ECONOMIC RESEARCH
FEDERAL RESERVE BANK OF ST. LOUIS

WORKING PAPER SERIES

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Authors	David Andolfatto, and Fernando M. Martin
Working Paper Number	2018-002A
Creation Date	January 2018
Citable Link	https://doi.org/10.20955/wp.2018.002
Suggested Citation	Andolfatto, D., Martin, F.M., 2018; Monetary Policy and Liquid Government Debt, Federal Reserve Bank of St. Louis Working Paper 2018-002. URL https://doi.org/10.20955/wp.2018.002

Published In	Journal of Economic Dynamics & Control
Publisher Link	https://doi.org/10.1016/j.jedc.2018.01.037

Monetary Policy and Liquid Government Debt*

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January 19, 2018

Abstract

We examine the conduct of monetary policy in a world where the supply of outside money is controlled by the fiscal authority–a scenario increasingly relevant for many developed economies today. Central bank control over the long-run inflation rate depends on whether fiscal policy is Ricardian or Non-Ricardian. The optimal monetary policy follows a generalized Friedman rule that eliminates the liquidity premium on scarce treasury debt. We derive conditions for determinacy under both fiscal regimes and show that they do not necessarily correspond to the Taylor principle. In addition, Non-Ricardian regimes may suffer from multiplicity of steady-states when the government runs persistent deficits.

Keywords: monetary policy; inflation; Taylor rule; determinacy; Ricardian; liquid bonds.

JEL codes: E40, E52, E60, E63

*The views expressed in this paper do not necessarily reflect official positions of the Federal Reserve Bank of St. Louis, the Federal Reserve System, or the Board of Governors.

1 Introduction

It has long been recognized that safe government bonds like United States Treasury securities possess money-like properties (e.g., Fried and Howitt, 1983). Given the rapid growth of the shadow banking sector in recent decades and its reliance on treasury securities as an exchange medium, the demand for government bonds as a form of wholesale money is greater today than ever. Moreover, as the spread between bond yields and interest on central bank reserves continues to narrow, the size of central bank treasury holdings is likely becoming increasingly irrelevant from an economic perspective.

The growing relative importance of government debt over central bank reserves as money has an interesting implication, namely, that the supply of base money is increasingly under the control of the fiscal authority. Because this is the case, fiscal policy is likely to play an important role in determining the rate of inflation. But even if open-market purchases of treasury securities are of little economic significance, central banks–like the Federal Reserve Bank of the United States–seem able to dictate the nominal and real interest rate on at least short-term government bonds. The central question we pursue in this paper is the following: How are we to think about the economic consequences of a central bank that can influence the yield on government bonds in a world where the fiscal authority controls the supply of base money?

To study this question, we develop an analytically tractable model where government bonds serve as an exchange medium between investors producing final goods and suppliers involved in the provision of new capital goods. Monetary and fiscal policies are linked through a consolidated government budget constraint. While we abstract from central bank money, seigniorage revenue is still possible because government debt is nominal. Monetary policy is modeled as a forward-looking Taylor rule. The fiscal authority determines the path of the real primary deficit and the rate of nominal debt-issuance. Two alternative fiscal policy regimes are considered: in a *Ricardian* regime, the primary deficit adjusts passively to satisfy the government budget constraint, whereas in a *non-Ricardian* regime, the fiscal authority sets the primary deficit independently of other policy variables.[1]

[1] We use these labels in the same way as Woodford (1995); see also Canzoneri, Cumby and Diba (2001). The labels "active" and "passive" used by Leeper (1991) and others to describe the policy stance of the fiscal and monetary authorities can be misleading in our

We begin by characterizing stationary equilibria and describing their properties. We find that there is a unique stationary equilibrium when the fiscal authority is Ricardian. Non-Ricardian fiscal regimes are problematic in the sense that they open the door to non-existence and a multiplicity of equilibria. In the case of multiple equilibria, a classic Laffer curve effect is at work; namely, the economy can find itself stuck in a high-inflation equilibrium that generates the same real primary deficit possible in a low-inflation equilibrium. Running a surplus (even weakly) eliminates this type of multiplicity.

Under a Ricardian fiscal policy, the long-run inflation rate is determined by the rate of nominal debt-issuance, which is outside the control of the monetary authority. By choosing the nominal interest rate, the central bank effectively controls the real rate of return on government debt. When the real yield on treasury debt is lower than the natural interest rate (the inverse of the discount factor), collateral is scarce so that government bonds possess a liquidity premium. The optimal monetary policy is to eliminate the liquidity premium on bonds; that is, to follow a generalized Friedman rule–a policy that equates the real rate of return on bonds to the natural interest rate. Interestingly, it is not necessary for the central bank to hit its inflation target to achieve this goal.

Under a Non-Ricardian fiscal policy, control of the long-run inflation rate returns to the central bank. The central bank policy rate affects the inflation rate through the effect that interest rates have on the interest expense of government debt. For a fixed real primary deficit, a higher nominal interest rate necessitates a faster pace of nominal debt-issuance from the treasury. In this way, a central bank can increase long-run inflation through a persistent increase in its policy rate. In the same way, a low-interest-rate policy leads to low inflation. These result are consistent with the claims made by Neo-Fisherian theory, though our results do not depend on inflation expectations being driven by the Fisher effect (Williamson, 2016).

The model makes clear that the collateral shortage cannot be alleviated through an injection of new debt (which is neutral) or persistent injections of new debt (which is inflationary and *worsens* the shortage). What is needed

setup, since even when the primary deficit adjusts passively (the Ricardian regime), the fiscal authority can still actively manage the growth rate of government debt. As we shall see, there are circumstances under which one of the two authorities has no choice but to accommodate the other.

is a sufficiently large and persistent real primary budget surplus to finance the carry cost of debt that yields the natural interest rate.

Our analysis then turns to studying the dynamic properties of equilibria. We find that the condition leading to determinacy does not correspond to the Taylor principle, which requires the central bank's policy rate to react more than one-for-one with expected inflation. Specifically, under a Ricardian regime any Taylor-rule coefficient up to some upper bound (determined by parameters) implies determinacy of equilibrium. This upper bound is greater than one, so typical estimates of this coefficient are still consistent with determinacy in our environment. We can recover the Taylor principle if government debt is illiquid and the central bank reacts to current rather than future inflation. Under a Non-Ricardian regime with a zero primary deficit, any forward-looking Taylor rule leads to determinacy, except one that reacts one-for-one with future inflation.[2] In this case, recovering the Taylor principle only requires that the central bank reacts to current rather than future inflation. Determinacy in this case is not affected by whether bonds are liquid or not.

In our model, there is no reason to believe *a priori* that a central bank's preferred long-run inflation rate is consistent with the actual inflation rate that is generated by fiscal policy. What, if anything, can a central bank do to achieve its preferred inflation target in the face of an uncooperative fiscal authority? The equilibrium inflation rate in our model is determined in part by treasury supply growth and in part by treasury demand growth. Because the central bank in our model can influence the path of the interest rate, it can influence the path of real treasury demand and, hence, the rate of inflation–even if it cannot control the supply of treasuries.

Achieving a lower inflation target entails an ever-increasing path for the nominal and real interest rates. Because this policy works to diminish the liquidity premium on bonds, it actually stimulates output in credit-constrained sectors. But because the natural interest rate serves as an upper bound on the real yield for treasury debt, the central bank cannot maintain its lower inflation target indefinitely. In the opposite scenario, achieving a higher inflation target requires an ever-decreasing nominal and real interest rate. In

[2]When the nominal interest rate reacts one-for-one with future inflation (i.e., the coefficient in the Taylor rule is equal to one), there is real determinacy, but nominal indeterminacy.

principle, there is no lower bound on the real interest rate. However, real-world policy constraints (like the zero lower bound) would imply once again that the central bank must ultimately fail in defending its inflation rate peg.

As a final exercise, we apply our model to interpret the consequences of recent developments in U.S. monetary policy; namely, the apparent desire to systematically raise the policy rate to a more historically normal level. As is well-known, the rate of inflation in the United States has consistently undershot the Federal Reserve Bank's official two-percent target since it was established in 2012. While a number of external factors are no doubt contributing to this phenomenon, we use our model to show that the present tightening cycle is likely contributing to the systematic undershooting of inflation. Nevertheless, when viewed through the lens of our model, the policy is likely stimulative for credit constrained firms as it increases the real return on, and hence demand for, high-grade assets (e.g., treasuries), effectively alleviating their scarcity. Although this result may appear counterintuitive, it is in fact a standard property in all monetary models: increasing the real rate of return on an exchange medium when liquidity is scarce expands real economic activity.

Our paper is related to a number of other works that investigate the implications of liquid government debt using the fiscal theory of the price-level (Leeper, 1991, Sims, 1994 and Woodford, 1995). Canzoneri and Diba (2005) demonstrate how the existence of liquid government debt can lead to dramatically different conditions for price-level determinacy. We extend their analysis to incorporate a production economy, which among other things, permits us to investigate optimal policy. Bassetto and Cui (2018) show how the price-level remains indeterminate when the government runs primary budget deficits and while its debt has a liquidity premium. Berentsen and Waller (2018) use a similar model to ours to demonstrate how shocks affecting the liquidity premium of government debt can change its market value even when fiscal policy is held constant. Domínguez and Gomis-Porqueras (2016) show that multiplicity of equilibria may arise when government debt is scarce and traded in secondary markets. Williamson (2018) shows that in a Fisherian model, central banks can control inflation independently of fiscal policy, even when it is constrained to monetize the debt.

Other related papers study the interactions between fiscal and monetary policy when debt provides liquidity services and taxation is distortionary.

These include Cui (2016), Angeletos, Collard and Dellas (2016) and Martin (2017).

2 Monetary and fiscal policy

Time, denoted t, is discrete and the horizon is infinite, $t = 0, 1, ..., \infty$. The legislative branch of the government chooses a path for net tax revenue and government purchases, $\{T_t, G_t\}_{t=0}^{\infty}$. Any such path generates a path for the primary budget deficit $D_t \equiv G_t - T_t$. Assume that the treasury finances the deficit with nominal, one-period bonds, denoted B_t, which yield a risk-free gross nominal yield R_t.

