RESEARCH DIVISION

Working Paper Series

Nominal GDP Targeting With Heterogeneous Labor Supply

James Bullard
and
Aarti Singh

Working Paper 2017-016B

January 2019

FEDERAL RESERVE BANK OF ST. LOUIS
Research Division
P.O. Box 442
St. Louis, MO 63166

Nominal GDP Targeting with Heterogeneous Labor Supply

James Bullard* **Aarti Singh†**

Abstract

We study nominal GDP targeting as optimal monetary policy in a model with a credit market friction following Azariadis, Bullard, Singh and Suda (2018), henceforth ABSS. As in ABSS, the macroeconomy we study has considerable income inequality which gives rise to a large private sector credit market. Households participating in this market use non-state contingent nominal contracts (NSCNC). We extend the ABSS framework to allow for endogenous and heterogeneous household labor supply among credit market participant households. We show that nominal GDP targeting continues to characterize optimal monetary policy in this setting. Optimal monetary policy repairs the distortion caused by the credit market friction and so leaves heterogeneous households supplying their desired amount of labor, a type of "divine coincidence" result. We also analyze the incomplete markets equilibrium that exists when the monetary policymaker pursues a suboptimal policy, and show how an extension to more general preferences can limit the ability of the policymaker to provide full insurance to households in this setting.

Keywords: Non-state contingent nominal contracting, optimal monetary policy, nominal GDP targeting, life cycle economies, heterogeneous households, credit market participation, labor supply. *JEL codes*: E4, E5.

*Federal Reserve Bank of St. Louis; james.b.bullard@stls.frb.org. Any views expressed are those of the authors and do not necessarily reflect the views of others on the Federal Open Market Committee.

†Corresponding author. University of Sydney; aarti.singh@sydney.edu.au.

1 Introduction

1.1 Overview

Recent papers by Sheedy (2014), Koenig (2013), and Azariadis, Bullard, Singh and Suda (2018), hereafter ABSS, provide analyses of optimal monetary policy in economies where the key friction is in the credit market in the form of non-state contingent nominal contracting (NSCNC).[1] They all show that optimal policy can be characterized as a version of nominal GDP targeting in that environment. The monetary policy provides a form of insurance to private sector credit-using households.

The ABSS model is based on credit-using households with inelastic labor supply. An open question is whether the optimal monetary policy they isolate could continue to be characterized as nominal GDP targeting if credit-using households were allowed to adjust labor supply in response to shocks. In principle, these (heterogeneous) households may be able to partially self-insure in this circumstance, thus altering the nature of the nominal GDP targeting policy or even rendering it unnecessary. Our goal is to study this issue in this paper.

We construct an extension of the life cycle framework of ABSS to a case of endogenous (and heterogeneous) labor supply.[2] In particular, credit-using households now have homothetic preferences defined over both consumption and leisure choices. We think this is a step toward realism in models of this type, but we do not consider this version of the model to be sophisticated enough to compare to data in a comprehensive way.[3]

Our main finding is that nominal GDP targeting continues to characterize the optimal monetary policy in the situation with endogenous and heterogeneous labor supply. The policy completely repairs the distortion caused by the NSCNC

[1] We thank two anonymous referees, David Andolfatto, Been-Lon Chen, Yunjong Eo, Thomas Sargent, Chung-Shu Wu, and seminar participants at Narodowy Bank Polski, Bank of Korea's Policy Conference, Academia Sinica, Society for Computational Economics and Finance and London School of Economics for their constructive input. We are grateful for the financial support from the Faculty of Arts and Social Sciences, University of Sydney FRSS grant (Singh). The usual disclaimers apply.

[2] Sheedy (2014) and Koenig (2013) also consider labor supply in their models. In our model we have more than two types of agents and so it is not clear how endogenous labor supply would impinge on the NGDP targeting result in that case.

[3] We hope to take up the challenge of matching data in future versions of the model.

friction and allows all credit-using households to consume equal amounts at each date. This is the hallmark of the NGDP targeting policy in this model—under this policy credit markets are characterized by "equity share" contracting, which is optimal when preferences are homothetic. In this respect the findings here are consistent with the findings of Koenig (2013) and Sheedy (2014).

Our main result is a version of the "divine coincidence" result familiar from the New Keynesian monetary policy literature.[4] In our model, there is a single friction, which is non-state contingent nominal contracting in the credit sector. Monetary policy can alter the price level to eliminate the distortion arising from this friction and restore the first-best allocation of resources. Our main result shows that, in this situation, households are able to choose their optimal level of labor supply as well, and in fact these heterogeneous labor supply choices are independent of the aggregate shock in the model. The divine coincidence is that, by completely mitigating the credit market friction, the monetary policy also allows for optimal labor supply choices.

1.2 Additional motivation

Aside from the issue of the circumstances under which the NGDP targeting policy remains optimal in the face of endogenous labor supply in this setting, we additionally motivate the paper with a contemporary issue in monetary policy. Since the 2007-2009 financial crisis, the labor force participation rate in the U.S. has been low and falling.[5] A key question for monetary policymakers has been whether the falling labor force participation rate is driven by business cycle factors, in which case monetary policymakers may want to attempt to increase the participation rate through monetary policy choices. But an alternative, and we think more traditional, view is that the labor force participation rate is driven by demographic factors, in which case policymakers will be unable to meaningfully change the participation rate via monetary policy. The results in this paper provide some support for the traditional view.

Figure 1 is the first figure from Erceg and Levin (2014). They suggest, based

[4] See Blanchard and Gali (2010) and Woodford (2003).

[5] The literature documenting the fall in the labor force participation rate is growing. See for example, Aaronson, Cajner, Fallick, Galbis-Reig, Smith and Wascher (2014), Van Zandweghe (2012), Daly, Elias, Hobijn and Jordà (2012), Hotchkiss and Rios-Avila (2013) among others.

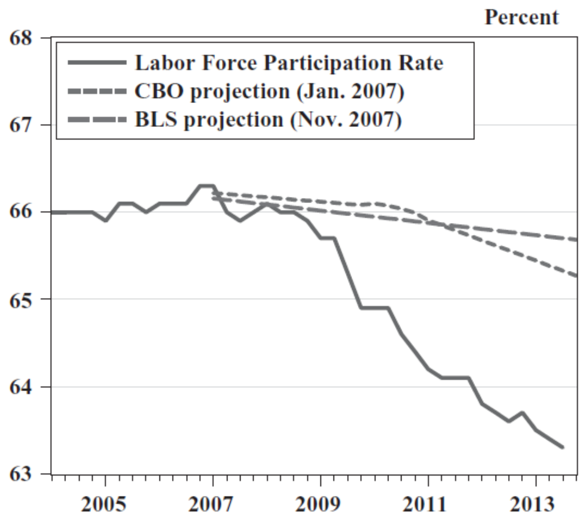

Figure 1: The first figure, left panel, from Erceg and Levin (2014). After the financial crisis and recession of 2007-2009, labor force participation in the U.S. fell more than its forecast by leading government agencies.

on this evidence from 2004-2013, that labor force participation fell much more than expected by some government agencies in the aftermath of the financial crisis and recession of 2007-2009 in the U.S. They construct a New Keynesian model of monetary policy in which the labor force participation rate would not be an important cyclical variable in normal times, but which may remain significantly depressed following a particularly large macroeconomic shock. They find that monetary policy may be able to help mitigate an inefficiently low level of labor force participation.

Figure 2 offers a different take on the same data. This Figure again shows the U.S. labor force participation rate, the solid line, this time from 1991-2015. The figure also shows a forecast labor force participation rate due to Aaronson, Fallick, Figura, Pingle, and Wascher (2006), represented by the dotted line in the figure. The Aaronson, *et al.*, (2006) model is based importantly on demographic effects. Their 2006 model successfully predicted the decline in labor force participation as of the 2013 to 2015 time frame, about 7 to 9 years in the future.[6] We think of the Aaronson, *et al.*, (2006) findings as consistent

[6] Hall and Petrosky-Nadeau (2016), Daly et. al. (2012) among others also attribute declin-

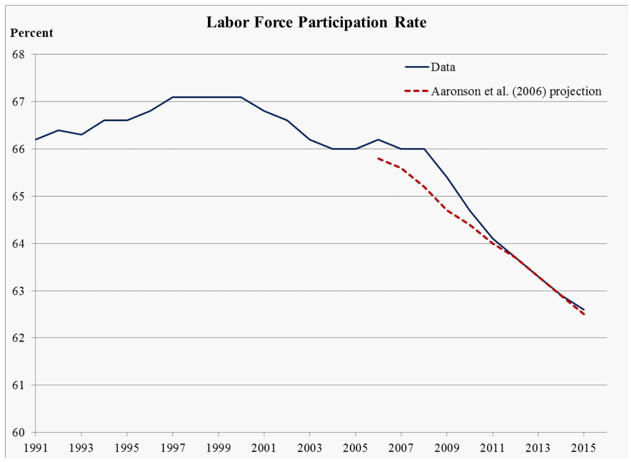

Figure 2: Labor force participation in the U.S., 1991-2015. The dotted line represents the forecast made by Aaronson et al (2006) using a model based largely on demographic factors. The forecast is essentially correct even 7 to 9 years in the future.

with what we see as a "traditional view" in macroeconomics, whereby labor force participation is an essentially acyclical variable, movements in which would not be indicative of business cycle developments but would rather reflect longer-run changes in desired labor supply by the workforce in the economy. These longer-run movements would then be viewed as largely independent of monetary policy choices.

In the current paper, we will present a model which is consistent with what we call the traditional view of labor force participation. In the model, monetary policy will have an important role to play in ensuring good credit market performance. If this monetary policy is carried out in the optimal manner (which will turn out to be our version of nominal GDP targeting), then labor supply choices, while heterogeneous, will indeed be independent of monetary policy and of the shocks buffeting the economy.[7,8]

ing labor force participation rates to ongoing secular change in trend. In a recent paper, Krueger (2016) plots a forecast of the labor force participation rate from 2007 to 2016. That plot exhibits a declining pattern similar to the one shown in Figure 2.

[7] Here the phrase "independent of monetary policy" is meant to convey that since the policymaker is changing the price level in response to aggregate productivity shocks, the households decide not to change their labor supply in response to the price level movements.

[8] For an extensive discussion of demographic trends and their associated effects on labor force participation, see Krueger (2016). For a review of some of the recent literature and

2 Environment

2.1 The private sector

2.1.1 Background on symmetry

ABSS work with a stylized general equilibrium life cycle economy with an aggregate shock to productivity growth. This means that the economy has heterogeneous households along with an aggregate shock, and therefore that the equilibrium includes tracking the asset-holding distribution in the economy. However, to keep the heterogeneity manageable and the equilibrium calculable based on closed-form solutions, ABSS made certain "stylized symmetry assumptions." In the life cycle model, much depends on the productive capacity of older cohorts, the suppliers of credit, versus the productive capacity of younger cohorts, the demanders of credit. The goal is to keep this aspect of the model in balance via simplifying assumptions. Accordingly, the productivity endowments of credit-using households are assumed to be perfectly symmetric, peaking in the middle period of life. In addition, there is no discounting of the future in the preferences (that is, the discount factor $\beta = 1$), which would otherwise tend to favor consumption today over consumption tomorrow.[9] Combined with time-separable log preferences, these assumptions help make the key quantities in the model—including the net asset positions of all households—linear in the real wage. This means that the asset-holding distribution can be tracked easily. The equilibrium real interest rate on consumption loans is then exactly equal to the real output growth rate each period, even in the stochastic case when the monetary policy is optimal. Altogether, this creates a particularly tractable framework in which the intuition behind the nominal GDP targeting policy is brought into clearer relief.

We will maintain these same stylized symmetry assumptions in this paper as well.

connections to current monetary policy issues, see Bullard (2014).

[9] We could add discounting, at the cost of minor complications. It is unnecessary in this type of model and so we omit it for simplicity.

