An Approximate Dual-Self Model and Paradoxes of Choice under Risk

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Abstract

We derive a simplified version of the model of Fudenberg and Levine [2006, 2011] and show how this approximate model is useful in explaining choice under risk. We show that in the simple case of three outcomes, the model can generate indifference curves that "fan out" in the Marshack-Machina triangle, and thus can explain the well-known Allais and common ratio paradoxes that models such as prospect theory and regret theory are designed to capture. At the same time, our model is consistent with modern macroeconomic theory and evidence and generates predictions across a much wider set of domains than these models.

Keywords: *common ratio effect; dual-self process; Allais paradox; self-control*

JEL Classification Codes: *D01, D03*

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1. Introduction

Fudenberg and Levine [2006, 2011, 2012] develop a model of costly self-control that can explain many ways that observed individual choice departs from the predictions

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of the "standard model" of maximizing expected discounted utility. Their self-control model is based on the idea that a more rational "long run self" controls the impulses of a "short run self" that is very tempted by immediate rewards.[2] Fudenberg and Levine [2006] points out that the self-control model can explain "time-domain" phenomena, such as a preference for commitment and time-inconsistent choice. Moreover, when the model is enriched with the assumption of mental accounts or "pocket cash constraints" it can also explain the very high levels of small-stakes risk aversion seen in the lab, a quantitative puzzle that has become known as the Rabin paradox, after Rabin [2000]. Fudenberg and Levine [2011] show that the same model can also explain the interaction of risk and delay seen in such experiments as Baucells and Heukamp [2010] and Keren and Roelofsma [1995]. Moreover they move beyond the qualitative matching of theories and facts that is typical in this literature to a quantitative calibration of the model to both Rabin-paradox data and the Allais paradox.[3]

Unfortunately the model of Fudenberg and Levine [2011] is fairly complex, which may obscure some of the key insights and make it difficult for others to apply the model. Our purpose here it to develop a somewhat simpler approximation to this model that is still accurate enough to be useful in applied work. After developing this approximation, we characterize its theoretical properties and show how it helps explain

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[2] While the model is intended as a very precise model of the internal processes underlying self-control, at a very rough level the model is consistent with fMRI evidence, since the "long run self" is identified with activation in the prefrontal cortex while the short run self corresponds to more primitive and faster-acting parts of the brain. See e.g. McClure et al. [2004, 2007] for fMRI evidence that support this general idea.
[3] One caveat is that the model described here, like Fudenberg and Levine [2006, 2011], assumes a short run self who lives only for a single period. For this reason the model, like quasi-hyperbolic discounting, implies that only the current period's rewards are tempting. This stark conclusion is not suitable for analyzing some aspects of the timing of decisions, such as the marginal interest rates found in the experiment of Myerson

observed behavior in the Allais paradox and common ratio paradoxes and examine the implications of the theory for intransitivity.

To study decision makers who act as if tempted by money winnings but also manage to save, as well as to explain the level of risk aversion observed in lab experiments, Fudenberg and Levine [2006] use the idea of mental accounting: A decision maker reduces the cost of restraining impulsive decisions by using mental accounting to commit to daily expenditures – the idea is that the mental account is set when the decision maker is in a "cool state" and not subject to temptation. By assumption, the commitment is to net expenditures, and not to consumption per se, so small losses must be born out of daily expenditures and small gains create a self-control problem. This results in the marginal propensity to consume out of small gains being large – one in fact.[4] Because small losses and small gains are applied entirely to daily expenditures and not spread over the lifetime, the decision maker is much more risk averse over small unexpected lotteries than under the classical model, where any change in wealth results in a much smaller permanent change in consumption over the individual's lifetime.

A second consequence of this theory is that if there is an increasing marginal cost of self-control then the decision maker's utility is not linear in probabilities and so this is not an expected utility theory. Moreover, while any form of nonlinearity makes the model depart from expected utility, the increasing-marginal-cost specification predicts the

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and Green [1995]. A more realistic version of the model, in which short run selves are less patient than the long run self without being completely myopic, is developed in Fudenberg and Levine [2012].

[4] This is consistent with psychological evidence that people need justification in order to spend money on "vices", which offer short-term gratification but low long-term benefit (Kivetz and Zheng, 2006). Earning small unexpected amounts provides such justification. Accordingly, people will tend to spend these amounts immediately on temptation goods, which they would not otherwise purchase.

particular violations of the independence axiom seen in, for example, the Allais, common-ratio and other related paradoxes, as detailed in Fudenberg and Levine [2011].

Our goal here is to introduce a simple approximate version of the theory that can be used to study data on risk and lotteries. A key simplification is the assumption that the long-run value function is risk neutral, that is, that the marginal utility of savings is a constant. This is a good approximation to decisions that have little impact on lifetime wealth; it simplifies the model by replacing an unknown non-linear value function with a known linear value function. We also assume that the interest received over a single period (the "temptation horizon" of the short run self) is small enough to be ignored; this fits with the usual calibration of this period length to be one to three days. We use the simplified model to explain how the theory ranks general small-stakes money lotteries, and illustrate this in the context of lotteries with only three possible outcomes in the gains domain using the classic Marschak-Machina depiction of indifference curves in the corresponding probability simplex. We also illustrate how the model leads to intransitivity.

The structure of the paper is as follows. In Section 2 we derive the approximate version of the dual-self model. In Section 3 we study its properties using a series of propositions. In Section 4 we examine the special case of a single gamble with a unique positive prize. Section 5 addresses the very interesting case of choices in menus of two lotteries, with three possible outcomes. We show how the model predicts well-known paradoxes that violate expected utility and illustrate this in the Marschak-Machina triangle. Section 6 provides a general discussion and concludes.

2. Deriving an Approximate Dual-Self Model

We will remind the reader of the main ingredients of the Fudenberg and Levine [2006, 2011] model, and then show how the approximation of risk neutrality for wealth leads to a much more tractable model. There is an individual who makes a consumption-savings decision, with a short-run utility function $u^x(x + c; x)$ each period. Here $x + c$ represents total consumption, x is the planned level of consumption under the mental account ("pocket cash"), and c denotes "incremental consumption": the additional (possibly negative if money is lost) consumption made possible by unexpected windfalls.

When studying a fixed individual and holding fixed that individual's initial wealth and preferences, we can suppress the dependence on x and take short-run utility to be $u(c) = u^x(x + c; x)$, where $u' > 0$ and $u'' < 0$. In the remainder of the paper, the term "consumption" will refer to this incremental consumption.

We are primarily interested in how the agent chooses lotteries Z from a fixed menu $\Im$, where each of the lotteries resolves in the current period. For this, an important intermediate step is to analyze the agent's optimal consumption *ex post* after a particular lottery has been chosen.[5] Let $u^* = \max_{Z \in \Im} Eu(Z)$ be the greatest available short-run utility. This "temptation" represents what the short-run self would like – to spend all the gains immediately. If the lottery Z has n outcomes, the choice of optimal consumption entails choosing an optimal random consumption plan $\tilde{c}$ with outcomes $(c_1, ..., c_n)$, specifying one consumption level for each possible lottery realization. Overall first period utility is given by $Eu(\tilde{c}) - g(u^* - Eu(\tilde{c}))$ where g is the cost of self-control.

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[5] Notice that by assumption the self-control cost is incurred when the agent determines the consumption plan, so it depends on the expected utility of this plan.

In other words, current utility depends on current consumption and the cost of self-control, which is increasing in "foregone utility," the difference between the utility the "short-run self" would have liked and what he actually got: $u* - Eu(\tilde{c})$. For convenience, we define the argument $\Delta \equiv Eu(\tilde{c}) - u*$, which is non-positive. We assume that g is a smooth, non-decreasing, weakly convex function satisfying $g(0) = 0, g(-\Delta) \geq 0$.

