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Optimal Ramsey Capital Income Taxation —A Reappraisal*

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Abstract

This paper uses a tractable model to address a long-standing problem in the optimal Ramsey capital taxation literature. The tractability of our model enables us to solve the Ramsey problem analytically along the entire transitional path. We show that the conventional wisdom on Ramsey tax policy and its underlying intuition and rationales do not hold in our model and may thus be misrepresented in the literature. We uncover a critical trade off for the Ramsey planner between aggregate allocative efficiency in terms of the modified golden rule and individual allocative efficiency in terms of self-insurance. Facing the trade off, the Ramsey planner prefers issuing debt rather than taxing capital to correct the capital-over-accumulation problem. In particular, the planner always intends to supply enough bonds to relax individuals' borrowing constraints and through which to achieve the modified golden rule by crowding out capital. Public debt is financed by labor income tax as it is less distortionary than capital tax. Thus, capital taxation is not the optimal tool to achieve aggregate allocative efficiency despite over-accumulation of capital. Hence, the optimal capital tax can be zero, positive, or even negative, depending on the Ramsey planner's ability to issue debt. In a Ramsey equilibrium the modified golden rule can fail to hold whenever the government encounters a debt limit. Finally, the desire to relax individuals' borrowing constraints by the planner may lead to an increasing debt accumulation, resulting in a dynamic path featuring no steady state.

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1 Introduction

The seminal work of Aiyagari (1995) has inspired a large literature. However, despite several important revisits, such as Chamley (2001), Conesa, Kitao, and Krueger (2009), Dávila, Hong, Krusell, and Ríos-Rull (2012), and many others, the issue regarding optimal capital income taxation in the heterogeneous-agent incomplete-market economy remains unsettled.

One of the daunting challenges in unlocking the precise mechanisms behind many results in this literature is tractability and transparency. When models are not tractable, not only the Ramsey problem becomes difficult to solve, but the proofs also become highly non-transparent. Consequently, intuitions often get lost and dubious claims may be made.

For example, the competitive market equilibrium in the Aiyagari model (Aiyagari (1994)) may appear to be dynamically inefficient due to excessive accumulation of capital under precautionary saving motives, which results in excessively low marginal product of capital at the aggregate. Thus, the modified golden rule (MGR hereafter) is violated and leads to an aggregate inefficiency. This observation provides the key intuition of Aiyagari (1995) that the Ramsey planner should tax capital income to restore the aggregate efficiency — the MGR.

On the other hand, the above intuition for justifying positive capital income tax is counter-intuitive. By taxing capital income and thus reducing each individual's optimal buffer stock of savings, the planner is hampering and even destroying individuals' ability to self-insure themselves against idiosyncratic risks. Since taxing capital does not directly address the lack-of-insurance problem for households (if anything, it intensifies the borrowing constraint problem), why would taxing capital always be an optimal thing to do for the social planner? Or why would the MGR matter to individuals' welfare more than the need of self insurance? This question is particularly intriguing since lack of insurance is the only friction in the economy and hence the ultimate concern for households regarding idiosyncratic risks. In other words, achieving the MGR through capital taxation does not help at all to alleviate the primal friction in the model — borrowing constraints, thus using the MGR principle to justify positive capital income tax regardless of model parameters is not at all clear and convincing.

In short, two margins of inefficiency exist in the Aiyagari model: aggregate inefficiency in light of MGR, and individual inefficiency in light of borrowing constraints or insufficient self insurance. Clearly, the aggregate inefficiency is a consequence of the individual inefficiency (due to externalities of household savings on the rate of return to aggregate capital), suggesting that restoring individual efficiency is the correct way to achieve aggregate efficiency. Hence, intuition tells us a social planner can improve welfare more likely through addressing the individual inefficiency problem by relaxing borrowing constraints. Yet Aiyagari (1995) argues that the social planner should directly address the aggregate inefficiency problem by hampering individual efficiency, or by sacrificing individual's buffer stock of savings, regardless of model parameters. This conclusion seems dubious and puzzling, for it completely ignores the trade off between the two margins of inefficiency in terms of benefits and costs.

The goal of this paper is to investigate the trade off between the two types of inefficiency in a transparent manner. Our contributions are four-fold. First, we construct a tractable heterogeneous-agent incomplete-market model with closed-form solutions, in which the distributions of market allocations can be derived analytically. The analytical solution of the competitive equilibrium simplifies the Ramsey problem dramatically. As a result, the optimal Ramsey allocation can be derived analytically both along the transitional path and in the steady state. Armed with the closed-form solutions of the Ramsey problem, the mechanism and intuition of optimal capital taxation can be seen clearly.

Second, our model has the special property that given an exogenously specified interest rate identical to the time discount rate, household's asset demand may remain finite and in such a case the borrowing constraint never binds for any households. However, if the interest rate is endogenously determined by the marginal product of capital, then under sufficiently large idiosyncratic risks, the equilibrium interest rate is always below the time discount rate, suggesting over-accumulation of capital and failure of the MGR. However, since the government is able to manipulate the market interest rate through bond supply, we show whether the modified golden rule holds under the Ramsey planner depends critically on the government's ability to issue bond as an alternative store of value for households to buffer the idiosyncratic risk. In particular, the MGR holds if and only if the government can issue sufficient amount of bond to completely relax households' borrowing constraints. Once

borrowing constraints are completely relaxed with zero binding probability, the equilibrium interest rate equals the time discount rate. In such a case, the social (aggregate) efficiency is achieved, simply because the individual efficiency is fully restored. As a result, there is no need to tax capital. This result provides a strong rationale for the supply of public debt.

Third, we demonstrate that if the government has limited capacity to issue debt, then MGR does not hold under Ramsey because individual efficiency cannot be fully achieved. In such a case, the optimal tax rate on capital income can be either positive or negative, depending on model parameters. When the optimal capital tax rate is negative, the social planner is encouraging precautionary saving despite over-accumulation of capital. In this case individual efficiency outweighs aggregate efficiency for the Ramsey planner, so it is not optimal to target MGR by tightening household borrowing constraints with positive capital tax, in sharp contrast to Aiyagari's (1995) analysis. In other words, there exist situations where the Ramsey planner opts not to achieve the MGR even if he/she is fully capable of doing so by taxing capital income.

Finally, we show that there may not even exist a Ramsey steady state if the magnitude of the idiosyncratic shock is not bounded above and if there is no limit on issuing government bond. This finding suggests the possibility of having a non-steady-state Ramsey outcome. Yet the existence of a steady state is a common assumption in the extant literature. Our finding provides a cautionary note to such a dangerous assumption.

Aiyagari (1995) proves his theoretical results based on a nonstandard feature — endogenous government spending in household utilities, which equalizes all agents' marginal utilities of government spending. Aiyagari's analysis utilizes this nonstandard feature and exclusively focuses on the efficiency of aggregate resource allocation without worrying about income or wealth distributions. Yet the main motivation for considering such heterogeneous agent models is precisely that distributions matter for social welfare.¹ In fact, allowing endogenous government spending in our model (the same way as in Aiyagari (1994)) does not change our results if the distributional effects of government policies are taken into account.

The intuition behind our results is rather simple and transparent. Government bond

¹Wen (2015) and Camera and Chien (2014) emphasize the distributional impact of inflation on welfare in incomplete market economies.

meets individuals' demand for precautionary saving without creating externalities on the marginal product of capital. Hence, by substituting for (crowding out) capital, government debt can satisfy households' buffer stock saving motive and, at the same time, correct the aggregate inefficiency due to over-accumulation of capital. This is why the Ramsey planner opts to flood the asset market with sufficient amount of bonds to relax borrowing constraints for all households in all states. As a result, if there is no debt limit, the risk-free rate of bond can approach (and become equal) to household's time discount rate, thus achieving the first-best allocation.² However, when debt limits exist, the Ramsey planner is unable to issue enough bond to raise the rate of return to household saving to fully alleviate borrowing constraints; but, in spite of this, the planner will not necessarily opt to levy tax on capital simply to achieve the MGR. Instead, the benevolent government may subsidize capital to encourage even more savings to relax borrowing constraints. Thus, the optimal capital income tax rate can be negative or positive, depending on the burden of interest payments on government debts and the availability of labor income tax. In addition, with unbounded idiosyncratic shocks and without restrictions on the debt limit, the Ramsey planner may tempt to issue infinite amount of bonds to achieve the first best allocation, thus destroying the Ramsey steady state commonly assumed in the existing literature.

2 A Brief Literature Review

The literature related to the optimal capital taxation is vast. Here we review only the most relevant papers in the incomplete market literature.

The work of Aiyagari (1994) is the first attempt at investigating optimal Ramsey taxation in incomplete-market heterogeneous-agent economies. With the assumption of endogenous government spending that equalizes all agents' marginal utilities, Aiyagari shows that the MGR should hold in the Ramsey steady state (if it indeed exists). On the other hand, since the risk free rate is below the time preference rate under precautionary saving motives, he argues that a positive capital tax should be levied by the Ramsey planner to correct the

²In this paper, we define the first-best allocation as a situation where both the aggregate efficiency and individual efficiency are achieved; namely, the MGR holds and borrowing constraints do not bind for all households in all states.

problem of over-accumulation of capital.

Our paper suggests otherwise. The distributional effects on welfare from government policies are specifically spelled out in our analysis. The Ramsey planner faces a non-trivial trade off between aggregate efficiency and individual efficiency, which is ignored entirely or assumed away by Aiyagari's analysis. Hence, common perceptions or intuitions derived from Aiyagari's theory are no longer valid in our setup, as discussed in the Introduction of this paper.

