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Investment and Bilateral Insurance*

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Abstract

Private information may limit insurance possibilities when two agents get together to pool idiosyncratic risk. However, if there is capital accumulation, bilateral insurance possibilities may improve because misreporting distorts investment. We show that if one of the Pareto weights is sufficiently large, that agent does not have incentives to misreport. This implies that, under some conditions, the full information allocation is incentive compatible when agents have equal Pareto weights. In the long run, either one of the agents goes to immiseration, or both agents' lifetime utilities are approximately equal. The second case is only possible with capital accumulation.

JEL Classifications: D82, D86, G32

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1 Introduction

Private information limits insurance possibilities when two agents get together to pool idiosyncratic risk. We show that if their total income depends on capital accumulation and production, bilateral insurance possibilities improve because misreporting distorts investment. This occurs because cheating implies that the misrepresenting agent will receive higher utility today at the expense of reducing the other agent's current consumption and of reducing investment. The latter reduces both agents' continuation values, and it therefore provides incentives to prevent cheating. This mechanism is more significant when the agent's Pareto weight is larger, as he cares more about the total level of capital in the economy. In the long run, either one of the agents is driven to immiseration, or both agents' lifetime utilities are approximately equal. We show that the second case is only possible in economies with capital accumulation.

We study the optimal bilateral insurance arrangement between two risk-averse agents facing preference shocks.[1] We compare two alternative setups: (i) Agents share the ownership of production and investment (Capital Accumulation Economy, CAE), and (ii) Total income is constant and exogenous (Endowment Economy, EE). The optimal contract maximizes the weighted sum of the agents' lifetime utility. Ideally, each agent's consumption should depend on his preference shock. However, if shocks are private information, the optimal arrangement must also deal with incentives to misreport. Thus, private information may limit insurance capabilities.

To understand the role of capital accumulation for incentives under private information, we show that there is a clear difference between the CAE and EE. In the EE, incentives to cheat exist because reporting a high-preference shock increases consumption of the reporting agent at the expense of reducing consumption of the other agent. Thus, unless promises about future consumption are modified as a function of the report, agents always want to report the highest value of the preference shock. In contrast, in the CAE, if an agent chooses to cheat, she will consume more this period at the cost (at least partially) of reducing investment, which will reduce her future consumption. We show that this force is more important for providing incentives for agents with larger Pareto weights, as they more strongly internalize the effect on future available resources. Thus, in addition to incorporating capital accumulation, it is crucial to consider a small number of agents. To the best of our knowledge, this result was not previously demonstrated or understood, because past work looked at only a continuum of agents or an endowment economy.[2]

[1] As in Diamond and Dybvig (1983) and Atkeson and Lucas (1992).

[2] A notable exception is Marcet and Marimon (1992). The key difference is that in their work one of the agents has linear preferences and deep pockets. As we show in the online appendix this assumption eliminates the disinvestment mechanism studied here.

We made two additional assumptions that help simplify the analysis: Agents are ex-ante identical, and the preference shock can be either high or low. Under these conditions we derive three key results as summarized by Propositions 1, 2, and 3. First, Proposition 1 shows that in the CAE an agent has no incentives to cheat under the full information allocation when his weight is above a threshold smaller than one. To understand this result, note that if an agent's weight is equal to one, all the extra funds she receives after cheating are only "financed" by disinvestment because the consumption of the other agent is already zero. She also fully internalizes the effect of the investment distortion because given that her weight is one, she will receive all future output. Thus, she would be strictly worse off misrepresenting her preference shock when her weight is exactly one. Now, as a small change in the weight changes consumption and investment only slightly (by continuity of the full information allocation), cheating will not be desirable even if the weight were slightly smaller than one. Although this result demonstrates that at some point the incentives to cheat vanish completely, the numerical solution of the model suggests a more general interpretation. In the CAE, there is a force created by the behavior of investment in the full information allocation that helps provide incentives for truthful revelation, and this force is increasing in the value of the agent's Pareto weight. In sharp contrast, Proposition 1 also shows that in the EE, if both agents have positive Pareto weights, their incentive compatibility constraints must be binding.

The next two propositions extend the analysis of the CAE. Proposition 2 shows that, under certain conditions, both agents have no incentives to cheat under the full information allocation when Pareto weights are equal across agents. In particular, the additional assumption is that the spread between the low and the high value of the preference shock is sufficiently large. To understand the result, consider the extreme case in which consumption is not valued if the low preference shock is realized. In this case, agents with the low value of the preference shock would be strictly worse off misrepresenting their shock; they would obtain extra consumption today, when it is not valued, at the expense of lower consumption tomorrow (because investment decreases), when it is valued. Similarly, if their valuation of consumption in the low state is close to zero and each agent weight is sufficiently high, the same forces are present and there are no incentives to cheat; i.e., the full information plan is incentive compatible for both agents.

Proposition 3 shows that in the long run private information becomes irrelevant and the agents' weights remain unchanged in the CAE. This may happen because either one of the agents has her Pareto weight equal to 0, or the weights are approximately equal. This proposition hinges on the existence of a region of weights such that both incentive compatibility constraints are not binding. This result is important because it implies that the immiseration result, which has been widely studied in private information problems, does not hold. More generally, when the condition in Proposition 2 is not satisfied, for instance, because the preference shock is less coarse, the main takeaway is that due to the incentives provided by capital accumulation, the economy spends more time in the surroundings of equal weights than in an EE.

This paper is related to a long standing literature studying private information problems. Previous work has made at least one of three assumptions: (i) resources available are not affected by the agents' decisions, as in the case of the endowment economy (Thomas and Worral, 1990); (ii) there is a continuum of agents (Atkeson and Lucas, 1992); and (iii) one of the agents is risk neutral with deep pockets (Marcet and Marimon, 1992; Clementi and Hopenhayn, 2006).[3] We find that these assumptions are not innocuous. When we relax these assumptions and consider a production economy with a finite number of risk-averse agents, we find that investment provides incentives to prevent cheating, in particular for agents with large Pareto weights.

The paper is organized as follows. Section 2 describes the model and shows how the problem can be represented recursively using Pareto weights and capital as state variables. Section 3 briefly characterizes the optimal allocation under full information. Sections 4 and 5 study how capital accumulation shapes the incentives to cheat and the long-run dynamics, respectively. Section 6 concludes. The recursive formulation of the problem is provided in Appendix A, the rest of the proofs are provided in Appendices B and C, and additional results are gathered in the online appendix.

2 Model

Time is discrete and the time horizon is infinite. At date 0, two agents, indexed by $i = 1, 2$, start operating a decreasing returns to scale technology that delivers a profits function $f(K)$ with $f'(K) >$ and $f''(K) < 0$. They start with capital $K_0 = K_{0,1} + K_{0,2}$, contributed by agents 1 and 2, respectively.[4] Capital depreciates at the rate $\delta \in (0, 1)$. Given technological assumptions, there exists some $\overline{K}$ such that $X = [0, \overline{K}]$ denotes the sustainable levels of capital.[5]

Agents face liquidity shocks as in Diamond and Dybvig (1983).[6] At the beginning of date t, agent i privately observes his shock $s_{i,t} \in S_{i,t} = \{s_L, s_H\}$, with $s_H > s_L$; denote by $S_i^t = \Pi_{h=0}^t S_{i,h}$, $S_t = S_{1,t} \times S_{2,t}$ and $S^t = \Pi_{j=0}^t S_t$.

[3]See also Green (1987), Spear and Srivastava (1987), Quadrini (2004), Clementi, Cooley, and Giannatale (2010), and Cole, Greenwood, and Sanchez (2012).

[4]This paper studies neither the formation nor the break-up of the contract. The formation could be determined by Nash bargaining between agents who would split the benefits of getting together (raising capital and providing insurance). To allow for break-ups, the model could be extended by adding some type of lack of commitment, along the lines of Wang (2011) or Amador, Werning, Hopenhayn, and Aguiar (2015).

[5]All the analyses regarding the solution method also applies to the general case in which there is an arbitrary number of agents I. Without loss of generality we assume that agents are symmetric. In the online appendix we argue that obtain similar results are obtained if agents are asymmetric.

[6]This class of preferences is standard in the literature; see Tirole (2005), Chapter 12. Moreover, liquidity shocks are multiplicative as in Atkeson and Lucas (1992).

The preference shocks are i.i.d. across time and agents, and $\pi(s_{i,t}) > 0$ denotes the probability of $s_{i,t}$. Let $s_t = (s_{1,t}, s_{2,t}) \in S_t$ be the joint shock at date t with probability $\pi(s_t) = \pi(s_{1,t}) \times \pi(s_{2,t})$; s_{-i} denotes a liquidity shock that excludes agent i's element (e.g., $s_{-1} = s_2$) and $s^t = (s_0, ..., s_t) \in S^t$ denotes a partial history of events from date 0 to date t. The probability at date 0 of a partial history s^t is $\pi(s^t) = \pi(s_0)...\pi(s_t)$.

Given a consumption path $\{C_{i,t}\}_{t=0}^{\infty}$ such that $C_{i,t} : S^t \to \mathbb{R}_+$, agent i's preferences are represented by

$$E\left\{\sum_{t=0}^{\infty} \beta^t \; s_{i,t} \; u(C_{i,t}(s^t))\right\} = \sum_{t=0}^{\infty}\sum_{s^t} \beta^t \pi(s^t) \; s_{i,t} \; u(C_{i,t}(s^t)),$$

where $u : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing, strictly concave, and twice differentiable; $\lim_{c\to 0} u'(c) = +\infty$; and $\beta \in (0,1)$. A higher value of the agent's liquidity needs implies that he is willing to take more resources to consume more today compared with the future.

Let $K' = \{K_{t+1}\}_{t=0}^{\infty}$ be an *investment plan* that every period allocates next-period capital, given a history of joint reports (i.e., $K_{t+1} : S^t \to \mathbb{R}_+$ for all t), and initial capital K_0. Similarly, let $C = \{(C_{1,t}, C_{2,t})\}_{t=0}^{\infty}$ be a *distribution plan*.

A plan (C, K') satisfying these properties is a *sequential plan*, and it is feasible if

$$K_{t+1}(s^t) + \sum_{i=1}^{2} C_{i,t}(s^t) \leq f(K_t(s^{t-1})) + (1-\delta)K_t(s^{t-1}) \tag{1}$$

for all t and all s^t. Feasibility means that part of the output is reinvested in the economy, $K_{t+1}(s^t) - (1-\delta)K_t(s^{t-1})$, and part is distributed for consumption, $C_{1,t}(s^t) + C_{2,t}(s^t)$. Importantly, note that there is no external finance.[7]

Given a sequential plan (C, K'), agent i's utility at date t given the partial history s^t is

$$U_{i,t}(C, K' \| s^{t-1}) = \sum_{j=0}^{\infty} \beta^j \sum_{(s_t,...,s_{t+j})} \pi(s_t, ..., s_{t+j}) \; s_{i,t+j} \; u(C_{i,t+j}(s^{t-1}, s_t, ..., s_{t+j})).$$

As liquidity shock realizations are privately observed, any mechanism for allocating investment and consumption must be incentive compatible.[8] A sequential plan (C, K') is *incentive compatible* if no agent has incentives to misreport his liquidity shocks, so truth-telling is the

[7] It is important for our results that there is no external finance. See more on the role of external finance in the online appendix.

[8] This restriction is without loss of generality since the Revelation Principle holds and allows us to restrict attention to mechanisms that rely on truthful reports of these shocks.

best response; i.e., for each i,

$$\sum_{s_{-i}} \pi(s_{-i}) \left(s_i \; u(C_{i,t}(s^{t-1}, s_i, s_{-i})) + \beta \; U_{i,t+1}(C, K' \| (s^{t-1}, s_i, s_{-i}))\right) \tag{2}$$
$$\geq \sum_{s_{-i}} \pi(s_{-i}) \left(s_i \; u(C_{i,t}(s^{t-1}, \widetilde{s}_i, s_{-i})) + \beta \; U_{i,t+1}(C, K' \| (s^{t-1}, \widetilde{s}_i, s_{-i}))\right)$$

for all $t \geq 0$, all s^{t-1}, and all $(s_i, \widetilde{s}_i)$.[9]

To solve for the constrained efficient arrangement, we consider a fictitious planner who chooses among plans that are incentive compatible and feasible. Let $\Psi^*(k)$ be the utility possibility correspondence defined as the levels of utility of the agents that can be attained by a corresponding plan that is incentive compatible and feasible at initial capital k,

$$\Psi^*(k) \equiv \big\{ w \in \mathbb{R}^2_+ : \exists \; (C, K') \text{ satisfying } (1) - (2)$$
$$\text{and } w_i \leq U_i(C, K') \; \forall \; i, \;\; K_0 = k \big\}.$$

As Ψ^* is a continuous, compact-valued, and convex correspondence, the set of constrained efficient plans can be parameterized by Welfare weights $(\theta_1, \theta_2) \in \mathbb{R}^2_+$ (Espino, 2005). We say that (C^*, K'^*) is constrained efficient if it is the corresponding plan sustaining the levels of lifetime utility that solves

$$h^*(k, \theta) = \max_{w \, \in \, \Psi^*(k)} (\theta_1 w_1 + \theta_2 w_2) \tag{3}$$

for some $(\theta_1, \theta_2) \in \mathbb{R}^2_+$. That is, $h^*(k, \theta)$ captures the Pareto frontier of the utility possibility correspondence Ψ^* by solving for the highest level of weighted lifetime utility given weights (θ_1, θ_2). Importantly, it can be shown that $w \in \Psi^*(k)$ if and only if[10]

$$\min_{\theta' \in \Delta} \left[h^*(k, \theta) - \sum_{i=1}^{2} \theta_i w_i \right] \geq 0.$$

Recursive formulation

It is convenient to write this problem recursively. Our recursive representation of the problem adapts the method developed by Spear and Srivastava (1987) and Abreu, Pearce, and Stacchetti (1990). We characterize the constrained efficient frontier by giving a Pareto weight to

[9]We restrict to temporary incentive compatibility. A more general concept of Bayesian implementation can be shown as equivalent in this particular dynamic environment. See Espino (2005).

[10]See Lemma 4 and its corresponding remark in Appendix A.

each agent.[11] These weights, together with capital, become endogenous state variables that summarize the history. In a nutshell, our approach can be interpreted as a combination of Abreu, Pearce, and Stacchetti (1990) and Marcet and Marimon (1994)'s Lagrangean method.[12]

In Appendix A we provide an algorithm capable of finding the value function h^* (and its corresponding policy functions) and argue that, for all $(k, \theta) \in X \times \Delta$, h^* satisfies[13]

$$h^*(k, \theta) = \max_{(c, w', k')} \sum_{i=1}^{2} \theta_i \left\{ \sum_s \pi(s) \left[s_i \ u(c_i(s)) + \beta \ w_i'(s) \right] \right\}, \tag{4}$$

subject to

$$k'(s) + \sum_{i=1}^{2} c_i(s) = f(k) + (1 - \delta) k \tag{5}$$

$$\begin{aligned} &\sum_{s_{-i}} \pi(s_{-i}) \left(s_i u(c_i(s_i, s_{-i})) + \beta \ w_i'(s_i, s_{-i}) \right) \\ &\geq \sum_{s_{-i}} \pi(s_{-i}) \left(s_i u(c_i(\widetilde{s}_i, s_{-i})) + \beta \ w_i'(\widetilde{s}_i, s_{-i}) \right) \end{aligned} \tag{6}$$

for all $(s_i, \widetilde{s}_i)$ and

$$c_i(s) \geq 0, \qquad w_i'(s) \geq 0 \qquad \text{for all } s \text{ and all } i, \tag{7}$$

$$\min_{\theta'(s) \in \Delta} \left[h^*(k'(s), \theta'(s)) - \sum_{i=1}^{2} \theta_i'(s) \ w_i'(s) \right] \geq 0 \text{ for all } s. \tag{8}$$

Note that the optimization problem takes as given (k, θ) to distribute output between current consumption to the agents and investment, and it assigns continuation utility levels. The optimization problem defined in condition (8) characterizes the set of continuation utility levels attainable at (k', θ').[14] The values of θ' that attain the minimum in (8) at (k, θ) for state s, denoted by $\theta'(k, \theta; s)$, are the next-period weights that are consistent with the entitlement of continuation utilities.

Note that in this setup we implicitly have a promised keeping constraint as in the usual alternative formulation. On the one hand, necessary and sufficient first-order conditions of

[11] The idea of substituting utility levels with Pareto weights is borrowed from Lucas and Stokey (1984).

