathsf{D}_1 P(\mathbf{w}_d, d)$, as desired. *Q.E.D.*

The main implication of Lemma C.18 is that, along the optimal path, the marginal cost martingale is non-constant with strictly positive probability at every state of the form $(\mathbf{w}_d, d)$. That is, the martingale "splits" with strictly positive probability at such histories.

C.4. *Main Proof of Theorem 1*

We use the following notation throughout the proof. Recall from Section 3 that the space of *paths* of type reports is $\mathcal{H} := S^\infty$ with generic element $h := \left(s^{t+1}\right)_{t=0}^\infty$, where s^t denotes the (truthfully) reported type in period $t-1$. Let $\tau^{(t)}$ denote the *random* time defined pathwise by $\tau^{(t)}(h) := \sup\left\{T \leq t : \ s^T = d\right\}$. That is, given path h, $\tau^{(t)}(h)$ is the last date (i) that precedes or equals t and (ii) that was immediately preceded by a realized endowment ω_d. In particular, the state in period $\tau^{(t)}$ is $(\mathbf{v}, d)$ for some $\mathbf{v} \in E_d$ (by Lemma C.8), where E_d is defined in Appendix C.3.1. It is easy to see that $\tau^{(t)}$ is a well-defined stopping time, that the process $\left(\tau^{(t)}\right)_{t=0}^\infty$ is non-decreasing, and that $\lim_{t\to\infty} \tau^{(t)} = \infty$, P-a.s.

Martingale Convergence. Define the events $\mathcal{J}, \mathcal{H}^*, \mathcal{J}^* \subseteq \mathcal{H}$ as follows:

$$\mathcal{J} := \left\{h \in \mathcal{H} : \forall i \in S, \, (s^t, s^{t+1}) = (d, i) \text{ occurs for infinitely many } t\right\},$$

$$\mathcal{H}^* := \left\{h \in \mathcal{H} : \lim_{t\to\infty} \mathsf{D}_1 P(\mathbf{v}^{(t)}(h), s^t) \text{ exists}\right\}, \text{ and } \mathcal{J}^* := \mathcal{J} \cap \mathcal{H}^*.$$

We will repeatedly make use of the following facts. First, $\mathcal{J}$ satisfies two properties: (i) $\lim_{t\to\infty} \tau^{(t)}(h) = +\infty$ for all $h \in \mathcal{J}$ by construction, and (ii) $\mathsf{P}(\mathcal{J}) = 1$ by standard facts about Markov chains (see Corollary G.3 in SA-G). Second, $\mathcal{H}^*$ satisfies $\mathsf{P}(\mathcal{H}^*) = 1$ by Doob's Martingale Convergence Theorem (see Theorem 2 in Shiryaev (1995, p. 517)), because the process $\left(\mathsf{D}_1 P(\mathbf{v}^{(t)}, s^{(t)})\right)_{t=0}^\infty$ is a strictly positive martingale (Proposition 4.4). Finally, $\mathcal{J}^*$ satisfies $\mathsf{P}(\mathcal{J}^*) = 1$, being the finite intersection of full-measure events.

LEMMA C.19: *The marginal cost martingale* $\mathsf{D}_1 P(\mathbf{v}^{(t)}, s^{(t)}) \to 0$ *almost surely.*

PROOF: It suffices to show that $\mathsf{D}_1 P(\mathbf{v}^{(t)}, s^{(t)}) \to 0$ on the event $\mathcal{J}^*$. So, fix a path $h = \left(s^{t+1}\right)_{t=0}^\infty \in \mathcal{J}^*$. Since the path is fixed, let $\tau^t := \tau^{(t)}(h)$ and $\mathbf{v}^t := \mathbf{v}^{(t)}(h)$ for all $t \in \mathbb{N}$. For all $i \in S$ and $t \in \mathbb{N}$, define $\tau_i^t := \sup\left\{T \leq t : (s^T, s^{T+1}) = (d, i)\right\}$. By construction, $s^{\tau_i^t + 1} = i$ for all $i \in S$ and $t \in \mathbb{N}$, and $\lim_{t\to\infty} \tau_i^t = +\infty$ for all $i \in S$.

Suppose, towards a contradiction, that $\mathsf{D}_1 P(\mathbf{v}^t, s^t) \to C > 0$. By construction, $\mathbf{v}^{\tau^t} = \xi^c(\mathbf{v}^{\tau^t - 1}, s^{\tau^t - 1}, d)$ and $s^{\tau^t} = d$. Thus, Lemma C.8 and ($\tilde{\mathrm{E}}_i$) imply that $\mathbf{v}^{\tau^t} \in E_d$ and $\mathsf{D}P(\mathbf{v}^{\tau^t}, s^{\tau^t} = d) \in \tilde{E}_d$ for all $t \in \mathbb{N}$. This and the supposition then imply that $\mathsf{D}P(\mathbf{v}^{\tau^t}, s^{\tau^t} = d) \to \mathbf{y}^* := (Cf_{d1}, \ldots, Cf_{dd}) \in \tilde{E}_d$. Since $\mathsf{D}P(\cdot, d)$ is a homeomorphism (Lemma C.5), it follows that $\mathbf{v}^{\tau^t} \to \mathbf{w}_d^* := [\mathsf{D}P(\cdot, d)]^{-1}(\mathbf{y}^*) \in E_d$. Thus, for every $i \in S$, we have $\xi^c(\mathbf{v}^{\tau^t}, d, i) \to \xi^c(\mathbf{w}_d^*, d, i) =: \tilde{\mathbf{w}}_i^*$ because the policy functions are continuous (Proposition 3.2). Since $\mathbf{v}^{\tau_i^t + 1} = \xi^c(\mathbf{v}^{\tau_i^t}, s^{\tau_i^t} = d, s^{\tau_i^t + 1} = i)$ by construction, it follows that $\lim_{t\to\infty} \mathsf{D}_1 P(\mathbf{v}^{\tau_i^t + 1}, s^{\tau_i^t + 1} = i) = \mathsf{D}_1 P(\tilde{\mathbf{w}}_i^*, i)$ for each $i \in S$, because each $P(\cdot, i)$ is continuously differentiable (Proposition 3.2). By the supposition, it follows that $\mathsf{D}_1 P(\mathbf{w}_d^*, d) = C = \mathsf{D}_1 P(\tilde{\mathbf{w}}_i^*, i)$ for all $i \in S$. But this violates Lemma C.18, yielding the desired contradiction. Thus, $\mathsf{D}_1 P(\mathbf{v}^t, s^t) \to 0$, as desired. *Q.E.D.*

Convergence of Multipliers. We next establish convergence of the Lagrange multipliers. For every state $(\mathbf{v}, s) \in D \times S$, define the following vectors. Denote by $\boldsymbol{\lambda}(\mathbf{v}, s) := \big(\lambda_1(\mathbf{v}, s), \ldots, \lambda_d(\mathbf{v}, s)\big) \in \mathbb{R}^d$ the vector of multipliers on the (PK$_i$) constraints. For each $j \in S$, denote by $\boldsymbol{\mu}_{*,j}(\mathbf{v}, s) := \big(\mu_{j+1,j}(\mathbf{v}, s), \ldots, \mu_{d,j}(\mathbf{v}, s)\big) \in \mathbb{R}_+^{d-j}$ the vector of multipliers on the (IC$_{ij}$) constraints (for all $i > j$). Let $\boldsymbol{\mu}(\mathbf{v}, s) := \big(\boldsymbol{\mu}_{*,j}(\mathbf{v}, s)\big)_{j \in S} \in \mathbb{R}^{d(d-1)/2}$ denote the vector that stacks the $\boldsymbol{\mu}_{*,j}(\mathbf{v}, s)$. Finally, let $\boldsymbol{\nu}(\mathbf{v}, s) := (\boldsymbol{\lambda}(\mathbf{v}, s), \boldsymbol{\mu}(\mathbf{v}, s)) \in \mathbb{R}^{d(d+1)/2}$ denote the vector that stacks all of the multipliers. We proceed through a series of lemmas.

LEMMA C.20: *It holds that $\boldsymbol{\nu}^{(\tau^{(t)})} \to \mathbf{0}$ and $\mathsf{D}P(\tilde{\mathbf{w}}_i^{(\tau^{(t)})}, i) \to \mathbf{0}$ for all $i \in S$ almost surely, where $\tilde{\mathbf{w}}_i^{(\tau^{(t)})} := \xi^c(\mathbf{v}^{(\tau^{(t)})}, s^{(\tau^{(t)})} = d, i)$.*

PROOF: Again, it suffices to show convergence on $\mathcal{J}^*$. So, fix a path $h = \big(s^{t+1}\big)_{t=0}^{\infty} \in \mathcal{J}^*$ and let $\tau^t := \tau^{(t)}(h)$ and $\mathbf{v}^t := \mathbf{v}^{(t)}(h)$. As shown in the proof of Lemma C.19, we have $\mathsf{D}_1 P(\mathbf{v}^t, s^t) \to 0$ along this path. Since $\mathbf{v}^{\tau^t} = \xi^c(\mathbf{v}^{\tau^t - 1}, s^{\tau^t - 1}, d)$ and $s^{\tau^t} = d$ by construction, Lemma C.8 and ($\tilde{\mathrm{E}}_i$) further imply that $\mathsf{D}P(\mathbf{v}^{\tau^t}, s^{\tau^t} = d) \to \mathbf{0}$. Then (Env$_i$) implies that

$$\boldsymbol{\lambda}^{\tau^t} := \boldsymbol{\lambda}(\mathbf{v}^{\tau^t}, s^{\tau^t} = d) \to \mathbf{0}. \tag{C.20}$$

We next show that $\boldsymbol{\mu}_{*,i}^{\tau^t} \to \mathbf{0}$ and $\mathsf{D}P(\tilde{\mathbf{w}}^{\tau_i^t}, i) \to \mathbf{0}$ for all $i \in S$. To do so, we proceed by induction through the type space, starting from the bottom.

Base step: The first-order condition (FOC$\mathbf{w}_{ij}$) with $i = 1$ at state $(\mathbf{v}^{\tau^t}, d)$ is $f_{d1} P_j(\tilde{\mathbf{w}}_1^{\tau^t}, 1) = f_{1j}\big(\lambda_1(\mathbf{v}^{\tau^t}, d) + 0\big) - \sum_{k=2}^{d} f_{kj} \mu_{k1}(\mathbf{v}^{\tau^t}, d)$. Because $\boldsymbol{\mu}(\mathbf{v}^{\tau^t}, d) \geq \mathbf{0}$ (dual feasibility), it follows from (C.20) and full connectedness (Assumption Markov) that $\limsup_{t \to \infty} P_j(\tilde{\mathbf{w}}_1^{\tau^t}, 1) \leq 0$ for all $j \in S$. But since $\mathsf{D}_1 P(\tilde{\mathbf{w}}_1^{\tau^t}, 1) \geq 0$ for all $t \in \mathbb{N}$ by Lemma C.1, we obtain $P_j(\tilde{\mathbf{w}}_1^{\tau^t}, 1) \to 0$ for all $j \in S$. It then follows from the FOC that $\sum_{k=2}^{d} f_{kj} \mu_{k1}(\mathbf{v}^{\tau^t}, d) \to 0$ for all $j \in S$, and hence $\mu_{k1}(\mathbf{v}^{\tau^t}, d) \to 0$ for all $2 \leq k \leq d$. Putting this together, we obtain

$$\mathsf{D}P(\tilde{\mathbf{w}}_1^{\tau^t}, 1) \to \mathbf{0} \quad \text{and} \quad \boldsymbol{\mu}_{*,1}^{\tau^t} := \boldsymbol{\mu}_{*,1}(\mathbf{v}^{\tau^t}, d) \to \mathbf{0}. \tag{C.21}$$