2.1.2 Cohorts and segmented markets

We now turn to describing the model in more detail. Following ABSS, we use a life cycle model with segmented markets. Cohorts are collections of identical, atomistic households entering and later exiting the economy at the same date. Cohorts are divided into two types, a large group of "credit market participants" of measure m and a small group of "credit market non-participants" with measure n, and $m + n = 1$. In the model population is constant.[10] We also refer to these two groups as "credit users" and "cash users," respectively. Households live in discrete time for $T + 1$ periods. Our results will hold for any integer $T \geq 2$, but for the main analysis in this paper we use the value $T + 1 = 241$ so that we can interpret results as corresponding to a quarterly model in which households begin and end economic life with zero assets, starting at age 20, and continuing until death at age 80. Choosing $T + 1$ to be an odd number allows for a convenient and specific peak period for participant productivity endowment profiles. The economy itself continues into the infinite past and into the infinite future, with discrete time denoted by t. The assets in the economy are privately-issued nominal debt and currency. Credit market participant households can hold either of these assets, but in the equilibria we study, they will only hold only privately-issued nominal debt, which will pay a real rate of return which is always higher than the real rate of return on holding currency.[11] Cash-using households are excluded from credit markets altogether and only hold currency. All loan contracts are for one period, are not state-contingent, and are expressed in nominal terms—we call this the non-state contingent nominal contracting (NSCNC) friction.

We follow the notational convention that subscript t indicates the cohort (the "birth date") and that t in parentheses denotes real time. The exception to this is the productivity endowment notation.

[10] For a version of the model with population growth included, see the working paper version of this paper.

[11] That is, nominal interest rates will always be positive.

2.1.3 Technology

The technology is a simple extension of the endowment economy idea that "one unit of labor produces one unit of the good," but with appropriate adjustments for productivity endowments e and labor supply $1 - \ell$. We denote the level of TFP as $Q(t)$. The gross growth rate of Q follows a stochastic process such that

$$Q(t) = \lambda(t-1,t)\,Q(t-1), \tag{1}$$

where $\lambda(t-1,t)$ is the growth rate of productivity between date $t-1$ and date t. The stochastic process driving the growth rate of productivity is $AR(1)$ with mean $\bar{\lambda}$,

$$\lambda(t,t+1) = (1-\rho)\,\bar{\lambda} + \rho\lambda(t-1,t) + \sigma\epsilon(t+1) \tag{2}$$

where $\bar{\lambda} > 1$ is the mean growth rate, serial correlation is $\rho \in (0,1)$, $\sigma > 0$ is a scale factor, and $\epsilon(t)$ is drawn from a truncated normal distribution with bounds $\pm b$, and where the bounds are set such that the zero lower bound on the nominal interest rate is not encountered.[12]

Aggregate output is given by

$$Y(t) = Q(t)\,L(t). \tag{3}$$

where $L(t) = L^p(t) + L^{np}(t)$ is the total supply of labor in this economy coming from both credit market participants $L^p(t)$ and non-participants $L^{np}(t)$. The marginal product of labor is

$$w(t) = Q(t) \tag{4}$$

and we conclude that

$$w(t) = \lambda(t-1,t)\,w(t-1) \tag{5}$$

as in ABSS. If we denote $1 - \ell \in (0,1)$ as the fraction of participant household time spent working per period, the participant labor input at date t is given by

$$L^p(t) = n[e_0(1-\ell_t(t)) + e_1(1-\ell_{t-1}(t)) + \cdots + e_T(1-\ell_{t-T}(t))] \tag{6}$$

and the supply of labor by non-participants is simply

$$L^{np}(t) = m\,[T/2]\,\gamma. \tag{7}$$

[12] We will not address ZLB issues in this paper. However, a version of the nominal GDP targeting monetary policy can address it when $\eta = 1$ as discussed in ABSS.

The aggregate output growth rate is then

$$\frac{Y(t)}{Y(t-1)} = \frac{Q(t)\,L(t)}{Q(t-1)\,L(t-1)} = \lambda(t-1,t)\,\frac{L(t)}{L(t-1)}. \tag{8}$$

What is $L(t)/L(t-1)$? Our baseline result is that the leisure choices ℓ are independent of the λ shocks and hence of the wage, so they are constants in this formula, meaning in particular that various cohorts will make the same leisure choice at the same stage of the life cycle, represented by $\ell_t(t) = \ell_{t-1}(t-1)$, $\ell_{t-1}(t) = \ell_{t-2}(t-1)$, and so on. This implies that $L(t)/L(t-1) = 1$. We conclude that we can write

$$\frac{Y(t)}{Y(t-1)} = \lambda(t-1,t) \tag{9}$$

for the equilibria we wish to study. Along the nonstochastic balanced growth path the gross output growth rate would be $\bar{\lambda}$. We will show below that the real interest rate equals the real output growth rate period-by-period in the stochastic equilibria we study in Section 3.

2.1.4 Timing protocol

The timing protocol in the credit market is as follows. At any period t, participant households enter with one-period nominal contracts carrying an interest rate $R^n(t-1,t)$ that were based on the expected growth rate between period $t-1$ and t, that is, $E_{t-1}[\lambda(t-1,t)]$, as well as expected inflation between period $t-1$ and t. Nature moves first and draws a value for $\epsilon(t)$ implying a value of $\lambda(t-1,t)$, the productivity growth rate between date $t-1$ and date t. The monetary policymaker moves next and chooses a value for its monetary policy instrument, the price level $P(t)$. Given these choices, credit-using households make decisions to consume, supply labor and save via non-state contingent nominal consumption loan contracts for the following period, carrying a nominal interest rate $R^n(t,t+1)$.

2.1.5 Participant households

Credit market participant households enter the economy endowed with a known sequence of productivity units given by $e = \{e_s\}_{s=0}^{T}$. We assume that this productivity endowment sequence is hump-shaped and symmetric (that is, $e_0 =$

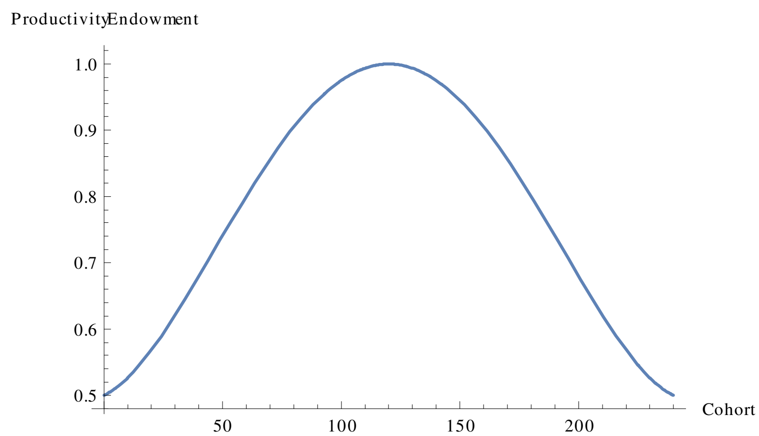

Figure 3: A schematic productivity endowment profile for credit market participant households. The profile is symmetric and peaks in the middle period of the life cycle. Households are about twice as productive in the middle periods of economic life as compared to the beginning and end of economic life.

e_T, $e_1 = e_{T-1}$, $e_2 = e_{T-2}$...) and that it peaks exactly at the middle period of life. We use the following profile:

$$e_s = f(s) = \mu_0 + \mu_1 s + \mu_2 s^2 + \mu_3 s^3 + \mu_4 s^4 \tag{10}$$

such that $f(0) = 0.5$, $f(60) = 0.8$, $f(120) = 1$, $f(180) = 0.8$, and $f(240) = 0.5$. This is a stylized endowment profile which emphasizes that near the beginning and end of the life cycle productivity is low, while in the middle of the life cycle it is high. This endowment profile is displayed in Figure 1. The profile is such that while productivity is relatively low at the beginning and end of economic life, it is not so low that households choose to supply zero labor in those stages of their life-cycle. This allows us to restrict attention to interior solutions for labor supply for the equilibria we study. Participant households sell the productivity units they are endowed with each period on a labor market at a economy-wide competitive wage per efficiency unit. Given the nature of the endowment profile, participant households want to borrow when young and save for later periods in their life-cycle. Borrowing and lending in the credit market is through one period NSCNC debt contracts.

Credit market participant households entering the economy at date t with no asset holdings have preferences given by

$$U_t = \sum_{s=0}^{T} \eta \ln c_t\left(t+s\right) + \left(1-\eta\right) \ln \ell_t\left(t+s\right) \tag{11}$$

where $c_t\left(t+s\right) > 0$ is the date $t+s$ real consumption of the household and $\ell_t\left(t+s\right) \in \left(0,1\right)$ is the date $t+s$ leisure of the household entering the economy at date t. Each household has one unit of time in each period of life. The discount factor $\beta = 1$ and $\eta \in \left(0,1\right]$ is the weight on consumption and $1-\eta$ is the weight on leisure in the utility function. The date t participant household is subject to a sequence of budget constraints expressed in real terms is given by

$$c_t\left(t\right) \leq e_0 w\left(t\right)\left(1-\ell_t(t) - \frac{a_t\left(t\right)}{P\left(t\right)}, \tag{12}$$

$$c_t\left(t+1\right) \leq e_1 w\left(t+1\right)\left(1-\ell_t(t+1)\right) + R^n\left(t,t+1\right)\frac{a_t\left(t\right)}{P\left(t+1\right)} - \frac{a_t\left(t+1\right)}{P\left(t+1\right)}, \tag{13}$$

...

$$c_t\left(t+T\right) \leq e_T w\left(t+T\right)\left(1-\ell_t(t+T)\right) + R^n\left(t+T-1,t+T\right)\frac{a_t\left(t+T-1\right)}{P\left(t+T-1\right)}, \tag{14}$$

where $R^n\left(t,t+1\right)$ is the one-period gross nominal rate of return on loans originated at date t and maturing at date $t+1$ in the credit sector of the economy and $P\left(t\right)$ is the price level at date t. The net nominal loan amounts of the participant cohort i at date t is denoted by $a_i\left(t\right)$, and we interpret negative values as borrowing.

Note that in this economy, there are also credit market participant households that entered the economy at date $t-1$. These households have preferences given by

$$U_{t-1} = \sum_{s=0}^{T-1} \eta \ln c_{t-1}\left(t+s\right) + \left(1-\eta\right) \ln \ell_{t-1}\left(t+s\right). \tag{15}$$

These households will also have a net asset position which we denote by $a_{t-1}\left(t-1\right)$, which indicates the net asset holdings carried into the current period from date $t-1$ by the cohort that entered the economy at date $t-1$. Preferences for participant households that entered the economy at dates $t-2$, ..., $t-T$ are defined analogously, with net asset positions $a_{t-2}\left(t-1\right)$, ..., $a_{t-T}\left(t-1\right)$.

From the optimization problem of the participant household born at date t (cohort t), based on the Euler equation, the non-state contingent nominal interest rate between date t and $t+1$ is given by[13]

$$R^n(t,t+1)^{-1} = E_t\left[\frac{c_t(t)}{c_t(t+1)}\frac{P(t)}{P(t+1)}\right]. \tag{16}$$

The E_t operator indicates that households use information available as of the end of period t before the realization of $\epsilon(t+1)$. For cohorts born at date $t-1$, Euler equation implies that the nominal interest rate is given by

$$R^n(t,t+1)^{-1} = E_t\left[\frac{c_{t-1}(t)}{c_{t-1}(t+1)}\frac{P(t)}{P(t+1)}\right]. \tag{17}$$

This would similarly be true for all other households entering the economy at earlier dates up to date $t-T+1$. However, in the equilibria we study, it will turn out that

$$\frac{c_t(t)}{c_t(t+1)} = \frac{c_{t-1}(t)}{c_{t-1}(t+1)} = \cdots = \frac{c_{t-T+1}(t)}{c_{t-T+1}(t+1)}, \tag{18}$$

so that these expectations will all be the same and hence (16) suffices to determine the non-state contingent nominal interest rate.

2.1.6 Non-participant households

The non-participant households are precluded from using the credit market. They provide a demand for currency in this economy. These households live for $T+1$ periods like the credit market participant households, and we will discuss them in terms of their stage of life $0,1,2,...,T$. Their productivity endowment pattern is quite different from credit market participant households. In the first period of life they are inactive. Thereafter, in odd-dated stages of life, these households have a productivity endowment $\gamma \in (0,1)$. We think of this as being a low value, and there is no life cycle aspect to it. The cash-using households entering the economy at date t then supply labor inelastically and earn income $\gamma w(t+s)$ for $s = 1,3,5,...,T-1$. In even-dated stages of life, the non-participant households consume. Their period utility is $\ln c_t(t+s)$, $s = 2,4,6,...,T$. The core idea is that these agents work intermittently, carrying

[13] See Chari and Kehoe (1999) for more details.

the value of their labor income via currency holdings into the period when they wish to consume.

The non-participant household problem generates a conventional currency demand. For brevity, we omit further description of this problem here and refer readers to ABSS. The cash-using households are motivated by an appeal to the unbanked sector of the U.S. economy, which has been estimated to be on the order of 10 to 15 percent of U.S. households.