An important recent paper by Gottardi, Kajii, and Nakajima (2015) revisits optimal Ramsey taxation in an incomplete market model with uninsurable human capital risk. As in our model, tractability in their model enables them to provide transparent analysis on Ramsey taxation and facilitates intuitive interpretations for their results. When government spending and bond supply are both set to zero, they find that the Ramsey planner should tax human capital and subsidize physical capital, despite the over-accumulation of physical capital. The purpose or the benefit of taxing human capital here is to reduce uninsurable risk from human capital returns; and the rationale for subsidizing physical capital despite over-accumulation is to satisfy households' demand for a buffer stock, similar to our finding but in contrast to Aiyagari (1995)'s results. However, they solve the Ramsey problem indirectly and can characterize analytically the properties of optimal taxes only in a neighborhood of zero government bond and zero government spending.

In contrast, we can solve the Ramsey problem analytically and directly along the entire dynamic path of the model, which permits transparent examinations on how the Ramsey planner takes into account the impact of his policies on the dynamic distributions of household resources and aggregate efficiency. Our model also enables us to show analytically the exact roles played by government debt and how such roles are hindered by debt limits.

Aiyagari and McGrattan (1998) study optimal government debt in the Aiyagari model. Similar to our finding, government bond is shown to play an important role in providing liquidity for households and help to relax their borrowing constraints. However, they restrict their analysis to the special case of same tax rates across capital and labor incomes and analyze welfare only in the steady state. In an overlapping generations model with uninsured individual risk, Conesa, Kitao, and Krueger (2009) conduct a numerical exercise to derive

optimal capital tax and non-linear labor tax. As in Aiyagari and McGrattan (1998), they only consider welfare in the steady state, the transitional path is therefore ignored.

However, Domeij and Heathcote (2004) show that welfare along the transitional path is an important concern for the Ramsey planner. Their findings indicate that steady-state welfare maximization could be misleading when designing optimal policies. But, instead of solving optimal tax policies, they numerically evaluate the welfare consequence of tax changes.

Two recent works by Acikgoz (2013) and Dyrda and Pedroni (2015) numerically solve optimal fiscal policies along the transitional path in an Aiyagari-type economy. In contrast to our findings, their results are consistent with Aiyagari's analysis. The sources of difference between their results and ours could be the implicit assumptions of unbounded debt limits and the existence of a steady state in their numerical analyses. Such assumptions may not be well justified in light of our studies. In fact, Straub and Werning (2014) also argue that the assumption commonly made in the numerical literature that the endogenous multipliers associated with the Ramsey problem converge to a steady state may be incorrect and could be misleading. Our model demonstrates exactly such a possibility. Specifically, under certain parameter values the endogenous multipliers in our model do not necessary converge over time.

Instead of using the Ramsey approach, Dávila, Hong, Krusell, and Ríos-Rull (2012) characterize constrained efficient allocations in an Aiyagari-type economy where the government can levy individual specific labor tax, which is not allowed in the Ramsey framework. They found that in competitive equilibrium the capital stock could be too high or too low compared to the constraint efficient allocation, thus optimal capital income tax rate can be either positive or negative.

Finally, Park (2014) considers Ramsey taxation in a complete-market environment featuring enforcement constraints, a la Kehoe and Levine (1993). She shows that capital accumulation improves the outside option of default, which is not internalized by household decisions. Therefore, capital income should be taxed in order to internalize such an adverse externality.

3 The Model

This section introduces a heterogeneous-agent model with incomplete markets, following Bewley (1980), Lucas (1980), Huggett (1993), Aiyagari (1994), and especially Wen (2009, 2015). An important feature of the model is its analytical tractability with closed-form solutions, which enables solving optimal Ramsey allocations analytically and directly over infinite horizon at any point in time.

3.1 Environment

A representative firm produces output according to the constant-returns-to-scale Cobb-Douglas technology, $Y_t = F(K_t, N_t) = K_t^\alpha N_t^{1-\alpha}$, where Y , K and L denote aggregate output, capital and labor, respectively. The firm rents capital and hires labor from households by paying competitive rental rate and real wage, denoted by q_t and w_t , respectively. The firm's optimal conditions for profit maximization at time t satisfy

$$w_t = \frac{\partial F(K_t, N_t)}{\partial N_t}, \tag{1}$$

$$q_t = \frac{\partial F(K_t, N_t)}{\partial K_t}. \tag{2}$$

There is a unit measure of *ex ante* identical households that face idiosyncratic preference shocks, denoted by θ . The shock is identically and independently distributed over time and across households, and has the mean $\bar{\theta}$ and the cumulative distribution $\mathbf{F}(\theta)$ with support $[\theta_L, \theta_H]$, where $\theta_H > \theta_L > 0$.

Time is discrete and indexed by $t = 0, 1, 2, \dots$. There are two sub-periods in each period t . The idiosyncratic preference shock θ_t is realized only in the second sub-period, and labor supply decision must be made in the first sub-period before observing θ_t . Namely, the idiosyncratic preference shock is uninsurable despite linearity in leisure cost. Let $\theta^t \equiv (\theta_1, \dots, \theta_t)$ denote the history of shocks. Period 0 is the planning period and there is no shock in period 0. All households are endowed with the same asset holdings at time 0.

Households are infinitely-lived with quasi-linear utility function and face borrowing con-

straints. Their lifetime expected utility is given by

$$V = E_0 \sum_{t=1}^{\infty} \beta^t [\theta_t \log(c_t(\theta^t)) - n_t(\theta^{t-1})], \quad (3)$$

where $\beta \in (0, 1)$ is the discount factor, $c_t(\theta^t)$ and $n_t(\theta^{t-1})$ denote consumption and labor supply for a household with history θ^t at time t . Note that labor supply in period t is only measurable with respect to θ^{t-1} , reflecting the assumption that labor supply decision is made in the first sub-period before observing the preference shock θ_t .

The government needs to finance an exogenous stream of purchases $\{G_t\}_{t=1}^{\infty}$, and it can issue bond and levy flat-rate, time-varying labor and capital taxes at rates $\tau_{n,t}$ and $\tau_{k,t}$, respectively. The flow government budget constraint in period t is

$$\tau_{n,t} w_t N_t + \tau_{k,t} q_t K_t + B_{t+1} \geq G_t + r_t B_t, \quad (4)$$

where B_{t+1} is the level of government bond chosen in period t and r_t is the gross risk free rate.

3.2 Household Problem

We assume there is no aggregate uncertainty and that government bond and capital are perfect substitute as stores of value for households. As a result, the after-tax gross rate of return to capital must equal the gross risk free rate: $1 + (1 - \tau_{k,t})q_t - \delta = r_t$, which constitutes a no-arbitrage condition for capital and bond.

Given the sequence of interest rates $\{r_t\}_{t=1}^{\infty}$, and after tax wage rates, $\{\bar{w}_t \equiv (1 - \tau_{n,t})w_t\}_{t=1}^{\infty}$, a household chooses a plan of consumption, labor and asset holdings $\{c_t(\theta^t), n_t(\theta^{t-1}), a_{t+1}(\theta^t)\}$ to solve

$$\max_{\{c_t(\theta^t), n_t(\theta^{t-1}), a_{t+1}(\theta^t)\}} E_0 \sum_{t=1}^{\infty} \sum_{\theta^t} \beta^t \{ \theta_t \log c_t(\theta^t) - n_t(\theta^{t-1}) \} \quad (5)$$

subject to

$$c_t(\theta^t) + a_{t+1}(\theta^t) \leq \bar{w}_t n_t(\theta^{t-1}) + r_t a_t(\theta^{t-1}), \quad (6)$$

$$a_{t+1}(\theta^t) \geq 0, \quad (7)$$

with $a_1 > 0$ given and $n_t(\theta^{t-1}) \in [0, \bar{N}]$. The solution of the household problem can be characterized analytically by the following proposition.

Proposition 1. *Denoting household gross income (or total liquidity in hand) by $x_t(\theta^{t-1}) \equiv r_t a_t(\theta^{t-1}) + \bar{w}_t n_t(\theta^{t-1})$, the optimal decisions for $x_t(\theta^{t-1})$, consumption $c_t(\theta^t)$, savings $a_{t+1}(\theta^t)$, and labor supply $n_t(\theta^{t-1})$ are given, respectively, by the following cutoff-policy rules:*

$$x_t = \bar{w}_t R(\theta_t^*) \theta_t^* \quad (8)$$

$$c_t(\theta_t) = \min \left\{ 1, \frac{\theta_t}{\theta_t^*} \right\} x_t \quad (9)$$

$$a_{t+1}(\theta_t) = \max \left\{ \frac{\theta_t^* - \theta_t}{\theta_t^*}, 0 \right\} x_t \quad (10)$$

$$n_t(\theta_{t-1}) = \frac{1}{\bar{w}_t} [x_t - r_t a_t(\theta_{t-1})], \quad (11)$$

where the cutoff θ_t^* is independent of individual history and determined by the following Euler equation,

$$\frac{1}{\bar{w}_t} = \beta \frac{r_{t+1}}{\bar{w}_{t+1}} R(\theta_t^*), \quad (12)$$

and the function $R(\theta_t^*)$ denotes

$$R(\theta_t^*) \equiv \int_{\theta \leq \theta_t^*} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) \geq 1. \quad (13)$$

Proof. See Appendix A.1 □

Notice that individual's consumption function in equation (9) is concave in gross income or liquidity in hand (as noted by Deaton (1991)), and the saving function in equation (10) exhibits a buffer stock behavior. When the urge to consume is low ($\theta_t < \theta_t^*$), the individual opts to consume only $\frac{\theta_t}{\theta_t^*} < 1$ fraction of total income and save the rest, anticipating that future consumption demand may be high. On the other hand, when the urge to consume is high ($\theta_t \geq \theta_t^*$), the agent opts to consume all gross income, up to the limit where the borrowing constraint binds, so the saving stock is reduced to zero. The function $R(\theta_t^*) \geq 1$ reflects the extra rate of return to saving due to the option value (liquidity premium) of the

buffer stock.