Inductive step: Let $2 \leq m \leq d$. Suppose we have shown that $\mathsf{D}P(\tilde{\mathbf{w}}_k^{\tau^t}, k) \to \mathbf{0}$ and $\boldsymbol{\mu}_{*,k}(\mathbf{v}^{\tau^t}, d) \to \mathbf{0}$ for all $1 \leq k < m$. The first order condition (FOC$\mathbf{w}_{ij}$) with $i = m$ at state $(\mathbf{v}^{\tau^t}, d)$ is $f_{dm} P_j(\tilde{\mathbf{w}}_m^{\tau^t}, m) = f_{mj}\big(\lambda_m(\mathbf{v}^{\tau^t}, d) + \sum_{k=1}^{m-1} \mu_{mk}(\mathbf{v}^{\tau^t}, d)\big) - \sum_{k=m+1}^{d} f_{kj} \mu_{km}(\mathbf{v}^{\tau^t}, d)$. The inductive hypothesis and (C.20) imply that $\lambda_m(\mathbf{v}^{\tau^t}, d) + \sum_{k=1}^{m-1} \mu_{mk}(\mathbf{v}^{\tau^t}, d) \to \mathbf{0}$. As in the base step, invoking dual feasibility, full connectedness, and Lemma C.1 then implies that $P_j(\tilde{\mathbf{w}}_m^{\tau^t}, m) \to 0$ and $\mu_{km}(\mathbf{v}^{\tau^t}, d) \to 0$ for all $m + 1 \leq k \leq d$. We conclude that

$$\mathsf{D}P(\tilde{\mathbf{w}}_m^{\tau^t}, m) \to \mathbf{0} \quad \text{and} \quad \boldsymbol{\mu}_{*,m}^{\tau^t} := \boldsymbol{\mu}_{*,m}(\mathbf{v}^{\tau^t}, d) \to \mathbf{0}. \tag{C.22}$$

This completes the induction. Combining (C.20), (C.21), and (C.22) yields the lemma. *Q.E.D.*

LEMMA C.21: *For every $k \in \mathbb{N} \cup \{0\}$, $\boldsymbol{\nu}^{(\tau^{(t)}+k)} \to \mathbf{0}$ and $\mathsf{D}P(\tilde{\mathbf{w}}_i^{(\tau^{(t)}+k)}, i) \to \mathbf{0}$ as $t \to \infty$ for all $i \in S$ almost surely, where $\tilde{\mathbf{w}}_i^{(\tau^{(t)}+k)} := \xi^c(\mathbf{v}^{(\tau^{(t)}+k)}, s^{(\tau^{(t)}+k)}, i)$.*

PROOF: Again, it suffices to show convergence on $\mathcal{J}^*$. So, fix a path $h = \big(s^{t+1}\big)_{t=0}^{\infty} \in \mathcal{J}^*$ and let $\tau^t := \tau^{(t)}(h)$ and $\mathbf{v}^t := \mathbf{v}^{(t)}(h)$. We proceed by induction on k, with Lemma C.20 serving as the base $(k = 0)$ step.

For the inductive step, let $k \in \mathbb{N}$ be given. Suppose we have shown that, for every $0 \leq m < k$, (i) $\lim_{t\to\infty} \boldsymbol{\nu}^{\tau^t+m} = \mathbf{0}$ and (ii) $\lim_{t\to\infty} \mathsf{D}P(\tilde{\mathbf{w}}_i^{\tau^t+m}, i) = \mathbf{0}$ for all $i \in S$. Since $\mathbf{v}^{\tau^t+k} = \xi^c(\mathbf{v}^{\tau^t+k-1}, s^{\tau^t+k-1}, s^{\tau^t+k})$ by construction, we obtain that $\mathbf{v}^{\tau^t+k} = \tilde{\mathbf{w}}^{\tau^t+k-1}_{s^{\tau^t+k}}$ from the definition of $(\tilde{\mathbf{w}}_i^{\tau^t+k-1})_{i\in S}$. Thus, since S is a finite set, part (ii) of the inductive hypothesis yields $\lim_{t\to\infty} \mathsf{D}P(\mathbf{v}^{\tau^t+k}, s^{\tau^t+k}) = \mathbf{0}$. The envelope condition (Env$_i$) then yields

$$\boldsymbol{\lambda}^{\tau^t+k} := \boldsymbol{\lambda}(\mathbf{v}^{\tau^t+k}, s^{\tau^t+k}) \to \mathbf{0} \quad \text{as } t \to \infty. \tag{C.23}$$

We can then show that

$$\mathsf{D}P(\tilde{\mathbf{w}}_i^{\tau^t+k}, i) \to \mathbf{0} \quad \text{and} \quad \boldsymbol{\mu}_{*,i}^{\tau^t+k} := \boldsymbol{\mu}_{*,i}(\mathbf{v}^{\tau^t+k}, s^{\tau^t+k}) \to \mathbf{0} \quad \forall i \in S \quad \text{as} \quad t \to \infty \tag{C.24}$$

by replicating the "induction through the type space" argument from the proof of Lemma C.20, with two minor modifications: (a) replace the date τ^t with the new date $\tau^t + k$ everywhere the former appears, and (b) replace the state $(\mathbf{v}^{\tau^t}, s^{\tau^t} = d)$ with the new state $(\mathbf{v}^{\tau^t+k}, s^{\tau^t+k})$ wherever the former appears. The details are straightforward, and thus omitted.

Combining (C.23) and (C.24) completes the induction on $k \in \mathbb{N}$, and the proof. *Q.E.D.*

LEMMA C.22: *The vector of multipliers $\boldsymbol{\nu}^{(t)} \to \mathbf{0}$ in probability.*

PROOF: Define the stochastic processes $\left(\delta^{(t)}\right)_{t=0}^{\infty}$ and $\left(L^{(t)}\right)_{t=0}^{\infty}$ by $\delta^{(t)} := \|\boldsymbol{\nu}^{(t)}\|$, where $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^{d(d+1)/2}$, and $L^{(t)} := t - \tau^{(t)}$. Thus, along path $h \in \mathscr{H}$, $\delta^{(t)}(h) \geq 0$ denotes the distance of $\boldsymbol{\nu}^{(t)}(h)$ from the zero vector and $L^{(t)}(h) \in \mathbb{N} \cup \{0\}$ denotes the lag since the last time that $s = d$.

To show convergence in probability, we must show that $\lim_{t\to\infty} \mathbf{P}(\delta^{(t)} > \varepsilon) = 0$ for all $\varepsilon > 0$. So, let $\varepsilon > 0$ be given. For every $t, k \in \mathbb{N}$, define the events $A_{\varepsilon,t} := \left\{h \in \mathscr{H} : \delta^{(t)}(h) > \varepsilon\right\}$, $B_{k,t} := \left\{h \in \mathscr{H} : L^{(t)}(h) > k\right\}$, and $C_{\varepsilon,k,t} := \bigcup_{T\geq t} \left[A_{\varepsilon,T} \cap B_{k,T}^c\right]$. For every $t, k \in \mathbb{N}$, we have $C_{\varepsilon,k,t+1} \subseteq C_{\varepsilon,k,t}$ and $A_{\varepsilon,t} \cap B_{k,t}^c \subseteq C_{\varepsilon,k,t}$. Consequently, we have:

$$\forall k, t \in \mathbb{N}, \quad \mathbf{P}(A_{\varepsilon,t}) = \mathbf{P}\left(A_{\varepsilon,t} \cap B_{k,t}^c\right) + \mathbf{P}(A_{\varepsilon,t} \cap B_{k,t}) \leq \mathbf{P}(C_{\varepsilon,k,t}) + \mathbf{P}(B_{k,t}). \tag{C.25}$$

To complete the proof, we must show that $\lim_{t\to\infty} \mathbf{P}(A_{\varepsilon,t}) = 0$. We do this in two steps, corresponding to the two terms on the RHS of (C.25).

Step 1: We claim that $\lim_{t\to\infty} \mathbf{P}(C_{\varepsilon,k,t}) = 0$ for every $k \in \mathbb{N}$. In particular, we have

$$\begin{aligned}
\lim_{t\to\infty} \mathbf{P}(C_{\varepsilon,k,t}) &= \mathbf{P}(\cap_{t\in\mathbb{N}} C_{\varepsilon,k,t}) \\
&= \mathbf{P}\left(\left\{h \in \mathscr{H} : \delta^{(t)}(h) > \varepsilon \text{ and } L^{(t)}(h) \leq k \text{ for infinitely many } t\right\}\right) = 0,
\end{aligned}$$

where the first equality is by continuity of probability (as the sequence $(C_{\varepsilon,k,t})_{t\in\mathbb{N}}$ is decreasing), the second equality is by definition, and the third equality is by the definition of $L^{(t)}$ and Lemma C.21. This proves the claim.

Step 2: Together, (C.25) and Step 1 deliver the following fact:

$$\forall k \in \mathbb{N}, \quad \limsup_{t\to\infty} \mathbf{P}(A_{\varepsilon,t}) \leq \limsup_{t\to\infty} \mathbf{P}(B_{k,t}). \tag{C.26}$$

Define the map $H : \mathbb{N} \to [0,1]$ by $H(k) := 1 - \limsup_{t\to\infty} \mathbf{P}(B_{k,t})$. Note that H is non-decreasing (since $B_{k+1,t} \subseteq B_{k,t}$ for every $k, t \in \mathbb{N}$). Consequently, if we can show that

$\lim_{k\to\infty} H(k) = 1$, we are done. In particular, since (C.26) holds for all k and only its RHS depends on k, it will then follow that $\limsup_{t\to\infty} \mathbf{P}(A_{\varepsilon,t}) \le \lim_{k\to\infty}(1 - H(k)) = 0$, and hence that $\lim_{t\to\infty} \mathbf{P}(A_{\varepsilon,t}) = 0$ (since probability is non-negative), as desired. Thus, the remainder of the proof is dedicated to showing that, in fact, $\lim_{k\to\infty} H(k) = 1$.

To this end, we appeal to facts about Markov chains. Under Assumption Markov, the type process $\left(s^{(t)}\right)$ is ergodic; in particular, there exists a unique stationary distribution $\boldsymbol{\pi} \in \Delta(S)$ such that $\lim_{t\to\infty} \mathbf{P}(s^{(t)} = i) = \pi_i > 0$ for all $i \in S$. Denote by $\left(r^{(t)}\right)$ the *time-reversed* type process, i.e., the S-valued Markov chain with transition probabilities $\mathbf{Q}\left(r^{(t+1)} = j \;\middle|\; r^{(t)} = i\right) = g_{ij} := \frac{\pi_j}{\pi_i} \cdot f_{ji}$ and induced measure over paths $\mathbf{Q} \in \Delta(S^\infty)$.[13] Denote by T_d^R the first hitting time of state d for the time-reversed chain, i.e., the random variable $T_d^R := \inf\left\{t \in \mathbb{N} \cup \{0\} : r^{(t)} = d\right\}$. For each $i \in S$, define the map $H_i : \mathbb{N} \to [0,1]$ as $H_i(k) := \mathbf{Q}\left(T_d^R \le k \;\middle|\; r^{(0)} = i\right)$. We claim that the following three properties hold:

(a) For every $i \in S$, $\lim_{k\to\infty} H_i(k) = 1$.
(b) For every $i \in S$ and $k \in \mathbb{N}$, $H_i(k) = 1 - \lim_{t\to\infty} \mathbf{P}\left(B_{k,t} \mid s^{(t)} = i\right)$.
(c) For every $k \in \mathbb{N}$, $H(k) = \sum_{i\in S} \pi_i H_i(k)$.

Together, (a)–(c) imply that $\lim_{k\to\infty} H(k) = 1$, so establishing (a)–(c) completes the proof.

For property (a), note that $\lim_{k\to\infty} H_i(k) = \mathbf{Q}\left(T_d^R < \infty \;\middle|\; r^{(0)} = i\right)$ for all $i \in S$. Furthermore, $\mathbf{Q}\left(T_d^R < \infty \;\middle|\; r^{(0)} = i\right) = 1$ for all $i \in S$ because the time-reversed chain is fully connected, and hence all states communicate with each other and are recurrent. This establishes property (a).

