Implementing the Modified Golden Rule? Optimal Ramsey Capital Taxation with Incomplete Markets Revisited

Yunmin Chen,
YiLi Chien
and
C.C. Yang

Working Paper 2017-003G
https://doi.org/10.20955/wp.2017.003

January 2018

FEDERAL RESERVE BANK OF ST. LOUIS
Research Division
P.O. Box 442
St. Louis, MO 63166

The views expressed are those of the individual authors and do not necessarily reflect official positions of the Federal Reserve Bank of St. Louis, the Federal Reserve System, or the Board of Governors.

Federal Reserve Bank of St. Louis Working Papers are preliminary materials circulated to stimulate discussion and critical comment. References in publications to Federal Reserve Bank of St. Louis Working Papers (other than an acknowledgment that the writer has had access to unpublished material) should be cleared with the author or authors.
Implementing the Modified Golden Rule?
Optimal Ramsey Taxation with Incomplete Markets Revisited

Yunmin Chen
Shandong University

YiLi Chien*
Federal Reserve Bank of St. Louis

C.C. Yang
Academia Sinica

October 14, 2019

Abstract

What is the prescription of Ramsey capital taxation in the long run? Aiyagari (1995) addressed the question in a heterogeneous-agent incomplete-markets (HAIM) economy, showing that a positive capital tax should be imposed to implement the so-called modified golden rule (MGR). This paper revisits the long-standing issue. We first show that the Aiyagari result holds if the shadow price of raising government revenues through distorting taxes converges to zero in the limit at the Ramsey optimum. This “if” is clearly a strong condition. As long as the condition fails to hold, we show (i) there is no Ramsey steady state when the elasticity of intertemporal substitution (EIS) is weakly less than 1, and (ii) a Ramsey steady state is possible if EIS is larger than 1 but the MGR does not hold and the corresponding capital tax is non-positive. The key to our non-existence result is embedded in the hallmark of the HAIM economy: the risk-free gross interest rate is lower than the inverse of the preference discount factor in steady state.

JEL Classification: C61; E22; E62; H21; H30
Key Words: Capital Taxation; Modified Golden Rule; Ramsey Problem; Incomplete Markets

*Corresponding author, YiLi Chien. Email: yilichien@gmail.com. This paper is a complete rewrite of our previous work under the title “Aiyagari Meets Ramsey: Optimal Capital Taxation with Incomplete Markets.” While we find the non-existence of a Ramsey steady state in the current version, our previous work builds on the incorrect premise that a Ramsey steady state always exists. We thank Andrew Atkeson, Dirk Krueger, Tomoyuki Nakajima, Yena Park, and participants at various seminars and conferences for useful comments. The views expressed are those of the individual authors and do not necessarily reflect the official positions of the Federal Reserve Bank of St. Louis, the Federal Reserve System, or the Board of Governors.
1 Introduction

The heterogeneous-agent incomplete-markets (HAIM hereafter) model considers an environment in which households are subject to uninsurable idiosyncratic shocks and borrowing restrictions; in response, households buffer their consumption against adverse shocks via precautionary savings. During the past two decades, the HAIM model has become a standard workhorse for policy evaluations in the current state-of-the-art macroeconomics that jointly addresses aggregate and inequality issues.\(^1\)

Given the importance and popularity of the HAIM model, it is natural to ask: what is the prescription of Ramsey capital taxation in the long run for the HAIM economy? The first attempt to answer this question is the work of Aiyagari (1995). Assuming the existence of a Ramsey steady state, Aiyagari (1995) showed that the so-called “modified golden rule” (MGR hereafter) has to hold in the (assumed) Ramsey steady state.\(^2\) On the other hand, in steady state, the after-tax gross return on capital, which is equated to the risk-free gross interest rate, \(R\), is always less than the inverse of the discount factor, \(1/\beta\). Aiyagari (1995) thus reached the conclusion that a positive capital tax should be imposed to implement the steady-state allocation that satisfies the MGR. In the absence of government intervention, agents overaccumulate capital relative to the level implied by the MGR because of their precautionary savings motive. The imposition of positive capital taxation therefore provides a remedy to restore production efficiency—the MGR. The finding by Aiyagari (1995) is important in the optimal taxation literature, and it represents a distinct departure from the classical result of no permanent capital tax prescribed by Chamley (1986) and Judd (1985).

This paper revisits the long-standing issue with respect to the existence of a Ramsey steady state and the implementation of the MGR. Working with the power utility function, we first demonstrate that the Aiyagari result holds if the shadow price of raising government revenues through distorting taxes converges to zero in the limit at the Ramsey optimum. This “if” is clearly a strong condition. As long as the condition fails to hold, we demonstrate (i) there is no Ramsey steady state if the elasticity of intertemporal substitution (EIS hereafter) is weakly less than 1, and (ii) a Ramsey steady state is possible if the EIS is larger than 1, but the shadow price of resources

\(^1\)It is also known as the Bewley-Hugget-Aiyagari model. For surveys of the literature, see Heathcote, Storesletten, and Violante (2009), Guvenen (2011), Ljungqvist and Sargent (2012, chapter 18), Quadrini and Ríos-Rull (2015) and Krueger, Mitman, and Perri (2016).

\(^2\)The Ramsey steady state is defined as a situation where the optimal Ramsey allocation features the steady-state property in the long run. See Definitions 3 and 4 for the detail.
must diverge in steady state and the MGR does not hold. Result (i) questions the existence of a Ramsey steady state, the basic premise of the Aiyagari (1995) analysis. Result (ii) contradicts Aiyagari’s (1995) implicit assumption on the convergence of the shadow price of resources.3 Both results indicate that the Ramsey steady state described and assumed by Aiyagari (1995) does not emerge at the optimum. Consequently, the subsequent results derived, including the MGR and positive capital tax, could be problematic.

As explained below, our adopted methodology permits us to analytically derive all the necessary first-order conditions (FOCs) of the Ramsey problem in the HAIM economy. Based on these FOCs, we investigate the long-run properties of the Ramsey allocation. In particular, it is critical to our results to include the FOC with respect to aggregate consumption in the analysis. This margin over aggregate consumption is overlooked in the analysis by Aiyagari (1995).4 After incorporating the margin, we show that the social benefit of having one extra unit of aggregate consumption must diverge in the long run once the “if” condition mentioned above does not hold. This divergence could upset the existence of Ramsey steady states. Our analysis highlights the shortcoming of the analysis in Aiyagari (1995): only a subset of all necessary Ramsey FOCs are derived and considered in his study; as a result, his conclusion could be dubious since it contradicts other necessary Ramsey FOCs.

Aiyagari (1995) obtained his results mainly on the setting of endogenous rather than exogenous government spending. We demonstrate that our main result remains robust, regardless of whether government spending is endogenously determined or exogenously given.

It is well-known that the steady-state outcome in a competitive equilibrium, \( R < 1/\beta \), represents the signature feature of the HAIM model.5 The fundamental divergent force underlying our results is exactly embedded in this feature. We show that the divergent force will exist (vanish) if and only if \( R < 1/\beta \) \((R = 1/\beta)\) holds in steady state. Intuitively, unlike individual households in the face of earnings risk, the Ramsey planner in the HAIM economy (without aggregate shocks) faces no uncertainty in allocating aggregate resources. Given that the planner discounts the future by \( \beta \), the strict inequality of \( R < 1/\beta \) then dictates that the market discounts resources at a

---

3 The footnote 15 in Aiyagari (1995) implicitly assumed that the shadow price of resources converges to a finite limit in the Ramsey steady state. Aiyagari (1995) did express his concern in footnote 14: “It seems quite difficult to guarantee that a solution to the optimal tax problem converges to a steady state.”

4 The analysis of Aiyagari (1995) only derives the FOCs with respect to capital and endogenous government spending; see his key result, equation (20). Other FOCs are ignored possibly because of their difficulties or complications in the derivation. Our methodology enables us to remedy the shortcoming by deriving these other FOCs.

5 Ljungqvist and Sargent (2012, p.9) explained that the outcome of \( R < 1/\beta \) in the steady state can be thought of as follows: it lowers the rate of return on savings enough to offset agents’ precautionary savings motive so as to make their asset holdings converge rather than diverge in the limit.
lower rate than the planner discounts utility, implying the existence of planner’s desire to improve welfare by front-loading aggregate consumption through policy tools. This desire persists as long as $R < 1/\beta$ holds in steady state. Interestingly, we show that whether it is feasible for the planner to bring about the desire depends on the value of the EIS. It is feasible if the EIS is weakly less than 1; otherwise, it may not be feasible.

The existence of a Ramsey steady state is commonly assumed for the Ramsey problem in the extant literature. However, this assumption may be problematic for the HAIM environment, according to our findings. The warning is particularly relevant and strong since the key driving force behind our results exactly underlies the hallmark feature of the HAIM economy—the risk-free gross interest rate is lower than the inverse of the discount factor in steady state.

1.1 Methodology

In order to explicitly account for the social benefit of having one extra unit of aggregate consumption, the primal approach to the Ramsey problem à la Lucas and Stokey (1983) is adopted. Lucas and Stokey (1983) considered a representative-agent setting. Our method follows the work of Werning (2007) and Park (2014) to extend the Lucas-Stokey formulation to the setting of heterogeneous households.

Our methodology first formulates the household problem as a time-zero trading problem of the Arrow-Debreu complete-market economy; however, we impose two additional constraints—one for incomplete markets and the other for borrowing constraints—to take into consideration the key features of the HAIM economy. Due to the fact that the Ramsey planner also encounters the same incomplete-markets frictions faced by households, the typical implementability condition is not sufficient and hence additional constraints are needed for the characterization of the Ramsey problem. This causes our HAIM Ramsey problem to become a generalization of the RA (representative-agent) Ramsey problem.

The methodology adopted by this paper results in several contributions to the literature on the Ramsey problem. First, our approach is capable of analytically deriving all FOCs of the primal Ramsey problem in the typical HAIM economy, which to our knowledge is unprecedented. As mentioned above, accounting for all the necessary optimal Ramsey conditions, especially the margin over aggregate consumption, is critical to our analysis and findings. Second, our approach

---

6This approach of modeling incomplete markets is pioneered by Aiyagari, Marcet, Sargent, and Seppala (2002), who named the additional constraints for incomplete markets as measurability conditions. The later work by Chien, Cole, and Lustig (2011) extends this approach to heterogeneous-agent models in the context of asset pricing.
offers an advantage in that the Ramsey problem of our HAIM economy would reduce to that of a RA economy if markets were complete rather than incomplete. Given that the meaning and intuition of the Ramsey problem in the RA economy are well-understood, this advantage makes the model mechanism that drives our main results transparent and intuitive. Finally, our methodology allows us to investigate the existence of a Ramsey steady state instead of assuming its existence as in the extant literature as well as to characterize the properties of a Ramsey steady state if it does exist.