Denote $\Lambda_t \equiv \frac{1}{\bar{w}_t}$ as the expected marginal utility of income. Then the left-hand side of equation (12) is the average marginal cost of consumption in the current period, and the right-hand side is the discounted expected next-period return to saving (augmented by r_{t+1}), which takes two possible values in light of the two components in equation (13) for the liquidity premium: The first is simply the discounted next-period marginal utility of consumption Λ_{t+1} in the case that borrowing constraint does not bind, which has probability $\int_{\theta \leq \theta_t^*} d\mathbf{F}(\theta)$. The second is the discounted marginal utility of consumption $\Lambda_{t+1} \frac{\theta_t}{\theta_t^*}$ in the case of high demand ($\theta_t > \theta_t^*$) with a binding borrowing constraint, which has probability $\int_{\theta > \theta_t^*} d\mathbf{F}(\theta)$. When the borrowing constraint binds, additional saving can yield a higher shadow marginal utility $\frac{\theta_{t+1}}{\theta_{t+1}^*} \Lambda_{t+1} > \Lambda_{t+1}$. The optimal cutoff θ_t^* is then determined at the point where the marginal cost of saving equals the expected marginal gains. Here, savings play the role of a buffer stock and the rate of return to saving is determined by the real interest rate r_t compounded by a liquidity premium $R(\theta_t^*)$, rather than just by r_t as in a representative-agent model without borrowing constraint. Notice that $\frac{\partial R}{\partial \theta^*} < 0$, and $R(\theta^*) > 1$ as long as $\theta^* < \theta_H$.

Equation (12) also suggests that the cutoff θ_t is independent of individual history. This property holds in this model because of the quasi-linear utility function and the assumption that labor supply is determined in the first sub-period. In other words, the optimal level of liquidity in hand in period t is determined by a “target” income level given by $x_t = \theta_t^* \bar{w}_t R(\theta_t^*)$, which is also independent of the history of realized values of θ_t but depends only on the distribution of θ_t . This target is essentially the optimal consumption level when the borrowing constraint binds. This target policy (uniform to all households) obtains because labor income ($\bar{w}_t n_t(\theta_{t-1})$) can be adjusted elastically to meet an optimal target, given (and regardless of) the initial asset holdings $a_t(\theta_{t-1})$.³ Hence, in the beginning of each period all households will choose the same level of gross income. Thus, the history-independent cutoff variable θ_t^* uniquely and fully characterizes the distributions of the economy.

³It is shown in Appendix A.1 that with reasonable parameter values, hours worked are bounded in the open interval $(0, \bar{N})$ as long as \bar{N} is sufficiently large.

3.3 Competitive Equilibrium

Denote C_t , N_t and K_{t+1} as the level of aggregate consumption, aggregate labor and aggregate capital, respectively. The competitive equilibrium allocation can be defined as follows:

Definition 1. *Given initial aggregate capital K_1 and bond B_1 , and a sequence of taxes, government spending and government bond, $\{\tau_{n,t}, \tau_{k,t}, G_t, B_{t+1}\}$, a competitive equilibrium is a sequence of prices $\{w_t, q_t\}$, allocations $\{c_t(\theta^t), n_t(\theta^{t-1}), a_{t+1}(\theta^t), K_{t+1}, N_t\}$, and distribution statistic θ_t^* such that*

1. *given the sequence of $\{w_t, q_t, \tau_{n,t}, \tau_{k,t}\}$, the sequences $\{c_t(\theta^t), a_{t+1}(\theta^t), n_t(\theta^{t-1})\}$ solve the household problem;*
2. *given the sequence of $\{w_t, q_t\}$, the sequences $\{N_t, K_t\}$ solve the firm's problem;*
3. *the no-arbitrage condition holds for each period: $r_t = 1 + (1 - \tau_{k,t})q_t - \delta$;*
4. *government budget constraint in equation (4) holds for each period;*
5. *all markets clear for all t :*

$$K_{t+1} = \int a_{t+1}(\theta_t) d\mathbf{F}(\theta_t) - B_{t+1} \quad (14)$$

$$N_t = \int n_t(\theta_{t-1}) d\mathbf{F}(\theta_{t-1}) \quad (15)$$

$$\int c_t(\theta_t) d\mathbf{F}(\theta_t) + G_t \leq F(K_t, N_t) + (1 - \delta)K_t - K_{t+1}. \quad (16)$$

Proposition 2. *In the Laissez-faire competitive equilibrium, the steady-state risk free rate is lower than the time discount rate, $r < 1/\beta$, and there exists over-accumulation of capital with a positive liquidity premium, $R(\theta^*) > 1$, if the size of the idiosyncratic shock (the upper bound θ_H) is sufficiently large such that the following condition holds:*

$$\frac{\alpha\beta}{1 - \frac{\theta}{\theta_H}} + \beta(1 - \alpha)(1 - \delta) + \alpha\beta g_k < 1, \quad (17)$$

where g_k is defined as the steady-state value of G_t/K_t .

Proof. See Appendix A.2 □

Notice that when $\theta_H \rightarrow \infty$, as in the case of Pareto distribution, the above condition is clearly satisfied if g_k is small enough. The intuition of Proposition 2 is straightforward. Since labor income is determined (ex anti) before the realization of the idiosyncratic preference shock θ_t , household's total income may be insufficient to provide full insurance for large enough preference shocks under condition (17). In this case, precautionary saving motives lead to over-accumulation of capital, which reduces the equilibrium interest rate below the time discount rate. This outcome is (constrained-) inefficient. It emerges because of the negative externalities of household savings on aggregate interest rate (due to diminishing marginal product of capital), as noted by Aiyagari (1994).

However, unlike the Aiyagari (1994) model, a competitive equilibrium can be efficient in our model if the idiosyncratic risk is sufficiently small (e.g., the upper bound θ_H is close enough to the mean $\bar{\theta}$ such that condition (17) is violated). In this case, households' borrowing constraints will never bind and individual savings are sufficiently large to fully buffer preference shocks. Clearly, under full self-insurance, it must be true that $\theta^* = \theta_H$, $R(\theta^*) = 1$, and $r = 1/\beta$.

Full self-insurance is impossible in the Aiyagari (1994) model because every individual household's marginal utility of consumption (or the shadow price of consumption goods) follows a supermartingale when $r = 1/\beta$. This implies that household's savings (or asset demand) can diverge to infinity in the long run, which cannot constitute an equilibrium.⁴ In our model, however, because the household's utility function is quasi-linear, the expected shadow price of consumption goods is thus the same across agents and given by $\frac{1}{\bar{w}_t}$ (as revealed by equations (37) and (39) in Proof of Proposition 1 in the Appendix), which kills the supermartingale property of household's marginal utility of consumption. As a result, household savings (or asset demand) are bounded away from infinity even at the point $r = 1/\beta$. More specifically, equations (8) and (10) show that individual's asset demand is always bounded above by $(\theta_H - \theta_t)\bar{w}_t$ for any shock θ_t when $r = 1/\beta$. This upper bound is finite as long as the support of θ_t is bounded. This special property renders our model

⁴Please refer to Ljungqvist and Sargent (2012, Chapter 17) for details.

analytically tractable with closed-form solutions, and it implies that the Ramsey planner has the potential to use government debt to achieve individual allocative efficiency in this economy.

Nonetheless, the trade-off mechanism uncovered in this paper does not hinge on this special property of our model. The trade off between individual allocative efficiency (pertaining to self-insurance) and aggregate allocative efficiency (pertaining to MGR) is driven entirely by capital taxation under precautionary saving motives. Because capital tax discourages households to save, it mitigates the over-accumulation of capital but at the same time tightens individuals' borrowing constraints. Hence, such a trade off should exist in any incomplete-market heterogeneous-agent models with capital accumulation and endogenously determined interest rate. We show that government debt is an ideal tool to address this trade-off problem, as implicitly revealed in the model of Gottardi, Kajii, and Nakajima (2015), but made clear in this paper.

3.4 Conditions to Support a Competitive Equilibrium

Since government policies are in the aggregate state space of the competitive equilibrium and hence affect the distributions (such as the average) of all endogenous economic variables, the Ramsey problem is to pick a competitive equilibrium through policies that attains the maximum of the expected household lifetime utility V defined in (3). Note that V depends on the distributions (see below and Wen (2015)).