For property (b), let $i \in S$ and $k \in \mathbb{N}$ be given. Since $\lim_{t\to\infty} \Pr(s^{(t)} = i) = \pi_i > 0$, there exists $T \in \mathbb{N}$ such that $\Pr(s^{(t)} = i) > 0$ for all $t \ge T$. Then for $t \ge T$, we have

$$\mathbf{P}\left(B_{k,t} \mid s^{(t)} = i\right) = \frac{\mathbf{P}\left(s^{(t-m)} \ne d \;\; \forall\, m = 0, \ldots, k \text{ and } s^{(t)} = i\right)}{\mathbf{P}\left(s^{(t)} = i\right)}$$

by definition of $B_{k,t}$ and conditional probability. For $i = d$, we have $\mathbf{P}(B_{k,t} \mid s^{(t)} = i) = 0 = 1 - H_i(k)$ for all $t \ge T$, so we are done. For any $i \ne d$, we have

$$\begin{aligned}
&\mathbf{P}\left(s^{(t-m)} \ne d \;\; \forall\, m = 0, \ldots, k \text{ and } s^{(t)} = i\right) \\
&\quad= \sum_{(j_1,\ldots,j_k)\in\{1,\ldots,d-1\}^k} \mathbf{P}\left(s^{(t-k)} = j_k\right) \cdot f_{j_k,j_{k-1}} \cdots f_{j_2,j_1} \cdot f_{j_1,i} \\
&\quad= \sum_{(j_1,\ldots,j_k)\in\{1,\ldots,d-1\}^k} \pi_i \cdot \left(g_{i,j_1} \cdots g_{j_{k-1},j_k}\right) \cdot \frac{\mathbf{P}\left(s^{(t-k)} = j_k\right)}{\pi_{j_k}}
\end{aligned}$$

where the first equality uses the Markov property (of the forward chain) to sum over all paths from potential starting points $s^{(t-k)} \in S$ to the endpoint $s^{(t)} = i$, and the second equality is by definition of the time-reversed transition probabilities. Consequently,

$$\lim_{t\to\infty} \mathbf{P}\left(B_{k,t} \mid s^{(t)} = i\right) = \sum_{(j_1,\ldots,j_k)\in\{1,\ldots,d-1\}^k} \left(g_{i,j_1} \cdots g_{j_{k-1},j_k}\right) = \mathbf{Q}\left(T_d^R > k \mid r^{(0)} = i\right)$$

[13] The backward transition probabilities g_{ij} are defined from the forward transition probabilities f_{ij} via Bayes' Rule, with the stationary distribution of the forward chain, $\boldsymbol{\pi}$, serving as the "prior."

where the first equality is by the preceding two displays and ergodicity of $(s^{(t)})$, and the second equality is by definition of T_d^R. Since $H_i(k) = 1 - \mathbf{Q}\left(T_d^R > k \mid r^{(0)} = i\right)$ by construction, this establishes property (b).

Finally, for property (c), note that $\mathbf{P}(B_{k,t}) = \sum_{i \in S} \mathbf{P}(s^{(t)} = i)\, \mathbf{P}(B_{k,t} \mid s^{(t)} = i)$. Ergodicity of $(s^{(t)})$ and property (b) imply that $1 - H(k) = \lim_{t\to\infty} \mathbf{P}(B_{k,t}) = \sum_{i\in S} \lim_{t\to\infty} \mathbf{P}(s^{(t)} = i) \cdot \lim_{t\to\infty} \mathbf{P}(B_{k,t} \mid s^{(t)} = i) = \sum_{i \in S} \pi_i\,(1 - H_i(k)) = 1 - \sum_{i\in S} \pi_i H_i(k)$, as desired. *Q.E.D.*

Convergence of Allocations. Our final lemma shows that convergence of the multipliers implies convergence of the allocation. Recall that $u_i(\mathbf{v}, s) = \xi^f(\mathbf{v}, s, i)$ is the flow utility given to a type-i agent in state $(\mathbf{v}, s)$. To ease notation, for each $i \in S$, define the process $(u_i^{(t)})_{t=0}^{\infty}$ by $u_i^{(t)} := u_i(\mathbf{v}^{(t)}, s^{(t)})$.

LEMMA C.23: *Under the optimal contract, $u_i^{(t)} \to -\infty$ in probability for all $i \in S$.*

PROOF: Let $i \in S$ be given. The first-order condition (FOCu_i) at state $(\mathbf{v}^{(t)}, s^{(t)})$ is

$$f_{s^{(t)}i} C'\left(u_i^{(t)}, i\right) = \underbrace{\lambda_i^{(t)} + \sum_{k=1}^{i-1} \mu_{ik}^{(t)}}_{=:\, A_i^{(t)}} - \underbrace{\sum_{k=i+1}^{d} \psi'\left(u_i^{(t)}, k, i\right) \mu_{ki}^{(t)}}_{=:\, B_i^{(t)}}$$

Note that $B_i^{(t)} \geq 0$ because $\mu_{ki}^{(t)} \geq 0$ and $\psi'(\cdot, k, i) > 0$ for all $k, i \in S$. Since $C'(\cdot, i) > 0$, it follows that $0 < C'(u_i^{(t)}, i) \leq A_i^{(t)}/f_{s^{(t)}i}$. Lemma C.22 and full connectedness (Assumption Markov) imply that $A_i^{(t)}/f_{s^{(t)}i} \to 0$ in probability. Thus, we obtain $C'(u_i^{(t)}, i) \to 0$ in probability. Finally, because $C'(\cdot, i) : \mathcal{U} \to \mathbb{R}_{++}$ is a strictly increasing homeomorphism (by Assumption DARA), the Continuous Mapping Theorem (for convergence in probability) applied to the inverse of $C'(\cdot, i)$ delivers that $u_i^{(t)} \to -\infty$ in probability. *Q.E.D.*

Wrapping Up. We now complete the proof of Theorem 1.

PROOF OF THEOREM 1: We use Lemma C.23 to prove each part of the theorem in turn.

Part (a). Let $i \in S$ be given. Define $\mathbf{w}_i^{(t)} := \xi^c(\mathbf{v}^{(t)}, s^{(t)}, i)$. The promise keeping constraint (PK$_i$) requires that $v_i^{(t)} = u_i^{(t)} + \alpha\, \mathbf{E}^{\mathbf{f}_i}[\mathbf{w}_i^{(t)}]$. Because $\mathbf{w}_i^{(t)} \in D \subseteq \mathcal{U}^d$ (by definition of the domain) and $\mathcal{U} = (-\infty, 0)$ (by Assumption DARA), it follows that $v_i^{(t)} \leq u_i^{(t)}$. Lemma C.23 then implies that $v_i^{(t)} \to -\infty$ in probability, as desired.

Part (b). Immediate from Lemma C.23.

Part (c). As $u^{(t)} = U(c^{(t)} + \omega^{(t)})$ by definition and $U : (\underline{c}, \infty) \to \mathcal{U}$ is a strictly increasing homeomorphism (by Assumption DARA), the result follows from part (b) and the Continuous Mapping Theorem (for convergence in probability) applied to $U^{-1}(\cdot)$.

Nonexistence of limiting and stationary distributions. The nonexistence of limiting distributions follows directly from parts (a)–(c), proved above. For instance, for any compact $K \subseteq D$, part (a) implies that $\lim_{t\to\infty} \mathbf{P}(\mathbf{v}^{(t)} \in K) = 0$, from which it follows that $\mathbb{P}(\cdot) := \lim_{t\to\infty} \mathbf{P}(\mathbf{v}^{(t)} \in \cdot)$ cannot define a (Borel) probability measure on D.[14] The arguments for the $u^{(t)}$ and $c^{(t)} + \omega^{(t)}$ processes are analogous. We show by contradiction

[14]Note that D is σ-compact, i.e., can be written as the countable union of compact subsets $K \subseteq D$.

that there does not exist a stationary distribution for the (time-homogeneous, Markovian) state process $(\mathbf{v}^{(t)}, s^{(t)})$. Let $\Psi : D \times S \to \Delta(D \times S)$ denote the Markov kernel induced by P and the optimal contract, i.e., for all $t \in \mathbb{N}$, $(\mathbf{v}, s) \in D \times S$, and Borel $A \subseteq D \times S$, $\Psi((\mathbf{v}, s), A) := \mathsf{P}\left((\mathbf{v}^{(t+1)}, s^{(t+1)}) \in A \mid (\mathbf{v}^{(t)}, s^{(t)}) = (\mathbf{v}, s)\right)$. By induction, for every $k, t \in \mathbb{N}$, the k-step transition probabilities are given by

$$\begin{aligned}\Psi^k((\mathbf{v}, s), A) &:= \int_{D\times S} \Psi^{k-1}((\mathbf{v}', s'), A)\Psi((\mathbf{v}, s), \mathrm{d}(\mathbf{v}', s')) \\ &= \mathsf{P}\left((\mathbf{v}^{(t+k)}, s^{(t+k)}) \in A \mid (\mathbf{v}^{(t)}, s^{(t)}) = (\mathbf{v}, s)\right).\end{aligned}$$

Suppose there exists a stationary distribution $\nu \in \Delta(D \times S)$, i.e., for all Borel $A \subseteq D \times S$, $\nu(A) = \int_{D\times S} \Psi((\mathbf{v}, s), A)\, \mathrm{d}\nu(\mathbf{v}, s)$. Induction and Fubini's Theorem imply that, for every $k \in \mathbb{N}$, $\nu(A) = \int_{D\times S} \Psi^k((\mathbf{v}, s), A)\, \mathrm{d}\nu(\mathbf{v}, s)$. For every compact $A \subseteq D \times S$, part (a) and Dominated Convergence imply that $\nu(A) = \int_{D\times S} \lim_{k\to\infty} \Psi^k((\mathbf{v}, s), A)\, \mathrm{d}\nu(\mathbf{v}, s) = 0$. But this contradicts $\nu \in \Delta(D \times S)$, as desired. *Q.E.D.*

APPENDIX D: PROOF OF COROLLARY 4.1

Herein, we assume that the environment is (TVC)-Regular (as in the statement of Theorem 1) and continue to use the notation developed in SA–C (especially SA–C.4) above. To prove Corollary 4.1, it suffices to show that the vector of multipliers $\boldsymbol{\nu}^{(t)} \to \mathbf{0}$ almost surely; once we have shown this, a simple adaptation of the arguments from the proof of Lemma C.23 and the final step in the proof of Theorem 1 (with "almost surely" replacing "in probability" everywhere the latter appears) implies that promised utility, flow utility, and consumption all converge almost surely. This adaptation is straightforward, so we omit the details.

Thus, we focus on showing that $\boldsymbol{\nu}^{(t)} \to \mathbf{0}$ almost surely. To this end, note that plugging the expression for the directional derivative $\mathsf{D}_1 P(\mathbf{w}_i(\mathbf{v}, s), i)$ in (C.2) (from SA-C.2) into (FOC$\mathbf{w}_{ij}$) implies that, for every $(\mathbf{v}, s) \in D \times S$ and $i \in S$, we have:

$$f_{si}\left(\frac{P_j(\mathbf{w}_i(\mathbf{v}, s), i)}{f_{ij}} - \mathsf{D}_1 P(\mathbf{w}_i(\mathbf{v}, s), i)\right) = \sum_{k=i+1}^{d}\left(1 - \frac{f_{kj}}{f_{ij}}\right)\mu_{ki}(\mathbf{v}, s). \tag{D.1}$$

We now use (D.1) to prove each part of the corollary in turn.