2.2 Monetary policy

There are no taxes or government expenditures in this model other than what is described below.

2.2.1 Equilibrium in the cash market

We denote the nominal currency stock issued by the monetary authority as $H(t)$. Consideration of the household problem indicates that $T/2$ of the cohorts demand currency and that these cohorts each have income $\gamma w(t)$. The real demand for currency is given by

$$h^d(t) = m[T/2]\gamma w(t). \tag{19}$$

Equality of supply and demand in the currency market means

$$\frac{H(t)}{P(t)} = m[T/2]\gamma w(t). \tag{20}$$

The central bank chooses the rate of currency creation between any two dates $t-1$ and t, $\theta(t-1,t)$, as

$$H(t) = \theta(t-1,t) H(t-1). \tag{21}$$

This implies

$$m[T/2]\gamma w(t) P(t) = \theta(t-1,t) m[T/2]\gamma w(t-1) P(t-1) \tag{22}$$

or

$$\theta(t-1,t) = \frac{P(t)}{P(t-1)} \frac{w(t)}{w(t-1)}. \tag{23}$$

The timing protocol implies that $P(t-1)$, $w(t-1)$, and, because nature moves first, $w(t)$, are all known to the policy authority at the moment when θ is chosen. The choice of θ will therefore determine $P(t)$. We conclude that the central bank can in effect choose the date t price level directly under the assumptions we have outlined.

We also assume that the revenue from seigniorage is rebated lump-sum to even-dated cash-using households at each date. Therefore, the level of inflation has no impact on the real allocations of the non-participant households. This means that there will be no distortion in the cash sector.[14]

2.2.2 The complete markets policy rule

The monetary policymaker uses the ability to set the price level at each date t to establish a fully credible policy rule $\forall t$. This state-contingent policy rule is a version of the one discussed in ABSS and is given by

$$P(t+1) = \frac{R^n(t,t+1)}{\lambda(t,t+1)} P(t). \tag{24}$$

The term $\lambda(t,t+1)$ is the ex post realized rate of productivity growth between date t and date $t+1$, that is, the realization of the growth rate for λ observed at date $t+1$. Because $\epsilon(t+1)$, the realized value of the shock, appears in the denominator, this rule calls for countercyclical price level movements. This is a hallmark of nominal GDP targeting as discussed in Sheedy (2014) and Koenig (2013).

2.3 Equilibrium

Given the timing assumption, stationary equilibrium can be described as a sequence of $\{R^n(t-1,t), P(t)\}_{t=-\infty}^{\infty}$ in which households, participant and non-participant, maximize utility subject to the constraints, markets clear, the monetary policymaker credibly adheres to a given rule which determines $P(t)$ and earns seigniorage which is rebated to cash-using households. Given this $P(t)$ sequence, the currency market clears. The credit market, which faces a NSCNC friction, will clear at a gross real interest rate $R(t-1,t)$ at every date. We

[14] For more details, see ABSS.

will conjecture and verify that in this equilibrium $R(t-1,t) = \lambda(t-1,t)$. See Appendix A for more details.

3 Main result

3.1 Overview

First, we describe the nature of the equilibrium of the economy where the monetary authority implements the policy rule (24). We will focus on the credit market which features the NSCNC friction.[15] In this equilibrium, the price rule overcomes the friction in the credit market. Each participant household consumes an equity share of the total credit market output at each date. Labor supply in this case does not depend on the shock, but it is heterogeneous across different cohorts at each date.

Second, we analyze an extension of the baseline case in which the policymaker does not follow the policy rule (24) and instead adheres to a price stability rule $P(t) = 1$ $\forall t$. Credit markets will now be incomplete. In this analysis, for simplicity, we will focus on the participant households. We analyze the implications of this policy rule by linearizing the economy around a non-stochastic steady state. In this case, the labor supply of the participant household will depend, heterogeneously, on the aggregate shock as households will attempt to self-insure because the monetary authority is no longer providing full insurance. We also show how consumption no longer has the equity share contracting feature.

Third, we analyze another extension of the baseline case, this time with more general preferences than the logarithmic case that formed the basis of our main result. We use preferences that nest logarithmic preferences as a special case. We show, analytically, that the monetary authority, despite using the baseline policy rule (24), can no longer fully insure households against the aggregate shock.

[15] The equilibrium in the cash market is described in section 2.2.1.

3.2 Fully optimal monetary policy

To characterize the equilibrium under optimal monetary policy, we first describe the consumption and income of all the cohorts of the participant households. We then turn to labor supply in the next subsection. The detailed steps are provided in Appendix A. The figures in this section are generated using the productivity endowment profile (10) described above. In addition, we have set $w(t) = 1$ for all time periods and $\eta = 0.5$.

Consider the problem of a participant household entering the economy at date t described in Section 2.1.3. The household faces a stochastic optimization problem with a finite sequence of budget constraints. These sequence of budget constraints can be combined into a single lifetime budget constraint. We substitute the state-contingent monetary policy rule (24) into the lifetime budget constraint of the participant household. This problem can then be solved analytically, as is shown in Appendix A.

The solution to the household problem gives a state-contingent plan for consumption and leisure choices. For consumption, the plan is described by

$$c_t(t+s) = \left[\prod_{j=0}^{s-1} \lambda(t+j, t+j+1)\right] c_t(t). \tag{25}$$

The state-contingent plan is that the individual's consumption growth rate is equal to the realized real output growth rate of the economy. The level of consumption of date t cohort of participant household is

$$c_t(t) = w(t) \frac{\eta}{T+1} \sum_{j=0}^{T} e_j. \tag{26}$$

This, like all key quantities in this model, is linear in the real wage $w(t)$. In the special case when $\eta = 1$ (inelastic labor supply) this expression indicates that initial consumption is the cohort share $(1/(T+1))$ of total real income in the credit sector of the economy at date t, which is $w(t)\sum_{s=0}^{T} e_s$. Other values for η make an adjustment for the desirability of leisure.

Other participant cohorts that entered the economy at earlier dates solve a similar problem to the one faced by a household in the date t cohort, except that they carry net assets into the period and that they solve a problem with a

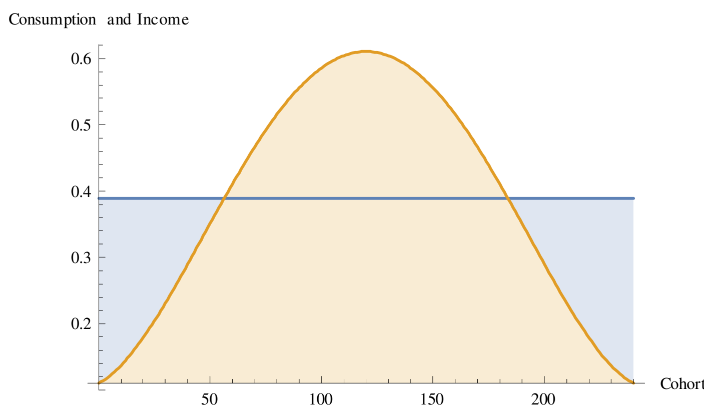

Figure 4: Consumption and labor income by cohort.

shorter horizon. The solution to this problem is similar as shown in Appendix A.

Due to logarithmic preferences, and other symmetry assumptions made in the analysis, even the level of consumption is equalized across households. As a result, when monetary policy is fully optimal, both the level and the growth rate of consumption for different cohorts are in fact equalized in the credit sector and therefore each household has an equity share in the output produced. The proposed monetary policy has completely mitigated the NSCNC friction. Therefore, under this policy credit markets are characterized by "equity share" contracting, which is optimal when preferences are homothetic. In this respect the findings here are consistent with the findings of Koenig (2013) and Sheedy (2014). Figure 4 shows real labor income and real consumption of the participant households in the economy by cohort when the price rule is optimal.

In the figure, best interpreted as a cross section, cohorts are arrayed from youngest to oldest on the horizontal axis. Real labor income by cohort, the hump-shaped line in the figure, is simply the productivity endowment profile multiplied by the real wage at date t, which we have assumed to be unity here, and adjusted by the labor supply of the households at each stage of life. Consumption by cohort, in contrast, is a flat line. This indicates that the credit arrangements in the economy, in conjunction with the monetary policy characterized by the policy rule (24), are allowing the households in the credit sector to completely mitigate the NSCNC friction. Consumption and labor income are both linear in the real wage. As the real wage increases stochastically over time, the curve representing income in the economy would increase proportionately, and the flat line representing consumption would increase but remain flat each period. This is consistent with the idea that households in the credit sector would split the income produced in that sector equally at each date, but that, as the technology improves, there would be more output at each date.

A schematic of equilibrium net asset holding is given in Figure 5. This figure can be viewed as a cross section of net asset holding in the economy at date t. Cohorts are arrayed along the horizontal axis from youngest to oldest, that is, those entering the economy in the current period on the left versus those entering the economy 240 periods in the past on the right. The net asset position is on the vertical axis. Cohorts before the peak earning date at the middle of life are net borrowers and so have negative net asset positions in the figure, while those after the middle period of life are net lenders. The positive and negative areas of the S-shaped curve sum to zero. Net asset holding is exactly zero in the middle period of life due to the symmetry assumptions underlying the model. The peak borrowers are those at period 60 in the life cycle, which corresponds approximately to age 35; we think of this as schematically representing households wishing to take on mortgages to move housing services consumption forward in the life cycle. The peak lenders are those at period 180 in the life cycle, which corresponds approximately to age 65. We think of this as schematically representing households on the verge of retirement. Net asset-holding positions are linear in the real wage $w(t)$, which is itself stochastic. As the real wage rises, the negative and positive net asset positions increase in proportion to the change in the real wage, in such a way as to keep the positive and negative areas summing to zero each period.

There is considerable financial wealth inequality in this economy. If the figure were perfectly triangular, meaning that the two areas were triangles, then 25 percent of the households would hold 75 percent of the financial assets. The actual figure, while not triangular, is close to this.[16]

[16] For an exploration of the extent to which this type of model can generate income, consumption, and financial wealth inequality relatively close to the U.S. data, see Bullard and

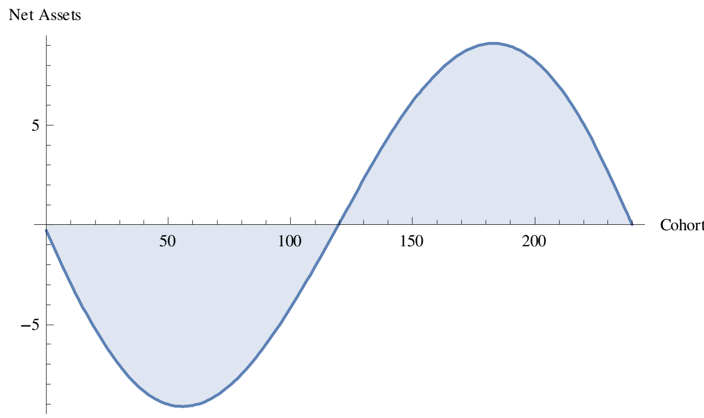

Figure 5: Net asset holding by cohort.

A final step is to ask whether the conjectured sequence of real rates of return clears the loan market in each period. In Appendix A we show that this condition is also met.

3.3 Labor supply in the first best allocation

A key result in this paper is that under the proposed monetary policy rule (24), the labor supply choices of participant households depend on demographics alone and are independent of the productivity shock and the real wage. These households supply labor based on their stage in the life cycle, and so the various cohorts do offer differing amounts of labor, but those differing amounts are not dependent on the realization of the shock in any particular period.

Appendix A shows that the leisure choices of a household in cohort t can be characterized as

$$\ell_t\left(t+s\right) = \frac{1-\eta}{\eta} \frac{c_t\left(t\right)}{w\left(t\right) e_s}, \tag{27}$$

for $s = 0, 1, ..., T$. If we substitute in the expression (26) for $c_t\left(t\right)$, the real wage cancels and the choices depend on the parameters η and e alone. This expression is given by

$$\ell_t\left(t+s\right) = \left(\frac{1-\eta}{T+1}\right)\left(\frac{1}{e_s}\right)\sum_{j=0}^{T} e_j. \tag{28}$$

DiCecio (2018).