The model of Fudenberg and Levine [2006, 2011] is not static, but considers an infinite horizon problem. It is well known that the recursive structure of the maximization problem allows us to represent future utility by means of a "value function" v, computed by optimizing beginning in period 2. This function has as its argument total wealth[6] w_2 at the beginning of the second period, which will be distributed optimally over the lifetime as consumption. If the realization of the lottery is z_i and consumption is $c_i \leq z_i$ then the realized wealth beginning in period 2 is $w_2 + z_i - c_i$. The present value of utility starting in period 2 is $Ev(w_2 + Z - \tilde{c})$. If δ is the discount factor, the overall objective function is

$$(1) \qquad V(\tilde{c}, u*, Z, w_2) = Eu(\tilde{c}) - g(u* - Eu(\tilde{c})) + \delta Ev(w_2 + Z - \tilde{c}).$$

Following Fudenberg and Levine [2006, 2011] we assume that the lottery is unanticipated.[7] Since pocket cash is chosen when the agent is not subject to temptation, the agent can obtain the optimal consumption path of someone who faces no self-control

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[6] Strictly speaking what matters is not total wealth and consumption but discretionary wealth and consumption, that is, net of expenditures such as rent and medical care that are committed in advance and do not pose a temptation.
[7] Or that the probability is small enough not to have had an appreciable impact on the choice of pocket cash.

costs, without using any self-control. This is achieved by setting $c = 0$: in this case absent any windfalls the agent is not able to consume more than the first-best consumption level, and so faces no temptation to consume more. Hence $c = 0$ is the optimal level, and so $u'(0) = \delta v'(w_2)$. Define the function $h(\Delta) \equiv \Delta - g(-\Delta)$, the "self-control gain function". This function is non-positive, smooth, strictly increasing, weakly concave function on $\Re_-$ satisfying $h'(0) \geq 1$. It inherits these properties from g. Also define

$$(2) \qquad U^c(\tilde{c}, u^*, Z) = h\left(Eu(\tilde{c}) - u^*\right) + u'(0)(EZ - E\tilde{c}).$$

We will now approximate V for small gambles. Recall that $o(y)$ denotes a function such that $\lim_{y \to 0} o(y) / y \to 0$, and that $\max \mid Z \mid$ is the largest value in the support of $\mid Z \mid$.

Lemma 1: *The objective function satisfies the equality*

$$V(\tilde{c}, u^*, Z, w_2) = U^c(\tilde{c}, u^*, Z) + u^* + \delta v(w_2) + o(\max \mid Z \mid).$$

Proof: Recall that $V(\tilde{c}, u^*, Z, w_2) = Eu(\tilde{c}) - g(u^* - Eu(\tilde{c})) + \delta Ev(w_2 + Z - \tilde{c})$. Since v is differentiable under standard conditions a first-degree Taylor approximation gives

$$\begin{aligned} Ev(w_2 + Z - \tilde{c}) &= \\ E\{v(w_2) + v'(w_2)(Z - \tilde{c}) + o(\max \mid Z \mid)\} &= \\ v(w_2) + v'(w_2)(EZ - E\tilde{c}) + o(\max \mid Z \mid) \end{aligned}$$

Substituting this and the definition of U^c into the objective function gives the desired result.

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In comparison to Fudenberg and Levine [2006, 2011], this model assumes that no interest is paid on money found, earned, or saved during the first period, which fits with the idea that the length of a period is measured in days. Further, as an approximation, first period savings are assumed not to change the marginal present value of period two value. If in fact savings are an appreciable portion of lifetime wealth, this approximation understates risk aversion.

Observe that for a given menu $\Im$ of lotteries $u* + \delta v(w_2)$ is a constant, and since the agent will pick the optimal consumption plan, for small Z the agent's preferences over lotteries can be represented by the approximate objective function

$$\max_{\tilde{c}} U^c(\tilde{c}, u^*, Z) = \max_{\tilde{c}} [h(Eu(\tilde{c}) - u^*) + u'(0)(EZ - E\tilde{c})].$$

Since consumption cannot exceed the amount earned by the prize, the problem is: choose $\tilde{c} = (c_1, ..., c_n)$ to maximize $h(Eu(\tilde{c}) - u^*) + u'(0)(EZ - E\tilde{c})$ subject to the constraints $c_i \leq z_i, i = 1, 2, ..., n$. Define

$$U(u^*, Z, z) \equiv h(Eu(Z) - u^* - E\max\{u(Z) - u(z), 0\}) + u'(0)E\max\{Z - z, 0\}.$$

Theorem 2: *The approximate objective function* $\max_{\tilde{c}} U^c(\tilde{c}, u^*, Z)$ *is equal to* $\max_z U(u^*, Z, z)$ *and there is a threshold* $\hat{z}$ *such that if* $z_i \leq \hat{z}$ *then* $c_i^* = z_i$ *(so all of the unexpected winnings are saved) while if* $z_i \geq \hat{z}$ *then* $c_i^* = \hat{z}$ *(so any amount over* $\hat{z}$ *is saved, regardless of the size of* z_i.[8])

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[8] This stark conclusion comes from our simplifying assumption that the marginal utility of savings is constant, which is a good approximation only if the winnings are not in fact too large.

Proof: Maximizing with respect to c_i for a given realization of the lottery z_i, gives the consumption function $\tilde{c}\,*$ as the implicit solution of

$$u'(c_i*)h'\big(Eu(\tilde{c}*) - u\,*\big) \geq u'(0)$$

with equality if $c_i* < z_i$.[9] For any specification of $\tilde{c}$, let $\hat{z}(\tilde{c})$ be the unique solution of

$$u'(\hat{z}(\tilde{c}))h'\big(Eu(\tilde{c}) - u\,*\big) = u'(0) \qquad (*)$$

in $(-x, \infty]$, where we assign $\hat{z} = \infty$ if there is no solution, that is, if $u'(\hat{z}(\tilde{c}))h'\big(Eu(\tilde{c}) - u\,*\big) > u'(0)$ for all $\hat{z} \in (-x, \infty]$. Then we see that $\tilde{c}\,*$ itself must have the property that if $z_i \leq \hat{z}$ then $c_i* = z_i$, while if $z_i \geq \hat{z}$ then $c_i* = \hat{z}$.

Notice that the marginal propensity to consume out of income above the threshold is zero (this is a consequence of our approximation assumption). Thus $c_i* = \min\{z_i, \hat{z}\}$, and therefore $Eu(\tilde{c}*) = E\min\{u(z_i), u(\hat{z})\}$ is non-decreasing in $\hat{z}$ and $u'(\hat{z})h'\big(Eu(\tilde{c}*) - u\,*\big)$ is strictly decreasing in $\hat{z}$. Notice that at $\hat{z} = -x$ we have $u'(-x)h'(u(-x) - u*) > u'(0)$, because $u'(-x) > u'(0)$ and $h'(u(-x) - u*) \geq 1$.[10] Our objective is to express the consumer's preferences for lotteries in terms of this unique threshold. Adding and subtracting $Eu(Z)$ inside of h and using the linearity of expectation we have:

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[9] Notice that the aforementioned condition is exactly the same for every $i = 1, 2, ..., n$. Hence, if for two optimal c_i^*, c_j^* the constraint is not binding (that is $c_i* < z_i, c_j* < z_j$) then it must be the case that $c_i* = c_j\,*$.

[10] This is because $h'(0) \geq 1$ and h is weakly concave.

$$h\big(Eu(\tilde{c}^*) - u\,^*\big) + u'(0)(EZ - E\tilde{c}^*) =$$
$$h\big(Eu(Z) - u\,^* - E(u(Z) - u(\tilde{c}^*))\big) + u'(0)E(Z - \tilde{c}^*).$$

Using $c_i^* = \min\{z_i, \hat{z}\}$ we see that $u(Z) - u(\tilde{c}^*) = \max\{u(Z) - u(\hat{z}), 0\}$ and

$Z - \tilde{c}^* = \max\{Z - \hat{z}, 0\}$. Substituting in gives the desired result.

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3. Choice from Menus of Lotteries

We now suppose that the decision maker faces a menu $\Im$ of lotteries on $[-x, y]$

where $x, y > 0$ and y is "small" relative to lifetime wealth. As above, we suppose that

the agent does not expect to face this menu.