This subsection expresses the necessary conditions, in terms of the aggregate variables and distributions characterized by the cutoff θ_t^* , that the Ramsey planner must respect in order to construct a competitive equilibrium. We first show that the individual allocations and prices in the competitive equilibrium can be expressed as a function of the aggregate variables and the cutoff θ_t^* . The idea of such an expression is straightforward since the cutoff θ_t^* is a sufficient statistics for describing the distribution of individual variables. To facilitate the expression, we first show the properties of aggregate consumption (or average consumption across households). By aggregating the individual consumption decision rules

(9), the aggregate consumption is determined by

$$C_t = D(\theta_t^*)x_t, \quad (18)$$

where the aggregate marginal propensity of consumption (the function D) is given by

$$D(\theta_t^*) \equiv \int_{\theta \leq \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} d\mathbf{F}(\theta) > 0. \quad (19)$$

Then, we can express the individual consumption and individual asset holding as a functions of C_t and θ_t^* by plugging equation (18) back to equations (9) and (10). To fully describe the conditions necessary for constructing a competitive equilibrium, we rely on the following proposition:

Proposition 3. *Given initial capital K_1 , initial government bond B_1 , and initial capital tax rate $\{\tau_{k,1}\}$, the sequences of aggregate allocations $\{C_t, N_t, K_{t+1}, B_{t+1}\}$, and a sequence of distribution statistics $\{\theta_t^*\}$ (with the associated R and D functions defined in equation (13) and (19), respectively) can be supported as a competitive equilibrium if and only if the resource constraint (16), asset market clearing condition*

$$B_{t+1} = \left(\frac{1}{D(\theta_t^*)} - 1 \right) C_t - K_{t+1}, \quad (20)$$

and the following implementability condition holds:

$$R(\theta_t^*)\theta_t^* \geq N_t + \frac{\theta_{t-1}^*}{\beta} (1 - D(\theta_{t-1}^*)). \quad (21)$$

Proof. See Appendix A.3 □

Note that the implementability condition essentially enforces the flow government budget constraint. This proposition demonstrates that the Ramsey planner can construct a competitive equilibrium by choosing sequences of aggregate allocations $\{C_t, N_t, K_{t+1}, B_{t+1}\}$ as well as a sequence of distribution statistics $\{\theta_t^*\}$ subject to the aggregate resource constraint, asset market clearing condition and implementability condition.

4 Optimal Ramsey Allocation

Armed with Proposition 3, we are ready to write down the Ramsey planner's problem.

4.1 Ramsey Problem

We first rewrite the lifetime utility function, V , as a function of aggregate variables and θ_t^* by utilizing equations (9) and (18):

$$V = \sum_{t=1}^{\infty} \beta^t [W(\theta_t^*) + \bar{\theta} \log C_t - N_t], \quad (22)$$

where $W(\theta_t^*)$ is defined as

$$W(\theta_t^*) \equiv \bar{\theta} \log \frac{1}{D(\theta_t^*)} + \int_{\theta \leq \theta_t^*} \theta \log \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta). \quad (23)$$

Based on Proposition 3, the Ramsey problem can be represented as maximizing the lifetime utility (22) by choosing the sequences of $\{\theta_t^*, N_t, C_t, K_{t+1}, B_{t+1}\}$ subject to the resource constraint (16), the asset market clearing condition (20), and the implementability condition (21). In addition, a debt limit $B_{t+1} \leq \bar{B}$ is imposed on the Ramsey planner to facilitate our analysis on the role of government debt, which most of the existing literature has ignored and assumed away by implicitly letting $\bar{B} = \infty$.

Therefore, the Lagrangian of the Ramsey problem is given by

$$\begin{aligned} L = & \max_{\{\theta_t^*, N_t, C_t, K_{t+1}, B_{t+1}\}} \sum_{t=0}^{\infty} \beta^t [W(\theta_t^*) + \bar{\theta} \log C_t - N_t] \\ & + \beta^t \mu_t (F(K_t, N_t) + (1 - \delta)K_t - G_t - C_t - K_{t+1}) \\ & + \beta^t \lambda_t \left(R(\theta_t^*)\theta_t^* - N_t - \frac{\theta_{t-1}^*}{\beta} (1 - D(\theta_{t-1}^*)) \right) \\ & + \beta^t \phi_t \left(K_{t+1} + B_{t+1} - \left(\frac{1}{D(\theta_t^*)} - 1 \right) C_t \right) \\ & + \beta^t \nu_t^B (\bar{B} - B_{t+1}) \end{aligned} \quad (24)$$

where μ_t , λ_t and ϕ_t denote the multipliers for the resource constraints, the implementability

conditions and the asset market clearing conditions, respectively. In addition, the multiplier of the bond boundary condition is denoted by ν_t^B .

The first-order conditions with respect to $K_{t+1}, N_t, C_t, B_{t+1}$ and θ_t^* for $t \geq 1$ yield, respectively

$$\mu_t - \phi_t = \beta\mu_{t+1}(MP_{K,t+1} + 1 - \delta) \quad (25)$$

$$1 + \lambda_t = \mu_t MP_{N,t} \quad (26)$$

$$\mu_t = \frac{\bar{\theta}}{C_t} - \phi_t \left(\frac{1}{D(\theta_t^*)} - 1 \right) \quad (27)$$

$$\beta^t \phi_t - \beta^t \nu_t^B = 0 \quad (28)$$

$$\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} + \lambda_t H(\theta_t^*) - \lambda_{t+1} J(\theta_t^*) + \phi_t \frac{C_t}{D(\theta_t^*)^2} \frac{\partial D(\theta_t^*)}{\partial \theta_t^*} = 0 \quad (29)$$

where

$$H(\theta_t^*) \equiv \left(R(\theta_t^*) + \frac{\partial R(\theta_t^*)}{\partial \theta_t^*} \theta_t^* \right) \quad (30)$$

$$J(\theta_t^*) \equiv \left(1 - D(\theta_t^*) - \theta_t^* \frac{\partial D(\theta_t^*)}{\partial \theta_t^*} \right) \quad (31)$$

The following three lemma are useful to characterize the optimal Ramsey allocation:

Lemma 1. $H(\theta_t^*) = J(\theta_t^*) = \mathbf{F}(\theta_t^*)$

Proof. Refer to Appendix A.4 □

Lemma 2. $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} > 0$ for all $\theta_t^* \in (\theta_L, \theta_H)$ and $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} = 0$ if $\theta_t^* = \theta_L$ or θ_H

Proof. Refer to Appendix A.5 □

Lemma 3. $\frac{\partial D(\theta_t^*)}{\partial \theta_t^*} < 0$ for all $\theta_t^* \in (\theta_L, \theta_H)$

Proof. Refer to Appendix A.6 □

4.2 Characterization of Long-term Optimal Ramsey Allocation

We consider and discuss three possible cases of long term Ramsey allocation below. In all cases, the condition (17) is assumed to hold; namely, the competitive equilibrium is inefficient. The first case characterizes the first-best allocation, which we define as follows:

Definition 2. *In our model economy, the first-best allocation is defined as a situation where both the aggregate efficiency and individual efficiency are achieved; namely, the MGR holds and the borrowing constraints do not bind for all households in all states.*

4.2.1 Case 1: First-best Steady State

Proposition 4. *Suppose $\theta_H < \infty$ and \bar{B} is sufficiently large such that the debt limit constraint $B \leq \bar{B}$ does not bind. Then there exists a first-best steady-state Ramsey allocation, which features the following properties:*

1. *Individual efficiency is achieved —the optimal choice of θ_t^* is a corner solution at $\theta_t^* = \theta_H$ so that no households face a positive probability of binding borrowing constraints.*
2. *Aggregate efficiency is achieved —the MGR holds and there is no liquidity premium ($R = 1$). The equilibrium interest rate equals $1/\beta$.*
3. *Capital tax is zero and labor tax is positive at rate $\frac{\lambda}{1+\lambda} < 1$. The government expenditures as well as bond interest payments are financed solely by revenues from labor income tax.*

Proof. See Appendix A.7 □

The above proposition states that if the support of θ is bounded and the debt limit for government bond does not bind, then the Ramsey planner can pick an competitive equilibrium that achieves both aggregate efficiency and individual efficiency — the first-best allocation under our definition. More importantly, the Ramsey planner achieves the MGR without the need to tax capital — the buffer stock of households, as capital taxation would undermine individual efficiency by discouraging households' marginal propensity to save. Instead, the Ramsey planner provides enough incentives for households to save by picking a sufficiently high level of interest rate on government bond such that all households are fully self-insured with zero probability of a binding liquidity constraint.

Obviously, the first-best allocation can be archived only if the Ramsey planner is capable of supplying enough bonds to satisfy the liquidity demand by each household across all states.

To shed light on this issue further, we study what happens if the government's ability to issue bond is limited.

4.2.2 Case 2: Steady State with a Binding Debt Limit

Now consider the case in which the constraint $B \leq \bar{B}$ binds.

Proposition 5. *There exists a steady-state Ramsey allocation with the following properties:*

1. *No individual efficiency — the optimal choice of θ_t^* is interior and there is always a non-zero fraction of households facing binding borrowing constraints in every period.*
2. *No aggregate efficiency — the modified golden rule does not hold and there is a positive liquidity premium ($R > 1$). The equilibrium interest rate is less than $1/\beta$.*
3. *Capital tax rate could be positive, zero, or negative, depending on the tightness of the constraint on government debt.*

Proof. See Appendix A.8 □

Obviously, a special sub-case is when government cannot issue bonds, $\bar{B} = 0$. This sub-case is analogous to the situation discussed in Proposition 2, where the equilibrium interest rate is strictly less than the time discount rate. In such a situation, the Ramsey planner cannot use government bond to manipulate the market interest rate and divert household savings away from capital formation, it hence faces a trade off between achieving the MGR by taxing capital (and thus sacrificing individual efficiency) and achieving full self-insurance by subsidizing capital (and thus sacrificing the MGR).

