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Working Paper Series

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Working Paper 2017-003F
https://doi.org/10.20955/wp.2017.003

January 2018

FEDERAL RESERVE BANK OF ST. LOUIS
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Implementing the Modified Golden Rule? Optimal Ramsey Taxation with Incomplete Markets Revisited

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February 20, 2019

Abstract

What is the prescription of Ramsey capital taxation in the long run? Aiyagari (1995) addressed the question in a heterogeneous-agent incomplete-markets (HAIM) economy, showing that a positive capital tax should be imposed to implement the so-called modified golden rule (MGR). This famous capital taxation result is built on a critical assumption that a Ramsey steady state (featuring a non-binding natural government debt limit) exists. This paper revisits and checks the validity of this critical assumption. We first show that an optimal Ramsey allocation may feature no steady state if the government's natural debt limit never binds. Hence, the Ramsey steady state described and assumed by Aiyagari (1995) turns out to be incorrect. We further show that the welfare of any steady state of the HAIM economy can be improved by issuing more government bonds to front-load consumption. The key to both results is embedded in the hallmark of the HAIM economy that the steady-state risk-free rate is lower than the time discount rate in a competitive equilibrium. On the basis of our findings, we argue that the most likely long-run Ramsey outcome should feature the coexistence of a Ramsey steady state with (i) a binding government debt limit and (ii) the failure of the MGR.

JEL Classification: C61; E22; E62; H21; H30
Key Words: Capital Taxation; Modified Golden Rule; Ramsey Problem; Incomplete Markets

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*Corresponding author, YiLi Chien. Email: yilichien@gmail.com. This paper is a complete rewrite of our previous work under the title "Aiyagari Meets Ramsey: Optimal Capital Taxation with Incomplete Markets." While we find the non-existence of a Ramsey steady state in the current version, our previous work builds on the incorrect premise that a Ramsey steady state always exists. We thank Andrew Atkeson, Dirk Krueger, Tomoyuki Nakajima, Yena Park, and participants at various seminars and conferences for useful comments. The views expressed are those of the individual authors and do not necessarily reflect the official positions of the Federal Reserve Bank of St. Louis, the Federal Reserve System, or the Board of Governors.

1 Introduction

The heterogeneous-agent incomplete-markets (HAIM hereafter) model considers an environment in which households are subject to uninsurable idiosyncratic shocks and borrowing restrictions; in response, households buffer their consumption against adverse shocks via precautionary savings. During the past two decades, the HAIM model has become a standard workhorse for policy evaluations in the current state-of-the-art macroeconomics that jointly addresses aggregate and inequality issues.[1]

Given the importance and popularity of the HAIM model, it is natural to ask: what is the prescription of Ramsey capital taxation in the long run for the HAIM economy? The first attempt to answer this question is the work of Aiyagari (1995). Assuming the existence of a Ramsey steady state, Aiyagari (1995) showed that the so-called "modified golden rule" (MGR hereafter) has to hold in the (assumed) Ramsey steady state.[2] On the other hand, in the steady state, the after-tax gross return on capital, which is equated to the risk-free gross interest rate, R, is always less than the time discount rate, $1/\beta$. Aiyagari (1995) thus reached the conclusion that a positive capital tax should be imposed to implement the steady-state allocation that satisfies the MGR. In the absence of government intervention, agents overaccumulate capital relative to the level implied by the MGR because of their precautionary-savings motive. The imposition of positive capital taxation therefore provides a remedy to restore production efficiency—the MGR. The finding by Aiyagari (1995) is important in the optimal taxation literature, and it represents a distinct departure from the classical result of no permanent capital tax prescribed by Chamley (1986) and Judd (1985).

This paper revisits the long-standing issue with respect to the existence of a Ramsey steady state and the implementation of the MGR. In his analysis Aiyagari (1995) made an implicit but standard assumption that the government's natural debt limit never binds.[3] Conditional on this standard assumption and working with the power utility function, our first main result demonstrates that (i) there is no Ramsey steady state with $R < 1/\beta$ if the elasticity of intertemporal substitution (EIS hereafter) is weakly less than 1 and (ii) if the EIS is larger than 1, a Ramsey steady state with $R < 1/\beta$ is possible, but the shadow price of resources must diverge in the steady state.

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[1] It is also known as the Bewley-Huggett-Aiyagari model. For surveys of the literature, see Heathcote, Storesletten, and Violante (2009), Guvenen (2011), and Quadrini and Ríos-Rull (2015).

[2] The Ramsey steady state is defined as a situation where the optimal Ramsey allocation features the steady-state property in the long run. See Definitions 3 and 4 for the detail.

[3] See Equation (19) of Aiyagari (1995) and the discussion about it. The so-called natural debt limit is defined to be the maximum level of indebtedness for which the debt can be repaid almost surely; see Aiyagari, Marcet, Sargent, and Seppala (2002).

Result (i) questions the existence of a Ramsey steady state, the basic premise of the Aiyagari (1995) analysis. Result (ii) contradicts Aiyagari's (1995) implicit assumption on the convergence of the shadow price of resources.[4] Both results indicate that the Ramsey steady state described and assumed by Aiyagari (1995) does not emerge at the optimum and hence is incorrect. Consequently, the subsequent results derived, including the MGR and positive capital tax, could be problematic.

As explained below, our adopted methodology permits us to analytically derive all the necessary first-order conditions (FOCs) of the Ramsey problem in the HAIM economy. Based on these FOCs, we investigate the long-run properties of the Ramsey allocation. In particular, it is critical to our results to include the FOC with respect to aggregate consumption in the analysis. This margin over aggregate consumption is overlooked in the analysis by Aiyagari (1995).[5] After incorporating the margin, we show that the social benefit of having one extra unit of aggregate consumption must diverge in the long run. This divergence could make the Ramsey steady state fail to exist, conditional on the standard assumption that the government's natural debt limit never binds. To put it differently, we show that once the additional margin over aggregate consumption is taken into account in the analysis, the assumed Ramsey steady state in Aiyagari (1995) never arises at the optimum, since it contradicts other necessary Ramsey FOCs. Our analysis highlights the shortcoming of the analysis in Aiyagari (1995): only a subset of all necessary Ramsey FOCs are derived and considered in his study; as a result, his conclusions could be dubious.

Aiyagari (1995) obtained his results mainly on the setting of endogenous rather than exogenous government spending. We demonstrate that our first main result remains robust, regardless of whether government spending is endogenously determined or exogenously given.

By exploiting the necessary FOCs of the Ramsey problem, our first main result addresses the existence issue of Ramsey steady states and finds that the Ramsey steady state described and assumed by Aiyagari (1995) is incorrect. However, it does not tell us why the Ramsey steady state may fail to exist. Our second main result intends to shed light on this question. Given $R < 1/\beta$ in the steady state, we show that when the EIS is weakly less than 1, the welfare of any steady state of the HAIM economy can be improved by the implementation of an earlier aggregate consumption through issuing more government bonds. Obviously, this second main result is consistent with the first one. The aforementioned force that makes the Ramsey allocation diverge away from its steady

[4]The footnote 15 in Aiyagari (1995) implicitly assumed that the shadow price of resources converges to a finite limit in the Ramsey steady state.

[5]The analysis of Aiyagari (1995) only derives the FOCs with respect to capital and endogenous government spending; see his key result, equation (20). Other FOCs are ignored possibly because of their difficulties or complications in the derivation. Our methodology enables us to remedy the shortcoming by deriving these other FOCs.

state will not cease to exist as long as the Ramsey planner is capable of issuing more government debt. This result has an important implication: the existence of a Ramsey steady state must come along with some form of binding government debt limit (either natural or ad hoc) at the optimum if the EIS is weakly less than 1.[6] It is worth noting that our second main result is proved without utilizing any optimal condition of the Ramsey problem and, therefore, it provides a robustness check to our first main result.

It is well-known that the steady-state outcome in a competitive equilibrium, $R < 1/\beta$, represents the signature feature of the HAIM model.[7] The fundamental divergent force underlying our two main results is exactly embedded in this feature. Conditional on the assumption that the government's natural government debt limit never binds, we show that the divergent force will exist (vanish) if and only if $R < 1/\beta$ ($R = 1/\beta$) holds in the steady state. Intuitively, unlike individual households in the face of uncertain labor income, the Ramsey planner in the HAIM economy (without aggregate shocks) faces no uncertainty in allocating aggregate resources. Given that the planner discounts the future by β, the strict inequality of $R < 1/\beta$ then dictates that the market discounts resources at a lower rate than the planner discounts utility, implying the existence of planner's desire to improve welfare by front-loading aggregate consumption through policy tools. This desire persists as long as $R < 1/\beta$ holds in the steady state. Interestingly, we show that whether it is feasible for the planner to bring about the desire depends on the value of the EIS. It is feasible if the EIS is weakly less than 1; otherwise, it may not be feasible.

Putting our two main results together leads to a natural conjecture: the government's debt limit (either the ad hoc or the natural one) should ultimately bind in the long-run Ramsey solution when the EIS is weakly less than 1. Once the debt limit is binding, the Ramsey planner can no longer increase welfare by issuing more government debt and the driving force of diverging away from the steady state is stopped even with $R < 1/\beta$. Therefore, a Ramsey steady state with the binding government debt limit could and should exist. Importantly, the binding of the government's debt limit could invalidate the MGR in the Ramsey steady state and hence the optimal capital tax in the long-run would not be necessarily positive. This is a sharp contrast to the permanent positive capital taxation obtained by Aiyagari (1995). The reason for the result is as follows. To clear the asset market, the household's total asset holdings must be equal to the sum of capital stock and

[6]Following Aiyagari, Marcet, Sargent, and Seppala (2002), we call a debt limit ad hoc if it is more stringent than the natural one.

[7]Ljungqvist and Sargent (2012, p.9) explained that the outcome of $R < 1/\beta$ in the steady state can be thought of as follows: it lowers the rate of return on savings enough to offset agents' precautionary-savings motive so as to make their asset holdings converge rather than diverge in the limit.

government bonds. If the government's debt limit never binds, government bonds can serve as a residual to clear the asset market; as such, it enables the Ramsey planner to implement the capital allocation independently of the household's asset holdings. This is the key underpinning as to why the MGR could and should hold in the analysis of Aiyagari (1995) if a Ramsey steady state exists. However, once the government's debt limit binds, the Ramsey planner has to consider the impact on the household's asset holdings when choosing an optimal capital stock. As a result, the MGR could and should fail in Ramsey steady states.

The non-binding of the government's natural debt limit along with the existence of a Ramsey steady state are commonly assumed for the Ramsey problem in the extant literature. However, these assumptions may be problematic for the HAIM environment, according to our findings. The warning is particularly relevant and strong since the key driving force behind our results exactly underlies the hallmark feature of the HAIM economy—the steady-state risk-free rate is lower than the time discount rate in any competitive equilibrium.

1.1 Methodology

In order to explicitly account for the social benefit of having one extra unit of aggregate consumption, the primal approach to the Ramsey problem is adopted. One difficulty encountered in formulating the Ramsey problem in the HAIM economy regards how to properly derive the implementability condition. Werning (2007) extends the Ramsey primal approach from the representative-agent (RA hereafter) to the heterogeneous-agent framework. However, agent types are permanently fixed in Werning's model, while agent types evolve stochastically over time in our economy. Park (2014) extends the work of Werning (2007) to a complete-market environment in which agent types evolve stochastically. We extend her approach to the incomplete-markets environment, or more specifically, to the HAIM economy.

Our methodology first formulates the household problem as a time-zero trading problem of the Arrow-Debreu complete-market economy; however, we impose two additional constraints—one for incomplete markets and the other for borrowing constraints—to take into consideration the key features of the HAIM economy.[8] The advantage of this methodology is that it allows us to trace the evolution of stochastic agent types over time through the Lagrangian multipliers associated with these additional constraints. Moreover, it helps set up the Ramsey problem and makes possible the

[8] This approach of modeling incomplete markets is pioneered by Aiyagari, Marcet, Sargent, and Seppala (2002), who named the additional constraints for incomplete markets as measurability conditions. The later work by Chien, Cole, and Lustig (2011) extends this approach to heterogeneous-agent models in the context of asset pricing.

analytical derivation of all necessary Ramsey FOCs. Due to the fact that the Ramsey planner also encounters the same incomplete-markets frictions faced by households, the typical implementability condition is not sufficient and hence additional constraints are needed for the characterization of the Ramsey problem. This causes our HAIM Ramsey problem to become a generalization of the RA Ramsey problem.

The methodology adopted by this paper results in several contributions to the literature on the Ramsey problem. First, our approach is capable of analytically deriving all FOCs of the primal Ramsey problem in the typical HAIM economy, which to our knowledge is unprecedented. As mentioned above, accounting for all the necessary optimal Ramsey conditions, especially the margin over aggregate consumption, is critical to our analysis and findings. In fact, we show that the erroneous assumption of an existing Ramsey steady state made and described by Aiyagari (1995) cannot be detected if some of the necessary Ramsey FOCs are ignored or missing in the analysis. Second, our approach offers an advantage in that the Ramsey problem of our HAIM economy would reduce to that of a RA economy if markets were complete rather than incomplete. Given that the meaning and intuition of the Ramsey problem in the RA economy are well-understood, this advantage makes the model mechanism that drives our main results transparent and intuitive. Finally, our two main results describing the long-term properties of the Ramsey steady state are built on the analytically-derived conditions that either support Ramsey equilibrium or characterize the constraint set of the Ramsey problem. In other words, our methodology allows us to investigate the existence of a Ramsey steady state instead of assuming its existence as in the extant literature as well as to characterize the properties of a Ramsey steady state if it does exist.

1.2 Related Literature

The literature on optimal capital taxation is vast. Here we focus only on a very limited subset of the studies framed in a heterogeneous-agent environment with incomplete markets or market frictions.

Our work is closely related to the recent study by Chien and Wen (2018), who utilized an analytically tractable heterogeneous-agent model with idiosyncratic preference shocks to address the same issue. They demonstrated that the Ramsey planner intends to increase the government bond supply until full self-insurance is achieved or an exogenous debt limit binds. Given that the full self-insurance outcome is infeasible in a canonical HAIM model, our result of no Ramsey steady state featuring a non-binding debt limit is consistent with their findings. However, in order to have an analytical solution, their model makes a few special assumptions and deviates from the standard HAIM model. Hence, their study cannot prove the incorrectness of the Ramsey steady state assumption made by Aiyagari (1995).

Conesa, Kitao, and Krueger (2009) considered optimal capital taxation in a HAIM-type economy but in a life-cycle framework. The quantitative part of their study largely focuses on the steady-state welfare. In an overlapping generations model with two-period-lived households, Krueger and Ludwig (2018) characterized the optimal capital tax of the Ramsey problem. In their analysis, the planner lacks government bonds as a policy tool. In contrast, government bonds play an essential role in our results. Hence, their results do not contradict ours. According to our results, there could exist a Ramsey steady state with a binding government debt limit.

Papers including Acikgoz, Hagedorn, Holter, and Wang (2018), Dyrda and Pedroni (2018), and Ragot and Grand (2017) numerically solve optimal Ramsey fiscal policy for the transition path and the steady state of the HAIM economy. In contrast to our findings, the numerical results of these papers are consistent with those of Aiyagari (1995) and feature a Ramsey steady state with a non-binding government debt limit. However, the sources of the difference could be due to the unjustified assumption of the existence of a Ramsey steady state and the implicit assumption of a non-binding natural government debt limit. In particular, our paper signals a warning about applying these common assumptions to the HAIM economy.

The work of Straub and Werning (2014) points out that the common assumption that endogenous multipliers associated with the Ramsey problem converge in the limit is not necessarily true and could thus lead to incorrect optimal policy prescriptions in the long-run. Aiyagari's (1995) assumption of a Ramsey steady state may be subject to the same problem. Conditional on the assumption of a non-binding natural government debt limit, we explicitly address it. It should be noted that the mechanism for our non-convergence of endogenous multipliers originates from the HAIM environment. Such a mechanism is absent from the environment studied by Straub and Werning (2014). Moreover, we point out that the Ramsey steady state could actually exist whenever the government debt limit binds.

Gottardi, Kajii, and Nakajima (2015) considered an environment deviating from the standard HAIM economy, in that there is risky human capital in addition to physical capital. They derived qualitative and quantitative properties for the solution to the Ramsey problem, showing that the interaction between market incompleteness and risky human capital accumulation provides a justification for taxing physical capital. In this paper, we stick to the standard HAIM economy with idiosyncratic labor income risk and show that assuming the existence of a Ramsey steady state with no consideration of government debt limit could be incorrect due to incomplete-markets frictions.

Dávila, Hong, Krusell, and Ríos-Rull (2012) characterized constrained efficiency for the HAIM economy. To decentralize the constrained efficient allocation, the planner is required to know each agent's realized shocks in order to impose individual-specific capital taxes. We consider flat tax rates applied uniformly to all agents as in the typical Ramsey problem and, as such, the constrained efficient allocation is infeasible to the Ramsey planner.

The rest of the paper is organized as follows. Section 2 and Section 3 introduce our model economy and characterize its competitive equilibrium, respectively. Section 4 formulates the Ramsey problem. Our main findings are demonstrated in Section 5. Section 6 checks and shows the robustness of our results to the endogenous government spending setting, and Section 7 concludes.

2 Model Economy

The model economy mainly builds on Aiyagari (1994). There is a unit measure of infinitely-lived households that are subject to idiosyncratic labor productivity shocks. There are no aggregate shocks. Markets are incomplete in that there are no state-contingent securities for idiosyncratic shocks. In addition, all households are subject to exogenous borrowing constraints at all times.

Time is discrete and the horizon is infinity, indexed by $t = 0, 1, 2, ...$ Time 0 is a planning period and actions begin in time 1. All households are ex ante identical and endowed with the same asset holdings at time 0. Ex post heterogeneity arises from time 1 on because households experience different histories of the idiosyncratic shock realization. Let $\theta_t \in \Theta$ (a finite set) denote the incidence of the idiosyncratic labor productivity shock at time t, and let θ^t denote the history of events for the idiosyncratic shock of a household up through and until time t. The shock θ_t is independently and identically distributed across households, and the sequence $\{\theta_t\}$ follows a first-order Markov process over time. We let $\pi_t(\theta^t)$ denote the unconditional probability of θ^t and $\pi(\theta_t|\theta_{t-1})$ denote the conditional probability. Because of the independence of productivity shocks across households, a law of large numbers applies so that the probability $\pi_t(\theta^t)$ also represents the fraction of the population that experiences θ^t at time t. We call a household that has the history θ^t simply "the household θ^t." We also introduce additional notations: $\theta^{t+1} \succ \theta^t$ means that the left-hand-side node is a successor node to the right-hand-side node; and for $s > t$, $\theta^s \succeq \theta^t$ ($\theta^s \succ \theta^t$)

represents the set of successor shocks after θ^t up to θ^s including (excluding) θ_t.

Households maximize their lifetime utility

$$U = \sum_{t=1}^{\infty} \beta^t \sum_{\theta^t} \left[u(c_t(\theta^t)) - v\left(\frac{l_t(\theta^t)}{\theta_t}\right)\right] \pi_t(\theta^t),$$

where $\beta \in (0,1)$ is the discount factor; $c_t(\theta^t)$ and $l_t(\theta^t)$ denote the consumption and the labor supply for household θ^t at time t; and $l_t(\theta^t)/\theta_t$ is the corresponding "raw" labor supply (hours worked). The assumptions on the functions $u(.)$ and $v(.)$ are standard.