In each period we have the flow government budget constraint, $B_t = D_t + R_{t-1}B_{t-1}$ for all $t \geq 0$ with $B_{-1} \geq 0$ given. Define $\mu_t \equiv B_t/B_{t-1}$ and rearrange the government budget constraint as $D_t = [1 - R_{t-1}/\mu_t]\, B_t$. Let P_t denote the price-level and define $d_t \equiv D_t/P_t$, $b_t \equiv B_t/P_t$. Expressed in real terms, the government budget constraint is given by,

$$d_t = [1 - R_{t-1}/\mu_t]\, b_t \tag{1}$$

In a steady-state, the budget constraint (1) reduces to $d = [1 - R/\mu]\, b$. Note that perpetual primary budget deficits are possible here as long as $R < \mu$. If instead $R > \mu$, the government will have to run a perpetual primary budget surplus to service the carrying cost of the debt b.

In what follows, we make no distinction between bonds and reserves, so the composition of the central bank's balance sheet is inconsequential. Monetary policy corresponds to a rule governing the path for the nominal interest rate $\{R_t\}_{t=0}^{\infty}$. Define the inflation rate, $\Pi_{t+1} \equiv P_{t+1}/P_t$. We assume the monetary policy follows a forward-looking Taylor rule,

$$R_t = R_t^* \left(\Pi_{t+1}/\Pi_t^*\right)^{\alpha} \tag{2}$$

where Π_t^* is the central bank's inflation target and $R_t^* \equiv r_t^* \Pi_t^*$ is the central bank's nominal interest rate target (where r_t^* is the central bank's assessment of the so-called "natural rate of interest"). The parameter $\alpha \geq 0$ governs how strongly the central bank raises its policy rate when expected inflation departs from target inflation.

The inflation target is the central bank's desired inflation rate, but is not necessarily associated with the actual inflation rate even in the long-run. The target does not provide a nominal anchor, as inflation will depend on fundamentals; in this case, the growth rate of government debt. As we shall see, there are circumstances under which the fiscal authority has full control of the inflation rate and the central bank's target is irrelevant.

When the time comes, we will be more explicit about the policy rule governing fiscal policy. We consider two scenarios: a *Ricardian* regime in which the real primary deficit is assumed to adjust passively to satisfy the government budget constraint (1); and a *non-Ricardian* regime in which the primary deficit is an exogenous process, so that other elements in (1) must adjust to satisfy the government budget constraint.

3 Private sector

3.1 Preferences and technology

The economy is populated by an equal mass of two types of agents, which we label *suppliers* and *investors*. Their preferences are respectively given by,

$$\sum_t \beta^t \left[u(c_t) - k_t\right] \tag{3}$$

$$\sum_t \beta^t x_t \tag{4}$$

where $u'' \leq 0 < u'$ and $0 < \beta < 1$. Here, c and x denote consumption of a final good on the part of suppliers and investors, respectively. The variable k represents effort expended by the suppliers to provide goods and services to investors, a process we describe in more detail below. We assume that one unit of effort creates one unit of "capital" goods, so that k represents both effort and capital investment.

Investors operate investment projects that convert supplier goods and services at date t into a nonstorable final good at date $t+1$. In particular, k_t units of supplier services at date t yields $f(k_t)$ units of the final good at date $t+1$. Assume, $f'' < 0 < f'$ and $f(0) = 0$, $f'(0) = \infty$. In addition, assume $f'(k)k = \theta f(k)$ where $0 < \theta < 1$. The resource constraint at date t is given by

$$c_t + x_t + g_t = f(k_{t-1}) \tag{5}$$

An efficient (first-best) allocation $\bar{k}$ satisfies $f'(\bar{k}) = 1/\beta$, which implies a level of potential output $\bar{y} = f(\bar{k})$. Define $\bar{r} \equiv 1/\beta$ as the "natural" rate of interest.

3.2 Exchange media

Investors need to acquire inputs k_t from suppliers at date t to produce a final good that is available for consumption at date $t+1$. We assume that suppliers do not fully trust unsecured promises made by investors. We model this lack of trust as a standard exogenous debt constraint, namely, investors can promise to repay up to a fraction λ_t of the total payment.[3] Consequently, an exchange medium is generally necessary. We assume that government debt serves as an acceptable exchange medium, either indirectly as collateral to secure a promise, or directly as a payment instrument.

Note that if $\lambda_t = 1$, then there are no frictions in this economy (suppliers fully trust investors) and the first welfare theorem applies. Thus, we focus on the case where there is some level of mistrust between agents.

Assumption 1 $\lambda_t \in [0,1)$ *for all* $t = 0, \ldots, \infty$.

3.3 Market constraints

Investors enter period t with capital goods k_{t-1} which generate $f(k_{t-1})$ units of the final good. Out of nominal sales revenue $P_t f(k_{t-1})$, investors repay loans to suppliers $R_{t-1}L_{t-1}$, pay a lump-sum tax T_t, purchase B_t dollars of bonds, and spend $P_t x_t$ dollars on consumption. Investors may also carry over some amount of unspent bonds, which deliver an additional flow income $R_{t-1}[B_{t-1} - (W_{t-1}k_{t-1} - L_{t-1})]$. The budget constraint for investors is given by,

$$P_t x_t + R_{t-1}L_{t-1} + B_t + T_t = P_t f(k_{t-1}) + R_{t-1}(B_{t-1} + L_{t-1} - W_{t-1}k_{t-1}) \tag{6}$$

[3] We index λ_t with a time subscript for generality. Shifts in this parameter can be interpreted as "money demand shocks." For our present analysis, however, we keep λ fixed.

for all $t \geq 0$. Notice that since loans and bonds share the same risk and liquidity characteristics, they both yield the same rate of return.

Although investors ultimately want to borrow money for inputs, they are motivated here to accumulate bond holdings for the purpose of securing their period input expenditures $W_t k_t$, where W_t denotes the nominal price of supplier services k_t. Thus, investors are subject to the liquidity constraints,

$$B_t \geq W_t k_t - L_t \tag{7}$$

$$\lambda_t W_t k_t \geq L_t \tag{8}$$

for all $t \geq 0$.

The lemma below is a version of a standard result in monetary economies and greatly simplifies the investor's problem. All proofs are in the Appendix.

Lemma 1 *(i) The real interest rate cannot exceed the natural rate,* $r_t \equiv R_t/\Pi_{t+1} \leq \bar{r} \equiv 1/\beta$*; (ii) liquidity constraints* (7) *and* (8) *bind if* $r_t < \bar{r}$ *and are satisfied with equality without loss of generality when* $r_t = \bar{r}$*; and (iii) investors carry no unspent bonds across periods.*

As we shall see below, the real interest rate on bonds is inversely related to the marginal product of capital. Since government debt is used to finance the purchase of capital goods, if bonds earn a high real rate of return (carry a low liquidity premium) then their purchase capacity is high; hence, capital accumulation is high and the marginal product of capital is low. Policy can distort the real rate of return on bonds downwards (i.e., r_t can be driven below $\bar{r} \equiv 1/\beta$), but not upwards as bonds cannot carry a negative liquidity premium (i.e., r_t cannot exceed $\bar{r} \equiv 1/\beta$).

Lemma 1 implies $B_t = (1 - \lambda_t)W_t k_t$, $L_t = \lambda_t W_t k_t$ and the investor's budget constraint (6) simplifies to

$$P_t x_t + R_{t-1}\lambda_{t-1}W_{t-1}k_{t-1} + (1 - \lambda_t)W_t k_t + T_t = P_t f(k_{t-1}) \tag{9}$$

Suppliers enter period t with maturing loans and bonds $R_{t-1}(L_{t-1} + B_{t-1}) = R_{t-1}W_{t-1}k_{t-1}$. To simplify exposition, we assume they do not pay taxes. We anticipate that in equilibrium suppliers will spend this entire source of income on the nonstorable final good. Consequently, the budget constraint for a supplier is given by,

$$P_t c_t = R_{t-1}W_{t-1}k_{t-1} \tag{10}$$

In contrast to monetary economies along the lines of Lagos and Wright (2005), suppliers in our model cannot use their current effort to finance current consumption. The reason is that suppliers cannot produce their own consumption goods and therefore, need to rely on past effort to generate the income necessary to finance the purchase of contemporaneous final goods.[4] Note that while suppliers spend all of their maturing assets on consumption, they are nevertheless engaging in an act of saving as well. In particular, when they supply k_t units of capital goods at date t, they are in effect saving $W_t k_t$ dollars for future consumption.

3.4 Decision making

It is convenient to rewrite the budget constraints above in real terms. Define $w_t \equiv W_t/P_t$ and $\tau_t \equiv T_t/P_t$. Deflating the investor's budget constraint (9) by the price level implies

$$x_t = f(k_{t-1}) - (R_{t-1}/\Pi_t)\,\lambda_{t-1} w_{t-1} k_{t-1} - (1-\lambda_t) w_t k_t - \tau_t \tag{11}$$

for all $t \geq 0$ with $k_{-1} > 0$ given. Maximizing (4) subject to (11) implies that investment demand satisfies,

$$\beta f'(k_t) = [1 - \lambda_t + \beta \lambda_t (R_t/\Pi_{t+1})] w_t \tag{12}$$

for all $t \geq 0$. With k_t so determined, the real demand for bonds generated by investors is given residually by,

$$b_t = (1-\lambda_t) w_t k_t \tag{13}$$

Condition (13) implies that the demand for real bond holdings is proportional to the level of desired investment spending. This property follows from the fact that expenditures that require money-finance will be highly sensitive to the level of money-financing. In our model, the relevant expenditure takes the form of investment and the relevant monetary instrument takes the form of government bonds. We could alternatively have focussed on money-financed consumption expenditure. Either way, the upshot here is that more real money balances is associated with more real economic activity, at least,

[4] This is also the reason why suppliers are willing to extend loans to investors at real rates below $1/\beta$.

for credit-constrained decision-makers. We stress this point here because in other types of environments, government debt frequently crowds out capital instead of crowding it in, the way it does here.[5]

In light of the discussion above, note that condition (12), which defines the investment demand function, is *increasing* in the real interest rate $r_t \equiv R_t/\Pi_{t+1}$. While this result may seem counterintuitive, it is in fact a standard property in all monetary models–namely, increasing the real rate of return on an exchange medium when liquidity is scarce expands real economic activity. The celebrated Friedman rule exploits this basic property.

