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Truncated Firm Productivity Distributions and Trade Margins[*]

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Abstract

A standard theoretical prediction is that average exports are independent of tariff rates when the underlying distribution of firm productivities is assumed to be the widely-used Pareto distribution. Assuming that the underlying distribution has no upper bound is undoubtedly inaccurate and produces theoretical results at odds with empirical results. In contrast, we show that upper-truncation of the Pareto distribution makes average exports rise with trade liberalization. This result is derived analytically, and is supported by simulations. We extend our analysis to the cases of lognormal and Fréchet distributions, which are also frequently used by trade economists. Our findings for lognormal and Fréchet distributions are qualitatively similar to the findings using the truncated Pareto.

JEL Codes: F1.

Keywords: Truncated probability distributions; Extensive and intensive margins of trade; Import tariff, Pareto; Lognormal; Fréchet.

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1. Introduction

Since the path-breaking work of Melitz (2003), models of firm heterogeneity have been a central pillar in the literature on international trade. An implication of the Melitz model is that to overcome fixed costs associated with exporting to another country, the productivity level of the exporting firm has to exceed a certain critical level. Moreover, this productivity threshold affects extensive (i.e., number of firms) and intensive (i.e., the scale of individual firms) trade margins. Interest in these margins, especially in the context of trade liberalization, is growing rapidly.[1]

To analyze the effects of various parameters, such as fixed trade costs, variable trade costs, and foreign demand, on trade margins, the literature has relied on specific functional forms for characterizing the distribution of firm productivities. With tractability being a key consideration, a widely used distribution is the Pareto distribution. In spite of its popularity, the use of this distribution generates a prediction that is not supported empirically (e.g., Lawless 2010). It predicts that the average exports are independent of the import tariff rate (henceforth referred to as the independence result).[2] In contrast, another popular distribution, the lognormal probability distribution, is shown to yield the more sensible prediction that average exports rise with tariff liberalization. Excluding other considerations, this suggests that in a comparison of the Pareto and lognormal distributions, the latter should be the preferred distribution.[3] The rest of this paper examines why such a conclusion may not be appropriate.

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[1] See Lawless (2010), Markusen (2010), Kehoe and Ruhl (2013), and Berthou and Fontagné (2008).

[2] Average firm exports (i.e., total exports divided by the number of exporting firms) are affected by both the exporting scale of individual firms as well as by the total number of exporting firms (i.e., the extensive margin). Accordingly, when average exports are used to measure the intensive margin of trade, the extensive margin is also involved. When trade costs fall, one would expect less productive firms to enter the market, pulling down average exports. On the other hand, declining trade costs likely increase the scale of production of pre-existing exporting firms, pushing up aggregate as well as average exports. When these different effects arising out of the extensive and the intensive margins exactly offset each other, we have the independence result.

[3] Head, Mayer, and Thoening (2014) compare Pareto and lognormal distributions in the context of gains from trade.

One implication of the standard untruncated probability distributions is that there exists some infinitely productive firm. In turn, this implies that there cannot be zero trade flows between any pair of nations because there must be some firm whose productivity exceeds the critical level for exporting. As pointed out by Helpman, Melitz and Rubinstein (2008, henceforth referred to as HMR) this assumption is unrealistic, and empirically untenable, given the existence of zero trade flows in the data for many pairs of nations. As a result, it is more reasonable to work with upper-truncated probability distributions for firm productivity. Accordingly, this paper focuses on truncated distributions and explores how trade margins may be impacted by different types of trade costs and also by the level of upper truncation itself.

Our analysis focuses on truncated Pareto, lognormal and Fréchet distributions. Pareto and lognormal are commonly used in Melitz (2003) type models, while the Fréchet distribution is more commonly used in the context of Eaton and Kortum (2002) type models. Therefore, our analysis, which is more closely tied to the heterogeneous firm models, also relates to the Eaton-Kortum type models. We find that when we consider a truncated Pareto distribution, the independence result disappears and average exports are indeed rising in tariff liberalization. Such a negative relationship between average exports and tariffs has been previously noted in the literature for the case of the lognormal distribution, but, to our knowledge, this has not yet been shown for the case of a Pareto distribution.[4]

Digging deeper, we consider the effects of tariffs, truncation levels and foreign income on average exports for all three types of truncated probability distributions. Although we are able to make much progress with analytical results in the truncated Pareto case and some

[4] See Bas, Mayer, and Thoenig (2015), Fernandes et al. (2015), among others, for detailed discussions on the comparisons between these distributions with a focus on the untruncated cases. HMR as well as Melitz and Redding (2015) use truncated Pareto, but they do not focus on the dependence between average firm exports and tariffs using the distribution.

progress with the lognormal case, we rely on simulations for the comparative statics for the Fréchet case. However, for all these cases we first derive analytical expressions for average exports and the moments of the respective truncated distributions. We then match the mean of these three distributions by appropriate choice of parameters that characterize these distributions. After matching the means, we simulate average exports with respect to differing tariff, foreign income and truncation levels. All distributions exhibit negative relationships between average exports and tariff rates, and positive relationships between average exports and truncation and/or foreign income levels (see Figures 1-9 at the end).

The remainder of the paper is arranged as follows. Section 2 presents the model and analysis. Section 3 presents the simulations. Section 4 summarizes our key results.

2. The Basic Model: Heterogeneous Firms, Trade Costs, and Exports

The basic model is adapted from HMR and Lawless (2010). The origins of these models can be found in Melitz (2003) and in Chaney (2008). Key factors driving the results are that firms differ in their productivity and, to export, must incur fixed as well as variable export costs. As a result, not all (empirically few) domestic producers are also exporters and, in fact, trade flows between potential origins and destinations are often zero.[5]

In the tradition of the recent trade literature, we consider heterogeneous monopolistically competitive firms producing different varieties within an industry. Origin nations are indexed by i and foreign nations by j. Following HMR, we assume that consumers in nation j consume a continuum of products indexed by k, where the set of products available for consumption in that

[5] Bernard et al. (2007) found that 4 percent of 5.5 million U.S. firms in 2000 were exporters.

nation is B_j. The standard utility function characterizing consumer preferences in the foreign nation is

$$U_j = \left[\int_{k \in B_j} x_j\left(k\right)^{\frac{\varepsilon-1}{\varepsilon}} dk \right]^{\frac{\varepsilon}{\varepsilon-1}}, \ \varepsilon > 1, \tag{1}$$

where ε is a constant elasticity of substitution between products. Utility maximization subject to budget constraint yields the following demand function for product *k* in nation *j*

$$x_j\left(k\right) = \frac{p_j\left(k\right)^{-\varepsilon} Y_j}{P_j^{1-\varepsilon}}, \tag{2}$$

where Y_j is nation *j*'s total expenditure (given exogenously), and P_j is its Dixit-Stiglitz aggregate price index, such that:

$$P_j = \left[\int_{k \in B_j} p_j\left(k\right)^{1-\varepsilon} dk \right]^{\frac{1}{1-\varepsilon}}. \tag{3}$$

We assume that the mass of firms in the industry in *i* is given exogenously at $\overline{N}_i$, and that there is no fixed cost in producing for the domestic market.[6] We assume that the marginal input cost of any product produced by an *i* firm is a constant c_i. Also, assume that a producer of product *k* in nation *i* has a productivity level $a\left(k\right)$, such that its marginal cost of production is $c_i / a\left(k\right)$.

[6] This follows HMR, and, as in their paper, this immediately implies that all domestic firms produce for the domestic market because the monopolistically competitive domestic price includes a markup over marginal cost, and, in addition, there is no fixed cost for domestic sales.

For this firm to export its product to nation *j*, in addition to the standard variable cost, it has to incur two additional costs. First, there is an iceberg type tariff/transportation cost, such that for each unit reaching the foreign market, the firm needs to produce $t_{ij}\left(>1\right)$ units, where $t_{ij}-1$ units melt away between production and delivery in the export market. There is a fixed cost F_{ij} faced by all firms of nation *i* in order to export their product to nation *j*. Limiting our focus to exporting firms, the profit from exports to nation *j* of firm *k* is:

$$\pi_{ij}\left(k\right)=p_{ij}\left(k\right)x_{ij}\left(k\right)-\frac{c_i}{a\left(k\right)}t_{ij}x_{ij}\left(k\right)-F_{ij}\,, \tag{4}$$

where $x_{ij}\left(k\right)$ is the level of exports to nation *j* of firm *k* of nation *i*.