[12] As in Marcet and Marimon (1994), we sidestep the requirement that future utilities must lie in the utility correspondence next period by mapping utility levels into Pareto weights. See also Mele (2014), Messner, Pavoni, and Sleet (2012), and Beker and Espino (2013).

[13] Hereafter, we restrict the welfare weights to add up to 1; that is, $\Delta \equiv \{\theta \in \mathbb{R}_+^2 : \theta_1 + \theta_2 = 1\}$. This restriction is innocuous because solutions are homogeneous of degree 0 with respect to (θ_1, θ_2).

[14] The same condition was used for the same purpose by Lucas and Stokey (1984), equation (5.7), and more recently by Beker and Espino (2011), equation (15).

problem (8) imply that policy functions $(c(k, \theta; s), k'(k, \theta; s), w'(k, \theta; s), \theta'(k, \theta; s))$ must satisfy

$$\frac{\partial h^*(k'(k, \theta; s), \theta'(k, \theta; s))}{\partial \theta_i} = w_i'(k, \theta; s). \tag{9}$$

On the other hand, the envelope theorem applied to (4) evaluated at $(k'(k, \theta; s), \theta'(k, \theta; s))$ implies that these policy functions must also satisfy

$$\frac{\partial h^*(k'(k, \theta; s), \theta'(k, \theta; s))}{\partial \theta_i} = \sum_{s'} \pi(s') \left[s_i' u(c_i(k'(k, \theta; s), \theta'(k, \theta; s); s')) + \beta w_i'(k'(k, \theta; s), \theta'(k, \theta; s); s') \right]. \tag{10}$$

Coupled together, equations (9) and (10) imply the utility promised for tomorrow is kept; i.e.,

$$w_i'(k, \theta; s) = \sum_{s'} \pi(s') \left[s_i' u(c_i(k'(k, \theta; s), \theta'(k, \theta; s); s')) + \beta w_i'(k'(k, \theta; s), \theta'(k, \theta; s); s') \right]. \tag{11}$$

It is well-known that there is an issue regarding renegotiation-proofness in dynamic contracts, even in simpler settings (see Wang, 2000). However, in our setting with investment, capital can be manipulated to make sure that constrained, efficient plans are always renegotiation-proof.[15]

While the representation above focused on the economy with Capital Accumulation Economy (CAE), it can be simplified to represent the Endowment Economy (EE) as well. In that case, the constraint (5) is replaced by $\sum_{i=1}^{2} c_i(s) = y$, where y is the endowment. We assume a constant endowment y so that the only state variable for the value function h is θ. We study how private information has different implications for a CAE and an EE.

3 Insurance under Full Information

Consider the CAE with full information (i.e., liquidity shocks are perfectly observable and incentive constraints are not imposed). Because $\theta_1 = 1 - \theta_2$ and the model is symmetric between both agents, hereafter we refer to agent 1's weight directly as θ. The proofs of the results in this section are provided in Appendix B.

Lemma 1 characterizes the main features of agents' consumption and investment. First, efficiency dictates that welfare weights are kept constant. This result is especially interesting because it will contrast with the behavior of welfare weights under private information. To

[15] Details are available upon request. We thank an anonymous referee about the need to mention this property.

grasp the intuition of this result, think about a fictitious planner who wants to distribute utility across agents optimally. The valuations of delivering one more unit of utility to agents 1 and 2 are θ_1 and θ_2, respectively. On the other hand, the valuations of delivering one more unit of continuation utility to agents 1 and 2 at s are $\beta\pi(s)\theta_1$ and $\beta\pi(s)\theta_2$, respectively. Consequently, the relative valuation remains unchanged at θ_1/θ_2 and this implies that the normalized weights must satisfy $\theta'(s) = \theta$. This reasoning makes evident the difference with the case under private information. There, continuation utilities are additionally manipulated to provide incentives, so their valuations can differ.

Lemma 1 also describes how consumption depends on welfare weights and liquidity-needs shocks. Part of the increase in an agent's payout after reporting high liquidity needs is financed by means of disinvestment; i.e., next period capital is smaller when an agent reports high liquidity needs than when she reports low liquidity needs.

Lemma 1 (Full Information)**.** *Under full information:*

1. *Welfare weights do not change; i.e., $\theta'(k, \theta; s) = \theta$ for all s, all k and all θ.*
2. *If $u(c) = c^{1-\sigma}/(1-\sigma)$, the optimal investment and distribution policy of consumption satisfy:*
   (a) *The fraction of total consumption that is paid to agent* i *is increasing in his liquidity needs, decreasing in the other agent's liquidity needs, and increasing in his weight.*
   (b) *The level of investment is decreasing in the agent's needs of liquidity; i.e.,*

$$k'(k, \theta; s_L, s_2) > k'(k, \theta; s_H, s_2),$$
$$k'(k, \theta; s_1, s_L) > k'(k, \theta; s_1, s_H).$$

4 Incentives to cheat and the role of capital accumulation

This section studies insurance with privately observed liquidity shocks in two alternative setups: EE and CAE. Appendix C contains all the proofs in this section.

4.1 Individual incentives to cheat

We show that in the EE the incentive constraint is binding for Pareto weights $\theta \in (0, 1)$. In contrast, in the CAE, when one agent's welfare weight is relatively large, her corresponding incentive compatibility constraint becomes slack. Because incentives to cheat disappear when Pareto weights are big enough, we call this result "too-big-to-cheat." Next, we discuss the intuition of the driving mechanism.

Proposition 1 (Too-big-to-cheat)**.** *Incentives to cheat are different in the EE and CAE. In particular:*

1. *In the EE, the incentive compatibility constraints of agents 1 and 2 are binding for all* $\theta \in (0,1)$.

2. *In the CAE, given a value of k, there exists some value of the agent 1's welfare weight* $\overline{\theta}(k) \in [0,1)$ *such that the agent 1's incentive compatibility constraint does not bind for all* (k,θ) *with* $\theta \in \left[\overline{\theta}(k),1\right]$. *Similarly, the agent 2's incentive compatibility constraint does not bind for all* (k,θ) *with* $\theta \in \left[0,1-\overline{\theta}(k)\right]$.

In the EE, the full information plan violates the incentive compatibility constraints for both agents for all $\theta \in (0,1)$. To understand this result, consider the incentive compatibility constraint under the full information allocation in the endowment economy. Continuation utilities are independent of the reports about preference shocks, and therefore the incentive compatibility constraint only depends on consumption. Consumption is strictly increasing in the preference shock for all $\theta \in (0,1)$. As a result, the incentive compatibility constraint is always violated. At $\theta = 0$ and $\theta = 1$ the agent is indifferent between cheating or not cheating because future utility and consumption are independent of the report. In contrast, in the CAE the full information allocation dictates that investment, and as a consequence, continuation utilities depend on the reports. Hence, at $\theta = 0$ and $\theta = 1$, the agent would be strictly worse off by cheating.[16]

The underlying intuition for the result of the CAE can be grasped as follows. Cheating implies that the agent misrepresenting high liquidity needs will receive higher consumption. The resources for that extra consumption are obtained from two sources: (i) decreasing consumption of the other agent and (ii) reducing investment. Note that this higher consumption is not necessarily beneficial for this agent. It would be beneficial if it is financed with a reduction in the other agent's consumption, but it may not be beneficial if it is financed with a reduction in investment, because it implies that future output will be lower. The magnitude of the second force depends (and is increasing) in the value of the Pareto weight, which represents how much of future output will be consumed by the cheater.

If one agent's weight is equal to one, all the extra funds he receives after cheating are only financed by disinvestment (the decline in investment described by Lemma (1)) because the

[16]Technically, this explains why a continuity argument can be used to show Proposition 1 in the production economy but not in the endowment economy.

consumption of the other agent is already zero. He fully internalizes the effect of the investment distortion because given that the Pareto weight is one, he will own all future output. Therefore, when the agent's Pareto weight is equal to one, he would be *strictly* worse off misrepresenting his liquidity needs. Now, since small changes in the Pareto weight change consumption and investment only slightly (by continuity), cheating will not be desirable even if the Pareto weight were slightly smaller than 1.

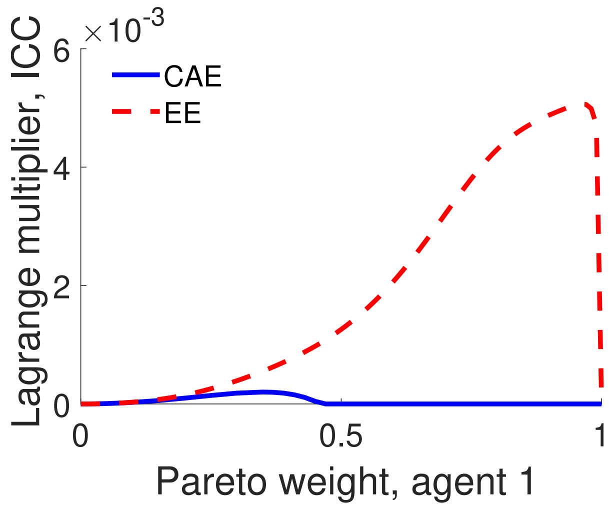
Note that the mechanism described above is quite general. For instance, if the preference shock takes finitely many values, the same logic would apply. The incentive compatibility constraints will remain strictly slack at $\theta = 1$, and the same result could be shown. Even in the extreme case, in which there is a continuous state space for the liquidity-needs shock, we argue that there are less incentives to misrepresent the shock once we allow for capital accumulation. To see this, we imagine an agent is offered the full information plan and compare the reported liquidity-needs shock both in the EE and in the CAE. First, in the EE, the agent would always have incentives to report the highest liquidity shock as consumption is increasing in the report, independently of his Pareto weight. In contrast, in the CAE, for $\theta = 1$ the same logic implies that the full information plan is actually incentive compatible because any increment in consumption must be financed with disinvestment. As a consequence, for a Pareto weight close to 1, by continuity of the allocation on the Pareto weight, the only misreport that can be desirable is a marginal one, which is in clear contrast with the EE.

Although Proposition 1 demonstrates that at some point the incentives to cheat vanish completely (the multiplier of the incentive compatibility constraint is zero), the numerical solution of the model presented in Figure 1, which shows agent 1's incentive compatibility constraint multipliers in the EE and the CAE, suggests a more general interpretation.[17] In the CAE, in contrast to the EE, there is a force created by the behavior of investment in the full information allocation that helps provide incentives for truthful revelation, and this force is increasing in the value of the agent's Pareto weights. This extra force toward truth-telling is illustrated in Figure 1, which shows that the multiplier is increasing in the EE until the Pareto weight is equal to 1, and it is non-monotone in the CAE. Below, in Figure 2, we compare these two economies in terms of the cost of private information.

Notably, this result does not depend on the assumption that capital is used for production. Under certain conditions, it also holds in a simpler case in which capital accumulation is replaced

[17]For all figures we use numerical results derived with the following assumptions. The utility function is $\frac{c^{(1-\sigma)}}{1-\sigma}$, the profit function is $f(K) = k^{\alpha}$ with $0 < \alpha < 1$, and the shocks are $s_L = 1 - \epsilon$ and $s_H = 1 + \epsilon$. The parameter $\sigma = 0.5$, as in other studies of private information such as those by Hopenhayn and Nicolini (1997) and Pavoni (2007). The parameter $\alpha = 0.7$ is in the range of estimations of Cooper and Ejarque (2003). We set $\delta = 0.07$ and $\beta = 0.97$ as is standard in the literature. We consider $\epsilon = 0.6$ just for illustrative purposes.

Figure 1: Incentive Compatibility Constraint's multipliers



by storage.[18] In particular, we have also shown the result in Proposition 1 in a simple two-period model with storage in which only one agent faces preference shocks in the first period of her life.[19] The key mechanism providing insurance is that the amount of resources saved in storage to be consumed in the second period are reduced if the agent reports a high-preference shock in the first period.

4.2 Too-big-to-cheat region

In this section, we show that in the CAE there exist examples of parameter and Pareto weights combinations for which the full information allocation is *strictly incentive compatible*: Both incentive compatibility constraints are slack. This would happen if the intersection of the too-big-to-cheat zone for agent 1 and agent 2 is non empty. We label this intersection as the "too-big-to-cheat region." As the economy is symmetric, this will be a symmetric ball around $\theta = 1/2$, and the existence and the radius of this ball is characterized by the preferences and technology of the economy. Of course, this is impossible in the EE because as Proposition 1 shows, the incentive compatibility constraints are binding for all $\theta \in (0, 1)$.[20]

The following proposition analyzes the role of one of the parameters that describes the characteristics of the model: the size of the dispersion in the values of liquidity-needs shocks.

[18]In a model with storage there is an additional constraint that storage cannot be negative. For the results to hold, we need that this additional constraint be not binding. Instead, with production this condition is immediately satisfied as production function satisfies the standard Inada conditions.

[19]Results available upon request.

[20]In the EE an agent with $\theta = 1$ would be indifferent between reporting a low or high shock, while he would be strictly worse off misreporting in the CAE.

In order to do that, we parameterize liquidity shocks so that $s_L = 1 - \epsilon$ and $s_H = 1 + \epsilon$ for both agents.

Proposition 2 (Existence of "too-big-to-cheat" region)**.** *There exists some $\epsilon^* \in (0, 1)$ such that for all $\epsilon \in (\epsilon^*, 1)$ the full information plan is strictly incentive compatible at $\theta = 1/2$ for all k.*

Proposition 2 states that both agents have no incentives to cheat with $\theta = 1/2$ if the relative value of the preference shocks is sufficiently large.[21] To understand the result, consider the case of $\epsilon = 1$, which implies that consumption is not valued if the low preference shock is realized; i.e., $s_L = 0$. In this case, agents with the low value of the preference shock would be *strictly* worse off misrepresenting their shock; they would obtain extra consumption today, when it is not valued, at the expense of lower consumption tomorrow (because investment decreases), when it is valued. The proposition shows that if ϵ is close to 1 and each agent weight is sufficiently high, the same forces are present and there are no incentives to cheat. Thus, for $\theta = 1/2$, there exists an $\epsilon^* < 1$ such that for all $\epsilon \in (\epsilon^*, 1)$ the full information plan is incentive compatible for both agents.

While Proposition 2 uses the difference between high and low values of the shock, similar forces are at work when other parameters change. For instance, while increasing ϵ helps to provide incentives because the agent assigns a very low value for consumption today, increasing β is similar because it increases the weight that agents assign to the distortion on investment. In the following section we show in a quantitative example that for a lower value of β the full information plan is not strictly incentive compatible at $\theta = 0.5$.

The existence of the too-big-to-cheat region, however, hinges on the assumption of the coarseness of the preference shock. If the preference shock could take more values, for instance $\{s_L, s_M, s_H\}$ with s_M close to s_H, the incentive constraint could still bind for s_M even if s_L were sufficiently close to 0.

4.3 Welfare costs of private information

Figure 2 illustrates the welfare cost of private information in the CAE and EE.[22] The left figure of the first row shows the result of Proposition 2. For large values of ϵ the welfare costs of private information around $\theta = 0.5$ are equal to zero, while for small values of ϵ the costs are positive for all $\theta \in (0, 1)$.

[21] Note that the same result holds if the shocks are $s_L = 1 - \epsilon$ and $s_H = 1$.

[22] We compute, in consumption-equivalent units, the cost of moving from private to full information. To facilitate the comparison in the examples below, the level of resources in the economy without capital accumulation are set at the output produced with the mean of the steady-state level of capital in the economy with capital accumulation.

The quantitative examples derive two main differences between CAE and EE. In the EE we find that (i) the maximum welfare gains are about 0.2 percent in terms of consumption-equivalent units,[23] and (ii) the maximum value of welfare gains occurs when both agents have relatively large weights (but not necessarily equal weights). In sharp contrast, in the production economy we find that (i) welfare gains are one order of magnitude smaller, and (ii) these gains are actually zero when both agents have the same weight and ϵ is large enough. In general, the last result means that equal weights minimize the cost of private information in a production economy.

