What Inventories Tell Us About Aggregate Fluctuations -- A Tractable Approach To (S,s) Policies

Pengfei Wang,
Yi Wen
and
Zhiwei Xu

Working Paper 2011-014A
https://doi.org/10.20955/wp.2011.014

March 2014

FEDERAL RESERVE BANK OF ST. LOUIS
Research Division
P.O. Box 442
St. Louis, MO 63166

When Do Inventories Destabilize the Economy?
— An Analytical Approach to (S,s) Policies*

Pengfei Wang[a] Yi Wen[b,c] Zhiwei Xu[a]

This Version: May 10, 2011

Abstract

Conventional wisdom has it that inventory investment destabilizes the economy because it is procyclical to sales. Khan and Thomas (2007) show that the conventional wisdom is wrong in a general equilibrium (S,s) model with capital. We argue that their finding is not robust—the conventional wisdom can still hold in general equilibrium if firms can adjust output by varying the capacity utilization rate. Our result also holds true if there exist investment adjustment costs. Unlike the existing (S,s) inventory literature that relies on the Krusell-Smith (1998) numerical solution methods, we characterize (S,s) inventory policies in closed form despite the large state space in our general equilibrium model. Standard log-linearization methods can be used to solve the model and generate impulse response functions.

Keywords: Inventory Investment, Great Moderation, Business Cycle, Capacity Utilization, (S,s) Policy, State-Dependent Decisions.

JEL codes: E13, E22, E32.

*a: Hong Kong University of Science & Technology; b: Federal Reserve Bank of St. Louis; c: Tsinghua University. Pengfei Wang acknowledges the financial support from the Hong Kong Research Grants Council (project #643908). The usual disclaimer applies. Correspondence: Pengfei Wang, Department of Economics, Hong Kong University of Science & Technology, Hong Kong, China. Phone: 852-2358-7612. Email: pfwang@ust.hk.

1 Introduction

A long-held belief in the business cycle literature has been that inventories amplify the variance of production and are consequently destabilizing to the economy (see, e.g., Blinder 1981, 1986, 1990; and Blinder and Maccini, 1991). This belief is based on the overwhelming empirical evidence that inventory investment is procyclical (or equivalently, production is more volatile than sales).[1]

This conventional wisdom is recently challenged by general equilibrium theory because it is based on a partial equilibrium argument: by the accounting identity, output equals sales plus inventory investment. Therefore, *given sales*, a positive covariance of inventory investment with sales increases the variance of output, making production more volatile than sales. Hence, procyclical inventory behavior is destabilizing.

Khan and Thomas (KT 2007a) present a general equilibrium model where this conventional argument is not true. In their model, firms hold inventories to economize on the fixed costs of ordering inputs and, consequently, inventory investment is procyclical when the aggregate price of intermediate input is lowered by technology shocks. However, this procyclical inventory investment does not lead to a more volatile economy because in general equilibrium inventories affect both production and sales. In the KT model, inventories facilitate final goods production by reducing the average ordering costs of intermediate goods. Thus, a positive productivity shock to the intermediate goods sector would enable final goods firms to expand production more than they could without intermediate goods inventories. On the other hand, producing more intermediate goods requires more resources such as labor, which would divert resources away from the final goods sector to the intermediate goods sector; so the final goods sector does not expand as much as it would otherwise. Hence, a procyclical inventory investment is buffered by a weakened final goods production due to resource reallocation, making the volatility of the aggregate economy essentially unchanged. Put alternatively, since both inventory investment and final sales effectively enter the same aggregate resource constraint, there is a tradeoff between inventory accumulation versus consumption and capital investment. Thus, larger fluctuations in inventory investment are accompanied by smaller fluctuations in aggregate sales, implying that the variability of gross domestic product (GDP) is essentially unaffected by the existence of inventories (Khan and

[1]See Blanchard (1983), Eichenbaum (1989), Haltiwanger and Maccini (1988), Kahn (1987, 1992), Ramey (1989, 1991), Ramey and West (1999), and West (1986), among many others.

Thomas, 2007a, p1166).

This finding of KT is provocative. It shows the power of general equilibrium analysis in reevaluating conventional wisdoms, and it suggests that inventories may be irrelevant for understanding the business cycle despite its seemingly large share (contribution) in GDP volatility. However, we show in this paper that the KT model is not robust: adding standard economic features into the KT model, such as variable capacity utilization or investment adjustment costs, can overturn their result and reproduce the conventional wisdom.

The intuition behind this is simple. Variable capital utilization is a purely "local input" that does not compete with resources in other sectors, and it makes the supply of goods more elastic, thus relaxing the tension between inventory investment and capital investment in the KT model. On the other hand, adjustment costs make fixed investment rigid (or more expensive to adjust relative to inventory investment), thus mitigating the crowding-out effect of inventory investment on capital investment. When a modified KT model with either capacity utilization or adjustment costs is calibrated to the U.S. data, we found that inventories can significantly destabilize the economy, raising the variance of GDP by 12-18% compared to a counterpart benchmark model without inventories.

The sales-smoothing role of inventories identified by KT therefore does not necessarily imply that inventory is not destabilizing to the economy. We can borrow intuitions from a standard real business cycle (RBC) model: introducing capital would make an RBC model's aggregate output much more volatile although capital can smooth consumption (final sales) dramatically. Introducing inventories can thus have similar consequences.

Empirical evidence indicates that the Great Moderation has been associated with lower inventory-to-sales ratios in the U.S. manufacturing sector. Based on this evidence, a segment of the existing literature has argued that the reduction of inventories (because of improvement in inventory management technology since the 1980s) is partially responsible for the Great Moderation (see, e.g., Kahn, McConnell, and Perez-Quiros, 2002; Davis and Kahn, 2008; among others). Although a positive association between inventories and economic volatility does not imply causality or necessary connections between the two, our result does provide a (micro-founded) general equilibrium rationale for the informal argument that lowered inventory-to-sales ratios may be partially responsible for the Great Moderation.

We derive (S,s) inventory policies in our model analytically, in contrast to KT who solved their model numerically. Solving (S,s) inventory policies in closed form in general equilibrium is nontrivial. The presence of fixed ordering costs in an (S,s) inventory model yields a discrete ordering decision, which makes a firm's dynamic programming problem nonconvex. In addition, the occasionally binding non-negative nature of inventory holdings imposes a nonlinear constraint on a firm's inventory stock, which makes a firm's value function not differentiable everywhere. General equilibrium analysis compounds the difficulties because in general equilibrium, one needs to track the distribution of inventory holdings at the firm level for any given macro state space (such as the aggregate capital stock, lagged aggregate investment, inventory distributions, and aggregate shocks), yet part of the macro state space is itself determined by the sum of individual firms' actions. Due to the curse of dimensionality, numerical computation methods, such as the one proposed by Krusell and Smith (1998) and adopted by KT, become increasingly difficult to implement if the state space is relatively large, as is the case with capital and investment adjustment costs (which introduce lagged investment into the model).