1.2 Related Literature

The literature on optimal capital taxation is vast. Here we focus only on a very limited subset of the studies framed in a heterogeneous-agent environment with incomplete markets or market frictions.

Our work is closely related to the recent study by Chien and Wen (2019), who utilized an analytically tractable heterogeneous-agent model with idiosyncratic preference shocks to address the same issue. They demonstrated that the Ramsey planner intends to increase the supply of government bonds until full self-insurance is achieved or an exogenous debt limit binds. However, in order to have an analytical solution, their model makes a few special assumptions and deviates from the standard HAIM model. Hence, their study cannot directly investigate the issue about the existence of the Ramsey steady state assumption made by Aiyagari (1995).

Conesa, Kitao, and Krueger (2009) considered optimal capital taxation in a HAIM-type economy but in a life-cycle framework. The quantitative part of their study largely focuses on the steady-state welfare. In an overlapping generations model with two-period-lived households, Krueger and Ludwig (2018) characterized the optimal capital tax of the Ramsey problem. In their analysis, the planner lacks government bonds as a policy tool. In contrast, government bonds play an essential role in our results. Hence, their results do not contradict ours. According to our analysis, there could exist a Ramsey steady state with a binding government debt limit.

Papers including Açıkgoz, Hagedorn, Holter, and Wang (2018), Dyrda and Pedroni (2018), and Le Grand, Ragot, et al. (2017) numerically solve optimal Ramsey fiscal policy for the transition path and the steady state of the HAIM economy. The numerical results of these papers are basically consistent with those of Aiyagari (1995). However, once our stated condition fails to hold so that a Ramsey steady state fails to exist, our paper signals a warning about applying the common Ramsey steady state assumption to the HAIM economy.
The work of Straub and Werning (2014) points out that the common assumption that endogenous multipliers associated with the Ramsey problem converge in the limit is not necessarily true and could thus lead to incorrect optimal policy prescriptions in the long-run. Aiyagari’s (1995) assumption of a Ramsey steady state may be subject to the same problem. It should be noted that the mechanism for our non-convergence of endogenous multipliers originates from the HAIM environment. Such a mechanism is absent from the environment studied by Straub and Werning (2014).

Gottardi, Kajii, and Nakajima (2015) considered an environment deviating from the standard HAIM economy, in that there is risky human capital in addition to physical capital. They derived qualitative and quantitative properties for the solution to the Ramsey problem, showing that the interaction between market incompleteness and risky human capital accumulation provides a justification for taxing physical capital. In this paper, we stick to the standard HAIM economy with idiosyncratic earnings risk and show that a Ramsey steady state can fail to exist.

Dávila, Hong, Krusell, and Ríos-Rull (2012) characterized constrained efficiency for the HAIM economy. To decentralize the constrained efficient allocation, the planner is required to know each agent’s realized shocks in order to impose individual-specific capital taxes. We consider flat tax rates applied uniformly to all agents as in the typical Ramsey problem and, as such, the constrained efficient allocation is infeasible to the Ramsey planner.

The rest of the paper is organized as follows. Section 2 and Section 3 introduce our model economy and characterize its competitive equilibrium, respectively. Section 4 formulates the Ramsey problem. Our main findings are demonstrated in Section 5. Section 6 checks and shows the robustness of our results to the endogenous government spending setting, and Section 7 offers a discussion of our findings.

\section{Model Economy}

The model economy mainly builds on Aiyagari (1994). There is a unit measure of infinitely-lived households who are subject to idiosyncratic labor productivity shocks. There are no aggregate shocks. Markets are incomplete in that there are no state-contingent securities for idiosyncratic shocks. In addition, all households are subject to exogenous borrowing constraints at all times.

Time is discrete and the horizon is infinity, indexed by $t = 0, 1, 2, \ldots$. Time 0 is a planning period and actions begin in time 1. All households are ex ante identical and endowed with the
same asset holdings. Ex post heterogeneity arises because households experience different histories of the idiosyncratic shock realization. Let $\theta_t$ (which takes a positive value in a finite set) denote the incidence of the idiosyncratic labor productivity shock at time $t$, and let $\theta^t$ denote the history of events for the idiosyncratic shock of a household up through and until time $t$. The shock $\theta_t$ is independently and identically distributed across households, and the sequence $\{\theta_t\}$ follows a first-order Markov process over time. We let $\pi_t(\theta^t)$ denote the unconditional probability of $\theta^t$ and $\pi(\theta_t|\theta_{t-1})$ denote the conditional probability. We have $\pi_t(\theta^t) = \pi(\theta_t|\theta_{t-1})\pi_{t-1}(\theta^{t-1})$. Because of the independence of productivity shocks across households at any time, a law of large numbers applies so that the probability $\pi_t(\theta^t)$ also represents the fraction of the population that experiences $\theta^t$ at time $t$. We let $\pi_1(\theta_1 = \theta_1) = 1$ for the initial value of $\theta_1$. The role of this assumption will become clear as we proceed. We call a household that has the history $\theta^t$ simply “the household $\theta^t$.” We also introduce additional notations: $\theta^{t+1} \succ \theta^t$ means that the left-hand-side node is a successor node to the right-hand-side node; and for $s > t$, $\theta^s \succeq \theta^t$ ($\theta^s \succ \theta^t$) represents the set of successor shocks after $\theta^t$ up to $\theta^s$ including (excluding) $\theta_t$.

Households maximize their lifetime utility

$$U = \sum_{t=1}^{\infty} \beta^t \sum_{\theta^t} \left[ u(c_t(\theta^t)) - v \left( \frac{l_t(\theta^t)}{\theta_t} \right) \right] \pi_t(\theta^t),$$

where $\beta \in (0, 1)$ is the discount factor; $c_t(\theta^t)$ and $l_t(\theta^t)$ denote the consumption and the labor supply for household $\theta^t$ at time $t$; and $l_t(\theta^t)/\theta_t$ is the corresponding “raw” labor supply (hours worked). The assumptions on the functions $u(.)$ and $v(.)$ are standard.

There is a standard neoclassical constant returns-to-scale production technology, denoted by $F(K, L)$, that is operated by a representative firm, where $K$ and $L$ are aggregate capital and labor, respectively. It is assumed that $F(K, L) \to 0$ if either $K \to 0$ or $L \to 0$. The firm produces output by hiring labor and renting capital from households. The firm’s optimal conditions for profit maximization at time $t$ satisfy

$$w_t = F_L(K_t, L_t),$$
$$r_t = F_K(K_t, L_t),$$

where $w_t$ and $r_t$ are the wage rate and the capital rental rate, and $F_L$ and $F_K$ denote the marginal product of labor and capital, respectively. All markets are competitive.

The government is required to finance an exogenous stream of government spending $\{G_t\}$ and
it can issue one-period government bonds and levy flat-rate, time-varying labor and capital taxes at rates $\tau_{l,t}$ and $\tau_{k,t}$, respectively. The flow government budget constraint at time $t$ is expressed as

$$
\tau_{l,t} w_L L_t + \tau_{k,t} (r_t - \delta) K_t + B_{t+1} = G_t + R_t B_t,
$$

where $R_t$ is the risk-free gross interest rate between time $t - 1$ and $t$, $\delta \in (0, 1)$ is the depreciation rate of capital, and $B_t$ is the amount of government bonds issued at time $t - 1$. The government is assumed to fully commit to a sequence of taxes imposed and debts issued, given the initial amount of government bonds $B_1$ at time 0. This setup for the government is standard for the Ramsey problem. Section 6 considers an alternative setup where $G_t$ becomes endogenously determined rather than exogenously given. This alternative setup is adopted by Aiyagari (1995).

3 Characterization of Competitive Equilibrium

This section characterizes the competitive equilibrium of the model economy, paving the way for the formulation of the Ramsey problem in the next section. We first describe the household problem.

3.1 Household Problem

We express the household problem as a time-zero trading problem as in an Arrow-Debreu economy but with the imposition of additional constraints to account for the key features of the HAIM economy. As noted in the Introduction, this method facilitates the formulation of the primal Ramsey problem for the HAIM economy.

Denote $P_t$ as the time-zero price of one unit of consumption delivered at time $t$. We set $P_0 = 1$ as a normalization. Given $K$ and $B$ are perfect substitutes in the mind of households, the after-tax return on capital has to equal to the risk-free rate:

$$
\frac{P_t}{P_{t+1}} = R_{t+1} = 1 + (1 - \tau_{k,t+1})(r_{t+1} - \delta),
$$

which constitutes a no-arbitrage condition for trades in capital and government bonds.

Let $p_t(\theta^t) = P_t \pi_t(\theta^t)$ be the state-contingent price of one unit of consumption delivered in the event of $\theta^t$ at time $t$. The household’s time-zero budget constraint in an Arrow-Debreu economy
is expressed as
\[ \tilde{a}_1 = \sum_{t \geq 1} \sum_{\theta^t} p_t(\theta^t) \left[ c_t(\theta^t) - \tilde{w}_t l_t(\theta^t) \right], \] (3)
where \( \tilde{w}_t = (1 - \tau_{t,t})w_t \) is the after-tax wage rate at time \( t \) and \( \tilde{a}_1 = K_1 + B_1 \), where \( K_1 \) and \( B_1 \) are the economy’s initial capital and government bond, respectively. All households by assumption have the same initial asset holdings \( \tilde{a}_1 \).

3.1.1 Measurability Conditions and Borrowing Constraints

Two key features of the HAIM economy are (i) incomplete markets—no state-contingent claims on idiosyncratic shocks, and (ii) borrowing constraints—a lower bound on household asset holdings. Both features impose restrictions on the choice of asset holdings across idiosyncratic states over time. We show how to embed these asset-holding restrictions into a time-zero trading problem for the household.

Given the history of shocks \( \theta^t \) at time \( t \), the asset holdings with complete markets can be written as
\[ p_t(\theta^t) a_t(\theta^t) = \sum_{s \geq t} \sum_{\theta^s \geq \theta^t} p_s(\theta^s) \left[ c_s(\theta^s) - \tilde{w}_s l_s(\theta^s) \right], \] (4)
where \( a_t(\theta^t) \) is the amount of the state-contingent claim held by household \( \theta^t \) at the beginning of time \( t \).

However, markets are incomplete rather than complete and households do not have access to state-contingent markets in the HAIM economy. This implies that the asset holdings at time \( t+1 \) are measurable only up to the events prior to the realization of shock \( \theta_{t+1} \). Formally, households face the following measurability conditions: for \( \forall t \geq 0 \) and \( \theta^t \),
\[ a_{t+1}(\theta^t, \theta_{t+1}) = a_{t+1}(\theta^t, \tilde{\theta}_{t+1}) \text{ for all } \tilde{\theta}_{t+1}, \theta_{t+1} \in \Theta, \]
which practically impose constraints on a household’s asset holdings.