Note that the extra welfare losses of private information on the EE as compared with the CAE are not simply because the planner has access to one more margin of adjustment. In particular, consider the cases in which the too-big-to-cheat region exists around $\theta = 0.5$. Recall that the difference between welfare losses in the EE and the CAE are the largest for $\theta = 0.5$. At that point, the allocations under full and private information coincide in the CAE. Thus, at that point, the planer does not "adjust" the extra margin (capital accumulation) to provide incentives. Therefore, it is the inclusion of investment—not access to one more margin of distortion—that drives this result. For regions in which the full and private information allocations do not coincide in the CAE, both the investment force and the additional margin of distortion are at work.[24]

The second and third rows of Figure 2 show the effects of other preference parameters on the welfare costs of private information. The second row repeats the exercise for different values of risk aversion. As risk aversion increases—higher σ—the welfare costs in the CAE economy also increase for all $\theta \in (0, 1)$. Interestingly, the region in which the welfare costs are equal to zero seems to be independent of σ. In the endowment economy, the welfare costs are non-monotone in risk aversion. For θ sufficiently close to zero or one, higher risk aversion implies larger costs, while for θ close to 0.5 the opposite is true.

The comparison between increasing risk (larger ϵ) and increasing risk aversion (larger σ) may be puzzling because although they look like similar changes, we obtain the opposite result. However, note that a larger ϵ facilitates the provision of incentives and reduces the cost of private information because with a larger ϵ, the full information economy implies a larger spread of consumption across state. Reducing risk aversion (smaller σ) also increases the dispersion of consumption across states under full information. This explains why similar results are obtained increasing ϵ and reducing σ.

[23] Note that this number is slightly more than 20 times higher than the gains from eliminating business cycles estimated by Lucas (1987) and is similar to those found by Krusell and Smith (1999) in an economy with large heterogeneity.

[24] Section D.2 in the Appendix discusses how investment is distorted under private information to provide incentives.

The third row shows the effect of the discount factor, β. A version of the folk theorem holds in this environment: It is harder to provide incentives when the discount factor diminishes. As a result, the welfare costs of private information are larger for low values of the discount factor, both in the CAE and in the EE. Importantly, in the CAE, the too-big-to-cheat region disappears when β is sufficiently low.

Finally, we also investigated the effects of production parameters in the CAE on the welfare costs of private information. For example, larger depreciation rates reduce the too-big-to-cheat region and increase the costs of private information for all θ. In contrast, in the EE, the costs of private information seem to be independent of the level of the endowment.

5 Capital accumulation and long-run dynamics

The previous section shows that having access to capital accumulation improves the degree of bilateral insurance. We now investigate the implications on long-run dynamics.

5.1 Convergence when the too-big-to-cheat region exists

The following result characterizes the long-run implications of private information with capital accumulation in this case. Let $k_{min}(\theta_t)$ and $k_{max}(\theta_t)$ be the lower and upper bounds for capital, respectively. They would be obtained under full information if the high shock, in the case of $k_{min}(\theta_t)$, and the low shock, in the case of $k_{max}(\theta_t)$, are realized forever.

Proposition 3. *Suppose that the full information plan is strictly incentive compatible at $\theta = 1/2$ for all k. Then, there exists $\theta^* \in (0, 1/2)$ such that*

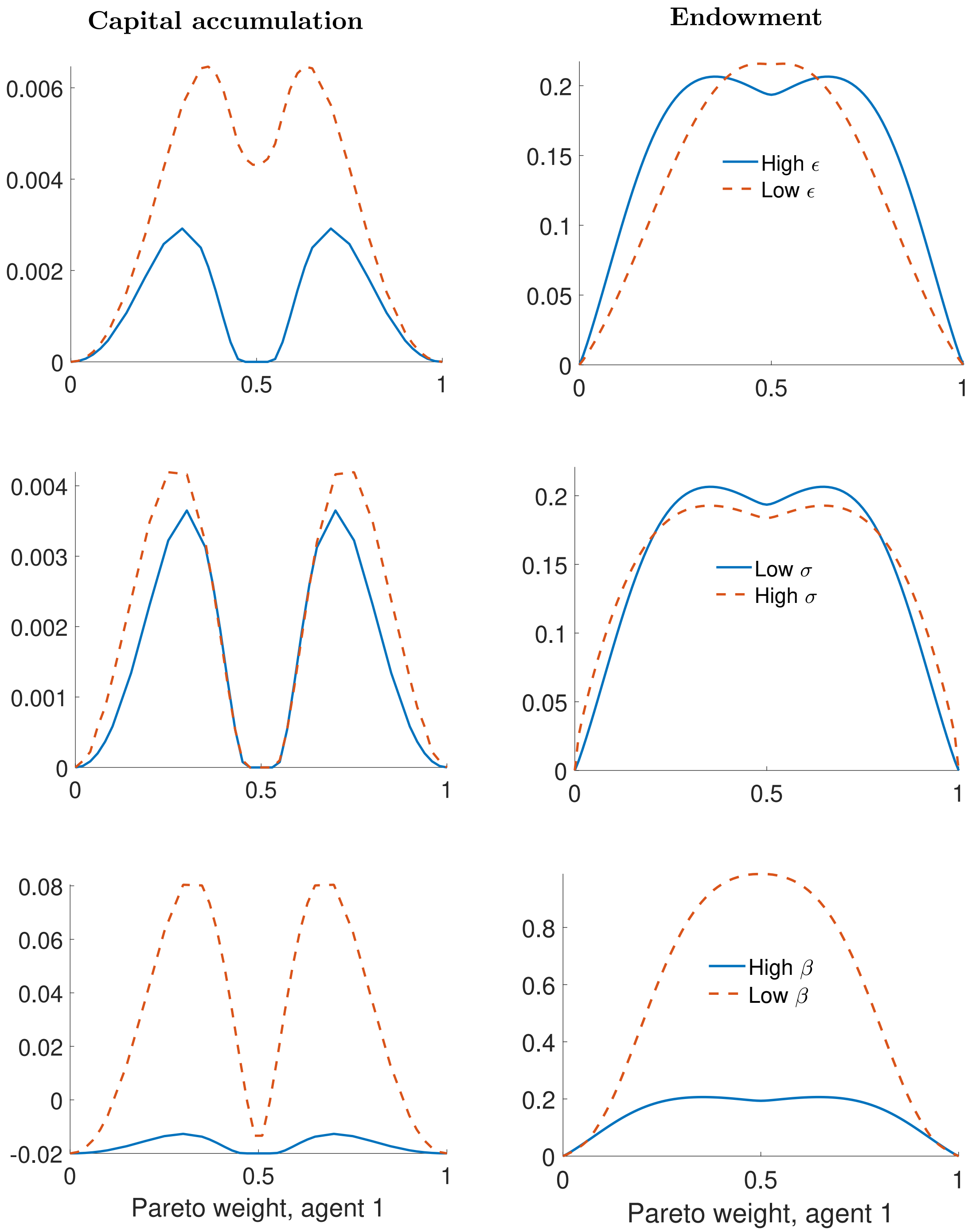
1. *If $(k_t, \theta_t) \in [k_{\min}(\theta_t), k_{\max}(\theta_t)] \times [\theta^*, 1-\theta^*]$ at some t, then the economy, (k_t, θ_t), stays in a region in which the constrained efficient plan and the full information plan coincide and $\theta_{t+n} = \theta_t$ for all $n \geq 0$.*

2. *If $(k_t, \theta_t) \in [k_{\min}(\theta_t), k_{\max}(\theta_t)] \times [0, \theta^*]$ at some t, the economy, (k_t, θ_t), reaches a region in which the constrained efficient plan and the full information plan coincide with probability 1; i.e., $\theta_t \to \{0, [\theta^*, 1-\theta^*]\}$ a.s.*

3. *If $(k_t, \theta_t) \in [k_{\min}(\theta_t), k_{\max}(\theta_t)] \times [1-\theta^*, 1]$ at some t, the economy, (k_t, θ_t), reaches a region in which the constrained efficient plan and the full information plan coincide with probability 1; i.e., $\theta_t \to \{[\theta^*, 1-\theta^*], 1\}$ a.s.*

As long as the full information plan is strictly incentive compatible at $\theta = 1/2$ for all k, Proposition 3 states that in the long run private information becomes irrelevant and the Pareto

Figure 2: **Welfare gains of moving from private to full information**



Note: Panels show the welfare gains of moving from private information to full information, measured in consumption-equivalent units. The left column shows the CAE and the right column shows the EE. The first row solves for ϵ equal to 0.4 and 0.6. The second row solves for σ equal to 0.5 and 0.6. The third row solves for β equal to 0.95 and 0.75.

weights remain unchanged. This may happen because: (i) one of the agents' Pareto weight is equal to one, or (ii) both Pareto weights are approximately equal to 1/2. Notice that the too-big-to-cheat region defined in Proposition 2 can be reached either immediately (as the initial weights and the initial capital stock starts there) or in the long run (as the weights and the capital stock converge as time and uncertainty unfold).[25]

The fact that in the long run private information does not matter resembles previous results. As in Thomas and Worral (1990), the simplified argument is the following: imagine Pareto weights converged and private information still matters. Then future Pareto weights must be spread to provide incentives, contradicting the initial statement that Pareto weights converged. What is new in our setup is that this may happen in a region in which both agents have positive weights. As we mentioned before, this is possible due to the inclusion of capital accumulation.

Finally, we provide some numerical examples to highlight the implications of Proposition 3. We considered different initial Pareto weights and simulated 1,000 economies until the weights converged. The left panel of Figure 3 considers the case in which the too-big-to-cheat region exists. This economy can converge either to the too-big-to-cheat region (TBTC), or immiseration of agent 1 or 2 (IM1, and IM2, respectively). To understand the results, imagine that the initial weight is drawn from a uniform distribution in $[0, 1]$. The figure shows that, ex-ante, the economy converges with roughly 50% probability to the TBTC region, with 25% probability to IM1, and with 25% probability to IM2. Hence, although the threshold θ^* is close to 0.5 (in this case, $\theta^* = 0.47$), the long-run implications are very stark. Moreover, if the initial draw of θ_0 is centered around $\theta = 0.5$ instead of uniformly distributed, the probability of convergence to the TBTC region increases. Recall that this result is not present in an endowment economy that converges to immiseration of one of the agents with probability one.

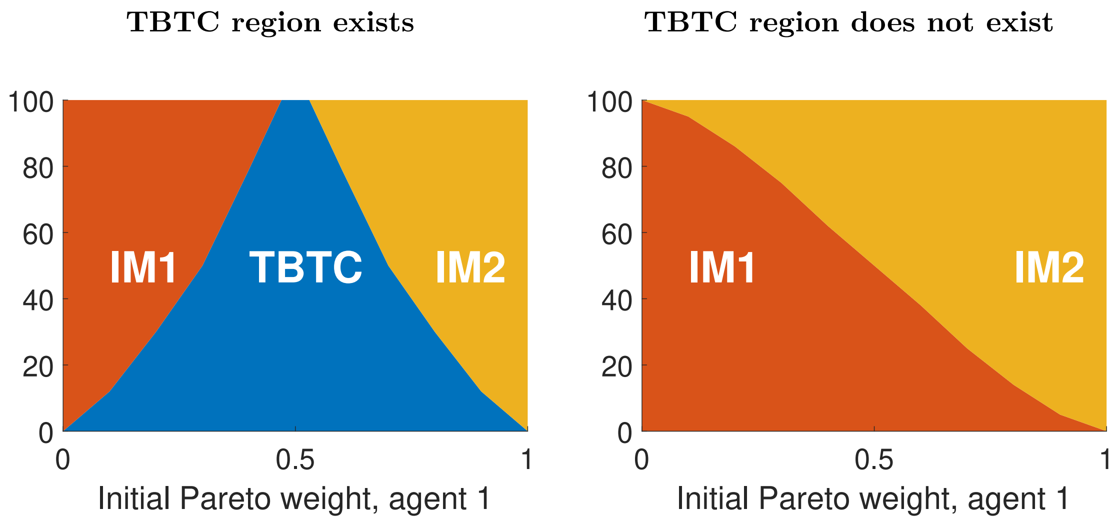
If the full information plan is not strictly incentive compatible at $\theta = 1/2$, then the TBTC does not exist and the economy converges to immiseration of either agent 1 or 2 (right panel of Figure 3). However, in the next section we show that capital accumulation is also important in this case for the dynamics of Pareto weights.

5.2 Dynamics when the too-big-to-cheat region does not exist

Proposition 3 hinges on the existence of a region in the space of capital and Pareto weights such that both incentive compatibility constraints are not binding. (Proposition 2 shows this case exists for some configuration of the preference shock.) This result is important because it implies that the immiseration result, which has been widely studied in private information

[25]Note that Proposition 3 assumes that k_t starts in the ergodic set for capital under full information, $[k_{min}(\theta_t), k_{max}(\theta_t)]$. We performed numerical exercises with very small or very large k_t and found that under private information, capital converges to this interval in about 65 periods.

Figure 3: **Long-run convergence**



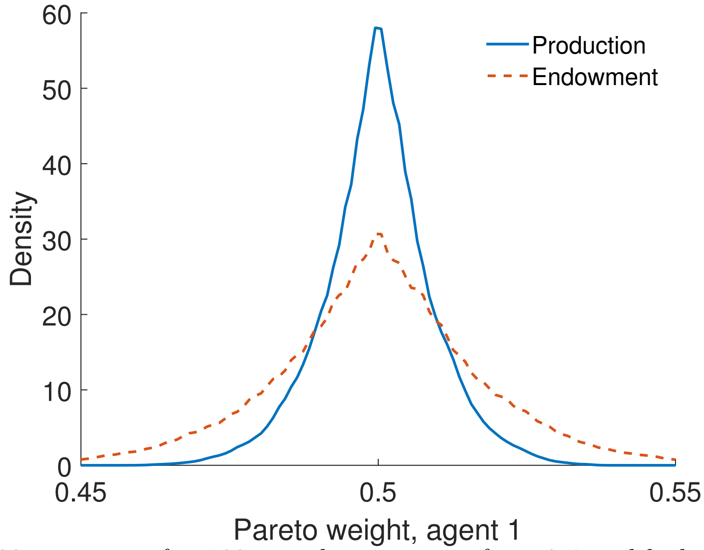
Note: We simulated N=1,000 economies of length M = 1,000,000, and kept the last t=1,000 periods. We defined too-big-to-cheat (TBTC) as $E[\theta] \in [0.45, 0.55]$, immiseration of agent 1 (IM1) as ($E[\theta] \leq 0.1$), and immiseration of agent 2 (IM2) as ($E[\theta] \geq 0.9$).

problems, does not hold when there is capital accumulation. More generally, when the condition in Proposition 2 is not satisfied (for instance because the preference shock is less coarse), the takeaway should be that due to the incentives provided by capital accumulation, the economy will spend more time in the surroundings of equal weights than in an endowment economy.

To show this, we simulated both the production and the endowment economy 1,000 times for 100 periods starting at $\theta = 0.5$ for the case in which $\epsilon < \epsilon^*$ and the full information plan is not incentive compatible at $\theta = 0.5$. Recall that under full information weights do not change, so θ would be constant at 0.5 forever. Figure 4 shows the resulting distribution of θ over those 100 periods for both the endowment and the production economy under private information. The production economy (solid blue line) spends more time close to $\theta = 0.5$ than the endowment economy (red dashed line). This result confirms that the main mechanism highlighted in this paper—that capital accumulation mitigates the role of private information—is at work even when the technical condition imposed in Proposition 2 is not satisfied.

Additionally, we simulate 1,000 times the CAE and the EE economies to compare the expected time to "almost immiseration" (defined as $\theta < 0.1$ or $\theta > 0.9$). The CAE converges 2.68 times slower than the EE to this threshold. This result highlights that the dynamics observed in Figure 4 are not only present during the initial periods but also in the long run.

Figure 4: **Evolution of Pareto weights**



Note: We simulated 1,000 economies for 100 periods starting at $\theta_0 = 0.5$ *and looked at the distribution of* θ *under both the production and the endowment economy. We consider* $\epsilon = 0.3$ *such that the too-big-too-cheat region does not exist at* $\theta = 0.5$.

6 Conclusion

Previous studies show that private information is important for determining the extent to which agents can get together and pool idiosyncratic risk. The analysis in this paper suggests that insurance capabilities improve as a consequence of the introduction of capital accumulation. Under certain conditions, we show that (i) when an agent's Pareto weight is sufficiently large, her incentives to cheat under the full information allocation disappear; (ii) the full information allocation may be incentive compatible when both agents have equal Pareto weights; and (iii) in the long run, either one of the agents is driven to immiseration, or both agents' lifetime utilities are approximately equal.