To overcome these hurdles, we adopt a strategy similar to that used by Dotsey, King, and Wolman (1999) and Thomas (2002).[2] Due to the i.i.d nature of fixed ordering costs, we show that all ordering firms have the same inventory target regardless of their inventory level in the previous period. And given a firm's inventory level in the previous period, the ordering decision follows a trigger (cutoff) strategy. Firms will order if and only if the fixed cost is below a unique threshold. Such a structure implies that firms are distinguished only by the time since their last order was made, so the distribution of inventories in the economy is discrete with finite support points and the optimal cutoff for each vintage firm group is history-independent. That is, regardless of the history of idiosyncratic shocks, firms that have placed orders in period $t - j$ will have the same amount of inventories if they have not ordered in the last j periods. In addition, firms that opt to order in the current period will replenish their inventory to the same level regardless of their existing inventory level. So we can group firms according to when their last order was made. This leads to a block-recursive structure in the model, which permits exact aggregation and closed-form characterization of the general equilibrium. Hence, the aggregate variables in the model form a system of nonlinear rational expectations equations that look identical to those in a standard representative-agent model. Standard solution methods available in the RBC

[2] However, in an (S,s) inventory model, the problem at the firm level is more complex than that in the Dotsey-King-Wolman model or in Thomas (2002). In our model, an inactive firm also needs to solve a dynamic optimization problem to determine the optimal inventory level, whereas in the state-dependent pricing model inactive firms simply set the current price to the previous level and in the lumpy-investment model inactive firms simply set the current investment to zero. For earlier literature on state-dependent (S,s) inventory policies, see Caplin (1985), Caplin and Leahy (1991), Caballero and Engel (1991), and Fisher and Hornstein (2000).

literature (such as log-linearization around the steady state and higher-order perturbation methods) can then be applied to solve for aggregate dynamics and generate impulse response functions.

Two caveats are in order. First, inventories can exist in an economy for many different reasons, such as to reduce fixed order costs (as studied by KT), to avoid stockout when demand is uncertain and production takes time (Kahn, 1987), to speculate for profits (Samuelson, 1971), and to smooth production (Blinder, 1986) or production costs (Eichenbaum, 1989).[3] As illustrated by Wang and Wen (2009) and Wen (2011), different incentives for inventory investment can have dramatically different implications for the (de)stabilizing role of inventories. Conclusions drawn from one type of inventory models therefore do not generalize to other types of inventory models. For example, although our analyses show that adding capacity utilization and investment adjustment costs into an (S,s) inventory model can make inventories significantly destabilizing, it is not so for the stockout-avoidance model of Wen (2011). In fact, adding either capacity utilization or investment adjustment costs into the general equilibrium model of Wen (2011) does not make inventories investment destabilizing to the economy, in contrast to the (S,s) inventory model studied in this paper.[4] Such diverse implications of different inventory models are interesting, as they may offer a potential litmus test to gauge which mechanisms are more important in explaining aggregate inventory dynamics. Therefore, it would be worthwhile to conduct systematic comparisons of these different inventory models and subject them to nested econometric tests. Because this task is beyond the scope of this paper, we leave it for the future.

Second, that capacity utilization or investment adjustment costs can improve the empirical fit of business cycle models is well known (see, e.g., Greenwood, Hercowitz, and Huffman, 1988; King and Rebelo, 1999; Christiano, Eichenbaum, and Evans, 2005; Jaimovich and Rebelo, 2009; and Wen, 1998). The KT model is no exception (see Engelhardt, 2011; and the results presented in Section 5 of this paper). However, improving the empirical fit of the KT model is not the focus of this paper. Rather, we show that once some standard features of business cycle models are added into the KT model, the major conclusion of KT regarding the role of inventories in the business cycle will no longer hold. We argue that our conclusion

[3] For recent literature on general equilibrium inventory models, see Alessandria, Kaboski, and Midrigan (2010), Fisher and Hornstein (2000), Khan and Thomas (2007b), Kryvtsov and Midrigan (2008), Wang and Wen (1009), and Wen (2008, 2011), among others.

[4] In the model of Wen (2011), firms hold inventories to avoid stockout under demand shocks. Because of an endogenously chosen procyclical probability of stockout, firms always charge higher prices in booms than in recessions, regardless of capacity utilization or investment adjustment costs. This procyclical relative price stabilizes aggregate demand and economic fluctuations.

will continue to hold in more general (S,s) inventory models than the one studied by KT because capacity utilization is a universal yet highly "localized" production factor.

The rest of the paper is organized as follows. Section 2 presents a generalized KT model with two new features: capacity utilization and investment adjustment costs. Section 3 characterizes general equilibrium and derives the firm's (S,s) inventory rules in closed form. Section 4 studies the steady state distributions of inventories and compares the results with those of KT. Section 5 studies the dynamics of the model and shows that inventories can significantly destabilize the economy under either capacity utilization or investment adjustment costs. Section 6 concludes the paper.

2 The Model

We extend the KT model by allowing for investment adjustment costs and variable capacity utilization. Notice that these features will render numerical solution techniques, such as the one proposed by Krusell and Smith (1998) and adopted by KT, difficult to implement because the state space is further enlarged by lagged investment. However, these features do not impose additional difficulties on our analytical solution method.[5]

The economy has three types of agents: households, intermediate goods producers, and final goods firms. Households derive utility from consumption and leisure according to a quasi-linear utility function with indivisible labor. Households supply labor to all the firms and purchase consumption goods from the final goods firms. Intermediate goods firms produce output using capital and labor. They also accumulate capital by making fixed investment, which is subject to investment adjustment costs. Intermediate goods producers can also vary the capital utilization rate to adjust production level. The final goods firms must pay fixed (stochastic) costs to order intermediate goods and they combine intermediate goods with labor to produce final goods. The final goods can be used as either consumption goods or investment goods. Given the structure of this economy, final goods firms have incentives to carry inventories to smooth ordering costs intertemporally.

2.1 Households

All households are identical (with a unit mass) and labor supply is indivisible. Households supply labor to both the final goods sector and the intermediate goods sector. Due to perfect

[5] We can also introduce lagged consumption (habit formation) and other lagged variables into our model without adding significantly to our computing task.

labor mobility, the real wages are equalized across the two sectors. The final good is used as the numeraire.

A representative household chooses consumption (C_t) and labor supply (N_t) to solve

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left[\log C_t + \tau(1 - N_t)\right] \tag{1}$$

subject to the budget constraint

$$C_t \leq W_t N_t + \Pi_t, \tag{2}$$

where W_t is the wage rate, and Π_t is the aggregate profits from all the firms. Households behave competitively, and their first-order conditions are

$$\Lambda_t = \frac{1}{C_t} \tag{3}$$

$$\tau = \Lambda_t W_t, \tag{4}$$

where the marginal utility Λ_t is also the shadow price of consumption goods. So $\beta \Lambda_{t+1}/\Lambda_t$ will be the pricing kernel for a firm's market value.