The demand function in Equation (2) implies that this firm perceives its price elasticity of demand in the export market as ε, which follows from the fact that for a sufficiently large set of consumption goods in the foreign nation, this firm's effect on the aggregate price level of the foreign nation is negligible. Using this fact, marginal revenue - marginal cost equalization yields the profit maximizing export and price levels for the firm as:

$$p_{ij}\left(k\right)\left(1-\frac{1}{\varepsilon}\right)=\frac{c_i}{a\left(k\right)}t_{ij}\Rightarrow p_{ij}\left(k\right)=\frac{\varepsilon\tau_{ij}}{\left(\varepsilon-1\right)a\left(k\right)},\tau_{ij}=t_{ij}c_i\,. \tag{5}$$

Plugging the price level obtained from Equation (5) into Equation (4), and using Equation (2), we get the firm's (optimized) profit from exports as:

$$\pi_{ij}\left(k\right)=-\left(\frac{\tau_{ij}}{a\left(k\right)}\right)^{1-\varepsilon}\frac{\mu Y_j}{P_j^{1-\varepsilon}}\quad F_{ij}\text{, where }\mu=\varepsilon^{-\varepsilon}\left(\varepsilon-1\right)^{\varepsilon-1}. \tag{6}$$

Using Equation (6), we find that positive (or zero) export profit (i.e., $\pi_{ij}\left(k\right)\geq 0$) can be obtained if and only if:[7]

$$a\left(k\right)\geq \tilde{a}_{ij}=\frac{\tau_{ij}}{P_j}\left(\frac{F_{ij}}{\mu Y_j}\right)^{\frac{1}{\varepsilon-1}}\Rightarrow\frac{\partial \tilde{a}_{ij}}{\partial \tau_{ij}}>0\text{, and, }\frac{\partial \tilde{a}_{ij}}{\partial F_{ij}}>0\text{,} \tag{7}$$

where $\tilde{a}_{ij}$ is the minimum (or threshold) productivity level required for a firm from nation *i* to profitably export to nation *j*. Now, the export revenue of a firm of nation *i* from its exports to nation *j* is:

$$s_{ij}\left(a\left(k\right)\right)=p_{ij}\left(k\right)x_{ij}\left(k\right)=\left[\frac{P_j\left(\varepsilon-1\right)a\left(k\right)}{\varepsilon\tau_{ij}}\right]^{\varepsilon-1}Y_j\Rightarrow\frac{\partial s_{ij}\left(a\left(k\right)\right)}{\partial \tau_{ij}}<0\text{,} \tag{8}$$

which means that as the variable costs of exporting from nation *i* to nation *j* increase, the export revenue of a nation *i* firm from its exports to nation *j* must fall.

We assume that the productivity level a is distributed as a truncated probability distribution, characterized by a probability density function $g\left(a\right)$, with support $\left(a_L, a_H\right)$, where $a_L<a_H$.[8] Let E_{ij} be the sum of export revenues from exports to nation *j* by firms of nation *i*. Noting that only firms above the threshold productivity level $\tilde{a}_{ij}$ export to nation *j*, and also that the mass of firms from nation *i* is $\bar{N}_i$, nation *i*'s aggregate export revenues are:

[7] We assume that the foreign nation's price index includes prices of a large basket of goods from its own firms as well as from firms from nations other than nation *i*, such that we can take this price level as given with respect to changes in τ_{ij}.

[8] This distribution, as noted in HMR, presents a more realistic picture of heterogeneous firms, such that no infinitely productive firm needs to be assumed.

$$E_{ij} = \int_{\tilde{a}_{ij}}^{a_H} s_{ij}(a)\bar{N}_i g(a)\,da \text{, for } \tilde{a}_{ij} < a_H;\ E_{ij} = 0 \text{, otherwise.} \tag{9}$$

Analogously, the total number of firms from nation *i* exporting to nation *j* is:

$$N_{ij} = \bar{N}_i \int_{\tilde{a}_{ij}}^{a_H} g(a)\,da = \bar{N}_i\left[1 - G\left(\tilde{a}_{ij}\right)\right], \tag{10}$$

Where $G(a)$ is the cumulative density function associated with $g(a)$. In other words, $1-G\left(\tilde{a}_{ij}\right)$ represents the fraction of firms of nation *i* who export to nation *j*.

Eqs. (1) through (10) describe an HMR and Lawless (2010) type environment. Following Lawless (2010), we have the following comparative static results.[9] The proofs are presented in the appendix.

Result1: *A rise in the fixed exporting cost* F_{ij} *should reduce nation i's aggregate export revenues from sales to nation j, and reduce the number of nation i firms exporting to nation j, but increase the average export revenues from sales to nation j.*

Comment: A rise in fixed costs does not affect the scale of a firm's production (see Eqs. 2 and 5), but it increases the productivity requirement to enter the export market (see Eq. 7). This leads to the exit of some firms from the export market bringing down total export revenues. Average

[9] Results 1-3 mirror the conclusions drawn by Lawless (2010) on page 1156. We state them here for completeness and for easy reference as we transition to the central part of our analysis, which is related to specific firm productivity distributions with truncation.

exporting firm revenues rise because the least productive firms with lower scales of operation must exit the export market.

Result2: *An increase of the variable cost parameter* τ_{ij} *should reduce nation i's export revenues from sales to nation j, and reduce the number of the nation's firms exporting to nation j, while its effect on average export revenues from sales to nation j is ambiguous, in general.*

Comment: Higher τ_{ij} amplifies the marginal cost of production and reduces scale for each firm. It also raises the cutoff productivity that is required to export, leading to the exit of the least productive firms. The effect on average exports is ambiguous because while the reduction in scale of each exporting firm tends to reduce average exports, the exit of least productive firms tends to raise average exports. If one assumes specific productivity distributions, such as Pareto or lognormal, this ambiguity is resolved under certain parameterizations.

Result3: *A rise in foreign income* Y_j *should increase each nation's aggregate export revenues from sales to nation j, and increase the number of each nation's firms who export to nation j, while the effect on a nation's average export revenue from sales to nation j is ambiguous.*

Comment: Higher foreign income will tend to raise demand for each firm. Eq. (5) suggests that price is not affected, but Eq. (2) implies demand rises. Thus, scale of production and revenues of each exporting firm must rise. In addition, the rise in revenues allows for a drop in the threshold productivity to export, bringing in less productive exporting firms. Therefore, there are opposing scale and composition effects, leading to ambiguity in average exports.

The general truncated distribution used until this point yields unambiguous results for all cases except for the effect of variable cost parameter τ_{ij} and of foreign income level Y_j on average export revenues E_{ij} / N_{ij}. To shed more light on this issue, we use three specific truncated distributions − Pareto, lognormal, and Fréchet. Our results indicate that unlike Lawless (2010), who used untruncated Pareto, truncation makes average export revenues dependent on both tariffs and foreign income. We also explore the effects of upper truncation and foreign income on average exports. The following relationships apply to all truncated distributions. If $f(a)$ denotes an untruncated distribution and $F(a)$ its cumulative distribution function, then the corresponding truncated distribution $g(a)$ with upper and lower truncation levels being respectively, a_H and a_L, is:

$$g(a)=\frac{f(a)}{F(a_H)-F(a_L)}, \text{ where } a_L \le a \le a_H, \text{ and } a_L < a_H. \tag{11}$$

The cumulative distribution function is:

$$G(A)=\int_{a_L}^{A} g(a)\,da=\frac{\int_{a_L}^{A} f(a)\,da}{F(a_H)-F(a_L)}=\frac{F(A)-F(a_L)}{F(a_H)-F(a_L)}. \tag{12}$$

2.1. Truncated Pareto Distribution

Consider a standard Pareto distribution (without upper truncation), such that:

$$f(a)=\frac{\phi a_L^{\phi}}{a^{\phi+1}},\ a \ge a_L,\ \phi > 0; \text{ and } F(a)=1-\left(\frac{a_L}{a}\right)^{\phi}. \tag{13}$$