It is of some interest to note the effect of an increase in financial market "confidence," as measured by λ_t. Condition (12) implies that investment demand depends positively on the willingness of creditors to extend unsecured credit. It is interesting to note that this confidence shock has an ambiguous impact on the real demand for bonds. That is, according to (13), the direct effect of a higher value for λ_t is to diminish the need for bonds as collateral, but the indirect effect is to expand planned capital spending, which increases the demand for bonds as collateral.

Deflating the supplier's budget constraint (10) by the price level yields,

$$c_t = (R_{t-1}/\Pi_t)\, w_{t-1}k_{t-1} \tag{14}$$

for all $t \geq 0$ with $k_{-1} > 0$ given. The necessary conditions describing optimal input supplies derived from maximizing (3) are given by,

$$1 = (R_t/\Pi_{t+1})\, w_t \beta u'(c_{t+1}) \tag{15}$$

The left-hand-side of (15) represents the real marginal cost of producing an extra unit of supplies. This effort is paid W_t dollars which are saved into the next period to become R_tW_t dollars. The future purchasing power of these savings is given by $R_tW_t/P_{t+1} = (R_t/\Pi_{t+1})\, w_t$. The future marginal utility of this purchasing power is given by $\beta u'(c_{t+1})$. Hence, the right-hand-side of (15) measures the expected discounted real marginal benefit of producing an extra unit of supplies.

In what follows, we impose linear utility in consumption.

Assumption 2 $u'(c) = 1$.

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[5]In Diamond (1965) and Aiyagari and McGrattan (1998), for example, government debt crowds out capital formation.

This has the effect of eliminating wealth effects and will not, as far as we know, affect any of the main conclusions that follow. At the same time, it greatly simplifies the analysis.

4 Equilibrium

We now gather the relevant restrictions to characterize an equilibrium. To begin, combine (12) with (15) to form

$$\beta f'(k_t) = \lambda_t + (1 - \lambda_t)\frac{\Pi_{t+1}}{\beta R_t} \tag{16}$$

Combining (12) and (15) also yields $(1-\lambda_t)w_t = \beta f'(k_t) - \lambda_t$, or $(1-\lambda_t)w_t k_t = \beta\left[f'(k_t) - \lambda_t\right]k_t$ which, by condition (13) implies,

$$b_t = [\beta f'(k_t) - \lambda_t]k_t \tag{17}$$

Next, invoke the market-clearing condition, $P_t b_t = B_t$ for all $t \geq 0$, which implies that the inflation rate must satisfy,

$$\Pi_{t+1} = \mu_{t+1}\left(b_t/b_{t+1}\right) \tag{18}$$

for $t \geq 0$. Note that here we can treat $b_{-1} > 0$ as given. Then, for a given (μ_0, b_0) we know $\Pi_0 = \mu_0\left(b_{-1}/b_0\right)$.

Finally, the government budget constraint (1) must also hold at every date,

$$d_t = \left[1 - R_{t-1}/\mu_t\right] b_t \tag{19}$$

where $R_{-1} > 0$ is given.

Definition 1 *Given R_{-1}, policy targets $\{R_t^*, \Pi_t^*\}_{t=0}^{\infty}$, and a sequence $\{\lambda_t\}_{t=0}^{\infty}$, an equilibrium is a sequence $\{k_t, b_t, d_t, R_t, \Pi_{t+1}, \mu_t\}_{t=0}^{\infty}$ that satisfies* (2) *and* (16)–(19) *for all $t = 0, 1, \ldots, \infty$.*

The policy parameters are normally regarded as fixed numbers, $(R_t^*, \Pi_t^*) = (R^*, \Pi^*)$, in which case $r_t^* = r^*$ as well. We think it might be reasonable, however, to imagine circumstances where (R_t^*, Π_t^*) are subject to change through a deliberate systematic changes in monetary policy. The same could hold true for α of course.

5 Stationary equilibrium

Assume $R_t^* = R^*, \Pi_t^* = \Pi^*$ and $\lambda_t = \lambda$ for all t. A stationary equilibrium satisfies $k_t = k, b_t = b, d_t = d, R_t = R, \Pi_{t+1} = \Pi$, and $\mu_t = \mu$ for all t. It follows immediately from (18) that $\Pi = \mu$. Since the debt growth rate μ is ultimately in the realm of fiscal policy, one interpretation of this result is that long-run inflation is ultimately a fiscal phenomenon (at least, if the world is stationary).

Substituting $\Pi = \mu$ into (16) implies

$$\beta f'(k) = \lambda + (1-\lambda)\frac{\mu}{\beta R} \tag{20}$$

Hence, the stationary equilibrium features an efficient level of investment if $R/\mu = 1/\beta$.[6] In models where money is a zero-interest security, $R = 1$ so that the latter condition is the standard Friedman rule prescription to deflate at the rate of time-preference, i.e., set $\mu = \beta$. Because exchange media generally earn interest, our condition can be thought of as a generalized Friedman rule which recommends that the real rate of return on all exchange media equal the rate of time-preference.

Since R is determined by (2) an efficient stationary allocation requires the following restrictions on the Taylor rule: $\mu = \beta R^* \left(\mu/\Pi^*\right)^{\alpha}$, or

$$r^* \left(\mu/\Pi^*\right)^{\alpha-1} = 1/\beta \tag{21}$$

where, recall, $r^* \equiv R^*/\Pi^*$.

If the equilibrium rate of inflation equals the central bank's inflation target, then efficiency requires $r^* = 1/\beta$. However, there is no logical requirement to impose $\mu = \Pi^*$ *a priori*. In cases for which $\mu \neq \Pi^*$, efficiency is still possible either by setting $r^* = 1/\beta$ and $\alpha = 1$ or, more generally, by setting $r^* = (1/\beta)\left(\mu/\Pi^*\right)^{1-\alpha}$.

Next, combine (17) with (19) to form the expression,

$$d = (1-r)[\beta f'(k) - \lambda]k \tag{22}$$

where $r = R/\mu$.

[6] As mentioned above, the first-best is also obtained when $\lambda = 1$, a situation we ruled out by Assumption 1.

Equations (20) and (22) characterize k and d for a given real interest $r = R/\mu$. Specifically, (20) defines a desired investment function $k = \kappa(r)$ with the following properties.

Lemma 2 *The function $\kappa(r)$ defined by (20) is non-negative and strictly increasing for all $r \in [0, 1/\beta]$, with $\kappa(0) = 0$ and $\kappa(1/\beta) = \bar{k}$.*

Using the fact that $\theta f(k) = f'(k)k$, express the right-hand-side of (22) as $S(r) \equiv (1 - r)[\beta\theta f(k) - \lambda k]$. Thus, (22) implies $d = S(r)$. The function $S(r)$ can be interpreted as (net) revenue from seigniorage, which is used to finance the primary deficit. If $S > 0$, then the government is in a position to run perpetual primary deficits financed with seigniorage revenue. If $S < 0$, then seigniorage revenue reflects the interest expense of carrying the public debt, a scenario that would require perpetual primary surpluses to balance the government budget.

Lemma 3 *The function $S(r)$ has the following properties: (i) $S(0) = 0$; (ii) $S(1) = 0$; (iii) $S(1/\beta) < 0$; and (iv) if $\lambda = 0$ then $S(r)$ is strictly increasing for all $r \in [0, \theta)$, peaks at $r = \theta$ and is strictly decreasing for all $r \in (\theta, 1/\beta]$.*

As shown in Lemma 3, when $\lambda = 0$, $S(r)$ peaks at $r = \theta$ and decreases for higher interest rates. Clearly, since $\theta < 1/\beta$ and k is strictly increasing in r by Lemma 2, $\kappa(\theta) < \kappa(1/\beta) = \bar{k}$. The calculus involved here is based on familiar Laffer curve logic. The tax base is given by the real demand for bonds, $b = \beta\theta f(k)$. The tax rate is given by $(1 - r)$ for $r < 1$ (a subsidy is involved if $r > 1$). Seigniorage revenue is initially increasing in the tax rate, but eventually the tax becomes so large as to diminish the tax base. If the debt here has a pegged nominal interest rate (like currency, for example), then the demand for real money balances approaches zero in very high inflation regimes. The revenue maximizing real interest rate on liquid debt balances these two countervailing effects.

To this point we have assumed the existence of equilibrium. To establish existence, we need to say more about the conduct of monetary and fiscal policy. In what follows, we assume two monetary policy regimes indexed by $\alpha \in \{0, 1\}$. The case of $\alpha = 0$ corresponds to a nominal interest rate peg and the case of $\alpha = 1$ implies that the central bank moves its policy rate one-for-one with the expected rate of inflation, i.e., a real interest rate peg. We also consider two fiscal regimes: Ricardian and Non-Ricardian. In the Ricardian regime, the fiscal authority chooses the rate of nominal debt-issuance μ and lets the real primary deficit d adjust passively to satisfy the government budget constraint. In the Non-Ricardian regime, the fiscal authority chooses the real primary deficit d and lets its treasury adjust the rate of nominal debt-issuance μ to satisfy the government's financing needs.

5.1 Nominal interest rate peg ($\alpha = 0$)

By the Taylor rule (2), setting $\alpha = 0$ implies $R = R^*$. In this case, the central bank follows a nominal interest rate peg.