The approximate utility for opportunity set $\Im$, threshold z and lottery $Z \geq 0$ is

given by Theorem 2 as

$$U(u^*, Z, z) = h\big(Eu(Z) - u\,^* - E\max\big\{u(Z) - u(z), 0\big\}\big) + u'(0)E\max\big\{Z - z, 0\big\}.$$

To reiterate, the first term h represents the combination of utility received from

immediate consumption and the cost of controlling the desire to spend even more. The

second term represents the long-term benefit of the amount that is saved: this is not

subject to a self-control problem, but is spread over the entire lifetime, and as indicated

we approximate the corresponding risk as negligible so that the utility from savings is

linear.

Notice that in general the ranking of lotteries is menu dependent, as it depends not

only on the lottery Z that is being assessed, but also the utility $u\,^*$ from the lottery that

yields the greatest short-run utility. This represents a temptation: spend all the money

right away, and choose the lottery that maximizes the expected utility from doing so. However, when h is linear the ranking does not depend on $u*$.

Properties of the Approximate Model

The next result shows how to determine the value of the threshold $\hat{z}$ above which all of the (unexpected) lottery payoff is saved.

Proposition 3: The $\arg\max_{z \geq 0} U(u^*, Z, z) \equiv \hat{z}$ *is characterized by*

$$h'\big(Eu(Z) - u* - E\max\{u(Z) - u(\hat{z}), 0\}\big) u'(\hat{z}) = u'(0)$$

which has a unique solution. If $h'(0) > 1$ then $\hat{z} > 0$. $U(u^*, Z, z)$ is differentiable with respect to z at $z = \hat{z}$ and the derivative is zero. The function

$$F(Z, z) = h'\big(Eu(Z) - u* - E\max\{u(Z) - u(z), 0\}\big) u'(z)$$

is strictly decreasing in z with left and right derivatives both bounded away from zero.

Proof: The expression follows from plugging the solution $c_i^* = \min\{z_i, \hat{z}\}$ into the necessary first order condition (*). The uniqueness of the solution follows from the fact that the LHS is strictly decreasing. If $h'(0) > 1$ then since h' is decreasing, it follows that the solution satisfies $u'(\hat{z}) < u'(0)$ which in turn implies $\hat{z} > 0$.

To see that $U(u^*, Z, z)$ is differentiable with respect to z at $z = \hat{z}$ we compute that its derivative is equal to zero. Observe first that $U(u^*, Z, \bullet)$ is certainly differentiable unless $z_i = \hat{z}$ for some i and regardless, the left and right derivatives exist. We will

complete the proof by showing that both are equal to zero at $z = \hat{z}$. The derivative has the form

$$\partial U(u^*, Z, z) / \partial z =$$
$$\sum_i p_i [-h'\big(Eu(Z) - u * - E\max\{u(Z) - u(z), 0\}\big)\partial\max\{u(z_i) - u(z), 0\} / \partial z$$
$$+ u'(0)\partial\max\{z_i - z, 0\} / \partial z]$$

where the derivative of the max is understood to depend on the direction if $z_i = z$. The key observation is that each individual term in the sum vanishes at $z = \hat{z}$. If $z_i < z$ this is immediate since near z the term does not depend on z. The same is true if $z_i = z$ for the right-hand side derivative. When $z_i > z$ the term is

$$-h'\big(Eu(Z) - u * - E\max\{u(Z) - u(z), 0\}\big)u'(z) + u'(0)$$

which vanishes at $\hat{z}$ by the earlier characterization of $\hat{z}$. The same applies when $z_i = z$ in the negative direction.

The properties of $F(Z, z)$ may be established by differentiating with respect to z.

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Proposition 4: *If h is linear then the decision-maker ranks lotteries according to*

$$E\Big[h'(0)\min\{u(z^L), u(Z)\} + u'(0)\max\{Z - z^L, 0\}\Big]$$

where z^L is the unique solution of $u'(z^L) = u'(0) / h'(0)$.

Proof: In this case the objective function is

$$h(0) + h'(0)\big(Eu(Z) - u * - E\max\{u(Z) - u(\hat{z}), 0\}\big) + u'(0)E\max\{Z - \hat{z}, 0\}.$$

Discarding the irrelevant constant term $h(0) - h'(0)u*$, and observing that

$$Eu(Z) - E\max\{u(Z) - u(\hat{z}), 0\} = E\min\{u(\hat{z}), u(Z)\}$$ gives the expression for ranking

lotteries. Substituting into (*) and solving gives the solution for z^L.

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In addition to menu independence, Proposition 4 shows that in the linear case the

independence axiom is satisfied: lotteries are ranked according to an expected utility and

the weak axiom of revealed preference is satisfied. However, the linear model cannot

explain choices such as the common ratio or Allais paradox that violate the independence

axiom, nor can it explain the interaction of risk and delay (Baucells and Heukamp, 2010;

Keren and Roelofsma, 1995), or the "compromise effect" (Simonson, 1989).

Even when h is not linear, the preferences still correspond to an expected utility

theory on particular pairs of lotteries, namely those with "all small outcomes" and those

all of whose outcomes are large. The next two propositions say this formally:

Proposition 5: *Define z^C as the unique solution of $h'\big(u(z^C) - u*\big)u'(z^C) = u'(0)$.*

Then $z^C \geq z^L$ and for all $Z \in \Im$ we have $\arg\max_{z\geq 0} U(u*, Z, z) \geq z^C$. Moreover, if

all $Z \geq z^C$ then $\arg\max_{z\geq 0} U(u*, Z, z) = z^C$ and lotteries are ranked according to EZ.

Proof: Uniqueness follows from strict monotonicity of the LHS of the expression;

$z^C \geq z^L$ follows from the fact that $u(z^C) - u*$, the argument of h', is non-positive.

Plugging the solution $c_i* = \min\{z_i, \hat{z}\}$ into (*) we have

$h'\big(E\min\{u(\hat{z}), u(Z)\} - u*\big)u'(\hat{z}) = u'(0)$. Observe also that $\min\{u(\hat{z}), u(Z)\} \leq u(\hat{z})$

so that $h'\big(u(\hat{z}) - u*\big)u'(\hat{z}) \leq u'(0)$ and thus $z^C \leq \hat{z}$. Finally, if all $Z \geq z^C$ then

$\min\{u(z^C), u(Z)\} = u(z^C)$, so $h'\big(E\min\{u(z^C), u(Z)\} - u*\big)u'(z^C) = u'(0)$ meaning that the first order condition is satisfied. Then, Proposition 3 implies that $\arg\max_{z\geq 0} U(u*, Z, z) = z^C$, and plugging back into the objective function, we get $\max_z U(u*, Z, z) = h\big(u(z^C) - u*\big) + u'(0)E(Z - z^C)$, which is increasing in EZ.

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In other words, the decision-maker is risk neutral with respect to relatively large positive lotteries.

Proposition 6: *If $Z \leq z^L$ for all $Z \in \Im$ then lotteries are ranked according to $Eu(Z)$.*

Proof: Observe that $\hat{z} \geq z^L$ since the argument in h' of Proposition 3 is non-positive. Hence $Z \leq z^L$ implies $Z \leq \hat{z}$ and so the objective function may be written as $U(u*, Z, \hat{z}) = h\big(Eu(Z) - u*\big)$ from which the result follows.

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This says that the decision maker uses the short-run utility function to evaluate sufficiently small lotteries.

The suggestion of these results is that if h is strictly concave rather than linear, then for lotteries with outcomes that do not lie entirely above the cutoff z^C or entirely below the cutoff z^L the theory need not be an expected utility theory, and so may exhibit reversals such as those exhibited in the Allais or common ratio paradoxes.

Proposition 7: *Preferences over lotteries are consistent with stochastic dominance.*

Proof: Quiggin [1989] has shown that one lottery first order stochastically dominates another if and only if the lotteries can be realized as random variables on a common probability space, such that every realization of the dominant lottery is at least as great as the corresponding realization of the dominated lottery**.** Hence it is sufficient to consider whether utility is non-decreasing in the vector of values of Z. Since the objective function is $U(u^*, Z, z) = h\left(E\min\{u(Z), u(z)\} - u^*\right) + u'(0)E\max\{Z - z, 0\}$, and this is non-decreasing in the vector of values of Z, the same is true for $\max_z U(u^*, Z, z)$.