Using Eqs. (11) through (13), the truncated Pareto distribution is

$$g\left(a\right)=\frac{\phi a_L^{\phi}a_H^{\phi}a^{-\phi-1}}{a_H^{\phi}-a_L^{\phi}}\text{; and } G\left(a\right)=\frac{a_H^{\phi}\left(1-a_L^{\phi}a^{-\phi}\right)}{a_H^{\phi}-a_L^{\phi}};\ \phi>0. \tag{14}$$

Using Eqs. (9) and (10):

$$\frac{E_{ij}}{N_{ij}}=\frac{\int_{\tilde{a}_{ij}}^{a_H}s_{ij}\left(a\right)g\left(a\right)da}{1-G\left(\tilde{a}_{ij}\right)}\quad. \tag{15}$$

Assuming that some, but not all, firms export, we can use Eqs. (7), (8) and (14) to reduce Eq. (15) to:

$$\frac{E_{ij}}{N_{ij}}=\frac{\phi\varepsilon F_{ij}\left(1-x^{1+\phi-\varepsilon}\right)}{\left(1+\phi-\varepsilon\right)\left(1-x^{\phi}\right)}\quad\psi\left(x\right)\text{, where } x=\tilde{a}_{ij}/a_H\text{, and } a_L/a_H<x<1. \tag{16}$$

Proposition 1

Average firm exports are not independent of variable trade costs for a truncated Pareto distribution. For a shape parameter ϕ, a sufficient condition for average exports to rise monotonically with a decline in the tariff rate (or with a rise in the foreign income) is that $\phi/\left(1+\phi-\varepsilon\right)$ assumes an integer value. Under the same condition, average exports fall as upper truncation of productivity becomes more severe.

Proof: Recall from Eq. (7) that $\tilde{a}_{ij}$ rises linearly with the tariff rate, while it falls with a rise in foreign income, given the foreign price level, and given other parameters. For a finite a_H,

$\tilde{a}_{ij} / a_H = x \neq 0$, thus a fall in the tariff rate which reduces x in Eq. (16) and changes the average export E_{ij} / N_{ij}, establishing dependence. This contrasts Lawless (2010), among others, who consider the untruncated Pareto case, and find Pareto distribution to yield independence of average exports to tariff levels. Their case is nested within ours, because as $a_H \to \infty$, for a finite $\tilde{a}_{ij}$, $x \to 0$, and we get $\frac{E_{ij}}{N_{ij}} \to \frac{\phi \varepsilon F_{ij}}{1+\phi-\varepsilon}$, which is the expression obtained in Lawless (2010), where the average export revenue is independent of variable cost parameter τ_{ij}. Now, Eq. (16) can be written as:

$$\psi(x) = \frac{\phi \varepsilon F_{ij}}{(1+\phi-\varepsilon) S(z(x))}, \text{ where, } S(z(x)) = \frac{1-z^n}{1-z}, \text{ and } 0 < z = x^{1+\phi-\varepsilon} < 1, \text{ and}$$

$$n = \phi / (1+\phi-\varepsilon). \tag{17a}$$

Now, for positive integer values of *n*,

$$\frac{1-z^n}{1-z} = 1 + z + z^2 + ... + z^{n-1}, \tag{17b}$$

It is easy to check that $S'(z) = 1 + 2z + 3z^2 + ... + (n-1) z^{n-2} > 0$, because $z > 0$, and $n - 1 = (\varepsilon - 1)/(1+\phi-\varepsilon) > 0$. Also, noting that $z'(x) = (1+\phi-\varepsilon) x^{\phi-\varepsilon} > 0$, we have,

$$\psi'(x) = \quad \frac{\phi \varepsilon F_{ij}}{(1+\phi-\varepsilon) S^2(z(x))} S'(z) z'(x) \quad 0. \tag{17c}$$

Eq. (17c) establishes that as tariff falls, or foreign income rises (both of which leads to a fall in x), average exports must rise. Similarly, as upper truncation becomes more severe (i.e., a_H falls and x rises), average exports must fall. This concludes the proof of proposition 1. ■

Comment: It is illustrative to consider a tractable special case for Eq. (16). If $\varepsilon = \phi = 3$, then $n = 3$, and we have

$$\psi(x) = \frac{9F_{ij}(1-x)}{(1-x^3)} = \frac{9F_{ij}}{1+x+x^2} \Rightarrow \psi'(x) = \frac{9F_{ij}(1+2x)}{(1+x+x^2)^2} \quad 0; \tag{18a}$$

$$\psi''(x) = \frac{54F_{ij}x(1+x)}{(1+x+x^2)^3} > 0 \Rightarrow \frac{d|\psi'(x)|}{dx} < 0. \tag{18b}$$

As τ_{ij} falls (or Y_j rises), $\tilde{a}_{ij}$ falls, and given a_H, x falls. If x falls, Eq. (18a) implies that average exports must rise. In addition, Eq. (18b) implies that the marginal effect of the tariff on average exports becomes larger as tariff becomes smaller. Similarly, given tariff rates, a fall in a_H raises x, and therefore must reduce average exports. Also, because of Eq. (18b), as the truncation becomes more severe, the marginal negative effect of truncation on average exports becomes smaller in magnitude.

Note that proposition 1 is restrictive in the sense that informs us only about integer values of $n = \phi/(1+\phi-\varepsilon)$. To shed more light on this issue, we have run simulations discussed in more detail later (where n is not necessarily an integer), all of which show a negative relationship between average exports and the tariff rate, and a positive relationship between average exports and foreign income. Similarly, the simulations show that more severe truncation reduces average exports. The intuition is the following. When we rule out some super-productive firms (i.e., when a_H is smaller), there is a loss in the extensive margin [it is easy to check that $1-G(\tilde{a})$ is positively related to a_H]. This tends to raise average exports. On the other hand, there is a decline in exports on the intensive margin because of the absence of super-productive firms. The intensive margin effect dominates in driving proposition 1 for the case of a truncated Pareto.

2.2. Truncated Lognormal Distribution

Let us now turn our attention to the analysis of the truncated lognormal case, and compare it with the untruncated lognormal case. Consider an untruncated lognormal with mean μ and variance σ^2. Using Eq. (8) in Eq. (15):

$$\frac{E_{ij}}{N_{ij}} = \left[\frac{P_j(\varepsilon-1)}{\varepsilon\tau_{ij}}\right]^{\varepsilon-1} Y_j\beta\left(\tilde{a}_{ij}, a_H\right),$$

$$\text{where, } \beta\left(\tilde{a}_{ij}, a_H\right) = \frac{\int_{\tilde{a}_{ij}}^{a_H} a^{\varepsilon-1} g(a)\,da}{1-G\left(\tilde{a}_{ij}\right)}. \tag{19}$$

Using the form of untruncated lognormal distribution, and using Eqs. (11) and (12) in Eq. (19), we get

$$\beta\left(\tilde{a}_{ij}\right) = \frac{\int_{\tilde{a}_{ij}}^{a_H} \frac{a^{\varepsilon-2}}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\ln a-\mu}{\sigma}\right)^2} da}{F(a_H) - F(\tilde{a})}$$

$$= \frac{e^{\frac{\sigma^2(\varepsilon-1)^2}{2}+\mu(\varepsilon-1)}\left[\Phi\left(\frac{\ln a_H-\mu}{\sigma}-\sigma(\varepsilon-1)\right)-\Phi\left(\frac{\ln\tilde{a}-\mu}{\sigma}-\sigma(\varepsilon-1)\right)\right]}{\Phi\left(\frac{\ln a_H-\mu}{\sigma}\right)-\Phi\left(\frac{\ln\tilde{a}-\mu}{\sigma}\right)}, \tag{20}$$

where $\Phi(.)$ is the distribution function of an untruncated standard normal distribution.

Proposition 2

At very high levels of the upper productivity limit, average exports must fall with more severe truncation.