2.2 Intermediate Goods Firms

A large number of identical intermediate goods firms combine capital K_t and labor L_t to produce intermediate goods and make investment to accumulate capital. A representative intermediate goods firm maximizes the discounted future dividends:

$$E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} (P_t X_t - W_t L_t - I_t), \tag{5}$$

where P_t is the price of intermediate goods, X_t is the output, and I_t is the total investment expenditure. Given its pre-determined capital stock K_t, the intermediate goods firm can vary its capital utilization rate e_t and labor input L_t to produce output according to the technology:

$$X_t = A_t \left(e_t K_t\right)^{\alpha} L_t^{1-\alpha}, \tag{6}$$

where the aggregate technology evolves according to

$$\log A_t = \rho \log A_{t-1} + v_t, \quad v_t \sim iid(0, \sigma_v^2). \tag{7}$$

We assume that the depreciation rate of capital is strictly increasing and convex in e_t:

$$\delta_t = \delta_0 + \delta_1 e_t{}^{\gamma}, \quad \gamma > 1. \tag{8}$$

Investment is subject to investment adjustment costs, so the law of capital accumulation is given by

$$K_{t+1} = \left[1 - \delta\left(e_t\right)\right] K_t + \left[1 - \varphi\left(\frac{I_t}{I_{t-1}}\right)\right] I_t. \tag{9}$$

The adjustment cost function $\varphi(\cdot)$ is strictly increasing and convex with the property that $\varphi(1) = \varphi^{'}(1) = 0$ and $\varphi^{''}(1) > 0$.

Denoting η_t as the Lagrangian multiplier for equation (9), the first-order conditions for $\{K_{t+1}, e_t, I_t, L_t\}$ are given, respectively, by

$$\beta E_t\left[\frac{\Lambda_{t+1}}{\Lambda_t}\left(\alpha P_{t+1}\frac{X_{t+1}}{K_{t+1}} + \left[1 - \delta\left(e_{t+1}\right)\right]\eta_{t+1}\right)\right] = \eta_t, \tag{10}$$

$$\alpha P_t \frac{X_t}{e_t K_t} = \eta_t \delta'\left(e_t\right), \tag{11}$$

$$1 = \beta E_t\left[\frac{\Lambda_{t+1}}{\Lambda_t}\eta_{t+1}\varphi'\left(\frac{I_{t+1}}{I_t}\right)\left(\frac{I_{t+1}}{I_t}\right)^2\right] + \eta_t\left[1 - \varphi\left(\frac{I_t}{I_{t-1}}\right) - \varphi'\left(\frac{I_t}{I_{t-1}}\right)\frac{I_t}{I_{t-1}}\right], \tag{12}$$

$$\left(1 - \alpha\right) P_t \frac{X_t}{L_t} = W_t. \tag{13}$$

2.3 Final Goods Firms

The key part of the model is the final goods sector where inventories are held. final goods firms combine intermediate goods with labor to produce output. There is a fixed cost involved for each firm when ordering intermediate goods. To minimize the average cost of ordering, firms opt to carry inventories to smooth ordering costs intertemporally according to an (S,s) rule. So final goods firms will be heterogenous in their inventory positions.

A typical final goods firm produces output y_t according to the production function,

$$y_t = m_t^{\theta_m} n_t^{\theta_n}, \tag{14}$$

where n_t denote labor and m_t denotes intermediate goods input. Following KT, the fixed ordering cost is paid in labor units. Denoting x_t as the size of an order, the total cost of an order is then given by $P_t x_t + \varepsilon_t W_t$, where P_t is the relative price of intermediate goods, ε_t is the fixed cost measured in labor units, and $\varepsilon_t W_t$ is the total fixed cost of placing an order. Following the existing (S,s) policy literature (e.g., Caballero and Engel, 1999), ε_t is assumed to be independently and identically distributed across time and firms, with the cumulative distribution function $F(\varepsilon)$. This distribution has a finite support in the positive domain with upper bound $\bar{\varepsilon}$. Denoting s_t as the existing inventory level carried over from the last period, the law of motion for inventory accumulation is given by

$$s_{t+1} = s_t + x_t - m_t. \tag{15}$$

There are storage costs involved in holding inventories. The storage cost is assumed to be proportional to the level of inventories, σs_{t+1}. The aggregate state of the economy is denoted by $\Omega_t = \{A_t, K_t, I_{t-1}, \mu_t\}$, which includes the aggregate technology shock A_t, the capital stock, and the lagged aggregate investment I_{t-1}, plus the distribution of firms' existing inventory stocks μ_t. Given the firm-level state $\{s_t, \varepsilon_t\}$ and the aggregate state Ω_t, the value function of a firm can be denoted by $V(s_t, \varepsilon_t, \Omega_t)$ or $V(s_t, \varepsilon_t)$ for short.

A final goods firm's profit maximization problem is to solve

$$V_t(s_t, \varepsilon_t) = \max_{m_t, s_{t+1}, x_t} \{m_t^{\theta_m} n_t^{\theta_n} - \sigma s_{t+1} - P_t x_t - W_t(n_t + \varepsilon_t \mathbf{1}_{x_t \neq 0}) + \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} V_{t+1}(s_{t+1}, \varepsilon_{t+1})\} \tag{16}$$

subject to equation (15) and the following non-negativity constraints:

$$s_{t+1} \geq 0 \tag{17}$$

$$m_t \geq 0 \tag{18}$$

$$n_t \geq 0, \tag{19}$$

where $\mathbf{1}_{x_t \neq 0}$ in the objective function is an index function, which equals 1 if an order is placed in period t and zero otherwise. The solution to (16) is a set of sequences, $n_t(s_t, \varepsilon_t)$, $x_t(s_t, \varepsilon_t)$, $m_t(s_t, \varepsilon_t)$, and $s_{t+1}(s_t, \varepsilon_t)$. Notice that it may be optimal for a firm not to produce in period t with $m_t = 0$ and $n_t = 0$.

2.4 Competitive Equilibrium

A competitive general equilibrium is a set of aggregate quantities for households and intermediate goods firms, $\{C_t, N_t, X_t, K_{t+1}, L_t, e_t, I_t\}$, market prices, $\{\eta_t, P_t, W_t, \Lambda_t\}$, firm level quantities for final goods firms, $\{n_t(s_t, \varepsilon_t), x_t(s_t, \varepsilon_t), m_t(s_t, \varepsilon_t), s_{t+1}(s_t, \varepsilon_t)\}$, and the distribution of firms' inventory stocks, $\{\mu_{t+1}\}$, such that households maximize utilities, firms maximize profits, and all markets clear. Namely, a general equilibrium is characterized by the following conditions:

1. C_t, N_t and Λ_t satisfy equations (3) and (4).
2. $X_t, K_{t+1}, L_t, e_t, \eta_t, I_t$ satisfy equations (6) to (13).
3. $\{n_t(s_t, \varepsilon_t), x_t(s_t, \varepsilon_t), m_t(s_t, \varepsilon_t), s_{t+1}(s_t, \varepsilon_t)\}$ solves (16).
4. Labor market clears

$$N_t = L_t + \int \int \left[n_t(s, \varepsilon) + \varepsilon \mathbf{1}_{x(s,\varepsilon) \neq 0} \right] d\mu_t dF. \tag{20}$$

5. intermediate goods market clears

$$X_t = \int \int x_t(s, \varepsilon) d\mu_t dF. \tag{21}$$

6. Final goods market clears

$$C_t + I_t = \int \int y_t(s, \varepsilon) d\mu_t dF, \tag{22}$$

where $y_t(s, \varepsilon) = m_t^{\theta_m} n_t^{\theta_n}$ is the production level of a final goods firm with inventory level s_t and fixed ordering cost ε_t.