In general, when the government is unable to supply enough bonds to meet the households' demand for buffer-stock savings, either because θ_H is sufficiently large or the debt limit \bar{B} is sufficiently small, the pursuit of individual efficiency by the Ramsey planner will necessarily lead to a binding constraint on government debt. In this case, there is a steady state where neither individual efficiency nor social efficiency is achieved.

More specifically, the result of this case shows that if the individual efficiency cannot be achieved, then the aggregate efficiency must also be sacrificed. This point can be seen clearly

by the set of first order conditions by the Ramsey planner. With a binding constraint on the debt limit, the Ramsey planner can no longer meet the household need for precautionary savings by supplying enough bonds. Hence, the planner has to face the trade-off between aggregate efficiency and individual efficiency. Once the debt limit binds, the planner could try to push for aggregate efficiency by taxing capital and thus deteriorating individual efficiency, or push for individual efficiency by subsidizing capital and further deteriorating aggregate efficiency, or leaving both margins unsatisfied. This trade-off shows up in the Ramsey optimal conditions (25) and (29), which represent aggregate and individual efficiency conditions, respectively. The strictly binding constraint on debt limit implies that the multiplier $\nu_t^B > 0$, which in turn implies a positive multiplier ϕ . The positive multiplier ϕ distorts both individual and social efficiency conditions.

This trade-off suggests that it is not optimal to tax capital income simply to achieve the aggregate efficiency, as Aiyagari would argue. In other words, the modified golden rule is not optimal in this case, and the capital tax could be either positive, zero or negative, depending on the model parameters.⁵ Intuitively, if the over-accumulation problem is not as severe as the self-insurance problem, the social planner may subsidize capital income to encourage more saving; otherwise, the social planner may tax capital to discourage saving. As shown in the proof A.8, the sign of $\tau_{k,t+1}$ depends on the value of the liquidity premium $R(\theta_t^*)$ and the ratio $\frac{\mu_{t+1}}{\mu_t - \phi_t}$, both are related to the tightness of the borrowing constraint and the debt limit constraint. This is in sharp contrast to Aiyagari's analysis where he argues that it is always optimal to tax capital as long as there is over-accumulation of capital. We show here that it is not the case because individual efficiency matters to the social planner and maybe more so than aggregate efficiency — as it is the root problem of aggregate efficiency. Hence, the tightness of the household borrowing constraints and government borrowing constraint matter for Ramsey planner, but such issues (especially the debt limit constraint) are commonly ignored and not studied in the literature.

⁵The sign of capital tax should depend on the trade off between individual efficiency and aggregate efficiency, or the elasticities of individual inefficient and aggregate inefficiency with respect to changes in the risk free interest rate (which relates to the after-tax capital return in equilibrium). These elasticities in turn depend on the utility function, production technology, and the density distribution function of idiosyncratic shocks.

Aiyagari (1994, 1995) argue that the equilibrium interest rate has to be lower than $1/\beta$. Otherwise, individual's asset demand goes to infinity, which cannot be a competitive equilibrium. But how close to $1/\beta$ the interest rate can be is never studied in the literature. In our model, however, household's asset demand remains finite even if the interest rate equals $1/\beta$, provided that the upper bound θ_H is finite. This property of our model makes individual allocative efficiency feasible (achievable) since household's marginal utility of consumption does not follow a supermartingale process when $r = 1/\beta$, thus a finite stock of precautionary savings may be sufficient for households to buffer idiosyncratic shocks without being borrowing-constrained. As aforementioned, this finitely-valued asset demand property at $r = 1/\beta$ has to do with an important feature of our model that the expected shadow price of consumption goods is the same across households under quasi-linear preferences. Nonetheless, household's asset demand can be unbounded in our model at $r = 1/\beta$ if preference shocks are unbounded. This situation would be similar to that in the Aiyagari model.

To shed light on the situation with infinite demand for government bond at $r = 1/\beta$ in the original Aiyagari model, we can simply let $\theta_H \rightarrow \infty$ in our model. As θ_H goes to infinity, the asset demand under $r = 1/\beta$ must also be infinite in order to achieve full self-insurance, as shown in the asset market clear condition below, which pins down the optimal level of aggregate bond demand B_{t+1} (see the Appendix A.7):

$$B_{t+1} = \left(\frac{\theta_H}{\bar{\theta}} - 1 \right) C_t - K_{t+1}, \quad (32)$$

which goes to infinity as θ_H goes to infinity under finite values of aggregate consumption and capital stock.

Next we discuss the Ramsey allocation under the conditions $\theta_H = \infty$ and $\bar{B} = \infty$.

4.2.3 Case 3: Possibility of No Ramsey Steady State

We define two concepts of steady state. A finitely-valued steady state and an infinitely-valued steady state. The former is the case where all variables in the model take finite values in the long run. Otherwise, if any variable in the model goes to infinity in the long run, we call it an infinitely-valued steady state. Based on these definitions, we have the following

Proposition 6. *There may not exist a Ramsey steady state if $\theta_H = \infty$ and $\bar{B} = \infty$.*

Proof. See Appendix A.9 □

In the Appendix A.9, we first prove that there does not exist any finitely-valued Ramsey steady state, and then we show that an infinitely-valued Ramsey steady state may also not exist.

The Ramsey planner may choose a long-term allocation featuring no steady state because of the strong motive to pursue individual efficiency, which is infeasible when $\theta_H = \infty$. Without debt limits, the Ramsey planner may attempt to issue an ever increasing amount of government debt to archive the allocation of individual efficiency, as shown in equation (29). Given that θ_H is infinite, the only path to approach the individual allocative efficiency is through supplying an ever-increasing amount of government bond. However, an increasing amount of bond also makes the government budget constraint harder and harder to satisfy because of the increasing burden of interest payment. This implies an increasing value of the multiplier on the implementability condition, λ_t . In addition, the increasing bond position also makes the resource constraint tighter and tighter for the Ramsey planner over time, so that the multiplier on resource constraint μ_t is also increasing over time. The everlasting endeavor of relaxing borrowing constraints may induce the Ramsey planner to play a Ponzi game. Also, as in the previous case, there is no clear direction on the sign of capital tax in the long run. It all depends on the trade off between individual and aggregate allocative efficiencies.

This possibility of no steady state provides a warning for the literature that uses numerical methods to compute Ramsey taxation, which often assumes the existence of a steady state and unlimited capacity to issue government debts.

5 Endogenous Government Spending

In this subsection we allow endogenous government spending in our model, as in Aiyagari (1995). With the assumption of endogenous government spending in household utilities, Aiyagari (1995) argues that the Ramsey planner cares only about MGR (aggregate effi-

ciency) in the steady state, and not about households' borrowing constraints and their lack of self-insurance, despite the fact that capital income tax can tighten households' borrowing constraints and severely affect the distributions of household income and wealth.

We show here that even with endogenous government spending, our previous results remain robust and unchanged. In particular, the introduction of endogenous government spending in household utilities does not eliminate the trade off problem, or automatically ensure the existence of finitely valued steady state under Ramsey planning. Namely, the Ramsey planner still cares about the trade off between individual efficiency (the tightness of household borrowing constraints) and aggregate efficiency (the MGR), despite the fact that the planner can effectively equalize all individuals' marginal utilities through endogenous government expenditures.

Following Aiyagari (1995), the household preference is modified into

$$E_0 \sum_{t=1}^{\infty} \beta^t [\theta_t \log(c_t(\theta^t)) - n_t (\theta^{t-1}) + U(G_t)], \quad (33)$$

where $U(G)$ is the utility function of government spending. The introduction of G_t into the household utility does not alter the household problem since they take the sequence of G_t as given. Hence, the definition of competitive equilibrium remains the same. However, the Ramsey problem changes slightly since G_t is now an endogenous choice variable for the government. The construction of the Ramsey problem follows the proof of Proposition 3 in exactly the same way except with one additional first-order condition with respect to $G_t \geq 0$, which is chosen by respecting the aggregate resource constraint. A non-negative G_t could be easily ensured by the assumption of $U'(0) = \infty$, which is common in the literature.

Therefore, the Lagrangian of the Ramsey problem is modified only slightly:

$$\begin{aligned}
L = & \max_{\{\theta_t^*, N_t, C_t, K_{t+1}, B_{t+1}, G_{t+1}\}} \sum_{t=0}^{\infty} \beta^t [W(\theta_t^*) + \bar{\theta} \log C_t - N_t + U(G_t)] \\
& + \beta^t \mu_t (F(K_t, N_t) + (1 - \delta)K_t - G_t - C_t - K_{t+1}) \\
& + \beta^t \lambda_t \left(R(\theta_t^*)\theta_t^* - N_t - \frac{\theta_{t-1}^*}{\beta} (1 - D(\theta_{t-1}^*)) \right) \\
& + \beta^t \phi_t \left(K_{t+1} + B_{t+1} - \left(\frac{1}{D(\theta_t^*)} - 1 \right) C_t \right) \\
& + \beta^t \nu_t^B (\bar{B} - B_{t+1})
\end{aligned} \tag{34}$$

The first order conditions with respect to K_{t+1} , N_t , C_t , B_{t+1} and θ_t^* are exactly identical to those in equations (25), (26), (27), (28) and (29), respectively. The additional first-order condition with respect to G_t is given by

$$U'(G_t) = \mu_t, \tag{35}$$

which together with equation (25) gives exactly the same Ramsey Euler equation as in Aiyagari (1995, eq.(20) p.1170). Suppose we follow Aiyagari by assuming that a Ramsey steady state exists, then equation (25) becomes the MGR. Given MGR, if we also pretend that the equilibrium interest rate in our model is below the time discount rate, we would immediately obtain that capital tax should be positive.