For ease of exposition, we rewrite the measurability conditions as follows: for \( \forall t \geq 0 \) and \( \theta^t \),
\[ \frac{a_{t+1}(\theta^t, \theta_{t+1})}{R_{t+1}} = \frac{a_{t+1}(\theta^t, \tilde{\theta}_{t+1})}{R_{t+1}} = \tilde{a}_{t+1}(\theta^t) \text{ for all } \tilde{\theta}_{t+1}, \theta_{t+1} \in \Theta, \] (5)
where \( R_{t+1} \) is the risk-free gross interest rate between time \( t \) and \( t+1 \). That is, \( \tilde{a}_{t+1}(\theta^t) \) is defined so that \( R_{t+1} \tilde{a}_{t+1}(\theta^t) = a_{t+1}(\theta^t, \theta_{t+1}) = a_{t+1}(\theta^t, \tilde{\theta}_{t+1}) \) for all \( \tilde{\theta}_{t+1}, \theta_{t+1} \in \Theta \). This makes sense because
households can hold only a one-period risk-free asset; and their asset holdings at the beginning of
time \( t + 1 \) deflated by their asset return, the risk-free gross interest rate, must be equal to the end
of time \( t \) asset holdings, which are denoted by \( \hat{a}_{t+1}(\theta^t) \).

Households also face the following ad hoc borrowing restrictions for \( \forall t \geq 0: \)
\[
\hat{a}_{t+1}(\theta^t) \geq 0, \forall \theta^t,
\]
which can be equivalently expressed as \( a_{t+1}(\theta^{t+1}) \geq 0 \) for all \( t \) and \( \theta^{t+1} \), according to (5).

### 3.1.2 Formulating and Solving the Household Problem

The asset-holding restrictions, such as the measurability conditions and borrowing constraints, are
equivalent to the restrictions imposed on the whole sequence of consumption and labor choices.

Using (4), we can restate the measurability conditions as
\[
P_{t-1} \hat{a}_t(\theta^{t-1}) \pi_t(\theta^t) = \sum_{s \geq t} \sum_{\theta^s \geq \theta^t} p_s(\theta^s) [c_s(\theta^s) - \hat{w}_s l_s(\theta^s)], \forall t \geq 1, \theta^t, (6)
\]
where we have replaced \( a_t(\theta^t) \) with \( R_t \hat{a}_t(\theta^{t-1}) \) as defined in (5) and used \( p_t(\theta^t) = P_t \pi_t(\theta^t) \) and the
result of \( P_{t-1} = P_t R_t \) in (2). Note that, given \( P_0 = 1 \) and \( \pi_1(\theta^1) = 1 \), the measurability conditions
(6) reduce to the household’s time-zero budget constraint (3) as \( t = 1 \). As to the borrowing
constraints, they can be expressed as
\[
\sum_{s \geq t} \sum_{\theta^s \geq \theta^t} p_s(\theta^s) [c_s(\theta^s) - \hat{w}_s l_s(\theta^s)] \geq 0, \forall t \geq 2, \theta^t. (7)
\]

Given that \( \pi_1(\theta^1) = 1 \), it is implicitly assumed that the borrowing constraints do not bite at \( t = 1 \).

If markets were complete, then households would only face a single constraint (3). The presence
of the additional constraints represented by (6) and (7) is due to the incomplete markets and
borrowing constraints, respectively.

Given prices \( \{\hat{w}_t, p_t(\theta^t)\} \), the household chooses a sequence of consumption \( \{c_t(\theta^t)\} \), labor
\( \{l_t(\theta^t)\} \), and asset holdings \( \{\hat{a}_{t+1}(\theta^t)\} \) to maximize the lifetime utility as of time zero, subject to the
time-zero budget constraint (3), the measurability conditions (6), and the borrowing constraints
(7). Let \( \chi \) be the multiplier on the time-zero budget constraint, \( \nu_t(\theta^t) \) the multiplier on the
measurability condition in the event of \( \theta^t \) at time \( t \), and \( \varphi_t(\theta^t) \) the multiplier on the borrowing
constraint in the event of \( \theta^t \) at time \( t \). Incorporating all the constraints through these multipliers
gives the household’s Lagrangian:

$$
\tilde{L} = \min_{\{\chi, \nu, \phi\} \{c, l, \tilde{a}\}} \max_{t=1}^{\infty} \sum_{s=1}^{\infty} \beta^t \sum_{\theta^t} \left[ u(c_t(\theta^t)) - v\left(\frac{l_t(\theta^t)}{\theta_t}\right)\right] \pi_t(\theta^t)
$$

$$
+ \chi \left\{ \hat{a}_1 - \sum_{t=1}^{\infty} \sum_{\theta^t} p_t(\theta^t) \left[ c_t(\theta^t) - \tilde{\omega}_t l_t(\theta^t) \right] \right\}
$$

$$
+ \sum_{t=2}^{\infty} \sum_{\theta^t} \nu_t(\theta^t) \left\{ \sum_{s=t}^{\infty} \sum_{\theta^s} p_s(\theta^s) \left[ c_s(\theta^s) - \tilde{\omega}_s l_s(\theta^s) \right] - P_{t-1} \hat{a}_t(\theta^{t-1}) \pi_t(\theta^t) \right\}
$$

$$
+ \sum_{t=2}^{\infty} \sum_{\theta^t} \phi_t(\theta^t) \left\{ \sum_{s=t}^{\infty} \sum_{\theta^s} p_s(\theta^s) \left[ c_s(\theta^s) - \tilde{\omega}_s l_s(\theta^s) \right] \right\}.
$$

Note that the constraints associated with the multipliers \(\{\nu_t(\theta^t)\}\) and \(\{\phi_t(\theta^t)\}\) start from \(t = 2\) rather than \(t = 1\). This is due to that the measurability conditions (6) reduce to the household’s time-zero budget constraint (3) as \(t = 1\), and that, given that \(\pi_1(\theta^1) = 1\), it is implicitly assumed the non-binding of the borrowing constraints at \(t = 1\).

Using Abel’s summation formula, the Lagrangian \(\tilde{L}\) can be rewritten as\(^7\)

$$
L = \min_{\{\chi, \nu, \phi\} \{c, l, \tilde{a}\}} \sum_{t=1}^{\infty} \beta^t \sum_{\theta^t} \left[ u(c_t(\theta^t)) - v\left(\frac{l_t(\theta^t)}{\theta_t}\right)\right] \pi_t(\theta^t)
$$

$$
- \sum_{t=1}^{\infty} \sum_{\theta^t} \zeta_t(\theta^t) p_t(\theta^t) \left[ c_t(\theta^t) - \tilde{\omega}_t l_t(\theta^t) \right] + \chi \hat{a}_1
$$

$$
- \sum_{t=2}^{\infty} \sum_{\theta^t} \nu_t(\theta^t) P_{t-1} \hat{a}_t(\theta^{t-1}) \pi_t(\theta^t),
$$

where \(\zeta_t(\theta^t)\) is called the “cumulative multiplier,” and its law of motion is given by

$$
\zeta_{t+1}(\theta^{t+1}) = \zeta_t(\theta^t) - \nu_{t+1}(\theta^{t+1}) - \phi_{t+1}(\theta^{t+1}) \text{ with } \zeta_1 = \chi > 0. \quad (8)
$$

Obviously, \(\zeta_t(\theta^t)\) is a cumulative sum of all Lagrangian multipliers in the past history from the measurability conditions and the borrowing constraints; it encodes the frequency and severity of both types of constraints over time.\(^8\)

\(^7\)See Ljungqvist and Sargent (2012, p.821) for the formula.

\(^8\)Note that the household problem is a standard convex programming problem since the constraint set is convex even with the incorporation of the measurability conditions and the borrowing constraints. Thus, the resulting first-order conditions are necessary and sufficient. In addition, this approach of defining recursive multipliers as in (8) was proposed and developed by Marce and Marimon (1999, 2019) for solving dynamic problems with forward-looking
From the Lagrangian $L$, the FOCs with respect to consumption $c_t(\theta^t)$ and labor supply $l_t(\theta^t)$ are given by

\begin{align}
\beta^t u'(c_t(\theta^t)) & = \zeta_t(\theta^t) P_t, \\
\beta^t v' \left( \frac{l_t(\theta^t)}{\theta_t} \right) \frac{1}{\theta_t} & = \zeta_t(\theta^t) \tilde{w}_t P_t,
\end{align}

while the FOC with respect to asset holdings $\hat{a}_{t+1}(\theta^t)$ is given by

\[ \sum_{\theta^{t+1} > \theta^t} \nu_{t+1}(\theta^{t+1}) \pi(\theta_{t+1}|\theta_t) = 0. \]

From the FOCs (9) and (10), we see that the value of $\zeta_t(\theta^t)$ cannot be negative.

The last FOC requires that the mean of multipliers on the measurability condition across idiosyncratic states $\theta^{t+1}$ be equal to zero, given $\theta^t$. If markets were complete instead, households could have a short position on consumption claims at time $t$ contingent on shock $\theta_{t+1}$ being high at time $t+1$ (“save less for a high state,” which is associated with $\nu_{t+1}(\theta^{t+1}) > 0$ in the Lagrangian $\tilde{L}$), and could have a long position on consumption claims at time $t$ contingent on shock $\theta_{t+1}$ being low at time $t+1$ (“save more for a low state,” which is associated with $\nu_{t+1}(\theta^{t+1}) < 0$ in the Lagrangian $\tilde{L}$). However, markets are incomplete and households cannot save at time $t$, depending on whether shock $\theta_{t+1}$ at time $t+1$ is high or low. As such, the best choice for $\hat{a}_{t+1}(\theta^t)$ at time $t$ is to satisfy an average—that is, the condition (11). Putting together (9), (11) and (2), the motion (8) actually enforces the household’s Euler equation

\[ u'(c_t(\theta^t)) \geq \beta R_{t+1} \sum_{\theta^{t+1} > \theta^t} u'(c_{t+1}(\theta^{t+1})) \pi(\theta_{t+1}|\theta_t), \]

where the equality holds if $\hat{a}_{t+1}(\theta^t) > 0$.

Note that the household’s Euler equation given by (12) can also be expressed equivalently as

\[ \sum_{\theta^{t+1} > \theta^t} \zeta_{t+1}(\theta^{t+1}) \pi(\theta_{t+1}|\theta_t) \leq \zeta_t(\theta^t), \]

where the equality holds if the borrowing constraint of the state-contingent asset, $a_{t+1}(\theta^{t+1}) \geq 0$, does not bind for all possible subsequent $\theta_{t+1}$ states. To see this, using (11), the summation of the constraints. Both Aiyagari, Marcet, Sargent, and Seppala (2002) and Chien, Cole, and Lustig (2011) adopted this approach.
motion (8) over $\theta^{t+1}$ gives

$$\sum_{\theta^{t+1} \succ \theta^t} \zeta_{t+1}(\theta^{t+1})\pi(\theta^{t+1}|\theta_t) = \zeta_t(\theta^t) - \sum_{\theta^{t+1} \succ \theta^t} \varphi_{t+1}(\theta^{t+1})\pi(\theta^{t+1}|\theta_t),$$  \hspace{1cm} (14)

and we know that $\varphi_{t+1}(\theta^{t+1}) \geq 0$ for all $\theta^{t+1}$. Thus, to uphold the equality part of (13), it is required that $\varphi_{t+1}(\theta^{t+1}) = 0$ for all $\theta^{t+1}$ in (14). This feature is caused by the measurability condition (5), which effectively ensures that $\varphi_{t+1}(\theta^{t+1}) = 0$ for all $\theta^{t+1}$, provided $\hat{a}_{t+1}(\theta^t) > 0$.