Proof: Define $\frac{\ln a_H-\mu}{\sigma}\equiv\rho_H$, and, $\frac{\ln\tilde{a}-\mu}{\sigma}\equiv\tilde{\rho}$. Also, let f^{Φ} denote the standard normal density function. Then, using Eqs. (20) and (21) we get:

$\frac{\partial(E_{ij}/N_{ij})}{\partial a_H}>0$, if and only if,

$$\frac{f^{\Phi}(\rho_H-\sigma(\varepsilon-1))}{f^{\Phi}(\rho_H)}>\frac{\left[\Phi(\rho_H-\sigma(\varepsilon-1))-\Phi(\tilde{\rho}-\sigma(\varepsilon-1))\right]}{\Phi(\rho_H)-\Phi(\tilde{\rho})}. \tag{21}$$

Noting that $\Phi(.)$ is bounded above by one, as $a_H\to\infty$, $\rho_H\to\infty$, and the right-hand-side of Eq. (21) tends to the positive finite expression below, which is independent of a_H:

$$\frac{\left[1-\Phi(\tilde{\rho}-\sigma(\varepsilon-1))\right]}{1-\Phi(\tilde{\rho})}>0. \tag{22}$$

Using the form of the standard normal function, the left-hand-side of Eq. (22) tends to:

$$\lim_{\rho^H \to \infty} \frac{f^{\Phi}\left(\rho_H - \sigma(\varepsilon - 1)\right)}{f^{\Phi}\left(\rho_H\right)} = \lim_{\rho^H \to \infty} e^{\sigma(\varepsilon-1)\rho^H - \frac{\sigma^2(\varepsilon-1)^2}{2}} \quad . \tag{23}$$

Eq. (22) yields a finite positive expression independent of ρ^H. Eq. (23) shows that the left-hand-side of Eq. (21) goes to infinity. Thus, inequality (21) must be satisfied for sufficiently high values of a_H. Accordingly, $\frac{\partial\left(E_{ij} / N_{ij}\right)}{\partial a_H} > 0$ for sufficiently large a_H, which implies that as a_H falls from very high levels, average exports must also fall. ■

Comment: As in the case of truncated Pareto, ruling out super-productive firms reduces average exports, at least starting from very high upper truncation levels. As a_H falls, the decline in extensive margin is more than offset by the decline on the intensive margin, to lead to a fall in average exports. The analytical relationship between average export and tariff (or foreign income) is too complicated, and we rely on simulations reported below to throw more light on the issue. Simulations also help us see how average exports are affected by truncation – and it seems that the limiting results of proposition 2 are qualitatively supported even at more moderate truncation levels.

2.3 Truncated Fréchet Distribution

First consider an untruncated Fréchet distribution (e.g., Rodriguez-Clare, 2010), where the density function of the productivities takes the form:

$$f(a) = T\theta a^{-\theta-1} e^{-Ta^{-\theta}}, \; T > 0, \; \theta > 1, \tag{24}$$

where T is a scale parameter, such that a higher T is associated with better productivity draws, and θ is a shape parameter. The cumulative distribution function is:

$$F(A)=\int_0^A f(a)\,da = e^{-TA^{-\theta}}. \tag{25}$$

Using Eqs. (11), (12) and (19):

$$\frac{E_{ij}}{N_{ij}}=\frac{\left[\frac{P_j(\varepsilon-1)}{\varepsilon\tau_{ij}}\right]^{\varepsilon-1} Y_j \int_{\tilde{a}_{ij}}^{a_H} a^{\varepsilon-1} f(a)\,da}{F(a_H)-F(\tilde{a})}. \tag{26}$$

Notice that:

$$\int_{\tilde{a}_{ij}}^{a_H} a^{\varepsilon-1} f(a)\,da = \int_{\tilde{a}_{ij}}^{\infty} a^{\varepsilon-1} f(a)\,da - \int_{a_H}^{\infty} a^{\varepsilon-1} f(a)\,da. \tag{27}$$

Now, for any $a=\overline{a}>0$, and $r>0$, we can show that

$$\int_{\overline{a}}^{\infty} a^r f(a)\,da = \int_{\overline{a}}^{\infty} a^r T\theta a^{-\theta-1} e^{-Ta^{-\theta}}\,da = T^{\frac{r}{\theta}}\gamma\left(1-\frac{r}{\theta}, T\overline{a}^{-\theta}\right), \text{ where } \frac{r}{\theta}<1, \text{ and} \tag{28}$$

where $\gamma(\bullet)$ is a standard *lower incomplete gamma function* of the form $\gamma(\rho,x)=\int_0^x t^{\rho-1}e^{-t}dt$, where $\rho>0$. Using Eqs. (25), (27) and (28), and using $\theta>r=\varepsilon-1>0$, Eq. (26) yields:

$$\frac{E_{ij}}{N_{ij}}=\frac{\left[\frac{P_j(\varepsilon-1)}{\varepsilon\tau_{ij}}\right]^{\varepsilon-1} Y_j T^{\frac{\varepsilon-1}{\theta}}\left[\gamma\left(1-\frac{\varepsilon-1}{\theta}, T\tilde{a}^{-\theta}\right)-\gamma\left(1-\frac{\varepsilon-1}{\theta}, Ta_H^{-\theta}\right)\right]}{e^{-Ta_H^{-\theta}}-e^{-T\tilde{a}^{-\theta}}}. \tag{29}$$

3. Simulations: Truncated Pareto, Lognormal and Fréchet with Matched Means

The r^{th} moment of the truncated Pareto distribution described in Eq. (14) can be derived as:

$$E\left(a^r\right)=\frac{\phi a_L^r\left[1-\left(a_L/a_H\right)^{\phi-r}\right]}{\left(\phi-r\right)\left[1-\left(a_L/a_H\right)^{\phi}\right]}\text{, where } r\neq\phi,\ r>0. \tag{30}$$

Similarly, the r^{th} moment of the truncated lognormal defined in section 2.2. can be derived to be

$$E\left(a^r\right)=\frac{e^{\frac{1}{2}\left(2r\mu+r^2\sigma^2\right)}\left[\Phi\left(\frac{\ln a_H-\mu}{\sigma}-r\sigma\right)-\Phi\left(\frac{\ln a_L-\mu}{\sigma}-r\sigma\right)\right]}{\Phi\left(\frac{\ln a_H-\mu}{\sigma}\right)-\Phi\left(\frac{\ln a_L-\mu}{\sigma}\right)}. \tag{31}$$

Finally, the relationships in section 2.3 can be used to show that the r^{th} moment of truncated Fréchet distribution is:

$$E\left(a^r\right)=\frac{T^{\frac{r}{\theta}}\left[\gamma\left(1-\frac{r}{\theta},Ta_L^{-\theta}\right)-\gamma\left(1-\frac{r}{\theta},Ta_H^{-\theta}\right)\right]}{e^{-Ta_H^{-\theta}}-e^{-Ta_L^{-\theta}}}. \tag{32}$$

Using $r=1$ in Eqs. (30), (31) and (32) we get the respective means of the three truncated distributions. We match these means to obtain the respective means of the corresponding untruncated distributions.[10]

For all three distributions average exports rise with tariff liberalization (see Figures 1-3), suggesting that the dampening effect of the entry of new less productive firms (the extensive margin) is more than compensated by gains on the intensive margin of exporting firms. This is an important contrast with the untruncated cases, where Pareto delivers independence. Turning to the effect of foreign income (Figures 4-6), we find that average exports rise with greater foreign income in all cases. Here too, the scale expansion effect dominates the dampening effect

[10] All the parametrization details are provided at the end of the manuscript following the Figures.

of entry of less productive firms. Finally, we show that more severe upper truncation must reduce average exports for all these distributions (Figures 7-9), although for Fréchet the rate of this change is qualitatively different. The overall message is that although there are some quantitative differences, qualitatively there is no stark contrast between these three widely used distributions. Accordingly, empirics that find average exports to rise with trade liberalization do not suggest one distribution to be preferred over the other.

4. Discussion and Concluding Comments

A novel contribution of this paper is to explore the analytical relationship between average firm exports and trade costs under truncation. As the paper by HMR points out, it is more realistic to assume that there is no infinitely productive firm. Accordingly, the discussion on effects of firm productivity on exports should be framed in the context of truncated distributions.

The qualitative comparative static effects for trade margins are markedly different between truncated and untruncated distributions. In particular, for the widely-used Pareto distribution, the well-known independence of average exports to variable trade costs, such as a change in tariff rates, is broken by truncation. A combination of analytical and simulation results show that in the presence of truncation, average exports rise with tariff liberalization for Pareto, lognormal and Fréchet distributions. The underlying economics is straightforward. Trade liberalization allows more firms to become exporters, which is an increase in the extensive margin. These new exporters are relatively less productive than existing exporters, so average exports tend to decline. On the other hand, existing exporters increase their export sales, driving average exports higher. Our findings using various distributions of firm productivity reveal that this latter effect dominates the former effect. In a similar vein, for all the three types of truncated distributions, greater foreign income raises average exports in spite of encouraging entry of less productive firms. Furthermore, we find that more severe upper truncation reduces average exports. Eliminating the possibility of super-productive firms brings down the average scale of exporting firms.