5.1.1 Ricardian fiscal policy

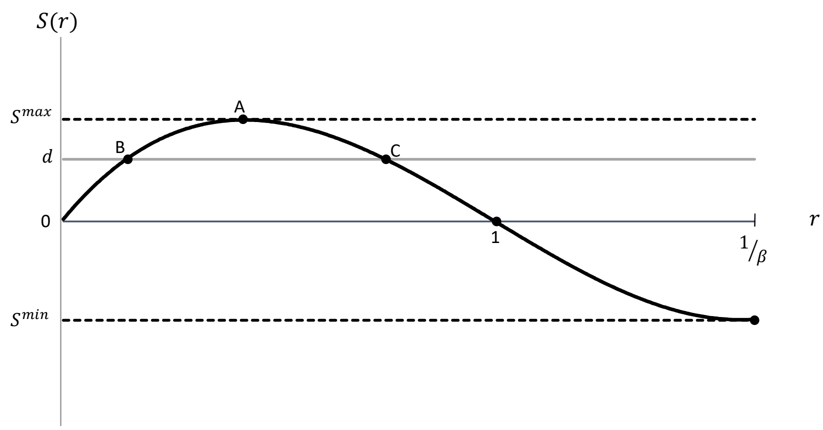
A Ricardian fiscal policy is a choice of μ. Together with monetary policy, this implies a real rate of interest $r = R^*/\mu$. It is easy to see that a *unique* stationary equilibrium with valued debt exists for any μ such that $r \in (0, 1/\beta]$. Note that while nominal debt is neutral, it is not superneutral. An increase in the rate of nominal debt-issuance leads to higher inflation and a lower real rate of return on the exchange medium, which contracts investment spending and real GDP. The rate of nominal debt-issuance that results in an efficient level of investment is given by $\mu = \beta R^*$, the generalized Friedman rule.

5.1.2 Non-Ricardian fiscal policy

A Non-Ricardian policy is a choice of d. As shown in Lemma 3, $S(r)$ peaks at $r = \theta$ when $\lambda = 0$. For general values of λ, S will peak at some interest rate $r < 1$; call this maximum seigniorage $S^{\max}$ (point A in Figure 1). The first thing to note is that a stationary equilibrium cannot exist for any $d > S^{\max}$. Second, a unique equilibrium with valued debt exists for any $d \in [S(1/\beta), 0]$ and for $d = S^{\max}$. Third, for any $d \in (0, S^{\max})$ there exist *two* stationary equilibria (points B and C in Figure 1).

The multiplicity of equilibria here follows from classic Laffer curve analysis. For any $d \in (0, S^{\max})$ there are two solutions to the budget condition (22). One solution entails a high rate of inflation (low value for r), which is needed to finance the deficit when the tax base is low (owing to the high

Figure 1: Seignorage revenue as a function of equilibrium real rate



inflation). The second solution entails a low rate of inflation (high value of r). In this latter case, the low rate inflation expands the tax base (the real demand for debt) sufficiently to make up for the revenue lost by printing nominal debt at a lower rate.

Consider an economy with $d \in (0, S^{\max})$. What is the effect of lowering the real primary deficit (austerity)? The answer depends on which side of the Laffer curve the economy starts off in.[7] In the high-inflation equilibrium, austerity leads to higher inflation and lower investment. In the low-inflation equilibrium, austerity leads to lower inflation and higher investment. Indeed, the efficient allocation is achieved under an austerity program that results in a persistent real primary budget surplus.[8]

What advice does this analysis suggest for a fiscal authority operating under a Non-Ricardian regime finding itself "stuck" in the high-inflation equilibrium? The analysis here suggests that the fiscal authority switch from a non-Ricardian regime to a Ricardian regime. That is, the recommendation is to take active control over the rate of nominal debt-issuance–in particular, by actively lowering it. Given the nominal interest rate peg, the fiscal authority would in this way increase the real interest rate and lower inflation, thus relaxing debt constraints and promoting investment, while keeping the

[7]The statements that follow assume that the economy remains in the neighborhood of its original equilibrium.

[8]We are of course abstracting from any costs associated with transition dynamics.

deficit at the same level.

5.2 Real interest rate peg ($\alpha = 1$)

By the Taylor rule (2), setting $\alpha = 1$ implies $R = R^* \left(\mu/\Pi^*\right)$, or $r = r^* \leq 1/\beta$. Note that since the central bank's policy rate moves one-for-one with the treasury nominal debt-issuance rate (the steady-state inflation rate), inflation is superneutral in this case–i.e., it does not affect the real interest rate. In other words, the central bank follows a real interest rate peg.

5.2.1 Ricardian fiscal policy

A Ricardian fiscal policy is a choice of μ. Together with monetary policy, this implies a real interest rate $r = r^* \leq 1/\beta$. A unique stationary equilibrium with valued debt exists for any $r^* \in (0, 1/\beta]$.

We saw how under a nominal interest rate peg, an increase in the rate of nominal debt issuance served to reduce the real interest rate and contract economic activity. Under the present configuration of monetary policy, an increase in the rate of nominal debt issuance continues to be inflationary, however, the economic consequences of the inflation are fully mitigated by a corresponding increase in the central bank's policy rate.

5.2.2 Non-Ricardian fiscal policy

A Non-Ricardian policy is a choice of d. The level of desired investment continues to be described by the function $\kappa(r)$. Since monetary policy implies $r = r^*$, the level of investment is given by $\kappa(r^*) \leq \bar{k}$. The level of seigniorage revenue is therefore given by $S(r^*) = (1 - r^*)[\beta\theta f(\kappa(r^*)) - \lambda\kappa(r^*)]$. For an arbitrary configuration of (d, r^*), the government budget constraint (22) will be violated. For a stationary equilibrium to exist, something must give.

If the fiscal authority remains resolute in its choice of the primary deficit under a real interest rate peg, responsibility for financing the deficit falls on the monetary authority. For a given d, the monetary authority must adjust its target real interest rate r^* in a manner that is consistent with (22). Whether this is done by changing R^* or Π^* is inconsequential.

The economics of this scenario is then identical to the case of a nominal interest rate peg under a Non-Ricardian fiscal policy analyzed above. The only difference here is that it is the monetary authority that must adjust the real return on debt, rather than the treasury department of the fiscal authority.

6 Dynamics

To study the dynamic properties of the model, we make the following two assumptions for analytical tractability.

Assumption 3 $\lambda_t = 0$ *for all* $t = 0, \ldots, \infty$.

Assumption 4 $f(k) = k^\theta$, $\theta \in (0, 1)$.

In what follows, we consider the general form of the Taylor rule (2).

6.1 Ricardian fiscal policy

Assumption 5 *The fiscal authority independently sets a sequence* $\{\mu_t\}_{t=0}^{\infty}$ *and lets* d_t *adjust to satisfy* (19) *for all* $t = 0, \ldots, \infty$.

Let $\hat{z} = \ln z$ for any variable or parameter z. Under Assumptions 3, 4 and 5 the equilibrium defined by (2) and (16)–(19) is the solution to a set of difference equations. Taking as given initial condition $\hat{R}_{-1}$, a sequence of monetary targets $\{\hat{R}_t^*, \hat{\Pi}_t^*\}_{t=0}^{\infty}$ and a fiscal policy $\{\hat{\mu}_t\}_{t=0}^{\infty}$, an equilibrium can be characterized by a sequence $\{\hat{k}_t, \hat{b}_t, \hat{\Pi}_{t+1}\}_{t=0}^{\infty}$ that solve:

$$\begin{aligned} (1-\theta)\hat{k}_t &= 2\hat{\beta} + \hat{\theta} + (\hat{R}_t^* - \alpha\hat{\Pi}_t^*) + (\alpha - 1)\hat{\Pi}_{t+1} && (23) \\ \hat{b}_t &= \hat{\beta} + \hat{\theta} + \theta\hat{k}_t && (24) \\ \hat{\Pi}_{t+1} &= \hat{\mu}_{t+1} + \hat{b}_t - \hat{b}_{t+1} && (25) \end{aligned}$$

The values for R_t and d_t can be recovered from (2) and (19), respectively. From (25), given some initial $\hat{b}_{-1}$ we obtain $\hat{\Pi}_0 = -(\hat{\beta} + \hat{\theta}) + \hat{\mu}_0 + \hat{b}_{-1} - \theta\hat{k}_0$.

An appropriate monetary policy can implement the first-best.

Proposition 1 (Generalized Friedman rule in Ricardian regimes) *Under Assumptions 3–5, a central bank following a Taylor rule* (2) *implements the first best for all* $t \geq 0$ *if and only if* $\hat{\beta} + \hat{R}_t^* - \hat{\Pi}_t^* = (1-\alpha)(\hat{\Pi}_{t+1} - \hat{\Pi}_t^*)$. *This can be achieved independently of fiscal policy by targeting the natural rate* $\hat{r}_t^* \equiv \hat{R}_t^* - \hat{\Pi}_t^* = -\beta$ *and responding one-for-one with inflation,* $\alpha = 1$.

In general, a central bank attempting to implement the first-best using a Taylor rule needs to appropriately adjust its policy targets in response to movements in fiscal policy. By responding one-for-one with inflation (setting $\alpha = 1$), it can implement the first-best independently of fiscal policy by targeting the natural rate.

Let us now study equilibrium dynamics. From (24), $\{\hat{b}_t\}_{t=0}^{\infty}$ depends only on $\{\hat{k}_t\}_{t=0}^{\infty}$. Combining (24) and (25) we can also express $\{\hat{\Pi}_{t+1}\}_{t=0}^{\infty}$ as a function of $\{\hat{k}_t\}_{t=0}^{\infty}$:

$$\hat{\Pi}_{t+1} = \hat{\mu}_{t+1} + \theta(\hat{k}_t - \hat{k}_{t+1}) \tag{26}$$

Combining (23) and (26), we can determine that the sequence $\{\hat{k}_t\}_{t=0}^{\infty}$ solves

$$\hat{k}_t = \frac{\hat{\Sigma}_t - \theta(\alpha-1)\hat{k}_{t+1}}{(1-\theta\alpha)} \tag{27}$$

where $\hat{\Sigma}_t \equiv 2\hat{\beta} + \hat{\theta} + (\hat{R}_t^* - \hat{\mu}_t) - \alpha(\hat{\Pi}_t^* - \hat{\mu}_t)$.

In order to study determinacy, we focus on the case with constant policy.