### 3.2 Competitive Equilibrium

A competitive equilibrium of the model economy is defined in the standard way.

**Definition 1.** Given the initial capital $K_1$ and initial government bonds $B_1$, a competitive equilibrium is defined as sequences of tax rates, government spending and government bonds $\{\tau_{l,t}, \tau_{k,t}, G_t, B_{t+1}\}_{t=1}^{\infty}$, and sequences of prices $\{w_t, r_t, P_t\}_{t=1}^{\infty}$, aggregate allocations $\{C_t, L_t, K_{t+1}\}_{t=1}^{\infty}$ and individual allocation plans $\{c_t(\theta^t), l_t(\theta^t), \hat{a}_{t+1}(\theta^t)\}_{t=1}^{\infty}$, such that

1. $\{c_t(\theta^t), l_t(\theta^t), \hat{a}_{t+1}(\theta^t)\}$ solve the household problem.
2. $\{L_t, K_t\}$ solve the representative firm's problem.
3. The no-arbitrage condition holds: $\frac{P_t}{P_{t+1}} = 1 + (1 - \tau_{k,t+1})(r_{t+1} - \delta)$.
4. The time-zero government budget constraint holds:\footnote{From the flow government budget constraint (1) to the time-zero one, the transversality condition, $\lim_{t \to \infty} P_{t-1}B_t = 0$, is imposed.}

$$B_1 = \sum_{t=1}^{\infty} P_t [\tau_{l,t}w_tL_t + \tau_{k,t}(r_t - \delta)K_t - G_t].$$  \hspace{1cm} (15)

5. All markets clear for all $t$:

$$B_{t+1} + K_{t+1} = \sum_{\theta^t} \hat{a}_{t+1}(\theta^t)\pi_t(\theta^t),$$

$$L_t = \sum_{\theta^t} l_t(\theta^t)\pi_t(\theta^t),$$

$$C_t = \sum_{\theta^t} c_t(\theta^t)\pi_t(\theta^t),$$

$$F(K_t, L_t) = C_t + G_t + K_{t+1} - (1 - \delta)K_t.$$
3.3 Characterizing the Competitive Equilibrium

This subsection characterizes the competitive equilibrium in terms of the aggregate allocations and the cumulative multipliers of the household problem. This step is critical for the primal Ramsey approach in the HAIM economy. To facilitate the characterization, we work with the commonly used power utility function:

**Assumption 1.**

\[ u(c) = \frac{c^{1-\alpha} - 1}{1 - \alpha}, \alpha > 0; \quad v\left(\frac{l}{\theta}\right) = \frac{1}{\gamma}\left(\frac{l}{\theta}\right)^\gamma, \gamma > 1. \]

It is known that \(1/\alpha\) represents the EIS. As will be seen, the value of the consumption EIS plays an important role for our result.

**Proposition 1.** Under Assumption 1, the consumption and labor sharing rules are given, respectively, by

\[ c_t(\theta^t) = \frac{\zeta_t(\theta^t)^{1-\frac{1}{\alpha}}}{H_t} C_t, \]  
\[ l_t(\theta^t) = \frac{\theta_t^\gamma \zeta_t(\theta^t)^{\frac{1}{\gamma}}}{J_t} L_t, \]

where \(H_t\) and \(J_t\) are defined as

\[ H_t = \sum_{\theta^t} \zeta_t(\theta^t)^{\frac{1}{\alpha}} \pi_t(\theta^t), \]
\[ J_t = \sum_{\theta^t} \theta_t^\gamma \zeta_t(\theta^t)^{\frac{1}{\gamma}} \pi_t(\theta^t). \]

\(H_t\) and \(J_t\), are referred to, respectively, as the consumption and labor aggregate multipliers, which are specific moments of the distribution of the individual cumulative multiplier \(\zeta_t(\theta^t)\). \(10\) In addition, \(P_t\) and \(\hat{w}_t\) can be expressed respectively as

\[ P_t = \beta^t C_t^{-\alpha} H_t^\alpha \]
\[ \hat{w}_t = \frac{L_t^{\gamma-1} J_t^{1-\gamma}}{C_t^{-\alpha} H_t^\alpha}. \]

\(10\) Similar expressions for consumption can be seen in Nakajima (2005), Werning (2007) and Park (2014).
Finally, with (18), the risk-free rate is given by

\[
\frac{1}{R_{t+1}} = \frac{P_{t+1}}{P_t} = \beta \left( \frac{C_{t+1}}{C_t} \right)^{-\alpha} \left( \frac{H_{t+1}}{H_t} \right)^{\alpha}.
\]  

(20)

The proofs of our results, including Proposition 1, are all relegated to the Appendix. Equations (16) through (19) show that one can express the individual allocations \(\{c_t(\theta^t), l_t(\theta^t)\}\) and the market prices \(\{P_t, \hat{w}_t\}\) of the competitive equilibrium in terms of the aggregate allocations \(\{C_t, L_t\}\) and the individual cumulative multipliers \(\{\zeta_t(\theta^t)\}\), and the aggregate multipliers \(\{H_t, J_t\}\). The following proposition demonstrates that the Ramsey planner can pick a competitive equilibrium by choosing aggregate allocations plus asset holdings and cumulative multipliers that satisfy a set of conditions.\(^{11}\)

For ease of exposition, we define

\[
\kappa_t(\theta^t) \equiv \beta^t \left[ C_t^{1-\alpha} H_t^{\alpha-1} \zeta_t(\theta^t) \frac{1}{\gamma} - L_t^\gamma J_t^{-\gamma} \tilde{\theta}_t \zeta_t(\theta^t)^{\frac{1}{\gamma-1}} \right] = P_t \left( c_t(\theta^t) - \hat{w}_t l_t(\theta^t) \right),
\]  

(21)

which represents the present value of the time-\(t\) net savings made by household \(\theta^t\). The second equality holds by utilizing equations (16) through (19).

**Proposition 2.** Impose Assumption 1. Given the initial capital \(K_1\), government bonds \(B_1\), the capital tax rate \(\tau_{k,1}\), and a stream of government spending \(\{G_t\}\), sequences of aggregate allocations \(\{C_t, L_t, K_{t+1}\}\), asset holdings \(\{\hat{a}_{t+1}(\theta^t)\}\), and cumulative multipliers \(\{\zeta_t(\theta^t)\}\) (with the associated aggregate multipliers, \(H_t\) and \(J_t\)) can be supported as a competitive equilibrium if and only if they satisfy the following conditions:\(^{12}\)

1. Resource constraints: \(F(K_t, L_t) + (1 - \delta)K_t - K_{t+1} \geq C_t + G_t, \forall t \geq 1\).

2. The implementability condition:

\[
\sum_{t=1}^{\infty} \sum_{\theta^t} \kappa_t(\theta^t) \pi_t(\theta^t) \geq \hat{a}_1.
\]

---

\(^{11}\)Results similar to Proposition 2 but in different contexts can be seen in Aiyagari, Marcet, Sargent, and Seppala (2002, Proposition 1), Werning (2007, Proposition 1), and Park (2014, Proposition 1).

\(^{12}\)The initial capital tax rate, \(\tau_{k,1}\), should be a choice variable for the Ramsey planner. However, given that the initial capital is pre-installed and that households are homogeneous at time zero, taxing the initial capital is essentially the same as allowing a lump-sum tax. As is standard in the literature, we restrict the planner’s ability to choose \(\tau_{k,1}\) in the Ramsey problem.
3. Measurability conditions:

\[ \sum_{s \geq t} \sum_{\theta^s \geq \theta^t} \kappa_s(\theta^s) \pi_s(\theta^s) = \beta^{t-1} C_{t-1}^{-\alpha} H_{t-1}^{\alpha} \hat{a}_t(\theta^{t-1}) \pi_t(\theta^t), \forall t \geq 2, \theta^t. \]

4. Borrowing constraints:

\[ \sum_{s \geq t} \sum_{\theta^s \geq \theta^t} \kappa_s(\theta^s) \pi_s(\theta^s) \geq 0, \forall t \geq 2, \theta^t. \]

5. The evolution of \( \zeta_t(\theta^t) \) satisfies \( \sum_{\theta^{t+1} \geq \theta^t} \zeta_{t+1}(\theta^{t+1}) \pi(\theta_{t+1}|\theta_t) \leq \zeta_t(\theta^t), \forall t \geq 1, \theta^t. \)

6. If the borrowing constraint does not bind for \( \hat{a}_{t+1}(\theta^t) \), then

\[ \sum_{\theta^{t+1} \geq \theta^t} \zeta_{t+1}(\theta^{t+1}) \pi(\theta_{t+1}|\theta_t) = \zeta_t(\theta^t), \]

and this property holds for all \( \theta^t \) and all \( t \geq 1. \)

Condition 2 of Proposition 2 corresponds to the time-zero household budget constraint (by Walras’ law, equivalently, the government time-zero budget constraint), which is conventionally called the “implementability condition” in the formulation of the primal Ramsey problem. When the market is complete without frictions, our model reduces to the RA economy and imposing Conditions 3-6 becomes unnecessary. In particular, since \( \zeta_t(\theta^t) \) in (8) equals \( \chi \) at all times, Conditions 5 and 6 become redundant since they are automatically satisfied.

### 4 Ramsey Problem

Different government policies result in different competitive equilibria. We define the Ramsey problem formally:

**Definition 2.** The Ramsey problem is to choose a competitive equilibrium that attains the maximization of the household’s lifetime utility \( U \).

On the basis of Proposition 2, the Ramsey problem can be represented as maximizing

\[ \sum_{t \geq 1} \beta^t \sum_{\theta^t} \left[ \frac{1}{1-\alpha} \left( \left( \frac{\zeta_t(\theta^t)}{H_t} C_t \right)^{1-\alpha} - 1 \right) - \frac{1}{\gamma} \left( \frac{\theta_{t-1}^{1-\gamma} \zeta_t(\theta^t) \gamma^{-1}}{J_t^{1-\gamma} L_t} \right)^\gamma \right] \pi_t(\theta^t) \]
by choosing $C_t, L_t, K_{t+1}, \{\tilde{a}_{t+1}(\theta^t)\}$, and $\{\zeta_t(\theta^t)\}$ subject to Conditions 1 to 6 stated in Proposition 2 and to $H_t$ and $J_t$ defined earlier, given $K_1, B_1, \tau_{k,1}$ and $\{G_t\}$. The objective of the Ramsey problem is derived by substituting the consumption sharing rule (16) and the labor sharing rule (17) into $U(.)$.