However, the assumptions of a steady state and a low interest rate are not necessarily consistent with all the first-order conditions. But these first-order conditions are ignored in Aiyagari's analysis. In particular, after taking all necessary first-order conditions into account, we have shown previously that a finitely-valued steady state consistent with the MGR exists if and only if θ_H is sufficiently small and the debt limit \bar{B} is sufficiently large so that the constraint $B_t \leq \bar{B}$ does not bind. And even in this case we have shown that the Ramsey planner achieves the MGR only by eliminating individual inefficiency instead of by taxing capital. It is straightforward to see that such results continue to hold here by following our analysis in Section 4 and the first-order conditions provided therein. It is also true here that a steady state may violate the MGR if the debt limit constraint binds, and

that a steady state may not even exist if $\theta_H = \infty$ and $\bar{B} = \infty$. These results are in sharp contrast to the claims of Aiyagari (1995) under the assumption of endogenous government spending.

6 Conclusion

This paper uses a tractable model to address a long-standing problem in the optimal Ramsey capital taxation literature. The tractability of our model enables us to solve the Ramsey problem analytically along the entire transitional path. We show that the conventional wisdom on Ramsey tax policy and its underlying intuition and rationales do not hold in our model and may thus be misrepresented in the literature in general. We reveal the trade-off mechanisms and conditions for the Ramsey planner to improve aggregate allocative efficiency and individual allocative efficiency through the tools of government debt and taxes. We show clearly in a transparent way that government debt plays an important role in determining the optimal Ramsey outcome. The planner always intends to supply enough bond to relax individuals' borrowing constraints and through which to achieve the modified golden rule by crowding out capital. Capital tax is not the vital tool to achieve aggregate allocative efficiency despite over-accumulation of capital. Thus the optimal capital tax can be zero, positive, or even negative, depending on the Ramsey planner's ability to issue debt. We prove that the modified golden rule fails to hold whenever the government encounters a debt limit and that taxing capital is not necessarily optimal even in this case. In contrast to the arguments of Aiyagari (1995), the planner does not levy capital tax solely to satisfy the modified golden rule. Instead, the planner may even subsidize capital to encourage more precautionary savings despite over-accumulation of capital. Finally, the desire to relax individuals' borrowing constraints by the planner may lead to an increasing debt accumulation, resulting in a dynamic path featuring no steady state. Hence, numerical approaches to solve the Ramsey problem by assuming the existence of a finitely-valued steady state may be incorrect.

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A Appendix

A.1 Proof of Proposition 1

Denoting $\{\beta^t \lambda_t(\theta^t), \beta^t \mu_t(\theta^t)\}$ as the Lagrangian multipliers for constraints (6) and (7), respectively, the first-order conditions for $\{c_t(\theta^t), n_t(\theta^{t-1}), a_{t+1}(\theta^t)\}$ are given, respectively, by

$$\frac{\theta_t}{c_t(\theta^t)} = \lambda_t(\theta^t) \quad (36)$$

$$1 = \bar{w}_t \int \lambda_t(\theta^t) d\mathbf{F}(\theta_t) \quad (37)$$

$$\lambda_t(\theta^t) = \beta r_{t+1} E_t [\lambda_{t+1}(\theta^{t+1})] + \mu_t(\theta^t), \quad (38)$$

where equation (37) reflects the fact that labor supply $n_t(\theta^{t-1})$ must be chosen before the idiosyncratic taste shocks (and hence before the value of $\lambda_t(\theta^t)$) are realized. By the law of iterated expectations and the *iid* assumption of idiosyncratic shocks, equation (38) can be written as (using equation (37))

$$\lambda_t(\theta^t) = \beta \frac{r_{t+1}}{\bar{w}_{t+1}} + \mu_t(\theta^t), \quad (39)$$

where $\frac{1}{\bar{w}}$ is the marginal utility of consumption in terms of labor income.

We adopt a guess-and-verify strategy to derive the decision rules. The decision rules for an individual's consumption and savings are characterized by a cutoff strategy, taking as given the aggregate states (such as interest rate and real wage). Anticipating that the optimal cutoff θ_t^* is independent of individual's history of shocks, consider two possible cases:

Case A. $\theta_t \leq \theta_t^*$. In this case the urge to consume is low. It is hence optimal to save so as to prevent possible liquidity constraints in the future. So $a_{t+1}(\theta^t) \geq 0$, $\mu_t(\theta^t) = 0$ and the shadow value

$$\lambda_t(\theta^t) = \beta \frac{r_{t+1}}{\bar{w}_{t+1}} \equiv \Lambda_t,$$

where Λ_t depends only on aggregate states. Notice that $\lambda_t(\theta^t) = \lambda_t$ is independent of the history of idiosyncratic shocks. Equation (36) implies that consumption is given by $c_t(\theta^t) = \theta_t \Lambda_t^{-1}$. Defining $x_t(\theta^{t-1}) \equiv r_t a_t(\theta^{t-1}) + \bar{w}_t n_t(\theta^{t-1})$ as the gross income of the

household, the budget identity (6) then implies $a_{t+1}(\theta^t) = x_t(\theta^{t-1}) - \theta_t \Lambda_t^{-1}$. The requirement $a_{t+1}(\theta^t) \geq 0$ then implies

$$\theta_t \leq \Lambda_t x_t \equiv \theta_t^*, \quad (40)$$

which defines the cutoff θ_t^* .

We conjecture that the cutoff is independent of the idiosyncratic state, then the optimal gross income x_t is also independent of the idiosyncratic state. The intuition is that x_t is determined before the realization of θ_t and all households face the same distribution of idiosyncratic shocks. Since the utility function is quasi-linear, the household is able to adjust labor income to meet any target level of liquidity in hand. As a result, the distribution of x_t is degenerate. This property simplifies the model tremendously.

Case B. $\theta_t > \theta_t^*$. In this case the urge to consume is high. It is then optimal not to save, so $a_{t+1}(\theta^t) = 0$ and $\mu_t(\theta^t) > 0$. By the resource constraint (6), we have $c_t(\theta^t) = x_t$, which by equation (40) implies $c_t(\theta^t) = \theta_t^* \Lambda_t^{-1}$. Equation (36) then implies that the shadow value is given by $\lambda_t(\theta^t) = \frac{\theta_t}{\theta_t^*} \Lambda_t$. Since $\theta_t > \theta_t^*$, equation (39) implies $\mu_t(\theta^t) = \Lambda_t \left[\frac{\theta_t}{\theta_t^*} - 1 \right] > 0$. Notice that the shadow value of goods (the marginal utility of income), $\lambda_t(\theta^t)$, is higher under case B than under case A because of the binding borrowing constraint.

The above analyses imply that the expected shadow value of income, $\int \lambda_t(\theta) d\mathbf{F}(\theta)$, and hence the optimal cutoff value θ^* , is determined by equation (37) by plugging in the expressions for $\lambda_t(\theta^t)$ under case A and B, which immediately gives equation (12). Specifically, combining Case A and Case B, we have

$$\begin{aligned} \lambda_t(\theta^t) &= \beta \frac{r_{t+1}}{\bar{w}_{t+1}} \text{ for } \theta \leq \theta_t^* \\ \lambda_t(\theta^t) &= \frac{\theta_t}{\theta_t^*} \beta \frac{r_{t+1}}{\bar{w}_{t+1}} \text{ for } \theta_t \geq \theta_t^*, \end{aligned}$$

the aggregate Euler equation is therefore given by

$$\frac{1}{\bar{w}_t} = \int \lambda_t(\theta) d\mathbf{F}(\theta) = \beta \frac{r_{t+1}}{\bar{w}_{t+1}} \left[\int_{\theta \leq \theta_t^*} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) \right] = \beta \frac{r_{t+1}}{\bar{w}_{t+1}} R(\theta_t^*),$$

which is equation (12). This equation reveals that the optimal cutoff depends only on

aggregate states and is independent of individual history.

We also immediately obtain

$$x_t = \theta_t^* \left(\beta \frac{r_{t+1}}{\bar{w}_{t+1}} \right)^{-1} = \theta_t^* R(\theta_t^*) \bar{w}_t,$$

which leads to equation (8). By the discussion of case A and B as well as utilizing equation (8), the decision rules of household consumption and saving can then be summarized by equations (9) and (10), respectively. Finally, the decision rule of household labor supply, equation (11), is decided residually to satisfy household budget constraint.

Furthermore, to ensure that $n_t(\theta^{t-1}) > 0$, consider the worst situation where $n_t(\theta^{t-1})$ takes its minimum value. Given $x_t = r_t a_t(\theta^{t-1}) + \bar{w}_t n_t(\theta^{t-1})$, $n_t(\theta^{t-1})$ is at its minimum if $\mu_t = 0$ and $a_t(\theta^{t-1})$ takes the maximum possible value, $a_t(\theta^{t-1}) = \frac{\theta_{t-1}^* - \theta_L}{\theta_{t-1}^*} x_{t-1}$. So $n_t(\theta^{t-1}) > 0$ if

$$x_t - r_t \frac{\theta_{t-1}^* - \theta_L}{\theta_{t-1}^*} x_{t-1} > 0, \quad (41)$$

which is independent of θ_t . In the steady state the condition for positive n becomes $1 - r \frac{\theta^* - \theta_L}{\theta^*} > 0$, or equivalently (by using equation (12)),

$$\beta R(\theta^*) > \frac{\theta^* - \theta_L}{\theta^*}, \quad (42)$$

which is satisfied if the precautionary saving motive in the model is strong enough so that the interest rate $r \equiv \frac{1}{\beta R(\theta^*)}$ is low enough. This condition is ensured as long as the variance of θ_t is large enough. This condition is assumed to hold throughout the paper.