Throughout a series of extensions and generalizations, we have learned that there are two key ingredients needed for Proposition 1 to hold. First, the economy must have a technology to transfer resources across time. In the benchmark model, we assume capital accumulation with decreasing returns to scale. The advantage is that we can solve the infinite horizon model and challenge the celebrated immiseration result. In a supplemental note, we consider a model with a linear storage technology and show that a result similar to Proposition 1 also holds in that environment. The second ingredient is that marginal utilities of consumption are private information. As is standard in the literature, e.g., Atkeson and Lucas (1992), we assume that agents face privately observed liquidity needs. In a supplemental note, we show that a similar result to Proposition 1 holds if agents have privately observed endowment shocks, as in Thomas and Worral (1990). What matters is that the shocks affect the marginal utility of consumption. When $\theta = 1$, private and aggregate marginal costs of operating the saving technology coincide and there are no incentives to cheat.

One possible application of our theory is to small-business partnerships given that we consider a small number of owners, shared ownership, production possibilities, liquidity shocks, and internal financing. Thus, we can derive the following predictions for the organization of these businesses: Equal shares of ownership facilitates the provision of incentives, ownership shares change more frequently when firm ownership is unequally distributed, and the distribution of ownership shares moves over time toward either equal distribution of ownership or sole-proprietorship. We argue in Espino, Kowloski, and Sanchez (2016) that these findings can be found in the data.

Our theory can also be applied to different settings. For instance, a partnership could be reinterpreted as an economic union among several countries. Then, the size of the countries (in terms of how much wealth they have relative to the union) would be important for determining the extent to which misreporting must be prevented by the union's structure and regulations. In an economic union between a large and a small country, our results suggest that the small country would have incentives to misreport if the union regulations are not carefully designed. Moreover, our theory predicts that adding more countries to the union—and thereby reducing each member's share—would exacerbate information problems.

References

Abreu, D., D. Pearce, and E. Stacchetti (1990): "Toward a Theory of Discounted Repeated Games with Imperfect Monitoring," *Econometrica*, 58(5), 1041–1064.

Amador, M., I. Werning, H. A. Hopenhayn, and M. Aguiar (2015): "Equilibrium Default," 2015 Meeting Papers 1539, Society for Economic Dynamics.

Atkeson, A., and R. E. J. Lucas (1992): "On Efficient Distribution with Private Information," *Review of Economics Studies*, 59(3), 427–453.

Beker, P. F., and E. Espino (2011): "The dynamics of efficient asset trading with heterogeneous beliefs," *Journal of Economic Theory*, 146(1), 189–229.

Beker, P. F., and E. Espino (2013): "Short-Term Momentum and Long-Term Reversal in General Equilibrium," Working Paper, University of Warwick.

Clementi, G. L., T. Cooley, and S. D. Giannatale (2010): "A Theory of Firm Decline," *Review of Economic Dynamics*, 13(4), 861–885.

Clementi, G. L., and H. A. Hopenhayn (2006): "A Theory of Financing Constraints and Firm Dynamics," *The Quarterly Journal of Economics*, 121(1), 229–265.

Cole, H. L., J. Greenwood, and J. M. Sanchez (2012): "Why doesn't technology flow from rich to poor countries?" Federal Reserve Bank of St. Louis Working Papers 2012-040.

Cooper, R., and J. Ejarque (2003): "Financial Frictions and Investment: Requiem in Q," *Review of Economic Dynamics*, 6(4), 710–728.

Diamond, D. W., and P. H. Dybvig (1983): "Bank Runs, Deposit Insurance, and Liquidity," *Journal of Political Economy*, 91(3), 401–19.

Espino, E. (2005): "On Ramsey's conjecture: efficient allocations in the neoclassical growth model with private information," *Journal of Economic Theory*, 121(2), 192–213.

Espino, E., J. Kowloski, and J. M. Sanchez (2016): "Stylized Facts on the Organization of Partnerships," Federal Reserve Bank of St. Louis Review.

Green, E. J. (1987): *Lending and the Smoothing of Uninsurable Income. In Contractual arrangements for intertemporal trade. Edited by Edward C.Prescott and Neil Wallace.* University of Minnesota Press, Minnesota Studies in Macroeconomics series, vol. 1.

Hopenhayn, H. A., and J.-P. Nicolini (1997): "Optimal Unemployment Insurance," *Journal of Political Economy*, 105(2), 412–438.

Krusell, P., and A. A. Smith (1999): "On the Welfare Effects of Eliminating Business Cycles," *Review of Economic Dynamics*, 2(1), 245 – 272.

Lucas, R. (1987): *Models of Business Cycles.* Oxford: Basil Blackwell.

Lucas, R. E., and N. L. Stokey (1984): "Optimal Growth with Many Consumers," *Journal of Economic Theory*, 32(1), 139–171.

Marcet, A., and R. Marimon (1992): "Communication, Commitment, and Growth," *Journal of Economic Theory*, 58, 219–249.

——— (1994): "Recursive contracts," Department of Economics and Business, Universitat Pompeu Fabra, Economics Working Papers.

Mele, A. (2014): "Repeated Moral Hazard and Recursive Lagrangeans," *Journal of Economic Dynamics and Control*, 42, 69–85.

Messner, M., N. Pavoni, and C. Sleet (2012): "Recursive Methods for Incentive Problems," *Review of Economic Dynamics*, 15(4), 501–525.

Pavoni, N. (2007): "On the Optimal Unemployment Compensation," *Journal of Monetary Economics*, 54(6), 1612–1630.

Quadrini, V. (2004): "Investment and liquidation in renegotiation-proof contracts with moral hazard," *Journal of Monetary Economics*, 51(4), 713–751.

Rockafellar, R. (1970): *Convex Analysis*, Princeton mathematical series. Princeton University Press.

Spear, S., and S. Srivastava (1987): "On Repeated Moral Hazard with Discounting," *Review of Economic Studies*, 53, 599–617.

Thomas, J., and T. Worral (1990): "Income Fluctuations and Asymmetric Information: An Example of Repeated Principal-Agent Problem," *Journal of Economic Theory*, 51(2), 367–390.

Tirole, J. (2005): *The Theory of Corporate Finance*. Princeton University Press.

Wang, C. (2000): "Renegotiation-Proof Dynamic Contracts with Private Information," *Review of Economic Dynamics*, 3(3), 396–422.

——— (2011): "Termination of dynamic contracts in an equilibrium labor market model," *Journal of Economic Theory*, 146(1), 74–110.

A Recursive formulation

In this Appendix we show how to write the problem recursively. Our analysis here generalizes to I agents with privately observed shocks to liquidity needs because our alternative recursive approach does not depend on our 2-agent assumption. Abusing our notation, we denote $s \in \{s_L, s_H\}^I$ and $(s_i, s_{-i}) \in \{s_L, s_H\} \times \{s_L, s_H\}^{I-1}$.

Let $\Delta^I \equiv \{\theta \in \mathbb{R}_+^I : \sum_{i=1}^I \theta_i = 1\}$ and $\|h\| = \sup_{(k,\theta)} \left\{| \ h(k,\theta) \ |: \theta \in \Delta^I\right\}$, define

$$F \equiv \{h : X \times \mathbb{R}_+^I \to \mathbb{R}_+ : \ h \text{ is continuous and } \ \|h\| < \infty\}$$

as the set of continuous and bounded functions mapping $X \times \mathbb{R}_+^I$ into $\mathbb{R}_+$, and denote

$$\overline{F} \equiv \{h \in F : h \text{ is HOD 1 and concave in } k\}$$

as the subset of functions that are homogeneous of degree 1 and concave.

Given the metric induced by $\|.\|$, observe that $\left(\overline{F}, \|.\|\right)$ is a closed subset of the Banach space $(F, \|.\|)$ and thus a Banach space itself. Define the operator T defined on $\overline{F}$ as follows

$$(Th)(k,\theta) = \sup_{(c,w',k')} \sum_{i=1}^{I} \theta_i \left\{ \sum_s \pi(s) \left[s_i \ u(c_i(s)) + \beta \ w_i'(s) \right] \right\}, \tag{12}$$

subject to

$$k'(s) + \sum_{i=1}^{I} c_i(s) = f(k) + (1-\delta)k \tag{13}$$

$$\begin{aligned} &\sum_{s_{-i}} \pi(s_{-i}) \left(s_i u(c_i(s_i, s_{-i})) + \beta \ w_i'(s_i, s_{-i}) \right) \\ &\geq \sum_{s_{-i}} \pi(s_{-i}) \left(s_i u(c_i(\widetilde{s}_i, s_{-i})) + \beta \ w_i'(\widetilde{s}_i, s_{-i}) \right) \end{aligned} \tag{14}$$

for all $(s_i, \widetilde{s}_i)$ and

$$c_i(s) \geq 0, \qquad w_i'(s) \geq 0 \qquad \text{for all } s \text{ and all } i, \tag{15}$$

$$h(k'(s), \theta') \geq \sum_{i=1}^{I} \theta_i'(s) \ w_i'(s) \qquad \text{for all } \theta' \text{and } s. \tag{16}$$

In what follows, we say that a sequential plan (C, K') is generated by the set of policy functions $(\widehat{c}_i(k,\theta;s), \widehat{k}'(k,\theta;s), \widehat{\theta}'(k,\theta;s))$ solving (4)-(8) as

$$C_i(s^t) = \widehat{c}_i(K(s^{t-1}), \theta(s^{t-1}); s_t) \tag{17}$$

$$\theta(s^{t-1}, s_t) = \widehat{\theta}'(K(s^{t-1}), \theta(s^{t-1}); s_t)$$
$$K(s^{t-1}, s_t) = \widehat{k}'(K(s^{t-1}), \theta(s^{t-1}); s_t)$$

for all t and all $s^t \in S^t$, given θ_0 and K_0.

Now define

$$\Psi(k)(h) \equiv \big\{w \in \mathbf{R}_+^I : \exists(c, k', w') \text{ such that (13)-(16) are satisfied}$$
$$\text{and } w_i = \sum_s \pi(s)[s_i u(c_i(s)) + \beta w_i'(s)]\big\}.$$

Given $h \in \overline{F}$, let $\mathcal{W}(k)(h)$ denote the constraint correspondence defined by (13)-(16) at $k \in X$. Any $(c, w', k') \in \mathcal{W}(k)(h)$ will be referred to as a *feasible, incentive-compatible recursive plan with respect to* h. We say that $h \in \overline{F}$ is *preserved under* T if $h(k,\theta) \leq (Th)(k,\theta)$ for all (k,θ). Importantly, notice that it is straightforward to check that

$$(Th)(k,\theta) = \sup_{w \in \Psi(k)(h)} \sum_{i=1}^{I} \theta_i w_i.$$

The following result establishes that the correspondence $\Psi(.)(h)$ is well behaved.[26]

Lemma 2. *$\Psi(.)(h)$ is a continuous compact-valued correspondence for all $h \in \overline{F}$.*

It follows that the sup in the operator T is attained. In the next lemma, we establish the convexity of $\Psi(k)(h)$, a property that is key to our approach.

Lemma 3. *$\Psi(k)(h)$ is convex for all $k \in X$ and all $h \in \overline{F}$.*

Proof. Let w and $\widetilde{w} \in \Psi(k)(h)$ as $(c, w', k'), (\widetilde{c}, \widetilde{w}', \widetilde{k}') \in \mathcal{W}(k)(h)$ are the corresponding feasible, incentive-compatible recursive plans with respect to h.

We need to show that $w^\lambda = \lambda w + (1-\lambda)\widetilde{w} \in \Psi(k)(h)$ for any $\lambda \in [0,1]$. In order to do that, define for each i and all s

$$u(c_i^\lambda(s)) = \lambda u(c_i(s)) + (1-\lambda)u(\widetilde{c}_i(s))$$

$$k'^\lambda(s) = \lambda k'(s) + (1-\lambda)\widetilde{k}'(s))$$

$$w'^\lambda(s) = \lambda w'(s) + (1-\lambda)\widetilde{w}'(s)).$$

Notice that the strict concavity of u implies that $c_i^\lambda(s) \leq \lambda c_i(s) + (1-\lambda)\widetilde{c}_i(s)$ for all i, all s.

[26]The proof is omitted as it follows standard arguments. Details are available upon request.

Step 1. Notice that by construction, it follows that

$$w_i^\lambda = \lambda w + (1-\lambda)\widetilde{w} = \sum_s \pi(s)[s_i u(c_i^\lambda(s)) + \beta\ w_i'^\lambda(s)]$$

for all i.

Step 2. Feasibility. As (c, w', k') and $(\widetilde{c}, \widetilde{w}', \widetilde{k}')$ are both feasible and $c_i^\lambda(s) \leq \lambda c_i(s) + (1-\lambda)\widetilde{c}_i(s)$ for all i, all s as mentioned, it follows immediately that

$$k'^\lambda(s) + \sum_{i=1}^{I} c_i^\lambda(s) \leq f(k) + (1-\delta)k$$

for all s, and so $(c^\lambda, w'^\lambda, k'^\lambda)$ is also feasible.

Step 3. Incentive Compatibility. As the liquidity shocks are multiplicative, it follows by the linear construction that $(c^\lambda(s), w'^\lambda(s), k'^\lambda(s))$ satisfies (14).

Step 4. Take any $\theta' \in \Delta^I$ and notice that since $h \in \overline{F}$ is concave in k, it follows that

$$\begin{aligned} h(k'^\lambda, \theta') &\geq \lambda h(k', \theta') + (1-\lambda) h(\widetilde{k}', \theta') \\ \geq \lambda \sum_{i=1}^{I} \theta_i'(s)\ w_i'(s) + (1-\lambda)\sum_{i=1}^{I}\theta_i'(s)\ \widetilde{w}_i'(s) &= \sum_{i=1}^{I}\theta_i'(s)\ w_i'^\lambda(s). \end{aligned}$$

Because $\theta' \in \Delta^I$ is arbitrary, condition (16) is satisfied. Therefore, we can conclude that $(c^\lambda, w'^\lambda, k'^\lambda)$ is a feasible incentive-compatible recursive plan with respect to h. $\square$

The next result is useful to characterize convex sets and used to make our alternative approach computationally simpler (see also Lucas and Stokey, 1984).

Lemma 4. *For any $h \in \overline{F}$, $w \in \Psi(k)(h)$* if and only if $w \geq 0$

$$Th(k,\theta) \geq \sum_{i=1}^{I} \theta_i\ w_i \qquad \textit{for all } \theta \in \Delta^I. \tag{18}$$

Proof. See Rockafellar (1970), Theorem 13.1. $\square$

Remark. For computational purposes, it is convenient to recall that condition (18) holds if and only if

$$\min_{\widetilde{\theta} \in \Delta^I} \left[Th\left(K, \widetilde{\theta}\right) - \sum_{i=1}^{I} \widetilde{\theta}_i w_i \right] \geq 0.$$

Our method complements the APS approach (Abreu, Pearce, and Stacchetti, 1990) as it identifies attainable levels of next-period utility by iterating directly on the utility possibility frontier without having to know the utility possibility correspondence that describes the utility possibility set a priori. Our alternative approach can be summarized as follows. Following the methods developed by Abreu, Pearce, and Stacchetti (1990), one would need to construct an operator defined on correspondences and iterate on that space. Taking advantage of the convexity of our problem, we instead iterate on the (convex) frontier similar to Marcet and Marimon (1994)'s Lagrangean method.

The next result below is similar in spirit to APS's celebrated self-generation.

Lemma 5 (Self-generating)**.** *If $h \in \overline{F}$ is preserved under T, then*

$$(Th)\,(k,\theta) \le h^*(k,\theta)$$

for all (k,θ).