7. The evolution of inventory stocks across firms is characterized by

$$\mu_{t+1}(S) = \int \int \mathbf{1}_{s_{t+1}(s,\varepsilon) \leq S} d\mu_t dF. \tag{23}$$

where $\mu_{t+1}(S) \equiv \Pr[s_{t+1} \leq S]$ denotes the commutative distribution function of inventory stocks across final goods firms in period $t+1$, and $\mathbf{1}_{s_{t+1}(s,\varepsilon) \leq S}$ is an index function.

3 Characterization of Inventory Decision Rules

The above discussions suggest that as long as we can analytically solve for final goods firms' decision rules, $\{n_t(s_t, \varepsilon_t), x_t(s_t, \varepsilon_t), m_t(s_t, \varepsilon_t), s_{t+1}(s_t, \varepsilon_t)\}$, the general equilibrium can then be characterized analytically. The purpose of this section is to show that the competitive equilibrium can be described by a system of closed-form nonlinear difference equations and thus solvable by standard techniques available in the representative-agent DSGE literature.

We call firms placing orders in period t "active firms" and those not placing orders "inactive firms". A final goods firm's inventory decision rule can be characterized by a cutoff strategy: placing an order if $\varepsilon_t \leq \varepsilon_t^*(s_t)$ and remaining inactive if $\varepsilon_t > \varepsilon_t^*(s_t)$.

Proposition 1 *Denoting an active firm's optimal level of intermediate goods input by $m_{0,t}$ and the optimal inventory stock carried over to the next period by $s_{1,t+1}$, a final goods firm's optimal decision rules for intermediate goods demand (m_t), labor demand (n_t), and inventory holdings (s_{t+1}) are given by*

$$m_t = \begin{cases} m_{0t} & \text{if } \varepsilon_t \leq \varepsilon_t^*(s_t) \\ m_t(s_t) & \text{if } \varepsilon_t > \varepsilon_t^*(s_t) \end{cases} \tag{24}$$

$$n_t = \begin{cases} \left(\frac{\theta_n}{W_t}\right)^{\frac{1}{1-\theta_n}} m_{0t}^{\frac{\theta_m}{1-\theta_n}} & \text{if } \varepsilon_t \leq \varepsilon_t^*(s_t) \\ \left(\frac{\theta_n}{W_t}\right)^{\frac{1}{1-\theta_n}} m(s_t)^{\frac{\theta_m}{1-\theta_n}} & \text{if } \varepsilon_t > \varepsilon_t^*(s_t) \end{cases} \tag{25}$$

$$s_{t+1} = \begin{cases} s_{1,t+1} & \text{if } \varepsilon \leq \varepsilon_t^*(s_t) \\ s_t - m_t(s_t) & \text{if } \varepsilon > \varepsilon_t^*(s_t) \end{cases}, \tag{26}$$

where $\{m_{0,t}, s_{1,t+1}\}$ are state-independent, i.e., independent of the firm's existing inventory stock s_t and the history of firm-specific cost shocks ε_t.

Proof. See Appendix I. ■

The inventory decision rule (26) implies that (i) all firms that decide to order intermediate goods in period t will replenish their inventories to the same level and thus carry the same amount of inventories forward into the next period regardless of their individual history; and (ii) all firms that have placed their last order in period $t-j$ will have the same existing inventory stock at the beginning of period t regardless of their history. The same logic applies to intermediate goods demand and labor demand since these variables depend on s_t. Therefore, firms are distinguished only by the time since their last order of intermediate goods was made. This property greatly simplifies the analysis and permits exact aggregation of final goods firms' decision rules. But because inactive firms' decisions are state-dependent, we need to characterize the distribution of firms based on the time since their last order was made.

In anticipation of the results, assume that there are finite types of final goods firms distinguished by their inventory holdings at the start of the period, s_t. We can divide all firms into vintage groups $j = 1, 2, ...$, where j is a positive integer. For example, $s_{j,t}$ denotes the inventory level at the beginning of period t for firms that placed their last order in period $t-j$, and analogously $s_{j+1,t+1}$ denotes the inventory level at the start of period $t+1$ for firms that placed their last order in period $t-j$. However, $s_{j+1,t+1} \leq s_{j,t}$ because of inventory depletion, unless a new order is placed.

As equation (26) suggests, a firm will eventually run out of stock if it has not ordered for a sufficiently long period of time due to consecutive bad shocks. Let J be the biggest possible value of j such that $s_{J,t} > 0$ in period t. This means that if some firms have not ordered for $J+1$ periods (or longer), they will have zero inventory in period t, so $s_{J+k,t} = 0$ for all $k \geq 1$. We can group those firms with zero inventory into the same vintage group and label this group as vintage $J+1$. The fraction of vintage j firms in the total population is denoted by $\omega_{j.t}$. Obviously, $\sum_{j=1}^{J+1} \omega_{j,t} = 1$.

Hence, the distribution of inventory stocks across firms is discrete. At the start of each period t, there exists a fraction $\omega_{j,t}$ of firms with inventory level $s_{j,t}$. The distribution $\omega_{j,t}$ evolves according to the following simple mechanism. In period t, firms will place an order if and only if the fixed cost facing them is below the threshold $\varepsilon_t^*(s_{j,t})$, or $\varepsilon_{j,t}^*$ for short. For these active firms, their inventory level will be adjusted immediately to $s_{1,t+1}$ after placing an order. So the total number of firms who have just placed an order in period t and hence have inventory stock $s_{1,t+1}$ in period $t+1$ is given by

$$\omega_{1,t+1} = \sum_{j=1}^{J+1} F(\varepsilon_{jt}^*)\omega_{jt}, \tag{27}$$

which is a discrete version of equation (23).

On the other hand, for each vintage j, there are $\left[1 - F(\varepsilon_{j,t}^*)\right] \omega_{j,t}$ number of firms that do not order in period t. These firms evolve according to

$$\omega_{j+1,t+1} = \left[1 - F(\varepsilon_{j,t}^*)\right] \omega_{j,t} \text{ for } j = 1, 2, ..., J-1. \tag{28}$$

The total fraction of firms with zero inventories at the start of period $t+1$ can consist of both vintage J firms and vintage $J+1$ firms (notice that a firm in vintage $J+1$ will remain in that way if it does not order):

$$\omega_{J+1,t+1} = \left[1 - F\left(\varepsilon_{J,t}^*\right)\right] \omega_{J,t} + \left[1 - F\left(\varepsilon_{J+1,t}^*\right)\right] \omega_{J+1,t}. \tag{29}$$

The graphical presentation of the evolution of the cross-sectional distribution of firms in our model is analogous to that of Thomas (2002, p.516, Figure 1).