PROOF OF PART (I): By hypothesis, there exists $\boldsymbol{\pi} \in \Delta(S)$ such that $f_{ij} = \pi_j$ for all $i, j \in S$.[15] Plugging this into (D.1) and invoking full connectedness (Assumption Markov) yields

$$\frac{P_j(\mathbf{w}_i(\mathbf{v}, s), i)}{f_{ij}} = \mathsf{D}_1 P(\mathbf{w}_i(\mathbf{v}, s), i) \tag{D.2}$$

for all $(\mathbf{v}, s) \in D \times S$ and $i, j \in S$.[16] By construction, $\mathbf{v}^{(t+1)} = \mathbf{w}_{s^{(t+1)}}(\mathbf{v}^{(t)}, s^{(t)})$ for all $t \in \mathbb{N}$. Combined with (D.2) and full connectedness (Assumption Markov), this implies that the martingale $\mathsf{D}_1 P(\mathbf{v}^{(t)}, s^{(t)}) \to 0$ almost surely if and only if the derivative $\mathsf{D}P(\mathbf{v}^{(t)}, s^{(t)}) \to \mathbf{0}$

[15]This notation is consistent with that from the proof of Lemma C.22: for i.i.d. Markov chains, the transition probability $\boldsymbol{\pi} \in \Delta(S)$ equals the chain's stationary distribution.

[16]Consistent with Lemma C.7, (D.2) states that $\mathsf{D}P(\mathbf{w}_i(\mathbf{v}, s), i) \in \tilde{E}_i$ as defined in ($\tilde{\mathrm{E}}_i$). That is, in the i.i.d. case, the optimal contract solves the efficiency problem (Eff$_i$) at each step.

almost surely. Thus, Lemma C.19 implies that the derivative $\mathsf{D}P(\mathbf{v}^{(t)}, s^{(t)}) \to \mathbf{0}$ almost surely. Replicating the proof of Lemma C.20 (with τ^t and $(\mathbf{v}^{\tau^t}, s^{\tau^t} = d)$ replaced everywhere by t and $(\mathbf{v}^t, s^t)$, respectively) then establishes that $\boldsymbol{\nu}^{(t)} \to \mathbf{0}$ almost surely. *Q.E.D.*

For part (ii), wherein $d = 2$, we consider the complementary cases of FOSD types ($f_{11} \geq f_{21}, f_{22} \geq f_{12}$) and non-FOSD types ($f_{11} < f_{21}, f_{22} < f_{12}$) separately.[17]

PROOF OF PART (II), FOSD CASE: We begin with two preliminary facts ((D.3) and (D.4) below). First, because $d = 2$, (D.1) for $i = 1$ reduces to

$$f_{s1}\left(\frac{P_j(\mathbf{w}_1(\mathbf{v}, s), 1)}{f_{1j}} - \mathsf{D}_1 P(\mathbf{w}_1(\mathbf{v}, s), 1)\right) = \left(1 - \frac{f_{2j}}{f_{1j}}\right)\mu_{21}(\mathbf{v}, s).$$

Because $f_{11} \geq f_{21}, f_{22} \geq f_{12}$ and $\mu_{21}(\mathbf{v}, s) \geq 0$, the above display and full connectedness (Assumption Markov) imply that

$$\frac{P_1(\mathbf{w}_1(\mathbf{v}, s), 1)}{f_{11}} \geq \mathsf{D}_1 P(\mathbf{w}_1(\mathbf{v}, s), 1) \geq \frac{P_2(\mathbf{w}_1(\mathbf{v}, s), 1)}{f_{12}}. \tag{D.3}$$

Second, because $d = 2$, the first-order condition (FOC$\mathbf{w}_{ij}$) for $i = j = 1$ reduces to

$$f_{s1} P_1(\mathbf{w}_1(\mathbf{v}, s), 1) = f_{11}\lambda_1(\mathbf{v}, s) - f_{21}\mu_{21}(\mathbf{v}, s).$$

Because $\lambda_1(\mathbf{v}, s) = P_1(\mathbf{v}, s)$ by (Env$_i$) and $\mu_{21}(\mathbf{v}, s) \geq 0$, the above display implies that

$$P_1(\mathbf{v}, s) \geq \frac{f_{s1}}{f_{11}} \cdot P_1(\mathbf{w}_1(\mathbf{v}, s), 1). \tag{D.4}$$

We now turn to the main proof. As in the proof of Lemma C.19, fix a path $h = (s^{t+1})_{t=0}^{\infty} \in \mathcal{J}^*$ and let $\tau^t := \tau^{(t)}(h)$ and $\mathbf{v}^t := \mathbf{v}^{(t)}(h)$. Denote by $\{t_k\}_{k\in\mathbb{N}}$ the range of the sequence $(\tau^{t+1})_{t=0}^{\infty}$, enumerated so that $t_k < t_{k+1}$ for all $k \in \mathbb{N}$. The proof of Lemma C.19 delivers that $\lim_{k\to\infty} \mathsf{D}_1 P(\mathbf{v}^{t_k}, s^{t_k} = 2) = 0$. Since $\mathbf{v}^{t_k} = \mathbf{w}_2(\mathbf{v}^{t_k - 1}, s^{t_k - 1})$ for all k, Lemma C.8 and ($\tilde{\mathrm{E}}_i$) then imply that the derivative satisfies $\lim_{k\to\infty} \mathsf{D}P(\mathbf{v}^{t_k}, s^{t_k} = 2) = \mathbf{0}$.

We claim that, in fact, the derivative converges along the full sequence, namely, it holds that $\lim_{t\to\infty} \mathsf{D}P(\mathbf{v}^t, s^t) = \mathbf{0}$. To this end, for every $k \in \mathbb{N}$, define $g_k := t_{k+1} - t_k$ (which is finite by definition of $\mathcal{J}^*$). By construction: (a) for every $t \in \mathbb{N}\backslash\{t_k\}_{k\in\mathbb{N}}$, there exists some $k \in \mathbb{N}$ and $1 \leq m < g_k$ such that $t = t_k + m$, and (b) for every k, we have $s^{t_k} = 2$ and $s^{t_k + m} = 1$ for all $1 \leq m < g_k$. With these facts in hand, we establish the claim in three steps.

Step 1: We first use (D.4) to show that, for every $k \in \mathbb{N}$, it holds that

$$P_1(\mathbf{v}^{t_k}, s^{t_k} = 2) \geq \frac{f_{21}}{f_{11}} \cdot \max_{1 \leq m < g_k} P_1(\mathbf{v}^{t_k + m}, s^{t_k + m} = 1). \tag{D.5}$$

To this end, consider any date t_k such that $g_k \geq 2$. (If $g_k = 1$, there is nothing to prove.) By definition of t_k and g_k, we have $\mathbf{v}^{t_k + m} = \mathbf{w}_1(\mathbf{v}^{t_k + m - 1}, s^{t_k + m - 1})$ for all $1 \leq m < g_k$. Thus, for $m = 1$, (D.4) yields $P_1(\mathbf{v}^{t_k}, s^{t_k} = 2) \geq \frac{f_{21}}{f_{11}} \cdot P_1(\mathbf{v}^{t_k + 1}, s^{t_k + 1} = 1)$. If $g_k = 2$, this immediately yields (D.5). If $g_k > 2$, then for every $2 \leq m < g_k$, (D.4) yields $P_1(\mathbf{v}^{t_k + m - 1}, s^{t_k + m - 1} = 1) \geq P_1(\mathbf{v}^{t_k + m}, s^{t_k + m} = 1)$. Stringing these inequalities together yields (D.5).

[17] When $d = 2$, these two cases are exhaustive because the identity $f_{11} + f_{12} = 1 = f_{21} + f_{22}$ implies that $f_{11} - f_{21} = f_{22} - f_{12}$.

Step 2: Next, we use (D.3) and (D.5) to show that, for every $k \in \mathbb{N}$, it holds that

$$P_2(\mathbf{v}^{t_k}, s^{t_k} = 2) \geq \frac{f_{22}}{f_{12}} \cdot \max_{1 \leq m < g_k} P_2(\mathbf{v}^{t_k+m}, s^{t_k+m} = 1). \tag{D.6}$$

To this end, note that because $\mathbf{v}^{t_k} = \mathbf{w}_2(\mathbf{v}^{t_k-1}, s^{t_k-1})$ by definition of t_k, Lemma C.8 and ($\tilde{\mathrm{E}}_i$) imply that $P_2(\mathbf{v}^{t_k}, s^{t_k} = 2) = \frac{f_{22}}{f_{21}} P_1(\mathbf{v}^{t_k}, s^{t_k} = 2)$. Plugging this in to (D.5) yields

$$\begin{aligned} P_2(\mathbf{v}^{t_k}, s^{t_k} = 2) &\geq \frac{f_{22}}{f_{21}} \cdot \frac{f_{21}}{f_{11}} \cdot \max_{1 \leq m < g_k} P_1(\mathbf{v}^{t_k+m}, s^{t_k+m} = 1) \\ &\geq \frac{f_{22}}{f_{21}} \cdot \frac{f_{21}}{f_{11}} \cdot \frac{f_{11}}{f_{12}} \max_{1 \leq m < g_k} P_2(\mathbf{v}^{t_k+m}, s^{t_k+m} = 1), \end{aligned}$$

where the second inequality follows from (D.3) applied to each term on the RHS of the first line. Canceling terms yields (D.6).

Step 3: We now prove the claim, i.e., $\lim_{t\to\infty} \mathsf{D}P(\mathbf{v}^t, s^t) = \mathbf{0}$. Recall that (as established above) the derivative satisfies $\lim_{k\to\infty} \mathsf{D}P(\mathbf{v}^{t_k}, s^{t_k} = 2) = \mathbf{0}$. When combined with (D.5) and (D.6), this implies that the partial derivatives satisfy $\limsup_{t\to\infty} P_i(\mathbf{v}^t, s^t) \leq 0$ for $i = 1, 2$. But since the directional derivative $\mathsf{D}_1 P(\mathbf{v}^t, s^t) \geq 0$ for all t by Lemma C.1, it follows that $\lim_{t\to\infty} P_i(\mathbf{v}^t, s^t) = 0$ for all $i \in S$. That is, $\lim_{t\to\infty} \mathsf{D}P(\mathbf{v}^t, s^t) = \mathbf{0}$, as claimed.

We now complete the proof. Since $\lim_{t\to\infty} \mathsf{D}P(\mathbf{v}^t, s^t) = \mathbf{0}$, replicating the proof of Lemma C.20 (with τ^t and $(\mathbf{v}^{\tau^t}, s^{\tau^t} = d)$ replaced everywhere by t and $(\mathbf{v}^t, s^t)$, respectively) delivers $\boldsymbol{\nu}(\mathbf{v}^t, s^t) \to \mathbf{0}$. Since $\mathsf{P}(\mathcal{J}^*) = 1$, we have $\boldsymbol{\nu}^{(t)} \to \mathbf{0}$ almost surely, as desired. *Q.E.D.*

Proof of Part (ii), non-FOSD case: The argument is analogous to the FOSD case above. We begin by deriving analogues of (D.3) and (D.4) ((D.7) and (D.8) below). First, because we now have $f_{11} < f_{21}$, $f_{22} < f_{12}$, the inequalities in (D.3) flip, delivering

$$\frac{P_1(\mathbf{w}_1(\mathbf{v}, s), 1)}{f_{11}} \leq \mathsf{D}_1 P(\mathbf{w}_1(\mathbf{v}, s), 1) \leq \frac{P_2(\mathbf{w}_1(\mathbf{v}, s), 1)}{f_{12}}. \tag{D.7}$$

Second, because $d = 2$, the FOC (FOC$\mathbf{w}_{ij}$) for $i = 1$ and $j = 2$ reduces to

$$f_{s1} P_2(\mathbf{w}_1(\mathbf{v}, s), 1) = f_{12}\lambda_1(\mathbf{v}, s) - f_{22}\mu_{21}(\mathbf{v}, s).$$

Because $\lambda_1(\mathbf{v}, s) = P_1(\mathbf{v}, s)$ by (Env$_i$) and $\mu_{21}(\mathbf{v}, s) \geq 0$, this FOC and (D.7) together imply

$$P_1(\mathbf{v}, s) \geq \frac{f_{s1}}{f_{12}} P_2(\mathbf{w}_1(\mathbf{v}, s), 1) \geq \frac{f_{s1}}{f_{11}} P_1(\mathbf{w}_1(\mathbf{v}, s), 1). \tag{D.8}$$