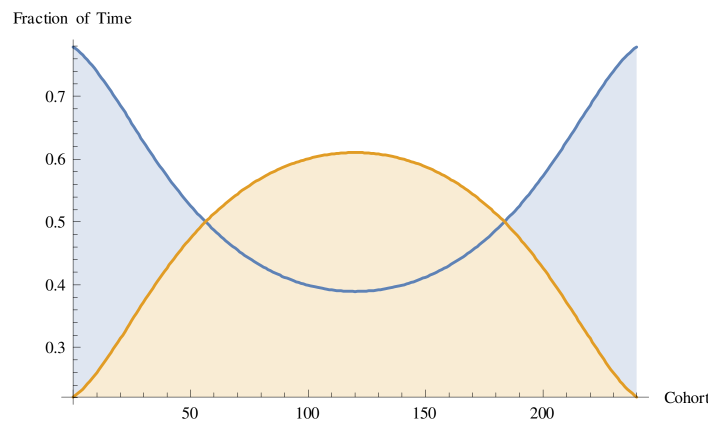

Figure 6: Time devoted to leisure, the u-shaped curve, and work, the hump-shaped curve, by cohort. These curves depend on η and $\{e_s\}$ alone, and not on the aggregate shock.

Figure 6 illustrates the leisure choices and labor supply by cohort for the productivity profile (10) with $\eta = 0.5$. The figure is best viewed as a cross section. Cohorts are arranged from youngest to oldest along the horizontal axis as in previous figures. The vertical axis is the fraction of the one unit time endowment allowed to each household each period which is devoted to leisure. The hump-shaped line in the figure represents the time each cohort devotes to market work, while the inverted u-shaped line represents the fraction of time each cohort spends on leisure. The two lines sum to one for each cohort. The households that are working the most are the ones at the exact middle period (=120) of the life cycle, the peak earners, approximately age 50. The households near the beginning and end of the life cycle devote relatively little time to market work, only about 20 percent versus the 60 percent devoted by the peak income earners. We interpret this as representative of the idea that labor force participation is known to be relatively low near the beginning and end of the life cycle and relatively high during peak earning years. The leisure curve reaches relatively high levels in this figure, but not all time is devoted to leisure even at the beginning and end of the life cycle. This is because we have restricted attention to interior solutions for the purposes of this paper.

The key difference in Figure 6, the labor supply figure, from those representing consumption, labor income, and asset holding, is that labor supply does not depend on the real wage. We interpret this as being consistent with a traditional interpretation of labor supply as discussed at the outset of this paper, in which the labor force participation of households depends most importantly on demographic factors which are unlikely to be influenced by monetary policy. In effect, this makes variables like labor force participation acyclical. In fact, in the economy of this paper, monetary policy is conducted optimally via the policy rule (24), which, in turn, enables households to make optimal labor supply choices independently of productivity shocks hitting the economy. One could view the optimal monetary policy here as allowing households to make labor supply decisions optimally without having to self-insure by adjusting labor supply in response to shocks.

4 Two extensions

4.1 Price stability policy and incomplete markets

In this subsection we evaluate how decisions of different participant cohorts, in particular consumption and labor supply, are impacted when the central bank does not pursue optimal policy and hence credit markets are incomplete. To analyze this, we maintain all the assumptions of the baseline model except that we now consider a monetary policy where the central bank pursues price stability such that $P(t) = 1 \; \forall \, t$. In order the understand the implications of such a policy on the endogenous decisions of participant households of each cohort, we linearize the model around the non-stochastic stationary steady state. To render the variables stationary, we divide them by $w(t)$. Appendix B describes the linear approximation of a $T+1 = 3$ model. Such a linearization can be easily extended to a $T+1 = 241$ period model. Note however, to facilitate a linear approximation of our model, we specify the stochastic process for productivity growth as

$$\lambda(t-1,t) = \bar{\lambda}\exp(\epsilon_t). \tag{29}$$

This does not alter the optimal monetary policy rule specified in equation (24). We set $\bar{\lambda} = 1.01$ and all other calibrated parameter values are same as before.

The results for the 241 period model are described below.

We first consider the case where households do not value leisure, $\eta = 1$. In Figure 7 we plot the impulse response function for consumption for cohorts 40, 100, 160 and 220 to a positive one standard deviation shock at date 0.[17] Note that the income of each cohort is scaled by their productivity profile. Cohorts also enter period 0 with different debt/asset positions, which would be the same as in the non-stochastic steady state. However, the real interest payments on these debt/asset holdings at date 0 will be different depending on the stance of the monetary policy. For instance, if the central bank was conducting an optimal policy, it would have reduced the price level at date 0 when the positive shock occurs, thereby increasing the real interest rate. However, in this case as prices are stable, with higher income, the real interest rate is lower relative to the case when the central bank pursues optimal monetary policy. This benefits the borrowers, younger cohorts, but hurts the savers, the older cohorts.

The overall impact on consumption, which is non-stationary, at date 0 for all the cohorts is plotted in Figure 8.

Now consider the case where households also value leisure so $\eta \in (0, 1)$. The response of deviations of consumption and leisure from their non-stochastic steady state to a one time, one standard deviation shock in this economy is plotted in Figure 9. Note that the deviation of consumption from its steady state value for each cohort is the same whether the agent cares about leisure or not. However the steady state consumption is different—the economy where labor supply is inelastic has higher aggregate consumption than an economy with an elastic labor supply. Therefore the level of consumption for each cohort is higher/lower in an economy with inelastic labor supply relative the consumption of that cohort when labor supply is elastic. As a result, the deviation, in absolute terms, will also be larger when households have an inelastic labor supply. As for leisure, the younger cohorts enjoy more leisure while the older cohorts devote a larger fraction of their time to work, see Figure 9, right panel. Note that the impulse response of consumption and leisure for different cohorts are completely

[17] These impulse response functions plot the deviation of consumption of cohort of a particular age from their respective non-stochastic steady state level to an exogeneous shock at date 0. They are computed using Dynare.

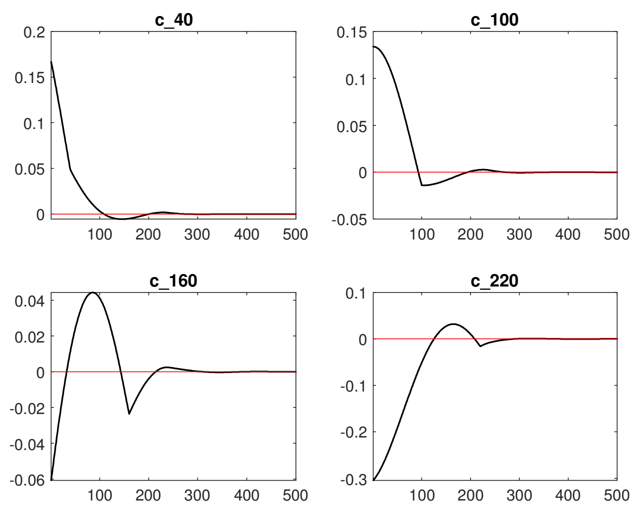

Figure 7: The impulse response functions of deviations of consumption from the non-stochastic steady state for four cohorts to a positive one standard deviation shock when $\eta = 1$.

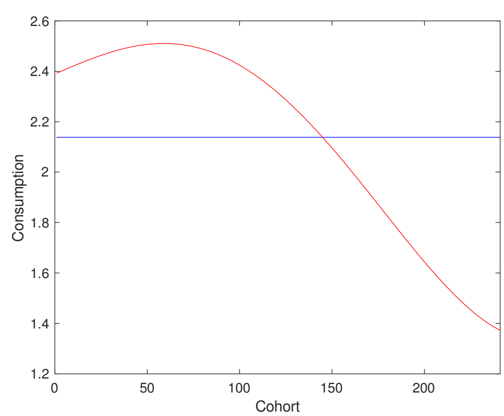

Figure 8: Cross-sectional consumption when $\eta = 1$. The blue line is for the case when the monetary policy is optimal while the red line is when the monetary policy stabilizes prices and credit markets are incomplete.

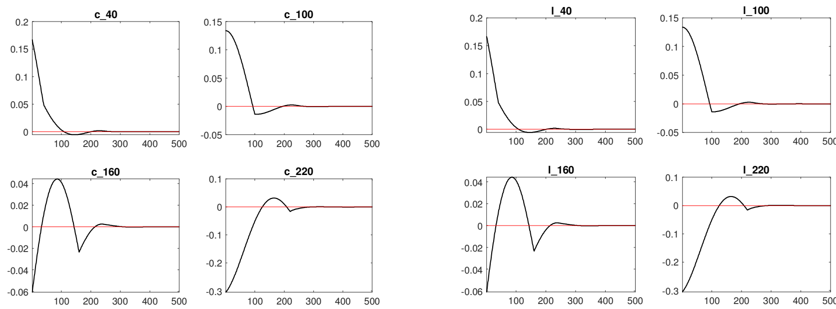

Figure 9: The impluse response functions of deviations of consumption, left panel, and leisure, right panel, from the non-stochastic steady state for four cohorts to a positive one standard deviation shock when $\eta = 0.5$.

identical because of the intratemporal condition given in equation (27).

Figure 10 plots the level of consumption and fraction of time devoted to leisure. Note that since $w(t) = 2.74$, when there in perfect risk sharing, the level of consumption (blue line in the left panel) is scaled by 2.74 relative to Figure 4 but the fraction of time devoted to leisure (blue line in the right panel) is the same as Figure (6). This analysis makes it clear that a monetary policy that does not provide optimal risk sharing between debtors and creditors has both aggregate and distributional implications. The aggregate leisure and aggregate consumption under price stability is lower relative to an economy where the central bank pursues optimal policy. In addition, the distribution of consumption gets distorted relative to the flat consumption across cohorts and it also distorts labor-leisure choice. For instance as seen in the right panel of Figure 10, when the shock is positive, the less productive older cohorts are working harder relative to the cohorts in the middle of the life-cycle that have higher productivity. This analysis also suggests that in an economy where labor supply is elastic, the gains from NGDP targeting are somewhat diminished but like Sheedy (2014), they still remain consequential. However, in this stylized model it is hard to quantify these gains.

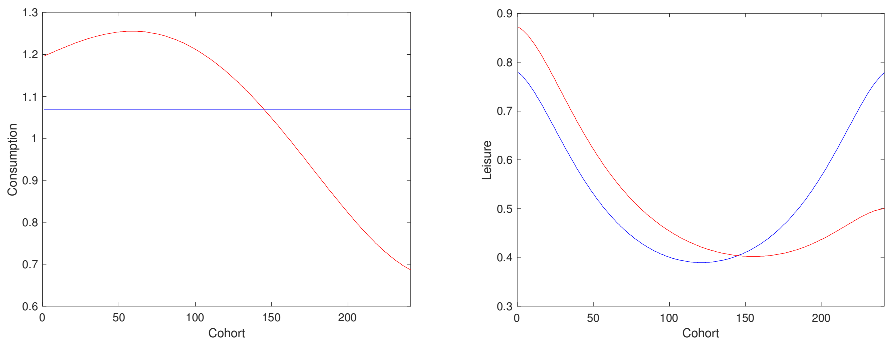

Figure 10: Cross-sectional consumption, left panel, and the fraction of time devoted to leisure, right panel, by different cohorts when there is a one time positive shock. The blue line in each plot is for the case when the monetary policy is optimal while the red line is when the monetary policy stabilizes prices and credit markets are incomplete.

4.2 Extended preferences

We now turn to consider the extent to which our main findings depend on the logarithmic preferences assumption. We find that the logarithmic assumption is important. The monetary policymaker can provide perfect insurance in the logarithmic case but only partial insurance in a case with more general preferences. Our solutions in this section are exact and do not depend on the linear approximation of the previous section.[18]

To answer this question we will restrict attention to the $T+1=3$ case, but generalization to the case of larger values for T is straightforward.

For the participant household cohort entering the economy at date t, consider the preferences given by

$$V=\sum_{s=0}^{2}\mathcal{S}_s\frac{\left[c_t\left(t+s\right)^{\eta}\ell_t\left(t+s\right)^{1-\eta}\right]^{1-\sigma}}{1-\sigma} \tag{30}$$

where $\eta\in(0,1)$ and $\sigma>0$. When $\sigma\to1$ and $\mathcal{S}_s=1$ $\forall s$, these preferences collapse to the preferences used for our main result in Section 3. The new element

[18] For simplicity, the optimization problem of the non-participant households is the same as described in Sections 2.1.4 and 2.2.1.

is a set of shift factors $\mathcal{S}$ which are defined as follows:

$$\mathcal{S}_0 = 1, \tag{31}$$

$$\mathcal{S}_1 = \lambda\left(t, t+1\right)^{\vartheta}, \tag{32}$$

$$\mathcal{S}_2 = \left[\left(\lambda\left(t, t+1\right)\lambda\left(t+1, t+2\right)\right]^{\vartheta}. \tag{33}$$

This definition means that the agent expects that the sequence of $\mathcal{S}$ factors will adjust lifetime utility somewhat each period in the future using a function of the realized growth rate of the economy defined by a power with the parameter ϑ. The value of this parameter is based on σ and η and is derived below. A household entering the economy at date $t-1$ has analogously defined preferences.