Since there are $J+1$ types of firms and each type has a unique cutoff, the next step is to determine vintage j firms' inventory stock $s_{j,t}$ $(j = 1, 2, ..., J+1)$, inputs of intermediate goods $m_{j,t}$ $(j = 0, 1, 2, ..., J)$, and the associated cutoff $\varepsilon^*_{j,t}$ $(j = 1, 2, ..., J+1)$.[6] Once we have determined $m_{j,t}$, we can then determine employment using equation (49) and the output level using production function. The complication involved is that all of these variables depend on the value functions of active firms and inactive firms.

Proposition 2 *Given the state of the aggregate economy Ω_t, the system of equations to jointly determine the following set of $3(J+1)+1$ variables, $\left\{\{s_{j,t+1}\}_{j=1}^{J+1}, \{m_{j,t}\}_{j=0}^{J}, \{\varepsilon^*_{j,t}\}_{j=1}^{J+1}, V_t^a\right\}$, is a set of value functions and firms' choices given by the following $3(J+1)+1$ equations:*

$$V_t^a = R_t m_{0t}^{\theta} - \sigma s_{1,t+1} - P_t\left(m_{0,t} + s_{1,t+1}\right) + \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\left[V_{t+1}^a + P_{t+1}s_{1,t+1} - W_{t+1}\int \min\left\{\varepsilon, \varepsilon^*_{1,t+1}\right\} dF(\varepsilon)\right] \tag{30}$$

$$V_t^a + P_t s_{j,t} - W_t \varepsilon^*_{j,t} = R_t m_{j,t}^{\theta} - \sigma s_{j+1,t+1} + \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\left[V_{t+1}^a + P_{t+1}s_{j+1,t+1} - W_{t+1}\int \min(\varepsilon, \varepsilon^*_{j+1,t+1}) dF(\varepsilon)\right], \quad j = 1, 2, ...J \tag{31}$$

$$V_t^a + P_t s_{J+1,t} - W_t \varepsilon^*_{J+1,t} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\left[V_{t+1}^a - W_{t+1}\int \min(\varepsilon, \varepsilon^*_{J+1,t+1}) dF(\varepsilon)\right] \tag{32}$$

$$\theta R_t m_{j,t}^{\theta-1} + \sigma = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\left[F(\varepsilon^*_{j+1,t+1})P_{t+1} + (1 - (F(\varepsilon^*_{j+1,t+1}))\theta R_{t+1} m_{j+1,t+1}^{\theta-1}\right], \quad j = 0, 1, ..., J-1 \tag{33}$$

$$s_{J+1,t+1} = 0 \tag{34}$$

$$\theta R_t m_{0t}^{\theta-1} = P_t \tag{35}$$

$$m_{jt} = s_{jt} - s_{j+1t+1}, \quad j = 1, 2, ..., J; \tag{36}$$

where $\theta = \frac{\theta_m}{1-\theta_n}$ and $R_t = (1-\theta_n)\left(\frac{\theta_n}{W_t}\right)^{\frac{\theta_n}{1-\theta_n}}$.

[6] Recall that $s_{J+1,t} = 0$ and $m_{J+1,t} = 0$. Firms with zero inventories also have a different cutoff, $\varepsilon^*_{J+1,t}$, in period t. This is why we let the index of cutoff run up to $J+1$.

Proof. See Appendix II. ■

Equation (30) is the value function of active firms in period t with zero beginning-period inventories. Equations (31) and (32) are the value functions of inactive firms ($V_{j,t}^n$) in vantage $j = 1, 2, ..., J+1$. In both equations, we have substituted $V_{j,t}^n$ with $V_t^a + P_t s_{j,t} - W_t \varepsilon_{j,t}^*$ using the cutoff equation (55) and the relation $V_{j,t}^a = V_t^a + P_t s_{j,t}$. Equations (35) and (36) are the policy functions for material input $m_{j,t}$, $j = 0, 1, 2, ..., J$.

Equations (33) and (34) are the optimality conditions for choosing the next-period inventory stock $s_{j+1,t+1}$, $(j = 0, 1, 2, ..., J)$. In particular, equation (34) is based on the definition for vantage $J+1$ and equation (33) is the Euler equation for intertemporal tradeoff between the marginal cost of increasing inventories today and the marginal benefit of having more inventories tomorrow.

Specifically, when $j = 0$, the left-hand side (LHS) of equation (33) equals $P_t + \sigma$ (based on equation (35)), which is the active firm's marginal cost of increasing the inventory stock by placing a new order: for each unit of additional inventories the firm pays P_t to order and σ to store the goods. The right-hand side (RHS) of equation (33) is the marginal gain of increasing the inventory stock. After ordering, the firm becomes a vintage $j = 1$ firm in the next period. It has a probability $F\left(\varepsilon_{1,t+1}^*\right)$ of placing a new order and in such a case one additional unit of inventories will save the firm by $P_{t+1} = \frac{\partial V_{t+1}^a(s_{1,t+1})}{\partial s_{1,t+1}}$ in ordering cost in period $t+1$. There is a probability $\left(1 - F(\varepsilon_{1t+1}^*)\right)$ that the firm will not order, in which case one additional unit of inventories generates $\theta R_{t+1} m_{1,t+1}^{\theta-1}$ units of profits. Equation (33) thus states that the optimal inventory level for an active firm $(s_{1,t+1})$ must be such that it makes the benefits and costs equal in the margin.

When $j = 1, 2, ..., J-1$, the LHS of equation (33) is the marginal cost of carrying one additional unit of inventories forward for an inactive firm of vintage j. Increasing the inventory stock by one unit (without ordering) reduces the firm's operating revenue by $\theta R_t m_{jt}^{\theta-1}$ units and incurs σ units of storage costs. On the other hand, the RHS is the benefit of having one additional unit of inventories available in the next period. With probability $F(\varepsilon_{j+1,t+1}^*)$ the firm will place a new order, in which case one additional unit of existing inventories can reduce the ordering cost by P_{t+1}. With probability $1 - F(\varepsilon_{j+1t+1}^*)$, this firm will not order and in such a case one additional unit of inventories can increase the firm's operating revenue by $\theta R_{t+1} m_{j+1t+1}^{\theta-1}$ units.