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Appendix

Consider any parameter θ that can affect the trading equilibrium. Also, let us focus on nations that have non-zero exports before and after the change in this parameter. The change in the number of exporting firms can be obtained by differentiating Equation (10), to yield:

$$\frac{\partial N_{ij}}{\partial \theta} = -\bar{N}_i g\left(\tilde{a}_{ij}\right)\frac{\partial \tilde{a}_{ij}}{\partial \theta}. \tag{A1}$$

Equation (A1) shows that a change in θ potentially alters the threshold productivity level $\tilde{a}_{ij}$, leading to a smaller (larger) equilibrium number of nation *i* firms exporting to nation *j*, if the threshold is raised (lowered).

The change in aggregate export revenues of nation *i*'s firms from exports to nation *j* is:

$$\frac{\partial E_{ij}}{\partial \theta} = -\bar{N}_i \int_{\tilde{a}_{ij}}^{a_H} \frac{\partial s_{ij}}{\partial \theta} g\left(a\right) da \quad \bar{N}_i s_{ij}\left(\tilde{a}_{ij}\right) g\left(\tilde{a}_{ij}\right)\frac{\partial \tilde{a}_{ij}}{\partial \theta} . \tag{A2}$$

The first term on the right-hand-side of Equation (A2) is the change in exports of nation *i* to nation *j* due to a change in the export revenues of an existing exporting firm, while the second term is the change in these exports due to the entry (or exit) of firms from nation *i* from nation *j*'s market, because of a change in the relevant threshold productivity level.

Finally, consider average export revenue of firms from a nation from their exports to nation *j*. This is given by E_{ij} / N_{ij}. The effect of a change in θ on this average is:

$$\frac{\partial\left(E_{ij} / N_{ij}\right)}{\partial \theta} = \frac{N_{ij}\frac{\partial E_{ij}}{\partial \theta} - E_{ij}\frac{\partial N_{ij}}{\partial \theta}}{N_{ij}^2}. \tag{A3}$$

Using Equations (A1) and (A2) in Equation (A3), we get:

$$\frac{N_{ij}^2}{\bar{N}_i}\left[\frac{\partial\left(E_{ij}/N_{ij}\right)}{\partial\theta}\right] = N_{ij}\int_{\tilde{a}_{ij}}^{a_H}\frac{\partial s_{ij}}{\partial\theta}g(a)\,da \quad \left[E_{ij} \quad N_{ij}s_{ij}\left(\tilde{a}_{ij}\right)\right]g\left(\tilde{a}_{ij}\right)\frac{\partial\tilde{a}_{ij}}{\partial\theta}, \tag{A4}$$

where $E_{ij} - N_{ij}s_{ij}\left(\tilde{a}_{ij}\right) > 0$. This is because $N_{ij}s_{ij}\left(\tilde{a}_{ij}\right)$ is the export revenue of the least productive firm of nation *i* scaled by the total number of the nation's firms exporting to *j*, and hence must be lower than the aggregate nation *i* export revenues from trade with nation *j* $\left(i.e., E_{ij}\right)$. This is because E_{ij} also includes export revenues of firms whose productivities are above the cut-off level $\tilde{a}_{ij}$.

Proof of Result1: *Change in fixed exporting cost* $F_{ij}\left(i.e., d\theta = dF_{ij}\right)$

From Equation (8), we can see that $\frac{\partial s_{ij}}{\partial F_{ij}} = 0$ for all firms. Therefore, there is no change in exports due to the scale of operation of existing exporting firms. Effects of the change of the fixed cost on the extensive margin can be analyzed using Equations (A1) and (A2), which yield:

$$\frac{\partial N_{ij}}{\partial F_{ij}} = \quad \bar{N}_i g\left(\tilde{a}_{ij}\right)\frac{\partial\tilde{a}_{ij}}{\partial F_{ij}} \quad 0, \text{ and, } \frac{\partial E_{ij}}{\partial F_{ij}} = \quad \bar{N}_i s_{ij}\left(\tilde{a}_{ij}\right)g\left(\tilde{a}_{ij}\right)\frac{\partial\tilde{a}_{ij}}{\partial F_{ij}} \quad 0. \tag{A5}$$

Thus, the number of firms exporting from nation *i* to nation *j*, as well as the nation's aggregate export revenues from sales to nation *j* must fall in response to a rise in the fixed cost F_{ij}. Using Equation (A4) we get:

$$\frac{\partial\left(E_{ij}/N_{ij}\right)}{\partial F_{ij}} => \frac{\bar{N}_i\left[E_{ij} - N_{ij}s_{ij}\left(\tilde{a}_{ij}\right)\right]g\left(\tilde{a}_{ij}\right)\frac{\partial\tilde{a}_{ij}}{\partial F_{ij}}}{N_{ij}^2} \quad 0, \tag{A6}$$

Proof of Result2: *Change in variable exporting cost due to change in transportation cost* t_{ij} *or change in variable production cost (i.e.,* $d\theta = d\tau_{ij}$ *).*

Using Equations (7) and (8) in Equations (A1) and (A2):

$$\frac{\partial N_{ij}}{\partial \tau_{ij}} = \quad \bar{N}_i g\left(\tilde{a}_{ij}\right)\frac{\partial \tilde{a}_{ij}}{\partial \tau_{ij}} \quad 0 \text{, and } \frac{\partial E_{ij}}{\partial \tau_{ij}} = \bar{N}_i \int_{\tilde{a}_{ij}}^{a_H} \frac{\partial s_{ij}}{\partial \tau_{ij}} g\left(a\right)da + s_{ij}\left(\tilde{a}_{ij}\right)\frac{\partial N_{ij}}{\partial \tau_{ij}} < 0 . \tag{A7}$$

Thus, a rise in variable cost τ_{ij} reduces both the number of nation *i* firms exporting to nation *j*, and also the aggregate export revenues of the nation from nation *j*. Using Eq. (A4), the effect on average export revenue from sales to nation *j* is:

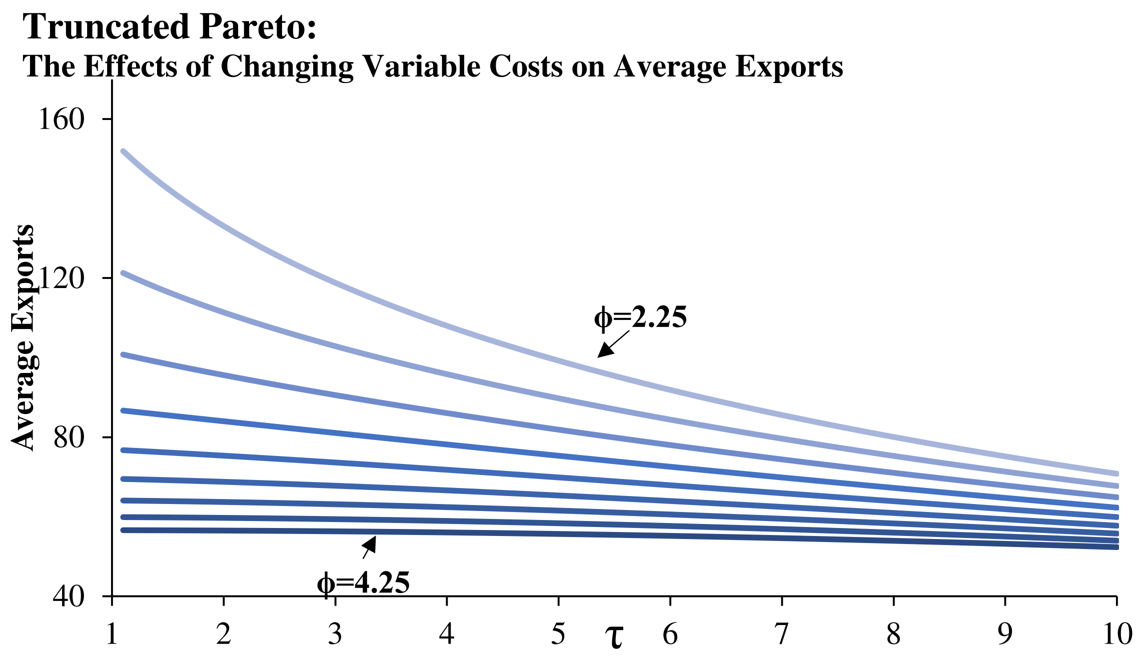
$$\frac{N_{ij}^2}{\bar{N}_i}\left[\frac{\partial\left(E_{ij} / N_{ij}\right)}{\partial \tau_{ij}}\right] = N_{ij}\int_{\tilde{a}_{ij}}^{a_H} \frac{\partial s_{ij}}{\partial \tau_{ij}} g\left(a\right)da \quad \left[E_{ij} \quad N_{ij}s_{ij}\left(\tilde{a}_{ij}\right)\right] g\left(\tilde{a}_{ij}\right)\frac{\partial \tilde{a}_{ij}}{\partial \tau_{ij}} . \tag{A8}$$