We are not making any claims about the realism of the shift factors $\mathcal{S}$. We merely wish to use these shifters to understand the residual amount of insurance that would have to be provided to agents to get to the complete markets solution when the policymaker is following the monetary policy rule (24) in the economy with more general preferences. Accordingly, in the exercise of this section, we will always choose the shifter values appropriately in order to obtain the complete markets solution. In the logarithmic case that we have emphasized in our main result, it will turn out that all $\mathcal{S} = 1$ which is an indication that no additional insurance needs to be provided other than that provided by the monetary policy rule (24). But in other cases the shift factors will be larger than one, which is an indication that some additional insurance would need to be provided over and above the amount achieved with the monetary policy rule (24).

The new timing protocol within a period t is as follows: (1) nature chooses the shock and therefore a value for the realized growth rate $\lambda\left(t-1, t\right)$, (2) the policymaker chooses $P\left(t\right)$, (3) the shifter values are set, (4) agents solve their date t problems taking $P\left(t\right)$ as given.

Let's assume the policymaker again uses the monetary policy rule (24). We can substitute this rule into the budget constraint of the household entering the economy at date t (see appendix for more details) and therefore write the budget constraint for the agent entering the economy at date t as

$$c_t(t) + \frac{c_t(t+1)}{R(t,t+1)} + \frac{c_t(t+2)}{R(t,t+1)\,R(t+1,t+2)} \leq$$
$$e_0[1-\ell_t(t)]\,w(t) + \frac{e_1[1-\ell_t(t+1)]\,w(t+1)}{R(t,t+1)} + \frac{e_2[1-\ell_t(t+2)]\,w(t+2)}{R(t,t+1)\,R(t+1,t+2)}, \tag{34}$$

where $R(t,t+1)$ is the gross real rate of interest. We will guess and verify that $R(t,t+1) = \lambda(t,t+1)\ \forall t$, understanding from the first main result that this is the complete markets solution for this economy. The issue now is what values do the shifters $\mathcal{S}$ have to take on in order to attain the complete markets solution. If all the shift factors turn out to be unity, then the monetary policymaker has provided full insurance with the policy rule (24). If the shift factors are greater than unity, then the policymaker is providing only partial insurance, and the shift factors are providing the rest.

We can use the first order conditions for the problem of this agent, along with the conjecture that $R(t,t+1) = \lambda(t,t+1)\ \forall t$ to obtain (see Appendix for details):

$$c_t(t) = \frac{\eta w(t)[e_0+e_1+e_2]}{1+d_1+d_2} \tag{35}$$

where

$$d_1 = \lambda(t,t+1)^{\frac{\vartheta}{\sigma}}\,\lambda(t,t+1)^{\frac{1-\sigma}{\sigma}}\left[\frac{e_1\lambda(t,t+1)}{e_0}\right]^{\frac{(1-\sigma)(\eta-1)}{\sigma}} \tag{36}$$

and

$$d_2 = [\lambda(t,t+1)\,\lambda(t+1,t+2)]^{\frac{\vartheta}{\sigma}}\,[\lambda(t,t+1)\,\lambda(t+1,t+2)]^{\frac{1-\sigma}{\sigma}}$$
$$\left[\frac{e_2\lambda(t,t+1)\,\lambda(t+1,t+2)}{e_0}\right]^{\frac{(1-\sigma)(\eta-1)}{\sigma}}. \tag{37}$$

Adding the exponents on $\lambda(t,t+1)$ in the d_1 term together and solving for ϑ yields

$$\vartheta = \eta(\sigma-1). \tag{38}$$

The same value is obtained considering the d_2 term. At this value for ϑ, all of the stochastic λ terms in d_1 and d_2 cancel exactly, leaving only terms not involving λ :

$$d_1 = \left[\frac{e_1}{e_0}\right]^{\frac{(1-\sigma)(\eta-1)}{\sigma}} \tag{39}$$

$$d_2 = \left[\frac{e_2}{e_0}\right]^{\frac{(1-\sigma)(\eta-1)}{\sigma}}. \tag{40}$$

We conclude that this household can choose the value of $c_t(t)$ given by equation (35). We also recall the symmetry assumptions of the model which state that, in the $T+1=3$ model, $e_2 = e_0$, implying that $d_2 = 1$. So it is only the d_1 term that is not unity in the $T+1=3$ model under complete markets with extended preferences.

The special cases of the model can now be seen clearly. The logarithmic preferences case featured in the main result section of the paper has $\sigma \to 1$. In that case, $\vartheta = 0$ implying $\mathcal{S}_1$ and $\mathcal{S}_2$ are both unity, and in addition $d_1 = d_2 = 1$. In this case we do not need the $\mathcal{S}$ terms in order to obtain the complete markets solution, as the use of the monetary policy rule (24) is sufficient to completely insure this agent against future shocks.[19] However, when $\sigma \neq 1$ we deviate from the logarithmic case and the terms d_1 and d_2 will no longer be equal to unity and will in fact involve random variables which will require the use of the $\mathcal{S}$ factors to provide the residual amount of insurance to get to the complete markets solution. The amount of residual insurance that must be provided is increasing in $\sigma > 1$ and $\eta \in (0,1)$, as larger values of $\vartheta = \eta(\sigma-1)$ increase the amount of adjustment made to future period period utility outcomes for the agent.

To get to these results we have conjectured that $R(t,t+1) = \lambda(t,t+1)$ $\forall t$ is the stochastic general equilibrium. We need to verify this conjecture by showing that with $\mathcal{S}_1$ and $\mathcal{S}_2$ set appropriately we obtain $A(t) = 0$.[20] We show the details of this calculation in the Appendix.

[19] Another special case is when $\eta \to 0$, which would mean that the household puts vanishingly little weight on consumption versus leisure in the consumption-leisure bundle. In this case, the monetary policymaker would again be able to provide perfect insurance through use of the price level rule.

[20] It may not equal zero since in the general preferences case we have seen that $d_1 \neq 1$, which in principle may prevent the equilibrium real interest rate being equal to the output growth rate.

5 Conclusion

We have provided an analysis of a version of nominal GDP targeting as optimal monetary policy in a model with a credit market friction (NSCNC) following ABSS. The extension here has been to add elastic labor supply as well as population growth. We wanted to study the case of elastic labor supply because, by giving households another margin on which to adjust to shocks, it is possible that nominal GDP targeting would no longer characterize optimal monetary policy as it does in ABSS.

Our main result is that optimal monetary policy is still characterized by a version of nominal GDP targeting even in the expanded setting of this paper. This result could be characterized as a "divine coincidence" result—by providing the optimal monetary policy through countercyclical price level movements, the monetary authority also allows the heterogeneous households to make optimal labor supply decisions. Those decisions do not depend on the aggregate productivity shock in the model and are instead demographically based.

We have also provided two extensions of the baseline model. In one extension, we considered the baseline model with a suboptimal monetary policy, and analyzed the resulting incomplete markets equilibrium. In that case households do alter their labor supply in response to an aggregate shock in order to partially insure. In another extension, we considered the baseline model but with more general preferences. We find that the monetary policymaker can only partially insure households against the aggregate shock in this case.

The life cycle framework used here is quite flexible and has been widely used in the literature to study income and wealth inequality, labor supply issues, as well as consumption and saving. The highly stylized version proposed here and in ABSS has an important role for monetary policy via a credit market friction as the centerpiece of the theory, similar to Sheedy (2014) and Koenig (2013). While the stylized version provides a lot of clarity as to what is going on in the model and the role for monetary policy, we also think the model is flexible enough to be taken to the data in a more comprehensive way in future versions. In addition, we think that the results here could be expanded to include additional assets, like capital, and also to include idiosyncratic labor income risk, as suggested in Werning (2014).

References

[1] Aaronson, S., B. Fallick, A. Figura, J. Pingle, and W. Wascher. 2006. "The Recent Decline in Labor Force Participation and Its Implications for Potential Labor Supply." *Brookings Papers on Economic Activity*, Spring, 69-154.

[2] Aaronson, S., T. Cajner, B. Fallick, F. Galbis-Reig, C. Smith, and W. Wascher. 2014. "Labor Force Participation: Recent Developments and Future Prospects." *Brookings Papers on Economic Activity*, Fall, 197-255.

[3] Aksoy, Y., R. P. Smith, T. Grasl, and H. Basso. 2015. "Demographic Structure and Macroeconomic Trends." Working paper, Banco de España.

[4] Azariadis, C., J. Bullard, A. Singh, and J. Suda. 2018. "Incomplete Credit Markets and Monetary Policy." Working paper, Federal Reserve Bank of St. Louis.

[5] Blanchard, O., and J. Gali. 2010. "Labor Markets and Monetary Policy: A New-Keynesian Model with Unemployment." *American Economic Journal: Macroeconomics*, 2, 1–30.

[6] Bullard, J. 2014. "The Rise and Fall of Labor Force Participation in the U.S." *Federal Reserve Bank of St. Louis Review*, 96, 1-12

[7] Bullard, J. 2014. "Discussion of 'Debt and Incomplete Financial Markets' by Kevin Sheedy." *Brookings Papers on Economic Activity*, Spring, 362-368.

[8] Bullard, J., and R. DiCecio. 2018. "Optimal Monetary Policy for the Masses." Working paper, Federal Reserve Bank of St. Louis.

[9] Chari, V., and P. Kehoe. 1999. "Optimal Fiscal and Monetary Policy." In J. Taylor and M. Woodford, eds., *Handbook of Monetary Economics*, Volume 1C, 26: 1671-1745. Elsevier.

[10] Daly, M., E. Elias, B. Hobijn, and Ò. Jordà. 2012. "Will the Jobless Rate Drop Take a Break?" *Federal Reserve Bank of San Francisco Economic Letter*.

[11] Diamond, P. 1965. "National Debt in a Neoclassical Growth Model." *American Economic Review,* 55, 1126-1150.

[12] Erceg, C., and A. Levin. 2014. "Labor Force Participation and Monetary Policy in the Wake of the Great Recession." *Journal of Money, Credit, and Banking*, Supplement, 46, 3-49.

[13] Eggertsson, G. B., and N. R. Mehrotra. 2015. "A Model of Secular Stagnation." Working paper # 20574, National Bureau of Economic Research, Inc.

[14] Fair, R., and K. Dominguez. 1991. "Effects of the Changing US Age Distribution on Macroeconomic Equations." *American Economic Review*, 81, 1276–1294.

[15] Feng, Z., and M. Hoelle. 2017. "Indeterminacy in Stochastic Overlapping Generations Models: Real Effects in the Long Run." *Economic Theory*, 63, 559-85.

[16] Hall, R., and N. Petrosky-Nadeau. 2016. "Changes in Labor Participation and Household Income." *Federal Reserve Bank of San Francisco Economic Letter.*

[17] Hotchkiss, J., and F. Rios-Avila. 2013. "Identifying Factors behind the Decline in the Labor Force Participation Rate." *Business and Economic Research*, 3, 257–75.

[18] Koenig, E. 2013. "Like a Good Neighbor: Monetary Policy, Financial Stability, and the Distribution of Risk." *International Journal of Central Banking* 9, 57–82.

[19] Krueger, A. 2016. "Where Have all the Workers Gone?" Working paper, Princeton.

[20] Miles, D. 1999. "Modelling the Impact of Demographic Change upon the Economy." *Economic Journal*, 109, 1–36.

[21] Nucci, F., and M. Riggi. 2016. "Labor Force Participation, Wage Rigidities, and Inflation." Working paper, Banca D'Italia.

[22] Sheedy, K. 2014. "Debt and Incomplete Financial Markets: A Case for Nominal GDP Targeting." *Brookings Papers on Economic Activity*, Spring, 301-361.

[23] Van Zandweghe, W. 2012. "Interpreting the Recent Decline in Labor Force Participation." *Economic Review*, Federal Reserve Bank of Kansas City, 5–34.

[24] Werning, I. 2014. "Discussion of 'Debt and Incomplete Financial Markets' by Kevin Sheedy." *Brookings Papers on Economic Activity*, Spring, 368-373.