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Hence, our model predicts that people will tend to choose a lottery that is "clearly better" than another (dominated) lottery. "Better" is in the sense that any given monetary payoff is offered with at least as high probability as in the dominated lottery.

4. Found Money

A key role in the analysis is played by the cutoff z^L, which is the solution to $u'(z^L) = u'(0) / h'(0)$. To get an idea of how this cutoff works, it is interesting to examine the simplest possible decision problem: that of found money. Here $\Im$ is a singleton containing a single lottery that delivers a certain amount ζ. This can correspond to finding the amount ζ on the street. While in standard theory such gains will be spread over the entire lifetime, here when the amount is small, that is, less than the cutoff z^L, it will all be spent. Although this sounds like the description of the threshold $\hat{z}$, the cutoff z^L is different, and we will try to explain why.

When the agent finds money ζ the temptation is to spend all of it, so $u^* = u(\zeta)$. If the amount found is small, then the approximation we have introduced is valid, and the agent will choose consumption c to maximize

$$U^c(\tilde{c}, u^*, \tilde{z}) = h\big(u(c) - u^*\big) + u'(0)(\zeta - c) = h\big(u(c) - u(\zeta)\big) + u'(0)(\zeta - c).$$

The corresponding first order condition is $h'(u(c) - u(\zeta))u'(c) \geq u'(0)$ with equality if $c < \zeta$. Notice that the threshold $\hat{z}$ is defined by $h'(u(\hat{z}) - u(\zeta))u'(\hat{z}) = u'(0)$, so it depends on the earned amount ζ.

Now we use an example to illustrate the role of the cutoff z^L, which, unlike $\hat{z}$, does not depend on ζ. Assume logarithmic utility $u(c) = \log(x + c)$, where x is the exogenously given pocket cash, and also that $h(\Delta) = -A\exp(-\gamma\Delta)$ with $A \geq 1$, $\gamma \geq 1$. In this case, $u'(0) = 1/x$, and the objective function is

$$-A\exp[\gamma(\log(x + \zeta) - \log(x + c))] + (\zeta - c)/x =$$

$$-A[(x + \zeta)/(x + c)]^\gamma + (\zeta - c)/x.$$

Setting the derivative of this with respect to c equal to zero gives $A\gamma(x + \zeta)^\gamma / (x + c)^{\gamma+1} - (1/x) = 0$. It follows that the value of the threshold $\hat{z}$ for each value of ζ is given by $\hat{z} = [Ax\gamma(x + \zeta)^\gamma]^{1/(\gamma+1)} - x$. This has slope

$$d\hat{z}/d\zeta = [Ax\gamma/(x + \zeta)]^{1/(\gamma+1)}\gamma/(\gamma + 1) > 0,$$

which is decreasing in ζ. So the $\hat{z}(\zeta)$ line is continuous, strictly increasing and strictly concave in ζ, and $\hat{z}(0) = (A\gamma)^{1/(\gamma+1)}x - x \geq 0$.

Now specialize further to the case where $\gamma = 1$, so that $\hat{z} = x\sqrt{A(1 + \zeta / x)} - x$, with $d\hat{z} / d\zeta = \sqrt{A} / 2\sqrt{(1 + \zeta / x)}$. Notice that if $\zeta = 0$, $\hat{z} = x(\sqrt{A} - 1)$. Optimal consumption for each level of ζ is depicted in the 2-dimensional plane in Figure 1.

[FIGURE 1 SOMEWHERE HERE]

Remember that the role of the threshold $\hat{z}$ is that if the *realization* of the lottery is higher than it, then consumption is $\hat{z}$, but if the realization is less than $\hat{z}$, then consumption is equal to the realization. Here since the lottery is deterministic, the realization corresponding to ζ is simply ζ. The $\hat{z}(\zeta)$ line starts above zero, and it crosses the 45-degree line once at a point greater than zero. The cutoff point $z^L = x(A - 1)$, where it crosses this line, is critical for determining consumption. For the values of ζ where $\hat{z} > \zeta$ (at the left of z^L) the threshold is higher than ζ so optimal consumption is equal to ζ. For the values of ζ where $\hat{z} < \zeta$ (at the right of z^L) optimal consumption is equal to the threshold $\hat{z}$. Accordingly, optimal consumption is given by the thick grey line in Figure 1. Finally, notice that the cutoff satisfies $h'(u(z^L) - u(\zeta))u'(z^L) = u'(0)$, and $z^L = \zeta$, so that indeed $u'(z^L) = u'(0) / h'(0)$. Note in particular that in this case consumption goes to infinity as found money goes to infinity, even though the fraction that is spent goes to zero.

5. Three-Outcome Gambles

To better understand the implications of self-control preferences for choices among lotteries we examine lotteries with just three outcomes. This very simple case is

sufficient to illustrate some of the best-known departures from expected utility theory, namely as the Allais paradox[11] and the common ratio effect[12] (Allais, 1953; Kahneman and Tversky, 1979).

Assume that the possible lottery outcomes are $z_1 < z_2 < z_3$, with probabilities p_1, p_2, p_3 and corresponding short-run utilities $u_1 < u_2 < u_3$, where $u_i = u(z_i), i = 1, 2, 3$. As is traditional for lotteries of this type, we will work in the Marschak-Machina triangle: we take $p_2 = 1 - p_1 - p_3$ and plot p_1, p_3. Machina [1987] illustrates how the above stated anomalies, which violate the expected utility benchmark, can be captured by indifference curves that "fan out", or become steeper as one moves towards the northwest part of the triangle. Figures 2 and 3 illustrate and explain how fanning-out curves can generate the Allais paradox and the common ratio effect (respectively). Our objective in this section is to show that indifference curves with these characteristics can also be generated by our simple model, which therefore captures these departures from expected utility.

[FIGURE 2 SOMEWHERE HERE]

[FIGURE 3 SOMEWHERE HERE]

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[11] The paradox concerns choices between pairs of lotteries. We will remind readers of the original Allais experiment, which involves two choice scenarios. In Scenario I, Lottery s gives \$1m (one million) with probability 1.00 and Lottery r gives \$1m with probability 0.89, \$5m with probability 0.10 and \$0 with probability 0.01. In Scenario II, Lottery s' gives \$1m with probability 0.11 and \$0 with probability 0.89 and Lottery r' gives \$5m with probability 0.10 and \$0 with probability 0.90. Some subjects choose Lottery s in Scenario I but Lottery r' in Scenario II, which violates expected utility. The Allais paradox is part of a more general effect, called the "common consequence" effect.

[12] In experiments of the common ratio effect, there is also choice among pairs of lotteries, where each lottery involves the zero outcome and either outcome A or B, where $0 < A < B$. In Scenario 1, the small outcome (A) offered with a high probability (π) is preferred over the large outcome (B) with a low probability (ρ). However, when in Scenario 2 the probability for both positive outcomes is multiplied by the same number $0 < \xi < 1$, the choice is reversed in favor of the lottery with the large outcome.

For simplicity and because the Allais and common ratio paradoxes have this form, we consider only the gains domain, and moreover assume that the worst possible outcome is zero ($z_1 = 0$). We assume that $h'(0) > 1$ so that by Proposition 3 $z_1 < \hat{z}$.[13] Also recall from Proposition 6 that if $z_3 \leq z^L$ then preferences are just those of the short-run self. To avoid this uninteresting case we assume that $z_3 > z^L$ so that $u'(z_3) < u'(0) / h'(0)$ and $z_3 > \hat{z}$.[14]

As benchmarks, consider first constant expected value curves (those of a risk neutral agent) so that indifference curves in the triangle are given by lines of the form $p_3(z_3 - z_2) - p_1(z_2 - z_1) = k$. As a second benchmark, consider indifference curves for the short-run self which are given by lines of the form $p_3(u_3 - u_2) - p_1(u_2 - u_1) = k$. Because the short run self is risk averse, its indifference curves have a steeper slope than those of the risk neutral agent.