4 Steady State

4.1 The System of Aggregate Equations

Denoting the aggregate variables by capital letters, we have a dynamic system consisting $4(J+1)+15$ variables,

$$V_t^a, \{\varepsilon_{jt+1}^*\}_{j=1}^{J+1}, \{s_{jt}\}_{j=1}^{J+1}, \{m_{jt}\}_{j=0}^{J}, \{\omega_{jt}\}_{j=1}^{J+1}, C_t, N_t, X_t, S_t, M_t, L_t, e_t, \eta_t, I_t, P_t, W_t, R_t, \Lambda_t, K_t.$$

Among these variables, 14 are aggregate variables and $4(J+1)+1$ are firm-specific variables pertaining to inventory distributions. To solve for the competitive general equilibrium, we thus need $4(J+1)+15$ equations, which are listed below.

Labor market clearing implies

$$N_t = L_t + \sum_{j=0}^{J-1} n_{jt}\omega_{j+1t+1} + n_{Jt}\left[1 - F\left(\varepsilon_{Jt}^*\right)\right]\omega_{Jt} + \sum_{j=1}^{J+1}\omega_{jt}\int_{\varepsilon<\varepsilon_{jt}^*}\varepsilon dF\left(\varepsilon\right), \tag{37}$$

where $n_{jt} = \left(\frac{\theta_n}{W_t}\right)^{\frac{1}{1-\theta_n}} m_{jt}^{\frac{\theta_m}{1-\theta_n}}$ for $j = 0, 1, 2, ..., J$. The aggregate inventory at the beginning of period t is

$$S_t = \sum_{j=1}^{J+1}\omega_{j,t}s_{j,t}. \tag{38}$$

The total intermediate goods input is

$$M_t = \sum_{j=0}^{J-1} m_{jt}\omega_{j+1t+1} + m_{Jt}\left[1 - F\left(\varepsilon_{Jt}^*\right)\right]\omega_{Jt}. \tag{39}$$

intermediate goods market clearing requires

$$X_t = S_{t+1} + M_t - S_t. \tag{40}$$

final goods market clearing implies

$$C_t + I_t = Y_t \equiv \sum_{j=0}^{J-1} y_{j,t}\omega_{j+1,t+1} + y_{J,t}\left[1 - F\left(\varepsilon_{J,t}^*\right)\right]\omega_{J,t}, \tag{41}$$

where $y_{j,t} = R_t m_{j,t}^{\theta}/(1-\theta_n) - \sigma s_{j+1,t+1}$, with $s_{J+1,t+1} = 0$. In the intermediate goods firm's profit function, R_t is defined by

$$R_t = (1-\theta_n)\left(\frac{\theta_n}{W_t}\right)^{\frac{\theta_n}{1-\theta_n}}. \tag{42}$$

In addition, we have the first-order conditions for households in equations (3) and (4), and the first-order conditions for intermediate goods firms in equations (6)-(13). These together constitute 14 equations. The remaining $4(J+1)+1$ equations come from equations (27) to (36).

4.2 Steady-State Distributions

A steady state is a situation without aggregate uncertainty (i.e., $A_t = \bar{A}$), in which all aggregate variables and the distribution of inventories are constant over time. Under the assumptions that $\varphi'(1) = \varphi(1) = 0$ for adjustment costs and $e = 1$ for the capacity utilization rate in the steady state, the steady state of our model is identical to that of KT model. Hence, these assumptions facilitate comparisons between the results in our model and those in their model.[7]

The detailed steps for solving the steady state, especially the steady-state distribution of inventories, are provided in the Appendix. The key is to determine the relative price of intermediate goods, P. Given P, we can solve for the steady-state wage W using equation (13) and the value of R. Then equations (27)-(36) can be used to solve for $\left\{V^a, \{\varepsilon_j^*\}_{j=1}^{J+1}, \{s_j\}_{j=1}^{J+1}, \{m_j\}_{j=0}^{J}, \{\omega_j\}_{j=1}^{J+1}\right\}$. Given these firm-level variables, the aggregate variables can be solved easily using equations (37)-(42).

Calibration. The parameters specific to our model include γ and $\varphi''(1)$. The rate of capacity utilization is variable if $\gamma < \infty$ and investment incurs adjustment costs if $\varphi''(1) > 0$. Since these parameters do not affect the steady state, we defer their calibrations to the later sections.

The fixed order cost is assumed to follow a uniform distribution with support $[0, \bar{\varepsilon}]$, as in the KT model. All of the common structural parameters between our model and the KT model are set to values as those by KT and summarized in Table 1.

[7] Assuming $e < 1$ in the steady state does not affect our results significantly.

Table 1. Common Parameters and Their Values

β	τ	α	θ_m	θ_n	δ	σ	$\bar{\varepsilon}$	ρ	σ_v
0.984	2.128	0.374	0.499	0.328	0.017	0.012	0.220	0.956	0.015

Table 2 shows the steady-state distributions of inventory-holding firms in our model, which replicate the results of KT (2007a, Table 2, p. 1177) up to the third digit.[8] Using the words of KT, this table shows that firms are distributed over six levels of inventories at the start of the period, reflecting six vintage groups. This vintage distribution is in columns labeled from 1 to 6, while the first column (labeled active firms) represents those firms from each of these six groups that undertake inventory adjustment prior to production. The inventory level selected by all adjusting (active) firms is 1.702 in the steady state. Firms that adjusted their inventory holdings in the last period (those in column 1) begin the current period with 1.163 units of the intermediate good and a low probability of adjustment, $F(\varepsilon_1^*) = 0.034$. Because inventory holdings decline with the time since the last order, firms are willing to accept larger adjustment costs as they move from vintage 1 across the distribution to vintage 6. The existence of fixed order costs implies that the adjustment probability is less than one for all vintage groups. In fact, even among the 0.017 firms that begin the period with no inventory, only 83.5 percent adjust prior to production. The remainder forego production in the current period and await lower adjustment costs.

Table 2. Steady-State Distribution of final goods Firms

Vintage (j)	Active firms	1	2	3	4	5	6
Distribution $\omega(s_j)$		0.266	0.257	0.224	0.160	0.076	0.017
Inventories s_j	1.702	1.163	0.712	0.349	0.098	0.002	0.000
Fraction adjusting $F(\varepsilon_j^*)$		0.034	0.129	0.287	0.526	0.807	0.835

5 The Destabilizing Force of Inventories

5.1 Control Model

To examine whether inventories are destabilizing in our general model with variable capacity utilization or investment adjustment costs, we compare several stochastic versions of our model with counterpart control models in which there are no inventories (i.e., $\bar{\varepsilon} = 0$).[9] In

[8] The minor difference may be due to numerical approximations. KT adopted a cubic spline approximation for solving the value functions of firms, while we compute the value functions recursively by a set of closed-form nonlinear equations.