The first term on the right-hand side of Equation (A8) is negative, reflecting shrinkage of the average scale of firm exports to nation *j* due to a reduction on the intensive margin of each firm. The last term in Equation (A8) is positive, reflecting an expansion of the average scale due to exit of the least productive firms - which tends to raise average scale of surviving firms. The net effect of these two terms is ambiguous in general, but it may be possible to determine which effect dominates if we use some specific probability density functions, such as the one associated with the truncated Pareto distribution. We present that analysis at the end of this section.

Proof of Result3: *Change in foreign income (i.e.,* $d\theta = dY_j$ *)*

The analysis of this case is similar to Case 2, and hence we focus our discussion on the underlying intuition and results rather than providing the formal proof, which is available upon request. The first thing to notice is that the profit maximizing export price for each product sold to foreign nation *j* must be independent of foreign income Y_j. This is because, given ε, Eq. (5) ties down the export price of product *k* to a constant markup above the effective marginal cost of the exporting the marginal unit to nation *j*, which is $\tau_{ij} / a\left(k\right)$. Furthermore, Equation (2) shows that demand for each product rises proportionally with foreign income. Thus, a rise in Y_j must raise both the scale of exports and the export revenues of each existing domestic firm selling to foreign nation *j* (i.e., an increase on the *intensive margin*). In addition, Eq. (6) reveals that all domestic firms' (including previously non-exporting ones) potential profit from exporting to nation *j* must rise because price, marginal cost, and fixed cost do not change, but the scale of potential export to nation *j* (i.e., x_{ij}) increases. This rise in potential profit brings some previously non-exporting firms of each nation into the market for exports to nation *j* (an increase on the *extensive margin*). However, analogous to the previous case explored, average exports E_{ij} / N_{ij} may or may not rise, because relaxation of the extensive margin introduces some less productive firms, which pulls average scale in the negative direction.

Figure 1



Note: Parameterization details for all figures are provided at the end of the manuscript.