Assumption 6 $R_t^* = R^*$, $\Pi_t^* = \Pi^*$ *and* $\mu_t = \mu$ *for all* $t \geq 0$.

Let us first solve for the steady state. By Assumption 6, $\hat{\Sigma}_t = \hat{\Sigma} = 2\hat{\beta} + \hat{\theta} + (\hat{R}^* - \hat{\mu}) - \alpha(\hat{\Pi}^* - \hat{\mu})$. Setting $\hat{k}_t = \hat{k}_{t+1} = \hat{k}$ in (27) we obtain

$$\hat{k} = \frac{\hat{\Sigma}}{1-\theta} \tag{28}$$

Note that when the central bank's inflation target $\hat{\Pi}^*$ and the fiscal authority's long-run debt growth objective $\hat{\mu}$ coincide, the steady state $\hat{k}$ is independent from the coefficient in the Taylor rule α (since $\hat{\Sigma}$ no longer depends on α).

Next, let us rearrange (27) to write $\hat{k}_{t+1}$ a function of $\hat{k}_t$:

$$\hat{k}_{t+1} = \frac{-\hat{\Sigma}}{\theta(1-\alpha)} + \frac{(1-\theta\alpha)\hat{k}_t}{\theta(1-\alpha)} \tag{29}$$

This is a linear first-order difference equation with constant coefficient $A \equiv \frac{(1-\theta\alpha)}{\theta(1-\alpha)}$. It has a known solution: $\hat{k}_{t+1} = \hat{k} + A^t(\hat{k}_0 - \hat{k})$ if $A \neq 0$ and $A \neq 1$; and $k_{t+1} = \hat{k}$ if $A = 0$, i.e., $\alpha = 1/\theta$. The case $A = 1$ is ruled-out since $\theta < 1$.

When $|A| \in (0, 1)$, there are infinite possible values of $\hat{k}_0$ that imply solutions for $\hat{k}_{t+1}$ that converge to $\hat{k}$. Since $\hat{k}_0$ itself is an element of the equilibrium sequence, rather than a given initial condition, the equilibrium in this case is indeterminate.

When $|A| > 1$, the solution of $\hat{k}_{t+1}$ diverges from $\hat{k}$, as long as $\hat{k}_0 \neq \hat{k}$. In particular, if $A > 1$ the solution diverges monotonically and if $A < -1$ the solution oscillates, with increasing amplitude. The only exception is when $\hat{k}_0 = \hat{k}$, in which case, $\hat{k}_{t+1} = \hat{k}_0$ $(= \hat{k})$ for all $t \geq 0$ is the unique solution converging to $\hat{k}$.

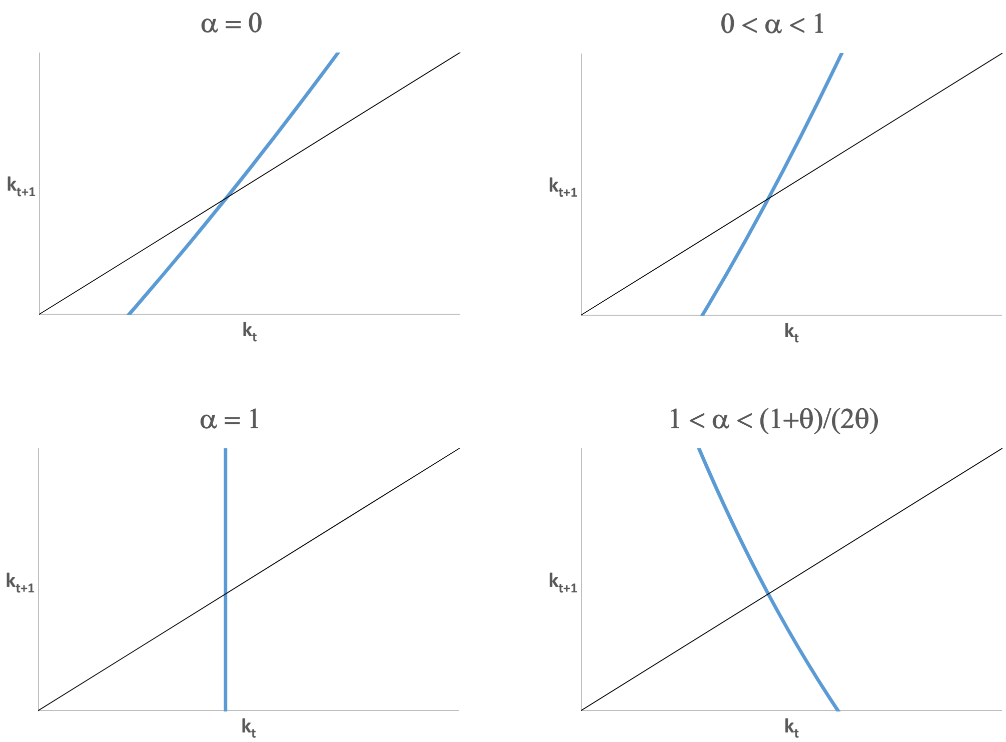
When the sequence diverges from $\hat{k}$ ($|A| > 1$ and $\hat{k}_0 \neq \hat{k}$), there are two other equilibria which converge to one of two bounds. First, the equilibrium can converge to $\hat{k}_t = -\infty$ (i.e., $k_t = 0$), in which case, the economy steadily contracts and shuts-down in the long-run. A stricter (i.e., finite) lower bound on $\hat{k}_t$ could be set by imposing a zero-lower-bound on the nominal interest rate: $\hat{R}_t \geq 0$.[9] Second, the equilibrium can converge to the upper bound $\hat{k}_t = \bar{k}$ (given by the upper bound on the real interest rate $\bar{r} = 1/\beta$, as stated in Lemma 1). In this case, the economy converges to the first-best. Note that hitting this upper bound also requires modifying the Taylor rule, in this case by imposing an upper bound on the nominal interest rate: $\hat{R}_t \leq \hat{\Pi}_{t+1} - \hat{\beta}$.

The following result establishes under what conditions for α we obtain (local) determinacy, i.e., uniqueness of the equilibrium sequence that converges to the steady state $\hat{k}$.

Proposition 2 *Under Assumptions 3–6, if $\alpha \in [0, \frac{1+\theta}{2\theta}]$ then there is a unique equilibrium sequence $\{\hat{k}_t\}_{t=0}^{\infty}$ that converges to $\hat{k}$: $\hat{k}_t = \hat{k}$ for all $t \geq 0$.*

[9] See also Benhabib, Schmitt-Grohé and Uribe (2001) and Andolfatto and Williamson (2015).

Figure 2: Investment dynamics



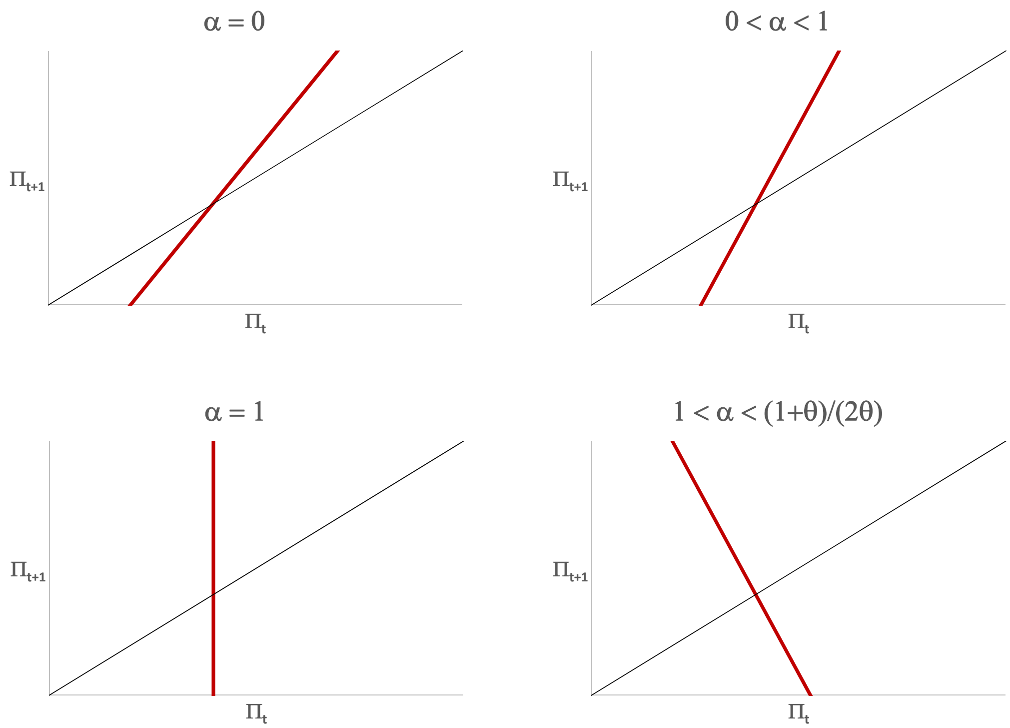
Note that the upper bound on α defined in Proposition 2, $\frac{1+\theta}{2\theta}$, is consistent with the values commonly used in the literature. For example, as long as $\theta \leq 1/2$, $\alpha = 1.5$ implies there is a unique equilibrium converging to $\hat{k}$.

Figures 2 and 3 display the phase diagrams for investment (k_t vs k_{t+1}) and inflation (Π_t vs Π_{t+1}), under various assumptions for α.

6.2 Non-Ricardian fiscal policy

We can similarly analyze dynamics for the Non-Ricardian regime. Dynamics are tractable if we focus on the case when the primary deficit is zero in every period. In this case, we avoid the area with multiple steady states identified in the previous section. Furthermore, monetary policy cannot achieve the first-best as this would require targeting the natural rate $\bar{r} = 1/\beta$, which requires

Figure 3: Inflation dynamics



a surplus. To simplify exposition, we also also impose fixed monetary targets from the start.

Assumption 7 $R_t^* = R^*$, $\Pi_t^* = \Pi^*$ *and* $d_t = 0$ *for all* $t = 0, \ldots, \infty$.