Turning to the self-control case, we will consider choices between pairs of gambles, that is menus with two items, $p = (p_1, p_3), q = (q_1, q_3)$, which we shall index $k \in \{p, q\}$. The utility to the gamble p in the menu $\{p, q\}$ is given by

$$V(p, q) \equiv \max_z U(\max\{Eu(Z_p), Eu(Z_q)\}, Z_p, z).$$

We may now define the indifference set $I(p) \equiv \{q \mid V(p, q) = V(q, p)\}$ and the corresponding indifference relation $q \sim p$ if $q \in I(p)$. Notice that this relation is

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[13] This highlights a special role of 0: it is less than the threshold $\hat{z}$ except in the non-self-control case where $h'(0) = 0$.

[14] To see that the latter inequality follows, note that if $z_3 \leq \hat{z}$ then plugging in (*) would imply that $z^L = \hat{z}$, which entails a contradiction to our assumption that $z_3 > z^L$.

reflexive but need not be transitive. To understand more clearly the indifference set, consider that it is defined implicitly by

$$\max_z U(\max\{Eu(Z_p), Eu(Z_q)\}, Z_p, z) - \max_z U(\max\{Eu(Z_p), Eu(Z_q)\}, Z_q, z) = 0.$$

We wish to examine the slope of this indifference set in the Marschak-Machina triangle. For this, we need to invoke the implicit function theorem and therefore need to show that this expression is continuously differentiable, at least in some neighborhood. The key fact is that it is indeed differentiable with respect to q at least in an open neighborhood of $q = p$ and at all points where $Eu(Z_p) \neq Eu(Z_q)$. To see this, evaluate the utility difference at the optimal value of z, that is, $\hat{z}_p, \hat{z}_q$:

$$\Phi(q_1, q_3, u*(q_1, q_3), \hat{z}_p(q_1, q_3), \hat{z}_q(q_1, q_3)) = U(u*, Z_p, \hat{z}_p) - U(u*, Z_q, \hat{z}_q). \quad (2)$$

We shall show that the indirect effects of a marginal change of q through its effect on $u*$ and $\hat{z}_p, \hat{z}_q$ can be ignored (in a neighborhood of $q = p$). First, the derivative of Φ with respect to $u*$ is zero at $q = p$, since

$$dU(u*, Z_p, \hat{z}_p) / du* - dU(u*, Z_q, \hat{z}_q) / du* \big|_{q=p} = 0.$$

Second, the derivative with respect to $\hat{z}_k$ is zero by Proposition 3.[15] Note that $\hat{z}_k$ is implicitly determined by the relation $F(Z_k, z_k) = u'(0)$ from Proposition 3. The function F is differentiable in $k = p, q$ by inspection and is strictly decreasing with left and right derivatives in z_k bounded away from zero by Proposition 3. Thus, the implicit

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[15] Note that now we need to ensure that $du*/dq$ and $d\hat{z}_k/dq$ are not infinity in order to avoid the indeterminate form $0 \times \infty$. It is clear that $du*/dq < \infty$, but we now need to make sure that $d\hat{z}_k/dq < \infty$.

function theorem applies to $\hat{z}_k$ as a function of q, hence since $\hat{z}_k$ is determined optimally, the envelope theorem implies that we need consider only the derivative with respect to Z_q. This dependence is differentiable by inspection.

In order for the indifference curves to fan out in such a manner that they explain the paradoxes, we need their slope to increase as we move towards the northwest of the triangle. We shall show that this is the case, because as we move in that direction the threshold $\hat{z}$ increases. Intuitively we expect that higher $\hat{z}$ corresponds to a more difficult self-control problem and that this should result in preferences – that is slopes of indifference curves – less like that of the long-run risk neutral self and more like the steeper sloped short-run indifference curves. This is verified by the next result.

Proposition 8: *The slope of $I(p)$ at the point $q = p$, denoted $S(p)$, is positive and greater than the slope of risk neutral indifference curves. The slope depends on p only through $\hat{z}$ and is increasing in $\hat{z}$.[16]*

Proof: It will be convenient to normalize so that $u_1 = 0, u_2 = z_2$. Then the slope of the risk neutral indifference curves is

$$\frac{z_2}{z_3 - z_2}.$$

We now compute the slope of the actual indifference curves at p, that is $S(p)$. First, the expressions for $U(u^*, Z_q, \hat{z}_q)$ may be written as

$$U(u^*, Z_q, \hat{z}_q) = h\left(E \min \left\{ u(Z_q), u(\hat{z}_q) \right\} - u^* \right) + u'(0) E \max \left\{ Z_q - \hat{z}_q, 0 \right\}$$

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[16] In particular, it can be written as a function $S(p) = \Sigma(\hat{z}(p))$ where Σ is strictly increasing in $\hat{z}$.

$$= h\Big((1 - q_1 - q_3)\min\big\{u(z_2), u(\hat{z}_q)\big\} + q_3 u(\hat{z}_q) - u^*\Big)$$
$$+ u'(0)[(1 - q_1 - q_3)\max\big\{z_2 - \hat{z}_q, 0\big\} + q_3(z_3 - \hat{z}_q)].$$

From this we may compute

$$\partial U(u^*, Z_q, \hat{z}_q) / \partial q_1 \mid_{u^*} =$$
$$-h'\big(E\min\big\{u(Z_q), u(\hat{z}_q)\big\} - u^*\big)[\min\{u(z_2), u(\hat{z}_q)\}] - u'(0)[\max\big\{z_2 - \hat{z}_q, 0\big\}]$$

$$= -\min\big\{u(z_2), u(\hat{z}_q)\big\} h'\Big((1 - q_1 - q_3)\min\big\{u(z_2), u(\hat{z}_q)\big\} + q_3 u(\hat{z}_q) - u^*\Big)$$
$$-u'(0)(\max\big\{z_2 - \hat{z}_q, 0\big\}).\ ^{17}$$

$$\partial U(u^*, Z_q, \hat{z}_q) / \partial q_3 \mid_{u^*} = h'\big(E\min\big\{u(Z_q), u(\hat{z}_q)\big\} - u^*\big)[u(\hat{z}_q) - \min\{u(z_2), u(\hat{z}_q)\}]$$
$$+ u'(0)[z_3 - \hat{z}_q - \max\big\{z_2 - \hat{z}_q, 0\big\}].$$

Using $h'\big(E\min\big\{u(Z_q), u(\hat{z}_q)\big\} - u^*\big) u'(\hat{z}_q) = u'(0)$ we have the proportionality where

the common factor $u'(0) / u'(\hat{z}_q)$ omitted:

$$\partial U(u^*, Z_q, \hat{z}_q) / \partial q_1 \mid_{u^*} = -[\min\{u(z_2), u(\hat{z}_q)\}] - u'(\hat{z}_q)[\max\big\{z_2 - \hat{z}_q, 0\big\}]$$

$$\partial U(u^*, Z_q, \hat{z}_q) / \partial q_3 \mid_{u^*} = [u(\hat{z}_q) - \min\{u(z_2), u(\hat{z}_q)\}] + u'(\hat{z}_q)[z_3 - \hat{z}_q - \max\big\{z_2 - \hat{z}_q, 0\big\}]$$

There are two cases.

Case 1: $z_3 > \hat{z}_q > z_2$. In this case, the partial derivatives are:

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[17] Notice that we are ignoring the partial derivative of U with respect to u^* since $d\Phi / du^* = 0$ at $q = p$ so that it will cancel out in the final computation of the slope.

$$\partial U(u^*, Z_q, \hat{z}_q)/\partial q_1 \mid_{u^*} = -[\min\{u(z_2), u(\hat{z}_q)\}] - u'(\hat{z}_q)[\max\{z_2 - \hat{z}_q, 0\}]$$
$$= -z_2$$

$$\partial U(u^*, Z_q, \hat{z}_q)/\partial q_3 \mid_{u^*} = [u(\hat{z}_q) - z_2] + u'(\hat{z}_q)[z_3 - \hat{z}_q].$$

This gives for the slope of indifference curve

$$dp_3 / dp_1 = \frac{z_2}{[u(\hat{z}_q) - z_2] + u'(\hat{z}_q)[z_3 - \hat{z}_q]} =$$
$$\frac{z_2}{z_3 - z_2 - (z_3 - u(\hat{z}_q)) + u'(\hat{z}_q)[z_3 - \hat{z}_q]}.$$

This will be steeper than in the risk neutral case provided that

$$z_3 - u(\hat{z}_q) > u'(\hat{z}_q)[z_3 - \hat{z}_q].$$

Since $u(\hat{z}_q) < \hat{z}_q$ it follows that $z_3 - u(\hat{z}_q) > z_3 - \hat{z}_q$. We normalized so that $u_1 = 0, u_2 = z_2$ and since $\hat{z}_q > z_2$ it follows that $u'(\hat{z}_q) < 1$. This gives the desired inequality.