Proof. Now, take any arbitrary $\left(\widehat{c}_0(s_0), \widehat{w}_0'(s_0), \widehat{k}_1'(s_0)\right)_{s_0\in S} \in \mathcal{W}(k_0)(h)$ and notice that this implies, in particular, that

$$\sum_{i=1}^{I} \theta_i' \; w_{i,0}'(s_0) \le h(\widehat{k}_1'(s_0), \theta') \tag{19}$$

for all $\theta' \in \Delta^I$. On the other hand, as h is preserved under T, this last condition implies that given $\widehat{k}_1'(s_0)$

$$h(\widehat{k}_1'(s_0), \theta') \le (Th)(\widehat{k}_1'(s_0), \theta') \tag{20}$$

for all $\theta' \in \Delta^I$. Hence, as we couple conditions (19) and (20), we conclude that

$$\theta' \widehat{w}_0'(s_0) \le (Th)(\widehat{k}_1'(s_0), \theta')$$

for all $\theta' \in \Delta^I$ and therefore $\widehat{w}_0'(s_0) \in \Psi(k'(s_0))(h)$ as a direct implication of Lemma 4. Importantly, this implies that there exists some $\left(\widehat{c}_1(s_0,s_1), \widehat{w}_1'(s_0,s_1), \widehat{k}_2'(s_0,s_1)\right)_{s_1\in S} \in \mathcal{W}(\widehat{k}_1'(s_0))(h)$ such that

$$\widehat{w}_0'(s_0) = \sum_{s_1} \pi(s_1)\left[s_i \; u(\widehat{c}_{i,1}(s_0,s_1)) + \beta \; \widehat{w}_{i,1}'(s_0,s_1)\right]$$

for each $s_0 \in S$.

As we repeat this strategy N times, we can conclude that for any arbitrary $\theta_0 \in \Delta^I$

$$\sum_{i=1}^{I} \theta_{i,0} \; \left\{ \sum_{s_0} \pi(s_0)\left[s_i \; u(\widehat{c}_{i,0}(s_0)) + \beta \; \widehat{w}_{i,0}'(s_0)\right] \right\}$$
$$= \sum_{i=1}^{I} \theta_{i,0} \; \left\{ \sum_{s_0} \pi(s_0) s_i \; u(\widehat{c}_{i,0}(s_0)) \right.$$

$$
\left. +\beta \sum_{s_0} \pi(s_0) \ \sum_{s_1} \pi(s_1) \left[s_i \ u(\widehat{c}_{i,1}(s_0, s_1)) + \beta \ \widehat{w}'_{i,1}(s_0, s_1) \right] \right\}
$$

$$
= \sum_{i=1}^{I} \theta_{i,0} \ E \left(\sum_{t=0}^{N} \beta^t \ s_{i,t} u(\widehat{c}_{i,t}) \right) + \beta^{N+1} \sum_{i=1}^{I} \theta_{i,0} \ E \left(\widehat{w}'_{i,N+1} \right).
$$

Condition (16) implies that

$$
\sup \sum_{i=1}^{I} \theta_{i,0} \ E \left(\widehat{w}'_{i,N+1} \right) \le \|h\| ,
$$

and so, taking limits on both sides as $N \to \infty$, it follows from the Dominated Convergence Theorem that

$$
\begin{aligned}
& \sum_{i=1}^{I} \theta_{i,0} \ \left\{ \sum_{s_0} \pi(s_0) \left[s_i \ u(\widehat{c}_{i,0}(s_0)) + \beta \ \widehat{w}'_{i,0}(s_0) \right] \right\} \\
& \le \sum_{i=1}^{I} \theta_{i,0} \ E \left(\sum_{t=0}^{\infty} \beta^t \ s_{i,t} u(\widehat{c}_{i,t}) \right)
\end{aligned} \tag{21}
$$

as $\beta \in (0, 1)$.

Consider the sequential plan $(\widehat{c}, \widehat{k}')$ stemming from above. It is immediate that this plan is sequentially feasible by construction. We now argue that it is incentive compatible as well. To see this, denote recursively $W_{i,t}(s^t) = \widehat{w}'_{i,t}(s_0, ..., s_t)$ and observe that, by construction,

$$
\begin{aligned}
& \left| U_{i,t}(\widehat{c}, \widehat{k}'; s^t) - W_{i,t}(s^t) \right| \\
& = \beta \left| \sum_{s_{t+1}} \pi(s_{t+1}) \left(U_{i,t}(\widehat{c}, \widehat{k}'; s^t, s_{t+1}) - W_{i,t+1}(s^t, s_{t+1}) \right) \right| \\
& \le \beta \sup_{s_{t+1}} \left| U_{i,t}(\widehat{c}, \widehat{k}'; s^t, s_{t+1}) - W_{i,t+1}(s^t, s_{t+1}) \right| \\
& \le \beta^k \sup_{(s_{t+1}, ... s_{t+k})} \left| U_{i,t}(\widehat{c}, \widehat{k}'; s^t, s_{t+1}, ..., s_{t+k}) - W_{i,t+k}(s^t, s_{t+1}, ..., s_{t+k}) \right| .
\end{aligned}
$$

Observe that $0 \le W_{i,t}(s^t) \le \|h\| < \infty$ for all i and all s^t while $\widehat{c}$ is uniformly bounded by construction. Taking the $\limsup$ as $k \to \infty$ for this last expression, we can conclude that $U_{i,t}(\widehat{c}, \widehat{k}')(s^t) = W_{i,t}(s^t)$ for all i and all s^t, and so sequential incentive compatibility follows immediately.

Because both $\left(\widehat{c}_0(s_0), \widehat{w}'_0(s_0), \widehat{k}'_1(s_0) \right)_{s_0} \in \mathcal{W}(k_0)(h)$ and the corresponding sequential plan $(\widehat{c}, \widehat{k})$ are arbitrary, we take the sup on both sides of (21) to conclude that

$$
\begin{aligned}
Th(k,\theta) &= \sup_{\left(\widehat{c},\widehat{w}',\widehat{k}'\right)\in\mathcal{W}(k_0)(h)} \sum_{i=1}^{I} \theta_{i,0} \ \left\{ \sum_{s_0} \pi(s_0) \left[s_i \ u(\widehat{c}_i) + \beta \ \widehat{w}_i' \right] \right\} \\
&\leq \sup \sum_{i=1}^{I} \theta_i \ E \left(\sum_{t=0}^{\infty} \beta^t \ s_{i,t} u(\widehat{c}_{i,t}) \right) \\
&= h^*(k,\theta),
\end{aligned}
$$

and this completes the proof. □

We now are prepared to prove our two main results of this section.

Proposition 4. *h^* is a fixed point of T.*

Proof. Given (K,θ), take any $w \in \Psi^*(K)$ for which (C,K') denotes the corresponding feasible incentive-compatible plan. Observe that

$$
\sum_{i=1}^{I} \theta_i \ U_i(C,K') = \sum_{i=1}^{I} \theta_i \sum_{s_0\in S_0} \pi(s_0) \ [s_{i,0} \ u(C_i(s_0)) + \beta \ U_{i,1}(C,K'\|(s_0))].
$$

Notice that $(U_{i,1}(C,K'\|(s_0)))_{i=1}^{I} \in \Psi^*(K(s_0))$ for all s_0. It follows by definition of h^* (see (3)) that

$$
h^*(K'(s_0),\theta') \geq \sum_{i=1}^{I} \theta_i' \ U_{i,1}(C,K'\|(s_0))
$$

for all $\theta' \in \Delta^I$ and all s_0. Therefore, $(C_i, U_{i,1}(C,K'),K')_{i=1}^{I} \in \mathcal{W}(K_0)(h^*)$ and then

$$
\sum_{i=1}^{I} \theta_i \ U_i(C,K') \leq (Th^*)(K,\theta).
$$

Because weak inequalities are preserved in the limit, we can conclude that

$$
h^*(K,\theta) = \sup_{(C,K')} \sum_{i=1}^{I} \theta_i U_i(C,K') \leq (Th^*)(K,\theta)
$$

for all (K,θ) (i.e., h^* is preserved under T). Thus, Lemma 5 implies that $h^*(K,\theta) = (Th^*)(K,\theta)$ for all (K,θ). □

Importantly, it can be shown that the following version of the Principle of Optimality holds.[27]

[27] Proof available upon request.

Remark 1. *A plan* (C^*, K'^*) *is constrained efficient at* K_0 *if and only if it is generated by the set of policy functions*

$$\begin{aligned} \widehat{C}_i(s^t) &= \widehat{c}_i(\widehat{K}(s^{t-1}), \theta(s^{t-1}); s_t) \\ \theta(s^{t-1}, s_t) &= \widehat{\theta}'(\widehat{K}(s^{t-1}), \theta(s^{t-1}); s_t) \\ \widehat{K}(s^{t-1}, s_t) &= \widehat{k}'(\widehat{K}(s^{t-1}), \theta(s^{t-1}); s_t). \end{aligned} \tag{22}$$

Thus, the value of any plan that can be attained with an incentive-compatible, feasible sequential plan (C, K') can also be attained by splitting output between total current payouts and investment and then by delivering current payouts and contingent future ownership shares to each agent.

We now provide an algorithm capable of finding the value function h^* and its corresponding policy functions.

Let $\widehat{T}$ be the operator solving the recursive problem for the full information case, discussed in Section 3 (i.e., the incentive compatibility constraints (14) are ignored), and h^{**} be the corresponding value function such that $h^{**} = \widehat{T}h^{**}$. Evidently, $Tf \leq \widehat{T}f$ for all $f \in \overline{F}$ and $h^*(k, \theta) \leq h^{**}(k, \theta)$ for all (k, θ).

Proposition 5. *Let* $h_0 = h^{**}$ *and denote* $h_n = T^n\left(h^{**}\right)$. *Then,* $\{h_n\}$ *is a monotone decreasing sequence of continuous functions and* $\lim_{n\to\infty} h_n = h^*$ *uniformly.*

Proof. It is a routine exercise to show that T is a monotone operator (i.e., if $f \geq g$, then $Tf \geq Tg$). This property and Proposition 4 imply that $h^* = Th^* \leq Th^{**} \leq \widehat{T}h^{**} = h^{**}$.

Step 1. Since $h_n = T^n h^{**}$, then monotonicity implies that $h_n \geq h_{n+1} \geq h^*$ for all n.

Step 2. T preserves concavity with respect to k. In addition, it can be easily checked that $\mathcal{W}(k)(h)$ is a continuous correspondence as $f \in \overline{F}$, a standard application of the Theorem of the Maximum makes it possible to conclude that $T : \overline{F} \to \overline{F}$; i.e., h_n is continuous for all n.

Step 3. As $\{h_n\}$ is a monotone decreasing sequence of uniformly bounded continuous functions, we can conclude that there exists h_∞ such that $h_n \to h_\infty \geq h^*$ pointwise.

Step 4. We claim that h_∞ is preserved under T.

Given (k, θ), $h_\infty(k, \theta) \leq h_n(k, \theta) \leq h_m(k, \theta)$ for all $n \geq m$. Fix m and notice that this implies that there exists $(c^n, w'^n, k'^n) \in \mathcal{W}(k)(h_n) \in \mathcal{W}(k)(h_m)$ such that for all n

$$h_\infty(k, \theta) \leq h_{n+1}(k, \theta) = \sum_{i=1}^{I} \theta_i \sum_s \pi(s) \left[u(c_i^n(s)) + \beta\ w_i'^n(s)\right] \tag{23}$$

and

$$h_m(k'^n(s), \theta') \geq h_n(k'^n(s), \theta') \geq \sum_{i=1}^{I} \theta'_i \; w_i'^n(s), \quad \text{for all } s \text{ and all } \theta'. \tag{24}$$

Because (c^n, w'^n, k'^n) lies in a compact set, it has a convergent subsequence with limit point $\left(\widehat{c}, \widehat{w}', \widehat{k}'\right)$. Suppose for notational simplicity that the convergent subsequence is the sequence itself and notice that both feasibility and incentive compatibility are preserved in the limit. Given m, because h_m is continuous, condition (24) implies that

$$\lim_{n\to\infty} h_m(k'^n(s), \theta') = h_m(\widehat{k}'(s), \theta') \geq \sum_{i=1}^{I} \theta'_i \; \widehat{w}'_i(s), \quad \text{for all } s \text{ and all } \theta'. \tag{25}$$

In addition, as h_m converges pointwise to h_∞ it follows that

$$\lim_{m\to\infty} h_m(\widehat{k}'(s), \theta') = h_\infty(\widehat{k}'(s), \theta') \geq \sum_{i=1}^{I} \theta'_i \; \widehat{w}'_i(s), \quad \text{for all } s \text{ and all } \theta'.$$

Therefore, $\left(\widehat{c}, \widehat{w}', \widehat{k}'\right) \in \mathcal{W}(k)(h_\infty)$ and condition (23) implies that in the limit

$$h_\infty(k, \theta) \leq \sum_{i=1}^{I} \theta_i \sum_s \pi(s) \left[u(\widehat{c}_i(s)) + \beta \; \widehat{w}'_i(s)\right].$$

Finally, because the recursive allocation plan $\left(\widehat{c}, \widehat{w}', \widehat{k}'\right) \in \mathcal{W}(k)(h_\infty)$ is arbitrary, we can conclude that for all (k, θ)

$$\begin{aligned}\left(T h_\infty\right)(k, \theta) &\geq \sum_{i=1}^{I} \theta_i \sum_s \pi(s) \left[u(\widehat{c}_i(s)) + \beta \; \widehat{w}'_i(s)\right] \\ &\geq h_\infty(k, \theta),\end{aligned}$$

and thus h_∞ is preserved under T by definition.

Step 5. This implies that $h_\infty \leq h^*$ due to Lemma 5 and therefore $h_\infty = h^*$.

Step 6. Finally, as the limit function h^* is continuous, Dini's theorem implies that $h_n \to h_\infty$ uniformly.[28]

$\square$

[28] Rudin, Walter R. (1976) Principles of Mathematical Analysis, Third Edition, McGraw-Hill. See Theorem 7.13 on page 150 for the monotone decreasing case.

B Full Information - Proofs

This Appendix provides the proof of Lemma 1.

Proof of Lemma 1. 1. To show this result, we use the first-order condition with respect to w'. To save space, this condition can be found in equations (39)-(42) below by setting the multipliers of the incentive compatibility constraint, ϕ, equal to zero. As shown there, because $\theta_2 = 1 - \theta_1$, we can derive equations 43-46, which, given that all ϕ's are equal to zero, imply that $\theta'(s) = \theta$.

2. In addition:

   (a) The necessary and sufficient first-order conditions of the full information problem imply

$$c_1(s) = \frac{(\theta_1 s_1)^{1/\sigma}}{(\theta_1 s_1)^{1/\sigma} + (\theta_2 s_2)^{1/\sigma}} \left(f(k) + (1-\delta)k - k'(s)\right) \tag{26}$$

$$c_2(s) = \frac{(\theta_2 s_2)^{1/\sigma}}{(\theta_1 s_1)^{1/\sigma} + (\theta_2 s_2)^{1/\sigma}} \left(f(k) + (1-\delta)k - k'(s)\right). \tag{27}$$

   Hence the payout of agent i is increasing in his liquidity needs, decreasing in the other agent's liquidity needs, and increasing in his ownership shares.

   (b) Under full information the problem reduces to choosing total payouts and investment in a sole proprietorship with an "aggregate" investor with preferences

$$S(\theta, s)\frac{C^{1-\sigma}}{1-\sigma},$$

   where $C = c_1 + c_2$ denotes total payouts and

$$S(\theta, s) = \left((\theta_1 s_1)^{1/\sigma} + (\theta_2 s_2)^{1/\sigma}\right)^{\sigma}$$

   is an "aggregate" liquidity shock. The problem reduces to

$$h^{**}(k, \theta) = \max_{c,k'} \left\{ \sum_s \pi(s) \left[S(\theta, s) u(C(s)) + \beta h^{**}(k'(s), \theta)\right] \right\},$$

   subject to

$$k'(s) + C(s) = f(k) + (1-\delta) k\ .$$

The Euler equation is

$$S(\theta,s)\,C(s)^{-\sigma}=\beta\left(f'(k'(s))+1-\delta\right)\sum_{s'}S(\theta,s')\,C(s')^{-\sigma}.$$

We can write the Euler equation as

$$S(\theta,s)=\beta\frac{\left(f'(k'(s))+1-\delta\right)}{\left(f(k)+(1-\delta)k-k'(s)\right)^{-\sigma}}\sum_{s'}S(\theta,s')\,C(k'(s)\,s')^{-\sigma} \tag{28}$$

and note that the right-hand side is decreasing in k'. Hence, as $S(\theta,s)$ increases, $k'(s)$ decreases. Moreover, recall that $\theta_2=1-\theta_1$. Hence, the ratio of aggregate shocks is

$$\frac{S(\theta_1,s_H,s_2)}{S(\theta_1,s_L,s_2)}=\frac{\left((\theta_1 s_H)^{1/\sigma}+((1-\theta_1)\,s_2)^{1/\sigma}\right)^{\sigma}}{\left((\theta_1 s_L)^{1/\sigma}+((1-\theta_1)\,s_2)^{1/\sigma}\right)^{\sigma}}, \tag{29}$$

greater than one and increasing in θ_1, which concludes the proof.