From the government budget constraint (1), $d_t = 0$ implies $\mu_t = R_{t-1}$. Thus, μ_0 is pinned down by the initial condition R_{-1} and μ_t is equal to the nominal interest rate in all subsequent periods.

As in the previous section, we express the equilibrium conditions in logs. Taking as given initial condition $\hat{R}_{-1}$, an equilibrium under a zero primary deficit policy can be characterized by a sequence $\{\hat{k}_t, \hat{b}_t, \hat{\Pi}_{t+1}, \hat{\mu}_t\}_{t=0}^{\infty}$ that solves:

$$\begin{aligned}(1-\theta)\hat{k}_t &= 2\hat{\beta}+\hat{\theta}+(\hat{R}^*-\alpha\hat{\Pi}^*)+(\alpha-1)\hat{\Pi}_{t+1} &(30)\\ \hat{b}_t &= \hat{\beta}+\hat{\theta}+\theta\hat{k}_t &(31)\\ \hat{\Pi}_{t+1} &= \hat{\mu}_{t+1}+\hat{b}_t-\hat{b}_{t+1} &(32)\\ \hat{\mu}_{t+1} &= \hat{R}^*+\alpha(\hat{\Pi}_{t+1}-\hat{\Pi}^*) &(33)\end{aligned}$$

for all $t \geq 0$ and $\hat{\mu}_0 = \hat{R}_{-1}$. Note, again, that we can retrieve the initial inflation rate for a given $\hat{b}_{-1}$: $\hat{\Pi}_0 = \hat{R}_{-1} + \hat{b}_{-1} - \hat{b}_0$.

Combining equations (31)–(33) and rearranging, we can express $\{\hat{\Pi}_{t+1}\}_{t=0}^{\infty}$ as a function of $\{\hat{k}_t\}_{t=0}^{\infty}$:

$$\hat{\Pi}_{t+1} = \frac{\hat{R}^* - \alpha\hat{\Pi}^* + \theta(\hat{k}_t - \hat{k}_{t+1})}{1-\alpha} \tag{34}$$

Replacing (34) into (30) we obtain that the sequence $\{\hat{k}_t\}_{t=0}^{\infty}$ solves

$$\hat{k}_t = 2\hat{\beta} + \hat{\theta} + \theta\hat{k}_{t+1} \tag{35}$$

The steady state for the Non-Ricardian regime with zero primary deficits is thus

$$\hat{k} = \frac{2\hat{\beta}+\hat{\theta}}{1-\theta}$$

If we write (35) as $\hat{k}_{t+1}$ as a function of $\hat{k}_t$, we get a linear first-order difference equation with constant coefficient $1/\theta > 1$. Thus, there is a unique equilibrium sequence converging to $\hat{k}$: $\hat{k}_t = \hat{k}$ for all $t \geq 0$. In other words, there is real determinacy for all values of α.

We summarize the results in the following statement.

Proposition 3 *Under Assumptions 3, 4 and 7, for all $\alpha \geq 0$, there is a unique equilibrium sequence $\{\hat{k}_t\}_{t=0}^{\infty}$ that converges to $\hat{k}$: $\hat{k}_t = \hat{k}$ for all $t \geq 0$.*

What about nominal variables? Given $\hat{k}_t = \hat{k}$, condition (34) implies $\hat{\Pi}_{t+1} = \frac{\hat{R}^*-\alpha\hat{\Pi}^*}{1-\alpha}$ for all $t \geq 0$. Thus, there is nominal determinacy for any $\alpha \neq 1$. It is easy to see that this result depends on the forward-looking nature of the Taylor rule. If instead the central bank reacted to current inflation, $\hat{\Pi}_{t+1}$ would depend on $\hat{\Pi}_t$ and α would play a role. In the appendix we show that the Taylor principle applies in this case.

6.3 Determinacy: fiscal policy regime, nature of the Taylor-rule and liquidity of debt

Table 1 summarizes our findings for determinacy of the equilibrium, according to the fiscal policy regime, the nature of the Taylor-rule and the liquidity of government debt. In the sections above we focused on our benchmark economy with a forward-looking Taylor rule and liquid bonds, under both Ricardian and Non-Ricardian (zero primary deficit) fiscal policies. In the Appendix, we derive results for current-looking Taylor-rules and illiquid bonds.

Table 1: Conditions for Determinacy

	Ricardian		Zero primary deficit	
	Forward	Current	Forward	Current
Liquid	$\alpha \in [0, \frac{1+\theta}{2\theta}]$	$\alpha \geq 0$	$\alpha \neq 1$	$\alpha > 1$
Illiquid	$\alpha \neq 1$	$\alpha > 1$	$\alpha \neq 1$	$\alpha > 1$

Note: "Forward" and "Current" refer to whether Π_{t+1} or Π_t enters the Taylor-rule, respectively. "Liquid" and "Illiquid" correspond to $\lambda = 0$ and $\lambda = 1$, respectively. Determinacy is defined as uniqueness of the equilibrium sequence converging to the (unique) interior steady state, for both real and nominal variables.

As in the preceding analysis, determinacy is defined as uniqueness of the equilibrium sequence converging to the (unique) interior steady state, for both real and nominal variables. When the central bank reacts to future inflation, what we label (adopting the language from Carlstrom and Fuerst, 2000) a forward-looking Taylor rule, any non-negative value of α is consistent with real determinacy and all non-negative values different from 1 are consistent with nominal determinacy. The exception is when debt is liquid, which is our benchmark. In this case, as we showed above, there is an upper bound on α, determined by the curvature of the production function.

Instead, when the central bank reacts to current inflation, what we label a current-looking Taylor rule, we recover the Taylor principle ($\alpha > 1$) for equilibrium determinacy, in most cases. The exception again is when the debt is liquid; in this case, any $\alpha \geq 0$ is consistent with determinacy. These results complement those by Carlstrom and Fuerst (2000), who find that how determinacy is related to the responsiveness of the nominal interest rate to inflation (the parameter α), depends on whether the Taylor-rule is forward-, current- or backward-looking.

Relative to standard models, the benchmark economy we focus on in this paper departs in two important ways: (i) the Taylor rule is forward-looking, i.e., the central bank reacts to Π_{t+1} rather than Π_t; and (ii) government bonds are liquid. Under a Ricardian fiscal policy, we need both a Taylor rule that reacts to current inflation and illiquid bonds to recover the Taylor principle for determinacy. Under a zero primary deficit (non-Ricardian) fiscal rule, we only need a Taylor rule that reacts to current inflation to recover the Taylor principle.

7 Monetary vs fiscal control of inflation

Recall that in this model, under a Ricardian regime, long-run inflation is determined by the fiscal authority's choice of the debt growth rate, μ. For an arbitrary monetary policy, there is nothing to guarantee that the central bank's inflation target satisfies $\Pi^* = \mu$. When this is the case, the question arises as to whether there is any monetary policy the central bank could pursue to ensure that the equilibrium inflation rate corresponds to its own preferred target in those circumstances where $\mu \neq \Pi^*$.

From the Taylor rule (2), if the central bank hits its target, $\Pi_{t+1} = \Pi^*$, then $R_t = R_t^*$ for any α. Note that although we assume the inflation target is constant, we permit the central bank to alter its policy rule by choosing a sequence of "interest rate targets" that vary systematically over time.

If the central bank hits its inflation target, then condition (23) implies $\hat{R}_t^* = -(2\hat{\beta} + \hat{\theta}) + \hat{\Pi}^* + (1-\theta)\hat{k}_t$, so that $\hat{R}_t^* - \hat{R}_{t-1}^* = (1-\theta)(\hat{k}_t - \hat{k}_{t-1})$. Combining this latter expression with (26), we obtain

$$\hat{R}_t^* = \hat{R}_{t-1}^* + \left(\frac{1-\theta}{\theta}\right)(\hat{\mu} - \hat{\Pi}^*) \tag{36}$$

Condition (36) describes what it takes for a central bank to hit its preferred inflation target against an unaccommodative fiscal authority. If the central bank wants a lower inflation target ($\hat{\mu} > \hat{\Pi}^*$), then the policy that delivers this result entails an ever-increasing policy target rate R_t^*. As this policy path brings monetary policy ever-closer to operating at the Friedman rule, this tightening cycle is associated with an expanding level of economic activity in credit-constrained sectors. Of course, if the central bank wants a higher inflation target ($\hat{\mu} < \hat{\Pi}^*$), then an ever-decreasing policy target rate is in order.

There is the question of what, if anything, might prevent a central bank from "bucking against the fiscal winds" forever. Let us first consider the quest to lower inflation. The requisite policy requires an ever-increasing nominal interest rate and, since inflation is fixed, an ever-increasing real interest rate (on liquid debt) as well. As shown in Lemma 1, the real interest rate is bounded above by the "natural" interest rate, $r_t^* \equiv R_t^*/\Pi^* \leq 1/\beta$.[10] Once this upper bound is reached, the central bank must abandon its inflation peg. On the other hand, if the central bank continues to keep its real target rate equal to the natural rate, then the liquidity premium on debt vanishes, so that in the end the outcome is greater efficiency, in spite of the higher rate of inflation.

Consider next the policy of raising the inflation target. The requisite policy here requires an ever-decreasing nominal interest rate and, since inflation is being held fixed, the real interest rate declines over time. The economic consequence of this policy is to exacerbate the collateral shortage, so that production at credit-constrained firms declines over time. The question is whether there is any natural lower bound on the real rate of interest. Apart from the fact that $r^* > 0$, the answer for this economy appears to be no. The central bank in this model can drive the economy—or at least a part of it that depends on bonds as an exchange medium—progressively into the ground (k_t will shrink to zero as $r_t^* \searrow 0$) for a very long time.

In reality, there are likely limits to a central bank implementing an inflation target higher than what the treasury is willing to support. To begin, there would be political pressure from groups charging the central bank with "financial repression" (Reinhart and Sbrancia, 2015). Second, firms that rely on government bonds as collateral are likely to find cheaper substitutes. Third, the central bank may be constrained through fear or legislation from lowering its nominal policy rate too far. This latter constraint is referred to as the "zero lower bound".