A.2 Proof of Proposition 2

In the Laissez-faire economy, the capital taxes, labor taxes and government bonds are all equal to zero. In this Laissez-faire competitive equilibrium, the capital-labor ratio $\frac{K_t}{N_t}$ satisfies two conditions. The first condition is derived from the resource constraint (16), which can be expressed as

$$F(K_t, N_t) + (1 - \delta)K_t = C_t + K_{t+1} + G_t = x_t + G_t,$$

where the last equality utilizes the definition of x_t . Dividing both sides of the equation by K_t gives

$$\left(\frac{K_t}{N_t}\right)^{\alpha-1} + (1 - \delta) = \frac{1}{1 - D(\theta_t^*)} + g_{k,t}, \quad (43)$$

where x_t/K_t is substituted out by $\frac{1}{1-D(\theta_t^*)}$ and $g_{k,t}$ is defined as G_t/K_t .

The second condition is derived by combining equation (12) and the no-arbitrage condition, $r_t = 1 + q_t - \delta$, which gives

$$1 = \beta \left(1 + \alpha \left(\frac{K_t}{N_t} \right)^{\alpha-1} - \delta \right) R(\theta_t^*), \quad (44)$$

where the marginal product of capital q_t is replaced by $\alpha \left(\frac{K_t}{N_t} \right)^{\alpha-1}$. Since the capital-to-labor ratio must be the same in both equations, conditions (43) and (44) imply the following equation in the steady state:

$$\frac{\alpha\beta}{(1 - D(\theta^*))} + \beta(1 - \alpha)(1 - \delta) + \alpha\beta g_k = \frac{1}{R(\theta^*)}, \quad (45)$$

which solves for the steady-state value of θ^* .

It can be shown easily that both $R(\theta^*)$ and $D(\theta^*)$ are monotonically decreasing in θ^* , thus the right hand side (RHS) of equation (45) increases monotonically in θ^* and the left hand side (LHS) of equation (45) decreases monotonically in θ^* .

It remains to show if the RHS and the LHS cross each other at an interior value of $\theta^* \in [\theta_L, \theta_H]$. The RHS of equation (45) reaches its minimum value of 1 when $\theta^* = \theta_H$ and its maximum value of $\bar{\theta}/\theta_L > 1$ when $\theta^* = \theta_L$. The LHS of equation (45) takes the maximum value of infinity when $\theta^* = \theta_L$ and the minimum value of $\frac{\alpha\beta}{1 - \frac{\bar{\theta}}{\theta_H}} + \beta(1 - \alpha)(1 - \delta) + \alpha\beta g_k$ when $\theta^* = \theta_H$. Thus, an interior solution exists if and only if

$$\frac{\alpha\beta}{1 - \frac{\bar{\theta}}{\theta_H}} + \beta(1 - \alpha)(1 - \delta) + \alpha\beta g_k < 1.$$

Clearly, $\theta^* = \theta_L$ cannot constitute a solution for any positive value $\theta_L > 0$. On the other hand, $\theta^* = \theta_H$ may constitute a solution if the above condition is violated. For example, if

θ_H is small and close enough to the mean $\bar{\theta}$, then the above condition does not hold since its LHS approaches infinity when $\theta_H \rightarrow \bar{\theta}$. On the other hand, if $\theta_H \rightarrow \infty$, as in the case of Pareto distribution, then $1 - \frac{\bar{\theta}}{\theta_H} \rightarrow 1$. In this case, the LHS is less than 1 if $g_k = 0$ or small enough.

Therefore, an interior solution for θ^* exists if the upper bound of the idiosyncratic shock is large enough. Otherwise we have the corner solution $\theta^* = \theta_H$. Finally, if θ^* is an interior solution, then $R(\theta^*) > 1$ and $r < 1/\beta$ by equation (12).

A.3 Proof of Proposition 3

A.3.1 The "only if" Part

Assume that we have the allocation $\{\theta_t^*, C_t, N_t, K_{t+1}, B_{t+1}\}$. With this allocation, we can directly construct the prices, taxes and individual allocations in the competitive equilibrium in the following 7 steps:

1. w_t and q_t are set by (1) and (2), which are $w_t = MP_{N,t}$ and $q_t = MP_{K,t}$, respectively.
2. Given C_t and θ_t^* , the total liquidity in hand can be set by equation (18), $x_t = \frac{C_t}{D(\theta_t^*)}$.
3. The individual consumption and asset holdings, $c_t(\theta_t)$ and $a_{t+1}(\theta_t)$, are pinned down by equation (9) and (10).
4. $\tau_{n,t}$ is determined by equation (8), which implies $\tau_{n,t} = 1 - \frac{x_t}{R(\theta_t^*)\theta_t^* MP_{N,t}}$. Hence, \bar{w}_t can be expressed as:

$$\bar{w}_t = \frac{x_t}{R(\theta_t^*)\theta_t^*} = \frac{C_t}{D(\theta_t^*)R(\theta_t^*)\theta_t^*}.$$

Given \bar{w}_t , the interest rate r_{t+1} can be backed out by the Euler equation (12):

$$r_{t+1} = \frac{\bar{w}_{t+1}}{\bar{w}_t} \frac{1}{\beta R(\theta_t^*)} = \frac{D(\theta_t^*)R(\theta_t^*)\theta_t^*}{D(\theta_{t+1}^*)R(\theta_{t+1}^*)\theta_{t+1}^*} \frac{C_{t+1}}{C_t} \frac{1}{\beta R(\theta_t^*)} = \frac{D(\theta_t^*)\theta_t^*}{D(\theta_{t+1}^*)R(\theta_{t+1}^*)\theta_{t+1}^*} \frac{C_{t+1}}{\beta C_t}.$$

Given the expression of r_{t+1} , the capital tax $\tau_{k,t+1}$ is chosen to satisfy the no-arbitrage condition: $r_{t+1} = 1 + (1 - \tau_{k,t+1})MP_{K,t+1} - \delta$.

5. Finally, set $n_t(\theta_{t-1})$ to satisfy equation (11), which implied by the individual household budget constraint.

6. Define A_{t+1} as the aggregate asset holding in period t . Integrating (10) gives

$$A_{t+1} \equiv \int a_{t+1}(\theta_t) dF(\theta_t) = \int \max \left\{ \frac{\theta_t^* - \theta_t}{\theta_t^*}, 0 \right\} x_t dF(\theta_t) = \left(\frac{1}{D(\theta_t^*)} - 1 \right) C_t,$$

where the last equality utilizes equation (18). The above equation together with the condition defined in equation (20) gives the competitive equilibrium asset market clearing condition (14).

7. The implementability condition suggests that

$$R(\theta_t^*)\theta_t^* \geq N_t + \frac{\theta_{t-1}^*}{\beta} (1 - D(\theta_{t-1}^*)).$$

Multiplying both side of the above equation with $\frac{C_t}{D(\theta_t^*)R(\theta_t^*)\theta_t^*}$ leads to

$$\frac{C_t}{D(\theta_t^*)} \geq \frac{C_t}{D(\theta_t^*)R(\theta_t^*)\theta_t^*} N_t + \frac{\theta_{t-1}^*}{D(\theta_t^*)R(\theta_t^*)\theta_t^*} \frac{1}{\beta} (1 - D(\theta_{t-1}^*)) C_t.$$

Using the relationship constructed in steps 2, 4 and 6 for x_t , C_t , A_{t+1} , \bar{w}_t and r_t into the above equation gives

$$C_t + A_{t+1} \geq \bar{w}_t N_t + r_t A_t,$$

which together with the resource constraint and asset market clearing condition enforce the government budget constraint.

Step 1 ensures that the representative firm's problem is solved. Steps 2 to 5 guarantee the individual household problem is solved. Steps 6 and 7 ensure the asset market clearing condition and government budget constraint are satisfied, respectively. The labor market clearing condition is satisfied by the Walras law.

A.3.2 The "if" Part

Note that the resource constraint and asset market clearing condition are trivially implied by a competitive equilibrium, since they are part of the definition. The implementability condition is constructed as follows. First we rewrite the government budget constraint as

$$G_t \leq F(K_t, N_t) - (1 - \tau_{k,t})q_t K_t - (1 - \tau_{n,t})w_t N_t + B_{t+1} - r_t B_t$$

Combining this equation with the resource constraint (16), no-arbitrage condition and asset market clearing condition (14) implies

$$(1 - \tau_{n,t})w_t N_t + r_t A_t \leq C_t + A_{t+1},$$

and hence leads to the following equation:

$$C_t + A_{t+1} \geq \bar{w}_t N_t + r_t A_t. \tag{46}$$

The equilibrium conditions (8), (18) and (12) suggest that \bar{w}_t and r_t can be expressed as

$$\bar{w}_t = \frac{x_t}{R(\theta_t^*)\theta_t^*} = \frac{C_t}{D(\theta_t^*)R(\theta_t^*)\theta_t^*}$$

and

$$r_t = \frac{\bar{w}_t}{\bar{w}_{t-1}} \frac{1}{\beta R(\theta_{t-1}^*)} = \frac{D(\theta_{t-1}^*)R(\theta_{t-1}^*)\theta_{t-1}^*}{D(\theta_t^*)R(\theta_t^*)\theta_t^*} \frac{C_t}{C_{t-1}} \frac{1}{\beta R(\theta_{t-1}^*)} = \frac{D(\theta_{t-1}^*)\theta_{t-1}^*}{D(\theta_t^*)R(\theta_t^*)\theta_t^*} \frac{C_t}{C_{t-1}} \frac{1}{\beta}.$$

Substituting above two equations into (46) and rearranging terms, we get the implementability condition (21).