Differentiating the denominator with respect to $\hat{z}_q$ we see that

$$d\{[u(\hat{z}_q) - z_2] + u'(\hat{z}_q)[z_3 - \hat{z}_q]\} / d\hat{z}_q = u'(\hat{z}_q) + u''(\hat{z}_q)[z_3 - \hat{z}_q] - u'(\hat{z}_q) = u''(\hat{z}_q)[z_3 - \hat{z}_q] < 0.$$

Thus, the indifference curves indeed get steeper as $\hat{z}$ increases.

Case 2: $z_2 > \hat{z}_q$. Now we have that:

$$\partial U(u^*, Z_q, \hat{z}_q)/\partial q_1 \mid_{u^*} = -u(\hat{z}_q) - u'(\hat{z}_q)[z_2 - \hat{z}_q]$$

$$\partial U(u^*, Z_q, \hat{z}_q)/\partial q_3 \mid_{u^*} = u'(\hat{z}_q)[z_3 - z_2].$$

It follows that the slope of an indifference curve is

$$\frac{z_2 - \hat{z}_q + u(\hat{z}_q) \,/\, u'(\hat{z}_q)}{z_3 - z_2} \;.$$

Since $\hat{z}_q < z_2$ and $u(z_2) = z_2$ by strict risk aversion $u(\hat{z}_q) > \hat{z}_q$. Again using the mean value theorem, since $u(0) = 0$ and by strict risk aversion, it follows that $u'(\hat{z}_q)\hat{z}_q < u(\hat{z}_q)$. This implies slope steeper than risk neutral.

Rewriting the slope we have

$$\frac{z_2 - \hat{z}_q + u(\hat{z}_q) \,/\, u'(\hat{z}_q)}{z_3 - z_2} = \frac{z_2 + \hat{z}_q\left(\dfrac{u(\hat{z}_q)}{u'(\hat{z}_q)\hat{z}_q} - 1\right)}{z_3 - z_2}$$

$$d[-\hat{z}_q + u(\hat{z}_q) \,/\, u'(\hat{z}_q)] \,/\, d\hat{z}_q = -1 + 1 - u(\hat{z}_q)u''(\hat{z}_q) \,/\, [u'(\hat{z}_q)]^2 = -u(\hat{z}_q)u''(\hat{z}_q) \,/\, [u'(\hat{z}_q)]^2 > 0.$$

This gives the desired result.

$$\text{☑}$$

Now that we have shown that the slope is increasing in $\hat{z}$, we need to examine how $\hat{z}$ behaves as p_3 increases and p_1 decreases (as we move in the northwest direction). In the following, we will be evaluating $\hat{z}$ at the point where $p = q$.

Proposition 9: The $\hat{z}_p$ *(corresponding to the singleton menu* $\{p\}$*) is increasing in* p_3, *and there is a critical value* $\hat{p}_3$ *such that if* $p_3 \leq \hat{p}_3$ *then* $\hat{z}_p$ *is decreasing in* p_1*, while if* $p_3 > \hat{p}_3$ *then* $\hat{z}_p$ *is independent of* p_1.

Proof: We work with the alternative version of the condition defining $\hat{z}$

$$h'\Big(E\min\{u(Z_p), u(\hat{z}_p)\} - u* \Big) u'(\hat{z}_p) = u'(0).$$

Writing that out in terms of p_1, p_3 we get

$$h'\Big((1 - p_1 - p_3)\min\{0, u(\hat{z}_p) - z_2\} + p_3 \min\{0, u(\hat{z}_p) - u_3\}\Big) u'(\hat{z}_p) = u'(0). \quad (**)$$

First we examine the dependence of $\hat{z}_p$ on p_3. Differentiating (**) with respect to p_3 we find

$$h''\Big((1 - p_1 - p_3)\min\{0, u(\hat{z}_p) - z_2\} + p_3 \min\{0, u(\hat{z}_p) - u_3\}\Big)$$
$$\times \Big(\min\{0, u(\hat{z}_p) - u_3\} - \min\{0, u(\hat{z}_p) - z_2\}\Big) u'(\hat{z}_p) \geq 0.$$

The inequality follows from the fact that $u_3 > z_2 = u_2$ and is strict since $\hat{z}_p < z_3$. Since the derivative of (**) is negative with respect to $\hat{z}_p$ (by Proposition 3) we can apply the implicit function theorem to conclude that

$$\frac{\partial \hat{z}_p}{\partial p_3} > 0.$$

Finally, we consider the derivative with respect to p_1. We shall show that this differs depending on whether or not $\hat{z}$ lies above or below z_2. To do this, we first solve

for the curve where $\hat{z} = z_2$. There is always a solution to $h'\left(E\min\{u(Z), u(\hat{z})\} - u*\right)u'(\hat{z}) = u'(0)$, (i.e. $\hat{z}$ is interior and the relevant first-order condition holds with equality) ; thus if $\hat{z} = z_2$ then

$$h'\left(p_3(z_2 - u_3)\right)u'(z_2) = u'(0). \qquad (***)$$

When (***) holds it implicitly defines a unique value of $\hat{p}_3$, with $\hat{z} > z_2$ for $p_3 > \hat{p}_3$ and $\hat{z} < z_2$ for $p_3 \leq \hat{p}_3$. If there is no solution to (***) then either $\hat{z} > z_2$ for all p_3, and we set $\hat{p}_3 = 0$, or $\hat{z} < z_2$ for all p_3, and we set $\hat{p}_3 = 1$.

Consider the case $p_3 > \hat{p}_3$; here since $u(z_2) = z_2$ (**) becomes

$$h'\left(p_3(u(\hat{z}) - u_3)\right)u'(\hat{z}) = u'(0).$$

This is indeed independent of p_1. Hence, in this region it holds that $\partial \hat{z}_p / \partial p_1 = 0$ as asserted. When $p_3 < \hat{p}_3$ we differentiate (**) with respect to p_1 to find

$$-h''((1 - p_1 - p_3)\min\{0, u(\hat{z}) - z_2\} + p_3\min\{0, u(\hat{z}) - u_3\})u'(\hat{z})$$
$$\times\min\{0, u(\hat{z}) - z_2\}$$

This expression is negative for $z_2 > \hat{z}$, hence $\hat{z}$ is decreasing in p_1 in this case.

☑

Now we discuss how these results imply that the approximate dual-self model can explain behavior such as the Allais paradox, using Figure 4 to illustrate the ideas. Scenario I in Allais-paradox experiments juxtaposes a lottery s with certain gain z_2

(located at the origin) with a risky lottery r that has a positive return, so it lies above the risk neutral indifference curve (the thick line crossing s). The data indicate that the latter tends to get rejected as too risky, so it has to lie below the actual indifference curve – the dashed line crossing s. In the figure the alternative Lottery r lies to the upper right in between the two indifference curves; so indeed, Lottery s will be preferred.

FIGURE 4 SOMEWHERE HERE]

[FIGURE 5 SOMEWHERE HERE]

Scenario II entails reducing the probability, for both lotteries, of the middle outcome z_2 and adding this probability to outcome $z_1 = 0$. This holds fixed the probability of z_3, so in the diagram it simply shifts both s and r the same distance to the right, resulting in the new lotteries s' and r'. If the probability of z_3 in Lottery s is less than $\hat{p}_3$, shifting Lottery s to the right (to become s') causes the actual indifference curve to get flatter. Depending on the exact magnitude of the change, it could shift preference, so that s' is now below r' rather than above r'. Hence an Allais reversal can occur, with s being chosen in Scenario I and the riskier alternative r' being chosen in Scenario II. This case is the one illustrated in the figure. Note that this reversal would not be possible if the indifference lines were parallel as in the standard model.