From (6), the strict inequality of the borrowing constraints (7) can be equivalently expressed as $P_{t-1} \tilde{a}_t(\theta^{-1}) \pi_t(\theta^t) > 0, \forall t \geq 2, \theta^t$. Using (18), Condition 6 of Proposition 2 can be captured by

$$
\beta^{t-1} C_{t-1}^{-\alpha} H_{t-1}^{\alpha} \tilde{a}_t(\theta^{-1}) \pi_t(\theta^t) \left[ \sum_{\theta^t \geq \theta^{t-1}} \zeta_t(\theta^t) \pi_t(\theta_t \theta_{t-1}) - \zeta_{t-1}(\theta^{-1}) \right] = 0, \forall t \geq 2, \theta^t.
$$

Namely, if $\tilde{a}_t(\theta^{-1}) > 0$, then the square brackets shown above must equal zero. Thus the Ramsey problem is given by

$$
\max_{\{C_t, L_t, K_{t+1}, \{\tilde{a}_{t+1}(\theta^t)\}, \{\zeta_t(\theta^t)\}\}} \sum_{t \geq 1} \beta^t \sum_{\theta^t} \frac{1}{1 - \alpha} \left( \left( \frac{\zeta_t(\theta^t)^{1/\alpha}}{H_t^{\alpha}} C_t \right)^{1-\alpha} - \frac{1}{\gamma} \left( \frac{\tilde{a}_t(\theta^{-1})^{1/\gamma}}{J_t} K_t \right)^{1-\gamma} \right) \pi_t(\theta^t),
$$

subject to

$$
\{\beta^t \mu_t\} : F(K_t, L_t) + (1 - \delta) K_t \geq C_t + G_t + K_{t+1}, \forall t \geq 1,
$$

$$
\nu^P_t(\theta^t) : \sum_{s \geq t} \sum_{\theta^s \geq \theta^t} \kappa_s(\theta^s) \pi_s(\theta^s) = \beta^{t-1} C_{t-1}^{-\alpha} H_{t-1}^{\alpha} \tilde{a}_t(\theta^{-1}) \pi_t(\theta^t), \forall t \geq 2, \theta^t,
$$

$$
\varphi^P_t(\theta^t) : \sum_{s \geq t} \sum_{\theta^s \geq \theta^t} \kappa_s(\theta^s) \pi_s(\theta^s) \geq 0, \forall t \geq 2, \theta^t,
$$

$$
\xi_t(\theta^t) : \sum_{\theta^t+1 \geq \theta^t} \zeta_{t+1}(\theta^{t+1}) \pi(\theta_{t+1}|\theta_t) \leq \zeta_t(\theta^t), \forall t \geq 1, \theta^t,
$$

$$
\phi_t(\theta^t) : \beta^{t-1} C_{t-1}^{-\alpha} H_{t-1}^{\alpha} \tilde{a}_t(\theta^{-1}) \pi_t(\theta^t) \left[ \sum_{\theta^t \geq \theta^{t-1}} \zeta_t(\theta^t) \pi_t(\theta_t \theta_{t-1}) - \zeta_{t-1}(\theta^{-1}) \right] = 0, \forall t \geq 2, \theta^t.
$$

where $\{\beta^t \mu_t\}, \nu^P_t(\theta^t), \{\varphi^P_t(\theta^t)\}, \{\xi_t(\theta^t)\}, \{\phi_t(\theta^t)\}$ are the corresponding multipliers.

Using Abel’s summation formula and $\pi_t(\theta^t) = \pi(\theta_t | \theta_{t-1}) \pi_{t-1}(\theta^{-1})$, the Lagrangian for the
Ramsey problem gives

\[ L = \max_{\{C_t, L_t, K_{t+1}, \{\eta_t(\theta^t)\}, \{\xi_t(\theta^t)\}\}} \sum_{t \geq 1} \beta^t W(t) + \sum_{t \geq 1} \beta^t \mu_t [F(K_t, L_t) + (1 - \delta)K_t - K_{t+1} - C_t - G_t]
\]

\[ + \sum_{t \geq 1} \sum_{\theta^t} \xi_t(\theta^t) \left[ \zeta_t(\theta^t) - \sum_{\theta^{t+1} > \theta^t} \zeta_{t+1}(\theta^{t+1}) \pi(\theta_{t+1}|\theta_t) \right] - \chi^P \hat{a}_1 \]

\[ - \sum_{t \geq 2} \beta^{t-1} C_{t-1}^{-\alpha} H_{t-1}^\alpha \sum_{\theta^{t-1}} \hat{\alpha}_t(\theta^{t-1}) \left( \sum_{\theta^{t-1} > \theta^t-1} \nu_t^P(\theta^t) \pi(\theta_t|\theta_{t-1}) \right) \pi_{t-1}(\theta^{t-1}) \]

\[ - \sum_{t \geq 2} \beta^{t-1} C_{t-1}^{-\alpha} H_{t-1}^\alpha \sum_{\theta^{t-1}} \hat{\alpha}_t(\theta^{t-1}) \left( \sum_{\theta^{t-1} > \theta^t-1} \phi_t(\theta^t) \pi(\theta_t|\theta_{t-1}) \times \right. \]

\[ \left. \sum_{\theta^{t-1} > \theta^t-1} \zeta_t(\theta^t) \pi(\theta_t|\theta_{t-1}) - \zeta_{t-1}(\theta^{t-1}) \right) \pi_{t-1}(\theta^{t-1}), \]

with

\[ W(t) \equiv \sum_{\theta^t} \pi_t(\theta^t) \]

\[ = \frac{1}{1 - \alpha} \left( \frac{\zeta_t(\theta^t)^{-\frac{1}{\alpha}} C_t}{H_t} \right)^{1-\alpha} - \frac{1}{\gamma} \left( \frac{\theta_t^{-\frac{1}{\gamma}} \zeta_t(\theta^t)^{\frac{1}{\gamma}}}{J_t} \right)^\gamma L_t \]

\[ + \beta^{-1} \eta_t(\theta^t) \left( \sum_{\theta^t} \alpha_t(\theta^t) \right) \]

where

\[ \eta_{t+1}(\theta^{t+1}) = \eta_t(\theta^t) + \nu_{t+1}^P(\theta^{t+1}) + \varphi_{t+1}^P(\theta^{t+1}), \quad \eta_1 = \chi^P > 0, \]

which is the motion of the Ramsey planner’s cumulative multiplier. The Ramsey planner cannot complete the market as typically assumed and is thereby subject to the same market structure of the HAIM economy—that is, the same measurability conditions and borrowing constraints as those facing the household. These market frictions are summarized by the multipliers \( \nu_{t+1}(\theta^{t+1}) \) and \( \varphi_{t+1}(\theta^{t+1}) \) in the household problem and by \( \nu_{t+1}^P(\theta^{t+1}) \) and \( \varphi_{t+1}^P(\theta^{t+1}) \) in the planner problem. However, note that while we have the term \( \chi \hat{a}_1 \) in the household Lagrangian \( L \), we have the term \( -\chi^P \hat{a}_1 \) in the planner Lagrangian \( \mathcal{L} \). The opposite sign is due to the fact that the implementability condition in the Ramsey problem represents the government budget constraint rather than the household budget constraint. As such, while increasing \( \hat{a}_1 \) relaxes the household budget constraint, it tightens the government budget constraint.
4.1 Comparison with the Representative-Agent Model

When the market is complete without frictions as in the RA model, $\zeta_t(\theta^t)$ in (8) equals $\chi$ for all $t$ and $\theta^t$. As such, $H_t$ equals $\chi^{-1/\alpha}$, $P_t$ is reduced to $\beta^tC_t^{-\alpha}\chi^{-1}$ and $J_t$ becomes $\chi^{-1} \sum_\theta^{\gamma_{-1}} \pi_t(\theta^t)$. In addition, from (23), we know that $\eta_t(\theta^t)$ in (25) reduces to $\eta_0 = \chi^P$. Hence, $W(t)$ defined in (22) reduces to

$$W^{RA}(t) = \frac{C_t^{1-\alpha} - 1}{1 - \alpha} - \frac{L_t^\gamma}{\gamma} + \chi P^{-1} (C_t^{-\alpha} C_t - L_t^{1-1} L_t),$$

which is the corresponding pseudo-utility function in the RA model under Assumption 1.\(^{13}\) Part 1 of $W^{RA}(t)$ represents the current-period utility. Its Part 2, in terms of $\beta^t W^{RA}(t)$, is given by

$$\chi P^{-1} \beta^t (C_t^{-\alpha} C_t - L_t^{1-1} L_t) = \chi P^{RA}(C_t - \tilde{w}^{RA}_t L_t),$$

where $P^{RA}_t = \beta^t C_t^{-\alpha} \chi^{-1}$ is the time-zero price of one unit of consumption at time $t$, and $\tilde{w}^{RA}_t = L_t^{1-1} C_t^{\alpha}$ is the after-tax wage rate at time $t$. Thus, the term $P^{RA}_t (C_t - \tilde{w}^{RA}_t L_t)$ shown in (24) represents the time-$t$ net savings evaluated at the time-zero price in the RA model. The time-$t$ net savings of households also represents the amount of net revenues collected by the government in period $t$ because of Walras’ law; hence, the implementability condition multiplier, $\chi^P$, “measures the utility costs of raising government revenues through distorting taxes” (Ljungqvist and Sargent (2012, p.629)) in the RA framework.

Part 1 of $W(t)$ in our HAIM model also represents the current-period utility. Its Part 2, in terms of $\beta^t W(t)$, is given by

$$\eta_t(\theta^t) \kappa_t(\theta^t) = \eta_t(\theta^t) P_t (c_t(\theta^t) - \tilde{w}_t l_t(\theta^t)), $$

where the equality holds according to (21). Thus, the term $P_t (c_t(\theta^t) - \tilde{w}_t l_t(\theta^t))$ represents the time-$t$ net savings of household $\theta^t$ evaluated at time zero in the HAIM economy. The taxes imposed by the Ramsey planner alter the household $\theta^t$’s consumption and labor supply and, consequently, distort his/her net savings. The shadow price of this distortion on the household $\theta^t$’s net savings is given by the multiplier $\eta_t(\theta^t)$. Note that $\eta_t(\theta^t)$ is no longer a time-invariant constant $\chi^P$, as in the RA model. From the evolution of $\eta_t(\theta^t)$ governed by (23), we see that $\eta_t(\theta^t)$ starts from $\chi^P (\eta_1 = \chi^P)$, but in a sequence it varies not only across households but also over time, meaning

\(^{13}\)Under the complete-market assumption, our Ramsey planner problem is identical to the one in the RA model, which can be seen in subsection 16.6.1 in Ljungqvist and Sargent (2012, p. 626).
that the utility cost of collecting government revenues is not only household specific but also time varying.