[25] Woodford, M. 2003. Interest and Prices. Princeton University Press.

A Details of model solution when monetary policy is optimal

The model features heterogeneous households and an aggregate shock, so that the evolution of the asset-holding distribution in the economy is part of the description of the equilibrium. This would normally require numerical computation. However, symmetry, log preferences, and other simplifying assumptions allow solution by "pencil and paper" methods. In this appendix we outline this solution in some detail. A key feature of the solution is that the asset-holding distribution will be linear in the current real wage $w(t)$, and so will simply shift up and down with changes in $w(t)$. Another key feature of the solution will be that the stochastic real rate of return on asset-holding will be equal to the stochastic real output growth rate period-by-period. We do not claim uniqueness of this equilibrium, but we regard the equilibrium we isolate as a natural focal point for this analysis.[21]

We guess-and-verify a solution given a particular price rule for P employed by the monetary authority.

(1) We first propose the state-contingent policy rule for the price level P.

(2a) We then solve the problem of a household entering the economy at date t under the proposed policy and determine their state-contingent plan for consumption and non-state contingent plan for leisure.

(2b) Step 2a also applies for all other households entering the economy at earlier dates with shorter horizons and asset holdings from the previous period. We show how these households will adjust their asset holdings as the real wage evolves.

(3) We then establish that per capita consumption is equal, the optimal "equity share" contract.

(4) Finally we verify that under the proposed policy rule and the derived household behavior, the loan market clearing condition is satisfied. This establishes the equilibrium in the credit sector of the economy.

We will assume interior solutions and verify later.

[21] See Feng and Hoelle (2017) for a recent discussion and analysis. Typical quantitative-theoretic applications in the area of stochastic OLG would be unable to address the issues brought out by the Feng and Hoelle (2017) analysis.

Step 1. The household entering the economy at date t faces uncertainty about income over their life cycle because it does not know what the real wage level is going to be in the future. In addition, these households can borrow and lend using one period NSCNC debt contracts. The proposed policy rule is such that it make the price level state contingent and is given by

$$P(t+1) = \frac{R^n(t,t+1)}{\lambda(t,t+1)} P(t) \tag{41}$$

for all t, with $P(0) > 0$.

Step 2a.

First consider the optimization problem of the households entering the economy at date t

$$\max_{\{c_t(t+s),\ell_t(t+s)\}_{s=0}^{T}} E_t[\sum_{s=0}^{T} [\eta \ln c_t(t+s) + (1-\eta)\ln \ell_t(t+s)] \tag{42}$$

subject to life-time budget constraint

$$\begin{aligned} c_t(t) + \sum_{s=1}^{T} \left(\frac{P(t+s)}{P(t)} \frac{c_t(t+s)}{\prod_{j=0}^{s-1} R^n(t+j,t+j+1)} \right) \\ \leq e_0 w(t)[1-\ell_t(t)] + \sum_{s=1}^{T} \left(\frac{P(t+s)}{P(t)} \frac{e_s w(t+s)(1-\ell_t(t+s))}{\prod_{j=0}^{s-1} R^n(t+j,t+j+1)} \right). \end{aligned} \tag{43}$$

Substitution of the state-contingent policy rule into the budget constraint, along with the conjectured solution $R(t,t+1) = \lambda(t,t+1)$ $\forall t$ for these households yields

$$c_t(t) + \sum_{s=1}^{T} \left(\frac{c_t(t+s)}{\prod_{j=0}^{s-1} \lambda(t+j,t+j+1)} \right) \leq w(t) \sum_{s=0}^{T} e_s (1-\ell_t(t+s)). \tag{44}$$

Because $w(t)$ is known by the household at the time when this problem is solved, the uncertainty about future income has been eliminated by the state-contingent policy. The household then solves this problem where μ is the multiplier on the life-time budget constraint. The following sequence of first order conditions with respect to consumption are

$$\frac{\eta}{c_t(t)} = \mu, \tag{45}$$

and for for $s = 1, 2, ..., T$

$$\frac{\eta}{c_t(t+s)} = \frac{\mu}{\prod_{j=0}^{s-1} \lambda(t+j, t+j+1)} \tag{46}$$

which implies

$$c_t(t+s) = \left[\prod_{j=0}^{s-1} \lambda(t+j, t+j+1)\right] c_t(t) \tag{47}$$

The sequence of first order conditions for $s = 0, 1, ..., T$ with respect to ℓ are

$$\frac{1-\eta}{\ell_t(t+s)} = \mu w(t) e_s \tag{48}$$

Using the FOC for first period consumption to substitute out μ gives leisure for $s = 0, 1, ..., T$

$$\ell_t(t+s) = \frac{1-\eta}{\eta} \frac{c_t(t)}{w(t) e_s} \tag{49}$$

We can then substitute (47) back into the budget constraint, which is

$$(T+1)c_t(t) \leq w(t) \sum_{s=0}^{T} e_s (1 - \ell_t(t+s)). \tag{50}$$

Substituting for leisure gives

$$c_t(t) = w(t) \frac{\eta}{T+1} \sum_{j=0}^{T} e_j \tag{51}$$

We conclude that the choice for $c_t(t)$, first period consumption depends on today's wage $w(t)$ alone. The household has a state-contingent consumption plan for the future. It is to consume

$$c_t(t+s) = \left[\prod_{j=0}^{s-1} \lambda(t+j, t+j+1)\right] c_t(t). \tag{52}$$

depending on the realizations of future shocks to the TFP growth rate λ.

The amount of leisure chosen at date t and in the future depends on where they are in the life cycle. They are given by

$$\ell_t\left(t+s\right) = \left(\frac{1-\eta}{T+1}\right)\left(\frac{1}{e_s}\right)\sum_{j=0}^{T} e_j. \tag{53}$$

If $\eta = 1$ the household will choose no leisure. If $\eta \to 0$ and $e_0 = e_T$ are small enough, then $\ell_t\left(t\right)$ and $\ell_t\left(t+T\right)$ could be larger than one, meaning the households would supply no labor on those dates. This would violate our interior solution assumption. We assume $e_0 = e_T \gg 0$ and η sufficiently large to maintain interior leisure choices.

This household will carry some real asset position into the next period. The date t real value of this position is given by

$$\begin{aligned}
\frac{a_t\left(t\right)}{P\left(t\right)} &= e_0\left[1-\ell_t\left(t\right)\right]w\left(t\right) - c_t\left(t\right) && (54)\\
&= e_0\left[1-\frac{1-\eta}{T+1}\frac{\left[\sum_{j=0}^{T} e_j\right]}{e_0}\right]w\left(t\right) - \frac{\eta}{T+1}w\left(t\right)\left[\sum_{j=0}^{T} e_j\right] && (55)\\
&= w\left(t\right)\left[e_0 + \frac{\eta-1}{T+1}\left[\sum_{j=0}^{T} e_j\right] - \frac{\eta}{T+1}\left[\sum_{j=0}^{T} e_j\right]\right] && (56)\\
&= w\left(t\right)\left[e_0 - \frac{1}{T+1}\left[\sum_{j=0}^{T} e_j\right]\right]. && (57)
\end{aligned}$$

Step 2b.

There are also households that entered the economy at date $t-1, t-2, \cdots, t-T$ that would solve a similar problem. These households would have brought nominal asset holdings $a_{t-1}\left(t-1\right), a_{t-2}\left(t-1\right), \cdots, a_{t-T}\left(t\right)$, respectively, into the current period, and have a shorter remaining horizon in their life cycle. Here we will show the solution to a household problem for a household that entered the economy at date $t-1$. In particular, we will show that asset holdings $a_{t-1}\left(t\right)$ continues to be linear in the current real wage $w\left(t\right)$. We will then infer solutions for all of the other household problems for households entering the economy at dates $t-2, \cdots, t-T$.

The household entering the economy at date $t-1$ solve this problem at date t :

$$\max_{\{c_{t-1}(t+s-1), \ell_{t-1}(t+s-1)\}_{s=1}^{T}} E_t[\sum_{s=1}^{T} [\eta \ln c_{t-1}(t+s-1) + (1-\eta) \ln \ell_{t-1}(t+s-1)]$$

subject to life-time budget constraint

$$c_{t-1}(t) + \sum_{s=2}^{T} \left(\frac{P(t+s-1)}{P(t)} \frac{c_{t-1}(t+s-1)}{\prod_{j=0}^{s-2} R^n(t+j, t+j+1)} \right)$$
$$\leq e_1 w(t)(1-\ell_{t-1}(t)) + \sum_{s=2}^{T} \left(\frac{P(t+s-1)}{P(t)} \frac{e_s w(t+s-1)(1-\ell_{t-1}(t+s-1))}{\prod_{j=0}^{s-2} R^n(t+j, t+j+1)} \right)$$
$$+ \frac{R^n(t-1,t) a_{t-1}(t-1)}{P(t)}.. \quad (58)$$

In this "remaining lifetime" budget constraint, we can see from section $2a$ that the nominal value of $a_{t-1}(t-1)$ is

$$a_{t-1}(t-1) = P(t-1) w(t-1) \left[e_0 - \frac{1}{T+1} \sum_{j=0}^{T} e_j \right]. \quad (59)$$

We can therefore find the value of $R^n(t-1,t) a_{t-1}(t-1)$ as

$$R^n(t-1,t) a_{t-1}(t-1) = R^n(t-1,t) P(t-1) w(t-1) \left[e_0 - \frac{1}{T+1} \sum_{j=0}^{T} e_j \right]. \quad (60)$$

We can use the policy rule $P(t) = \frac{R^n(t-1,t)}{\lambda(t-1,t)} P(t-1)$ and the law of motion for

$w(t)$ to simplify the RHS as

$$= R^n(t-1,t)\frac{P(t)\lambda(t-1,t)}{R^n(t-1,t)}\frac{w(t)}{\lambda(t-1,t)}\left[e_0 - \frac{1}{T+1}\sum_{j=0}^{T} e_j\right] \tag{61}$$

$$= P(t)w(t)\left[e_0 - \frac{1}{T+1}\sum_{j=0}^{T} e_j\right]. \tag{62}$$

Therefore the entire real-valued term is given by

$$\frac{R^n(t-1,t)a_{t-1}(t-1)}{P(t)} = w(t)\left[e_0 - \frac{1}{T+1}\sum_{j=0}^{T} e_j\right] \tag{63}$$

$$= w(t)\left[\frac{T}{T+1}e_0 - \frac{1}{T+1}\sum_{j=1}^{T} e_j\right]. \tag{64}$$

Since $w(t)$ and $P(t)$ are known at date t when the consumption-saving decision is made, these decisions are linear in $w(t)$.