Another possible outcome is that the pain inflicted by monetary policy

[10] The same restriction in standard monetary models (which assume $R = 1$) entails $\mu \geq \beta$, since an equilibrium otherwise fails to exist.

on the economy results in the fiscal authority relenting and raising μ to Π^*. This can also be the case if the treasury sets a goal for debt expansion based on observations about past inflation rates. Regardless of the motive, once μ increases to Π^* the economy jumps immediately to a new steady state, with lower output than in the initial steady state. Though the central bank may in this case win the policy battle, it does so here at the cost of lower welfare for private agents.

8 Raising the policy rate

In December of 2008, the Federal Open Market Committee (FOMC) of the U.S. Federal Reserve lowered its policy rate–the interest-on-excess reserves (IOER) rate–to 25 bp. The IOER remained at 25bp for the next seven years and then, on December 16, 2015, the FOMC raised the IOER rate 50bp. It has since raised the IOER rate four more times to 1.50%.

Many FOMC members seem convinced that further rate hikes are needed. Their concern is rooted firmly in the Phillips curve theory of inflation, which suggests an inverse causal relationship between unemployment and future inflation. Since the unemployment rate is presently so low, inflation is evidently lurking around the corner. Hence, despite inflation falling below the Fed's official two-percent target for several years, policy rate hikes now are prescribed by those eager to "get ahead of the curve."

There is no Phillips curve theory of inflation in the model developed above. Inflation in our model is a monetary-fiscal phenomenon: it depends on the relative growth rates of the supply and demand for nominal securities. In this section, we evaluate the implications of a central bank embarking on a "tightening cycle" when long-run inflation appears well-anchored and when the real yield on treasury debt is very low (that is, the circumstances prevailing at the date of "lift-off" in December 2015).

From the analysis in the preceding sections, we know that if the economy is not at the first-best, it must be because the real interest rate is below the natural rate. That is, the nominal interest rate is too low or the inflation rate is too high or both.

Motivated by the results in Proposition 1 on how to achieve efficiency, suppose the Fed sets $\alpha = 1$, so that it moves nominal interest rates one-for-one with anticipated inflation, and then gradually raises the target for the real rate $r_t^* \equiv R_t^*/\Pi_t^*$ up to $\bar{r} \equiv 1/\beta$. We know this will bring the economy up to "potential" in this model, but there is still the question of what the transition looks like.

From the system (23)–(25), we can obtain the evolution of k_t and Π_{t+1} given $\alpha = 1$ and a constant μ:

$$\begin{aligned} \hat{k}_t &= \frac{1}{1-\theta}(2\hat{\beta} + \hat{\theta} + \hat{r}_t^*) \\ \hat{\Pi}_{t+1} &= \hat{\mu} + \frac{\theta}{1-\theta}(\hat{r}_t^* - \hat{r}_{t+1}^*) \end{aligned}$$

Hence, if $\hat{r}_t^* < \hat{r}_{t+1}^*$ then $\hat{\Pi}_{t+1} < \hat{\mu}$. In other words, as the Fed gradually raises the real interest rate, inflation will be below its steady state value. If $\mu \leq \Pi^*$, the Fed will be missing on its inflation target along the transition, even though it is raising the policy rate.

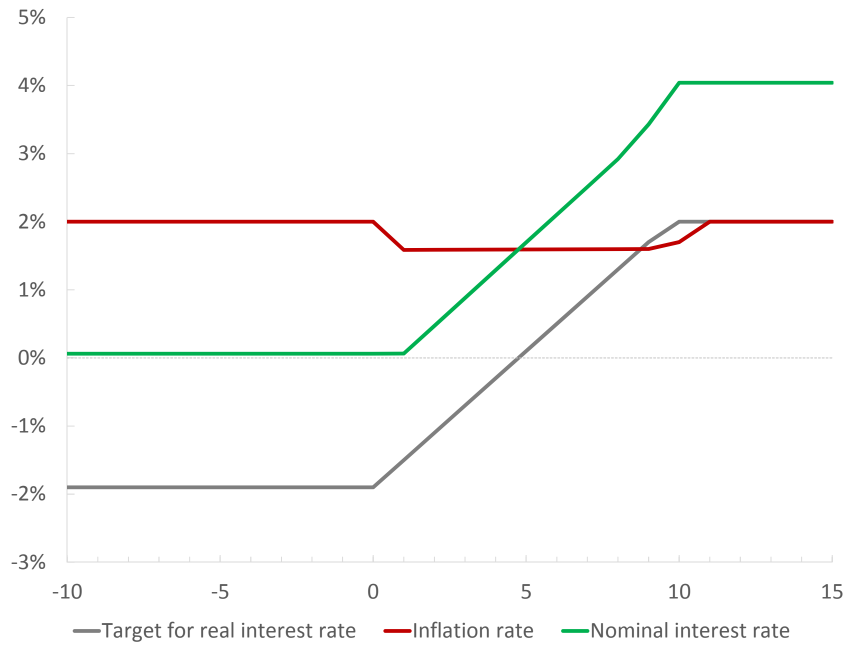
Figures 4 and 5 illustrate the dynamics of raising the policy rate to restore efficiency. Parameters are chosen so that the natural real rate, the Fed's inflation target and the actual long-run inflation are 2% annual, which are the standard assumptions to describe for U.S. monetary policy. The implication of "lift-off" is that one should expect the normalization policy to be associated with a below-target level of inflation for as long as the policy persists.

9 Summary and conclusions

The spread on short-term treasuries and interest on reserves today is much smaller than has historically been the case. An open-market operation involving a swap of reserves for treasuries today is less consequential than a similar-sized operation in the past. An economic force driving down the yield on bonds is their increasing popularity as a form of wholesale money. Viewed in this light, the supply of money remains as consequential as ever in determining the rate of inflation. All that has changed is the agency responsible for managing the supply of base money.

Monetary policy through interest rate control has its most obvious direct impact on short-term treasury debt–the most liquid government securities. As such, monetary policy boils down to influencing the liquidity-premium on

Figure 4: Policy dynamics when raising the policy rate



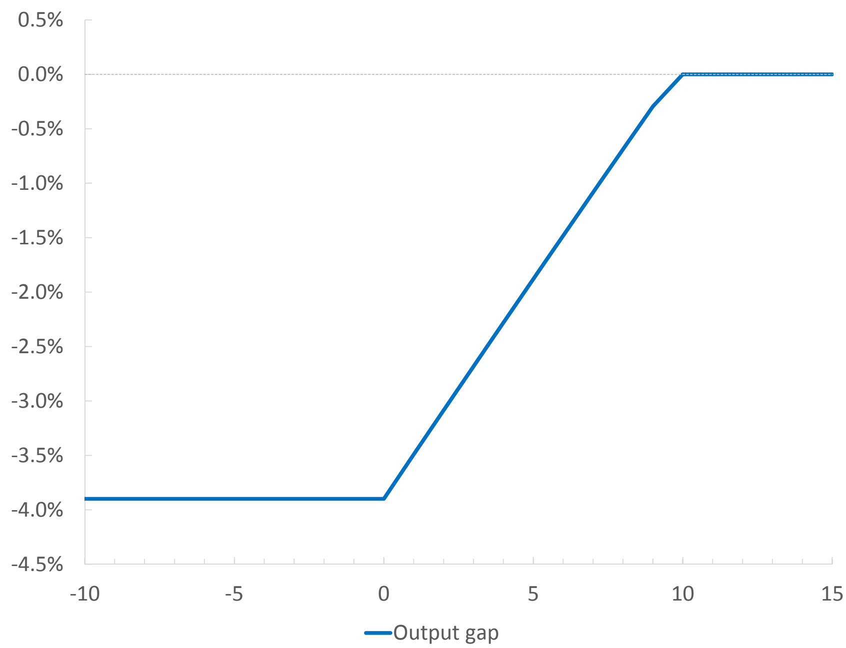
Note: $\beta^{-1} = \Pi^* = \mu = 1.02$, $r^* = 1.01$, $\alpha = 1.0$, $\theta = 0.5$.

high-grade collateral. A low central bank policy rate in this context could be interpreted as a form of "financial repression" in the sense of making government debt excessively scarce and expensive. Assuming that limited commitment is the only financial market friction, an optimal monetary policy follows a generalized Friedman rule.[11]

It is worth repeating that a shortage of government debt is not solved by increasing its nominal supply. In fact, increasing the nominal supply of treasury debt actually exacerbates a collateral shortage for the same reason that a higher inflation rate exacerbates the liquidity shortage in any monetary model. The standard Friedman rule–the prescription to deflate at the rate of time-preference–is financed through a real primary budget surplus. The

[11] See also Azariadis (2016), who specifies the Taylor rule as managing the size of the public debt rather than its yield.

Figure 5: Output dynamics when raising the policy rate



Note: $\beta^{-1} = \Pi^* = \mu = 1.02$, $\alpha = 1.0$, $\theta = 0.5$.

same prescription applies in our model (see also Berentsen and Waller, 2018).

Our analysis is built on the pioneering work of Leeper (1991), Sims (1994) and Woodford (1995) on the fiscal theory of the price level. Our work extends this line of enquiry by examining how liquid public debt potentially alters the conclusions reached in this literature. Our paper is closely related to the work of Canzoneri, Cumby, Diba and López-Salido (2008, 2011) who study the likely quantitative importance of the mechanisms highlighted above. We view our analysis complementary to theirs as it permits a high degree of analytical tractability.