There is a standard neoclassical constant returns-to-scale production technology, denoted by $F(K,L)$, that is operated by a representative firm, where K and L are aggregate capital and labor, respectively. Also assume that $F(K,L) \to 0$ if either $K \to 0$ or $L \to 0$. The firm produces output by hiring labor and renting capital from households. The firm's optimal conditions for profit maximization at time t satisfy

$$\begin{aligned} w_t &= F_L(K_t, L_t), \\ r_t &= F_K(K_t, L_t), \end{aligned}$$

where w_t and r_t are the wage rate and the capital rental rate, and F_L and F_K denote the marginal product of labor and capital, respectively. All markets are competitive.

The government is required to finance an exogenous stream of government spending G_t and it can issue one-period government bonds and levy flat-rate, time-varying labor and capital taxes at rates $\tau_{l,t}$ and $\tau_{k,t}$, respectively. The flow government budget constraint at time t is expressed as

$$\tau_{l,t} w_t L_t + \tau_{k,t}(r_t - \delta)K_t + B_{t+1} = G_t + R_t B_t, \tag{1}$$

where R_t is the (gross) risk-free gross interest rate between time $t-1$ and t, and B_t is the amount of government bonds issued at time $t-1$. The government is assumed to fully commit to a sequence of taxes imposed and debts issued, given the initial amount of government bonds B_1 at time 0. This setup for the government is standard for the Ramsey problem. Section 6 considers an alternative setup where G_t becomes endogenously determined rather than exogenously given. This alternative setup is adopted by Aiyagari (1995).

3 Characterization of Competitive Equilibrium

This section characterizes the competitive equilibrium of the model economy, paving the way for the formulation of the Ramsey problem in the next section. We first describe the household problem.

3.1 Household Problem

We express the household problem as a time-zero trading problem as in an Arrow-Debreu economy but with the imposition of additional constraints to account for the key features of the HAIM economy. As noted in the Introduction, this method facilitates the formulation of the primal Ramsey problem for the HAIM economy.

Denote P_t as the time-zero price of one unit of consumption delivered at time t. We set $P_0 = 1$ as a normalization. Given K and B are perfect substitutes in the mind of households, the after-tax return on capital has to equal to the risk-free rate:

$$\frac{P_t}{P_{t+1}} = R_{t+1} = 1 + (1 - \tau_{k,t+1})(r_{t+1} - \delta), \tag{2}$$

which constitutes a no-arbitrage condition for trades in capital and government bonds.

Let $p_t(\theta^t) = P_t \pi_t(\theta^t)$ be the state-contingent price of one unit of consumption delivered in the event of θ^t at time t. The household's time-zero budget constraint in an Arrow-Debreu economy is expressed as

$$\widehat{a}_1 = \sum_{t \geq 1} \sum_{\theta^t} p_t(\theta^t) \left[c_t(\theta^t) - \widehat{w}_t l_t(\theta^t) \right], \tag{3}$$

with $\widehat{a}_1 = K_1 + B_1$, where K_1 and B_1 are the economy's initial capital and government bond, respectively. All households by assumption have the same initial asset holdings $\widehat{a}_1$.

3.1.1 Measurability Conditions and Borrowing Constraints

Two key features of the HAIM economy are (i) incomplete markets—no state-contingent claims on idiosyncratic shocks, and (ii) borrowing constraints—a lower bound on household asset holdings. Both features impose restrictions on the choice of asset holdings across idiosyncratic states over time. We show how to embed these asset-holding restrictions into a time-zero trading problem for the household.

Given the history of shocks θ^t at time t, the asset holdings with complete markets can be written as

$$p_t(\theta^t)a_t(\theta^t) = \sum_{s\geq t}\sum_{\theta^s\succeq\theta^t} p_s(\theta^s)\left[c_s(\theta^s) - \widehat{w}_s l_s(\theta^s)\right], \tag{4}$$

where $\widehat{w}_s \equiv (1-\tau_{l,s})w_s$ is defined as the after-tax wage rate at time s and $a_t(\theta^t)$ is the amount of the state-contingent claim held by household θ^t at the beginning of time t.

However, markets are incomplete rather than complete and households do not have access to state-contingent markets in the HAIM economy. This implies that the asset holdings at time $t+1$ are measurable only up to the events prior to the realization of shock θ_{t+1}. Formally, households face the following measurability conditions: for $\forall t \geq 0$ and θ^t,

$$a_{t+1}(\theta^t, \theta_{t+1}) = a_{t+1}(\theta^t, \widetilde{\theta}_{t+1}) \text{ for all } \widetilde{\theta}_{t+1}, \theta_{t+1} \in \Theta,$$

which practically impose constraints on a household's asset holdings.

For ease of exposition, we rewrite the measurability conditions as follows: for $\forall t \geq 0$ and θ^t,

$$\frac{a_{t+1}(\theta^t, \theta_{t+1})}{R_{t+1}} = \frac{a_{t+1}(\theta^t, \widetilde{\theta}_{t+1})}{R_{t+1}} \equiv \widehat{a}_{t+1}(\theta^t) \text{ for all } \widetilde{\theta}_{t+1}, \theta_{t+1} \in \Theta, \tag{5}$$

where R_{t+1} is the risk-free gross interest rate between time t and $t+1$. That is, $\widehat{a}_{t+1}(\theta^t)$ is defined so that $R_{t+1}\widehat{a}_{t+1}(\theta^t) = a_{t+1}(\theta^t, \theta_{t+1}) = a_{t+1}(\theta^t, \widetilde{\theta}_{t+1})$ for all $\widetilde{\theta}_{t+1}$, $\theta_{t+1} \in \Theta$. This makes sense because households can hold only a one-period risk-free asset; and their asset holdings at the beginning of time $t+1$ deflated by their asset return, the risk-free gross interest rate, must be equal to the end of time t asset holdings, which are denoted by $\widehat{a}_{t+1}(\theta^t)$.

Households also face the following borrowing restrictions for $\forall t \geq 0$:

$$\widehat{a}_{t+1}(\theta^t) \geq 0, \text{for all } \theta^t,$$

which can be equivalently expressed as $a_{t+1}(\theta^{t+1}) \geq 0$ for all t and θ^{t+1}, according to (5).

3.1.2 Formulating and Solving the Household Problem

The asset-holding restrictions, such as the measurability conditions and borrowing constraints, are equivalent to the restrictions imposed on the whole sequence of consumption and labor choices.

Using (4), we can restate the measurability conditions as

$$P_{t-1}\widehat{a}_t(\theta^{t-1})\pi_t(\theta^t) = \sum_{s\geq t}\sum_{\theta^s\succeq\theta^t} p_s(\theta^s)\left[c_s(\theta^s) - \widehat{w}_s l_s(\theta^s)\right],\ \forall t\geq 1, \theta^t, \tag{6}$$

where we have replaced $a_t(\theta^t)$ with $R_t\widehat{a}_t(\theta^{t-1})$ as defined in (5) and used $p_t(\theta^t) = P_t\pi_t(\theta^t)$ and the result of $P_{t-1} = P_t R_t$ in (2). As to the borrowing constraints, they can be expressed as

$$\sum_{s\geq t}\sum_{\theta^s\succeq\theta^t} p_s(\theta^s)\left[c_s(\theta^s) - \widehat{w}_s l_s(\theta^s)\right] \geq 0,\ \forall t\geq 1, \theta^t. \tag{7}$$

If markets were complete, then households would only face a single constraint (3). The presence of the additional constraints represented by (6) and (7) is due to the incomplete markets and borrowing constraints, respectively.

Given prices $\{\widehat{w}_t, p_t(\theta^t)\}$, the household chooses a sequence of consumption $\{c_t(\theta^t)\}$, labor $\{l_t(\theta^t)\}$, and asset holdings $\{\widehat{a}_{t+1}(\theta^t)\}$ to maximize the lifetime utility as of time zero, subject to the time-zero budget constraint (3), the measurability conditions (6), and the borrowing constraints (7). Let χ be the multiplier on the time-zero budget constraint, $\nu_t(\theta^t)$ the multiplier on the measurability condition in the event of θ^t at time t, and $\varphi_t(\theta^t)$ the multiplier on the borrowing constraint in the event of θ^t at time t. Incorporating all the constraints through these multipliers gives the household's Lagrangian:

$$\begin{aligned}
\tilde{L} \ = \ & \min_{\{\chi,v,\varphi\}}\max_{\{c,l,\widehat{a}\}} \sum_{t=1}^{\infty}\beta^t\sum_{\theta^t}\left[u(c_t(\theta^t)) - v\left(\frac{l_t(\theta^t)}{\theta_t}\right)\right]\pi_t(\theta^t) \\
& +\chi\left\{\widehat{a}_1 - \sum_{t=1}^{\infty}\sum_{\theta^t} p_t(\theta^t)\left[c_t(\theta^t) - \widehat{w}_t l_t(\theta^t)\right]\right\} \\
& +\sum_{t=1}^{\infty}\sum_{\theta^t}\nu_t(\theta^t)\left\{\sum_{s\geq t}\sum_{\theta^s\succeq\theta^t} p_s(\theta^s)\left[c_s(\theta^s) - \widehat{w}_s l_s(\theta^s)\right] - P_{t-1}\widehat{a}_t(\theta^{t-1})\pi_t(\theta^t)\right\} \\
& +\sum_{t=1}^{\infty}\sum_{\theta^t}\varphi_t(\theta^t)\left\{\sum_{s\geq t}\sum_{\theta^s\succeq\theta^t} p_s(\theta^s)\left[c_s(\theta^s) - \widehat{w}_s l_s(\theta^s)\right]\right\}.
\end{aligned}$$

Using Abel's summation formula, the Lagrangian $\tilde{L}$ can be rewritten as[9]

$$\begin{aligned} L &= \min_{\{\chi,\upsilon,\varphi\}} \max_{\{c,l,\widehat{a}\}} \sum_{t=1}^{\infty} \beta^t \sum_{\theta^t} \left[u(c_t(\theta^t)) - v\left(\frac{l_t(\theta^t)}{\theta_t}\right)\right] \pi_t(\theta^t) \\ & - \sum_{t=1}^{\infty} \sum_{\theta^t} \zeta_t(\theta^t) p_t(\theta^t) \left[c_t(\theta^t) - \widehat{w}_t l_t(\theta^t)\right] \\ & + \chi \widehat{a}_1 - \sum_{t=1}^{\infty} \sum_{\theta^t} \nu_t(\theta^t) P_{t-1} \widehat{a}_t(\theta^{t-1}) \pi_t(\theta^t), \end{aligned}$$

where $\zeta_t(\theta^t)$ is called the "cumulative multiplier," and its law of motion is given by

$$\zeta_{t+1}(\theta^{t+1}) = \zeta_t(\theta^t) - \nu_{t+1}(\theta^{t+1}) - \varphi_{t+1}(\theta^{t+1}) \text{ with } \zeta_0 = \chi > 0. \tag{8}$$

Obviously, $\zeta_t(\theta^t)$ is a cumulative sum of all Lagrangian multipliers in the past history from the measurability conditions and the borrowing constraints; it encodes the frequency and severity of both types of constraints over time.[10]

From the Lagrangian L, the FOCs with respect to consumption $c_t(\theta^t)$ and labor supply $l_t(\theta^t)$ are given by

$$\beta^t u'(c_t(\theta^t)) = \zeta_t(\theta^t) P_t, \tag{9}$$

$$\beta^t v'\left(\frac{l_t(\theta^t)}{\theta_t}\right)\frac{1}{\theta_t} = \zeta_t(\theta^t) \widehat{w}_t P_t, \tag{10}$$

while the FOC with respect to asset holdings $\widehat{a}_{t+1}(\theta^t)$ is given by

$$\sum_{\theta^{t+1} \succ \theta^t} \nu_{t+1}(\theta^{t+1}) \pi(\theta_{t+1}|\theta_t) = 0. \tag{11}$$

From the FOCs (9) and (10), we see that the value of $\zeta_t(\theta^t)$ cannot be negative.

The last FOC requires that the mean of multipliers on the measurability condition across idiosyncratic states θ^{t+1} be equal to zero, given θ^t. If markets were complete instead, households could have a short position on the consumption claims at time t contingent on shock θ_{t+1} being

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[9]See Ljungqvist and Sargent (2012, p.821) for the formula.

[10]Note that the household problem is a standard convex programming problem since the constraint set is convex even with the incorporation of the measurability conditions and the borrowing constraints. Thus, the resulting first-order conditions are necessary and sufficient. In addition, this approach of defining recursive multipliers as in (8) was proposed and developed by Marcet and Marimon (1999, 2017) for solving dynamic problems with forward-looking constraints. Both Aiyagari, Marcet, Sargent, and Seppala (2002) and Chien, Cole, and Lustig (2011) adopted this approach.

high at time $t+1$ ("save less for a high state"), and could have a long position on the consumption claims at time t contingent on shock θ_{t+1} being low at time $t+1$ ("save more for a low state"). However, markets are incomplete and households cannot save at time t, depending on whether shock θ_{t+1} at time $t+1$ is high or low. As such, the best choice for $\widehat{a}_{t+1}(\theta^t)$ at time t is to satisfy an average—that is, the condition (11). Combining the FOCs (9) and (11) with the motion (8) actually enforces the household's Euler equation

$$u'(c_t(\theta^t)) \geq \beta R_{t+1} \sum_{\theta^{t+1} \succ \theta^t} u'(c_{t+1}(\theta^{t+1}))\pi(\theta_{t+1}|\theta_t), \tag{12}$$

where the equality holds if $\widehat{a}_{t+1}(\theta^t) > 0$.

Note that the household's Euler equation given by (12) can also be expressed equivalently as

$$\sum_{\theta^{t+1} \succ \theta^t} \zeta_{t+1}(\theta^{t+1})\pi(\theta_{t+1}|\theta_t) \leq \zeta_t(\theta^t), \tag{13}$$

where the equality holds *only* if the borrowing constraint of the state-contingent asset, $a_{t+1}(\theta^{t+1}) \geq 0$, does not bind for all possible subsequent θ_{t+1} states. To see this, using (11), the summation of the motion (8) over θ^{t+1} gives

$$\sum_{\theta^{t+1} \succ \theta^t} \zeta_{t+1}(\theta^{t+1})\pi(\theta_{t+1}|\theta_t) = \zeta_t(\theta^t) - \sum_{\theta^{t+1} \succ \theta^t} \varphi_{t+1}(\theta^{t+1})\pi(\theta_{t+1}|\theta_t), \tag{14}$$

and we know that $\varphi_{t+1}(\theta^{t+1}) \geq 0$ for all θ^{t+1}. Thus, to uphold the equality part of (13), it is required that $\varphi_{t+1}(\theta^{t+1}) = 0$ for all θ^{t+1} in (14). This feature is caused by the measurability condition (5), which effectively ensures that $\varphi_{t+1}(\theta^{t+1}) = 0$ for all θ^{t+1}, provided $\widehat{a}_{t+1}(\theta^t) > 0$.

3.2 Competitive Equilibrium

A competitive equilibrium of the model economy is defined in the standard way.

Definition 1. *Given the initial capital K_1 and initial government bonds B_1, a competitive equilibrium is defined as sequences of tax rates, government spending and government bonds $\{\tau_{l,t}, \tau_{k,t}, G_t, B_{t+1}\}_{t=1}^{\infty}$, and sequences of prices $\{w_t, r_t, P_t\}_{t=1}^{\infty}$, aggregate allocations $\{C_t, L_t, K_{t+1}\}_{t=1}^{\infty}$ and individual allocation plans $\{c_t(\theta^t), l_t(\theta^t), \widehat{a}_{t+1}(\theta^t)\}_{t=1}^{\infty}$, such that*

1. *$\{c_t(\theta^t), l_t(\theta^t), \widehat{a}_{t+1}(\theta^t)\}$ solve the household problem.*

2. $\{L_t, K_t\}$ *solve the representative firm's problem.*

3. The no-arbitrage condition holds: $\frac{P_t}{P_{t+1}} = 1 + (1 - \tau_{k,t+1})(r_{t+1} - \delta)$.

4. The time-zero government budget constraint holds:[11]

$$B_1 = \sum_{t=1}^{\infty} P_t \left[\tau_{l,t} w_t L_t + \tau_{k,t} \left(r_t - \delta\right) K_t - G_t\right].$$

5. All markets clear for all t*:*

$$\begin{aligned} B_{t+1} + K_{t+1} &= \sum_{\theta^t} \widehat{a}_{t+1}(\theta^t)\pi_t(\theta^t), \\ L_t &= \sum_{\theta^t} l_t(\theta^t)\pi_t(\theta^t), \\ C_t &= \sum_{\theta^t} c_t(\theta^t)\pi_t(\theta^t), \\ F(K_t, L_t) &= C_t + G_t + K_{t+1} - (1-\delta)K_t. \end{aligned}$$

3.3 Characterizing the Competitive Equilibrium

This subsection characterizes the competitive equilibrium in terms of the aggregate allocations and the cumulative multipliers of the household problem. This step is critical for the primal Ramsey approach in the HAIM economy. To facilitate the characterization, we work with the popular power utility function:

Assumption 1.

$$u(c) = \frac{c^{1-\alpha} - 1}{1 - \alpha},\ \alpha > 0;\ v\left(\frac{l}{\theta}\right) = \frac{1}{\gamma}\left(\frac{l}{\theta}\right)^{\gamma},\ \gamma > 1.$$

It is known that $1/\alpha$ represents the EIS. As will be seen, the value of the consumption EIS plays an important role for our result.

Proposition 1. *Under Assumption 1, the consumption and labor sharing rules are given, respec-*

[11]From the flow government budget constraint (1) to the time-zero one, the transversality condition, $\lim_{t\to\infty} P_{t-1}B_t = 0$, is imposed.

tively, by

$$c_t(\theta^t) = \frac{\zeta_t(\theta^t)^{\frac{-1}{\alpha}}}{H_t} C_t, \tag{15}$$

$$l_t(\theta^t) = \frac{\theta_t^{\frac{\gamma}{\gamma-1}} \zeta_t(\theta^t)^{\frac{1}{\gamma-1}}}{J_t} L_t, \tag{16}$$

where H_t and J_t are defined as

$$H_t \equiv \sum_{\theta^t} \zeta_t(\theta^t)^{\frac{-1}{\alpha}} \pi_t(\theta^t),$$

$$J_t \equiv \sum_{\theta^t} \theta_t^{\frac{\gamma}{\gamma-1}} \zeta_t(\theta^t)^{\frac{1}{\gamma-1}} \pi_t(\theta^t).$$

H_t and J_t, are referred to, respectively, as the consumption and labor aggregate multipliers, which are specific moments of the distribution of the individual cumulative multiplier $\zeta_t(\theta^t)$.[12] *In addition, P_t and $\widehat{w}_t$ can be expressed respectively as*

$$P_t = \beta^t C_t^{-\alpha} H_t^{\alpha} \tag{17}$$

and

$$\widehat{w}_t = \frac{L_t^{\gamma-1} J_t^{1-\gamma}}{C_t^{-\alpha} H_t^{\alpha}}. \tag{18}$$

Finally, with (17), the risk-free rate is given by

$$\frac{1}{R_{t+1}} = \frac{P_{t+1}}{P_t} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\alpha} \left(\frac{H_{t+1}}{H_t}\right)^{\alpha}. \tag{19}$$

The proofs of our results, including Proposition 1, are all relegated to the Appendix. Equations (15) through (18) show that one can express the individual allocations $\{c_t(\theta^t), l_t(\theta^t)\}$ and the market prices $\{P_t, \widehat{w}_t\}$ of the competitive equilibrium in terms of the aggregate allocations $\{C_t, L_t\}$ and the individual cumulative multipliers $\{\zeta_t(\theta^t)\}$, and the aggregate multipliers $\{H_t, J_t\}$. The following proposition demonstrates that the Ramsey planner can pick a competitive equilibrium by choosing aggregate allocations plus asset holdings and individual multipliers that satisfy a set of conditions.[13]

[12]Similar expressions for consumption can be seen in Nakajima (2005), Werning (2007), and Park (2014).