Our results show that if the probability of z_3 was larger than $\hat{p}_3$ this reversal could not occur (but remember that for the Allais paradox the probability of z_3 is in fact zero). This is illustrated in Figure 5, where the lotteries in the initial scenario have probability of the best outcome that exceeds $\hat{p}_3$. In this case, a mere shift of the lottery to

the right leads to a new lottery s', whose indifference curve does not have a different slope than $I(s)$, hence no reversal occurs.

This shows the importance of the fact that in the Allais experiments there is a large difference in the short-run expected payoffs across the two scenarios. It is the convexity in the self-control function that leads to reversals when the difference in these expected payoffs is sufficiently high. Now consider the common ratio paradox as depicted in Figure 3. In Scenario I the agent has a choice between a Lottery s with a high probability s_2 of winning outcome z_2 (or else yields zero) and a more risky Lottery r, which has a certain chance $r_3 < s_2$ of winning z_3 (or else zero). Again, the choice of s, which is observed in the data, corresponds to the case where lottery r lies between the risk-neutral indifference curve and the actual (steeper) indifference curve crossing s. In Scenario II, r shifts down and to the right, while s shifts to the right (notice that the vector from s to r gets shorter but continues to point in the same direction). If the indifference curve gets flatter there can again be a reversal.

This time the reversal occurs whether or not p_3 is smaller than $\hat{p}_3$, because not only does p_1 get larger, but also p_3 gets smaller and that always flattens the indifference curve. Note the implication here: this theory predicts that common ratio paradoxes hold for a wider variety of parameter values than common consequence, since the latter only occur below $\hat{p}_3$. The reason that common ratio scenarios always generate reversals in the approximate dual self model is that the short-run payoffs in the second scenario are always a fixed fraction of the payoffs in the initial scenario (assuming $u(0) = 0$), regardless of the position of the initial-scenario lotteries in the triangle.

Finally we consider the issue of whether preferences are transitive. Transitivity holds if and only if for every choice of lotteries p, q such that $q \in I(p)$ we have $I(p) = I(q)$. We now show that this need not be true. So, fix a lottery p and a lottery $q' \in I(p)$ and recall that when $z_2 > \hat{z}_q$ the slope of $I(p)$ at the point $q = p$ is given by

$$\frac{z_2 - \hat{z}_q + u(\hat{z}_q) / u'(\hat{z}_q)}{z_3 - z_2} \Big|_{q=p} = \frac{z_2 - \hat{z}_p + u(\hat{z}_p) / u'(\hat{z}_p)}{z_3 - z_2}.$$

We would like to show that when h is not linear the slope of $I(q')$ at p is different from this.[18] Recall that

$$U(u^*, Z_q, \hat{z}_q) =$$
$$h\left((1 - q_1)u(\hat{z}_q) - u^*\right) + u'(0)[(1 - q_1 - q_3)z_2 + q_3 z_3 - (1 - q_1)\hat{z}_q].$$

Notice that the FOC for a maximum with respect to $\hat{z}_q$ is

$$h'\left((1 - q_1)u(\hat{z}_q) - u^*\right)u'(\hat{z}_q) = u'(0).$$

Let $u(p')$ denote the short-run expected utility from a generic lottery p'. We may then rewrite the previous slope as

$$\frac{z_2 - \hat{z}_p + h'((1 - p_1)u(\hat{z}_p) - u(p))u(\hat{z}_p) / u'(0)}{z_3 - z_2}.$$

Without loss of generality, consider the case where $u(q') > u(p)$. We are then interested in what $I(q')$ looks like for lotteries r near p, so we may assume $u(q') > u(r)$; the indifference curve we are interested in is defined locally by the following relation:

$$G \equiv h\big((1 - q_1)u(\hat{z}_q) - u^*\big) + u'(0)[(1 - q_1 - q_3)z_2 + q_3 z_3 - (1 - q_1)\hat{z}_q] -$$
$$h\big((1 - r_1)u(\hat{z}_r) - u^*\big) + u'(0)[(1 - r_1 - r_3)z_2 + r_3 z_3 - (1 - r_1)\hat{z}_r] = 0.$$

We want to find dr_3 / dr_1 at $r = p$. Note that $u^* = u(q')$ is constant, and that we may

ignore the dependence of $\hat{z}_r$ on r by the envelope theorem as described above. Taking

the derivatives of G with respect to r_1, r_3 we have that:

$$dG / dr_1 = -h'\big((1 - r_1)u(\hat{z}_p) - u^*\big)u(\hat{z}_r) + u'(0)[\hat{z}_r - z_2]$$

$$dG / dr_3 = u'(0)[z_3 - z_2]$$

Accordingly, the slope of the indifference curve $I(q')$ at $r = p$ is equal to:

$$\frac{z_2 - \hat{z}_p + h'\big((1 - p_1)u(\hat{z}_p) - u(q')\big)u(\hat{z}_p) / u'(0)}{z_3 - z_2}.$$

Remember that the slope of $I(p)$ at $r = p$ was equal to:

$$\frac{z_2 - \hat{z}_p + h'((1 - p_1)u(\hat{z}_p) - u(p))u(\hat{z}_p) / u'(0)}{z_3 - z_2}.$$

These are the same if and only if

$$h'((1 - p_1)u(\hat{z}_p) - u(q')) = h'((1 - p_1)u(\hat{z}_p) - u(p)).$$

But since $u(q') > u(p)$ by assumption, this equation requires that h' be locally constant.

The "expected regret" models of Loomes and Sugden [1982] and Fishburn [1982]

also generate intransitive preferences. Like the dual-self model, the key characteristic of

these models is the dependence of preferences on the choice menu. Like our model, these

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[18] Notice that if p does not belong to $I(q')$ there is nothing to prove.

models generate "indifference curves" that cross under certain assumptions; they can also explain the common ratio and common consequence effects.

6. Discussion and Conclusion

We have examined how the assumption of a linear long-run value function can lead to a more tractable model than the model of Fudenberg and Levine [2011], and we showed how this simplified model is useful in explaining choice among lotteries. The model respects stochastic dominance, and for lotteries with very low and very high possible prizes, lottery choice corresponds to the maximization of short-run expected utility and expected value, respectively. Restricting attention to the case of two lotteries with three outcomes, we show how the model can generate indifference curves that "fan out" and thus can explain the well-known Allais and common ratio paradoxes.

As we have pointed out, models such as "expected regret" can capture the static risk-based anomalies that we have discussed here, and in addition generate the classic preference reversal phenomenon and other paradoxes. However, the general dual-self model has a wider scope, in the sense that it is consistent with a large number of facts, across different domains. In particular, the model has predictions about time-related phenomena (such as preference reversals for delayed rewards), risk-related phenomena (such as the ones described here), contextual psychological phenomena (such as the effect of cognitive load), etc. At the same time, the model is consistent with modern macroeconomic theory and evidence. In addition, the derivation of risk preference and reversals from an underlying model of self-control has implications about correlations between an individual's choices between lotteries and her choices in other domains that

are not present in alternative theories such as prospect theory. It also makes predictions that the same individual may make different choices between lotteries depending on other decision problems that have recently been faced. In effect the h function is determined by past behavior and personal characteristics.

Psychological evidence indicates that self-control depends on a resource that resembles "strength" or a "muscle" (Muraven, and Baumeister 1998, Baumeister et al. 1998). In particular, repeated use of self-control within short time intervals depletes the "stock of willpower", and rest is needed in order to for this stock to recover. Further, like a muscle, the ability to exercise self-control can be enlarged by repeatedly exercising it. This time dimension in self-control is analyzed in Fudenberg and Levine [2012]. In particular an individual who has recently faced difficult self-control problems – and so depleted her stock of self-control – will exhibit a higher cost of self-control as measured by h.

Second, different individuals have different degrees of past exercise of self-control, and therefore different capacities of using it. There is some evidence that the ability to exercise self-control is heterogeneous and correlated with such positive outcomes as scholarly achievement, interpersonal skills, and less alcohol abuse (Tagney, Baumeister and Boone, 2004). So for example we may expect individuals who have a history of addiction and alcohol abuse to have a higher cost of self-control as measured by h. Finally, as shown in O' Donoghue and Rabin [1999] self-control costs imply an individual exhibits present bias. Hence we may expect individuals who exhibit greater present bias to have a higher cost of self-control as measured by h.