We now turn to the main proof. Let h, τ^t, $\mathbf{v}^t$, t_k, and g_k be as defined above in the proof for the FOSD case. We obtain (D.5) using the same argument as in Step 1 from the FOSD case, except with (D.8) replacing (D.4) everywhere the latter appears. To obtain (D.6), we modify Step 2 from the FOSD case as follows:

Step 2′: For every $k \in \mathbb{N}$, Lemma C.8 and ($\tilde{\mathrm{E}}_i$) imply that $P_2(\mathbf{v}^{t_k}, s^{t_k} = 2) = \frac{f_{22}}{f_{21}} P_1(\mathbf{v}^{t_k}, s^{t_k} = 2)$. For every k such that $g_k \geq 2$ (there is nothing to prove if $g_k = 1$), plugging this into (the first inequality in) (D.8) then delivers

$$P_2(\mathbf{v}^{t_k}, s^{t_k} = 2) \geq \frac{f_{22}}{f_{12}} P_2(\mathbf{v}^{t_k+1}, s^{t_k+1} = 1).$$

If $g_k = 2$, this immediately yields (D.6). If $g_k > 2$, then for every $2 \leq m < g_k$, we have

$$P_2(\mathbf{v}^{t_k+m-1}, s^{t_k+m-1} = 1) \geq \frac{f_{12}}{f_{11}} P_1(\mathbf{v}^{t_k+m-1}, s^{t_k+m-1} = 1)$$
$$\geq \frac{f_{12}}{f_{11}} \cdot \frac{f_{11}}{f_{12}} P_2(\mathbf{v}^{t_k+m}, s^{t_k+m} = 1) = P_2(\mathbf{v}^{t_k+m}, s^{t_k+m} = 1),$$

where the first inequality is by (D.7), the second inequality is by (the first inequality in) (D.8), and the final equality is by canceling terms. Stringing the inequalities in the above two displays together yields (D.6).

Step 3 and the final paragraph from the proof of the FOSD case then carry over verbatim, completing the present proof. *Q.E.D.*

APPENDIX E: PROOF OF THEOREM 2

We first prove Theorem 3(d) (from Appendix B.1), which we then use to prove Theorem 2.

E.1. *Proof of Theorem 3(d)*

For every $i, k \in S$, denote by $F^*(k \mid i) := \sum_{\ell \geq k} f_{i\ell}$ the probability of transitioning from type i to some type weakly greater than k. For every $i \in S \backslash \{1\}$, define $\Delta F^*_{i,i-1}(k) := F^*(k \mid i) - F^*(k \mid i-1)$. By FOSD, $\Delta F^*_{i,i-1}(k) \geq 0$ for all $i \in S \backslash \{1\}$ and $k \in S$ (with equality for $k = 1$). For every $i \in S \backslash \{1\}$ and $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, a standard calculation delivers

$$\mathbf{E}^{\mathbf{f}_i}[\mathbf{x}] - \mathbf{E}^{\mathbf{f}_{i-1}}[\mathbf{x}] = \sum_{k=1}^{d} (f_{ik} - f_{i-1,k}) x_k = \sum_{k=1}^{d-1} \Delta F^*_{i,i-1}(k+1)[x_{k+1} - x_k]. \tag{E.1}$$

Now, recall the sets $D^* \subseteq D$ and $\Xi^*(\mathbf{v}) \subseteq \Xi$ defined in Appendix B.1. Let $\mathbf{v} \in D^*$ and a contract $\xi \in \Xi^*(\mathbf{v})$ be given. Let $(\mathbf{v}^{(t)})_{t=0}^{\infty}$ be the induced promise processes (initialized at $\mathbf{v}^{(0)} := \mathbf{v}$) and let $(u_i^{(t)}, \mathbf{w}_i^{(t)}) := \xi(\mathbf{v}^{(t)}, s^{(t)}, i)$ for each $i \in S$.[18] For every $t \in \mathbb{N} \cup \{0\}$ and $i \in S \backslash \{1\}$, the (IC^*_{ij}) constraint (for $j = i-1$) yields

$$v_i^{(t)} - v_{i-1}^{(t)} \geq \psi(u_{i-1}^{(t)}, i, i-1) - u_{i-1}^{(t)} + \alpha \left[\mathbf{E}^{\mathbf{f}_i}[\mathbf{w}_{i-1}^{(t)}] - \mathbf{E}^{\mathbf{f}_{i-1}}[\mathbf{w}_{i-1}^{(t)}] \right]$$
$$= Z_i(u_{i-1}^{(t)}) + \alpha \sum_{k=1}^{d-1} \Delta F^*_{i,i-1}(k+1)[w_{i-1,k+1}^{(t)} - w_{i-1,k}^{(t)}],$$

where in the second line we let $Z_i(u) := \psi(u, i, i-1) - u$ for all $u \in \mathcal{U}$ and invoke (E.1) for $\mathbf{x} = \mathbf{w}_{i-1}^{(t)}$. For every $i \in S \backslash \{1\}$ and $t \in \mathbb{N}$, let $\hat{H}_i^t := \{h^t = (s^1, \ldots, s^t) \in S^t : s^1 = i-1,\ s^\tau \neq d\ \forall \tau = 1, \ldots, t\}$. Then, for each $i \in S \backslash \{1\}$, iterating the above display forward T times starting from $t = 0$ yields

$$v_i - v_{i-1} \geq Z_i(u_{i-1}^{(0)}) + \sum_{t=1}^{T} \alpha^t \sum_{h^{t+1} \in \hat{H}_i^{t+1}} \left(\prod_{\tau=1}^{T} \Delta F^*_{s^\tau+1, s^\tau}(s^{\tau+1}+1) \right) Z_{s^{t+1}+1}(u_{s^{t+1}}^{(t)}(h^t))$$

[18]Two remarks on notation: (i) we drop the ξ subscript on the induced promises for simplicity, and (ii) unlike in SA–C above, ξ need not be optimal and the policies $(u_i^{(t)}, \mathbf{w}_i^{(t)})$ need not satisfy the FOCs.

$$+\,\alpha^{T+1}\sum_{h^{T+2}\in\hat{H}_i^{T+2}}\left(\prod_{\tau=1}^{T+1}\Delta F^*_{s^\tau+1,s^\tau}(s^{\tau+1}+1)\right)\left[v^{(T+1)}_{s^{T+2}+1}(h^{T+1})-v^{(T+1)}_{s^{T+2}}(h^{T+1})\right],$$

where at each step of the iteration we use the identity $\mathbf{v}^{(t+1)}((h^t,j))=\mathbf{w}_j^{(t)}(h^t)$ for all $h^t\in S^t$ and $j\in S$. Note that the first line of the RHS is bounded below by some $M_i>0$ because $Z_k(\cdot)>0$ for all $k\in S\backslash\{1\}$, and FOSD implies that each $\Delta F^*_{s^\tau+1,s^\tau}(s^{\tau+1}+1)\geq 0$. To bound below the second line of the RHS, let $\varepsilon>0$ be given and note that, since $\xi\in\Xi^*(\mathbf{v})$ and hence $\lim_{t\to\infty}\inf_{h\in\mathscr{H}}\alpha^t\mathbf{v}^{(t)}(h)=\mathbf{0}$ (i.e., (TVC) holds), there exists $T_\varepsilon\in\mathbb{N}$ such that, for all $T\geq T_\varepsilon$ and paths $h\in H$, we have $\alpha^T\mathbf{v}^{(T)}(h)\geq-\varepsilon\mathbf{1}$ and hence $\alpha^T\left(v_{i+1}^{(T)}(h)-v_i^{(T)}(h)\right)\geq-\varepsilon$ for all $i\in S\backslash\{d\}$. Plugging these bounds into the above display and using FOSD on the second line, we conclude that, for all $T\geq T_\varepsilon$,

$$v_i-v_{i-1}\geq M_i-\varepsilon\cdot\sum_{h^{T+2}\in\hat{H}_i^{T+2}}\left(\prod_{\tau=1}^{T+1}\Delta F^*_{s^\tau+1,s^\tau}(s^{\tau+1}+1)\right). \tag{E.2}$$

By repeated use of (E.1) and the Law of Iterated Expectations, backward induction yields

$$\sum_{h^{T+2}\in\hat{H}_i^{T+2}}\left(\prod_{\tau=1}^{T+1}\Delta F^*_{s^\tau+1,s^\tau}(s^{\tau+1}+1)\right)=\mathbf{E}[s^{(T+2)}\mid s^{(1)}=i]-\mathbf{E}[s^{(T+2)}\mid s^{(1)}=i-1],$$

which vanishes as $T\to\infty$ because the type process is ergodic. Thus, sending $T\to\infty$ in (E.2) delivers $v_i-v_{i-1}\geq M_i>0$. We conclude that $\mathbf{v}\in V_d$. Hence, $D^*\subseteq V_d$, as desired.

E.2. *Main Proof of Theorem 2*

Herein, we maintain the hypotheses that the environment is (TVC)-Regular and the type process is FOSD, and continue to use the notation developed in SA–C above.

PROOF OF PART (A): Let $i\in S\backslash\{1\}$ be given. Since the environment is (TVC)-Regular, Proposition 3.2(b) implies that the unique optimal contract is (TVC)-implementable. Thus, the continuation utility process $\mathbf{w}_{i-1}^{(t)}:=\xi^c(\mathbf{v}^{(t)},s^{(t)},i-1)$ satisfies $\mathbf{w}_{i-1}^{(t)}\in D^*$ pathwise. Since the type process is FOSD, Theorem 3(d) then implies that $\mathbf{w}_{i-1}^{(t)}\in V_d$, i.e., the map $k\mapsto w_{i-1,k}^{(t)}$ is increasing. Thus, FOSD implies that $\mathbf{E}^{\mathbf{f}_i}\left[\mathbf{w}_{i-1}^{(t)}\right]-\mathbf{E}^{\mathbf{f}_{i-1}}\left[\mathbf{w}_{i-1}^{(t)}\right]\geq 0$ pathwise. Plugging this inequality into the IC constraint (IC^*_{ij}) (for $j=i-1$) from Section 4.2 yields

$$\begin{aligned}v_i^{(t)}-v_{i-1}^{(t)}&\geq\psi(u_{i-1}^{(t)},i,i-1)-u_{i-1}^{(t)}+\alpha\left(\mathbf{E}^{\mathbf{f}_i}\left[\mathbf{w}_{i-1}^{(t)}\right]-\mathbf{E}^{\mathbf{f}_{i-1}}\left[\mathbf{w}_{i-1}^{(t)}\right]\right)\\&\geq\psi(u_{i-1}^{(t)},i,i-1)-u_{i-1}^{(t)}.\end{aligned}\tag{E.3}$$

Define $Z_i(u):=\psi(u,i,i-1)-u$. Assumption DARA implies that the map $Z_i:\mathscr{U}\to(0,\infty)$ is strictly decreasing, convex, and continuously differentiable.[19] Let $a_i:=Z_i'(-1)<0$ and

[19]The Inverse Function Theorem delivers $Z_i'(u)=\frac{U'(\omega_i-\omega_{i-1}+U^{-1}(u))}{U'(U^{-1}(u))}-1$. Since $\omega_i>\omega_{i-1}$ and U is strictly concave, $Z_i'(u)<0$. Part (c) of Assumption DARA implies that the map $u\mapsto-\log\left(Z_i'(u)+1\right)$ is weakly decreasing. Thus, $Z_i'(\cdot)$ is non-decreasing, i.e., $Z_i(\cdot)$ is convex.