□

C Private Information - Proofs

In this Appendix we prove Propositions 1, 2, and 3. First, we describe the problem and the system of equations that characterizes the solution:

$$h^*(k,\theta)=\max_{c,w',k'}\left\{\sum_{s^1\in\{L,H\}}\sum_{s^2\in\{L,H\}}\sum_{i=1}^{2}\theta_i\pi(s_1)\,\pi(s_2)\left[s_i u(c_i(s_1,s_2))+\beta w_i'(s_1,s_2)\right]\right\}, \tag{30}$$

subject to

$$k'(s_1,s_2)+\sum_{i=1}^{2}c_i(s_1,s_2)=f(k)+(1-\delta)k \tag{31}$$

$$\begin{aligned}&\pi(s_L)\left(s_L u(c_1(s_L,s_L))+\beta w_1'(s_L,s_L)\right)+\pi(s_H)\left(s_L u(c_1(s_L,s_H))+\beta w_1'(s_L,s_H)\right)\\ \geq\;&\pi(s_L)\left(s_L u(c_1(s_H,s_L))+\beta w_1'(s_H,s_L)\right)+\pi(s_H)\left(s_L u(c_1(s_H,s_H))+\beta w_1'(s_H,s_H)\right)\end{aligned} \tag{32}$$

$$\begin{aligned}&\pi(s_L)\left(s_L u(c_2(s_L,s_L))+\beta w_2'(s_L,s_L)\right)+\pi(s_H)\left(s_L u(c_2(s_H,s_L))+\beta w_2'(s_H,s_L)\right)\\ \geq\;&\pi(s_L)\left(s_L u(c_2(s_L,s_H))+\beta w_2'(s_L,s_H)\right)+\pi(s_H)\left(s_L u(c_2(s_H,s_H))+\beta w_2'(s_H,s_H)\right)\end{aligned} \tag{33}$$

$$\min_{\theta' \in \Delta} \left[h\left(k'\left(s_1, s_2\right), \theta'\left(s_1, s_2\right)\right) - \sum_{i=1}^{2} \theta_i'\left(s_1, s_2\right) w_i'\left(s_1, s_2\right) \right] \geq 0. \tag{34}$$

Let $\lambda\left(k, \theta; s_1, s_2\right)$, $\phi_1(k, \theta)$, $\phi_2(k, \theta)$, and $\mu\left(k, \theta; s_1, s_2\right)$ be the Lagrange multipliers of (31), (32), (33) and (34) respectively. To simplify the exposition, we abuse notation when possible and eliminate the dependence of policy functions on state variables (k, θ). We only explicitly incorporate this dependence when it is important for the proof.

The necessary and sufficient first-order conditions are as follows.

First, the first-order conditions with respect to consumption for agents 1 and 2, respectively, imply that for all s_1 and s_2

$$\left(\theta_1 \pi\left(s_L\right) s_L + \phi_1 s_L\right) \pi\left(s_2\right) u'\left(c_1\left(s_L, s_2\right)\right) = \lambda\left(s_L, s_2\right) \tag{35}$$

$$\left(\theta_1 \pi\left(s_H\right) s_H - \phi_1 s_L\right) \pi\left(s_2\right) u'\left(c_1\left(s_H, s_2\right)\right) = \lambda\left(s_H, s_2\right) \tag{36}$$

$$\left(\theta_2 \pi\left(s_L\right) s_L + \phi_2 s_L\right) \pi\left(s_1\right) u'\left(c_2\left(s_1, s_L\right)\right) = \lambda\left(s_1, s_L\right) \tag{37}$$

$$\left(\theta_2 \pi\left(s_H\right) s_H - \phi_2 s_L\right) \pi\left(s_1\right) u'\left(c_2\left(s_1, s_H\right)\right) = \lambda\left(s_1, s_H\right). \tag{38}$$

Second, the first-order conditions with respect to continuation utilities for agents 1 and 2, respectively, imply that for all s_1 and s_2

$$\left(\theta_1 \pi\left(s_L\right) + \phi_1\right) \pi\left(s_2\right) \beta = \mu\left(s_L, s_2\right) \theta_1'\left(s_L, s_2\right) \tag{39}$$

$$\left(\theta_1 \pi\left(s_H\right) - \phi_1\right) \pi\left(s_2\right) \beta = \mu\left(s_H, s_2\right) \theta_1'\left(s_H, s_2\right) \tag{40}$$

$$\left(\theta_2 \pi\left(s_L\right) + \phi_2\right) \pi\left(s_1\right) \beta = \mu\left(s_1, s_L\right) \theta_2'\left(s_1, s_L\right) \tag{41}$$

$$\left(\theta_2 \pi\left(s_H\right) - \phi_2\right) \pi\left(s_1\right) \beta = \mu\left(s_1, s_H\right) \theta_2'\left(s_1, s_H\right). \tag{42}$$

Finally, the first-order conditions with respect to capital for an interior solution and making use of the corresponding envelope condition deliver the Euler equation

$$\lambda\left(s_1, s_2\right) \quad = \quad \mu\left(s_1, s_2\right)\left(f'\left(k'\left(s_1, s_2\right)\right) + (1-\delta)\right) \sum_{\left(s_1', s_2'\right)} \lambda\left(k'\left(s_1, s_2\right), \theta'\left(s_1, s_2\right); s_1', s_2'\right).$$

As $\theta_2 = 1 - \theta_1$, we get from (39)-(42) that

$$\theta_1'\left(s_L, s_L\right) = \frac{\left(\theta_1 + \frac{\phi_1}{\pi(s_L)}\right)}{\left(1 + \frac{\phi_1}{\pi(s_L)} + \frac{\phi_2}{\pi(s_L)}\right)} \tag{43}$$

$$\theta_1'\left(s_L, s_H\right) = \frac{\left(\theta_1 + \frac{\phi_1}{\pi(s_L)}\right)}{\left(1 + \frac{\phi_1}{\pi(s_L)} - \frac{\phi_2}{\pi(s_H)}\right)} \tag{44}$$

$$\theta_1^{'}\left(s_H, s_L\right) = \frac{\left(\theta_1 - \frac{\phi_1}{\pi(s_H)}\right)}{\left(1 - \frac{\phi_1}{\pi(s_H)} + \frac{\phi_2}{\pi(s_L)}\right)} \tag{45}$$

$$\theta_1^{'}\left(s_H, s_H\right) = \frac{\left(\theta_1 - \frac{\phi_1}{\pi(s_H)}\right)}{\left(1 - \frac{\phi_1}{\pi(s_H)} - \frac{\phi_2}{\pi(s_H)}\right)} \ . \tag{46}$$

C.1 Proof of Proposition 1

Proof of Proposition 1. **Step 1: Endowment Economy (EE):**

By contradiction, suppose that the multiplier of the incentive constraint of agent 1 is zero, i.e., $\phi_1 = 0$, for some $\theta \in (0,1)$. It follows by the law of motion of welfare weights (see equations (43)-(46)) that θ' is independent of agent 1's shock, s_1.[29] This implies that w_1' does not depend on s_1. To see this, notice that necessary and sufficient first-order conditions in problem (34) deliver w_1',

$$\frac{\partial h}{\partial \theta_1'}\left(\theta'\left(s_1, s_2\right)\right) = w_1'\left(s_1, s_2\right). \tag{47}$$

Therefore, $w_1'\left(s_H, s_2\right) = w_1'\left(s_L, s_2\right)$ for all s_2.

Incentive compatibility for agent 1 reads

$$\sum_{s_2} \pi\left(s_2\right)\left(s_L u\left(c_1\left(s_L, s_2\right)\right) + \beta w_1'\left(s_L, s_2\right)\right) \geq \sum_{s_2} \pi\left(s_2\right)\left(s_L u\left(c_1\left(s_H, s_2\right)\right) + \beta w_1'\left(s_H, s_2\right)\right)$$

As continuation utilities are equivalent, we get that

$$\sum_{s_2} \pi\left(s_2\right) s_L \left[u\left(c_1\left(s_L, s_2\right)\right) - \left(c_1\left(s_H, s_2\right)\right)\right] \geq 0.$$

Consequently, it must hold that

$$c_1\left(s_L, s_2\right) \geq c_1\left(s_H, s_2\right) \tag{48}$$

for some s_2. Without loss of generality, suppose that (48) holds for $s_2 = s_L$.

As we evaluate FOC's for consumption with $\phi_1 = 0$, we obtain

$$\frac{u'\left(c_1\left(s_L, s_L\right)\right)}{u'\left(y - c_1\left(s_L, s_L\right)\right)} = \left(\frac{1-\theta}{\theta} + \frac{\phi_2}{\theta \pi\left(s_L\right)}\right)$$

[29] Notice that in general $\phi_i(k, \theta)$ for $i = 1, 2$; i.e., the Lagrange multipliers are functions of the state variables but not of the preference shocks, because the corresponding constraint is ex-ante.

and

$$\frac{s_H}{s_L}\frac{u'\left(c_1\left(s_H,s_L\right)\right)}{u'\left(y-c_1\left(s_H,s_L\right)\right)}=\left(\frac{1-\theta}{\theta}+\frac{\phi_2}{\theta\pi\left(s_L\right)}\right).$$

Because $\frac{s_H}{s_L}>1$, we have that

$$\frac{u'\left(c_1\left(s_H,s_L\right)\right)}{u'\left(y-c_1\left(s_H,s_L\right)\right)}<\frac{u'\left(c_1\left(s_L,s_L\right)\right)}{u'\left(y-c_1\left(s_L,s_L\right)\right)},$$

and this implies that $c_1\left(s_H,s_2\right)>c_1\left(s_L,s_2\right)$, which leads to a contradiction to (48) evaluated at $s_2=s_L$. The same contradiction is obtained if it is evaluated at $s_2=s_H$.

Step 2: Capital Accumulation Economy (CAE).

First, we show that the full information plan satisfies the agent 1's incentive compatibility constraints at $\theta=1$.

Consider the recursive problem (30)-(34) for the case in which the incentive compatibility constraints are absent, and let $(c(k,\theta;s),c_2(k,\theta;s),k'(k,\theta;s),\theta'(k,\theta;s),w_1'(k,\theta;s),w_2'(k,\theta;s))$ be the set of continuous policy functions such that $\phi_i=0$ for $i=1,2$.

Notice that in this case Lemma 1 implies that $\theta'(k,\theta;s)=\theta$ for all s and all (k,θ).

Because $\theta'(k,\theta;s)=\theta=1$, then $h(k'(s),1)=w_1'(s)$ for all (s,θ,k) and the value function reduces to

$$\begin{aligned}h(k,1)=\pi(s_L)\left[s_Lu(c_1(k,1;s_L,s_2))+\beta w_1'(k,1;s_{L,}s_2)\right]\\+\pi(s_H)\left[s_Hu(c_1(k,1;s_H,s_2))+\beta w_1'(k,1;s_H,s_2)\right]\end{aligned}$$

for all s_2. Notice that as $\theta'(k,\theta;s)=\theta=1$, then s_2 plays no allocative role. Therefore, consumption, future promised utilities, and capital accumulation are independent of s_2.

Suppose that the corresponding full information plan is not incentive compatible; i.e.,

$$s_L\ u(c_1(k,1;s_L,s_2))+\beta\ h(k'(k,1;s_L,s_2),1)<s_L\ u(c_1(k,1);s_H,s_2)+\beta\ h(k'(k,1;s_H,s_2),1) \tag{49}$$

while

$$c_1(k,1;s_L,s_2)+k'(k,1;s_L,s_2)=f\left(k\right)+(1-\delta)k$$

$$c_1(k,1;s_H,s_2)+k'(k,1;s_H,s_2)=f\left(k\right)+(1-\delta)k.$$

This implies that $(c_1(k,1;s_H,s_2),k'(k,1;s_H,s_2))$ is feasible at $s_1=s_L$ and

$$w_1'(k,1;s_H,s_2)=h(k'(k,1;s_H,s_2)).$$

This contradicts that $(c_1(k,1;s_L,s_2), k'(k,1;s_L,s_2), w_1'(k,1;s_L,s_2))$ is the unique solution at $(k,1)$.

Now, suppose that equation (49) holds with equality. We know that $(c_1(k,1;s_H,s_2), k'(k,1;s_H,s_2))$ is feasible. Moreover, by Lemma 1 $(c_1(k,1;s_H,s_2), k'(k,1;s_H,s_2)) \neq (c_1(k,1;s_L,s_2), k'(k,1;s_L,s_2))$. Hence, we find another feasible allocation that delivers the same utility, which violates the uniqueness of the maximum. Hence, at $\theta = 1$ the agent 1's incentive compatibility constraint must hold with strict inequality. This implies that there exists some $\overline{\theta}(k) < 1$ such that the agent 1's incentive compatibility constraint does not bind for $\theta \in [\overline{\theta}(k), 1]$.

It follows by symmetry that the agent 2's incentive compatibility constraint does not bind for all (θ, k) with $(1-\theta) \in \left[0, 1-\overline{\theta}(k)\right]$. □

C.2 Proof of Proposition 2

Proof of Proposition 2. Consider the solution to the full information problem evaluated at $\theta = 1/2$ and let $s_L = 1-\epsilon$ and $s_H = 1+\epsilon$. Given ϵ, consider the incentive compatibility constraint of agent 1 that needs to be satisfied to complete the proof

$$\begin{aligned} &\textstyle\sum_{s_2} \pi(s_2)\left((1-\epsilon)u(c_1(k,1/2)(1-\epsilon,s_2)) + \beta\, w_1'(k,1/2)(1-\epsilon,s_2)\right) \\ \geq &\textstyle\sum_{s_2} \pi(s_2)\left((1-\epsilon)u(c_1(k,1/2)(1+\epsilon,s_2)) + \beta\, w_1'(k,1/2)(1+\epsilon,s_2)\right). \end{aligned} \tag{50}$$

As we consider the full information plan, Lemma 1 implies that $\theta'(k,1/2;s_1,s_2) = 1/2$ for all (s_1,s_2) and all k. This implies that $w_1'(k,1/2;s_1,s_2) = w_2'(k,1/2;s_1,s_2)$ for all (s_1,s_2) and all k. Hence,

$$h(k'(k,1/2;s_1,s_2),1/2) = w_1'(k,1/2;s_1,s_2). \tag{51}$$

As h is strictly increasing in k and investment is decreasing in the liquidity shocks (Lemma 1), it follows by (51) that $w_1'(1-\epsilon,s_2) > w_1'(1+\epsilon,s_2)$ for all s_2. By the Theorem of the Maximum, policy functions can be parameterized continuously with respect to ϵ. For each k, because (50) holds with strict inequality as ϵ goes to 1, we can conclude that there exists some $\epsilon^*(k) \in (0,1)$ such that the full information plan is strictly incentive compatible for agent 1 for all $\epsilon \in (\epsilon^*(k), 1)$ at k.

Finally, standard arguments prove that $\epsilon^*(k)$ varies continuously

$$\epsilon^* \equiv \max\left\{\epsilon^*(k) : k \in [k_{\min}(1/2), k_{\max}(1/2)]\right\} \in (0,1)$$

and ϵ^* is well-defined. Therefore, the full information plan is strictly incentive compatible for agent 1 for all $\epsilon \in (\epsilon^*, 1)$ *for all* k.

It follows by symmetry it is also strictly incentive compatible for agent 2.

$\square$

C.3 Proof of Proposition 3

To prove Proposition 3, the following result is key. Let $\{\theta_t\}_{t=0}^{\infty}$ be the stochastic process for ownership shares generated by the set of policy functions as in (22). That is, $\theta_t : S^{\infty} \to [0, 1]$, where $\theta_t(s^{\infty})$ denotes a particular realization at date t.