[9] As pointed out by KT, when $\bar{\varepsilon} = 0$, any final goods firm can order the exact quantity of intermediate goods it will use in current production without suffering delivery costs. In this case, the firm will opt not to carry any inventories.

the presence of aggregate technology shocks, our models can all be solved by log-linearizing around the steady state. We generate artificial time series from these models and use the HP filtered data to estimate the models' business cycle moments. Gross domestic product (GDP) in this paper is measured as the sum of aggregate final goods output plus the value of intermediate goods inventory investment based on the value-added approach:

$$GDP_t = C_t + I_t + P_t \left(X_t - M_t \right). \tag{43}$$

The general control model is specified below. Because final goods firms do not carry inventories, they are identical in equilibrium. So the control model can be cast as a social planner's problem:

$$\max_{C_t, e_t, I_t, M_t, N_t, L_t} E_0 \sum_{t=0}^{\infty} \beta^t \left[\log \left(C_t - \chi C_{t-1} \right) + \eta (1 - N_t - L_t) \right] \tag{44}$$

subject to

$$C_t + I_t \leq M_t^{\theta_m} N_t^{\theta_n} \tag{45}$$

$$M_t = A_t \left(e_t K_t \right)^{\alpha} L_t^{1-\alpha} \tag{46}$$

$$K_{t+1} = (1 - \delta(e_t)) + \left[1 - \varphi \left(\frac{I_t}{I_{t-1}} \right) \right] I_t. \tag{47}$$

Specific versions of the control model can be obtained by shutting down one or both of the two channels: capacity utilization and investment adjustment costs.

5.2 Replicating KT's Findings

We first check if our solution methods can successfully replicate the results of KT by turning off capacity utilization and investment adjustment costs (i.e., by setting $\gamma = \infty$ and $\varphi''(1) = 0$) in our inventory model. We find that the quantitative predictions of this model are very close to those reported by KT. In particular, the volatility of GDP is essentially the same with or without inventories. Table 3 reports the volatilities of selected model variables relative to the volatility of GDP.

Table 3. Business Cycle Moments ($\gamma = \infty$, $\varphi''(1) = 0$)

	Final Sales	Inventory Investment	Inventory-to-Sales Ratio
A. Volatility relative to GDP			
U.S. Data	0.710	0.295	0.545
Inventory Model	0.846	0.177	1.044
KT	0.839	0.188	0.742
B. Correlation with GDP			
U.S. Data	0.943	0.669	-0.381
Inventory Model	0.995	0.890	-0.935
KT	0.994	0.880	-0.991

The table shows that our results are very close to those computed by KT using entirely different solution methods. For example, in panel A the volatility of sales relative to GDP is 0.846 when our log-linearization method is used. This number is 0.839 when the Krusell-Smith numerical method reported by KT is used instead. The lower panel (B) shows that the inventory model successfully accounts for two major features of inventory dynamics: that inventory investment is strongly procyclical (middle column) and that the inventory-to-sales ratio is countercyclical (last column).

Table 4. Contribution of Inventories to Volatility ($\gamma = \infty$, $\varphi''(1) = 0$)

	GDP	FS	C	I	H	K	X	M
A. Volatility relative to GDP								
Control Model	(1.650)	1.000	0.407	7.086	0.644	0.404	1.597	1.597
Inventory Model	(1.676)	0.846	0.364	6.101	0.695	0.361	1.688	1.374
B. Correlation with GDP								
Control Model		1.000	0.922	0.978	0.970	0.407	0.998	0.998
Inventory Model		0.995	0.888	0.988	0.973	0.342	0.999	0.987

FS: final sales; C: consumption; I: investment; H: employment; K: capital; X: order of intermediate goods; M: intermediate goods input.

Table 4 reports the volatilities of selected model variables in our inventory model and the counterpart control model. As the second column (labeled GDP) in panel A shows, the existence of inventories barely changes the volatility of GDP. Compared with the control model without inventories, the inventory model increases the volatility of GDP by only 1.58% (this value is 1.51% as reported by KT). The main reason for this irrelevance result is that the excess volatility introduced by procyclical inventory investment is exactly offset by the reduced variability of final sales (consumption and capital investment), as shown in the column labeled FS where the volatility of final sales is reduced by 14% with inventories as opposed to without ($0.846 \times \frac{1.676}{1.650} = 0.86$).

5.3 Effects of Capacity Utilization

To isolate the effects of capacity utilization, we set $\gamma > 0$ and $\varphi''(1) = 0$, so only investment adjustment costs are shut off. Following Rebelo and Jaimovich (2009), we set $\gamma = 1.15$ for the depreciation elasticity of capacity utilization. As discussed before, allowing for variable capacity utilization does not change the steady state of the model. Hence, the model's ability to match the average inventory-to-stock ratio is not affected by capacity utilization.

Table 5 shows that adding capacity utilization to the model does not deteriorate the model's ability to account for the business cycle moments. Inventory investment remains procyclical and the inventory-to-sales ratio remains countercyclical. In fact, capacity utilization actually improves the performance of the benchmark model along some important dimensions (as also noted by Engelhardt, 2011). For example, it improves the positive correlation of inventory investment with GDP and the negative correlation of the inventory-to-sales ratio with GDP.

Table 5. Business Cycle Moments ($\gamma = 1.15$, $\varphi''(1) = 0$)

	Final Sales	Net inventory Investment	Inventory-Sales Ratio
A. Volatility relative to GDP			
U.S. Data	0.710	0.295	0.545
Capacity Model	0.792	0.346	0.393
KT	0.839	0.188	0.742
B. Correlation with GDP			
U.S. Data	0.943	0.669	-0.381
Capacity Model	0.951	0.711	-0.838
KT	0.994	0.880	-0.991

One of the most important findings of this paper is reported in Table 6 (panel A, first column labeled GDP), where it is shown that inventories significantly destabilize the economy. The standard deviation of GDP in the control model (with capacity utilization but without inventories) is 2.898 while this value is 3.428 in the counterpart inventory model. So the standard deviation of GDP increases by 18% with inventories as opposed to without inventories. This is in sharp contrast to the case with fixed capacity utilization reported in Table 4. That is, even though capacity utilization increases the volatility of GDP in the control model—2.898 in Table 6 versus 1.650 in Table 4, adding inventories into the capacity model increases the volatility of GDP significantly further.

Notice that the volatility of final sales in the capacity model is also reduced by the existence of inventories, but not to the same extent as in the benchmark model with fixed capacity. For example, the volatility of final sales relative to GDP in the inventory model is 0.792, implying a 20% reduction in the volatility of final sales relative to the volatility of GDP. However, the volatility of final sales in the inventory model is 94% of that in the control model ($\frac{3.428}{2.898} \times 0.792 = 93.6\,\%$), implying only a 7% reduction in the volatility of final sales relative to the control model. But this figure was 14% in the benchmark model with inventories. In the meantime, the volatility of input orders (the column labeled X) is 32% larger than that in the control model ($\frac{1.658 \times 3.428}{1.488 \times 2.898} = 1.32$). Hence, with variable capacity utilization, the excess volatility introduced by procyclical inventory investment to GDP is far larger than the negative contribution to GDP volatility from the reduced volatility of final sales. This shows that capacity utilization mitigates the tension between inventory investment and final sales (especially capital investment).