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Appendix

A Proofs

Proof of Lemma 1. Consider the problem of maximizing (4) subject to (6)–(8). Let $\beta^t \zeta_{b,t}$ and $\beta^t \zeta_{l,t}$ be the Lagrange multipliers associated with (7) and (8), respectively. The first-order conditions with respect to B_t and L_t imply: $\zeta_{b,t} = 1 - \beta r_t$ and $\zeta_{b,t} = \zeta_{l,t}$, where $r_t \equiv R_t/\Pi_{t+1}$. Given that both multipliers are non-negative, statements (i)–(iii) follow. ■

Proof of Lemma 2. Let $F(k,r) \equiv \lambda + (1-\lambda)(1/\beta)(1/r) - \beta f'(k)$. From $r \equiv R/\mu$ and (20), $F(k,r) = 0$ and so, by the Implicit Function Theorem, $dk/dr = -(dF/dr)/(dF/dk)$. We obtain:

$$\frac{dk}{dr} = -\frac{(1-\lambda)}{(\beta r)^2 f''(k)} > 0$$

When $r = 0$, the right-hand-side of (20) goes to infinity, which given $f'(0) = \infty$ implies $\kappa(0) = 0$. By Lemma 1, $r \leq 1/\beta$, which defines the admissible range of real interest rates. When $r = 1/\beta$, (20) implies $\beta f'(k) = 1$ and so, $k = \bar{k}$. ■

Proof of Lemma 3. (i) $S(0) = \beta\theta f(k) - \lambda k$. As argued in the proof of Lemma 2, by (20) $\kappa(0) = 0$ and so, $S(0) = 0$.

(ii) By (20), $\kappa(1)$ is positive and finite. Hence, $S(1) = 0$.

(iii) By (20), $\kappa(1/\beta) = \bar{k}$. Thus, $S(1/\beta) = (1 - 1/\beta)[\beta\theta f(\bar{k}) - \lambda\bar{k}] = (1 - 1/\beta)[\beta f'(\bar{k}) - \lambda]\bar{k} = (1 - 1/\beta)[1 - \lambda]\bar{k} < 0$.

(iv) When $\lambda = 0$, $S(r) = (1-r)\beta\theta f(k)$. So,

$$dS/dr = -\beta\theta f(k) + (1-r)\beta\theta f'(k)(dk/dr)$$

Since $dk/dr > 0$, $dS/dr < 0$ for all $r \geq 1$. Let us now verify at which values of r below 1, $dS/dr = 0$. Substituting the expression for dk/dr (see proof of Lemma 2) and rearranging, the condition is $f(k) = -\frac{(1-r)f'(k)}{(\beta r)^2 f''(k)}$. Using $f'(k)k = \theta f(k)$, which also implies $-f'(k)/f''(k) = \frac{k}{1-\theta}$, we get $f'(k) =$

$\frac{(1-r)\theta}{(\beta r)^2(1-\theta)}$. From (20) $f'(k) = \frac{1}{\beta^2 r}$. Hence, the unique solution to $dS/dr = 0$ is $r = \theta$. After some algebra we can show

$$\frac{d^2S}{dr^2} = \beta\theta \left\{ -f'(k) + \frac{(\beta r - 2)k}{(\beta r)^3(1-\theta)} + \frac{(1-r)(dk/dr)}{(\beta r)^2(1-\theta)} \right\}$$

Thus, $d^2S/dr^2 < 0$ for $r < 1$, so that $S(r)$ is strictly concave around $r = \theta$. Thus, $S(r)$ is strictly increasing for $r \in [0, \theta)$ and strictly decreasing in $r(\theta, 1/\beta]$. ■

Proof of Proposition 1. We need to show that $\ln \hat{k} = \frac{\hat{\beta}\hat{\theta}}{1-\theta}$ satisfies (23). Setting $\hat{k}_t = \ln \bar{k}$ we get: $0 = \hat{\beta} + (\hat{R}^*_t - \alpha\hat{\Pi}^*_t) + (\alpha - 1)\hat{\Pi}_{t+1}$. A simple rearrangement gives us the expression in the statement. Setting $\hat{r}^*_t \equiv \hat{R}^*_t - \hat{\Pi}^*_t = -\beta$ and $\alpha = 1$ satisfies the condition. ■

Proof of Proposition 2. First, note that when $A = -1$, i.e., $\alpha = \frac{1+\theta}{2\theta}$, $\hat{k}_{t+1}$ oscillates between $\hat{k}_0$ and $2\hat{k} - \hat{k}_0$. Thus, the only sequence converging to $\hat{k}$ is $\hat{k}_0 = \hat{k}$. As we mentioned above, the case $|A| = 1$ cannot happen since it would require $\theta = 1$.

From the discussion above, when $|A| < 1$ there are infinite sequences $\hat{k}^{\infty}_{t=0}$ that converge to $\hat{k}$. Thus, all we are left to show now is when $|A| > 1$, so that the equilibrium sequence $\{\hat{k_t}\}^{\infty}_{t=0}$ does not converge to $\hat{k}$ unless $\hat{k}_0 = \hat{k}$. Recall that $A \equiv \frac{(1-\theta\alpha)}{\theta(1-\alpha)}$. If $\alpha = 1$ then $A = \infty$. If $\alpha \in [0, 1)$ then A is positive; $A > 1$ implies $(1-\theta\alpha) > \theta(1-\alpha)$ and so $1 > \theta$. Thus, $A > 1$ for all $\alpha \in [0, 1]$. If $\alpha > 1$ then A is negative; hence, $A < -1$ implies $(1 - \theta\alpha) < \theta(1 - \alpha)$, i.e., $\alpha < (1 + \theta)/(2\theta)$. Thus, $A < -1$ for all $\alpha \in (1, \frac{1+\theta}{2\theta})$. ■

B Determinacy under alternative assumptions

B.1 Illiquid bonds

In the case of $\lambda = 1$, the demand for liquidity (money) vanishes. Recall that when $\lambda = 1$, $k_t = \bar{k}$ for all $t \geq 0$, so that we always get real determinacy. The only relevant interest rate is that for an illiquid bond. For this, we have the Fisher equation, $R_t = (1/\beta)\Pi_{t+1}$. In log form:

$$\hat{R}_t = -\hat{\beta} + \hat{\Pi}_{t+1} \tag{37}$$

The Taylor rule (2) under constant policy:

$$\hat{R}_t = \hat{R}^* + \alpha(\hat{\Pi}_{t+1} - \hat{\Pi}^*) \tag{38}$$

Combining these (37) and (38), we can solve for a unique value for $\hat{\Pi}_{t+1}$:

$$\hat{\Pi}_{t+1} = \frac{\hat{\beta} + \hat{R}^* - \alpha\hat{\Pi}^*}{1-\alpha}$$

for all $t \geq 0$. Thus, we obtain nominal determinacy, for any $\alpha \neq 1$.

Consider instead a Taylor rule that reacts to current inflation

$$\hat{R}_t = \hat{R}^* + \alpha(\hat{\Pi}_t - \hat{\Pi}^*) \tag{39}$$

Combining (37) with (39) yields,

$$\hat{\Pi}_{t+1} = \hat{\beta} + \hat{R}^* - \alpha\hat{\Pi}^* + \alpha\hat{\Pi}_t$$

Now we get the standard result: nominal determinacy requires an explosive path for this difference equation, i.e., $\alpha > 1$.

B.2 Liquid bonds and current-looking Taylor rule

Consider now what happens when we use the second Taylor rule (39) in our model (with $\lambda = 0$). In this case, we replace the Fisher equation (37) with

$$(1-\theta)\hat{k}_t = 2\hat{\beta} + \hat{\theta} + \hat{R}_t - \hat{\Pi}_{t+1}$$

Combining the condition above with the Taylor rule (39), we have

$$(1-\theta)\hat{k}_t = 2\hat{\beta} + \hat{\theta} + \hat{R}^* - \alpha\hat{\Pi}^* + \alpha\hat{\Pi}_t - \hat{\Pi}_{t+1} \tag{40}$$

In a Ricardian regime, we also have the quantity theory equations,

$$\begin{aligned} \hat{\Pi}_t &= \hat{\mu} + \theta(\hat{k}_{t-1} - \hat{k}_t) \\ \hat{\Pi}_{t+1} &= \hat{\mu} + \theta(\hat{k}_t - \hat{k}_{t+1}) \end{aligned}$$

Combining these latter two expressions with (40) we get

$$(1-\theta)\hat{k}_t = \hat{\Sigma} + \theta\alpha(\hat{k}_{t-1} - \hat{k}_t) - \theta(\hat{k}_t - \hat{k}_{t+1})$$

where $\hat{\Sigma}$ is as we defined in the main body of the text. This expression can be rearranged into a second-order difference equation:

$$\hat{k}_{t+1} - (1/\theta + \alpha)\,\hat{k}_t + \alpha\hat{k}_{t-1} = -\hat{\Sigma}/\theta$$

This system is stable if and only if $|-(1/\theta+\alpha)| < 1+\alpha$ and $|\alpha| < 1$. Since $\theta \in (0,1)$ the first of these two requirements fails. Thus, any α delivers determinacy. The unique equilibrium converging to the steady state has constant investment and inflation: $\hat{k}_t = \hat{k} = \hat{\Sigma}/(1-\theta)$ and $\hat{\Pi}_{t+1} = \mu$ for all $t \geq 0$.

In a non-Ricardian regime, the quantity theory equation implies

$$\hat{\Pi}_{t+1} = \hat{\mu}_{t+1} + \theta(\hat{k}_t - \hat{k}_{t+1})$$

Assume zero deficit, $d_t = 0$. Then, the government budget constraint implies $\hat{\mu}_t = \hat{R}_{t-1}$ for all $t \geq 0$, which combined with (39) and the expression above yields

$$\hat{\Pi}_{t+1} = \hat{R}^* + \alpha(\hat{\Pi}_t - \hat{\Pi}^*) + \theta(\hat{k}_t - \hat{k}_{t+1})$$

Replacing $\hat{\Pi}_{t+1}$ in(40) we get

$$\hat{k}_t = 2\hat{\beta} + \hat{\theta} + \theta\hat{k}_{t+1}$$

which is identical to (35), which we derived for a forward-looking Taylor rule. Thus, there is real determinacy for all values of α. Given $\hat{k}_t = \hat{k}$ for all $t \geq 0$, inflation follows the following difference equation:

$$\hat{\Pi}_{t+1} = \hat{R}^* + \alpha(\hat{\Pi}_t - \hat{\Pi}^*)$$

In this case, nominal determinacy requires the Taylor principle, $\alpha > 1$. We obtain, $\Pi_{t+1} = \frac{\hat{R}^* - \alpha\hat{\Pi}^*}{1-\alpha}$ for all $t \geq 0$.