[13]Results similar to Proposition 2 but in different contexts can be seen in Aiyagari, Marcet, Sargent, and Seppala (2002, Proposition 1), Werning (2007, Proposition 1), and Park (2014, Proposition 1).

For ease of exposition, we define

$$\begin{aligned}\kappa_t\left(\theta^t\right) &\equiv \beta^t\left[C_t^{1-\alpha}H_t^{\alpha-1}\zeta_t(\theta^t)^{\frac{-1}{\alpha}} - L_t^{\gamma}J_t^{-\gamma}\theta_t^{\frac{\gamma}{\gamma-1}}\zeta_t(\theta^t)^{\frac{1}{\gamma-1}}\right] \\ &= P_t\left(c_t(\theta^t) - \widehat{w}_t l_t(\theta^t)\right),\end{aligned} \tag{20}$$

which represents the present value of the time-t net savings made by household θ^t. The second equality holds by utilizing equations (15) through (18).

Proposition 2. *Impose Assumption 1. Given the initial capital K_1, government bonds B_1, the capital tax rate $\tau_{k,1}$, and a stream of government spending $\{G_t\}$, sequences of aggregate allocations $\{C_t, L_t, K_{t+1}\}$, asset holdings $\{\widehat{a}_{t+1}(\theta^t)\}$, and cumulative multipliers $\{\zeta_t(\theta^t)\}$ (with the associated aggregate multipliers, H_t and J_t) can be supported as a competitive equilibrium if and only if they satisfy the following conditions:*[14]

1. *Resource constraints:* $F(K_t, L_t) + (1-\delta)K_t - K_{t+1} \geq C_t + G_t,\ \forall t \geq 1$.

2. *The implementability condition:*

$$\sum_{t=1}^{\infty}\sum_{\theta^t}\kappa_t\left(\theta^t\right)\pi_t(\theta^t) \geq \widehat{a}_1.$$

3. *Measurability conditions:*

$$\sum_{s\geq t}\sum_{\theta^s\succeq\theta^t}\pi_s(\theta^s)\kappa_s\left(\theta^s\right) = \beta^{t-1}C_{t-1}^{-\alpha}H_{t-1}^{\alpha}\widehat{a}_t(\theta^{t-1})\pi_t(\theta^t),\ \ \forall t\geq 1, \theta^t.$$

4. *Borrowing constraints:*

$$\sum_{s\geq t}\sum_{\theta^s\succeq\theta^t}\pi_s(\theta^s)\kappa_s\left(\theta^s\right) \geq 0,\ \ \forall t\geq 1, \theta^t.$$

5. *The evolution of $\zeta_t(\theta^t)$ satisfies* $\sum_{\theta^{t+1}\succ\theta^t}\zeta_{t+1}(\theta^{t+1})\pi(\theta_{t+1}|\theta_t) \leq \zeta_t(\theta^t),\ \forall t\geq 0, \theta^t$.

6. *If the borrowing constraint does not bind for $\widehat{a}_{t+1}(\theta^t)$, then*

$$\sum_{\theta^{t+1}\succ\theta^t}\zeta_{t+1}(\theta^{t+1})\pi(\theta_{t+1}|\theta_t) = \zeta_t(\theta^t),$$

[14] The initial capital tax rate, $\tau_{k,1}$, should be a choice variable for the Ramsey planner. However, given that the initial capital is pre-installed and that households are homogeneous at time zero, taxing the initial capital is essentially the same as allowing a lump-sum tax. As is standard in the literature, we restrict the planner's ability to choose $\tau_{k,1}$ in the Ramsey problem.

and this property holds for all θ^t and all $t \geq 0$.

Condition 2 of Proposition 2 corresponds to the time-zero government budget constraint, which is conventionally called the "implementability condition" in the formulation of the primal Ramsey problem. When the market is complete without frictions, our model reduces to the RA economy and imposing Conditions 3-6 becomes unnecessary. In particular, since $\zeta_t(\theta^t)$ in (8) equals χ at all times, Conditions 5 and 6 become redundant since they are automatically satisfied.

4 Ramsey Problem

Different government policies result in different competitive equilibria. We define the Ramsey problem formally:

Definition 2. *The Ramsey problem is to choose a competitive equilibrium that attains the maximization of the household's lifetime utility U.*

On the basis of Proposition 2, the Ramsey problem can be represented as maximizing

$$\sum_{t\geq 1} \beta^t \sum_{\theta^t} \left[\frac{1}{1-\alpha} \left(\left(\frac{\zeta_t(\theta^t)^{\frac{-1}{\alpha}}}{H_t} C_t \right)^{1-\alpha} - 1 \right) - \frac{1}{\gamma} \left(\frac{\theta_t^{\frac{1}{\gamma-1}} \zeta_t(\theta^t)^{\frac{1}{\gamma-1}}}{J_t} L_t \right)^{\gamma} \right] \pi_t(\theta^t)$$

by choosing $C_t, L_t, K_{t+1}, \widehat{a}_{t+1}(\theta^t)$, and $\zeta_t(\theta^t)$ subject to Conditions 1 to 6 stated in Proposition 2 and to H_t and J_t defined earlier, given $K_1, B_1, \tau_{k,1}$ and $\{G_t\}$. The objective of the Ramsey problem is derived by substituting the consumption sharing rule (15) and the labor sharing rule (16) into $U(.)$.

Note that we have formulated the Ramsey problem in terms of the sequences of the aggregate allocations, $\{C_t, L_t, K_{t+1}\}$, the asset holdings, $\{\widehat{a}_{t+1}(\theta^t)\}$, and the cumulative multipliers, $\{\zeta_t(\theta^t)\}$.

4.1 Relaxed Ramsey Problem

The "if" statement of Condition 6 listed in Proposition 2 imposes a technical challenge to study Ramsey outcomes analytically. It is not trivial to form a Ramsey Lagrangian if Condition 6 must be accounted for. Fortunately, the optimal allocation chosen by the Ramsey planner that satisfies Conditions 1-5 of Proposition 2 will also satisfy Condition 6. That is, Condition 6 is not a binding constraint in the optimum of the Ramsey problem. The detail of the proof for the claim is relegated to Appendix A.3. Although the proof is lengthy, the intuition is relatively simple.

Consider an allocation that satisfies Conditions 1 to 5 but fails Condition 6. In this allocation, there must exist a household θ^t at time t and the borrowing constraint does not bind ($\widehat{a}_{t+1}(\theta^t) > 0$) but the Euler equation stated in Condition 6 holds as a strict inequality rather than an equality. That is, we have

$$\sum_{\theta^{t+1} \succ \theta^t} \zeta_{t+1}(\theta^{t+1}) \pi(\theta_{t+1}|\theta_t) < \zeta_t(\theta^t).$$

Applying the FOC (9) to the above inequality yields

$$u'(c_t(\theta^t)) > \beta \frac{P_t}{P_{t+1}} \sum_{\theta^{t+1} \succ \theta^t} u'(c_{t+1}(\theta^{t+1})) \pi(\theta_{t+1}|\theta_t),$$

which indicates that household θ^t's marginal payoff from time-t consumption is higher than the expected marginal payoff from time $t+1$ consumption given the risk-free rate. Since the borrowing constraints do not bind for $\widehat{a}_{t+1}(\theta^t)$, the welfare of household θ^t can be improved by a simple adjustment that tightens only the borrowing constraint for increasing $c_t(\theta^t)$ at the expense of decreasing $c_{t+1}(\theta^{t+1})$.

Obviously, this simple adjustment does change the consumption, labor supply and asset holdings of the household θ^t and hence indirectly affects the aggregate allocation. This raises the concern of whether this adjustment is feasible to the Ramsey planner, given the set of policy tools available in our framework. As shown in the proof, this simple adjustment is feasible. Most importantly, the government bonds play a key role in supporting its feasibility since any gap between capital and asset holdings resulting from the adjustment can be closed by changing the level of government bonds in order to respect the asset-market clearing condition. As such, the optimal Ramsey allocation remains unchanged in the relaxed Ramsey problem, even though the constraint set is enlarged by omitting Condition 6.

Thus, we can ignore Condition 6 and consider the following relaxed Ramsey problem:

$$\max_{\substack{\{C_t, L_t, K_{t+1}, \\ \{\widehat{a}_{t+1}(\theta^t)\}, \{\zeta_t(\theta^t)\}\}}} \sum_{t \geq 1} \beta^t \sum_{\theta^t} \left[\frac{1}{1-\alpha} \left(\left(\frac{\zeta_t(\theta^t)^{\frac{-1}{\alpha}}}{H_t} C_t \right)^{1-\alpha} - 1 \right) - \frac{1}{\gamma} \left(\frac{\theta_t^{\frac{1}{\gamma-1}} \zeta_t(\theta^t)^{\frac{1}{\gamma-1}}}{J_t} L_t \right)^{\gamma} \right] \pi_t(\theta^t),$$

subject to

$$\{\beta^t \mu_t\} : F(K_t, L_t) + (1-\delta) K_t \geq C_t + G_t + K_{t+1}, \ \forall t \geq 1,$$

$$\chi^P : \sum_{t \geq 1} \sum_{\theta^t} \kappa_t \left(\theta^t\right) \pi_t(\theta^t) \geq \widehat{a}_1,$$

$$\{v_t^P(\theta^t)\}: \sum_{s\geq t}\sum_{\theta^s\succeq\theta^t} \kappa_s\left(\theta^s\right)\pi_s(\theta^s) = \beta^{t-1}C_{t-1}^{-\alpha}H_{t-1}^{\alpha}\widehat{a}_t(\theta^{t-1})\pi_t(\theta^t),\ \forall t\geq 1,\theta^t,$$

$$\{\varphi_t^P(\theta^t)\}: \sum_{s\geq t}\sum_{\theta^s\succeq\theta^t} \kappa_s\left(\theta^s\right)\pi_s(\theta^s) \geq 0,\ \forall t\geq 1,\theta^t,$$

$$\{\beta^t\xi_t(\theta^t)\}: \sum_{\theta^{t+1}\succ\theta^t} \zeta_{t+1}(\theta^{t+1})\pi(\theta_{t+1}|\theta_t) \leq \zeta_t(\theta^t),\ \forall t\geq 0,\theta^t,$$

where $\{\beta^t\mu_t\}, \chi^P, \{v_t^P(\theta^t)\}, \{\varphi_t^P(\theta^t)\}$, and $\{\beta^t\xi_t(\theta^t)\}$ are the multipliers on the aggregate resource constraints, the implementability condition, the measurability conditions, the borrowing constraints, and the law of motion for the household's cumulative multipliers, respectively.

Forming the Lagrangian for the relaxed Ramsey problem and using Abel's summation formula gives

$$\begin{aligned}\mathcal{L} \;=\; & \max_{\{C_t,L_t,K_{t+1},\{\widehat{a}_{t+1}(\theta^t)\},\{\zeta_t(\theta^t)\}\}} \sum_{t\geq 1}\beta^t W(t) + \sum_{t\geq 1}\beta^t\mu_t\left[F(K_t,L_t)+(1-\delta)K_t-K_{t+1}-C_t-G_t\right] \\ & + \sum_{t\geq 0}\sum_{\theta^t}\beta^t\xi_t(\theta^t)\left[\zeta_t(\theta^t)-\sum_{\theta^{t+1}\succ\theta^t}\zeta_{t+1}(\theta^{t+1})\pi(\theta_{t+1}|\theta_t)\right]-\chi^P\widehat{a}_1 \\ & - \sum_{t\geq 1}\beta^{t-1}C_{t-1}^{-\alpha}H_{t-1}^{\alpha}\left[\sum_{\theta^t}\widehat{a}_t(\theta^{t-1})v_t^P(\theta^t)\pi_t(\theta^t)\right],\end{aligned}$$

with

$$W(t) \equiv \sum_{\theta^t}\pi_t(\theta^t)\left[\begin{array}{c}\underbrace{\frac{1}{1-\alpha}\left(\left(\frac{\zeta_t(\theta^t)^{\frac{-1}{\alpha}}}{H_t}C_t\right)^{1-\alpha}-1\right)-\frac{1}{\gamma}\left(\frac{\theta_t^{\frac{1}{\gamma-1}}\zeta_t(\theta^t)^{\frac{1}{\gamma-1}}}{J_t}L_t\right)^{\gamma}}_{Part\ 1} \\ +\underbrace{\beta^{-t}\eta_t(\theta^t)\kappa_t\left(\theta^t\right)}_{Part\ 2}\end{array}\right], \tag{21}$$

where

$$\eta_{t+1}(\theta^{t+1}) = \eta_t(\theta^t)+\nu_{t+1}^P(\theta^{t+1})+\varphi_{t+1}^P(\theta^{t+1}),\ \eta_0=\chi^P>0, \tag{22}$$

which is the motion of the Ramsey planner's cumulative multiplier. The Ramsey planner cannot complete the market as typically assumed and is thereby subject to the same market structure of the HAIM economy—that is, the same measurability conditions and borrowing constraints as those facing the household. These market frictions are summarized by the multipliers $\nu_{t+1}(\theta^{t+1})$ and $\varphi_{t+1}(\theta^{t+1})$ in the household problem and by $\nu_{t+1}^P(\theta^{t+1})$ and $\varphi_{t+1}^P(\theta^{t+1})$ in the planner problem.

However, note that while we have the term $\chi \widehat{a}_1$ in the household Lagrangian L, we have the term $-\chi^P \widehat{a}_1$ in the planner Lagrangian $\mathcal{L}$. The opposite sign is due to the fact that the implementability condition in the Ramsey problem represents the government budget constraint rather than the household budget constraint. As such, while increasing $\widehat{a}_1$ relaxes the household budget constraint, it tightens the government budget constraint.

4.2 Comparison with the Representative-Agent Model

To gain insights into the pseudo-utility function $W(t)$ defined in (21), we first look at its formula under the complete market assumption, which becomes identical to the one derived in the RA model. When the market is complete without frictions as in the RA model, $\zeta_t(\theta^t)$ in (8) equals χ for all t and θ^t. As such, H_t equals $\chi^{-1/\alpha}$, P_t is reduced to $\beta^t C_t^{-\alpha} \chi^{-1}$ and J_t turns into $\chi^{\frac{1}{\gamma-1}} \sum_{\theta^t} \theta_t^{\frac{\gamma}{\gamma-1}} \pi_t(\theta^t)$ under complete market, or equivalently, under the RA economy. In addition, from (22), we know that $\eta_t(\theta^t)$ in (24) reduces to $\eta_0 = \chi^P$. Hence, the RA pseudo-utility function, $W^{RA}(t)$, can be expressed as[15]

$$W^{RA}(t) = \underbrace{\frac{C_t^{1-\alpha} - 1}{1-\alpha} - \frac{L_t^{\gamma}}{\gamma}}_{Part\ 1} + \underbrace{\chi^P \chi^{-1} \left(C_t^{-\alpha} C_t - L_t^{\gamma-1} L_t \right)}_{Part\ 2},$$

which is the corresponding pseudo-utility function in the RA model under Assumption 1. Part 1 of $W^{RA}(t)$ represents the current-period utility. Its Part 2, in terms of $\beta^t W^{RA}(t)$, is given by

$$\chi^P \chi^{-1} \beta^t \left(C_t^{-\alpha} C_t - L_t^{\gamma-1} L_t \right) = \chi^P P_t^{RA} (C_t - \widehat{w}_t^{RA} L_t), \tag{23}$$

where $P_t^{RA} \equiv \beta^t C_t^{-\alpha} \chi^{-1}$ is the time-zero price of one unit of consumption at time t, and $\widehat{w}_t^{RA} = L_t^{\gamma-1} C_t^{\alpha}$ is the after-tax wage rate at time t. Thus, the term $P_t^{RA}(C_t - \widehat{w}_t^{RA} L_t)$ shown in (23) represents the time-t net savings evaluated at the time-zero price in the RA model. The time-t net savings of households also represents the amount of net revenue collected by the government in period t because of Walras' law; hence, the implementability condition multiplier, χ^P, "measures the utility costs of raising government revenues through distorting taxes"(Ljungqvist and Sargent (2012, p.629)) in the RA framework. In other words, Part 2 of $W^{RA}(t)$ measures the total utility cost (evaluated at time zero) of collecting period-t government revenue.

Given that the primal Ramsey problem in the RA model is well understood and analyzed, the

[15] Under the complete-market assumption, our Ramsey planner problem is identical to the one in the RA model, which can be seen in subsection 16.6.1 in Ljungqvist and Sargent (2012, p. 626).

following comparison of our W_t to that of the RA model makes its economic meaning transparent and very clear. Part 1 of $W(t)$ in our HAIM model also represents the current-period utility. Its Part 2, in terms of $\beta^t W(t)$, is given by

$$\eta_t(\theta^t)\kappa_t(\theta^t) = \eta_t(\theta^t)P_t\left(c_t(\theta^t) - \widehat{w}_t l_t(\theta^t)\right), \tag{24}$$

where the last equality holds according to equation (20). Thus, the term $P_t\left(c_t(\theta^t) - \widehat{w}_t l_t(\theta^t)\right)$ represents the time-t net savings of household θ^t evaluated at time zero in the HAIM economy. It also represents the period-t government revenue collected from households θ^t. The marginal utility cost of increasing one extra unit of government revenue through distorting taxes on household θ^t is evaluated by the multiplier $\eta(\theta^t)$. Thus, the utility costs of raising government revenue in period t through households θ^t is given by (24).

Although the structure of $W(t)$ is basically the same as that of $W^{RA}(t)$, $W(t)$ does deviate from $W^{RA}(t)$ in two important respects.

First, the multiplier in (24), $\eta_t(\theta^t)$, is no longer a time-invariant constant χ^P, as in the RA model. This is because the Ramsey planner has to consider the heterogeneous impact of her or his policies on each household. Given that all households face the same tax rates, the amount of tax revenue collected as well as the marginal utility cost associated with it are both heterogeneous. The associated price tag of collecting government revenue is captured by the multiplier $\eta_t(\theta^t)$. From the evolution of $\eta_t(\theta^t)$ governed by (22), we see that $\eta_t(\theta^t)$ starts from χ^P ($\eta_0 = \chi^P$), but in a sequence it varies not only across households but also over time, meaning that the utility cost of collecting government revenue is not only household specific but also time varying. The following lemma shows that the average value of $\eta_t(\theta^t)$ tends to increase over time, implying that it could stochastically diverge to infinity in the limit.

Lemma 1. *The average of $\eta_t(\theta^t)$, $\sum_{\theta^t} \eta_t(\theta^t)\pi_t(\theta^t)$, is positive, and, moreover, it is non-decreasing and becomes strictly increasing if $\varphi_t^P(\theta^t) > 0$ for some θ^t.*

The second important deviation stems from the behavior of intertemporal prices. While the time-zero price of consumption delivered at time t is $P_t^{RA} = \beta^t C_t^{-\alpha}\chi^{-1}$ in the RA economy, this price becomes $P_t = \beta^t C_t^{-\alpha} H_t^{\alpha}$ in the HAIM model. As such, in the steady state, the market discounting rate implied by P_t^{RA} is consistent with the time discount factor β, whereas the market discounting rate implied by P_t is lower than β, provided that H_t is increasing over time. Now consider the steady-state version of equation (19):

$$1 = \beta R \left(\frac{H_{t+1}}{H_t} \right)^{\alpha}, \tag{25}$$

which tells us that H_t is increasing over time and must diverge to infinity in the limit if the asymmetric discounting, $R < 1/\beta$, holds in the steady state.[16] Thus, the feature of an increasing and divergent H_t exactly underlies the hallmark of the competitive equilibrium in the HAIM model—the risk-free rate is lower than the time discount rate in the steady state.