Figure 2

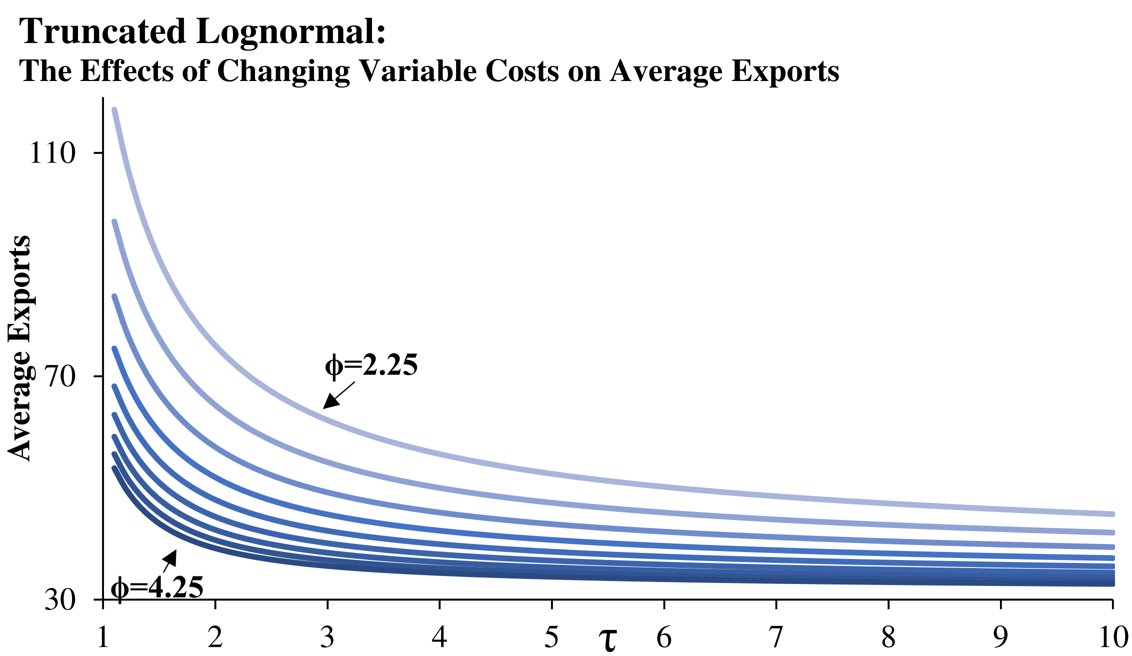


Figure 3

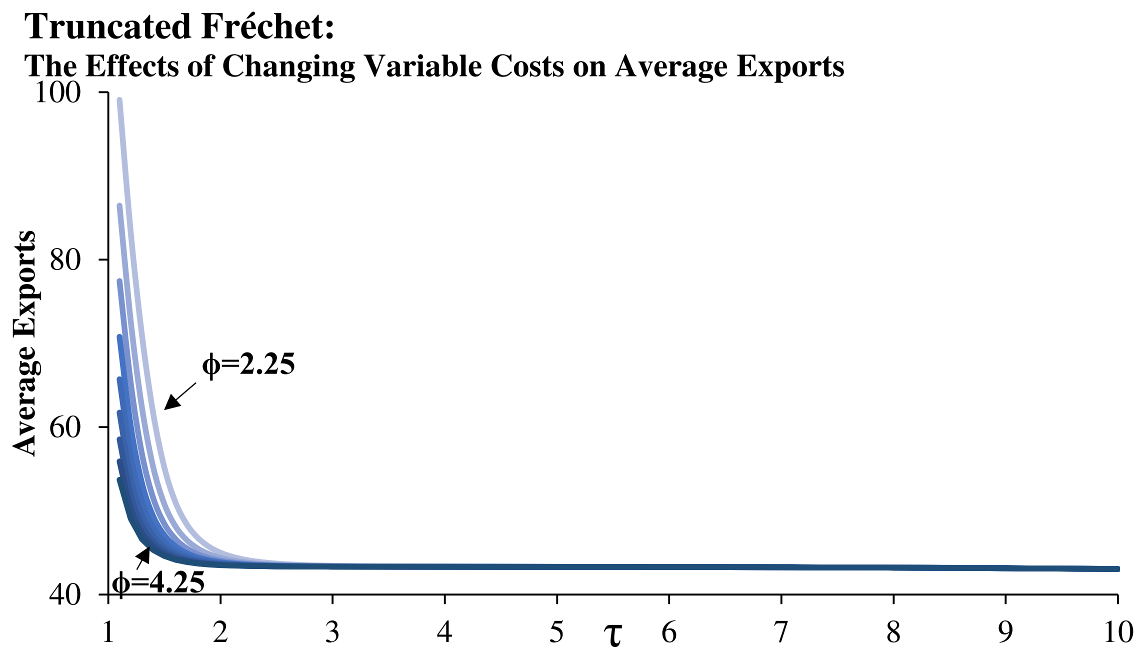


Figure 4

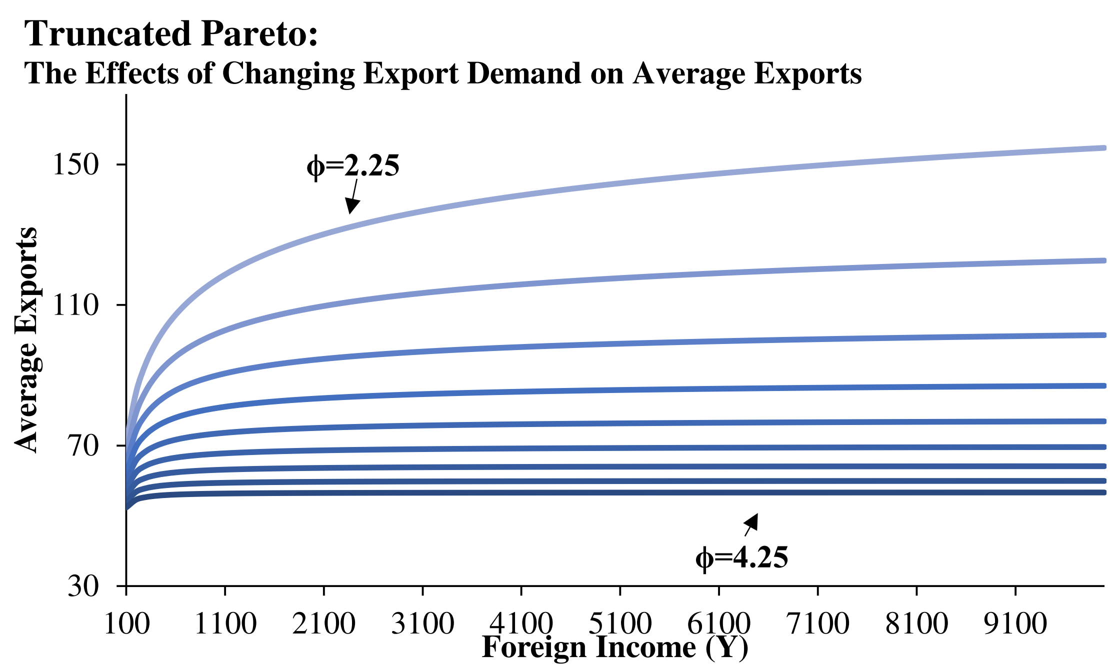


Figure 5

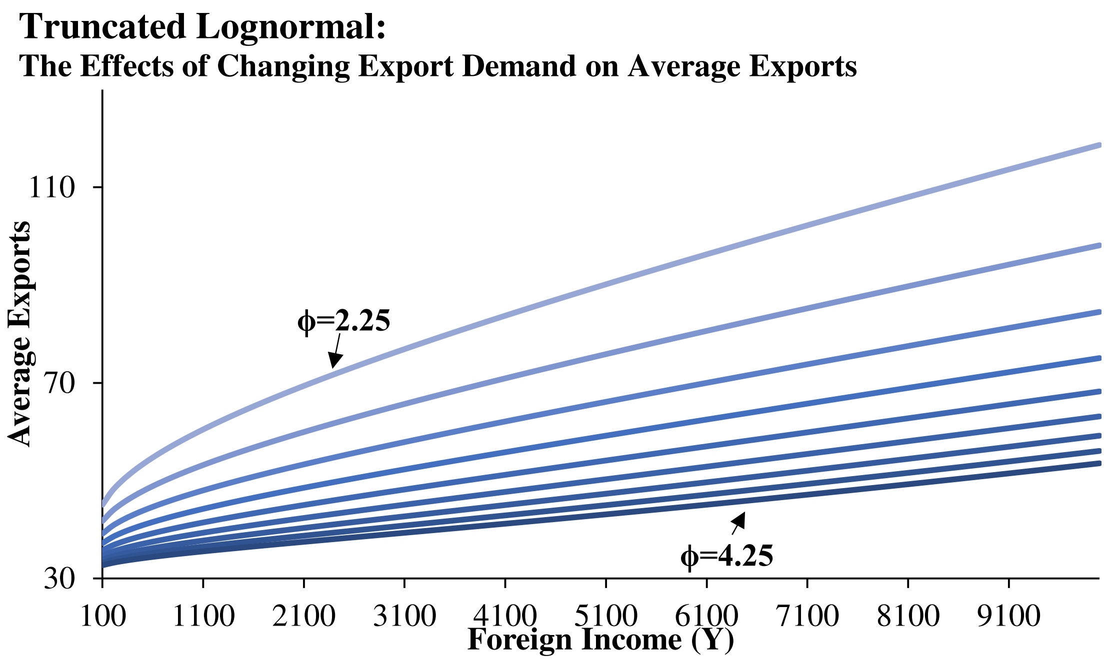


Figure 6

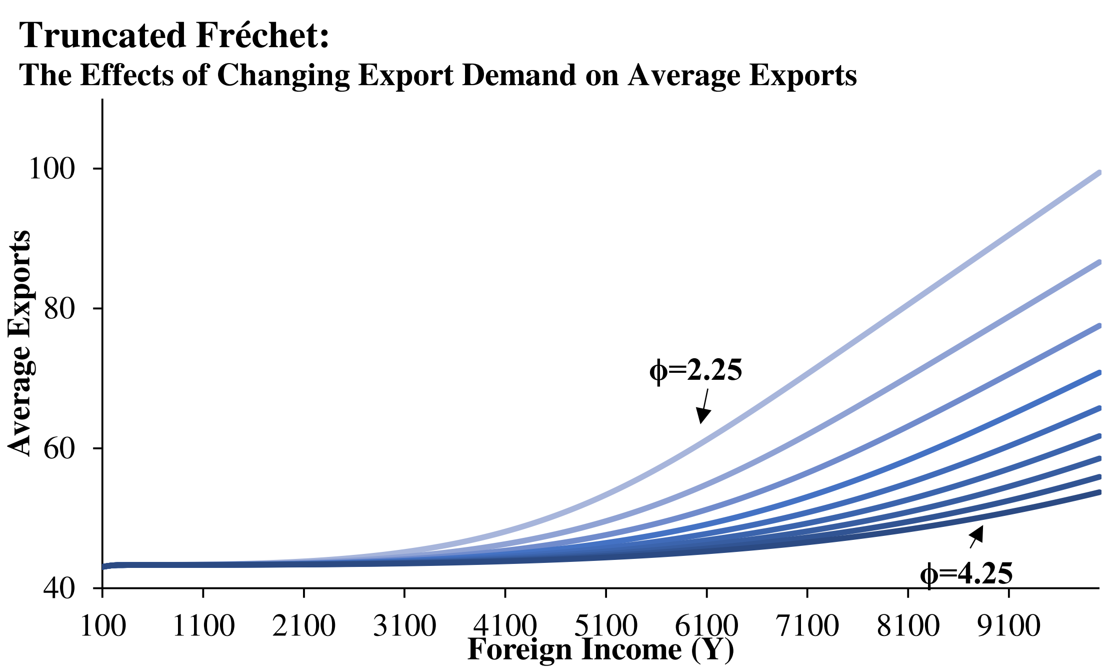


Figure 7

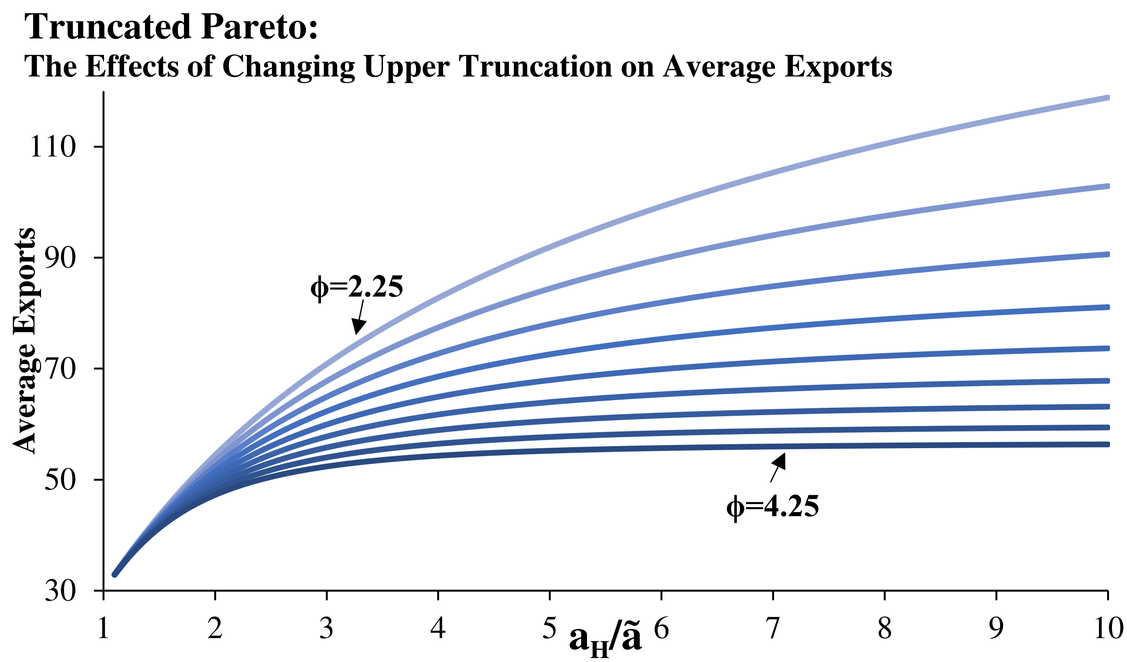


Figure 8

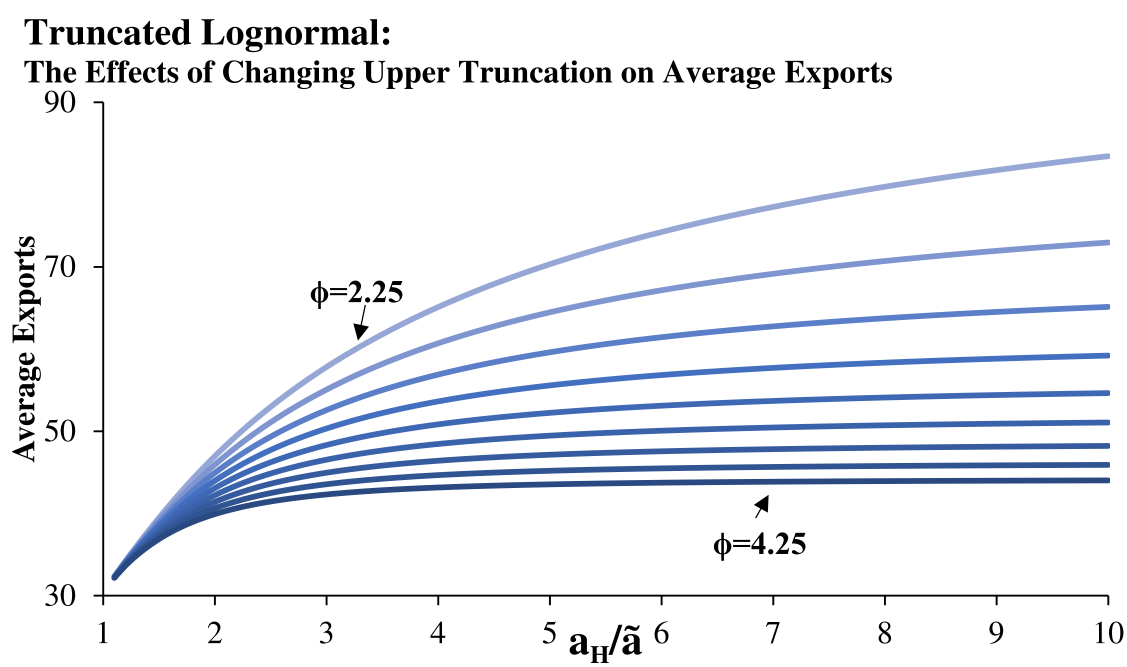
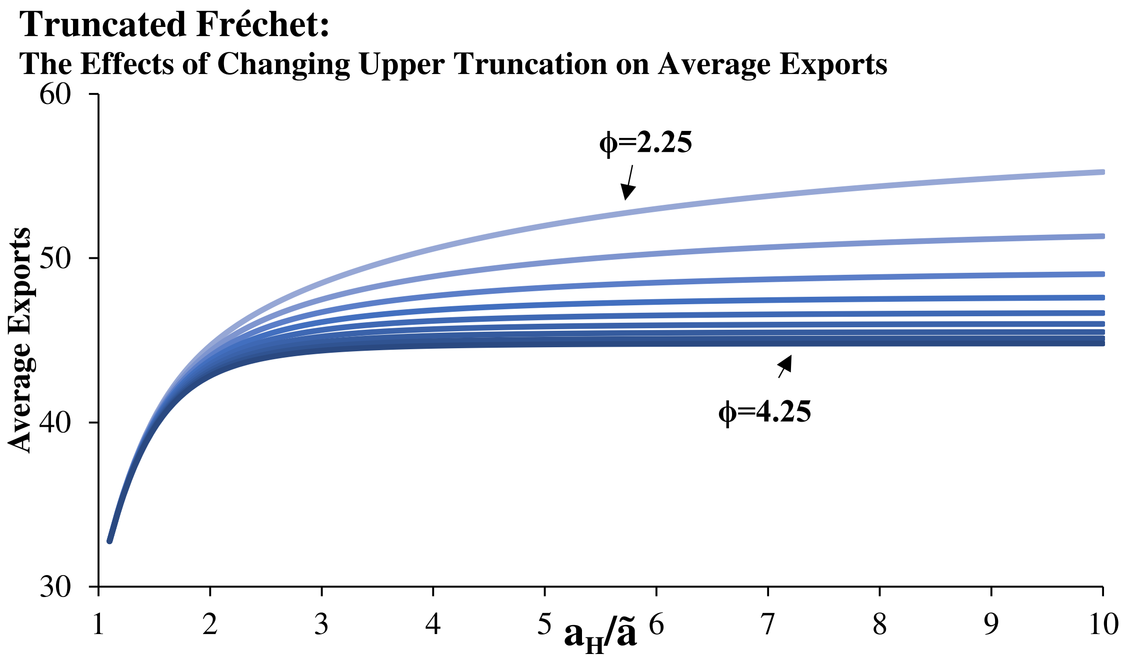


Figure 9



Truncated Fréchet:
The Effects of Changing Upper Truncation on Average Exports
Average Exports
60
50
40
30
1
2
3
4
5
6
7
8
9
10
a_H/ã
ϕ=2.25
ϕ=4.25

Notes on Parameterizations:

1. *Figures 1 through 3 use the following parameter values:*

$\tau = 1.1$ to 10; foreign income (Y)=1000; fixed cost (F)=10; foreign price level (P)=1; $\varepsilon = 3$; $\phi =$ 2.25 to 4.25; $a_L = \min\ \tilde{a} = \tilde{a}$ corresponding to $\tau = 1.1$; $a_H = 3(\max\ \tilde{a}) = 3(\tilde{a}$ corresponding to $\tau = 10$). In Figure 3, θ=6.5. ϕ values in Figures 2 and 3 refer to adjustments to the truncated lognormal shape parameters (μ and σ) and truncated Fréchet scale parameter (T) so that the first moments of the productivity distributions underlying the curves are equal to the first moments of the truncated Pareto distributions underlying the Figure 1 curves with the same ϕ values.