Lemma 6. *Suppose that the full information plan is strictly incentive compatible at $\theta = 1/2$ for all k.*
Suppose that $\theta_0 \in [0, 1/2]$. The ratio of ownership shares satisfies the following properties:

1. *It is a nonnegative martingale; i.e., for all t and all s^t,*

$$\mathbf{E}\left[\frac{\theta_{t+1}}{(1-\theta_{t+1})} \parallel s^t\right] = \frac{\theta_t(s^{\infty})}{(1-\theta_t(s^{\infty}))} \in [0,1] \qquad s^{\infty} - a.s.$$

2. *There exists a random variable $\widehat{\theta}$ on $(S^{\infty}, \mathcal{B}(S^{\infty}))$, such that*

$$\frac{\theta_t(s^{\infty})}{(1-\theta_t(s^{\infty}))} \to \frac{\widehat{\theta}(s^{\infty})}{\left(1-\widehat{\theta}(s^{\infty})\right)} \qquad s^{\infty} - a.s.$$

Proof of Lemma 6. Consider first the case in which $\theta \in [0, 1/2]$. As the full information plan is strictly incentive compatible at $\theta = 1/2$ for all k, conditions (39)-(42) for both agents and the fact that $\phi_2 = 0$ for $\theta \leq 1/2$ imply that[30]

$$\frac{\theta'(k,\theta,1-\theta;s_L,s_2)}{(1-\theta'(k,\theta,1-\theta;s_L,s_2))} = \frac{\theta + \phi_1(k,\theta,1-\theta)/\pi(s_L)}{(1-\theta)} = \frac{\theta}{(1-\theta)} + \frac{\phi_1(k,\theta,1-\theta)}{(1-\theta)\ \pi(s_L)}$$

$$\frac{\theta'(k,\theta,1-\theta;s_H,s_2)}{(1-\theta'(k,\theta,1-\theta;s_H,s_2))} = \frac{\theta - \phi_1(k,\theta,1-\theta)/\pi(s_H)}{(1-\theta)} = \frac{\theta}{(1-\theta)} - \frac{\phi_1(k,\theta,1-\theta)}{(1-\theta)\ \pi(s_H)}$$

for all s_2. This implies that

$$\mathbf{E}\left[\frac{\theta'(k,\theta,1-\theta;s_1,s_2)}{(1-\theta'(k,\theta,1-\theta;s_1,s_2))}\right] = \frac{\theta}{(1-\theta)}. \tag{52}$$

We argue now that this expectation (52) is bounded by 1.

[30] In the paper, the state variables were denoted by (k, θ). Here we abuse notation and make the state (k, θ_1, θ_2).

Note first that for all $\theta \in [\theta^*, 1/2]$ no incentive compatibility constraint binds, and so

$$\frac{\theta'(k,\theta,1-\theta;s_1,s_2)}{(1-\theta'(k,\theta,1-\theta;s_1,s_2))} = \frac{\theta}{(1-\theta)} \leq 1 \tag{53}$$

for all k, all (s_1, s_2).

Note that $\frac{\theta'(k,\theta,1-\theta;s_1,s_2)}{(1-\theta'(k,\theta,1-\theta;s_1,s_2))}$ is homogeneous of degree 0 with respect to $(\theta, 1-\theta)$. In addition, it is a standard exercise to show that $\frac{\theta'\left(k,\frac{\theta}{(1-\theta)},1;s_1,s_2\right)}{\left(1-\theta'\left(k,\frac{\theta}{(1-\theta)},1;s_1,s_2\right)\right)}$ is increasing in $\frac{\theta}{(1-\theta)}$. This implies that for all $\theta \leq \theta^*$

$$0 \leq \frac{\theta'(k,\theta,1-\theta;s)}{(1-\theta'(k,\theta,1-\theta;s))} \leq \frac{\theta'\left(k,\frac{\theta}{(1-\theta)},1;s\right)}{\left(1-\theta'\left(k,\frac{\theta}{(1-\theta)},1;s\right)\right)} \leq \frac{\theta'\left(k,\frac{\theta^*}{(1-\theta^*)},1;s\right)}{\left(1-\theta'\left(k,\frac{\theta^*}{(1-\theta^*)},1;s\right)\right)} = \frac{\theta^*}{(1-\theta^*)} \leq 1 \tag{54}$$

for all k and all s.

Conditions (53) and (54) imply that $\frac{\theta_t(s^\infty)}{(1-\theta_t(s^\infty))} \in [0,1]$ as $\theta_0 \in [0, 1/2]$ and (52) reads

$$\mathbf{E}\left[\frac{\theta_{t+1}}{(1-\theta_{t+1})} \parallel s^t\right] = \frac{\theta_t(s^\infty)}{(1-\theta_t(s^\infty))} \qquad s^\infty - a.s.$$

Hence, $\{\frac{\theta_t(s^\infty)}{(1-\theta_t(s^\infty))}\}_{t=0}^\infty$ follows a bounded martingale, and so it follows by the martingale convergence theorem that

$$\frac{\theta_t(s^\infty)}{(1-\theta_t(s^\infty))} \to \frac{\widehat{\theta}(s^\infty)}{\left(1-\widehat{\theta}(s^\infty)\right)} \qquad s^\infty - a.s.$$

for some random variable $\widehat{\theta}$ on $(S^\infty, \mathcal{B}(S^\infty))$. □

Proof of Proposition 3. Suppose that the full information plan is strictly incentive compatible at $\theta = 1/2$ for all k. Using the same arguments developed in the proof of Proposition 1, it follows by continuity of the full information policy functions that there exists $\theta^* \in (0, 1/2)$ such that if $(\theta, k) \in [\theta^*, 1-\theta^*] \times [k_{\min}(\theta), k_{\max}(\theta)]$, both agents' incentive compatibility constraints do not bind.

1. Suppose that $(\theta_t, k_t) \in [\theta^*, 1-\theta^*] \times [k_{\min}(\theta_t), k_{\max}(\theta_t)]$ at some t. It follows by definition of θ^* that no agent incentive compatibility constraint binds in this case. Consequently, the private information plan and the full information plan coincide, and that implies by Lemma 1 that $\theta'(s)(\theta, k) = \theta$ for all (s, θ, k) and $k' \in [k_{\min}(\theta), k_{\max}(\theta)] = [k_{\min}(\theta'), k_{\max}(\theta')]$, and so $\theta_{t+n} = \theta_t$ for all $n \geq 0$.

2. Suppose that $(\theta_t, k_t) \in [0, \theta^*) \times [k_{\min}(\theta_t), k_{\max}(\theta_t)]$ at some t. Notice that, by definition of

θ^*, the agent 2's incentive compatibility constraint does not bind. It follows by Lemma 6 that the ratio of ownership shares is a non-negative martingale.

Using the notation of Lemma 6, take any arbitrary $s^\infty \in \Omega = \{s^\infty \in S^\infty : \theta_t(s^\infty) \to \widehat{\theta}(s^\infty) \in [0, 1/2]\}$.

If $\widehat{\theta}(s^\infty) = 0$, then the limiting plan reaches a full information plan as it is a sole-proprietorship.

If $\widehat{\theta}(s^\infty) \in [\theta^*, 1/2]$, it follows by part 1 above that the limiting plan coincides with a full information plan as no incentive compatibility constraint binds.

We need to show that $\widehat{\theta}(s^\infty) \notin (0, \theta^*)$; i.e., the limiting plan can converge to a plan where some ICC is binding only for zero-probability sequences.

Step 2.1. $\widehat{\theta}(s^\infty) < \overline{\theta}(k) \leq \theta^*$ for all k.

It follows by definition that agent 1's incentive compatibility constraint binds for all k. Suppose that the state (s_H, s_L) occurs infinitely often and consider an infinite subsequence $\{(s_{1,t_n}, s_{2,t_n})\}_{n=0}^\infty$ in which $(s_{1,t_n}, s_{2,t_n}) = (s_H, s_L)$ for all n. Because $\{k_{t_n}\}_{n=0}^\infty$ is a sequence in a compact set, it must have a convergent subsequence with limit $\widehat{k}(s^\infty) \in \left[k_{\min}(\widehat{\theta}(s^\infty)), k_{\max}(\widehat{\theta}(s^\infty))\right]$. To simplify notation, suppose that it is the sequence $\{k_{t_n}\}_{n=0}^\infty$ itself. Because $\theta_{t_n+1} = \theta'(\theta_{t_n}, k_{t_n}; s_H, s_L)$, it follows by continuity that taking the limit gives

$$\widehat{\theta}(s^\infty) = \theta'(\widehat{\theta}(s^\infty), \widehat{k}(s^\infty); s_H, s_L);$$

i.e., the the agent 1's incentive compatibility constraint does not bind. But this contradicts that $\widehat{\theta}(s^\infty) \notin (0, \theta^*)$ and consequently, as in Thomas and Worral (1990), $\{\theta_t\}_{t=0}^\infty$ can converge to some number in the interval $(0, \theta^*)$ only for sequences where (s_H, s_L) occurs only finitely often. Those events occur with zero probability.

Step 2.2. $\widehat{\theta}(s^\infty) < \overline{\theta}(k) \leq \theta^*$ for some $k \in (k_{min}(\widehat{\theta}(s^\infty)), k_{max}(\widehat{\theta}(s^\infty)))$.

Let $\widehat{k}(s^\infty)$ be defined such that $\widehat{\theta}(s^\infty) = \overline{\theta}(\widehat{k}(s^\infty))$. Hence $\widehat{k}(s^\infty) \in (k_{min}(\widehat{\theta}(s^\infty)), k_{max}(\widehat{\theta}(s^\infty)))$.

As long as $\widehat{\theta}(s^\infty) \geq \overline{\theta}(k_t(s^\infty))$, then it follows that $\theta'\left(\widehat{\theta}(s^\infty), k_t(s^\infty)\right)(s_{1,t}, s_{2,t}) = \widehat{\theta}(s^\infty)$ for all $(s_{1,t}, s_{2,t})$ (i.e., no incentive compatibility constraint binds). Because s^∞ belongs to a set with positive probability and under the assumption that $\widehat{\theta}(s^\infty) < \overline{\theta}(k)$ for some $k \in (k_{min}(\widehat{\theta}(s^\infty)), k_{max}(\widehat{\theta}(s^\infty)))$, there exists some finite T such that $\widehat{\theta}(s^\infty) < \overline{\theta}(k_T(s^\infty))$. But then the full information plan does not satisfy the agent 1's incentive compatibility constraint at $(\widehat{\theta}(s^\infty), k_T(s^\infty))$, and so the argument follows as in Step 2.1.

3. Notice that symmetry implies that

$$c_1(k, \theta, 1-\theta; s) = c_2(k, 1-\theta, \theta; s)$$
$$k'(k, \theta, 1-\theta)(s) = k'(k, 1-\theta, \theta; s)$$

for all s, for all k, and for all $\theta \in [0,1]$.

Therefore, the analysis for the case in which $(\theta_t, k_t) \in [1-\theta^*, 1] \times [k_{\min}(\theta_t), k_{\max}(\theta_t)]$ at some t, is analogous to 2 above.

$\square$

Investment and Bilateral Insurance
ONLINE APPENDIX*

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1 Two-period model

In this note we solve for a simple two-period model to illustrate that the results in Proposition 1 of the paper are more general than for the production economy with preference shocks presented in the benchmark model.

Section 1.1 considers a Storage Economy (SE) with privately observed preference shocks, instead than a Capital Accumulation Economy (CAE) with decreasing returns to scale as in the benchmark model. We compare the results with an Endowment Economy (EE) without storage and show that a result similar to Proposition 1 holds in the SE but not in the EE.

Section 1.4 studies an economy in which agents have private information about their endowment instead of preference shocks as in the benchmark model. We assume that only the planner is able to reallocate resources from the first to the second period. We show that the storage technology provides incentives for truthful revelation of private information when the agent is sufficiently large. Specifically, the incentive compatibility constraint of agent 1 under the full information allocation at $\theta = 1$ is slack. Instead, when the planner does not have access to the storage technology, the incentive compatibility constraint for agent 1 is binding for all θ.

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1.1 Preference shocks and storage

Consider a two-period model with two agents, $i = A, B$, with preferences $u(c) = \log c$. Agent A faces a preference shock in period 1, $s = L, H$ and there is no preference shock in period 2. Agent B has no preference shock. As a result, there is only one shock in period 1, with $(\pi_L, 1 - \pi_L)$ denoting the probabilities of state L and H, respectively. Let $c^i_{t,s}$ be the consumption of agent $j = A, B$, in period $t = 1, 2$ and state $s = L, H$.

1.2 Full information

We first solve for the full information allocations in the economies with endowment and storage.

Endowment Let $(\theta, 1 - \theta)$ denote the welfare weights corresponding to agents A and B, respectively, and y_t the endowment in period $t = 1, 2$. The planner's problem under full information in the endowment economy is

$$\max_{c^j_{t,s}} \pi_L \left[\theta(s_L \log(c^A_{1,L}) + \beta \log(c^A_{2,L})) + (1-\theta)(\log(c^B_{1,L}) + \beta \log(c^B_{2,L}))\right]$$
$$+ (1 - \pi_L) \left[\theta(s_H \log(c^A_{1,H}) + \beta \log(c^A_{2,H})) + (1-\theta)(\log(c^B_{1,H}) + \beta \log(c^B_{2,H}))\right]$$

subject to

$$c^A_{t,s} + c^B_{t,s} = y_t \quad \text{for } t = 1, 2 \text{ and } s = L, H$$

The optimal consumptions for agent A are

$$c^A_{1,L} = \left(\frac{\theta s_L}{\theta s_L + (1-\theta)}\right) y_1, \tag{1}$$
$$c^A_{2,L} = \theta y_2, \tag{2}$$
$$c^A_{1,H} = \left(\frac{\theta s_H}{\theta_A s_H + (1-\theta)}\right) y_1, \tag{3}$$
$$c^A_{2,H} = \theta y_2. \tag{4}$$

Storage Suppose now that there is an storage technology that makes possible to transfer goods from the first to the second period. To make sure that agents want to move consumption toward the future and not the opposite (borrow, which is not technologically possible), we need to add the assumption that y_1 is sufficiently larger than y_2 so that the non-negativity constraint on storage is not binding.

The objective function once we add storage is the same but the feasibility constraints are

$$c^A_{1,S} + c^B_{1,S} + c^A_{2,S} + c^B_{2,S} = y_1 + y_2, \text{ for } s = L, H$$

In this case, the optimal consumptions under full information for agent A are

$$c^A_{1,L} = \left(\frac{\theta s_L}{\theta s_L + (1-\theta) + \theta\beta + (1-\theta)\beta} \right) (y_1 + y_2), \tag{5}$$

$$c^A_{2,L} = \left(\frac{\theta \beta}{\theta s_L + (1-\theta) + \theta\beta + (1-\theta)\beta} \right) (y_1 + y_2) \tag{6}$$

$$c^A_{1,H} = \left(\frac{\theta s_H}{\theta s_H + (1-\theta) + \theta\beta + (1-\theta)\beta} \right) (y_1 + y_2), \tag{7}$$

$$c^A_{2,H} = \left(\frac{\theta \beta}{\theta s_H + (1-\theta) + \theta\beta + (1-\theta)\beta} \right) (y_1 + y_2). \tag{8}$$

1.3 Incentive compatibility constraint

We write down the incentive compatibility constraint (ICC) for each setting and then check if it is possible that the full information allocation is incentive compatible. The incentive compatibility constraint for agent A is

$$s_L \log(c^A_{1,L}) + \beta \log(c^A_{2,L}) \geq s_L \log(c^A_{1,H}) + \beta \log(c^A_{2,H}). \tag{9}$$

Endowment Replacing the full information allocation for the endowment economy, (1)-(4) in the ICC (9) and find a contradiction for any $\theta_A \in (0, 1)$:

$$[\log(\theta s_H + (1-\theta)) - \log(\theta s_L + (1-\theta))] \geq [\log(\theta s_H) - \log(\theta s_L)].$$

Clearly, this occurs because $c^A_{2,L} = c^A_{2,H}$ and $c^A_{1,H} > c^A_{1,L}$.

The previous result implies that for any $\theta \in (0, 1)$, the full information plan is **never** incentive compatible.