$b_i := Z_i(-1) + a_i \in \mathbb{R}$. By convexity, $Z_i(u) \geq Z_i(-1) + (u+1) \cdot Z_i'(-1) = a_i u + b_i$ for all $u \in \mathcal{U}$. Plugging this inequality into (E.3) delivers

$$v_i^{(t)} - v_{i-1}^{(t)} \geq Z_i(u_{i-1}^{(t)}) \geq a_i \cdot u_i^{(t)} + b_i \to +\infty \text{ in probability,}$$

where the limit is by $a_i < 0$ and Lemma C.23. This completes the proof. *Q.E.D.*

PROOF OF PART (B): Let $\overline{f}_d := \max_{s\in S} f_{sd}$ and $\underline{f}_d := \min_{s\in S} f_{sd}$. By the same argument as in the proof of part (a) above, the promised utility process satisfies $\mathbf{v}^{(t)} \in D^* \subseteq V_d$, and thus

$$v_d^{(t)} \geq \overline{f}_d v_d^{(t)} + (1-\overline{f}_d) v_{d-1}^{(t)} \geq \sum_{i=1}^{d} f_{s^{(t)},i} v_i^{(t)}$$

along every path. Plugging the above into (4.1) from Section 4.2, we obtain

$$\begin{aligned} \mathsf{V}\left(v_{s^{(t+1)}}^{(t)} \mid \mathbf{v}^{(t)}, s^{(t)}\right) &= \sum_{i=1}^{d} f_{s^{(t)},i} \left(v_i^{(t)} - \sum_{i=1}^{d} f_{s^{(t)},i} v_i^{(t)}\right)^2 \\ &\geq \underline{f}_d \left(v_d^{(t)} - \sum_{i=1}^{d} f_{s^{(t)},i} v_i^{(t)}\right)^2 \geq \underline{f}_d (1-\overline{f}_d)^2 \cdot (v_d^{(t)} - v_{d-1}^{(t)})^2. \end{aligned}$$

Full connectedness (Assumption Markov) yields $\overline{f}_d, \underline{f}_d \in (0,1)$. Thus, part (a) implies $\mathsf{V}\left(v_{s^{(t+1)}}^{(t)} \mid \mathbf{v}^{(t)}, s^{(t)}\right) \to +\infty$ in probability, as desired. *Q.E.D.*

APPENDIX F: FIRST-BEST BENCHMARK

Herein, we characterize the first-best contract that is optimal under full information. We adopt a recursive formulation analogous to that for the second-best problem in Section 3.1, the only difference being that the incentive constraints (IC_{ij}) are no longer included.[20]

A *(recursive) full-information contract* is a map $\zeta : \mathcal{U}^d \times S \to (\mathcal{U} \times \mathcal{U}^d)^d$, and we say that ζ is *feasible* if $\zeta(\mathbf{v}, s) \in \Gamma^{\text{FB}}(\mathbf{v})$ for every $(\mathbf{v}, s) \in \mathcal{U}^d \times S$, where $\Gamma^{\text{FB}}(\mathbf{v}) := \{(u_i, \mathbf{w}_i)_{i\in S} \in (\mathcal{U} \times \mathcal{U}^d)^d : (\text{PK}_i) \text{ holds } \forall\, i \in S \text{ at } \mathbf{v} \in \mathcal{U}^d\}$.[21] Let Ξ^{FB} denote the set of feasible full-information contracts. The principal's *full-information recursive problem* is

$$Q^*(\mathbf{v}, s) := \inf_{\zeta\in\Xi^{\text{FB}}} \mathsf{E}\left[\sum_{t=0}^{\infty} \alpha^t C\left(u_\zeta^{(t)}, s^{(t+1)}\right) \,\middle|\, (\mathbf{v}_\zeta^{(0)}, s^{(0)}) = (\mathbf{v}, s)\right], \tag{FB}$$

where the processes of induced allocations $(u_\zeta^{(t)})_{t=0}^{\infty}$ and induced promises $(\mathbf{v}_\zeta^{(t)})_{t=0}^{\infty}$ are defined by iterating on ζ in the natural way (cf. the recursive problem (RP) in Section 3.1). A contract ζ^* is *first-best* if it attains the infimum in (FB) at every $(\mathbf{v}, s) \in \mathcal{U}^d \times S$. We represent the infimal cost in (FB) by the principal's *first-best value function* $Q^* : \mathcal{U}^d \times S \to \overline{\mathbb{R}}$.

[20]Under Condition R.2, this recursive formulation is equivalent to the full-information analogue of the sequential formulation in Appendix A because the first-best contract in Lemma F.1 below satisfies (TVC).

[21]In the full-information problem, every $\mathbf{v} \in \mathcal{U}^d$ is implementable (e.g., via the first-best contract in Lemma F.1(a) below). Thus, the (recursive) domain for the full-information problem is the entirety of $\mathcal{U}^d$.

LEMMA F.1: *Suppose that Condition R.2 holds. Then the first-best value function $Q^*: \mathcal{U}^d \times S \to \mathbb{R}$ is finite-valued and satisfies the functional equation*

$$Q^*(\mathbf{v}, s) = \min_{(u_i, \mathbf{w}_i)_{i \in S} \in \Gamma^{FB}(\mathbf{v})} \sum_{i \in S} f_{si} \left[C(u_i, i) + \alpha Q^*(\mathbf{w}_i, i) \right]. \tag{F.1}$$

Furthermore, $Q^*(\cdot, s)$ is strictly convex, strictly increasing in the direction $\mathbf{1}$, and continuously differentiable for each $s \in S$. Moreover:

(a) *There exists a unique first-best contract, which is given by $\zeta^*(\mathbf{v}, s) = ((1-\alpha)v_i, v_i \mathbf{1})_{i \in S}$ for every $(\mathbf{v}, s) \in \mathcal{U}^d \times S$.*

(b) *The full-information contract generated by the (unique) policy function from (F.1) is ζ^*.*

PROOF: For each $s \in S$, $Q^*(\cdot, s) < +\infty$ on $\mathcal{U}^d$ because the contract described in part (a) is feasible and has finite cost, and $Q^*(\cdot, s)$ is convex by (FB) because each $C(\cdot, i)$ is convex and $\Gamma^{\text{FB}}: \mathcal{U}^d \rightrightarrows (\mathcal{U} \times \mathcal{U}^d)^d$ has convex graph. Thus, $Q^*(\mathbf{v}, s) = -\infty$ for some $(\mathbf{v}, s) \in \mathcal{U}^d \times S$ only if $Q^*(\mathbf{v}', s) = -\infty$ for all $\mathbf{v}' \in \mathcal{U}^d$, which would violate Condition R.2. Hence, each $Q^*(\cdot, s)$ is finite-valued and convex. Standard arguments then imply that Q^* satisfies (F.1), the minimum in (F.1) is attained and, under Condition R.2, the policy functions of (F.1) generate first-best contracts, which therefore exist (cf. Lemmas J.7–J.10 in Section J of Bloedel, Krishna, and Leukhina (2025)). It is easy to see from (FB) that each $Q^*(\cdot, s)$ is non-decreasing in the direction $\mathbf{1}$. Moreover, the Benveniste and Scheinkman (1979) envelope theorem applied to (F.1) yields that each $Q^*(\cdot, s) \in \mathbf{C}^1(\mathcal{U}^d)$.[22] To complete the proof, it suffices to show that the contract described in part (a) is first-best; given this fact, the strict convexity and monotonicity of each $C(\cdot, i)$ imply via (F.1) that each $Q^*(\cdot, s)$ is strictly convex and strictly increasing in the direction $\mathbf{1}$, and strict convexity of each $Q^*(\cdot, s)$ then yields the uniqueness claims in parts (a) and (b).

Thus, we claim that the contract $\zeta^*(\mathbf{v}, s) := ((1-\alpha)v_i, v_i \mathbf{1})_{i \in S}$, which is feasible by construction, is first-best. To this end, let ζ be any first-best contract and let $(\mathbf{v}^{(0)}, s^{(0)}) \in \mathcal{U}^d \times S$ be given. Standard arguments imply that the induced allocation $u_\zeta^{(t)}$ under ζ is a constant process *conditional on the initial type* $s^{(1)}$, i.e., conditional on $s^{(1)} = i$, there exists some $z_i(\mathbf{v}^{(0)}, s^{(0)}) \in \mathcal{U}$ such that $u_\zeta^{(t)} = z_i(\mathbf{v}^{(0)}, s^{(0)})$ for all $t \geq 0$.[23] Iterating the (PK$_i$) constraints forward in time implies that, for each $i \in S$, we have

$$v_i^{(0)} = \lim_{T \to \infty} \left[\frac{1 - \alpha^{T+1}}{1-\alpha} z_i(\mathbf{v}^{(0)}, s^{(0)}) + \alpha^{T+1} \mathbf{E}\left[\mathbf{E}^{\mathbf{f}_{s^{(T+1)}}}[\mathbf{v}_\zeta^{(T+1)}] \mid s^{(1)} = i \right] \right] \leq \frac{z_i(\mathbf{v}^{(0)}, s^{(0)})}{1-\alpha}$$

where the inequality is by $\mathcal{U} = (-\infty, 0)$ (Assumption DARA). Then, since ζ^* is feasible and costs weakly more than ζ (which is first-best), it must be that $z_i(\mathbf{v}^{(0)}, s^{(0)}) = (1-\alpha)v_i^{(0)}$ for each $i \in S$, i.e., ζ^* and ζ induce the same allocations starting from $(\mathbf{v}^{(0)}, s^{(0)})$. Hence, ζ^* also attains the optimal value $Q^*(\mathbf{v}^{(0)}, s^{(0)})$ starting from $(\mathbf{v}^{(0)}, s^{(0)})$. Since the initial condition was arbitrary, we conclude that ζ^* is first-best, as claimed. *Q.E.D.*

[22]Formally, let $(\mathbf{v}, s) \in \mathcal{U}^d \times S$ be given. Since $Q^*(\cdot, s)$ is convex, it suffices to show that each partial derivative exists at $\mathbf{v}$. So, let $j \in S$ be given. Define $\mathbf{v}(t) := \mathbf{v} + t\hat{e}_j$ where $\hat{e}_j \in \mathbb{R}^d$ is the unit vector in the jth direction. Given a menu $(u_i, \mathbf{w}_i)_{i \in S} \in \Gamma^{\text{FB}}(\mathbf{v})$ that attains the minimum in (F.1) at $(\mathbf{v}, s)$, define $u_i(t) := u_i + t\mathbf{1}(i = j)$ and $\mathbf{w}_i(t) := \mathbf{w}_i$ for each $i \in S$. Since $\mathcal{U}$ is open, there is an $\varepsilon > 0$ such that $\mathbf{v}(t) \in \mathcal{U}^d$ and $u_j(t) \in \mathcal{U}$ for $|t| < \varepsilon$. By construction, $(u_i(t), \mathbf{w}_i(t))_{i \in S} \in \Gamma^{\text{FB}}(\mathbf{v}(t))$ for all such t. Then, since $C(\cdot, j) \in \mathbf{C}^1(\mathcal{U})$, Benveniste and Scheinkman (1979) implies that the partial derivative $Q_j^*(\mathbf{v}, s)$ exists.

[23]This can be seen either from (FB) directly or from the envelope and first-order conditions for (F.1) (which mirror those for the second-best problem in Appendix C.1, except with Q^* appearing in place of P and $\mu_{ij}(\mathbf{v}, s) := 0$ for all $i, j \in S$).

Lemma F.1(a) shows that the first-best contract is unique and fully insures the agent *conditional on his initial type*. Specifically, conditional on $\omega^{(0)} = \omega_i$ (i.e., $s^{(1)} = i$), the induced allocations $u^{(t)}_{\xi^*} \equiv (1-\alpha)v^{(0)}_i$ are constant for $t \geq 0$ and the induced promises $\mathbf{v}^{(t)}_{\xi^*} \equiv v^{(0)}_i \mathbf{1}$ are constant for $t \geq 1$. However, the first-best contract fully insures the agent against his initial type if and only if the initial $\mathbf{v}^{(0)}$ is on the diagonal (i.e., $v^{(0)}_1 = \cdots = v^{(0)}_d$).