We now substitute the policy rule into the rest of the budget constraint to obtain

$$\sum_{s=1}^{T}\left(\frac{c_{t-1}(t+s-1)}{\prod_{j=0}^{s-1}\lambda(t+j,t+j+1)}\right) \leq w(t)\left(\sum_{s=1}^{T} e_s(1-\ell_{t-1}(t+s-1)) + \frac{T}{T+1}e_0 - \frac{1}{T+1}\sum_{j=1}^{T} e_j\right). \tag{65}$$

Let's now turn to the FOC for this problem. We have for $s = 1, ..., T$ with respect to consumption are

$$\frac{\eta}{c_{t-1}(t+s-1)} = \frac{\mu}{\prod_{j=0}^{s-1}\lambda(t+j,t+j+1)} \tag{66}$$

which implies

$$c_{t-1}(t+s) = \left[\prod_{j=0}^{s}\lambda(t+j,t+j+1)\right]c_{t-1}(t) \tag{67}$$

The following sequence of first order conditions for $s = 1, ..., T$ with respect to ℓ are

$$\frac{1-\eta}{\ell_{t-1}\left(t+s-1\right)} = \mu w\left(t\right) e_s \tag{68}$$

We combine each of these with the corresponding FOC for consumption to give the following choices for leisure for $s = 1, ..., T$

$$\ell_{t-1}\left(t+s-1\right) = \frac{1-\eta}{\eta}\frac{c_{t-1}\left(t\right)}{w\left(t\right) e_s} \tag{69}$$

We can now substitute back into the "remaining life" budget constraint. This yields

$$c_{t-1}\left(t\right) = w\left(t\right)\frac{\eta}{T+1}\left[\sum_{j=0}^{T} e_j\right]. \tag{70}$$

This is linear in $w\left(t\right)$. This household would then have a desired asset position:

$$\frac{a_{t-1}\left(t\right)}{P\left(t\right)} = e_1 w\left(t\right)\left(1-\ell_{t-1}\left(t\right)\right) - c_{t-1}\left(t\right) + \frac{R^n\left(t-1,t\right) a_{t-1}\left(t-1\right)}{P\left(t\right)} \tag{71}$$

$$\begin{aligned}
&= e_1 w\left(t\right)\left[1+\frac{\eta-1}{\eta}\frac{c_{t-1}\left(t\right)}{e_1 w\left(t\right)}\right] - \frac{\eta}{\eta}c_{t-1}\left(t\right) + w\left(t\right)\left[\frac{T}{T+1}e_0 - \frac{1}{T+1}\sum_{j=1}^{T} e_j\right], \\
&= e_1 w\left(t\right) - \frac{1}{\eta}c_{t-1}\left(t\right) + w\left(t\right)\left[\frac{T}{T+1}e_0 - \frac{1}{T+1}\sum_{j=1}^{T} e_j\right], \\
&= e_1 w\left(t\right) - \frac{1}{\eta}w\left(t\right)\frac{\eta}{T+1}\left[e_0 + \sum_{s=1}^{T} e_s\right] + w\left(t\right)\left[\frac{T}{T+1}e_0 - \frac{1}{T+1}\sum_{j=1}^{T} e_j\right], \\
&= w\left(t\right)\left[e_1 - \frac{1}{T+1}\left[e_0 + \sum_{s=1}^{T} e_s\right] + \left[\frac{T}{T+1}e_0 - \frac{1}{T+1}\sum_{j=1}^{T} e_j\right]\right] \\
&= w\left(t\right)\frac{1}{T+1}\left[e_0 + e_1 - T\sum_{j=2}^{T} e_j\right].
\end{aligned}$$

For all other households at date t who entered the economy at date $t-2, t-3, ..t-T$, consumption and assets at date t will also be linear in $w(t)$.

Step 3.

As already seen, date t consumption is equalized among participant households. The consumption amounts are, for $j = 0, 1....T$

$$c_{t-j}(t) = w(t) \frac{\eta}{T+1} \sum_{j=0}^{T} ej, \tag{72}$$

Step 4.

To show that the asset market clears at the conjectured real rate which equals the growth rate of productivity, first consider the asset market clearing condition

$$\sum_{s=0}^{T-1} a_{t-s}(t) = 0$$

Consider the $T+1 = 3$ period case for simplicity where the asset market clearing condition is given by

$$a_{t-1}(t) + a_t(t) = 0 \tag{73}$$

which is

$$a_{t-1}(t) + \{e_0 w(t)[1 - \ell_t(t)] - c_t(t)\} = 0. \tag{74}$$

The term $a_{t-1}(t)$ can be written as

$$a_{t-1}(t) = R(t-1,t)\, a_{t-1}(t-1) \tag{75}$$

$$+ \{e_1 w(t)[1 - \ell_{t-1}(t)] - c_{t-1}(t)\}. \tag{76}$$

The term $a_{t-1}(t-1)$ must be

$$a_{t-1}(t-1) = e_0 w(t-1)[1 - \ell_{t-1}(t-1)] - c_{t-1}(t-1). \tag{77}$$

Thus the entire expression is

$$\begin{aligned} a_{t-1}(t) + a_t(t) = \\ & R(t-1,t)\{e_0 w(t-1)[1 - \ell_{t-1}(t-1)] - c_{t-1}(t-1)\} \\ & + \{e_1 w(t)[1 - \ell_{t-1}(t)] - c_{t-1}(t)\} \\ & + \{e_0 w(t)[1 - \ell_t(t)] - c_t(t)\} = 0. \end{aligned} \tag{78}$$

Substituting out leisure and rearranging gives

$$R(t-1,t)\left\{e_0 w(t-1) - \frac{1}{\eta} c_{t-1}(t-1)\right\} + \left\{e_1 w(t) - \frac{1}{\eta} c_{t-1}(t)\right\} + \left\{e_0 w(t) - \frac{1}{\eta} c_t(t)\right\} = 0. \quad (79)$$

The term $c_{t-1}(t)$ can be written as

$$c_{t-1}(t) = R(t-1,t)\, c_{t-1}(t-1).$$

In our conjectured solution

$$c_t(t) = \frac{\eta w(t)[e_0 + e_1 + e_2]}{3}, \quad (80)$$

and

$$c_{t-1}(t-1) = \frac{\eta w(t-1)[e_0 + e_1 + e_2]}{3} \quad (81)$$

so that

$$\lambda(t-1,t)\, c_{t-1}(t-1) = \frac{\eta w(t)[e_0 + e_1 + e_2]}{3}. \quad (82)$$

Substituting the consumption and imposing the symmetry of endowment such that $e_0 = e_2$, the asset market for the 3 period economy clears at the conjectured real interest rate.

B Linearization

Consider a $T+1 = 3$ version of the model with $P(t) = 1\ \forall t$. In this economy, at any date t,there are 3 generations alive: the agents born at date t who have 3 periods to live, agents born at date $t-1$ and have 2 periods to live and finally the old who were born at date $t-2$ who have only one period to live. Looking at the problem of each cohort alive at date t.

B.1 Agent born at date t

Preferences

$$U_t = \eta \ln c_t(t) + (1-\eta)\ln \ell_t(t) + E_t\left[\eta \ln c_t(t+1) + (1-\eta)\ln \ell_t(t+1)\right] + E_t\left[\eta \ln c_t(t+2) + (1-\eta)\ln \ell_t(t+2)\right]$$

such that each period, $s = 0, 1, 2$, budget constraint is

$$c_t(t) + a_t(t) = e_0 w(t)(1 - \ell_t(t)) \tag{83}$$

$$c_t(t+1) + a_t(t+1) = e_1 w(t+1)(1 - \ell_t(t+1)) + R(t, t+1)a_t(t) \tag{84}$$

$$c_t(t+2) = e_2 w(t+2)(1 - \ell_t(t+2)) + R(t+1, t+2)a_t(t+1) \tag{85}$$

Note that the timing is such that $R(t, t+1)$ is determined at date t and $R(t+1, t+2)$ at date $t+1$.

Suppose the wage growth process is given by

$$w(t) = \lambda(t-1, t)w(t-1) \tag{86}$$
$$\lambda(t-1, t) = \bar{\lambda} \exp(\epsilon(t))$$

Given these log preferences, we know at any point, for $s = 0, 1, 2$ the leisure choice is given by

$$\ell_t(t+s) = \frac{(1-\eta)\, c_t(t+s)}{\eta e_s w(t+s)}$$

We can substitute out leisure from the budget constraint so that we can write them simply as

$$c_t(t) = \eta\left(e_0 w(t) - a_t(t)\right) \tag{87}$$

$$c_t(t+1) = \eta\left[e_1 w(t+1) + R(t, t+1)a_t(t) - a_t(t+1)\right] \tag{88}$$

$$c_t(t+2) = \eta\left[e_2 w(t+2) + R(t+1, t+2)a_t(t+1)\right] \tag{89}$$

The two Euler equations are

$$c_t(t) E_t\left[\frac{1}{c_t(t+1)}\right] = R(t, t+1)^{-1} \tag{90}$$

$$E_t\left[\frac{c_t(t+1)}{c_t(t+2)}\right] = R(t+1, t+2)^{-1} \tag{91}$$

B.2 Agent born at date $t-1$

Preferences

$$U_t = \eta \ln c_{t-1}(t) + (1-\eta)\ln \ell_{t-1}(t) + E_t\left[\eta \ln c_{t-1}(t+1) + (1-\eta)\ln \ell_{t-1}(t+1)\right].$$

Substituting out leisure, we get

$$c_{t-1}(t) = \eta \left[e_1 w(t) + R(t-1,t) a_{t-1}(t-1) - a_{t-1}(t) \right], \tag{92}$$

and

$$c_{t-1}(t+1) = \eta \left[e_2 w(t+1) \right] + R(t,t+1) a_{t-1}(t) \right]. \tag{93}$$

For these agents, they enter the period with assets and hence the gross return on these assets is given by $R(t-1,t)a_{t-1}(t-1)$. They also choose choose $a_{t-1}(t)$ at date t. Their Euler equation can be written as

$$c_{t-1}(t) E_t \left[\frac{1}{c_{t-1}(t+1)} \right] = R(t,t+1)^{-1} \tag{94}$$

Note, asset market clearing condition at date t is

$$a_{t-1}(t) + a_t(t) = 0 \tag{95}$$

B.3 Agent born at date $t-2$

Preferences

$$U_t = \eta \ln c_{t-2}(t) + (1-\eta) \ln \ell_{t-2}(t)$$

where

$$c_{t-2}(t) = e_2 w(t)(1 - \ell_{t-2}(t)) + R(t-1,t) a_{t-2}(t-1) \tag{96}$$

$$c_{t-2}(t) = \eta \left[e_2 w(t) + R(t-1,t) a_{t-2}(t-1) \right].$$

There is no future uncertainty for these agents. They consume all their endowments and the assets that they brought in this period. Even the interest rate is predetermined at date t.

B.4 Relevant date t equations

$$c_t(t) E_t \left[\frac{1}{c_t(t+1)} \right] = R(t,t+1)^{-1} \tag{97}$$

$$c_{t-1}(t) E_t \left[\frac{1}{c_{t-1}(t+1)} \right] = R(t,t+1)^{-1} \tag{98}$$

$$\ell_t(t) = \frac{(1-\eta)\, c_t(t)}{\eta e_0 w(t)} \tag{99}$$

$$\ell_{t-1}(t) = \frac{(1-\eta)\, c_{t-1}(t)}{\eta e_1 w(t)} \tag{100}$$

$$\ell_{t-2}(t) = \frac{(1-\eta)\, c_{t-2}(t)}{\eta e_2 w(t)} \tag{101}$$

$$c_t(t) = e_0 w(t)(1-\ell_t(t)) - a_t(t) \tag{102}$$

$$c_{t-1}(t) = e_1 w(t)(1-\ell_{t-1}(t)) + R(t-1,t)a_{t-1}(t-1) - a_{t-1}(t) \tag{103}$$

$$c_{t-2}(t) = e_2 w(t)(1-\ell_{t-2}(t)) + R(t-1,t)a_{t-2}(t-1) \tag{104}$$

$$a_t(t) + a_{t-1}(t) = 0 \tag{105}$$

B.5 Solving the model using linearization

1. Re-define stationary variables with a hat, $\hat{x}(t) = x_t(t)/w(t)$ and re-write the relevant date t equations

$$\hat{c}_t(t) E_t \left[\frac{1}{\hat{c}_t(t+1)(w(t+1)/w(t))} \right] = R(t,t+1)^{-1} \tag{106}$$

$$\hat{c}_{t-1}(t) E_t \left[\frac{1}{\hat{c}_{t-1}(t+1)(w(t+1)/w(t))} \right] = R(t,t+1)^{-1} \tag{107}$$

$$\ell_t(t) = \frac{(1-\eta)\, \hat{c}_t(t)}{\eta e_0} \tag{108}$$

$$\ell_{t-1}(t) = \frac{(1-\eta)\, \hat{c}_{t-1}(t)}{\eta e_1} \tag{109}$$

$$\ell_{t-2}(t) = \frac{(1-\eta)\, \hat{c}_{t-2}(t)}{\eta e_2} \tag{110}$$

$$\hat{c}_t(t) = e_0(1-\ell_t(t)) - \hat{a}_t(t) \tag{111}$$

$$\hat{c}_{t-1}(t) = e_1(1-\ell_{t-1}(t)) + R(t-1,t)\hat{a}_{t-1}(t-1)\frac{w(t-1)}{w(t)} - \hat{a}_{t-1}(t) \tag{112}$$

$$\hat{c}_{t-2}(t) = e_2(1-\ell_{t-2}(t)) + R(t-1,t)\hat{a}_{t-2}(t-1)\frac{w(t-1)}{w(t)} \tag{113}$$

$$\hat{a}_t(t) + \hat{a}_{t-1}(t) = 0.$$

2. Compute the steady state of the stationary variables. To simplify notation, c_y refers to the young, agents in the first period of their life, c_m refers to the middle-aged, people in the second period of life and finally c_o refers to the old, agents in the final period of their life:

$$\hat{c}_y^* = \hat{c}_m^* = \hat{c}_o^* = \frac{\eta}{3}(e_0 + e_1 + e_2) \tag{114}$$

$$\ell_y^* = \frac{(1-\eta)\,\hat{c}_y^*}{\eta e_0} \tag{115}$$

$$\ell_m^* = \frac{(1-\eta)\,\hat{c}_m^*}{\eta e_1} \tag{116}$$

$$\ell_o^* = \frac{(1-\eta)\,\hat{c}_o^*}{\eta e_2} \tag{117}$$

$$\hat{a}_y^* = e_0(1-\ell_y^*) - \hat{c}_y^* \tag{118}$$

$$\hat{a}_m^* = e_1(1-\ell_m^*) - \hat{c}_m^* + R^* * \hat{a}_y^* * \bar{\lambda}^{-1}. \tag{119}$$