Now consider the steady-state version of equation (20):

$$1 = \beta R \left( \frac{H_{t+1}}{H_t} \right)^\alpha. \quad (26)$$

Given that $H_t = \chi^{-1/\alpha}$ all the time in the RA economy, we see that $H_{t+1}/H_t = 1$ and $\beta R = 1$ are the two sides of the same coin in steady state under the RA economy. In contrast, given that $\beta R < 1$ in steady state in the HAIM economy, we see that $H_{t+1}/H_t > 1$ and $\beta R < 1$ are the two sides of the same coin in steady state under the HAIM economy. Equation (26) tells us that $H_t$ is increasing over time and must diverge to infinity in the limit in the HAIM economy, since $\beta R < 1$ holds in steady state. Put simply, the feature of an increasing and divergent $H_t$ exactly underlies the hallmark of the competitive equilibrium in the HAIM model—the risk-free gross interest rate is lower than the inverse of the discount factor in steady state.

The divergent tendency of $H_t$, all else equal, makes Part 2 of $W(t)$ converge more slowly than Part 1. As will be seen, this asymmetric convergence between Part 1 and Part 2 of $W(t)$ is the key to our result showing the non-existence of a Ramsey steady state.

### 4.2 Optimal Conditions of the Ramsey Problem

From the Lagrangian $\mathcal{L}$, the necessary FOCs with respect to $\tilde{a}_{t+1}(\theta^t)$, $C_t$, $L_t$, and $K_{t+1}$ for $t \geq 1$ yield, respectively,

$$\sum_{\theta^{t+1} \geq \theta^t} \left[ \nu_{t+1}^P(\theta^{t+1}) + \phi_{t+1}(\theta^{t+1}) \left( \sum_{\theta^{t+1} \geq \theta^t} \zeta_{t+1}(\theta^{t+1}) \pi(\theta_{t+1} | \theta_t) - \zeta_t(\theta^t) \right) \right] \pi(\theta_{t+1} | \theta_t) = 0, \quad (27)$$

$$W_C(t) = \mu_t, \quad (28)$$

$$-W_L(t) = \mu_t F_L(K_t, L_t), \quad (29)$$

$$\mu_t = \beta \mu_{t+1} [F_K (K_{t+1}, L_{t+1}) - \delta + 1], \quad (30)$$

where the derivation of (28) has made use of (27), and $W_C(t)$ and $W_L(t)$ denote the derivatives of $W(t)$ with respect to $C_t$ and $L_t$, respectively.\footnote{The FOC with respect to $\zeta_t(\theta^t)$ will not be needed for the derivation of our main results.}

The explicit expressions of $W_C(t)$ and $W_L(t)$ in the FOCs of the Ramsey problem are crucial to
our analysis later. One can derive them from the pseudo-utility \( W(t) \) defined in (22). However, to facilitate the proof and discussion hereafter, it is convenient to express \( W_C \) and \( W_L \) in the following way. First, using the consumption sharing rule (16), \( W_C(t) \) in (28) is expressed as

\[
W_C(t) = C_t^{-\alpha} \left[ \sum_{\theta^t} \left( \frac{c_t(\theta^t)}{C_t} \right) \left( \frac{c_t(\theta^t)}{C_t} \right)^{-\alpha} \pi_t(\theta^t) \right] + \left( 1 - \alpha \right) H_t^\alpha M_t, \tag{31}
\]

and using (16)-(19), \( W_L(t) \) in (29) is expressed as

\[
-W_L(t) = \tilde{w}_t C_t^{-\alpha} \left[ \sum_{\theta^t} \left( \frac{l_t(\theta^t)}{L_t} \right) \left( \frac{c_t(\theta^t)}{C_t} \right)^{-\alpha} \pi_t(\theta^t) \right] + \gamma H_t^\alpha N_t, \tag{32}
\]

where \( M_t = \sum_{\theta^t} \left( \frac{c_t(\theta^t)}{C_t} \right) \eta_t(\theta^t) \pi_t(\theta^t) \) and \( N_t = \sum_{\theta^t} \left( \frac{l_t(\theta^t)}{L_t} \right) \eta_t(\theta^t) \pi_t(\theta^t) \).

Part 1 of \( W_C(t) \) and Part 1 of \( W_L(t) \) denote the sum of households’ “normalized” marginal utility of consumption, \( \left( \frac{c_t(\theta^t)}{C_t} \right)^{-\alpha} \), weighted by their consumption shares and labor shares, respectively. They represent the planner’s social evaluation of increasing \( C_t \) and \( L_t \), respectively. We next explain the meaning of the weighted sum of \( \eta_t(\theta^t) \) shown in Part 2 of \( W_C(t) \) and of \( W_L(t) \). Summing up (25) across all households at time \( t \) gives

\[
P_t \left( M_t C_t - N_t \tilde{w}_t L_t \right).
\]

Contrasting the above with the corresponding one in the RA model, namely, \( P_t^{RA} \chi^P (C_t - \tilde{w}_t^{RA} L_t) \), we see that the role of \( \chi^P \) (i.e., the utility costs of raising government revenues through distorting taxes) has been replaced either by \( M_t \) or by \( N_t \), depending on whether distorting the time-\( t \) aggregate net savings is through the margin of changing \( C_t \) or changing \( \tilde{w}_t L_t \). Since the issue is about collecting government revenues across all households and different households contribute differently to aggregate consumption and labor supply, it is intuitive that these utility costs or shadow prices are weighted (by the consumption or labor share depending on the changed margin) rather than unweighted as given by \( \sum_{\theta^t} \eta_t(\theta^t) \pi_t(\theta^t) \).
5 No Ramsey Steady State

Before presenting our main results, we define the steady state of the HAIM economy.

**Definition 3.** The steady state of the HAIM economy meets two conditions:

1. Each aggregate variable stays at a positive finite value.

2. The cross-sectional distributions of the consumption share $c_t(\theta^t)/C_t$ and of the labor share $l_t(\theta^t)/L_t$ are time invariant.

As to the Ramsey steady state, it is defined as follows:

**Definition 4.** The long-run optimal solution to the Ramsey problem is defined as a Ramsey steady state if it features the steady state of the HAIM economy.

As shown by many quantitative studies, the existence of a steady state is not a problem for the HAIM economy. Our investigation is about the existence of a Ramsey steady state. We need the following condition for the investigation:

**Condition X** At the Ramsey optimum, $\lim_{t \to \infty} M_t = \lim_{t \to \infty} N_t = 0$.

Our main results differ sharply, depending on whether the above condition holds or fails to hold.

We are ready to state our first main finding.

**Proposition 3.** Impose Assumption 1 and suppose Condition X holds. Then the planner implements the MGR and levies a positive capital tax in the Ramsey steady state.

Once Condition X holds, it is clear from (28) and (31) that $\mu_t = \mu_{t+1}$ in steady state. We then reach the conclusion of Proposition 3 from the FOC (30) and that $R < 1/\beta$ in steady state. This is the same conclusion as reached by Aiyagari (1995).

Is Condition X a strong condition? In our view, the answer is positive. First, Ljungqvist and Sargent (2012, p. 629) argued that when a government has to use distortionary taxes, the shadow price $\chi^P$ in the RA economy will be strictly positive, which reflects the welfare cost of distortionary taxes at the margin. Our $M_t$ and $N_t$, as explained earlier, represent a generalization of $\chi^P$ by including the bite of market frictions. Second, Aiyagari, Marcet, Sargent, and Seppala (2002) considered a RA economy in which markets on aggregate shocks are incomplete and the

---

15For the existence and uniqueness of steady states in the HAIM economy, see Açıkgöz (2018).
shadow prices of taxation are no longer a constant but varies over time (similar to $M_t$ and $N_t$ in our HAIM economy). Under this environment, Ljungqvist and Sargent (2012, p. 659) noted that the shadow price of raising government revenues “remains strictly positive so long as the government must resort to distortionary taxation either in the current period or for some realization of the state in a future period.” Although the context is not the same as ours, Condition X is indeed a strong condition in the light of this statement. It is so, at least as far as optimal taxation is concerned.\footnote{We do not deny that Condition X may hold in some special cases. For example, when household preferences take the form of quasi-linear in consumption, Aiyagari, Marcet, Sargent, and Seppala (2002) showed in their context that an economy can reach the first-best in the long run at the Ramsey optimum, despite erratic government spending. However, this seems represent an exception rather than the general.}

We state our second main finding.

**Proposition 4.** Impose Assumption 1 and suppose Condition X fails to hold in that $\lim_{t \to \infty} M_t > 0$ and $\lim_{t \to \infty} N_t > 0$.

1. If $\alpha \geq 1$, there is no Ramsey steady state.

2. If $\alpha < 1$, a Ramsey steady state is possible, but (i) the shadow price of resources, $\mu_t$, must diverge in the limit and (ii) the MGR does not hold in the Ramsey steady state and the corresponding capital tax is non-positive.

From (26), we see that the result of $R < 1/\beta$ and the result of $H_{t+1}/H_t > 1$ are the two sides of the same coin: $H_{t+1}/H_t > 1$ holds in steady state if and only if $R < 1/\beta$ holds in steady state. As shown in the proof of Proposition 4, the increasing and divergent behavior of the $H_t$ term (equivalently, the feature of $R < 1/\beta$ in steady state) is the exact force that undermines the existence of Ramsey steady states when $\alpha \geq 1$.

Even though a Ramsey steady state is possible when $\alpha < 1$, the result differs from that prescribed by Aiyagari (1995). In particular, it is contrary to the implicit assumption made by Aiyagari (1995) that $\mu_t$ converges.

It is important to recognize that if we were to confine the analysis only to the FOC (30) and assume incorrectly the convergence of $\mu_t$ in the Ramsey steady state, we would have the exact conclusion reached by Aiyagari (1995); namely, the MGR holds at the optimum and capital income should be taxed since $R < 1/\beta$ holds in the long run. This recognition highlights the importance of taking into account the necessary Ramsey FOCs other than (30). To our knowledge, the analytical form of the expression for the term $W_C(t)$ or $W_L(t)$ that appears in the Ramsey FOCs (28)-(29) has
never been derived before. Thanks to our methodology, the consistency or inconsistency between
the existence of a Ramsey steady state and other Ramsey FOCs can be clearly checked and the
shortcoming of the analysis in Aiyagari (1995) can be remedied.

The intuition underlying our second main result can be understood as follows. Unlike house-
holds in the face of idiosyncratic income shocks, the Ramsey planner faces no uncertainty in
allocating aggregate resources. The strict inequality \( R < 1/\beta \) in the steady state of the HAIM
economy then dictates an asymmetric discounting; that is, the market discounting rate is always
lower than the preference discounting rate. This feature of asymmetric discounting impels a desire
for the planner to front-load aggregate consumption. Such a desire persists permanently since the
strict inequality \( R < 1/\beta \) holds in the steady state of the HAIM economy.