These two deviations of $W(t)$ from $W^{RA}(t)$ are in fact the two sides of one coin. Both are rooted in the frictions of the HAIM economy and both will vanish once markets are complete, as in the RA model. More importantly, the divergent tendency of both $\eta_t(\theta^t)$ and H_t, all else equal, makes Part 2 of $W(t)$ converge more slowly than Part 1. As will be seen, this asymmetric convergence between Part 1 and Part 2 of $W(t)$ is the key to our first result showing the non-existence of a Ramsey steady state.

4.3 Optimal Conditions of the Ramsey Problem

From the Lagrangian $\mathcal{L}$, the necessary FOCs with respect to $\widehat{a}_{t+1}(\theta^t), C_t, L_t,$ and K_{t+1} for $t \geq 1$ yield, respectively,

$$\sum_{\theta^{t+1} \succ \theta^t} v^P_{t+1}(\theta^{t+1}) \pi(\theta_{t+1}|\theta_t) = 0, \tag{26}$$

$$W_C(t) = \mu_t, \tag{27}$$

$$-W_L(t) = \mu_t F_L(K_t, L_t), \tag{28}$$

$$\mu_t = \beta \mu_{t+1} \left[F_K \left(K_{t+1}, L_{t+1} \right) - \delta + 1 \right], \tag{29}$$

where the derivation of (27) has made use of (26), and $W_C(t)$ and $W_L(t)$ denote the derivatives of $W(t)$ with respect to C_t and L_t, respectively.[17]

The explicit expressions of $W_C(t)$ and $W_L(t)$ in the FOCs of the Ramsey problem are crucial to our analysis later. One can derive them from the pseudo-utility $W(t)$ defined in (21). However, to facilitate the proof and discussion hereafter, it is convenient to express W_C and W_L in the following

[16]The converse should hold as well. When H_t diverges, it means that households suffer from the frictions of incomplete markets as in the standard scenario of the HAIM economy. Then, according to footnote 7, the outcome of $R < 1/\beta$ will result in the steady state.

[17]The FOC with respect to $\zeta_t(\theta^t)$ will not be needed for the derivation of our main results.

way. First, using the consumption sharing rule (15), $W_C(t)$ in (27) is expressed as

$$W_C(t) = C_t^{-\alpha} \left[\underbrace{\sum_{\theta^t} \left(\frac{c_t(\theta^t)}{C_t}\right) \left(\frac{c_t(\theta^t)}{C_t}\right)^{-\alpha} \pi_t(\theta^t)}_{Part\ 1} + \underbrace{(1-\alpha) H_t^{\alpha} \sum_{\theta^t} \left(\frac{c_t(\theta^t)}{C_t}\right) \eta_t(\theta^t)\pi_t(\theta^t)}_{Part\ 2} \right]. \tag{30}$$

Second, using (15)-(18), $W_L(t)$ in (28) is expressed as

$$-W_L(t) = \widehat{w}_t C_t^{-\alpha} \left[\underbrace{\sum_{\theta^t} \left(\frac{l_t(\theta^t)}{L_t}\right) \left(\frac{c_t(\theta^t)}{C_t}\right)^{-\alpha} \pi_t(\theta^t)}_{Part\ 1} + \underbrace{\gamma H_t^{\alpha} \sum_{\theta^t} \left(\frac{l_t(\theta^t)}{L_t}\right) \eta_t(\theta^t)\pi_t(\theta^t)}_{Part\ 2} \right]. \tag{31}$$

Part 1 of $W_C(t)$ and Part 1 of $W_L(t)$ denote the sum of households' "normalized" marginal utility of consumption, $\left(\frac{c_t(\theta^t)}{C_t}\right)^{-\alpha}$, weighted by their consumption shares and labor shares, respectively. They represent the planner's social evaluation of increasing C_t and L_t, respectively, under the utilitarian objective. We next explain the meaning of the weighted sum of $\eta_t(\theta^t)$ shown in Part 2 of $W_C(t)$ and of $W_L(t)$. Summing up (24) across all households gives

$$P_t \left(\sum_{\theta^t} \eta_t(\theta^t) \frac{c_t(\theta^t)}{C_t} \pi_t(\theta^t) C_t - \widehat{w}_t \sum_{\theta^t} \eta_t(\theta^t) \frac{l_t(\theta^t)}{L_t} \pi_t(\theta^t) L_t \right).$$

Contrasting the above with the corresponding term in the RA model, namely, $P_t^{RA}(\chi^P C_t - \widehat{w}_t^{RA} \chi^P L_t)$, we see that the role of χ^P has been replaced either by $\sum_{\theta^t} \eta_t(\theta^t) \frac{c_t(\theta^t)}{C_t} \pi_t(\theta^t)$ or by $\sum_{\theta^t} \eta_t(\theta^t) \frac{l_t(\theta^t)}{L_t} \pi_t(\theta^t)$. In other words, they represent the marginal utility cost of collecting total government revenue by either changing the margin of aggregate consumption or aggregate labor supply. Since the issue is about collecting *total* government revenue, it is intuitive that these utility costs or shadow prices are weighted (by the consumption or labor share depending on the changed margin) rather than unweighted as given by $\sum_{\theta^t} \eta_t(\theta^t)\pi_t(\theta^t)$.

Ljungqvist and Sargent (2012, p. 629) argued that when a government has to use distortionary taxes, the shadow price χ^P in the RA economy will be strictly positive, which reflects the welfare cost of distortionary taxes at the margin. By analogy, with $\chi^P > 0$ at the optimum in the RA model, the utility costs of collecting government revenue through changing C_t or L_t at the margin will be strictly positive at the optimum in the HAIM economy; that is, we have $\sum_{\theta^t} \eta_t(\theta^t) \frac{c_t(\theta^t)}{C_t} \pi_t(\theta^t) > 0$ and $\sum_{\theta^t} \eta_t(\theta^t) \frac{l_t(\theta^t)}{L_t} \pi_t(\theta^t) > 0$ at the optimum. These two properties will be used in the proof of our first main result reported in the next section.

5 Ramsey Steady State and the Government Debt Limit

Before presenting our main results, we define the steady state of the HAIM economy.

Definition 3. *The steady state of the HAIM economy meets two conditions:*

1. *Each aggregate variable stays at a positive finite value.*
2. *The cross-sectional distributions of the consumption share $c_t(\theta^t)/C_t$ and of the labor share $l_t(\theta^t)/L_t$ are time invariant with finite bounded support.*

As to the Ramsey steady state, it is defined as follows:

Definition 4. *The long-run optimal solution to the Ramsey problem is defined as a Ramsey steady state if it features the steady state of the HAIM economy*

As shown by many quantitative studies, the existence of a steady state may not be a problem for the HAIM economy. Our investigation is about the existence of a Ramsey steady state.

In the proof of Proposition 2, an implicit but standard assumption is that, given $\{K_{t+1}\}_{t=1}^{\infty}$ and $\{\widehat{a}_{t+1}(\theta^t)\}_{t=1}^{\infty}$, it is always feasible to pick a sequence of government bonds, $\{B_{t+1}\}_{t=1}^{\infty}$, so as to clear the asset market in each time period. Although a standard assumption in the literature, it may be inconsistent with the concept of the so-called natural debt limit, which is defined as the maximum level of indebtedness for which the debt can be repaid almost surely. Not imposing a natural debt limit on the household problem is inessential even if there is no ad hoc household borrowing constraint, since it will never bind in the optimal solution to the household problem. The reason is that if a natural debt limit were to bind for a household, then, in order to honor its debt, the household would have to set consumption equal to zero from then on. This is incompatible with an Inada condition $\lim_{c_t(\theta^t)\downarrow 0} u'(c_t(\theta^t)) = +\infty$ (see Ljungqvist and Sargent (2012, pp.271-273)). In contrast, there does not exist a similar mechanism to prevent the government's natural debt limit from binding in the Ramsey problem.

Thus a standard but implicit assumption behind Proposition 2 is that the government's natural debt limit must never bind. As already mentioned in the Introduction, Aiyagari (1995) also implicitly made this standard assumption in his analysis. We let this standard but implicit assumption be explicit from now on.

5.1 No Ramsey Steady State

We are ready to state our first main finding.

Proposition 3. *Impose Assumption 1 and assume that the government's natural debt limit never binds.*

1. *If $\alpha \geq 1$, there is no Ramsey steady state with $R < 1/\beta$.*
2. *If $\alpha < 1$, a Ramsey steady state with $R < 1/\beta$ is possible, but (i) the corresponding shadow price of resources, μ_t, must diverge in the limit and (ii) the planner may not implement the MGR.*

As already mentioned, the key driving force of our first main result stems from the asymmetric converging rate between Part 1 and Part 2 of $W(t)$, which originated from the frictions of the HAIM economy. Specifically, given $R < 1/\beta$ in the steady state of the HAIM economy, H_t is increasing over time and must diverge to infinity in the limit according to (25). From (31) and the FOC (28), it is then implied that the multiplier μ_t must diverge in a Ramsey steady state. This result is contrary to the implicit assumption made by Aiyagari (1995) that μ_t converges.

When $\alpha = 1$, the FOC (27) reduces to $1/C_t = \mu_t$. The divergent μ_t immediately implies that $C_t \to 0$ in the limit. However, this cannot occur in a Ramsey steady state, since it is incompatible with an Inada condition $\lim_{c_t(\theta^t)\downarrow 0} u'(c_t(\theta^t)) = +\infty$. The result of $C_t \to 0$ in the limit also violates the steady state defined in Definition 3.

When $\alpha > 1$, the divergent H_t causes Part 2 of $W_C(t)$ expressed in (30) to diverge to negative rather than positive infinity in the limit. Given that μ_t must be non-negative in the limit according to the FOC (28), there is no possibility for the FOC (27) to hold in a Ramsey steady state.

In the case where $\alpha = 1$, the FOC (27) is incompatible with a Ramsey steady state with $\beta R < 1$. In the case where $\alpha > 1$, the FOC (27) cannot hold in a Ramsey steady state with $\beta R < 1$. In both cases, a Ramsey steady state fails to exist. This result, listed as the first part of Proposition 3, contrasts sharply with the implicit assumption made by Aiyagari (1995) that a Ramsey steady state (featuring a non-binding natural government debt limit) exists. Thanks to our methodology, the divergent force embedded in $W_C(t)$ or $-W_L(t)$ that drives the non-existence of a Ramsey steady state can be seen clearly by means of our derived W_t.

When $\alpha < 1$, both the divergent μ_t and a convergent μ_{t+1}/μ_t can co-exist and be consistent with the FOCs (27)-(29) in a Ramsey steady state with $\beta R < 1$. However, because it is the ratio

μ_{t+1}/μ_t rather than μ_t itself that converges, the MGR need not hold in the Ramsey steady state. This result, listed as the second part of Proposition 3, can be contrary to the finding of Aiyagari (1995) that the MGR must hold in the long-run at the optimum.

At any rate, under the same assumption as made by Aiyagari (1995) that the government's natural debt limit never binds, Proposition 3 shows that the Ramsey steady state described and assumed by Aiyagari (1995) does not arise at the optimum. It is important to observe that if we were to confine the analysis *only* to the FOC (29) and assume *incorrectly* the convergence of μ_t in the Ramsey steady state, we would have the exact conclusion reached by Aiyagari (1995); namely, the MGR holds at the optimum and capital income should be taxed since $R < 1/\beta$ holds in the long-run. This observation highlights the importance of taking into account the necessary Ramsey FOCs other than (29). To our knowledge, the analytical form of the expression for the term $W_C(t)$ or $W_L(t)$ that appears in the Ramsey FOCs (27)-(28) has never been derived before. Thanks to our methodology, the consistency or inconsistency between the existence of a Ramsey steady state and other Ramsey FOCs can be clearly checked and the shortcoming of the analysis in Aiyagari (1995) can be remedied.

The intuition underlying our first main result can be understood as follows. Unlike households in the face of idiosyncratic income shocks, the Ramsey planner faces no uncertainty in allocating aggregate resources. The strict inequality $R < 1/\beta$ in the steady state of the HAIM economy then dictates an asymmetric discounting; that is, the market discounting rate is always lower than the preference discounting rate. This feature of asymmetric discounting impels a desire for the planner to front-load aggregate consumption. Such a desire persists permanently as long as the asymmetric discounting holds in the steady state.

Proposition 3 indicates that the existence of a Ramsey steady state may depend on the value of α, which controls the EIS. The following intends to provide additional explanations and intuition for such a dependence. As discussed in Section 4.2, the utility costs of implementing a policy depend on its effects over the net savings of households (or equivalently, by Walras' law, the amount of net revenue collected by the government). Let us consider the impact of changing aggregate consumption on the net savings (government revenue) through consumption spending. There is only a term involving C_t in Part 2 of $W(t)$ given by (21). Expressed in $\beta^t W(t)$ and by omitting $\eta_t(\theta^t)$, this term equals

$$\beta^t C_t^{1-\alpha} H_t^{\alpha-1} \zeta_t(\theta^t)^{\frac{-1}{\alpha}} = P_t C_t \frac{\zeta_t(\theta^t)^{\frac{-1}{\alpha}}}{H_t},$$

which represents household θ^t's consumption spending at time t according to the consumption sharing rule (15). From (17), we have $P_t C_t = \beta^t C_t^{1-\alpha} H_t^{\alpha}$ and so $\partial(P_t C_t)/\partial C_t = (1-\alpha)\beta^t C_t^{-\alpha} H_t^{\alpha}$. Thus, all else equal, a drop in aggregate consumption C_t will raise, lower, or not change individual consumption spending via altering $P_t C_t$ if α is larger than, less than, or equal to 1, respectively. This implies that a reduction in aggregate consumption over time (front-loading consumption) will make the government constraint associated with $\eta_t(\theta^t)$ in (21) looser, tighter, or unchanged, depending on whether α is larger than, less than, or equal to 1, respectively. Since front-loading aggregate consumption relaxes the government constraint by increasing its revenue if $\alpha > 1$, it actually enforces the planner's desire to front-load aggregate consumption in the presence of $R < 1/\beta$ in the steady state. In contrast, since front-loading aggregate consumption tightens the government constraint by reducing its revenue if $\alpha < 1$, it counterbalances the planner's desire to front-load aggregate consumption in the presence of $R < 1/\beta$ in the steady state.

When $\alpha = 1$, neither enforcement (associated with $\alpha > 1$) nor counterbalance (associated with $\alpha < 1$) occurs. We then see a clean case of front-loading aggregate consumption in the presence of $R < 1/\beta$ in the steady state. From the proof of Proposition 3, we know that μ_t is increasing and divergent because H_t is increasing and divergent. If $\alpha = 1$, we have $W_C(t) = C_t^{-1}$ from (30). Thus, given that μ_t increases over time, it is apparent that the optimal C_t determined by the FOC (27), namely, $C_t^{-1} = \mu_t$, will decrease over time. In the limit we obtain $C_t \to 0$, which is incompatible with an Inada condition $\lim_{c_t(\theta^t)\downarrow 0} u'(c_t(\theta^t)) = +\infty$.

5.2 Why No Ramsey Steady State

After an allocation of competitive equilibrium has been found, the primal approach to the Ramsey problem backs out tax rates to support the allocation from the optimal conditions of households and firms. For convenience, we call these tax rates "supportive tax rates."

Proposition 3 addresses the existence of a Ramsey steady state by means of the FOCs of the Ramsey problem, showing that if the government's natural debt limit never binds (an implicit but standard assumption in the literature), the existence of a Ramsey steady state assumed by Aiyagari (1995) can be incorrect. However, Proposition 3 itself does not really tell us why there is no Ramsey steady state in the case of $\alpha \geq 1$. Our second main result aims to shed light on this question.

Proposition 4. *Impose Assumption 1 with $\alpha \geq 1$ (EIS ≤ 1). Then the welfare of any steady state in the HAIM economy can be improved by the implementation of an early aggregate consumption* *through issuing more government bonds and adjusting supportive tax rates if the issuance and the adjustment are both feasible.*

Adjusting supportive tax rates will be infeasible if, for example, the adjustment requires labor tax rates to be higher than 100%. Now assuming that adjusting supportive taxes is always feasible, then Proposition 4 explicitly explains why a Ramsey steady state fails to exist in the case of $\alpha \geq 1$. It is due to the fact that, as long as it is feasible, the Ramsey planner would always desire to issue more government bonds to improve social welfare by front-loading consumption in any steady state of the HAIM economy. It is worth noting that this second main result is proved without utilizing any FOC of the Ramsey problem and, therefore, it provides a robustness check for our first main result.

The steady-state feature of $R < 1/\beta$ dictates that the market discounts resources at a lower rate than the Ramsey planner discounts utility. This then implies that it is always desirable for the Ramsey planner to front-load aggregate consumption through policy tools in any steady state of the HAIM economy. The question concerns whether it is feasible for the planner to implement the desire by issuing more government bonds. The proof of Proposition 4 is mainly to show that it is indeed feasible if $\alpha \geq 1$. Leaving aside this main but tedious part of the proof, the logic underlying the proof is actually straightforward. We sketch it below.

Let us focus on the steady-state periods t and $t+1$ and consider the following adjustment made by the Ramsey planner. In period t, the issuance of more government bonds will crowd out capital formation. This leads to an increased aggregate consumption in period t, denoted by $\Delta_c > 0$. The reduction in capital is set equal to Δ_c so that the resource constraint continues to be satisfied in period t. The lower capital investment in period t results in a reduction in aggregate consumption in period $t+1$, which is denoted by $-\Delta_{c'} < 0$. The labor supply and the consumption/labor sharing rule can be left unchanged between period t and $t+1$ by choosing Δ_c and $\Delta_{c'}$ appropriately to satisfy all the relevant constraints. The objective of the Ramsey problem then dictates that the steady-state welfare change from inducing an early aggregate consumption is equal to

$$\frac{1}{1-\alpha}[(C_t + \Delta_c)^{1-\alpha} - C_t^{1-\alpha}] + \frac{\beta}{1-\alpha}[(C_{t+1} - \Delta_{c'})^{1-\alpha} - C_{t+1}^{1-\alpha}], \tag{32}$$

where the first and the second bracketed terms are associated with the welfare gain and loss from the consumption changes in periods t and $t+1$, respectively.