2. *Figures 4 through 6 use the following parameter values:*

$\tau = 3$; foreign income (Y)=100 to 10,000; fixed cost (F)=10; foreign price level (P)=1; $\varepsilon = 3$; $\phi =$ 2.25 to 4.25; $a_L = \min\ \tilde{a} = \tilde{a}$ corresponding to Y=10,000; $a_H = 3(\max\ \tilde{a}) = 3(\tilde{a}$ corresponding to Y=100). In Figure 6, θ=6.5. Figures 5 and 6 adjust for first moment matching as described in #1 above.

3. *Figures 7 through 9 use the following parameter values:*

$\tau = 3$; foreign income (Y)=1000; fixed cost (F)=10; foreign price level (P)=1; $\varepsilon = 3$; $a_L = 0.75(\tilde{a})$; $\phi = 2.25$ to 4.25; $a_H = 1.1(\tilde{a})$ to $10(\tilde{a})$. In Figure 9, θ=6.5. ϕ values in Figures 8 and 9 refer to adjustments to the truncated lognormal shape parameters (μ and σ) and truncated Fréchet scale parameter (T) so that the first moments of the productivity distributions underlying points along the curves are equal to the first moments of the truncated Pareto distributions underlying the corresponding points along the Figure 7 curves with the same ϕ values.