Proposition 4 indicates that the existence of a Ramsey steady state depends on the value of
\( \alpha \), which controls the EIS. The following intends to provide additional explanations and intuition for
such a dependence. As discussed in Subsection 4.1, the utility costs of implementing a policy hinge
on its effects over the net savings of households (or equivalently, by Walras’ law, the amount of
net tax revenues collected by the government). Let us consider the impact of changing aggregate
consumption on the net savings (government revenues) through consumption spending. There is
only a term involving \( C_t \) in Part 2 of \( W(t) \) given by (22). Expressed in \( \beta^t W(t) \) and by omitting
\( \eta_t(\theta^t) \), this term equals
\[
\beta^t C_t^{1-\alpha} H_t^{\alpha-1} \zeta_t(\theta^t)^{\frac{\alpha-1}{\alpha}} = P_t C_t \frac{\zeta_t(\theta^t)^{\frac{1}{\alpha}}}{H_t^{\frac{1}{\alpha}}},
\]
which represents household \( \theta^t \)'s consumption spending at time \( t \) according to the consumption
sharing rule (16). From (18), we have \( P_t C_t = \beta^t C_t^{1-\alpha} H_t^\alpha \) and so \( \partial(P_t C_t)/\partial C_t = (1-\alpha)\beta^t C_t^{\alpha-\alpha} H_t^\alpha \).
Thus a drop in aggregate consumption \( C_t \), all else equal, will raise, lower, or not change individual
collection spending via altering \( P_t C_t \) if \( \alpha \) is larger than, less than, or equal to 1, respectively.
This implies that a reduction in aggregate consumption over time (front-loading consumption)
will make the government constraint associated with \( \eta_t(\theta^t) \) in (22) looser, tighter, or unchanged,
depending on whether \( \alpha \) is larger than, less than, or equal to 1, respectively. Since front-loading
aggregate consumption relaxes the government constraint by increasing its revenues if \( \alpha > 1 \),
it actually enforces the planner’s desire to front-load aggregate consumption in the presence of
\( R < 1/\beta \) in steady state. In contrast, since front-loading aggregate consumption tightens the
government constraint by reducing its revenues if \( \alpha < 1 \), it counterbalances the planner’s desire to
front-load aggregate consumption in the presence of \( R < 1/\beta \) in steady state.

When \( \alpha = 1 \), neither enforcement (associated with \( \alpha > 1 \)) nor counterbalance (associated with
\( \alpha < 1 \) occurs. We then see a clean case of front-loading aggregate consumption in the presence of \( R < 1/\beta \) in steady state. From the proof of Proposition 4, we know that \( \mu_t \) is increasing and divergent because \( H_t \) is increasing and divergent. If \( \alpha = 1 \), we have \( W_C(t) = C_t^{-1} \) from (31). Thus, given that \( \mu_t \) increases over time, it is apparent that the optimal \( C_t \) determined by the FOC (28), namely, \( C_t^{-1} = \mu_t \), will decrease over time.

6 Endogenous Government Spending

This section checks the robustness of our Proposition 4 findings by altering the model setup from exogenous to endogenous government spending, which is the main setting considered by Aiyagari (1995). We show here that even with endogenous government spending, our results concerning the existence of a Ramsey steady state are robust and remain unchanged.

Following Aiyagari (1995), the household lifetime utility \( U \) is modified to

\[
U^G = \beta^t \sum_{t=1}^{\infty} \left[ u(c_t(\theta^t)) - v \left( \frac{l_t(\theta^t)}{\theta_t} \right) + V(G_t) \right] \pi_t(\theta^t),
\]

where \( V(.) \) is the utility function of public consumption \( G_t \), which is assumed to be common for all households. The usual assumptions are applied to \( V(.) \). This modification of the setup does not change the household problem, since the determination of \( G_t \) is exogenous to households. However, the Ramsey problem is only changed slightly because \( G_t \) is now a choice variable to the Ramsey planner. As long as \( G_t \) is non-negative (which could be ensured by assuming \( V'(0) = \infty \)), \( G_t \) can be chosen to satisfy the time-\( t \) resource constraint so that Proposition 2 still applies. The Lagrangian for the Ramsey problem is identical to the previous Lagrangian \( \mathcal{L} \), except for the replacement of \( W(t) \) by \( W(t) + V(G_t) \). As a result, the FOCs with respect to aggregate consumption, labor, and capital remain the same as before. The additional FOC with respect to \( G_t \) is given by

\[
V'(G_t) = \mu_t,
\]

which together with FOC (30) does imply the MGR if a Ramsey steady state is assumed. This is essentially the procedure for obtaining the MGR in Aiyagari (1995); see Equation (20) of his Proposition 1 on page 1170.

However, the introduction of endogenous \( G_t \) does not alter the fundamental force that drives the results of Proposition 4 nor does it help to justify the assumption of a Ramsey steady state.
The marginal social benefit of having one extra unit of aggregate consumption, namely, $W_C(t)$, could still diverge in the long-run given that the Ramsey outcome of $R\beta = 1$ is infeasible in steady state. With the additional government tool—endogenous government spending—the extra output can be expended either on government spending or on private consumption, and hence the marginal benefits to the social welfare by exercising these two options have to be equalized at the optimum. Indeed, putting (28) and (33) together gives rise to $V'(G_t) = W_C(t)$ and hence the optimal choice of $G_t$ has to respect and be consistent with the divergent behavior of $\mu_t$. This equality casts doubt on the convergence assumption of $G_t$ to a finite positive value made by Aiyagari (1995). In brief, it is the erroneously assumed Ramsey steady state, not the endogenous government spending assumption, which is the root of the problem.

7 Discussion

We show a Ramsey steady state can fail to exist in the HAIM economy, but short of explaining in terms of policy tools why it fails to exist. This section provides a brief discussion.

In the proof of Proposition 2, an implicit but standard assumption is that, given the sequence of capital stocks $\{K_{t+1}\}_{t=1}^{\infty}$ and households’ asset holdings $\{\hat{a}_{t+1}(\theta^t)\}_{t=1}^{\infty}$, it is always feasible to pick a sequence of government bonds, $\{B_{t+1}\}_{t=1}^{\infty}$, so as to clear the asset market in each time period. Aiyagari (1995) made the same assumption in his analysis; see equation (19) of his paper and the discussion about it. Although standard in the literature, this feasibility assumption may be problematic here since the planner may implement front-loading aggregate consumption induced by $R < 1/\beta$ via issuing an ever increasing amount of government bonds. In the same context as our paper, Chien and Wen (2019) utilized an analytically tractable HAIM model to demonstrate that the Ramsey planner intends to increase the supply of government bonds until full self-insurance is achieved or an exogenous debt limit binds. Their work suggests that the feasibility assumption with no quantity restriction on the planner’s issuing government bonds could be the culprit for the non-existence of a Ramsey steady state in our economy. If this is indeed true, then imposing an upper bound on the issuance of government bonds should provide a mechanism to restore the existence of Ramsey steady states. Following Aiyagari, Marcet, Sargent, and Seppala (2002), let us impose an exogenous upper bound $\bar{B} < \infty$ on the issuance of government bonds:

$$B_{t+1} \leq \bar{B}, \forall t \geq 1.$$
It can be shown that incorporating the above additional constraints into the Ramsey problem does provide an offsetting to the increasing and divergent force of the $H_t$ term and offer an opportunity for the existence of Ramsey steady states.

The constraints above are known as the ad hoc debt limit. By analogy with the household savings problem in Aiyagari (1994), Aiyagari, Marcet, Sargent, and Seppala (2002) also considered the so-called natural debt limit, which is defined as “the maximum debt that could be repaid almost surely under an optimal tax policy” (p. 1225). What will happen if incorporating the government’s natural debt limit into the Ramsey problem? This is an interesting question. However, answering the question is a formidable task, in that the government’s natural debt limit in the Ramsey problem is endogenously determined and evolves dynamically according to the policy choice in a non-trivial way. At any rate, exposing this and other related issues in details is beyond the scope of the present study and we leave it to future works.

\[\text{In the absence of capital, Aiyagari, Marcet, Sargent, and Seppala (2002) was able to derive the natural debt limit explicitly; see p. 1232 of their paper. However, in the presence of capital, the derivation becomes much more difficult.}\]
References


28

A Appendix

A.1 Proof of Proposition 1

With the imposition of Assumption 1, the FOC for consumption (9) yields

\[ c_t(\theta^t) = \left( \frac{\zeta_t(\theta^t)P_t}{\beta^t} \right)^{-\frac{1}{\alpha}}. \]

Summing \( c_t(\theta^t) \) over \( \theta^t \) gives the aggregate consumption at time \( t \):

\[ C_t = \sum_{\theta^t} c_t(\theta^t)\pi_t(\theta^t) = \sum_{\theta^t} \left( \frac{\zeta_t(\theta^t)P_t}{\beta^t} \right)^{-\frac{1}{\alpha}}\pi_t(\theta^t) \]

\[ = \left( \frac{P_t}{\beta^t} \right)^{-\frac{1}{\alpha}} \sum_{\theta^t} \zeta_t(\theta^t)^{-\frac{1}{\alpha}}\pi_t(\theta^t) = \left( \frac{P_t}{\beta^t} \right)^{-\frac{1}{\alpha}} H_t, \]

which gives (18). Plugging (18) back into (9) gives (16).

From (10), we have

\[ l_t(\theta^t) = \left( \frac{\theta^t\zeta_t(\theta^t)\hat{w}_tP_t}{\beta^t} \right)^{\frac{1}{\gamma-1}}. \]

Summing \( c_t(\theta^t) \) over \( \theta^t \) gives the aggregate consumption at time \( t \):

\[ L_t = \sum_{\theta^t} l_t(\theta^t)\pi_t(\theta^t) = \sum_{\theta^t} \left( \frac{\theta^t\zeta_t(\theta^t)\hat{w}_tP_t}{\beta^t} \right)^{\frac{1}{\gamma-1}}\pi_t(\theta^t) \]

\[ = \left( \frac{\hat{w}_tP_t}{\beta^t} \right)^{\frac{1}{\gamma-1}} \sum_{\theta^t} \theta^t\gamma^{-\frac{1}{\gamma-1}}\zeta_t(\theta^t)^{\frac{1}{\gamma-1}}\pi_t(\theta^t) = \left( \frac{\hat{w}_tP_t}{\beta^t} \right)^{\frac{1}{\gamma-1}} J_t, \]

which together with (18) gives (19). Plugging (19) back into (10) gives (17).

A.2 Proof of Proposition 2

“Only if” part: Condition 1 of Proposition 2—the resource constraints—is implied by a competitive equilibrium since it is part of the definition. Note also that Conditions 5 and 6 of Proposition 2 are implied by (8) and (11) from the household problem in a competitive equilibrium.