3. Linear approximation of the stationary variables from their steady state Note that $\tilde{c}_m = (\hat{c}_{m,t} - \hat{c}_m^*)/\hat{c}_m^*$.

$$\tilde{c}_m(+1) - \tilde{c}_y - \epsilon(+1) - \tilde{R} = 0 \tag{120}$$

$$\tilde{c}_o(+1) - \tilde{c}_m - \epsilon(+1) - \tilde{R} = 0 \tag{121}$$

$$\tilde{\ell}_y = \tilde{c}_y \tag{122}$$

$$\tilde{\ell}_m = \tilde{c}_m \tag{123}$$

$$\tilde{\ell}_o = \tilde{c}_o \tag{124}$$

$$\hat{c}_y^* \tilde{c}_y = -e_0 \ell_y^* \tilde{\ell}_y - \hat{a}_y^* \tilde{a}_y \tag{125}$$

$$\hat{c}_m^* \tilde{c}_m = -e_1 \ell_m^* \tilde{\ell}_m + \frac{R^* \hat{a}_y^*}{\bar{\lambda}} \left[\tilde{R}(-1) + \tilde{a}_y(-1) - \epsilon \right] - \hat{a}_m^* \tilde{a}_m \tag{126}$$

$$\hat{c}_o^* \tilde{c}_o = -e_2 \ell_o^* \tilde{\ell}_o + \frac{R^* \hat{a}_m^*}{\bar{\lambda}} \left[\tilde{R}(-1) + \tilde{a}_m(-1) - \epsilon \right] \tag{127}$$

$$\hat{a}_y^* \tilde{a}_y + \hat{a}_m^* \tilde{a}_m = 0.$$

C Extended preferences

C.1 First order conditions

In this appendix, we show the calculations leading to the solution for ϑ from the first order conditions for a household entering the economy at date t and using the more general preference specification discussed in the text.

The first order conditions for today's choice variables are, for $c_t(t)$

$$\mathcal{S}_0 \left[c_t(t)^\eta \, \ell_t(t)^{1-\eta} \right]^{-\sigma} \eta c_t(t)^{\eta-1} \, \ell_t(t)^{1-\eta} = \mu \tag{128}$$

and for $\ell_t(t)$

$$\mathcal{S}_0 \left[c_t(t)^\eta \, \ell_t(t)^{1-\eta} \right]^{-\sigma} c_t(t)^\eta \, (1-\eta) \, \ell_t(t)^{-\eta} = \mu e_0 w(t) \, . \tag{129}$$

These can be combined to obtain

$$\ell_t(t) = \frac{1-\eta}{\eta} \frac{c_t(t)}{e_0 w(t)}, \tag{130}$$

and similarly we can obtain

$$\ell_t(t+1) = \frac{1-\eta}{\eta} \frac{c_t(t+1)}{e_1 w(t+1)} \tag{131}$$

and

$$\ell_t(t+2) = \frac{1-\eta}{\eta} \frac{c_t(t+2)}{e_2 w(t+2)}. \tag{132}$$

Now substitute the leisure choices in the first order conditions for consumption and simplify to obtain

$$\mathcal{S}_0 c_t(t)^{-\sigma} \left[\frac{1-\eta}{\eta} \frac{1}{e_0 w(t)}\right]^{(1-\eta)(-\sigma)} \eta \left[\frac{1-\eta}{\eta} \frac{1}{e_0 w(t)}\right]^{1-\eta} = \mu \tag{133}$$

and

$$\mathcal{S}_1 c_t(t+1)^{-\sigma} \left[\frac{1-\eta}{\eta} \frac{1}{e_1 w(t+1)}\right]^{(1-\eta)(-\sigma)} \eta \left[\frac{1-\eta}{\eta} \frac{1}{e_1 w(t+1)}\right]^{1-\eta} = \frac{\mu}{R(t,t+1)}, \tag{134}$$

where μ is a Lagrange multiplier, and similarly for $c_t(t+2)$. Combining these yields

$$c_t(t+1) = c_t(t) \left[\frac{\mathcal{S}_1}{\mathcal{S}_0} R(t,t+1)\right]^{\frac{1}{\sigma}} \left[\frac{e_1 w(t+1)}{e_0 w(t)}\right]^{\frac{(1-\sigma)(\eta-1)}{\sigma}}, \tag{135}$$

and similarly

$$c_t(t+2) = c_t(t) \left[\frac{\mathcal{S}_1}{\mathcal{S}_0} \frac{\mathcal{S}_2}{\mathcal{S}_1} R(t,t+1) R(t+1,t+2)\right]^{\frac{1}{\sigma}} \left[\frac{e_2 w(t+2)}{e_0 w(t)}\right]^{\frac{(1-\sigma)(\eta-1)}{\sigma}}. \tag{136}$$

We can now substitute these terms back into the budget constraint to obtain an expression for first period consumption that has the following form:

$$c_t(t) = \frac{\eta \left[e_0 w(t) + \frac{e_1 w(t+1)}{R(t,t+1)} + \frac{e_2 w(t+2)}{R(t,t+1) R(t+1,t+2)}\right]}{1 + d_1 + d_2} \tag{137}$$

where

$$d_1 = (\mathcal{S}_1)^{\frac{1}{\sigma}} R(t,t+1)^{\frac{1-\sigma}{\sigma}} \left[\frac{e_1 w(t+1)}{e_0 w(t)}\right]^{\frac{(1-\sigma)(\eta-1)}{\sigma}} \tag{138}$$

and

$$d_2 = (\mathcal{S}_2)^{\frac{1}{\sigma}} \left[R(t,t+1) R(t+1,t+2)\right]^{\frac{1-\sigma}{\sigma}} \left[\frac{e_2 w(t+2)}{e_0 w(t)}\right]^{\frac{(1-\sigma)(\eta-1)}{\sigma}}. \tag{139}$$

We will now use this expression for first period consumption to deduce the required value of the exponent ϑ that appears in the definitions of the $\mathcal{S}$ terms. An aspect of the complete markets solution is that this agent will be able to choose $c_t(t)$ without reference to future uncertainty, since the agent is fully insured against that uncertainty under complete markets. We are also conjecturing, but have not yet completely verified, that the solution occurs at $R(t,t+1) = \lambda(t,t+1)\ \forall t$. If we substitute in the conjectured value for the gross real interest rate $R(t,t+1)$ and use the law of motion for the real wage, we can rewrite the expression for first period consumption as

$$c_t(t) = \frac{\eta w(t)[e_0 + e_1 + e_2]}{1 + d_1 + d_2} \tag{140}$$

where

$$d_1 = \lambda(t,t+1)^{\frac{\vartheta}{\sigma}} \lambda(t,t+1)^{\frac{1-\sigma}{\sigma}} \left[\frac{e_1 \lambda(t,t+1)}{e_0}\right]^{\frac{(1-\sigma)(\eta-1)}{\sigma}} \tag{141}$$

and

$$d_2 = [\lambda(t,t+1)\lambda(t+1,t+2)]^{\frac{\vartheta}{\sigma}} [\lambda(t,t+1)\lambda(t+1,t+2)]^{\frac{1-\sigma}{\sigma}} \left[\frac{e_2 \lambda(t,t+1)\lambda(t+1,t+2)}{e_0}\right]^{\frac{(1-\sigma)(\eta-1)}{\sigma}}. \tag{142}$$

This is the expression displayed in the text.

C.2 Asset market equilibrium

To get to these results we have conjectured that $R(t,t+1) = \lambda(t,t+1)\ \forall t$ is the real interest rate in this stochastic equilibrium. We need to verify this conjecture by showing that with $\mathcal{S}_1$ and $\mathcal{S}_2$ set appropriately we obtain $A(t) = 0$.[22] This means

$$a_{t-1}(t) + a_t(t) = 0 \tag{143}$$

which is

$$a_{t-1}(t) + \{e_0 w(t)[1 - \ell_t(t)] - c_t(t)\} = 0. \tag{144}$$

[22] It may not equal zero since in the general preferences case we have seen that $d_1 \neq 1$, which in principle may prevent the equilibrium real interest rate being equal to the output growth rate.

The term $a_{t-1}(t)$ can be written as

$$a_{t-1}(t) = R(t-1,t)\, a_{t-1}(t-1) \tag{145}$$

$$+ \{e_1 w(t)[1-\ell_{t-1}(t)] - c_{t-1}(t)\}. \tag{146}$$

The term $a_{t-1}(t-1)$ must be

$$a_{t-1}(t-1) = e_0 w(t-1)[1-\ell_{t-1}(t-1)] - c_{t-1}(t-1). \tag{147}$$

Thus the entire expression is

$$\begin{aligned} &a_{t-1}(t) + a_t(t) = \\ &\quad R(t-1,t)\{e_0 w(t-1)[1-\ell_{t-1}(t-1)] - c_{t-1}(t-1)\} \\ &\qquad + \{e_1 w(t)[1-\ell_{t-1}(t)] - c_{t-1}(t)\} \\ &\qquad\quad + \{e_0 w(t)[1-\ell_t(t)] - c_t(t)\} = 0. \end{aligned} \tag{148}$$

Substituting out leisure and rearranging gives

$$\begin{aligned} &R(t-1,t)\left\{e_0 w(t-1) - \frac{1}{\eta} c_{t-1}(t-1)\right\} \\ &\qquad + \left\{e_1 w(t) - \frac{1}{\eta} c_{t-1}(t)\right\} \\ &\qquad\quad + \left\{e_0 w(t) - \frac{1}{\eta} c_t(t)\right\} = 0. \end{aligned} \tag{149}$$

The term $c_{t-1}(t)$ can be written as

$$c_{t-1}(t) = c_{t-1}(t-1)\left[\{\mathcal{S}_1 R(t-1,t)\}^{\frac{1}{\sigma}} \left\{\frac{e_1 w(t)}{e_0 w(t-1)}\right\}^{\frac{(1-\sigma)(\eta-1)}{\sigma}}\right] \tag{150}$$

$$= c_{t-1}(t-1)[Z] \tag{151}$$

where Z is the bracketed term in the first line. Note that Z is familiar. In fact,

$$\frac{Z}{R(t-1,t)} = d_1. \tag{152}$$

Now use the conjectured solution $R(t,t+1) = \lambda(t,t+1)\ \forall t$ to obtain

$$c_t(t) = \frac{\eta w(t)[e_0+e_1+e_2]}{1+d_1+d_2}, \tag{153}$$

and

$$c_{t-1}\left(t-1\right)=\frac{\eta w\left(t-1\right)\left[e_0+e_1+e_2\right]}{1+d_1+d_2} \tag{154}$$

so that

$$\lambda\left(t-1,t\right)c_{t-1}\left(t-1\right)=\frac{\eta w\left(t\right)\left[e_0+e_1+e_2\right]}{1+d_1+d_2}. \tag{155}$$

We note that with $e_0=e_2$ by symmetry, setting the appropriate values for $\mathcal{S}_i$ means $d_2=1$, while $d_1=\left[e_1/e_0\right]^{(1-\sigma)(\eta-1)/\sigma}\neq 1$. Using this information and the conjecture $R\left(t,t+1\right)=\lambda\left(t,t+1\right)\ \forall t$ along with the appropriate values for $\mathcal{S}_i$ as well as symmetry we can write

$$\begin{aligned} w\left(t\right)\left\{e_0-\frac{e_0+e_1+e_0}{2+d_1}\right\} & \\ +w\left(t\right)\left\{e_1-\left[\frac{e_0+e_1+e_0}{2+d_1}\right]d_1\right\} & \\ +w\left(t\right)\left\{e_0-\frac{e_0+e_1+e_0}{2+d_1}\right\} &=0. \end{aligned} \tag{156}$$

This expression is equal to zero regardless of the value of d_1. It is also equal to zero in the special case of log preferences.

We conclude that $R\left(t,t+1\right)=\lambda\left(t,t+1\right)\ \forall t$ is the real interest rate.