Storage Now we argue that the incentive compatibility constraint is satisfied in the economy with storage for sufficiently large values of $\theta < 1$. Replace the consumption allocations under full information for the storage economy, (5)-(8) in the ICC (9). Then

$$\frac{(s_L + \beta)}{s_L} > \frac{[\log(\theta_A s_H) - \log(\theta_A s_L)]}{[\log(\theta_A s_H + (1-\theta_A) + \beta) - \log(\theta_A s_L + (1-\theta_A) + \beta)]}.$$

If we evaluate this condition at $\theta = 1$ we obtain

$$\frac{(s_L + \beta)}{s_L} > \frac{\log(\frac{S_H}{S_L})}{\log(\frac{S_H+\beta}{S_L+\beta})}.$$

We can show that the above condition is satisfied. Jensen's inequality implies

$$\log(tx_1 + (1-t)x_2) > t\log(x_1) + (1-t)\log(x_2)$$

Let $x_1 = \frac{S_H}{S_L}, x_2 = 1, u = \frac{\beta}{S_L}, t = \frac{1}{1+u}$. Then

$$\begin{aligned} \log(\frac{1}{1+u}x_1 + \frac{u}{1+u}1) &> \frac{1}{1+u}\log(x_1) + \frac{u}{1+u}log(1) \\ \log(\frac{x_1+u}{1+u}) &> \frac{1}{1+u}\log(x_1) \\ 1+u &> \frac{\log(x_1)}{\log(\frac{x_1+u}{1+u})} \end{aligned}$$

$$\frac{S_L + \beta}{S_L} > \frac{\log(\frac{S_H}{S_L})}{\log(\frac{S_H+\beta}{S_L+\beta})} \tag{10}$$

Therefore, we can conclude that the ICC is satisfied for values of θ that are sufficiently close to 1.

1.4 Endowment shocks

In this section we argue that the main result of the paper also holds in another alternative environment. In particular, we consider an economy in which the storage technology is only available at the aggregate level and agents have private information about their endowments, and there are no preference shocks. When agent A receives a large endowment shock he can report truthfully and the aggregate economy will store some of the resources, or he can misreport and consume today the extra endowment. We show that the when the Pareto weight of agent A is equal to 1, his incentive compatibility constraint evaluated at the full information allocation is strictly slack: He prefers to report a high shock and save, instead of reporting a low shock and consume more today at expenses or less storage.

Environment

- 2 periods: $t = 1, 2$.

- 2 agents: $i = A, B$.
- Agents have endowment y_t^i, with $y_1^B = y_2^B = 1$ and $y_1^A \in \{y_L, y_H\}$ and $y_2^A = 1$.
- Agent's A endowment is private information.
- Agents do not have access to an individual storage technology (alternatively, individual storage is observable and can be fully monitored). However, the planer has access to a storage technology to transfer $x \geq 0$ from period 1 to period 2 at rate 1 as before.
- Preferences are $u(c) = \log c$ and there is no discounting.

Full information allocation Aggregate feasibility for periods 1 and 2 are

$$\begin{aligned} c_1^A(y) + c_1^B(y) + x &= 1 + y \\ c_2^A(y) + c_2^B(y) &= 2 + x \end{aligned}$$

Second period To illustrate the idea more clearly, let's solve the problem backwards. Conditional on x, in the second period the planer's problem is

$$H_2(\theta, x) = \max\{\theta \ln\left(c_2^A(x)\right) + (1-\theta)\ln\left(c_2^B(x)\right)\}$$

subject to $c_2^A(x) + c_2^B(x) = 2 + x$

The solution is

$$\begin{aligned} c_2^A(x) &= (2+x)\,\theta \\ c_2^B(x) &= (2+x)(1-\theta) \end{aligned}$$

And the continuation value is

$$H_2(\theta, x) = \ln(2+x) + \theta\ln(\theta) + (1-\theta)\ln(1-\theta)$$

First period In period 1, the planner's problem is

$$H_1(\theta, y) = \max_{c_1^A, c_1^B, x} \theta\ln\left(c_1^A\right) + (1-\theta)\ln\left(c_1^B\right) + \ln(2+x) + \theta\ln(\theta) + (1-\theta)\ln(1-\theta)$$

subject to

$$
\begin{aligned}
c_1^A + c_1^B + x &= 1 + y \\
x &\geq 0
\end{aligned}
$$

Let λ and μ be the corresponding Lagrange multiplier of the constraints above. Optimality implies

$$
\begin{aligned}
\frac{\theta}{c_1^A(\theta, y)} &= \lambda \\
\frac{1-\theta}{c_1^B(\theta, y)} &= \lambda \\
\frac{1}{2 + x(\theta, y)} + \mu &= \lambda
\end{aligned}
$$

If $y = y_H > 1$, then it is easy to show that $x(\theta, y_H) > 0$ and so $\mu = 0$. Therefore, $\lambda = \frac{1}{2+x}$ and

$$
\begin{aligned}
c_1^A(\theta, y_H) &= (2 + x)\,\theta \\
c_1^B(\theta, y_H) &= (2 + x)(1-\theta) \\
x(\theta, y_H) &= \frac{y_H - 1}{2}
\end{aligned}
$$

So the solution is

$$
\begin{aligned}
c_1^A(\theta, y_H) &= \theta \frac{3 + y_H}{2} \\
c_1^B(\theta, y_H) &= (1-\theta) \frac{3 + y_H}{2} \\
x(\theta, y_H) &= \frac{y_H - 1}{2}
\end{aligned}
$$

Instead, if $y = y_L < 1$, then $x(\theta, y_L) = 0$ and

$$
\frac{\theta}{c_1^A(\theta, y_L)} = \frac{(1-\theta)}{c_1^B(\theta, y_L)}
$$

which implies

$$
\begin{aligned}
c_1^A(\theta, y_L) &= \theta(1 + y_L) \\
c_1^B(\theta, y_L) &= (1-\theta)(1 + y_L)
\end{aligned}
$$

$$x(\theta, y_L) = 0$$

Summarizing, agent A's consumption profile given θ is $\left(c_1^A(\theta, y_L), c_2^A(\theta, y_L)\right) = (\theta(1+y_L), \theta 2)$ and $\left(c_1^A(\theta, y_H), c_2^A(\theta, y_H)\right) = (\theta\frac{3+y_H}{2}, \theta\frac{3+y_H}{2})$.

Incentives? Now, consider agent A who has an endowment $y = y_H$ and is considering reporting y_L and consume privately $y_H - y_L$. Therefore, the ICC reads

$$u\left(c_1^A(\theta, y_H)\right) + u\left(c_2^A(\theta, y_H)\right) \geq u\left(c_1^A(\theta, y_L) + (y_H - y_L)\right) + u\left(c_2^A(\theta, y_L)\right)$$

which becomes

$$2\log\left(\theta\frac{3+y_H}{2}\right) \geq \log\left(\theta(1+y_L) + (y_H - y_L)\right) + \log(2\theta)$$

Now, evaluate this condition at $\theta = 1$ which would imply

$$2\log\left(\frac{3+y_H}{2}\right) \geq \log(1+y_H) + \log(2) \tag{11}$$

Now we argue that condition (11) above holds with strict inequality for all $y_H > 1$. In the limit, if $y_H = 1$, then both the left and right hand sides of the ICC are $2log2$ and it holds with equality. If $y_H > 1$, let $x_1 = 1 + y_H$, $x_2 = 2$ and $t = 1/2$. Then, it follows by strict concavity of log it follows that

$$\begin{aligned}
\log(tx_1 + (1-t)x_2) &> t\log x_1 + (1-t)\log x_2 \\
\log\left(\frac{3+y_H}{2}\right) &> \frac{1}{2}\log(1+y_H) + \frac{1}{2}\log 2 \\
2\log\left(\frac{3+y_H}{2}\right) &> \log(1+y_H) + \log 2
\end{aligned}$$

which proves the claim.

To conclude, as the ICC holds with strict inequality for $\theta = 1$, it follows by continuity that it holds true for θ sufficiently large.

The intuition is the following. Agent A has two sources in general to obtain extra consumption: reducing consumption of agent B today or reducing consumption of agent B tomorrow by means of reducing storage. When $\theta = 1$ agent B gets zero consumption, by misreporting the shock the agent consumes more today at the cost of reducing storage, which will be one of the sources tomorrow for her own consumption (remember that she cannot "hiddenly" store). But then the argument is as before: she does not want to distort her own savings.

2 Additional results and robustness

In this section we discuss the dynamics in the CAE under Private Information and the role played by internal financing and symmetry across agents.

2.1 Dynamics in the CAE under Private Information

When the Pareto weight is smaller than the threshold described in Proposition 1, the constrained efficient allocation changes with respect to the full information allocation to provide incentives for truthful revelation. In particular, it changes in three dimensions: consumption, continuation weights, and investment. The distortions in consumption are standard in the literature; i.e., the constraint efficient allocation provides less consumption insurance in order to provide incentives. We now study how continuation weights and investment are distorted in the constrained efficient allocation.

Continuation weights are manipulated to provide incentives for truthful revelation of the shock. Lemma 1 characterizes how future values of Pareto weights under private information depend on the reports. In general, to provide incentives to report low preference shocks, the constraint efficient allocation punishes the report of a high preference shock by assigning a lower future weight than for reports of low preference shocks. When agent 1's incentive compatibility constraint does not bind, there is no need to provide incentives for agent 1, and as a consequence future weights are independent of his report. But weights could still depend (and, in general, will) on the other agent's report. By symmetry, if agent 2 reports a high shock, his future weight will be lower than if she reports a low shock. Finally, note that when both incentive compatibility constraints are slack and there is no need to provide incentives, future weights are equal to current weights and independent of reports about preference shocks.[1]

Lemma 1 (Dynamics of weights). *Agent's 1 Pareto weight evolves as follows:*

1. $\theta_1'(k,\theta;s_L,s_2) \geq \theta_1'(k,\theta;s_H,s_2)$ *for* $s_2 \in \{s_L,s_H\}$.

2. *Moreover, if the incentive compatibility constraint of agent 1 does not bind, then* $\theta_1'(k,\theta;s_L,s_2) = \theta_1'(k,\theta;s_H,s_2)$ *for* $s_2 \in \{s_L,s_H\}$.

[1] There is a subtle difference here between future utility and future Pareto weights. Imagine the case in which both incentives constraints are slack. As we mentioned, in that case the future Pareto weights will be independent of the report. However, promised utilities will not be independent of the report, as capital accumulation does depend on the report even in the full information allocation.

3. *Moreover, if no incentive compatibility constraint bind, then* $\theta_1'(k,\theta;s_1,s_2)=\theta$ *for* $(s_1,s_2)\in\{s_L,s_H\}\times\{s_L,s_H\}$.

Proof of Lemma 1. Note that:

1. Equations (43)-(46) imply that if $\phi_1=\phi_2=0$, then $\theta'(k,\theta,s_1,s_2)=\theta$ for $(s_1,s_2)\in\{s_L,s_H\}\times\{s_L,s_H\}$.

2. Equations (43)-(46) imply that

$$\theta_1^{'}(s_L,s_L)-\theta_1^{'}(s_H,s_L)=\left(\frac{\phi_1}{\pi(s_H)\pi(s_L)}\right)\frac{(1-\theta)+\frac{\phi_2}{\pi(s_L)}}{\left(1+\frac{\phi_1}{\pi(s_L)}+\frac{\phi_2}{\pi(s_L)}\right)\left(1-\frac{\phi_1}{\pi(s_H)}+\frac{\phi_2}{\pi(s_L)}\right)}\tag{12}$$

$$\theta_1^{'}(s_L,s_H)-\theta_1^{'}(s_H,s_H)=\frac{\phi_1}{\pi(s_H)}\frac{\frac{(1-\theta)}{\pi(s_L)}-2\frac{\phi_2}{\pi(s_H)}}{\left(1+\frac{\phi_1}{\pi(s_L)}-\frac{\phi_2}{\pi(s_H)}\right)\left(1-\frac{\phi_1}{\pi(s_H)}-\frac{\phi_2}{\pi(s_H)}\right)}\tag{13}$$

   Hence, if $\phi_1=0$, then $\theta'(k,\theta,s_L,s_2)=\theta'(k,\theta,s_H,s_2)$ for all $s_2\in\{s_L,s_H\}$.

3. Note that if $s_2=s_L$, then (12) implies that $\theta'(k,\theta,s_L,s_L)\geq\theta'(k,\theta,s_H,s_L)$.

4. If $s_2=s_H$, then (44) and (46) implies that $\frac{\theta'(k,\theta,s_L,s_H)}{\theta'(k,\theta,s_H,s_H)}\geq 1$ if $\phi_2\leq(1-\theta)\pi(s_H)$ and this condition is satisfied due to (42).

□

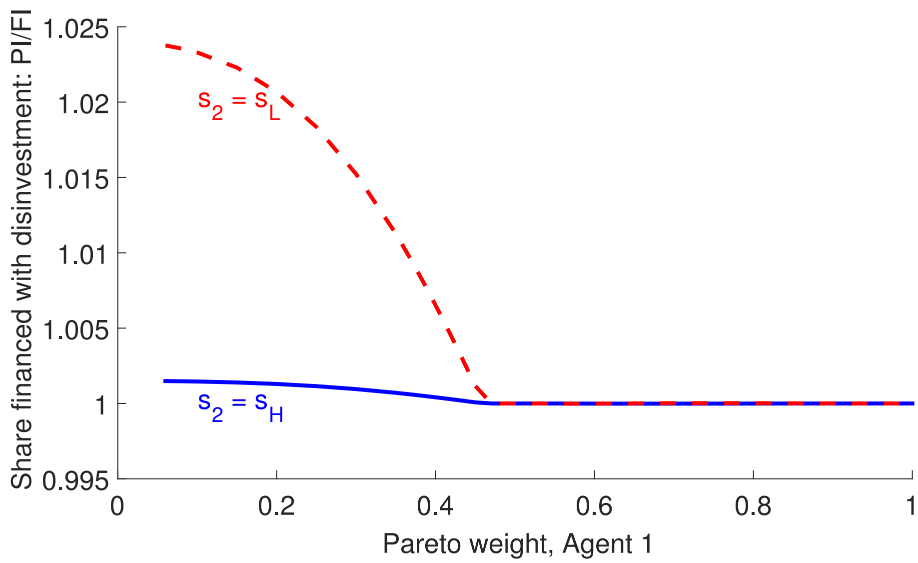
2.2 Investment distortions

Recall that there are two sources to finance the extra consumption that an agent receives after reporting high liquidity needs—namely, *redistribution* and *disinvestment*. Consider the increment of agent 1's consumption as he reports high liquidity needs compared with the case in which he reports low needs. His consumption is still conditional on agent 2's report, s_2. Define the share of this increment that is financed by means of disinvestment as

$$\text{DInv}(k,\theta,s_2)=\frac{k'(k,\theta;s_L,s_2)-k'(k,\theta;s_H,s_2)}{c_1(k,\theta;s_H,s_2)-c_1(k,\theta;s_L,s_2)};\tag{14}$$

that is, the fraction of the higher consumption financed by investing less. Figure 1 precisely shows how this additional margin (disinvestment) is distorted under private information to provide incentives. It displays the disinvestment share for private information relative to that of full information (i.e., the ratio of equation (14) in private information to full information). Note that this ratio is above one for low values of θ, when agent 1's incentive constraint is binding. To understand this, recall that the reduction in investment following a report of a high preference shock is the cost of cheating. Thus, to prevent cheating, it is natural that the optimal contract prescribes more disinvestment.

Figure 1: **Investment distortions under private information**



2.3 External Financing

To study the role of external financing, we consider an extreme case in which the agents have perfect access to capital market. To operationalize this, we contemplate an environment similar to Marcet and Marimon (1992), in which agent 2 is risk neutral, faces no shocks, and has deep pockets.[2]

In this setup, for all $\theta \in (0, 1)$, the first-order condition that characterizes optimal investment of the private information plan implies

$$1 = \beta \left(f'(k^*) + (1 - \delta)\right).$$

This finding is analogous to the result in Marcet and Marimon (1992), who find that private information does not distort optimal investment. This result is a direct consequence of evaluating the investment decision with the intertemporal marginal rate of substitution of the risk-neutral agent.

Note that the capital stock in this setup will jump directly from k_0 to k^* and remain constant forever (in particular, the reports of the agent with private information do not change investment). Therefore, under full information, future utility is independent of the reports, and

[2] The assumption that agent 2 does not face shocks is made only for simplicity.

as we explained in the case of the endowment economy, this implies that the full information plan violates the incentive compatibility constraints for all $\theta \in (0, 1)$.

2.4 Symmetry

Although throughout the paper we assume that agents are symmetric (the agents are ex-ante identical), our theory is certainly more general. This assumption is important for the result that the too-big-to-cheat region is around 50-50. If agents were asymmetric we would find two thresholds, θ_1 and θ_2, such that this region would be $[\theta_1, \theta_2]$. Note that if the asymmetry is sufficiently large this region might not include $\theta = 1/2$. For instance, if only agent 1 faces preference shocks, then the too-big-to-cheat region is $[\theta_1, 1]$.

References

Marcet, A., and R. Marimon (1992): "Communication, Commitment, and Growth," *Journal of Economic Theory*, 58, 219–249.