A.4 Proof of lemma 1

$$\begin{aligned}
H(\theta_t^*) &\equiv \left(1 - D(\theta_t^*) - \theta_t^* \frac{\partial D(\theta_t^*)}{\partial \theta_t^*}\right) = 1 - \left[\int_{\theta \leq \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} d\mathbf{F}(\theta) \right] + \int_{\theta \leq \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) \\
&= 1 - \int_{\theta > \theta_t^*} d\mathbf{F}(\theta) = \mathbf{F}(\theta_t^*)
\end{aligned}$$

$$\begin{aligned}
J(\theta_t^*) &\equiv \left(R(\theta_t^*) + \frac{\partial R(\theta_t^*)}{\partial \theta_t^*} \theta_t^*\right) = \int_{\theta \leq \theta_t^*} d\mathbf{F}(\theta) + \int_{\theta > \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) - \int_{\theta > \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) \\
&= \int_{\theta \leq \theta_t^*} d\mathbf{F}(\theta) = \mathbf{F}(\theta_t^*).
\end{aligned}$$

A.5 Proof of lemma 2

We first show that $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} = 0$ if $\theta_t^* = \theta_L$ or θ_H :

$$\begin{aligned}
\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} &= -\frac{\partial D(\theta_t^*)}{\partial \theta_t^*} \frac{\bar{\theta}}{D(\theta_t^*)} - \int_{\theta \leq \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) = \left[\frac{\bar{\theta}}{D(\theta_t^*) \theta_t^*} - 1 \right] \int_{\theta \leq \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) \\
&= \begin{cases} \left[1 - \frac{\bar{\theta}}{\frac{\bar{\theta}}{\theta_H} \theta_H}\right] \frac{\bar{\theta}}{\theta_H} = 0 & \text{if } \theta_t^* = \theta_H \\ \left[1 - \frac{\bar{\theta}}{\theta_L}\right] 0 = 0 & \text{if } \theta_t^* = \theta_L \end{cases}
\end{aligned}$$

Next, we show that $\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} > 0$ for any $\theta_t^* \in (\theta_L, \theta_H)$. Note that

$$\begin{aligned}
D(\theta_t^*) \theta_t^* &= \int_{\theta \leq \theta_t^*} \theta d\mathbf{F}(\theta) + \theta_t^* \int_{\theta > \theta_t^*} d\mathbf{F}(\theta) = \bar{\theta} - \int_{\theta > \theta_t^*} (\theta - \theta_t^*) d\mathbf{F}(\theta) < \bar{\theta} \\
&\rightarrow \frac{\bar{\theta}}{D(\theta_t^*) \theta_t^*} > 1
\end{aligned}$$

Hence,

$$\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} = \left[\frac{\bar{\theta}}{D(\theta_t^*) \theta_t^*} - 1 \right] \int_{\theta \leq \theta_t^*} \frac{\theta}{\theta_t^*} d\mathbf{F}(\theta) > 0$$

A.6 Proof of lemma 3

The definition of D is given by

$$D(\theta_t^*) = \int_{\theta \leq \theta_t^*} \frac{\theta}{\theta_t^*} dF(\theta) + \int_{\theta > \theta_t^*} dF(\theta) < 1$$

and hence the derivative is

$$\frac{\partial D(\theta_t^*)}{\partial \theta_t^*} = -1 - \int_{\theta \leq \theta_t^*} \frac{\theta}{\theta_t^{*2}} dF(\theta) + 1 = - \int_{\theta \leq \theta_t^*} \frac{\theta}{\theta_t^{*2}} dF(\theta) < 0$$

A.7 Proof of Proposition 4

From equation (28), $\nu_t^B = 0$ implies that $\phi_t = 0$. The first-order condition with respect to C_t leads to a constant multiplier μ_t in the steady state. Given that μ_t is constant in steady state, the modified golden rule holds by equation (25). A constant μ_t together with equation (26) makes the multiplier λ_t constant in the steady state. Given Lemma 1 and Lemma 2, the optimal decision of θ_t^* is a corner solution at θ_H .

In this case there is no liquidity premium. $\theta_t^* = \theta_H$ implies that $R(\theta_t^*) = R(\theta_H) = 1$. The steady state equilibrium interest rate is therefore $1/\beta$ by the Euler equation (12).

The labor tax rate, $\tau_{n,t}$, is decided by equation (8), which implies $(1 - \tau_{n,t}) = \frac{1}{1+\lambda_t} < 1$ in steady state. The steady state labor tax rate is $\frac{\lambda}{1+\lambda} < 1$. The capital tax is pinned down by comparing the aggregate Euler equation chosen by the Ramsey planner, which is the modified golden rule. Or equivalently the competitive equilibrium in equation (12) is

$$\frac{C_{t+1}}{C_t} = \frac{\bar{w}_{t+1}}{\bar{w}_t} = \beta R(\theta_t^*) [1 + q_{t+1} - \tau_{k,t+1} q_{t+1} - \delta],$$

Clearly, $R(\theta_t^*) = 1$ implies that $\tau_{k,t+1} = 0$ in steady state under the MGR. Therefore, the government expenditure as well as bond interest payment are financed solely by labor tax income. Finally, the steady state value of λ_t is chosen such that the steady state government budget constraint holds:

$$\left(1 - \frac{1}{1+\lambda}\right) MP_N N = G + \left(\frac{1}{\beta} - 1\right) B.$$

The government bond B_t is chosen by the asset market clearing condition (20). Hence, the steady state $B_t = B$ can be expressed as

$$B = \left(\frac{1}{D(\theta_H)} - 1 \right) C - K = \left(\frac{\theta_H}{\theta} - 1 \right) C - K.$$

Clearly, θ_H cannot be too large such that $B > \bar{B}$.

A.8 Proof of Proposition 5

In the steady state, the multiplier λ_t is constant. The first-order condition of θ_t^* becomes

$$\frac{\partial W(\theta_t^*)}{\partial \theta_t^*} = -\phi_t \frac{C_t}{D(\theta_t^*)^2} \frac{\partial D(\theta_t^*)}{\partial \theta_t^*}.$$

Given $\frac{\partial D(\theta_t^*)}{\partial \theta_t^*} < 0$, the optimal steady state θ_t^* is an interior solution, denoted by θ^* . Hence, by the household decision rule (10), there is a non-zero fraction of households encountering binding borrowing constraints.

Given the steady state $\phi_t = \phi > 0$, equation (25) suggests that the modified golden rule does not hold in steady state.

In this case, the labor tax as well as the capital tax could be either positive or negative. First, the steady state τ_n is therefore determined by equation (8), which implies

$$\tau_n = 1 - \frac{1}{R(\theta^*)\theta^*D(\theta^*)} \frac{C}{MP_N}.$$

The steady state capital tax τ_k is solved by comparing equation (12) and equation (25), which can be written, respectively, as the following two Euler equations in the steady state:

$$\frac{1}{\beta} = R(\theta^*) [(1 - \tau_k)MP_K + 1 - \delta]$$

$$\frac{1}{\beta} = \frac{\mu}{\mu - \phi} (MP_K + 1 - \delta)$$

The sign of steady state capital tax rate, τ_k , depends on the relatively value of $R(\theta^*)$ and $\frac{\mu}{\mu - \phi}$. More specifically, it depending on $\frac{\mu}{\mu - \phi} \stackrel{\leq}{\geq} R(\theta^*)$, suggesting that $\tau_k \stackrel{\leq}{\geq} 0$.

A.9 Proof of Proposition 6

First, suppose there exists a finitely-valued steady state with $\theta_t^* = \theta^* < \theta_H = \infty$. Then equation (29) in steady state implies that $\frac{\partial W}{\partial \theta^*} > 0$ and $\lambda_t \rightarrow \infty$, which implies that $\mu_t \rightarrow \infty$ by equation (26). However, $\mu_t \rightarrow \infty$ leads to $C_t \rightarrow 0$ by equation (27), which contradicts the assumption of a finitely-valued steady state.

Second, suppose $\theta_t^* \rightarrow \infty$, this implies that households must hold infinite amount of bond to buffer the possible infinitely large idiosyncratic shocks, namely, $B_{t+1} \rightarrow \infty$. To finance an infinite amount of debt, the government must constantly issue increasingly larger amount of new debts to rollover the increasingly larger amount of old debts. In such a case, the total debt must be growing forever such that strictly positive and finitely-valued aggregate allocations $\{C_t, x_t, K_{t+1}, N_t, Y_t\}$ cannot be pinned down. Also, since there is no way to pin down the growth rate of government debt, the no-Ponzi game condition may be violated, suggesting no equilibrium exists. Note that the bond supply could be infinity in the long run without violating the no-Ponzi condition. But in this case there is still the possibility that the steady state does not exist.