The government bonds B_t and capital stock K_t are given in period t, hence the household's total asset holdings in period t (i.e., $\sum_{\theta^{t-1}} \widehat{a}_t(\theta^{t-1})\pi_{t-1}(\theta^{t-1})$) must be unchanged in order to respect the asset-market clearing condition (i.e., $B_t + K_t = \sum_{\theta^{t-1}} \widehat{a}_t(\theta^{t-1})\pi_{t-1}(\theta^{t-1})$). Given that the labor supply and the consumption/labor sharing rule remain unchanged across periods t and $t+1$, Condition 3 of Proposition 2 then gives the following equality:

$$(C_t + \Delta_c)^{1-\alpha} H_t^{\alpha} + \beta \left(C_{t+1} - \Delta_{c'}\right)^{1-\alpha} H_{t+1}^{\alpha} = C_t^{1-\alpha} H_t^{\alpha} + \beta C_{t+1}^{1-\alpha} H_{t+1}^{\alpha}, \tag{33}$$

which is the same as equation (58) in the proof of Proposition 4. Using (17), one can rewrite (33) as

$$P_t^{\Delta} \left(C_t + \Delta_c\right) + P_{t+1}^{\Delta} \left(C_{t+1} - \Delta_{c'}\right) = P_t C_t + P_{t+1} C_{t+1}, \tag{34}$$

where P_t^{Δ} and P_{t+1}^{Δ} denote price P in periods t and $t+1$ respectively after the variation of Δ_c and $\Delta_{c'}$. Hence equation (33) is equivalent to stating that the sum of the aggregate consumption expenditure across the two periods t and $t+1$ remains the same, despite the variation of Δ_c and $\Delta_{c'}$. Note that issuing more government bonds to crowd out capital ensures that both Δ_c and $\Delta_{c'}$ are positive in (33).

The variational experiment of Δ_c and $\Delta_{c'}$ front-loads aggregate consumption by moving part of consumption from period $t+1$ to period t. Equation (32) shows that the Ramsey planner discounts this movement by β, whereas equation (33) or, equivalently, equation (34) shows that the market discounts this movement by $1/R$ according to (2). Using (33), we obtain

$$\begin{aligned} \frac{1}{1-\alpha}\left[(C_t + \Delta_c)^{1-\alpha} - C_t^{1-\alpha}\right] &= \frac{\beta}{1-\alpha}\frac{H_{t+1}^{\alpha}}{H_t^{\alpha}}\left(C_{t+1}^{1-\alpha} - (C_{t+1} - \Delta_{c'})^{1-\alpha}\right) \\ &> \frac{\beta}{1-\alpha}\left(C_{t+1}^{1-\alpha} - (C_{t+1} - \Delta_{c'})^{1-\alpha}\right), \end{aligned}$$

where the last inequality uses the steady-state properties of (25) and $\beta R < 1$ in the HAIM economy. From (32), we then see that front-loading aggregate consumption by the variation of Δ_c and $\Delta_{c'}$ improves the steady-state welfare. Again, as in the proof of Proposition 3, we utilize the steady-state hallmark of the HAIM economy, $R < 1/\beta$, in the proof of Proposition 4.

As we have shown in the previous subsection, a reduction in aggregate consumption over time (front-loading consumption) will make the household's net savings associated with $\eta_t(\theta^t)$ in (21) looser, tighter, or unchanged, depending on whether α is larger than, less than, or equal to 1, respectively. Thus, the implementation of front-loading aggregate consumption as described above may fail to meet the household's borrowing constraints if $\alpha < 1$, but not if $\alpha \geq 1$. This explains why we need to impose the restriction $\alpha \geq 1$ in the statement of Proposition 4.

5.3 Long-Run Ramsey Outcome

Proposition 3 does not tell us either what the possible long-run Ramsey outcome will be in the case of $\alpha \geq 1$. Our second main result, Proposition 4, also sheds light on this question.

The existence of feasible tax rates to support an allocation is a maintained assumption of the primal approach to the Ramsey problem. Proceeding under this maintained assumption, an immediate implication of Proposition 4 is the following:

Corollary 1. *Impose Assumption 1 with $\alpha \geq 1$ (EIS≤ 1). If a steady state of the HAIM economy constitutes a Ramsey steady state, then its associated government debt limit, either natural or ad hoc, must bind.*

This corollary is obviously important for studies that attempt to quantitatively characterize the Ramsey allocation, regardless of whether the focus is on transition paths or steady states of the HAIM economy. To carry out the quantitative analysis, the existence of a Ramsey steady state is typically assumed rather than proved in these studies. Corollary 1 tells us that such an assumption by itself may be questionable when studying the Ramsey problem in the HAIM economy. In particular, under Assumption 1 with $\alpha \geq 1$, a Ramsey steady state will not arise if neither natural nor ad hoc government debt limit is imposed on the Ramsey problem.

If the multiplier μ_t converges, the MGR will hold in the steady state according to (29). This is basically the result of Aiyagari (1995). However, as we have explained, the result of (29) critically relies on the assumption that the government's debt limit never binds so that the Ramsey planner could implement the capital allocation independently of the household's asset holdings. Once the government's debt limit binds, the Ramsey planner no longer has the degree of freedom to choose capital stock, K_{t+1}, and asset holdings, $\{\widehat{a}_{t+1}(\theta^t)\}$, independently as in our relaxed Ramsey problem. As a result, the Ramsey FOC with respect to K_{t+1} will be definitely more complicated than the expression given by (29) and the MGR could and should fail to hold in the Ramsey steady state.

Notice that Corollary 1 itself does not tell us whether a Ramsey steady state exists or not in the case where $\alpha \geq 1$. We argue below that the existence of a Ramsey steady state with a binding government debt limit is the most likely Ramsey outcome in the long run.

Putting our two main results and Corollary 1 together implies an either-or long-run solution to the Ramsey problem in the case where $\alpha \geq 1$. That is, either there is an ever increasing amount of government bonds but the government's natural debt limit never binds so that a Ramsey steady state fails to exist, or a Ramsey steady state exists along with the binding of the government's ad hoc or natural debt limit. The former outcome seems unlikely. An ever-increasing amount of government bonds enlarges the government's burden of bond interest payments, which must be financed by distorting taxes. As tax rates increase, regardless of labor or capital taxes, the distortion of the economy becomes more severe and the tax base of the economy shrinks. This then eventually tightens the government's natural debt limit because of the feature of the Laffer curve. Thus, the natural debt limit should bind before the economy diverges to some extreme to upset a Ramsey steady state.

To sum up, the most likely long-run Ramsey outcome in the case where $\alpha \geq 1$ should feature the coexistence of a Ramsey steady state with the binding of the government's debt limit and the failure of the MGR.

The arguments above lack a formal analysis. There are good reasons for this. First, the government's natural debt limit in the Ramsey problem is endogenously determined and evolves dynamically according to the policy choice in a non-trivial way. Thus, incorporating the government's natural debt limit into the analysis of the Ramsey problem is a formidable task, if not impossible. Next, even if the government's debt limit is ad hoc rather than natural, our approach may still be inapplicable. The proof of the relaxed Ramsey problem depends on the standard assumption that the government's debt limit never binds and therefore government bonds can always serve as a residual to clear the asset market in each time period. The binding of the ad hoc debt limit can fail this standard assumption and force us to incorporate the technically challenging Condition 6 of Proposition 2 into the analytical exposition of the Ramsey problem. In this case, the numerical rather than analytical approach could be useful to advance our knowledge further.

6 Endogenous Government Spending

This section checks the robustness of our Proposition 3 findings by altering the model setup from exogenous to endogenous government spending, which is the main setting considered by Aiyagari (1995). We show here that even with endogenous government spending, our results concerning the existence of a Ramsey steady state are robust and remain unchanged.

Following Aiyagari (1995), the household lifetime utility U is modified to

$$U^G = \sum_{t=1}^{\infty} \beta^t \sum_{\theta^t} \left[u(c_t(\theta^t)) - v\left(\frac{l_t(\theta^t)}{\theta_t}\right) + V(G_t) \right] \pi_t(\theta^t),$$

where $V(.)$ is the utility function of public consumption G_t, which is assumed to be common for all households. The usual assumptions are applied to $V(.)$. This modification of the setup does not change the household problem, since the determination of G_t is exogenous to households. However, the Ramsey problem is only changed slightly because G_t is now a choice variable to the Ramsey planner. As long as G_t is non-negative (which could be ensured by assuming $V'(0) = \infty$), G_t can be chosen to satisfy the time-t resource constraint so that Proposition 2 still applies. The Lagrangian for the Ramsey problem is identical to the previous Lagrangian $\mathcal{L}$, except for the replacement of $W(t)$ by $W(t) + V(G_t)$. As a result, the FOCs with respect to aggregate consumption, labor, and capital remain the same as before. The additional FOC with respect to G_t is given by

$$V'(G_t) = \mu_t, \tag{35}$$

which together with FOC (29) does imply the MGR if a Ramsey steady state is assumed. This is essentially the procedure for obtaining the MGR in Aiyagari (1995); see Equation (20) of his Proposition 1 on page 1170.

However, the introduction of endogenous G_t does not alter the fundamental force that drives the results of Proposition 3 nor does it help to justify the assumption of a Ramsey steady state. The marginal social benefit of having one extra unit of aggregate consumption, namely, $W_C(t)$, could still diverge in the long-run given that the Ramsey outcome of $R\beta = 1$ is infeasible in the steady state. With the additional government tool—endogenous government spending—the extra output can be expended either on government spending or on private consumption, and hence the marginal benefits to the social welfare by exercising these two options have to be equalized at the optimum. Indeed, putting (27) and (35) together gives rise to $V'(G_t) = W_C(t)$ and hence the optimal choice of G_t has to respect and be consistent with the divergent behavior of μ_t. This equality casts doubt on the convergence assumption of G_t to a finite positive value made by Aiyagari (1995). In brief, it is the erroneously assumed Ramsey steady state, not the endogenous government spending assumption, which is the root of the problem.

7 Conclusion

This paper investigates the critical and common assumption on the existence of a Ramsey steady state in the HAIM economy. Our first main result demonstrates that the optimal Ramsey allocation may feature no steady state if the government's natural debt limit never binds. This result shows that the Ramsey steady state described and assumed in Aiyagari (1995) does not arise at the optimum and thus the subsequent conclusions reached could be problematic. Our second main result further demonstrates that, if the EIS is weakly less than 1, then any steady state of the HAIM economy can be welfare-improved by issuing more government bonds. Both results question the conventional wisdom that the MGR has to hold in the long-run and also highlight the important role of the government's debt limit in the HAIM framework. The underpinning to both results is embedded in the signature feature of the HAIM economy, that the risk-free rate has to be lower than the time discount rate in the steady state of any HAIM competitive equilibrium.

Our analysis also makes a positive contribution to illuminating the most likely long-run Ramsey outcome in the HAIM economy. Putting our two main results together leads to a natural conjecture that the government's debt limit (either ad hoc or natural) should ultimately bind in the long-run Ramsey solution (given the EIS is weakly less than 1). Once the debt limit is binding, the Ramsey planner can no longer increase welfare by issuing more government debt; the force of diverging away from the steady state is stopped and hence a Ramsey steady state with a binding debt limit should exist. We also argue that the MGR should fail in such a Ramsey steady state because of the binding of the government's debt limit.

We conclude our study with two remarks with regard to extending our model to include lump-sum taxes/transfers and aggregate uncertainty. First, suppose that the Ramsey planner is equipped with the lump-sum tax/transfer as an additional policy tool. The presence of lump-sum taxes/transfers may not necessarily fail to invalidate the proof of Proposition 3. In addition, by fixing lump-sum taxes/transfers in the proof of Proposition 4, it is rather easy to check that the same proof still carries through. Thus, the welfare of any steady state in the HAIM economy can still be improved if it is feasible to issue more government bonds to front-load consumption. We conclude that, as far as our main result with regard to the non-existence of Ramsey steady states is concerned, the availability of lump-sum taxes/transfers may not invalidate the result. Second, our results are likely to change if aggregate uncertainty, such as government spending shocks, is included in our model. In the presence of aggregate shocks, capital becomes a risky asset, while government bonds remain risk free. As a result, they are no longer perfect substitutes in the mind of households and the households' precautionary-savings motive might not necessarily lead to capital overaccumulation with respect to the MGR. Moreover, allocating aggregate resources is no longer risk free to the Ramsey planner, as in our HAIM economy. In order to smooth tax distortions over time in response to aggregate shocks, the government may also have a precautionary savings motive to accumulate assets. As such, the government's debt limit may not bind in the long-run. Given that this second extension is interesting while non-trivial, we leave it to future research.

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A Appendix

A.1 Proof of Proposition 1

With the imposition of Assumption 1, the FOC for consumption (9) yields

$$c_t(\theta^t) = (\frac{\zeta_t(\theta^t)P_t}{\beta^t})^{-\frac{1}{\alpha}}.$$

Summing $c_t(\theta^t)$ over θ^t gives the aggregate consumption at time t:

$$\begin{aligned} C_t &= \sum_{\theta^t} c_t(\theta^t)\pi_t(\theta^t) = \sum_{\theta^t}(\frac{\zeta_t(\theta^t)P_t}{\beta^t})^{-\frac{1}{\alpha}}\pi(\theta^t) \\ &= (\frac{P_t}{\beta^t})^{-\frac{1}{\alpha}} \sum_{\theta^t} \zeta_t(\theta^t)^{-\frac{1}{\alpha}}\pi_t(\theta^t) = (\frac{P_t}{\beta^t})^{-\frac{1}{\alpha}} H_t, \end{aligned}$$

which gives (17). Plugging (17) back into (9) gives (15).

From (10), we have

$$l_t(\theta^t) = (\frac{\theta_t^{\gamma}\zeta_t(\theta^t)\widehat{w}_t P_t}{\beta^t})^{\frac{1}{\gamma-1}}.$$

Summing $c_t(\theta^t)$ over θ^t gives the aggregate consumption at time t:

$$\begin{aligned} L_t &= \sum_{\theta^t} l_t(\theta^t)\pi_t(\theta^t) = \sum_{\theta^t}(\frac{\theta_t^{\gamma}\zeta_t(\theta^t)\widehat{w}_t P_t}{\beta^t})^{\frac{1}{\gamma-1}}\pi(\theta^t) \\ &= (\frac{\widehat{w}_t P_t}{\beta^t})^{\frac{1}{\gamma-1}} \sum_{\theta^t} \theta_t^{\frac{\gamma}{\gamma-1}} \zeta_t(\theta^t)^{\frac{1}{\gamma-1}}\pi_t(\theta^t) = (\frac{\widehat{w}_t P_t}{\beta^t})^{\frac{1}{\gamma-1}} J_t, \end{aligned}$$

which together with (17) gives (18). Plugging (18) back into (10) gives (16).

A.2 Proof of Proposition 2

"Only if" part: Condition 1 of Proposition 2—the resource constraints—is implied by a competitive equilibrium since it is part of the definition. Note also that Conditions 5 and 6 of Proposition 2 are implied by (8) and (11) from the household problem in a competitive equilibrium.

The remaining proof is to show that the time-zero budget constraint (3), the measurability conditions (6), and the borrowing constraints (7) in the household problem can be re-expressed as Conditions 2-4 of Proposition 2. Substituting (2), (15)-(16) and (18)-(19), all of which build on the household's optimal behavior, into (3)-(6), we obtain Conditions 2-4.

"If" part: Suppose the sequence of asset holdings $\{\widehat{a}_{t+1}(\theta^t)\}_{t=1}^{\infty}$, aggregate allocations $\{C_t, K_{t+1}, L_t\}_{t=1}^{\infty}$, and cumulative multipliers $\{\zeta_t(\theta^t)\}_{t=1}^{\infty}$ with the associated aggregate multipliers $\{H_t, J_t\}_{t=1}^{\infty}$ satisfy Conditions 1-6 stated in Proposition 2. We show that a competitive equilibrium of the HAIM economy can be constructed in the following way.

First, we pick the prices and taxes defined below:

$$r_t = F_K(K_t, L_t), \tag{36}$$

$$w_t = F_L(K_t, L_t), \tag{37}$$

$$P_t = \beta^t C_t^{-\alpha} H_t^{\alpha}, \tag{38}$$

$$1 - \tau_{k,t+1} = \frac{\frac{P_t}{P_{t+1}} - 1}{F_K(K_{t+1}, L_{t+1}) - \delta} = \frac{\frac{1}{\beta}\left(\frac{C_t}{C_{t+1}}\right)^{-\alpha}\left(\frac{H_t}{H_{t+1}}\right)^{\alpha} - 1}{F_K(K_{t+1}, L_{t+1}) - \delta}, \tag{39}$$

$$1 - \tau_{l,t} = \frac{L_t^{\gamma-1} J_t^{1-\gamma}}{F_L(K_t, L_t) C_t^{-\alpha} H_t^{\alpha}}. \tag{40}$$

Note that (36)-(37) correspond to the profit-maximization conditions of the representative firm and that (39) ensures that the no-arbitrage condition (2) holds.

Second, we show that the household problem can be solved. Let the individual consumption and labor allocations be given by (15) and (16). Then, individual consumption and labor allocations together with prices defined in (36)-(40) satisfy the first-order conditions, (9) and (10), of the household problem. To derive the household's Euler equation, we combine the individual consumption allocations, prices defined in (36)-(40), and Conditions 5-6. The time-zero budget constraint (3), the measurability conditions (6), and the borrowing constraints (7) in the household problem can be obtained by using (36)-(40) plus Conditions 2-4.

Third, we need to make sure that all markets clear. Plugging in individual consumption allocations (15) into Condition 1 implies that the market clearing condition of the goods market is satisfied. The labor market clearing condition is achieved by aggregating (16) across all households. For the asset market, we pick $\{B_{t+1}\}_{t=1}^{\infty}$ such that

$$B_{t+1} = \sum_{\theta^t} \widehat{a}_{t+1}(\theta^t) - K_{t+1},$$

which ensures that the asset market clears in each time period.

The last condition to be met in competitive equilibrium is the government budget constraint.

From (3), we have

$$
\begin{aligned}
B_1 + K_1 &= \widehat{a}_1 = \sum_{t\geq 1} P_t \sum_{\theta^t} \left[c_t(\theta^t)\pi_t(\theta^t) - \widehat{w}_t l_t(\theta^t)\pi_t(\theta^t)\right] \\
&= \sum_{t\geq 1} P_t \left[C_t - w_t L_t + \tau_{l,t} w_t L_t\right] \\
&= \sum_{t\geq 1} P_t \left[C_t + q_t K_t - F(K_t, L_t) + \tau_{l,t} w_t L_t\right],
\end{aligned}
$$

where the derivation has made use of $\widehat{w}_t = w_t(1-\tau_{l,t})$ and $F(K_t, L_t) = w_t L_t + q_t K_t$. Utilizing the resource constraint and the no-arbitrage condition (2) then gives

$$
\begin{aligned}
B_1 + K_1 &= \sum_{t\geq 1} P_t \left[r_t K_t - K_{t+1} + (1-\delta) K_t + \tau_{l,t} w_t L_t - G_t\right] \\
&= \sum_{t\geq 1} P_t \left[(1 + (1-\tau_{k,t})(r_t - \delta)) K_t - K_{t+1} + \tau_{k,t}(r_t - \delta) K_t + \tau_{l,t} w_t L_t - G_t\right] \\
&= \sum_{t\geq 1} P_t \left[\frac{P_{t-1}}{P_t} K_t - K_{t+1} + \tau_{k,t}(r_t - \delta) K_t + \tau_{l,t} w_t L_t - G_t\right] \\
&= P_0 K_1 + \sum_{t\geq 1} P_t \left[\tau_{k,t}(r_t - \delta) K_t + \tau_{l,t} w_t L_t - G_t\right],
\end{aligned}
$$

which leads to the time-zero government budget constraint since we normalize $P_0 = 1$.

A.3 Proof of the Relaxed Ramsey Problem

We claim in the text that the optimal allocation chosen by the Ramsey planner that satisfies Conditions 1-5 of Proposition 2 will also satisfy Condition 6 of Proposition 2. This claim enables us to consider the relaxed rather than the original Ramsey problem. We now verify the claim. The proof is done by contradiction.