The remaining proof is to show that the time-zero budget constraint (3), the measurability conditions (6), and the borrowing constraints (7) in the household problem can be re-expressed as Conditions 2-4 of Proposition 2. Substituting (2), (16)-(17) and (18)-(20), all of which build on the household’s optimal behavior, into (3)-(7), we obtain Conditions 2-4.
“If” part: Suppose the sequence of asset holdings \( \{ \hat{a}_{t+1}(\theta^t) \}_{t=1}^{\infty} \), aggregate allocations \( \{ C_t, K_{t+1}, L_t \}_{t=1}^{\infty} \), and cumulative multipliers \( \{ \zeta_t(\theta^t) \}_{t=1}^{\infty} \) with the associated aggregate multipliers \( \{ H_t, J_t \}_{t=1}^{\infty} \) satisfy Conditions 1-6 stated in Proposition 2. We show that a competitive equilibrium of the HAIM economy can be constructed in the following way.

First, we pick the prices and taxes defined below:

\[
\begin{align*}
  r_t &= F_K(K_t, L_t), \\
  w_t &= F_L(K_t, L_t), \\
  P_t &= \beta_t C_t^{\gamma} - \alpha_t H_t^{\alpha_t}, \\
  1 - \tau_{k,t+1} &= \frac{P_t}{P_{t+1}} - 1 = \frac{1}{F_K(K_{t+1}, L_{t+1})} - \delta = \frac{1}{F_K(K_{t+1}, L_{t+1})} - \delta, \\
  1 - \tau_{l,t} &= \frac{L_t^{\gamma-1} J_t^{1-\gamma}}{F_L(K_t, L_t) C_t^{-\alpha} H_t^{\alpha}}.
\end{align*}
\]

Note that (34)-(35) correspond to the profit-maximization conditions of the representative firm and that (37) ensures that the no-arbitrage condition (2) holds.

Second, we show that the household problem can be solved. Let the individual consumption and labor allocations be given by (16) and (17). Then, individual consumption and labor allocations together with prices defined in (34)-(38) satisfy the first-order conditions, (9) and (10), of the household problem. To derive the household’s Euler equation, we combine the individual consumption allocations, prices defined in (34)-(38), and Conditions 5-6. The time-zero budget constraint (3), the measurability conditions (6), and the borrowing constraints (7) in the household problem can be obtained by using (34)-(38) plus Conditions 2-4.

Third, we need to make sure that all markets clear. Plugging in individual consumption allocations (16) into Condition 1 implies that the market clearing condition of the goods market is satisfied. The labor market clearing condition is achieved by aggregating (17) across all households. For the asset market, we pick \( \{ B_{t+1} \}_{t=1}^{\infty} \) such that

\[
B_{t+1} = \sum_{\theta^t} \hat{a}_{t+1}(\theta^t) - K_{t+1},
\]

which ensures that the asset market clears in each time period.

The last condition to be met in competitive equilibrium is the government budget constraint.
From (3), we have

\[ B_1 + K_1 = \hat{a}_1 = \sum_{t \geq 1} P_t \sum_{\theta^t} [c_t(\theta^t)\pi_t(\theta^t) - \hat{w}_t l_t(\theta^t)\pi_t(\theta^t)] \]

\[ = \sum_{t \geq 1} P_t [C_t - w_t L_t + \tau_{t,t} w_t L_t] \]

\[ = \sum_{t \geq 1} P_t [C_t + r_t K_t - F(K_t, L_t) + \tau_{t,t} w_t L_t], \]

where the derivation has made use of \( \hat{w}_t = w_t(1 - \tau_{t,t}) \) and \( F(K_t, L_t) = w_t L_t + r_t K_t \). Utilizing the resource constraint and the no-arbitrage condition (2) then gives

\[ B_1 + K_1 = \sum_{t \geq 1} P_t [r_t K_t - K_{t+1} + (1 - \delta) K_t + \tau_{t,t} w_t L_t - G_t] \]

\[ = \sum_{t \geq 1} P_t [(1 + (1 - \tau_{k,t}) (r_t - \delta)) K_t - K_{t+1} + \tau_{k,t} (r_t - \delta) K_t + \tau_{t,t} w_t L_t - G_t] \]

\[ = \sum_{t \geq 1} P_t \left[ \frac{P_{t-1}}{P_t} K_t - K_{t+1} + \tau_{k,t} (r_t - \delta) K_t + \tau_{t,t} w_t L_t - G_t \right] \]

\[ = P_0 K_1 + \sum_{t \geq 1} P_t [\tau_{k,t} (r_t - \delta) K_t + \tau_{t,t} w_t L_t - G_t], \]

which leads to the time-zero government budget constraint since we normalize \( P_0 = 1 \).

A.3 Proof of Proposition 4

Suppose there is a Ramsey steady state with \( \beta R < 1 \). Given the normalization of \( P_0 = 1 \), we obtain from (2) that \( P_t = \prod_{s=1}^t \frac{1}{R_s} \). Moreover, given (31)-(32) and \( P_t = \beta^t C_t^{-\sigma} H_t^\sigma \) according to (18), one can use the above result to rewrite the FOCs (28) and (29) as

\[
\left( \beta^t \prod_{s=1}^t R_s \right) \sum_{\theta^t} \left( \frac{c_t(\theta^t)}{C_t} \right) c_t(\theta^t)^{-\alpha} \pi_t(\theta^t) + (1 - \alpha) M_t \]

\[ = \left( \beta^t \prod_{s=1}^t R_s \right) \mu_t, \]

(39)
\[
\hat{w}_t \left( \sum_{\theta^t} \left( \frac{L_t(\theta^t)}{l_t(\theta^t)} \right) c_t(\theta^t)^{-\alpha} \pi_t(\theta^t) + \gamma N_t \right) = \left( \beta \prod_{s=1}^{t} R_s \right) \mu_t F_L(K_t, L_t) \quad \text{(40)}
\]

Note that both the term \( \sum_{\theta^t} \left( \frac{C_t(\theta^t)}{c_t(\theta^t)} \right) c_t(\theta^t)^{-\alpha} \pi_t(\theta^t) \) in (39) and the term \( \sum_{\theta^t} \left( \frac{L_t(\theta^t)}{l_t(\theta^t)} \right) c_t(\theta^t)^{-\alpha} \pi_t(\theta^t) \) in (40) converge due to the existence of a Ramsey steady state by presumption.

According to FOC (30), there are two possible cases for the existence of a Ramsey steady state according to Definition 3: (a) \( \mu_t \) itself converges, and (b) \( \mu_t \) diverges but its growth \( \frac{\mu_{t+1}}{\mu_t} \) converges.

First, let us consider Case (a). Given that the term \( N_t \) (the shadow price of collecting tax revenues via varying \( \hat{w}_t L_t \)) is positive, there is no possibility for Case (a) to uphold the FOC (40) in steady state. This is because (i) Part 1 of (40) vanishes in steady state because of \( \beta R < 1 \), and (ii) the term \( \left( \beta \prod_{s=1}^{t} R_s \right) \mu_t F_L(K_t, L_t) \) also vanishes in steady state since \( \mu_t \) itself converges and \( \beta R < 1 \).

Second, let us consider Case (b). Although \( \mu_t \) itself fails to converge, it is possible for the ratio \( \frac{\mu_{t+1}}{\mu_t} \) to converge so as to support a Ramsey steady state. There are three subcases for Case (b), depending on the value of \( \alpha \).

1. \( \alpha = 1 \). The term \( W_C(t) \) expressed in (31) reduces to \( C_t^{-1} \) and the divergence of \( \mu_t \) drives \( C_t \) to zero in the limit according to FOC (28). However, \( C_t \to 0 \) is incompatible with the steady state defined by Definition 3, which requires that \( C_t \) converge to a positive value in steady state. We thus have a contradiction with the existence of a Ramsey steady state.

2. \( \alpha > 1 \). Part 1 of (39) vanishes in steady state because of \( \beta R < 1 \). Given \( \alpha > 1 \), Part 2 of (39) is negative in the limit because the term \( M_t \) (the shadow price of collecting tax revenues via varying \( C_t \)) is positive in the limit. However, since \( \mu_t \) in (39) must be non-negative, it leads to the violation of the FOC (39) in steady state. Again, we have a contradiction with the existence of a Ramsey steady state.

3. \( \alpha < 1 \). Part 1 of both (39) and (40) vanish in steady state because of \( \beta R < 1 \). Part 2 of (40) is positive in steady state because of \( \lim_{t \to \infty} N_t > 0 \). Given \( \alpha < 1 \), Part 2 of (39) is also positive because of \( \lim_{t \to \infty} M_t > 0 \). Thus, given \( \beta R < 1 \), the divergent \( \mu_t \) contradicts neither (39) nor (40) in steady state. We conclude that both the divergent \( \mu_t \) and a convergent \( \mu_{t+1}/\mu_t \) can coexist and be consistent with the FOCs (28)-(30) in the Ramsey steady state.
Using (39), we have

\[
\frac{\mu_{t+1}}{\mu_t} = \frac{\sum_{\theta^{t+1}} \left( \frac{c_{t+1}(\theta^{t+1})}{c_t} \right) c_t(\theta^t) - \alpha \pi_t(\theta^t) + (1 - \alpha) M_t/\beta R}{\sum_{\theta^t} \left( \frac{c_t(\theta^t)}{c_t} \right) c_t(\theta^t) - \alpha \pi_t(\theta^t) + (1 - \alpha) M_t/\beta R}.
\]

Since both \( (\beta^t \prod_{s=1}^t R_s) \sum_{\theta^{t+1}} \left( \frac{c_{t+1}(\theta^{t+1})}{c_t} \right) c_t(\theta^t) - \alpha \pi_t(\theta^t) \) and \( (\beta^t \prod_{s=1}^t R_s) \sum_{\theta^t} \left( \frac{c_t(\theta^t)}{c_t} \right) c_t(\theta^t) - \alpha \pi_t(\theta^t) \) equal zero in the limit, we obtain

\[
\lim_{t \to \infty} \frac{\mu_{t+1}}{\mu_t} = \frac{1}{\beta R} \lim_{t \to \infty} \frac{M_{t+1}}{M_t},
\]

where \( \lim_{t \to \infty} \frac{M_{t+1}}{M_t} \) is a constant in steady state. The FOC (30) in steady state then yields

\[
1 = (1/R) \lim_{t \to \infty} \frac{M_{t+1}}{M_t} [F_K(K, L) - \delta + 1],
\]

which fails to satisfy the MGR, unless \( \lim_{t \to \infty} \frac{M_{t+1}}{M_t} = \beta R < 1 \). However, the result of \( \lim_{t \to \infty} \frac{M_{t+1}}{M_t} < 1 \) implies that \( M_t \) itself goes to zero in the limit, which contradicts our presumption that Condition X fails to hold. Given that \( \lim_{t \to \infty} \frac{M_{t+1}}{M_t} \geq 1 \) must hold, we obtain from (2) and (41)

\[
R = [1 + (1 - \tau_k)(r - \delta)] \geq [F_K(K, L) - \delta + 1],
\]

which shows that \( \tau_k \leq 0 \) in the Ramsey steady state.