Finally, recent evidence indicates that higher cognitive load, by reducing the psychological resources available for self-control, leads to higher self-control costs.[19] Like other things such as recent difficult self-control problems, an individual with a higher cognitive load will exhibit a higher h. Cognitive load can be easily controlled in the laboratory, typically by selectively assigning memory tasks to different treatment groups. This means that the theory implies that reversals like those of common ratio and common consequence can be induced by increasing cognitive load.[20]

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[19] For example, Shiv and Fedorikhin [1999] find that subjects who are under heavy cognitive load, having to remember a seven-digit number, tend to choose a - relatively unhealthy - chocolate cake dessert rather than a more healthy fruit salad. On the contrary, subjects who had to memorize a two-digit number tended to opt for the fruit salad more often. This indicates that higher cognitive load might have impaired subjects' ability to exercise self-control.

[20] Note that there is some initial evidence of this, such as the results from the experiment of Benjamin, Brown and Shapiro [2006] who find that cognitive load tends to exacerbate small stakes risk aversion and – to a lesser degree – short run impatience.

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FIGURES

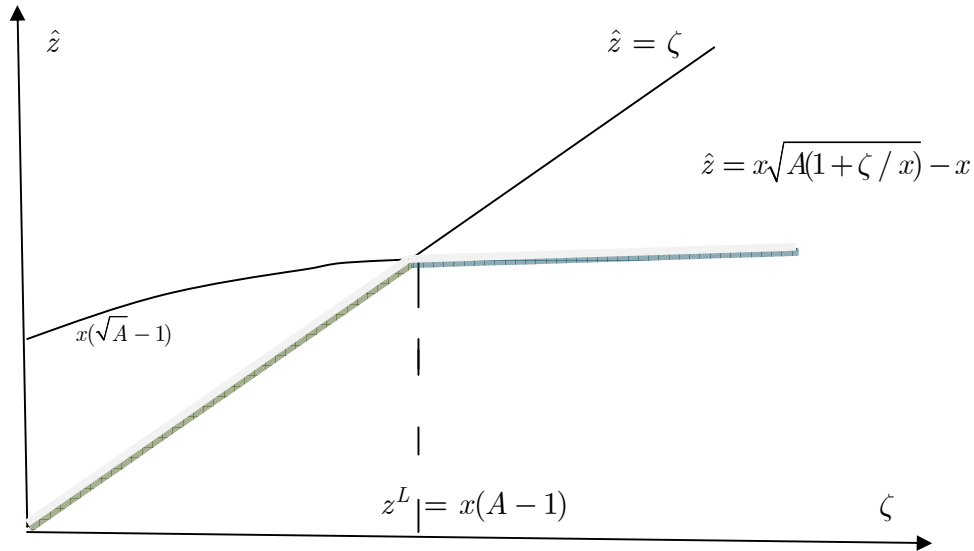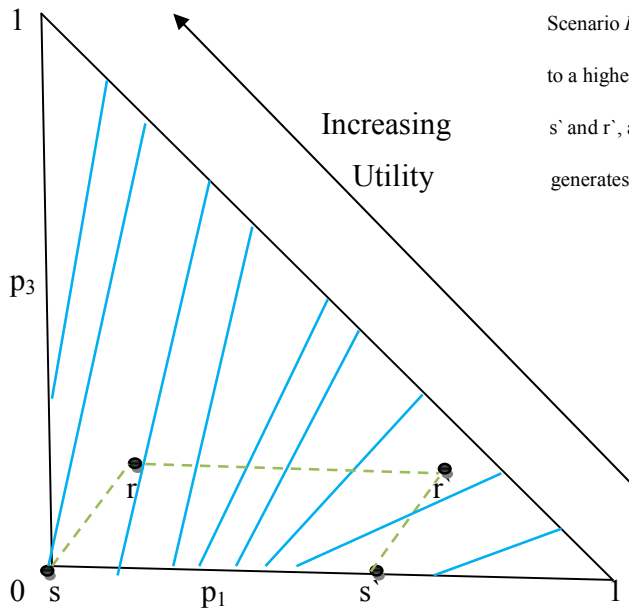*Figure 1:* **The Consumption Threshold as a Function of the Amount of Found Money**

$$\hat{z}$$

$$\hat{z} = \zeta$$

$$\hat{z} = x\sqrt{A(1+\zeta \,/\, x)} - x$$

$$x(\sqrt{A}-1)$$

$$z^{L} = x(A-1)$$

$$\zeta$$

Figure 2. **The Allais Paradox and Fanning-Out Curves**

Increasing
Utility

Scenario *I* concerns the choice between s and r, and s corresponds
to a higher indifference curve. Scenario *II* concerns choice between
s` and r`, and now r` is crossed by the higher indifference curve. This
generates the paradox: s is chosen in Scenario *I* and r` in Scenario *II*.

1

p_3

r

r

0   s

p_1

s

1

Figure 3. **The Common Ratio Paradox and Fanning-Out Curves**
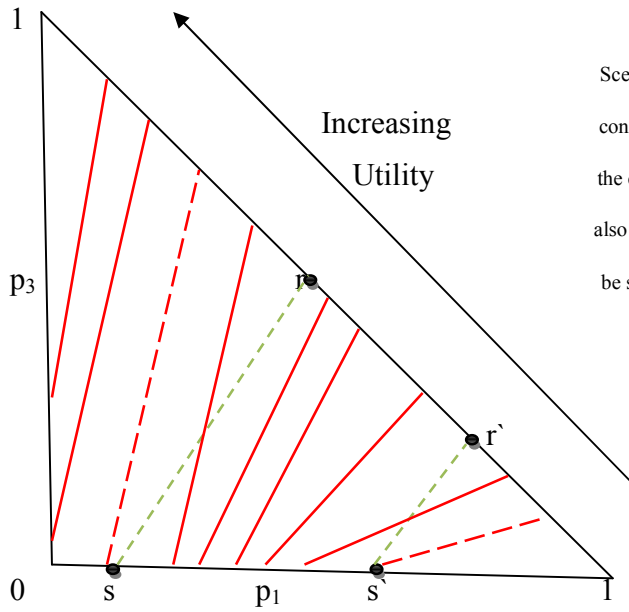
Scenario *I* concerns the choice between s and r and Scenario *II* concerns the choice between s` and r`. The paradox involves the choice of s in Scenario *I* and of r` in Scenario *II*. This can also be captured by indifference curves that "fan out", as can be seen in the figure.

Figure 4. **The Allais Paradox Reversal in the Approximate Model**
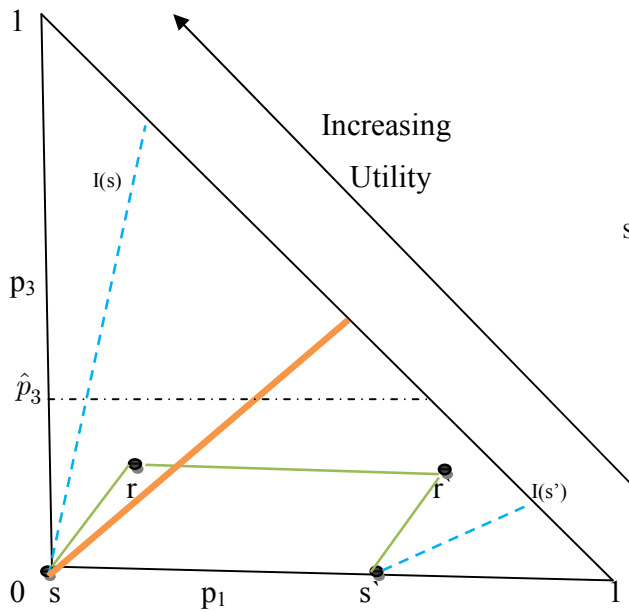
Scenario I concerns the choice between s and r

Scenario II concerns the choice between s` and r`

Figure 5. **The High Expected Payoff Case with No Reversal**

Scenario I concerns the choice between s and r

Scenario II concerns the choice between s` and r`