Suppose there is an original optimal allocation $\{\{\zeta_t(\theta^t)\}, H_t, J_t, C_t, L_t, K_{t+1}, \{\hat{a}_{t+1}(\theta^t)\}\}_{t=1}^{\infty}$ that satisfies Conditions 1-5 of Proposition 2 but fails to meet its Condition 6. That is, there exists a node $\hat{\theta}^{\widehat{s}}$ at time $\widehat{s}$ such that all its possible subsequent $\theta_{\widehat{s}+1}$ states at time $\widehat{s}+1$ have the property that the borrowing constraints do not bind, but

$$
x\left(\hat{\theta}^{\widehat{s}}\right) \equiv \zeta_{\widehat{s}}(\hat{\theta}^{\widehat{s}}) - \sum_{\theta_{\widehat{s}+1}} \zeta_{\widehat{s}+1}(\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1}) \pi(\theta_{\widehat{s}+1} | \hat{\theta}_{\widehat{s}}) > 0, \tag{41}
$$

where $(\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1})$ are successor nodes of $\hat{\theta}^{\widehat{s}}$ at time $\widehat{s}+1$.

We construct an alternative v allocation $\left\{\{\zeta_t^v(\theta^t)\}, H_t^v, J_t^v, C_t^v, L_t^v, K_{t+1}^v, \{\hat{a}_{t+1}^v(\theta^t)\}\right\}_{t=1}^{\infty}$ that deviates from the original allocation with the property that Conditions 1-5 of Proposition 2 still hold but tighten the constraint (41); that is,

$$x\left(\hat{\theta}^{\widehat{s}}\right) > x^v\left(\hat{\theta}^{\widehat{s}}\right) \equiv \zeta_{\widehat{s}}^v(\hat{\theta}^{\widehat{s}}) - \sum_{\theta_{\widehat{s}+1}} \zeta_{\widehat{s}+1}^v(\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1})\pi(\theta_{\widehat{s}+1}|\hat{\theta}_{\widehat{s}}) \geq 0. \tag{42}$$

We shall prove that the alternative v allocation surpasses the original optimal allocation in terms of the household's lifetime utility U. This then leads to a contradiction, implying that the optimal allocation of the relaxed Ramsey problem must also satisfy Condition 6 of Proposition 2; otherwise, there will be an alternative allocation to improve U.

To provide a clear exposition, we decompose the whole proof into three parts:

Part 1 Construct the v allocation by making a (ϵ_1, ϵ_2) variation to the original allocation.

The $\zeta_t^v(\theta^t)$ is chosen identically to $\zeta_t(\theta^t)$ except for the state $\hat{\theta}^{\widehat{s}}$ in period $\widehat{s}$ and its subsequent state $(\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1})$ in period $\widehat{s}+1$:

$$\zeta_t^v\left(\theta^t\right) = \begin{cases} \zeta_{\widehat{s}}(\hat{\theta}^{\widehat{s}}) - \epsilon_1 & \text{if } t = \widehat{s} \text{ and } \theta^t = \hat{\theta}^{\widehat{s}} \\ \zeta_{\widehat{s}+1}(\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1}) + \epsilon_2\left(\theta_{\widehat{s}+1}\right) & \text{if } t = \widehat{s}+1 \text{ and } \theta^t = (\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1}) \\ \zeta_t(\theta^t) & \text{otherwise} \end{cases}, \tag{43}$$

where $\epsilon_1 > 0$ and $\epsilon_2\left(\theta_{\widehat{s}+1}\right) > 0$ for all possible $\theta_{\widehat{s}+1}$. ϵ_1 is chosen such that equation (42) holds with equality or the borrowing constraint binds for $\hat{a}_{\hat{s}+1}(\hat{\theta}^{\widehat{s}})$. The $\epsilon_2\left(\theta_{\widehat{s}+1}\right)$ are consequently pinned down in order to satisfy the measurability conditions, as shown later in this proof.

Given (43), the aggregate multipliers associated with ζ_t^v, denoted by H_t^v and J_t^v, are identical to the original ones except for period $\widehat{s}$ and $\widehat{s}+1$; that is, $H_t^v = H_t$ and $J_t^v = J_t$ if $t \notin \{\widehat{s}, \widehat{s}+1\}$. Next, the C_t^v and L_t^v are chosen such that

$$\frac{C_t^v}{H_t^v} = \frac{C_t}{H_t} \text{ and } \frac{L_t^v}{J_t^v} = \frac{L_t}{J_t} \text{ for all } t. \tag{44}$$

Thus, $C_t^v = C_t$ and $L_t^v = L_t$ if $t \notin \{\widehat{s}, \widehat{s}+1\}$. In addition, note that the intratemporal and intertemporal shadow prices, $(L_t^v)^{\gamma-1}(J_t^v)^{1-\gamma} / \left(C_t^v\right)^{-\alpha}(H_t^v)^{\alpha}$ and $\beta^t\left(C_t^v\right)^{-\alpha}(H_t^v)^{\alpha}$, are identical to those in the original allocation for all t by our choice. Combining (43) and (44) shows that individual consumption and labor supply for all $\theta^t \neq \hat{\theta}^{\widehat{s}}$ and $\theta^t \neq \left(\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1}\right)$ remain unaltered in the v allocation. In addition, similar to the definition of $\kappa_t\left(\theta^t\right)$ in (20), define $\kappa_t^v(\theta^t)$ as

$$\kappa_t^v\left(\theta^t\right) \equiv \beta^t \left[(C_t^v)^{1-\alpha} (H_t^v)^{\alpha-1} \zeta_t^v(\theta^t)^{\frac{-1}{\alpha}} - (L_t^v)^{\gamma} (J_t^v)^{-\gamma} \theta_t^{\frac{\gamma}{\gamma-1}} \zeta_t^v(\theta^t)^{\frac{1}{\gamma-1}} \right].$$

Note that according to (43) and (44), we obtain

$$\kappa_t\left(\theta^t\right) = \kappa_t^v\left(\theta^t\right), \text{ for } \theta^t \neq \hat{\theta}^{\widehat{s}} \text{ and } \theta^t \neq \left(\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1}\right). \tag{45}$$

The asset holdings $\{\widehat{a}_{t+1}^v(\theta^t)\}$ are chosen as follows:

$$\widehat{a}_{t+1}^v\left(\theta^t\right) = \begin{cases} \widehat{a}_{\widehat{s}+1}\left(\hat{\theta}^{\widehat{s}}\right) + \Omega\left(\hat{\theta}^{\widehat{s}}\right) & \text{if } t = \widehat{s} \text{ and } \theta^t = \hat{\theta}^{\widehat{s}} \\ \widehat{a}_{t+1}\left(\theta^t\right) & \text{otherwise} \end{cases}, \tag{46}$$

where $\Omega\left(\hat{\theta}^{\widehat{s}}\right)$ is defined as

$$\Omega\left(\hat{\theta}^{\widehat{s}}\right) \equiv \frac{\kappa_{\widehat{s}}\left(\hat{\theta}^{\widehat{s}}\right) - \kappa_{\widehat{s}}^v\left(\hat{\theta}^{\widehat{s}}\right)}{\beta^{\widehat{s}} C_{\widehat{s}}^{-\alpha} H_{\widehat{s}}^{\alpha}}.$$

In addition, let the choice of $\epsilon_2\left(\theta_{\widehat{s}+1}\right)$ satisfy the following condition state by state:

$$\beta^{\widehat{s}} C_{\widehat{s}}^{-\alpha} H_{\widehat{s}}^{\alpha} \Omega\left(\hat{\theta}^{\widehat{s}}\right) = \kappa_{\widehat{s}+1}^v\left(\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1}\right) - \kappa_{\widehat{s}+1}\left(\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1}\right), \tag{47}$$

which together with the definition of $\Omega\left(\hat{\theta}^{\widehat{s}}\right)$ implies that

$$\kappa_{\widehat{s}}\left(\hat{\theta}^{\widehat{s}}\right) + \sum_{\theta_{\widehat{s}+1}} \kappa_{\widehat{s}+1}\left(\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1}\right) \pi(\theta_{\widehat{s}+1}|\hat{\theta}^{\widehat{s}}) = \kappa_{\widehat{s}}^v\left(\hat{\theta}^{\widehat{s}}\right) + \sum_{\theta_{\widehat{s}+1}} \kappa_{\widehat{s}+1}^v\left(\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1}\right) \pi(\theta_{\widehat{s}+1}|\hat{\theta}^{\widehat{s}}). \tag{48}$$

$\{K_{t+1}^v\}$ are chosen as

$$K_{t+1}^v = \begin{cases} K_{t+1} & \text{for all } t < \widehat{s} \\ K_{t+1} + F\left(K_t^v, L_t^v\right) - F\left(K_t, L_t\right) & \text{for } t = \widehat{s} \text{ or } \widehat{s}+1 \\ +\left(1-\delta\right)\left(K_t^v - K_t\right) - \left(C_t^v - C_t\right) & \\ F\left(K_t^v, L_t\right) + \left(1-\delta\right) K_t^v - C_t - G_t & \text{for all } t > \widehat{s}+1 \end{cases}. \tag{49}$$

Note that for $t > \widehat{s}+1$, the right-hand side of (49), $F\left(K_t^v, L_t\right) + (1-\delta) K_t^v - C_t - G_t$, is strictly concave in K^v, and that the left-hand side of (49) is exactly K^v. If $K_t \gtrless K_t^v$, then the marginal product of capital implies $F_K\left(K_t^v, L_t\right) \gtrless F_K\left(K_t, L_t\right)$, which makes $\left|K_{t+1} - K_{t+1}^v\right|$

less than $|K_t - K_t^v|$. As such, K^v adjusts toward K and converges to K eventually as long as $\{C, L\}$ stabilizes. This guarantees that the sequence of K^v is well-defined and feasible.

Part 2 Verify that the v allocation satisfies Conditions 1 to 5 of Proposition 2 and (42).

1. *Resource Constraints*: It is straightforward to see that the choice of C_t^v, L_t^v and K_{t+1}^v satisfies the resource constraint for all $t \geq 1$ periods.

2. *Implementability condition*: By (46), $\widehat{a}_1^v = \widehat{a}_1$ and hence

$$\widehat{a}_1^v = \widehat{a}_1 = \sum_{t\geq 1}\sum_{\theta^t} \kappa_t\left(\theta^t\right)\pi(\theta^t) = \sum_{t\geq 1}\sum_{\theta^t} \kappa_t^v\left(\theta^t\right)\pi(\theta^t),$$

where the second equality holds due to (48) and (45). Therefore, the implementability condition holds given the v allocation.

3. *Measurability conditions*: We verify the measurability conditions for the cases $\theta^t = \hat{\theta}^{\widehat{s}}$ and $\theta^t \neq \hat{\theta}^{\widehat{s}}$, respectively.

   - For any $\theta^t \neq \widehat{\theta^{\widehat{s}}}$, $\widehat{a}_{t+1}^v(\theta^t)$ is chosen to be $\widehat{a}_{t+1}(\theta^t)$. Hence,

$$\begin{aligned} & \beta^t \left(C_t^v\right)^{-\alpha}\left(H_t^v\right)^{\alpha}\widehat{a}_{t+1}^v(\theta^t)\pi_{t+1}(\theta^{t+1}) \\ = \;& \beta^t C_t^{-\alpha} H_t^{\alpha}\widehat{a}_{t+1}(\theta^t)\pi_{t+1}(\theta^{t+1}) = \sum_{j\geq t+1}\sum_{\theta^j\succeq\theta^{t+1}} \kappa_j\left(\theta^j\right)\pi_j(\theta^j) \\ = \;& \sum_{j\geq t+1}\sum_{\theta^j\succeq\theta^{t+1}} \kappa_j^v\left(\theta^j\right)_j \pi(\theta^j), \end{aligned}$$

   where the last equality holds due to (48) and/or (45). Therefore, given the v allocation, the measurability condition holds for any θ^t with $\theta^t \neq \hat{\theta}^{\widehat{s}}$ and $\theta^t \neq (\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1})$.

   - For $\theta^t = \hat{\theta}^{\widehat{s}}$, employing (46) and (44), we write the value of $\widehat{a}_{\widehat{s}+1}^v(\hat{\theta}^{\widehat{s}})$ as

$$\begin{aligned} & \beta^{\widehat{s}}\left(C_{\widehat{s}}^v\right)^{-\alpha}\left(H_{\widehat{s}}^v\right)^{\alpha}\widehat{a}_{\widehat{s}+1}^v(\hat{\theta}^{\widehat{s}})\pi_{\widehat{s}+1}(\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1}) \\ = \;& \beta^{\widehat{s}} C_{\widehat{s}}^{-\alpha} H_{\widehat{s}}^{\alpha}\left(\widehat{a}_{\widehat{s}+1}(\hat{\theta}^{\widehat{s}}) + \Omega\left(\hat{\theta}^{\widehat{s}}\right)\right)\pi_{\widehat{s}+1}(\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1}) \\ = \;& \kappa_{\widehat{s}+1}\left(\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1}\right)\pi_{\widehat{s}+1}(\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1}) + \beta^{\widehat{s}} C_{\widehat{s}}^{-\alpha} H_{\widehat{s}}^{\alpha}\Omega\left(\hat{\theta}^{\widehat{s}}\right) + \sum_{j>\widehat{s}+1}\sum_{\theta^j\succ(\hat{\theta}^{\widehat{s}},\theta_{\widehat{s}+1})} \kappa_j\left(\theta^j\right)\pi_j(\theta^j) \\ = \;& \sum_{j\geq\widehat{s}+1}\sum_{\theta^j\succeq\left(\hat{\theta}^{\widehat{s}},\theta_{\widehat{s}+1}\right)} \kappa_j^v\left(\theta^j\right)\pi_j(\theta^j), \end{aligned}$$

where the last equality holds due to (47). Therefore, the measurability condition for each $(\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1})$ state holds given the v allocation.

4. *Borrowing constraints*: We verify the borrowing constraints for the cases $\theta^t = \hat{\theta}^{\widehat{s}}$ and $\theta^t \neq \hat{\theta}^{\widehat{s}}$.

   - For any $\theta^t \neq \hat{\theta}^{\widehat{s}}$, the value of asset holdings is

   $$\sum_{j \geq t} \sum_{\theta^j \succeq \theta^t} \kappa_j^v \left(\theta^j\right) \pi(\theta^j) = \sum_{j \geq t} \sum_{\theta^j \succeq \theta^t} \kappa_j \left(\theta^j\right) \pi(\theta^j) \geq 0,$$

   where the first equality holds due to (48) and (45). Thus, the borrowing constraint for $\theta^t \neq \hat{\theta}^{\widehat{s}}$ holds.

   - For $\theta^t = \hat{\theta}^{\widehat{s}}$, we choose $(\epsilon_1, \epsilon_2(\theta_{\widehat{s}+1}))$ to render the borrowing constraint feasible.

5. *Condition 5*: By the construction of the v allocation, Condition 5 remains intact except for the three cases left to be verified below.

   - From period $\widehat{s}-1$ to period $\widehat{s}$, for $\hat{\theta}^{\widehat{s}-1}$ the realization prior to $\hat{\theta}^{\widehat{s}}$, due to $\zeta_{\widehat{s}}^v(\hat{\theta}^{\widehat{s}}) < \zeta_{\widehat{s}}(\hat{\theta}^{\widehat{s}})$, we obtain this inequality:

   $$\begin{aligned} \sum_{\hat{\theta}^{\widehat{s}} \succ \hat{\theta}^{\widehat{s}-1}} \zeta_{\widehat{s}}^v(\hat{\theta}^t)\pi(\hat{\theta}_{\widehat{s}}|\hat{\theta}_{\widehat{s}-1}) \quad &< \quad \sum_{\hat{\theta}^{\widehat{s}} \succ \hat{\theta}^{\widehat{s}-1}} \zeta_{\widehat{s}}(\hat{\theta}^{\widehat{s}})\pi(\hat{\theta}_{\widehat{s}}|\hat{\theta}_{\widehat{s}-1}) \\ &\leq \quad \zeta_{\widehat{s}-1}(\hat{\theta}^{\widehat{s}-1}) = \zeta_{\widehat{s}-1}^v(\hat{\theta}^{\widehat{s}-1}), \end{aligned}$$

   which suggests that for any $\hat{\theta}^{\widehat{s}-1}$ household who has a non-zero probability of encountering state $\hat{\theta}^{\widehat{s}}$, Condition 5 still holds.

   - From period $\widehat{s}$ to period $\widehat{s}+1$, for $\hat{\theta}^{\widehat{s}}$, Condition 5 holds by construction, as shown in equation (42).

   - From period $\widehat{s}+1$ to period $\widehat{s}+2$, for any given $(\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1})$, due to $\zeta_{\widehat{s}+1}^v(\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1}) > \zeta_{\widehat{s}+1}(\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1})$, we obtain this inequality:

   $$\begin{aligned} \sum_{\hat{\theta}^{\widehat{s}+2}} \zeta_{\widehat{s}+2}^v(\hat{\theta}^{\widehat{s}+2})\pi(\theta_{\widehat{s}+2}|\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1}) \quad &= \quad \sum_{\hat{\theta}^{\widehat{s}+2}} \zeta_{\widehat{s}+2}(\hat{\theta}^{\widehat{s}+2})\pi(\theta_{\widehat{s}+2}|\hat{\theta}^{\widehat{s}}, \theta_{\widehat{s}+1}) \\ &\leq \quad \zeta_{\widehat{s}+1}(\hat{\theta}^{\widehat{s}}, \theta^*_{\widehat{s}+1}) < \zeta_{\widehat{s}+1}^v(\hat{\theta}^{\widehat{s}}, \theta^*_{\widehat{s}+1}), \end{aligned}$$

   which suggests that for $\hat{\theta}^{\widehat{s}}$ and any $\theta_{\widehat{s}+1}$, Condition 5 still holds.

Part 3 Compare the lifetime utility U under the original allocation and the v allocation.

Note that the lifetime utility U is concave in $\zeta_t^v(\theta^t)^{\frac{-1}{\alpha}}\frac{C_t}{H_t}$ (individual consumption) and convex in $\theta_t^{\frac{1}{\gamma-1}}\zeta_t^v(\theta^t)^{\frac{1}{\gamma-1}}\frac{L_t}{J_t}$ (individual labor) and, therefore, the (ϵ_1, ϵ_2) variation in the v allocation enhances U.

A.4 Proof of Lemma 1

Using (26), we have from (22)

$$\sum_{\theta_{t+1}} \eta_{t+1}(\theta^{t+1})\pi(\theta_{t+1}|\theta_t) = \eta_t(\theta^t) + \sum_{\theta_{t+1}} \varphi_{t+1}^P(\theta^{t+1})\pi(\theta_{t+1}|\theta_t) \geq \eta_t(\theta^t),$$

which implies that $\eta_t(\theta^t)$ is non-decreasing on average, and this average becomes strictly increasing if $\varphi_t^P(\theta^t) > 0$ for some θ^t. Since $\eta_0 = \chi^P > 0$, the value of $\sum_{\theta^t} \eta_t(\theta^t)\pi_t(\theta^t)$ is clearly positive.