Naturally, if the principal could choose the initial $\mathbf{v}^{(0)}$, she would choose it to lie on the diagonal. To model this choice, consider for each $i \in S$ the *first-best efficiency problem*:

$$K^*(v,i) := \min_{\mathbf{v}\in\mathcal{U}^d} Q^*(\mathbf{v},i) \qquad \text{s.t.} \qquad \mathsf{E}^{\mathbf{f}_i}[\mathbf{v}] \geq v. \tag{$\mathrm{Eff}_i^{\mathrm{FB}}$}$$

This is the full-information analogue of the efficiency problem (Eff_i) from Appendix C.3.1.

LEMMA F.2: *Suppose that Condition R.2 holds. Then, for each $i \in S$, the unique solution $\mathbf{v}^*(v,i) \in \mathcal{U}^d$ and corresponding value of ($\mathrm{Eff}_i^{\mathrm{FB}}$) at $v \in \mathcal{U}$ are given by*

$$\mathbf{v}^*(v,i) = v \cdot \mathbf{1} \quad \text{and} \quad K^*(v,i) = \frac{U^{-1}((1-\alpha)v)}{1-\alpha} - \mathsf{E}\left[\sum_{t=0}^{\infty} \alpha^t \omega^{(t)} \,\middle|\, s^{(0)} = i\right]. \tag{F.2}$$

Moreover, the value function $K^*(\cdot,i) : \mathcal{U} \to \mathbb{R}$ is strictly increasing, strictly convex, continuously differentiable, unbounded above (i.e., $\lim_{v\to 0} K^*(v,i) = +\infty$), and satisfies the Inada conditions $\lim_{v\to-\infty}(K^*)'(v,i) = 0$ and $\lim_{v\to 0}(K^*)'(v,i) = +\infty$.

PROOF: Existence of a solution to ($\mathrm{Eff}_i^{\mathrm{FB}}$) follows from routine arguments (cf. Lemma J.10 in Section J of Bloedel, Krishna, and Leukhina (2025)). Uniqueness of the solution, the expressions in (F.2), and the strict monotonicity, strict convexity, and continuous differentiability of $K^*(\cdot,i)$ then follow from Lemma F.1. The limiting properties of $K^*(\cdot,i)$ follow from (F.2) and the fact that $v \mapsto U^{-1}((1-\alpha)v)$ satisfies the same properties (by Assumption DARA). *Q.E.D.*

APPENDIX G: PATHWISE PROPERTIES OF MARKOV CHAINS

Herein, we collect facts about the paths of Markov chains, which we use in the proof of Theorem 1 (see SA-C.4). The first fact is standard; see Shiryaev (1995, p. 577) for a proof.

LEMMA G.1: *Let $(X^{(t)})$ be a time-homogeneous Markov chain with countable state space $\mathcal{X}$ and law $\mathbb{P} \in \Delta(\mathcal{X}^\infty)$ over paths. If state $x \in \mathcal{X}$ is recurrent, then $\mathbb{P}(X^{(t)} = x$ for infinitely many $t \mid X^{(0)} = x) = 1$.*

The next result applies Lemma G.1 to our setting.

LEMMA G.2: *For all $i, j \in S$, we have* $\mathsf{P}\left((s^{(t-1)}, s^{(t)}) = (i,j) \text{ for infinitely many } t\right) = 1$.

PROOF: Consider the time-homogeneous Markov chain $X^{(t)} := (s^{(t)}, s^{(t+1)})$ with finite state space $\mathcal{X} := S \times S$, and law $\mathbb{P}$ induced by the initial distribution of $s^{(0)}$ under P and the transition probabilities $Q : \mathcal{X} \to \Delta(\mathcal{X})$ generated by P via $Q((i,j),(k,\ell)) := \mathbf{1}(j = k) \cdot f_{k\ell}$. The two-step transition probabilities are then $Q^{(2)}\big((i,j),(k,\ell)\big) = f_{jk} f_{k\ell} > 0$, so the chain is indecomposable. Let $i, j \in S$ be given. It follows that $\tau(i,j) := \inf\{t \in \mathbb{N} : X^{(t)} = (i,j)\}$ is finite $\mathbb{P}$-a.s. Then, Lemma G.1 and the Strong Markov Property (e.g., Norris (1997, Theorem 1.2, p. 20)) imply that $\mathbb{P}(X^{(k+\tau(i,j))} = (i,j) \text{ for infinitely many } k \mid X^{(\tau(i,j))} = (i,j)) = 1$. It follows that $\mathbb{P}\left(X^{(t)} = (i,j) \text{ for infinitely many } t\right) = 1$, as desired. *Q.E.D.*

The final result is a direct corollary of Lemma G.2 and the fact that the intersection of finitely-many full-measure events has full measure.

COROLLARY G.3: *Define the events* $\mathcal{I}_j, \mathcal{I} \subseteq \mathcal{H}$ *as*

$$\mathcal{I}_j := \{h \in \mathcal{H} : (s^{t-1}, s^t) = (d, j) \text{ infinitely often}\} \quad \text{for all } j \in S$$

and $\mathcal{I} := \bigcap_{j=1}^{d} \mathcal{I}_j$. *Then,* $\mathbf{P}(\mathcal{I}_j) = 1$ *for all* $j \in S$ *and hence* $\mathbf{P}(\mathcal{I}) = 1$.

APPENDIX H: DISCUSSION OF CONDITION R.5

Herein, we expand on the discussion of Condition R.5 in Section 3.2. We assume that Conditions R.1–R.4 hold (but do *not* assume R.5). Thus, P satisfies the Bellman equation (FE) and is strictly convex, and there exists a unique optimal contract (Proposition 3.2).[24]

Because our constraint set $\Gamma(\mathbf{v})$ includes both IC constraints and multiple interim promise keeping constraints, the standard envelope theorem for concave dynamic programs of Benveniste and Scheinkman (1979) does not directly apply.[25] We describe an alternative envelope theorem of Rincón-Zapatero and Santos (2009, Theorem 3.1) (henceforth RZS), which permits constraints of this form.

RZS's setup applies to our "interim" Bellman equation (FE-Q^i) from SA–C.3.3, once we have solved (PK_i) for the flow utility $u_i = v_i - \alpha\, \mathbf{E}^{\mathbf{f}_i}[\mathbf{w}_i]$, so that $\mathbf{w}_i \in D$ is the only choice variable. Substituting this into (IC^*_{ji}) yields the reduced-form incentive constraints

$$g^j(\mathbf{v}, \mathbf{w}_i, i) := v_j - \alpha\, \mathbf{E}^{\mathbf{f}_j}[\mathbf{w}_i] - \psi\left(v_i - \alpha\, \mathbf{E}^{\mathbf{f}_i}[\mathbf{w}_i], j, i\right) \geq 0 \qquad (\text{RIC}^*_{ji})$$

for all $j > i \in S$. We denote the derivative of $g^j(\mathbf{v}, \cdot, i)$ by

$$\nabla_2 g^j(\mathbf{v}, \mathbf{w}_i, i) = \alpha\left\{-\mathbf{f}_j + \psi'\left(v_i - \alpha\, \mathbf{E}^{\mathbf{f}_i}[\mathbf{w}_i], j, i\right)\mathbf{f}_i\right\} \in \mathbb{R}^d. \qquad (\text{H.1})$$

RZS require four technical conditions, D1–D4, to conclude that Condition R.5 holds.

- D1 requires that the principal's flow cost $C(\cdot, i)$ is $\mathbf{C}^1$. This holds in our setting because the agent's utility function U is $\mathbf{C}^1$ (Assumption DARA).
- D2 requires that, under the optimal contract, $(\mathbf{v}^{(t)})_{t=0}^{\infty}$ evolves in the interior of the domain D. This holds in our setting because D is open (Theorem 3).
- D3 is a constraint qualification requiring that for every $\mathbf{v} \in D$ and $i \in S$, *at the optimal* $\mathbf{w}_i(\mathbf{v})$ the derivatives corresponding to binding constraints $\{\nabla_2 g^j(\mathbf{v}, \mathbf{w}_i(\mathbf{v}), i) : g^j(\mathbf{v}, \mathbf{w}_i(\mathbf{v}), i) = 0\}_{j=i+1}^{d}$ are linearly independent, i.e., the matrix formed by using these

[24] In Proposition 3.2, Condition R.5 is only used to establish that $P(\cdot, s) \in \mathbf{C}^1(D)$ in part (b).

[25] With a single *ex ante* promised utility state $v \in \mathcal{U}$ and corresponding single *ex ante* promise keeping constraint, as in standard recursive formulations of the i.i.d. case, the Benveniste and Scheinkman (1979) theorem typically applies because we can implement any perturbation of v by varying only the flow utilities $(u_i)_{i \in S}$ and the flow cost $C(\cdot, i)$ is smooth. See Footnote 34 below for details. By contrast, with a vector of interim promised utilities $\mathbf{v} \in D$, one needs to perturb each component v_i separately; while this can be done by varying only the flow utilities in the first-best problem (SA–F), it generally cannot be done in the same manner in the second-best problem due to the presence of the IC constraints.

vectors as rows has full rank (of at most $d-i$).[26] Below, Lemma H.1 shows that D3 holds in two leading cases: (i) if only "local" incentive constraints ((RIC$^*_{ji}$) with $j=i+1$) bind, it holds for all utilities and transition matrices, and (ii) regardless of which constraints bind, it holds for CARA utility and generic transition matrices.

- D4 is an asymptotic condition requiring that the subgradients of $P(\mathbf{v}^{(t)}, s^{(t)})$ do not explode "too quickly" as $t \to \infty$ under the optimal contract.[27] Unfortunately, D4 is difficult to directly verify in our setting. At the end of this SA, we describe how this difficulty can be bypassed in the i.i.d. special case.

Overall, we conclude that RZS's D1 and D2 always hold, D3 holds under mild conditions, and that the main technical barrier to establishing Condition R.5 in general is verifying RZS's D4.[28] We note that D1–D4 are merely *sufficient* for Condition R.5. Indeed, we conjecture that Condition R.5 is implied by Regularity (Conditions R.1–R.3), and that this can be shown via the sequential problem (SP) by adapting Morand and Reffett (2015, Theorem 3). We leave further study of these issues as an important task for future work.

Sufficient Conditions for D3. Let $\mathcal{M} := \Delta(S)^S$ denote the set of transition matrices $\mathbf{F} := [\mathbf{f}_i]_{i=1}^d$ on $S=\{1,\dots,d\}$. Let $\mathcal{M}^\circ \subset \mathcal{M}$ denote the subset of fully connected transition matrices (i.e., those consistent with Assumption Markov). Note that, being the d-fold product of a $(d-1)$-dimensional simplex, $\mathcal{M}$ (respectively, $\mathcal{M}^\circ$) is a $(d-1)d$-dimensional compact (respectively, open) convex set. We denote the $(d-1)d$-dimensional Lebesgue measure as $\mathrm{Leb}_{(d-1)d}(\cdot)$. We have $\mathcal{L} := \mathrm{Leb}_{(d-1)d}(\mathcal{M}) = \mathrm{Leb}_{(d-1)d}(\mathcal{M}^\circ) > 0$.

LEMMA H.1: *For every $d \geq 2$, the following hold:*

(a) *For any $\mathbf{F} \in \mathcal{M}^\circ$ and utility function U satisfying Assumption DARA: If only local IC constraints bind,*[29] *then D3 holds.*

(b) *For any CARA utility function: There exists an $M \subseteq \mathcal{M}^\circ$ that is open, dense, and has full measure such that, regardless of which IC constraints bind, D3 holds for all $\mathbf{F} \in M$.*

The two parts of Lemma H.1 are complementary. Part (a) shows that, when the FOA is valid, D3 holds for all $\mathbf{F}$ and U. This covers all instances of the model with $d=2$, and also permits i.i.d. type process. Part (b) allows for any combination of binding IC constraints, but restricts attention to CARA utility and generic $\mathbf{F}$; when global IC constraints are binding, the genericity condition elides i.i.d. processes.