A.5 Proof of Proposition 3

Suppose there is a Ramsey steady state with $\beta R < 1$. Given the normalization of $P_0 = 1$, we obtain from (2) that $P_t = \prod_{s=1}^t \frac{1}{R_s}$. Moreover, given (30)-(31) and $P_t = \beta^t C_t^{-\sigma} H_t^{\sigma}$ according to (17), one can use the above result to rewrite the FOCs (27) and (28) as

$$\underbrace{\left(\beta^t \prod_{s=1}^t R_s\right) \sum_{\theta^t} \left(\frac{c_t(\theta^t)}{C_t}\right) c_t(\theta^t)^{-\alpha}\pi_t(\theta^t)}_{Part\ 1} + \underbrace{(1-\alpha)\sum_{\theta^t}\left(\frac{c_t(\theta^t)}{C_t}\right)\eta_t(\theta^t)\pi_t(\theta^t)}_{Part\ 2} = \left(\beta^t \textstyle\prod_{s=1}^t R_s\right)\mu_t, \tag{50}$$

$$\widehat{w}_t \left(\underbrace{\left(\beta^t \prod_{s=1}^t R_s\right) \sum_{\theta^t} \left(\frac{l_t(\theta^t)}{L_t}\right) c_t(\theta^t)^{-\alpha}\pi_t(\theta^t)}_{Part\ 1} + \underbrace{\gamma \sum_{\theta^t}\left(\frac{l_t(\theta^t)}{L_t}\right)\eta_t(\theta^t)\pi_t(\theta^t)}_{Part\ 2} \right) = \left(\beta^t \textstyle\prod_{s=1}^t R_s\right)\mu_t F_L(K_t, L_t). \tag{51}$$

Note that both the term $\sum_{\theta^t}\left(\frac{c_t(\theta^t)}{C_t}\right) c_t(\theta^t)^{-\alpha}\pi_t(\theta^t)$ in (50) and the term $\sum_{\theta^t}\left(\frac{l_t(\theta^t)}{L_t}\right) c_t(\theta^t)^{-\alpha}\pi_t(\theta^t)$ in (51) converge due to the existence of a Ramsey steady state.

According to FOC (29), there are two possible cases for the existence of a Ramsey steady state according to Definition 3: (a) μ_t itself converges, and (b) μ_t diverges but its growth $\frac{\mu_{t+1}}{\mu_t}$ converges.

First, let us consider Case (a). Given that the term $\sum_{\theta^t}\left(\frac{l_t(\theta^t)}{L_t}\right)\eta_t(\theta^t)\pi_t(\theta^t)$ (the shadow price of collecting tax revenue via varying L_t) is positive, there is no possibility for Case (a) to uphold the FOC (51) in the steady state. This is because (i) Part 1 of (51) vanishes in the steady state because of $\beta R < 1$, and (ii) the term $\left(\beta^t \prod_{s=1}^t R_s\right) \mu_t F_L(K_t, L_t)$ also vanishes in the steady state since μ_t itself converges.

Second, let us consider Case (b). Although μ_t itself fails to converge, it is possible for the ratio $\frac{\mu_{t+1}}{\mu_t}$ to converge so as to support a Ramsey steady state. There are three subcases for Case (b), depending on the value of α.

1. $\alpha = 1$. The term $W_C(t)$ expressed in (30) is reduced to C_t^{-1} and the divergence of μ_t drives C_t to zero in the limit according to FOC (27). However, given that $C_t \to 0$ is incompatible with an Inada condition $\lim_{c_t \downarrow 0} u'(c_t)) = +\infty$ and that the steady state defined by Definition 3 requires that C_t converge to a positive value in the steady state, we have a contradiction with the existence of a Ramsey steady state.

2. $\alpha > 1$. Part 1 of (50) vanishes in the steady state because of $\beta R < 1$. Given $\alpha > 1$, Part 2 of (50) is negative in the limit because the term $\sum_{\theta^t} \left(\frac{c_t(\theta^t)}{C_t}\right) \eta_t(\theta^t)\pi_t(\theta^t)$ (the shadow price of collecting tax revenue via varying C_t) is positive. However, since μ_t in (50) must be non-negative, it leads to the violation of the FOC (50) in the steady state. Again, we have a contradiction with the existence of a Ramsey steady state.

3. $\alpha < 1$. Part 1 of both (50) and (51) vanish in the steady state because of $\beta R < 1$. Part 2 of (51) is positive in the steady state because of $\sum_{\theta^t} \left(\frac{l_t(\theta^t)}{L_t}\right) \eta_t(\theta^t)\pi_t(\theta^t) > 0$. Given $\alpha < 1$, Part 2 of (50) is also positive because of $\sum_{\theta^t} \left(\frac{c_t(\theta^t)}{C_t}\right) \eta_t(\theta^t)\pi_t(\theta^t) > 0$. Thus, given $\beta R < 1$, the divergent μ_t contradicts neither (50) nor (51) in the steady state. We conclude that both the divergent μ_t and a convergent μ_{t+1}/μ_t can co-exist and be consistent with the FOCs (27)-(29) in the Ramsey steady state. However, depending on the value of μ_{t+1}/μ_t in the limit, we see from (29) that there is no guarantee that the Ramsey planner will implement the MGR in the steady state.

A.6 Proof of Proposition 4

Assume an allocation, denoted by $* = \left\{C_t^*, K_{t+1}^*, B_{t+1}^*, \hat{a}_{t+1}^*\left(\theta^t\right), L_t^*, \zeta_t^*\left(\theta^t\right)\right\}_{t=0}^{\infty}$, that has reached a steady state of the HAIM economy at $t = \widehat{s}$. Now consider an alternative allocation, denoted by

$$v = \left\{C_t^v, K_{t+1}^v, B_{t+1}^v, \hat{a}_{t+1}^v\left(\theta^t\right), L_t^*, \zeta_t^*\left(\theta^t\right)\right\}_{t=0}^{\infty},$$

which is identical to the $*$ allocation, except for the following variation:

For $t = \widehat{s}$ and $t = \widehat{s}+1$,

$$\begin{aligned} &C^v_{\widehat{s}} = C^*_{\widehat{s}} + \Delta_c,\ K^v_{\widehat{s}+1} = K^*_{\widehat{s}+1} - \Delta_k,\ B^v_{\widehat{s}+1} = B^*_{\widehat{s}+1} + \Delta_b, \\ &C^v_{\widehat{s}+1} = C^*_{\widehat{s}+1} - \Delta_{c'}, \end{aligned} \tag{52}$$

$$\begin{aligned} \hat{a}^v_{\widehat{s}}\left(\theta^{\widehat{s}-1}\right) &= \hat{a}^*_{\widehat{s}}\left(\theta^{\widehat{s}-1}\right) \\ &+ \frac{\beta\left(C^{v1-\alpha}_{\widehat{s}+1} - C^{*1-\alpha}_{\widehat{s}+1}\right)H^{*\alpha}_{\widehat{s}+1}}{C^{*-\alpha}_{\widehat{s}-1}H^{*\alpha}_{\widehat{s}-1}} \left(\sum_{\theta_{\widehat{s}+1}} \frac{c^*_{\widehat{s}+1}\left(\theta^{\widehat{s}+1}\right)}{C^*_{\widehat{s}+1}} \pi\left(\theta_{\widehat{s}+1}|\theta_{\widehat{s}}\right) - \frac{c^*_{\widehat{s}}\left(\theta^{\widehat{s}}\right)}{C^*_{\widehat{s}}}\right), \end{aligned} \tag{53}$$

$$\widehat{a}^v_{\widehat{s}+1}(\theta^{\widehat{s}}) = \frac{C^{*-\alpha}_{\widehat{s}}}{C^{v-\alpha}_{\widehat{s}}}\widehat{a}^*_{\widehat{s}+1}(\theta^{\widehat{s}}) + \frac{\beta\left(C^{v1-\alpha}_{\widehat{s}+1} - C^{*1-\alpha}_{\widehat{s}+1}\right)H^{*\alpha}_{\widehat{s}+1}}{C^{v-\alpha}_{\widehat{s}}H^{*\alpha}_{\widehat{s}}} \frac{c^*_{\widehat{s}+1}\left(\theta^{\widehat{s}+1}\right)}{C^*_{\widehat{s}+1}}, \tag{54}$$

and for $t > \widehat{s}+1$,

$$K^v_t = \begin{cases} F\left(K^v_{t-1}, L^*_{t-1}\right) + (1-\delta)K^v_{t-1} - C^v_{t-1} - G_{t-1}, & \text{if } t = \widehat{s}+2 \\ F\left(K^v_{t-1}, L^*_{t-1}\right) + (1-\delta)K^v_{t-1} - C^*_{t-1} - G_{t-1}, & \text{if } t > \widehat{s}+2 \end{cases}, \tag{55}$$

$$B^v_t = \sum_{\theta^{t-1}} \hat{a}^*_t\left(\theta^{t-1}\right)\pi_{t-1}\left(\theta^{t-1}\right) - K^v_t, \tag{56}$$

where $\Delta_c, \Delta_{c'}, \Delta_k$, and Δ_b in (52) are chosen to satisfy

$$\Delta_c = \Delta_k > 0, \tag{57}$$

$$\left(C^*_{\widehat{s}} + \Delta_c\right)^{1-\alpha} H^{*\alpha}_{\widehat{s}} + \beta\left(C^*_{\widehat{s}+1} - \Delta_{c'}\right)^{1-\alpha} H^{*\alpha}_{\widehat{s}+1} = C^{*1-\alpha}_{\widehat{s}} H^{*\alpha}_{\widehat{s}} + \beta C^{*1-\alpha}_{\widehat{s}+1} H^{*\alpha}_{\widehat{s}+1}, \tag{58}$$

$$\Delta_b = \Delta_k - K^*_{\widehat{s}+1} - B^*_{\widehat{s}+1} + \frac{1}{\beta^{\widehat{s}}\left(C^*_{\widehat{s}} + \Delta_c\right)^{-\alpha} H^{*\alpha}_{\widehat{s}}} \left\{ \begin{array}{c} \beta^{\widehat{s}+1}\left[\left(C^*_{\widehat{s}+1} - \Delta_{c'}\right)^{1-\alpha} H^{*\alpha}_{\widehat{s}+1} - L^{*\gamma}_{\widehat{s}+1} J^{*1-\gamma}_{\widehat{s}+1}\right] \\ + \sum_{s \geq \widehat{s}+2} \beta^s \left[C^{*1-\alpha}_s H^{*\alpha}_s - L^{*\gamma}_s J^{*1-\gamma}_s\right] \end{array} \right\}. \tag{59}$$

Given $\Delta_c > 0$ in (57), (58) implies $\Delta_{c'} > 0$. We show later that $\Delta_c = \Delta_k > 0$ and $\Delta_{c'} > 0$ together give $\Delta_b > 0$. Note that there is one degree of freedom in the choice of $\Delta_c, \Delta_{c'}, \Delta_k$, and Δ_b, given the imposition of (57)-(59). We shall choose $\Delta_{c'}$ and then let Δ_c, Δ_k, and Δ_b be pinned down by (57)-(59).

The rest of the proof consists of four parts. First, we show that (53)-(56) hold, given the choice of $\Delta_c, \Delta_{c'}, \Delta_k$, and Δ_b. Next, we verify that the v allocation satisfies Conditions 1 to 5 of

Proposition 2 (i.e., the constraints of the relaxed Ramsey problem are satisfied). We also verify that the asset-market clearing condition holds in the v allocation. Third, we show that the v allocation improves welfare relative to the $*$ allocation with $C^v_{\widehat{s}} = C^*_{\widehat{s}} + \Delta_c$ ($\Delta_c = \Delta_k > 0$), $C^v_{\widehat{s}+1} = C^*_{\widehat{s}+1} - \Delta_{c'}$ ($\Delta_{c'} > 0$), and $B^v_{\widehat{s}+1} = B^*_{\widehat{s}+1} + \Delta_b$ ($\Delta_b > 0$), as specified in (52); that is, the welfare of $*$ steady-state allocation can be improved by issuing more government bonds to front-load aggregate consumption, given that $\beta R < 1$ in the steady state. Finally, we show the adjusted taxes to support the change from the $*$ to the v allocation.

Part 1 We show that (53)-(56) hold.

First, by (6) and $P_t = \beta^t C_t^{-\alpha} H_t^{\alpha}$, $\hat{a}^v_{\widehat{s}}\left(\theta^{\widehat{s}-1}\right)$ satisfies

$$\begin{aligned}
&\beta^{\widehat{s}-1} C^{*-\alpha}_{\widehat{s}-1} H^{*\alpha}_{\widehat{s}-1} \hat{a}^v_{\widehat{s}}\left(\theta^{\widehat{s}-1}\right) \pi_{\widehat{s}}(\theta^{\widehat{s}}) \\
= &\left\{ \begin{array}{c}
\beta^{\widehat{s}} \left[C^{v1-\alpha}_{\widehat{s}} H^{*\alpha-1}_{\widehat{s}} \zeta^{*\frac{-1}{\alpha}}_{\widehat{s}}\left(\theta^{\widehat{s}}\right) - L^{*\gamma}_{\widehat{s}} J^{*-\gamma}_{\widehat{s}} \theta^{\frac{\gamma}{\gamma-1}}_{\widehat{s}} \zeta^{*\frac{1}{\gamma-1}}_{\widehat{s}}\left(\theta^{\widehat{s}}\right)\right] \pi_{\widehat{s}}(\theta^{\widehat{s}}) \\
+\beta^{\widehat{s}+1} \sum_{\theta_{\widehat{s}+1}} \left[C^{v1-\alpha}_{\widehat{s}+1} H^{*\alpha-1}_{\widehat{s}+1} \zeta^{*\frac{-1}{\alpha}}_{\widehat{s}+1}\left(\theta^{\widehat{s}+1}\right) - L^{*\gamma}_{\widehat{s}+1} J^{*-\gamma}_{\widehat{s}+1} \theta^{\frac{\gamma}{\gamma-1}}_{\widehat{s}+1} \zeta^{*\frac{1}{\gamma-1}}_{\widehat{s}+1}\left(\theta^{\widehat{s}+1}\right)\right] \pi_{\widehat{s}+1}(\theta^{\widehat{s}+1}) \\
+\sum_{\theta^s \succ \theta^{\widehat{s}}} \sum_{s \geq \widehat{s}+2} \kappa^*_s\left(\theta^s\right) \pi_s\left(\theta^s\right)
\end{array} \right\},
\end{aligned}$$

where, given (52), we uses $\kappa^v_s\left(\theta^s\right) = \kappa^*_s\left(\theta^s\right)$ for $s \geq \widehat{s}+2$ and all θ^s. Contrasting it with $\beta^{\widehat{s}-1} C^{*-\alpha}_{\widehat{s}-1} H^{*\alpha}_{\widehat{s}-1} \hat{a}^*_{\widehat{s}}\left(\theta^{\widehat{s}-1}\right) \pi_{\widehat{s}}(\theta^{\widehat{s}}) = \sum_{\theta^s \succeq \theta^{\widehat{s}}} \sum_{s \geq \widehat{s}} \kappa^*_s\left(\theta^s\right) \pi_s\left(\theta^s\right)$ yields

$$\begin{aligned}
&\beta^{\widehat{s}-1} C^{*-\alpha}_{\widehat{s}-1} H^{*\alpha}_{\widehat{s}-1} \left(\hat{a}^v_{\widehat{s}}\left(\theta^{\widehat{s}-1}\right) - \hat{a}^*_{\widehat{s}}\left(\theta^{\widehat{s}-1}\right)\right) \pi_{\widehat{s}}(\theta^{\widehat{s}}) \\
= \;& \beta^{\widehat{s}} \frac{\zeta^{*\frac{-1}{\alpha}}_{\widehat{s}}\left(\theta^{\widehat{s}}\right)}{H^*_{\widehat{s}}} \left(C^{v1-\alpha}_{\widehat{s}} H^{*\alpha}_{\widehat{s}} - C^{*1-\alpha}_{\widehat{s}} H^{*\alpha-1}_{\widehat{s}}\right) \pi_{\widehat{s}}(\theta^{\widehat{s}}) \\
&+\beta^{\widehat{s}+1} \frac{\sum_{\theta_{\widehat{s}+1}} \zeta^{*\frac{-1}{\alpha}}_{\widehat{s}+1}\left(\theta^{\widehat{s}+1}\right) \pi\left(\theta_{\widehat{s}+1}|\theta_{\widehat{s}}\right)}{H^*_{\widehat{s}+1}} \left(C^{v1-\alpha}_{\widehat{s}+1} H^{*\alpha}_{\widehat{s}+1} - C^{*1-\alpha}_{\widehat{s}+1} H^{*\alpha}_{\widehat{s}+1}\right) \pi_{\widehat{s}+1}(\theta^{\widehat{s}+1}) \\
= \;& \beta^{\widehat{s}} \left(C^{v1-\alpha}_{\widehat{s}+1} H^{*\alpha}_{\widehat{s}+1} - C^{*1-\alpha}_{\widehat{s}+1} H^{*\alpha}_{\widehat{s}+1}\right) \left(\sum_{\theta_{\widehat{s}+1}} \frac{c^*_{\widehat{s}+1}\left(\theta^{\widehat{s}+1}\right)}{C^*_{\widehat{s}+1}} \pi\left(\theta_{\widehat{s}+1}|\theta_{\widehat{s}}\right) - \frac{c^*_{\widehat{s}}\left(\theta^{\widehat{s}}\right)}{C^*_{\widehat{s}}} \right) \pi_{\widehat{s}}(\theta^{\widehat{s}}), \quad (60)
\end{aligned}$$

where the second equality holds because of (58) and $\frac{c_s(\theta^s)}{C_s} = \frac{\zeta_s^{\frac{-1}{\alpha}}(\theta^s)}{H_s}$ for all s. After some manipulation, we obtain (53) from (60).

Second, by (6) and $P_t = \beta^t C_t^{-\alpha} H_t^{\alpha}$, $\widehat{a}^v_{\widehat{s}+1}(\theta^{\widehat{s}})$ satisfies

$$\begin{aligned} & \beta^{\widehat{s}} \left(C^*_{\widehat{s}} + \Delta_c\right)^{-\alpha} H^{*\alpha}_{\widehat{s}} \widehat{a}^v_{\widehat{s}+1}(\theta^{\widehat{s}}) \pi_{\widehat{s}+1}(\theta^{\widehat{s}+1}) \\ = & \left\{ \begin{array}{c} \beta^{\widehat{s}+1} \left[\left(C^*_{\widehat{s}+1} - \Delta_{c'}\right)^{1-\alpha} H^{*\alpha-1}_{\widehat{s}+1} \zeta^{*\frac{-1}{\alpha}}_{\widehat{s}+1} - L^{*\gamma}_{\widehat{s}+1} J^{*-\gamma}_{\widehat{s}+1} \theta^{\frac{\gamma}{\gamma-1}}_{\widehat{s}+1} \zeta^{*\frac{1}{\gamma-1}}_{\widehat{s}+1} \right] \pi_{\widehat{s}+1}(\theta^{\widehat{s}+1}) \\ + \sum_{s \geq \widehat{s}+2} \kappa^*_s \left(\theta^s\right) \pi_s \left(\theta^s\right) \end{array} \right\}. \end{aligned} \tag{61}$$

Contrasting (61) with $\beta^{\widehat{s}} C^{*-\alpha}_{\widehat{s}} H^{*\alpha}_{\widehat{s}} \hat{a}^*_{\widehat{s}+1}\left(\theta^{\widehat{s}}\right) \pi_{\widehat{s}+1}(\theta^{\widehat{s}+1}) = \sum_{\theta^s \succeq \theta^{\widehat{s}+1}} \sum_{s \geq \widehat{s}+1} \kappa^*_s \left(\theta^s\right) \pi_s \left(\theta^s\right)$ leads to (54).

Third, we construct $\{K^v_t, B^v_t\}^{\infty}_{t>\hat{s}+1}$ to satisfy resource constraints and asset market clearing conditions, respectively. Note that, by (55), $K^v_{t+1} = F\left(K^v_t, L^*_t\right) + (1-\delta) K^v_t - C^*_t - G_t$, for $t > \widehat{s}+1$. Given that C^*_t and L^*_t stay at the same steady state as $t > \widehat{s}+1$, the right-hand side is strictly concave in K^v_t and the left-hand side in linear in K^v_{t+1}. It follows that K^v_t converges to K^* eventually. As such, the sequence of K^v_t is well-defined and feasible.

Part 2 We check Conditions 1 to 5 of Proposition 2 and the asset-market clearing conditions.

1. *Resource constraints*

   - For time $t < \widehat{s}$, all the resource constraints hold since the aggregate variables in these time periods remain unchanged.
   - For time $t > \widehat{s}$, by construction all the resource constraints hold, as shown in (55).
   - Given (52), the resource constraint at time $\widehat{s}$ is expressed as

   $$\begin{aligned} F\left(K^v_{\widehat{s}}, L^*_{\widehat{s}}\right) + (1-\delta) K^v_{\widehat{s}} &= F\left(K^*_{\widehat{s}}, L^*_{\widehat{s}}\right) + (1-\delta) K^*_{\widehat{s}} \\ &= C^v_{\widehat{s}} - \Delta_c + K^v_{\widehat{s}+1} + \Delta_k + G_{\widehat{s}}. \end{aligned}$$

   - Since $\Delta_k = \Delta_c$ by (57), we have $F\left(K^v_{\widehat{s}}, L^*_{\widehat{s}}\right) + (1-\delta) K^v_{\widehat{s}} = C^v_{\widehat{s}} + K^v_{\widehat{s}+1} + G_{\widehat{s}}$. Thus, the resource constraint at time $\widehat{s}$ is satisfied.

2. *Borrowing constraints*

   By (52), $\hat{a}^v_s = \hat{a}^*_s$ for $s \neq \{\widehat{s}, \widehat{s}+1\}$. Thus, we focus on $\hat{a}^v_s$ for $s = \{\widehat{s}, \widehat{s}+1\}$.

   - Consider $\hat{a}^v_s$ for $s = \widehat{s}$. First, suppose $\hat{a}^*_{\widehat{s}}\left(\theta^{\widehat{s}-1}\right) = 0$. Note that $C^{v1-\alpha}_{\widehat{s}+1} > C^{*1-\alpha}_{\widehat{s}+1}$ under $\alpha \geq 1$ due to (52). Hence, we see from (53) that $\hat{a}^v_{\widehat{s}}\left(\theta^{\widehat{s}-1}\right) > 0$ provided the condition that $\sum_{\theta_{\widehat{s}+1}} \frac{c^*_{\widehat{s}+1}\left(\theta^{\widehat{s}+1}\right)}{C^*_{\widehat{s}+1}} \pi\left(\theta_{\widehat{s}+1} | \theta_{\widehat{s}}\right) > \frac{c^*_{\widehat{s}}\left(\theta^{\widehat{s}}\right)}{C^*_{\widehat{s}}}$. This condition holds in the steady state because households who hit the borrowing constraint have the lowest consumption share among all households today and thus they must have a higher expected consumption share tomorrow. Next, suppose $\hat{a}^*_{\widehat{s}}\left(\theta^{\widehat{s}-1}\right) > 0$. It is then possible that $\sum_{\theta_{\widehat{s}+1}} \frac{c^*_{\widehat{s}+1}\left(\theta^{\widehat{s}+1}\right)}{C^*_{\widehat{s}+1}} \pi\left(\theta_{\widehat{s}+1}|\theta_{\widehat{s}}\right) < \frac{c^*_{\widehat{s}}\left(\theta^{\widehat{s}}\right)}{C^*_{\widehat{s}}}$. In such a situation, we can choose $\Delta_{c'}$ small enough so that the second right-hand term of (53) is also small enough in absolute value. This again ensures $\hat{a}^v_{\widehat{s}}\left(\theta^{\widehat{s}-1}\right) \geq 0$ according to (53).

- Consider $\hat{a}^v_s$ for $s = \widehat{s}+1$. Given that both $\Delta_c > 0$ and $\Delta_{c'} > 0$, we see from (54) that $\hat{a}^v_{\widehat{s}+1}(\theta^{\widehat{s}}) > \hat{a}^*_{\widehat{s}+1}(\theta^{\widehat{s}}) \geq 0$ under $\alpha \geq 1$.

3. *Implementability and measurability conditions*

Given (52), $\kappa^v_s\left(\theta^s\right) = \kappa^*_s\left(\theta^s\right)$ for all $\theta^s \preceq \theta^{\widehat{s}-1}$ and $\hat{a}^v_{\widehat{s}}\left(\theta^{\widehat{s}-1}\right) = \hat{a}^*_{\widehat{s}}\left(\theta^{\widehat{s}-1}\right)$ for all $\theta^{\widehat{s}-1}$. Thus the implementability condition holds. By construction, the measurability condition for all t holds. Since all $\zeta_t\left(\theta^t\right)$ remain unchanged, Condition 5 of Proposition 2 is satisfied under the v allocation.

4. *Asset-market clearing conditions*

- For time $t < \widehat{s}$, the asset-market clearing conditions hold since $\{K_t, B_t, \hat{a}_t\left(\theta^{t-1}\right)\}$ remains unchanged according to (52) and (53).

- For time $t > \widehat{s}+1$, by construction all the asset-market clearing conditions hold, as shown in (56).

- Aggregating asset demand $\hat{a}^v_{\widehat{s}}\left(\theta^{\widehat{s}-1}\right)$, as shown in (60), leads to

$$\sum_{\theta^{\widehat{s}-1}} \hat{a}^v_{\widehat{s}}\left(\theta^{\widehat{s}-1}\right) \pi_{\widehat{s}-1}\left(\theta^{\widehat{s}-1}\right) = \sum_{\theta^{\widehat{s}-1}} \hat{a}^*_{\widehat{s}}\left(\theta^{\widehat{s}-1}\right) \pi_{\widehat{s}-1}\left(\theta^{\widehat{s}-1}\right), \tag{62}$$

which equals $K^*_{\widehat{s}}+B^*_{\widehat{s}}$. Due to $K^*_{\widehat{s}}+B^*_{\widehat{s}} = K^v_{\widehat{s}}+B^v_{\widehat{s}}$, we have $\sum_{\theta^{\widehat{s}-1}} \hat{a}^v_{\widehat{s}}\left(\theta^{\widehat{s}-1}\right) \pi_{\widehat{s}-1}\left(\theta^{\widehat{s}-1}\right) = K^v_{\widehat{s}} + B^v_{\widehat{s}}$; i.e., the asset-market clearing condition at time $\widehat{s}$ holds.

- Using (54), the aggregate asset demand at time $\widehat{s}+1$ equals

$$\sum_{\theta^{\widehat{s}}} \hat{a}^v_{\widehat{s}+1}\left(\theta^{\widehat{s}}\right) \pi_{\widehat{s}}\left(\theta^{\widehat{s}}\right) = \frac{1}{\beta^{\widehat{s}} C^{v-\alpha}_{\widehat{s}} H^{*\alpha}_{\widehat{s}}} \left\{ \begin{array}{c} \beta^{\widehat{s}+1}\left[C^{v1-\alpha}_{\widehat{s}+1} H^{*\alpha}_{\widehat{s}+1} - L^{*\gamma}_{\widehat{s}+1} J^{*1-\gamma}_{\widehat{s}+1}\right] \\ + \sum_{s\geq\widehat{s}+2} \beta^s \left[C^{v1-\alpha}_s H^{*\alpha}_s - L^{*\gamma}_s J^{*1-\gamma}_s\right] \end{array} \right\}.$$

Using (52), it leads to

$$\sum_{\theta^{\widehat{s}}} \hat{a}^{v}_{\widehat{s}+1}\left(\theta^{\widehat{s}}\right)\pi_{\widehat{s}}\left(\theta^{\widehat{s}}\right) = \frac{1}{\beta^{\widehat{s}}\left(C^{*}_{\widehat{s}}+\Delta_{c}\right)^{-\alpha} H^{*\alpha}_{\widehat{s}}}\left\{\begin{array}{c}\beta^{\widehat{s}+1}\left[\left(C^{*}_{\widehat{s}+1}-\Delta_{c'}\right)^{1-\alpha} H^{*\alpha}_{\widehat{s}+1} - L^{*\gamma}_{\widehat{s}+1} J^{*1-\gamma}_{\widehat{s}+1}\right] \\ +\sum_{s\geq \widehat{s}+2}\beta^{s}\left[C^{*1-\alpha}_{s} H^{*\alpha}_{s} - L^{*\gamma}_{s} J^{*1-\gamma}_{s}\right]\end{array}\right\}$$
$$= K^{*}_{\widehat{s}+1} - \Delta_{k} + B^{*}_{\widehat{s}+1} + \Delta_{b},$$

where the second equality holds because of (59). We then have

$$\sum_{\theta^{\widehat{s}}} \hat{a}^{v}_{\widehat{s}+1}\left(\theta^{\widehat{s}}\right)\pi_{\widehat{s}}\left(\theta^{\widehat{s}}\right) = K^{*}_{\widehat{s}+1} - \Delta_{k} + B^{*}_{\widehat{s}+1} + \Delta_{b} \qquad (63)$$
$$= K^{v}_{\widehat{s}+1} + B^{v}_{\widehat{s}+1},$$

where the second equality uses (52). Thus, the asset-market clearing condition holds at time $\widehat{s}+1$.

Since $\hat{a}^{v}_{\widehat{s}+1}\left(\theta^{\widehat{s}}\right) > \hat{a}^{*}_{\widehat{s}+1}\left(\theta^{\widehat{s}}\right)$ for all $\theta^{\widehat{s}}$ as derived earlier, we have $\sum_{\theta^{\widehat{s}}} \hat{a}^{v}_{\widehat{s}+1}\left(\theta^{\widehat{s}}\right)\pi_{\widehat{s}}\left(\theta^{\widehat{s}}\right) > \sum_{\theta^{\widehat{s}}} \hat{a}^{*}_{\widehat{s}+1}\left(\theta^{\widehat{s}}\right)\pi_{\widehat{s}}\left(\theta^{\widehat{s}}\right)$ and, therefore, $K^{v}_{\widehat{s}+1}+B^{v}_{\widehat{s}+1} > K^{*}_{\widehat{s}+1}+B^{*}_{\widehat{s}+1}$ by the asset-market clearing condition. From (63), we obtain

$$\Delta_{b} = \left(K^{v}_{\widehat{s}+1} + B^{v}_{\widehat{s}+1}\right) - \left(K^{*}_{\widehat{s}+1} + B^{*}_{\widehat{s}+1}\right) + \Delta_{k},$$

which implies $\Delta_{b} > 0$ since $\Delta_{k} > 0$ and $K^{v}_{\widehat{s}+1} + B^{v}_{\widehat{s}+1} > K^{*}_{\widehat{s}+1} + B^{*}_{\widehat{s}+1}$.

Part 3 We show that the v allocation improves welfare relative to the $*$ allocation.

The proof of *Part 2* tells us that one can simply use the objective of the Ramsey problem to measure the welfare change from the $*$ to the v allocation. It is given by

$$\sum_{\theta^{\widehat{s}}}\left(\frac{\zeta^{*}_{\widehat{s}}(\theta^{\widehat{s}})^{\frac{-1}{\alpha}}}{H^{*}_{\widehat{s}}}\right)^{1-\alpha}\pi_{\widehat{s}}\left(\theta^{\widehat{s}}\right)\left(\frac{C^{v\,1-\alpha}_{\widehat{s}} - C^{*1-\alpha}_{\widehat{s}}}{1-\alpha}\right)$$
$$+\beta\sum_{\theta^{\widehat{s}+1}}\left(\frac{\zeta^{*}_{\widehat{s}+1}(\theta^{\widehat{s}+1})^{\frac{-1}{\alpha}}}{H^{*}_{\widehat{s}+1}}\right)^{1-\alpha}\pi_{\widehat{s}+1}\left(\theta^{\widehat{s}+1}\right)\left(\frac{C^{v1-\alpha}_{\widehat{s}+1} - C^{*1-\alpha}_{\widehat{s}+1}}{1-\alpha}\right)$$
$$= \sum_{\theta^{\widehat{s}}}\left(\frac{c^{*}_{\widehat{s}}(\theta^{\widehat{s}})}{C^{*}_{\widehat{s}}}\right)^{1-\alpha}\pi_{\widehat{s}}\left(\theta^{\widehat{s}}\right)\left[\begin{array}{c}\frac{\left(C^{*}_{\widehat{s}}+\Delta_{c}\right)^{1-\alpha} - C^{*1-\alpha}_{\widehat{s}}}{1-\alpha} \\ +\beta\frac{\left(C^{*}_{\widehat{s}+1}-\Delta_{c'}\right)^{1-\alpha} - C^{*1-\alpha}_{\widehat{s}+1}}{1-\alpha}\end{array}\right], \qquad (64)$$

where the equality uses $C^{v}_{\widehat{s}} = C^{*}_{\widehat{s}} + \Delta_{c}$ and $C^{v}_{\widehat{s}+1} = C^{*}_{\widehat{s}+1} - \Delta_{c'}$ as specified in (52), the consumption sharing rule (15), and the steady-state property with

$$\sum_{\theta^{\widehat{s}}} \left(\frac{c^*_{\widehat{s}}(\theta^{\widehat{s}})}{C^*_{\widehat{s}}} \right)^{1-\alpha} \pi_{\widehat{s}} \left(\theta^{\widehat{s}} \right) = \sum_{\theta^{\widehat{s}+1}} \left(\frac{c^*_{\widehat{s}+1}(\theta^{\widehat{s}+1})}{C^*_{\widehat{s}+1}} \right)^{1-\alpha} \pi_{\widehat{s}+1} \left(\theta^{\widehat{s}+1} \right).$$

Let $\Delta = \frac{C^{v\,1-\alpha}_{\widehat{s}} - C^{*1-\alpha}_{\widehat{s}}}{1-\alpha} + \beta \frac{C^{v1-\alpha}_{\widehat{s}+1} - C^{*1-\alpha}_{\widehat{s}+1}}{1-\alpha}$, which represents the square bracket in (64). Using (58) gives

$$\begin{aligned} \Delta &= \frac{1}{1-\alpha} \left\{ \begin{array}{c} C^{*1-\alpha}_{\widehat{s}} + \beta C^{*1-\alpha}_{\widehat{s}+1} \frac{H^{*\alpha}_{\widehat{s}+1}}{H^{*\alpha}_{\widehat{s}}} - \beta \left(C^*_{\widehat{s}+1} - \Delta_{c'} \right)^{1-\alpha} \frac{H^{*\alpha}_{\widehat{s}+1}}{H^{*\alpha}_{\widehat{s}}} - C^{*1-\alpha}_{\widehat{s}} \\ +\beta \left(C^*_{\widehat{s}+1} - \Delta_{c'} \right)^{1-\alpha} - \beta C^{*1-\alpha}_{\widehat{s}+1} \end{array} \right\} \\ &= \frac{\beta}{1-\alpha} \left\{ C^{*1-\alpha}_{\widehat{s}+1} \left(\frac{H^{*\alpha}_{\widehat{s}+1}}{H^{*\alpha}_{\widehat{s}}} - 1 \right) - \left(C^*_{\widehat{s}+1} - \Delta_{c'} \right)^{1-\alpha} \left(\frac{H^{*\alpha}_{\widehat{s}+1}}{H^{*\alpha}_{\widehat{s}}} - 1 \right) \right\} \\ &= \frac{\beta}{1-\alpha} \left(\frac{H^{*\alpha}_{\widehat{s}+1}}{H^{*\alpha}_{\widehat{s}}} - 1 \right) \left[C^{*1-\alpha}_{\widehat{s}+1} - \left(C^*_{\widehat{s}+1} - \Delta_{c'} \right)^{1-\alpha} \right]. \end{aligned}$$

Given $\beta R < 1$ in the steady state, we have $\frac{H^{*\alpha}_{\widehat{s}+1}}{H^{*\alpha}_{\widehat{s}}} > 1$ according to (25). This result together with $\Delta_{c'} > 0$ then implies that $\Delta > 0$.

Part 4 To support the change from the $*$ to the v allocation, tax rates need be adjusted accordingly. We show the adjustment.

From (2), (18), and (19), we have

$$\frac{1}{1 + \left(1 - \tau^*_{k,t+1}\right) \left(r^*_{t+1} - \delta\right)} = \beta \left(\frac{C^*_{t+1}}{C^*_t} \right)^{-\alpha} \left(\frac{H^*_{t+1}}{H^*_t} \right)^{\alpha},$$

$$\left(1 - \tau^*_{l,t}\right) w^*_t = \frac{L^{*\gamma-1}_t J^{*1-\gamma}_t}{C^{*-\alpha}_t H^{*\alpha}_t},$$

$$\frac{1}{1 + \left(1 - \tau^v_{k,t+1}\right) \left(r^v_{t+1} - \delta\right)} = \beta \left(\frac{C^v_{t+1}}{C^v_t} \right)^{-\alpha} \left(\frac{H^*_{t+1}}{H^*_t} \right)^{\alpha},$$

$$\left(1 - \tau^v_{l,t}\right) w^v_t = \frac{L^{*\gamma-1}_t J^{*1-\gamma}_t}{C^{v-\alpha}_t H^{*\alpha}_t},$$

which lead to

$$\frac{1 + \left(1 - \tau^v_{k,t+1}\right) \left(r^v_{t+1} - \delta\right)}{1 + \left(1 - \tau^*_{k,t+1}\right) \left(r^*_{t+1} - \delta\right)} = \left(\frac{C^v_{t+1}}{C^*_{t+1}} \right)^{\alpha} \left(\frac{C^v_t}{C^*_t} \right)^{-\alpha},$$

$$\frac{\left(1 - \tau^*_{l,t}\right) w^*_t}{\left(1 - \tau^v_{l,t}\right) w^v_t} = \left(\frac{C^*_t}{C^v_t} \right)^{\alpha}.$$

Therefore, from (52) we have

- for $t < \widehat{s}$, $\tau^*_{k,t} = \tau^v_{k,t}$, and $\tau^*_{l,t} = \tau^v_{l,t}$;
- at time $\widehat{s}$, $\tau^v_{k,\widehat{s}} < \tau^*_{k,\widehat{s}}$, and $\tau^v_{l,\widehat{s}} > \tau^*_{l,\widehat{s}}$;
- at time $\widehat{s}+1$, $\tau^v_{k,\widehat{s}+1} > \tau^*_{k,\widehat{s}+1}$, and $\frac{1-\tau^*_{l,\widehat{s}+1}}{1-\tau^v_{l,\widehat{s}+1}} = \frac{w^v_{\widehat{s}+1}}{w^*_{\widehat{s}+1}} \left(\frac{C^*_{\widehat{s}+1}}{C^v_{\widehat{s}+1}}\right)^\alpha$;
- for $t > \widehat{s}+1$, $\frac{1+\left(1-\tau^v_{k,t}\right)(r^v_t-\delta)}{1+\left(1-\tau^*_{k,t}\right)(r^*_t-\delta)} = \left(\frac{C^v_t}{C^*_t}\right)^\alpha \left(\frac{C^v_{t-1}}{C^*_{t-1}}\right)^{-\alpha}$, and $\frac{1-\tau^*_{l,t}}{1-\tau^v_{l,t}} = \frac{w^v_t}{w^*_t}$.