PROOF OF LEMMA H.1: For part (a), let $\mathbf{v} \in D$ and $i \in S \backslash \{d\}$ be given. By hypothesis, we have $g^j(\mathbf{v}, \mathbf{w}_i(\mathbf{v}), i) > 0$ for all $j > i+1$. If $g^{i+1}(\mathbf{v}, \mathbf{w}_i(\mathbf{v}), i) > 0$, there is nothing to prove. If $g^{i+1}(\mathbf{v}, \mathbf{w}_i(\mathbf{v}), i) = 0$, then the full-rank condition is violated iff $\nabla_2 g^{i+1}(\mathbf{v}, \mathbf{w}_i(\mathbf{v}), i) = \mathbf{0}$. By (H.1), the latter condition holds iff $\mathbf{f}_{i+1} = \psi'(v_i - \alpha\, \mathsf{E}^{\mathbf{f}_i}[\mathbf{w}_i], i+1, i)\mathbf{f}_i$, which is impossible because $\mathbf{f}_{i+1}, \mathbf{f}_i \in \Delta(S)$ and $\psi'(\cdot, i+1, i) < 1$ (recall Footnote 19). Thus, D3 holds.

For part (b), let $U(c) \equiv -e^{-\rho c}$ for some $\rho > 0$. For each $i \in S$, define $\theta_i := e^{-\rho\omega_i}$ so that, for all $j \geq i+1$, $\psi(u, j, i) \equiv \frac{\theta_j}{\theta_i} u$ and hence (H.1) becomes $\nabla_2 g^j(\cdot, \cdot, i) \equiv \alpha\{-\mathbf{f}_j + \frac{\theta_j}{\theta_i}\mathbf{f}_i\}$. For each

[26] For each $i \in S$, there are in total $d-i$ (RIC$^*_{ji}$) constraints with $j > i$, not all of which necessarily bind.

[27] Formally, D4 requires that that, for each $(\mathbf{v}^{(0)}, s^{(0)}) \in D \times S$, there exists a constant $B_0 \in \mathbb{R}$ such that $\lim_{t\to\infty} \mathsf{E}[\alpha^t G^{(t)} \zeta^{(t)} \mid (\mathbf{v}^{(0)}, s^{(0)})] = B_0$ for every measurable selection of subgradients $\zeta^{(t)} \in \partial P(\mathbf{v}^{(t)}, s^{(t)})$, where each $G^{(t)}$ is a (random) $d \times d$ matrix formed from pseudo-inverses of the matrices described in D3.

[28] We do not know of any examples in the literature where D1–D3 all hold but D4 fails. As discussed in RZS, known non-differentiabilities in dynamic contracting models are associated with violations of D1–D3.

[29] That is, if $g^j(\mathbf{v}, \mathbf{w}_i(\mathbf{v}), i) > 0$ for all $i, j \in S$ such that $j > i+1$ and $\mathbf{v} \in D$.

$i \in S \backslash \{d\}$ and $\mathbf{F} \in \mathscr{M}^\circ$, define the $(d-i) \times d$ matrix $B_i(\mathbf{F}) := [\mathbf{f}_j - \frac{\theta_j}{\theta_i}\mathbf{f}_i]_{j=i+1}^d$. If $B_i(\mathbf{F})$ has full (row) rank for every $i \in S\backslash\{d\}$, then D3 holds (regardless of which IC constraints bind). We claim that every $B_i(\mathbf{F})$ has full rank for all $\mathbf{F}$ in an open, dense, and full-measure subset $M \subseteq \mathscr{M}^\circ$. To this end, let $i \in S\backslash\{d\}$ be given. Let K_i be the (finite) index set of all the $(d-i)\times(d-i)$ submatrices of $B_i(\mathbf{F})$, with typical index $k \in K_i$ and corresponding submatrix $B_{ik}(\mathbf{F})$. For each $k \in K_i$, let $M_{ik} := \{\mathbf{F} \in \mathscr{M}^\circ : B_{ik}(\mathbf{F}) \text{ has full rank}\}$. Since $B_i(\mathbf{F})$ has full rank iff there exists a $k \in K_i$ for which $B_{ik}(\mathbf{F})$ has full rank, we have $\{\mathbf{F} \in \mathscr{M}^\circ : B_i(\mathbf{F}) \text{ has full rank}\} = \bigcup_{k\in K_i} M_{ik} =: N_i$. For each $k \in K_i$, note that $\mathbf{F} \in M_{ik}$ iff the determinant $\det(B_{ik}(\mathbf{F})) \neq 0$; since the mapping $\mathbf{F} \mapsto \det(B_{ik}(\mathbf{F}))$ is a non-constant polynomial on $\mathscr{M}^\circ$, $M_{ik} \subset \mathscr{M}^\circ$ is open and has full measure, i.e., $\text{Leb}_{(d-1)d}(M_{ik}) = \mathscr{L}$.[30] Thus, $N_i \subset \mathscr{M}^\circ$ is open and has full measure $\text{Leb}_{(d-1)d}(N_i) = \mathscr{L}$. Since $i \in S\backslash\{d\}$ was arbitrary, $M := \bigcap_{i=1}^{d-1} N_i = \{\mathbf{F} \in \mathscr{M}^\circ : B_i(\mathbf{F}) \text{ has full rank } \forall i \in S\backslash\{d\}\}$ is open and has full measure $\text{Leb}_{(d-1)d}(M) = \mathscr{L}$; hence, M is also dense in $\mathscr{M}^\circ$.[31] This proves the claim. *Q.E.D.*

Verifying Condition R.5 in the i.i.d. Case. Let $\boldsymbol{\pi} \in \Delta(S)$ denote the (type-independent) transition probabilities. Recall the efficiency problem (Eff_i) from SA–C.3.1. As noted in Lemma C.7, this problem has (type-independent) value function $K : \mathscr{U} \to \mathbb{R}$, which represents the restriction of the (type-independent) value function $P : D \to \mathbb{R}$ to the (type-independent) efficient set $E \subsetneq D$, which is parameterized by (one-dimensional) *ex ante* promised utility $v \in \mathscr{U}$.[32] Formally, Lemma C.6(a)–(b) implies that (i) there is a bijection between promised utility vectors $\mathbf{v} \in E$ and their corresponding *ex ante* promised utilities $v = \mathbf{E}^{\boldsymbol{\pi}}[\mathbf{v}] \in \mathscr{U}$, and (ii) $K(v) = \min\{P(\mathbf{v}) : \mathbf{v} \in D \text{ s.t. } \mathbf{E}^{\boldsymbol{\pi}}[\mathbf{v}] = v\}$ for all $v \in \mathscr{U}$, where the minimum is attained on E.[33] Moreover, the optimal contract always maps to E: in the notation of SA–C, $\xi^c(\mathbf{v}, s, i) \in E$ for all $\mathbf{v} \in D$ and $i, s \in S$.

We claim that, if only local IC constraints bind, then K and P are both $\mathbf{C}^1$. We sketch the proof below; it sidesteps the need to directly check RZS's D4 (cf. their Corollary 3.1).

To begin, note that the above discussion implies that P and K satisfy

$$\begin{aligned} P(\mathbf{v}) = &\min_{(u_i,w_i)_{i\in S}\in(\mathscr{U}\times\mathscr{U})^d} \sum_{i\in S} \pi_i \left[C(u_i, i) + \alpha K(w_i)\right] \\ &\text{s.t.} \quad v_i = u_i + \alpha w_i \qquad \forall i \in S, \\ &\qquad\quad v_i \geq \psi(u_{i-1}, i, i-1) + \alpha w_{i-1} \quad \forall i \in S, \end{aligned} \tag{H.2}$$

where $w_i := \mathbf{E}^{\boldsymbol{\pi}}[\mathbf{w}_i] \in \mathscr{U}$ is the *ex ante* continuation utility following report $i \in S$. Then, because $K(v) = \min\{P(\mathbf{v}) : \mathbf{v} \in D \text{ s.t. } \mathbf{E}^{\boldsymbol{\pi}}[\mathbf{v}] = v\}$ for all $v \in \mathscr{U}$, K satisfies

$$\begin{aligned} K(v) = &\min_{(u_i,w_i)_{i\in S}\in(\mathscr{U}\times\mathscr{U})^d} \sum_{i\in S} \pi_i \left[C(u_i, i) + \alpha K(w_i)\right] \\ &\text{s.t.} \quad v = \sum_{i\in S} \pi_i [u_i + \alpha w_i], \\ &\qquad\quad u_i + \alpha w_i \geq \psi(u_{i-1}, i, i-1) + \alpha w_{i-1} \quad \forall i \in S. \end{aligned} \tag{H.3}$$

[30] It is a standard fact that, for any open connected set $U \subseteq \mathbb{R}^{(d-1)d}$ and non-constant polynomial $f : U \to \mathbb{R}$, the zero set $Z(f) := \{x \in U : f(x) = 0\}$ is closed (in U) and satisfies $\text{Leb}_{(d-1)d}(Z(f)) = 0$. Applying this fact to $U = \mathscr{M}^\circ$ and $f(\cdot) = \det(B_{ik}(\cdot))$ and taking complements yields the desired conclusion.

[31] By a standard argument, every open subset of $\mathscr{M}^\circ$ with full measure is dense.

[32] Note that we implicitly redefine the domains of K and P to reflect the fact that, in the i.i.d. case, they do not depend on $i \in S$. This minor abuse of notation simplifies the subsequent presentation.

[33] In Lemma C.6, Condition R.5 is only used to show that $K \in \mathbf{C}^1(\mathscr{U})$ in part (c).

This is the Bellman equation from Thomas and Worrall (1990), which features a single *ex ante* promise keeping constraint (and we include only the local downward IC constraints). Thus, $K \in \mathbf{C}^1(\mathcal{U})$ by a standard application of Benveniste and Scheinkman (1979).[34]

Next, we use this fact to show via (H.2) that $P \in \mathbf{C}^1(D)$. In particular, in (H.2) we can use the promise keeping constraints to solve out for $(u_i)_{i\in S}$, yielding a minimization problem over $(w_i)_{i\in S} \in \mathcal{U}^d$ subject only to the reduced-form IC constraints

$$h^i(\mathbf{v}, w_1, \ldots, w_d) := v_i - \psi(v_{i-1} - \alpha w_{i-1}, i, i-1) - \alpha w_{i-1} \geq 0 \quad \forall i \in S.$$

An argument analogous to the proof of Lemma H.1(i) shows that the derivatives of the h^i functions with respect to $(w_i)_{i\in S}$ are linearly independent. Thus, RZS's D1–D3 hold for this reduced version of (H.2). But since $K \in \mathbf{C}^1(\mathcal{U})$, this suffices to show that $P \in \mathbf{C}^1(D)$ (cf. the special case of RZS's Proposition 3.1 in which the continuation value function is known to be smooth). This proves the claim.

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Co-editor Charles I. Jones handled this manuscript.

[34]Formally, suppose the menu $(u_i, w_i)_{i\in S}$ solves (H.3) at $v \in \mathcal{U}$. For $t \in \mathbb{R}$, define $x_d(t) := t$ and, for all $i < d$, inductively define $x_i(t) := \psi^{-1}(\psi(u_i, i+1, i) + x_{i+1}(t), i+1, i) - u_i$. (Since $\mathcal{U}$ is open, there is an $\varepsilon > 0$ such that every $u_i + x_i(t) \in \mathcal{U}$ for $|t| < \varepsilon$.) Then $(u_i + x_i(t), w_i)_{i\in S}$ is feasible in (H.3) at $v + g(t)$, where $g(t) := \sum_{i\in S} \pi_i x_i(t)$. (By construction, this is the essentially unique way to perturb the initial flow utilities without changing the amount of slack in any of the local downward IC constraints.) Each $x_i(\cdot)$ is $\mathbf{C}^1$ with $x_i'(\cdot) > 0$, and satisfies $x_i(0) = 0$. Then, since each $C(\cdot, i) \in \mathbf{C}^1(\mathcal{U})$, Benveniste and Scheinkman (1979) delivers $K \in \mathbf{C}^1(\mathcal{U})$. Thomas and Worrall (1990, Proposition 1) describe this argument informally; for CARA utility, it coincides with the construction in Golosov, Tsyvinski, and Werquin (2016, Lemma 1).