Table 6. Contribution of Inventories to Volatility ($\gamma = 1.15$, $\varphi''(1) = 0$)

	GDP	FS	C	I	H	K	X	M
A. Volatility relative to GDP								
Control Model	(2.898)	1	0.225	8.481	0.789	0.403	1.488	1.488
Inventory Model	(3.428)	0.792	0.193	6.869	0.836	0.345	1.658	1.191
B. Correlation with GDP								
Control Model		1.000	0.952	0.997	0.996	0.196	0.999	0.999
Inventory Model		0.951	0.914	0.950	0.995	-0.095	0.996	0.953

FS: final sale; C: consumption; I: investment; H: employment; K: capital; X: order of intermediate goods; M: intermediate goods input.

The intuition is as follows. A positive technology shock reduces the rpices of intermediate goods. This induces not only the active final goods firms to increase the size of their orders but also some of the inactive firms to place orders. This incentive for building up inventory stocks to reduce future fixed order costs increases the aggregate demand for intermediate goods more than in the case with the control model. However, with fixed capacity utilization, the only way to increase intermediate output is if labor input is increased because capital is predetermined. Thus, a sharp increase in the production of intermediate goods to satisfy inventory demand is possible if labor is diverted from the final goods sector to the intermediate goods sector so that the increase of labor input in the final goods sector is less than it would otherwise be. This reallocation of labor reduces the volatility of final goods production and thus offsets the positive contribution of inventory investment to GDP volatility, generating the KT result.

With variable capacity utilization, however, intermediate goods production can be increased without necessarily increasing labor input in this sector, regardless of inventories. So the general-equilibrium effect uncovered by KT—namely, labor is diverted from the final goods sector to the intermediate goods sector—is not an issue. That is, a rising inventory demand for intermediate goods can be met by a higher rate of capacity utilization even without labor reallocation. Given this, even if labor were reallocated from the final goods sector to the intermediate goods sector to the same extent as in the KT model, it would not completely offset the positive effect of inventory investment on GDP volatility.

In other words, because capacity utilization is a "local factor" of production, it does not compete with the final goods sector for resources. Hence, the general-equilibrium tradeoff between inventory investment and final sales in the original KT model is attenuated. This suggests that our result should continue to hold even in more general (S,s) models (such as a model in which both the final goods sector and the intermediate goods sector use capital in production), precisely because capacity utilization is a local input. Our finding thus suggests that inventories can still be significantly destabilizing to the economy even though they may reduce the volatility of final sales in general equilibrium (and they do as Table 6 shows).

Kahn, McConnell, and Perez-Quiros (2002) show that the inventory-to-sales ratio in the U.S. was 0.7 between 1953-1983, but started to decline since 1983 and became 0.4 by the end of 2000. They argue that improvements in information technology or inventory management during that period was responsible for the reduction in the inventory-to-sales ratio, which in turn could explain the Great Moderation that started in middle 1980s. Suppose we set the upper bound of the support of idiosyncratic shock in the model to $\bar{\varepsilon} = 0.142$ (this value was 0.22 in the previous analyses), then the steady-state inventory-to-sales ratio in the capacity model will decrease from 0.71 to 0.452. Consequently, the standard deviation of GDP in the model becomes 3.022, which is only 4.28% greater than that in the control model (see Table 6). In other words, a reduction in the inventory-to-sales ratio (with a magnitude similar to that observed in the U.S.) can lead to about 14% decrease in the volatility of GDP, which is about one third of the reduction in GDP volatility during the Great Moderation. Similar results hold for the model with IAC.

Sensitivity analysis. We can also show that the destabilizing effect of inventories on GDP increases with the variability of capacity utilization. In Table 7, the first row represents the values of γ and the second row the relative volatility of the inventory model to the control model. As the value of γ increases, it becomes more costly to adjust capacity utilization rate, so the destabilizing role of inventories diminishes. However, the lower part of Table 7 shows that regardless of the value of γ, the capacity model always implies a countercyclical inventory-to-sales ratio and a procyclical inventory investment, as in the data.

Table 7. Sensitivity Analysis (Capacity Utilization)

γ	1.01	1.05	1.15	1.4	2.0	3.0
$\left(\frac{\text{GDP volatility with inventory}}{\text{GDP volatility without inventory}}\right)$	1.35	1.27	1.18	1.10	1.05	1.03
$Corr$ (Inventory/sales, GDP)	-0.582	-0.726	-0.838	-0.896	-0.920	-0.928
$Corr$ (Inventory Investment, GDP)	0.689	0.697	0.710	0.736	0.776	0.811

5.4 Effects of Investment Adjustment Costs (IAC)

To isolate the effects of IAC, we set $\gamma = \infty$, and follow Jaimovich and Rebelo (2009) by setting $\varphi''(1) = 1.3$, so there exist costs of adjusting capital investment outside the steady state. Table 8 shows that, as in the case of capacity utilization, IAC improve the fit of the KT model in several dimensions, including the correlation with GDP for inventory investment and the inventory-to-sales ratio. IAC also reduce the volatility of final sales and boost the variance of inventory investment to better match the data. In particular, the relative volatility of the inventory-to-sales ratio is reduced from 0.742 (KT) to 0.536, matching the empirical counterpart almost exactly.

Table 8. Business Cycle Moments ($\varphi''(1) = 1.3$, $\gamma = \infty$)

	Final Sales	Net inventory investment	Inventory/sales
A. Volatility relative to GDP			
Data	0.710	0.295	0.545
IAC Model	0.776	0.377	0.536
KT	0.839	0.188	0.742
B. Correlation with GDP			
Data	0.943	0.669	-0.381
Inventory with IAC	0.941	0.716	-0.687
KT	0.994	0.880	-0.991

The improvements of empirical fit notwithstanding, the most important effect of IAC is that they make inventory investment significantly destabilizing. Table 9 shows that with IAC, the standard deviation of GDP is increased by 12.6% ($100 \times \frac{1.262-1.121}{1.121} = 12.6$ percent).

Table 9. Contribution of Inventories to Volatility ($\varphi''(1) = 1.3$, $\gamma = \infty$)

	GDP	FS	C	I	H	K	X	M
A. Volatility relative to GDP								
Control Model	(1.121)	1	0.783	4.008	0.339	0.333	1.845	1.845
IAC Model	(1.262)	0.776	0.582	3.323	0.439	0.283	1.964	1.441
B. Correlation with GDP								
Control Model		1	0.957	0.843	0.741	-0.042	0.997	0.996
IAC Model		0.941	0.993	0.679	0.990	-0.174	0.997	0.964

The intuition is that IAC imply that firms want to smooth out capital investment over time to avoid the adjustment costs. In this case inventories will play a more strategic role for final goods firms to reduce fixed order costs than when there are no IAC, because the total demand of final goods is now expected to persist for a longer period of time after a technology shock. Given the lowered intermediate goods price after the shock and the anticipated persistence in final sales in the future, firms will opt to increase inventory investment sharply, more so than they would otherwise without IAC (the relative standard deviation of inventory investment is 0.377 in the IAC model while it is 0.188 in the KT model). This increased procyclicality and volatility of inventory investment significantly raises the overall volatility of GDP. So the dampening effect of labor reallocation from the final goods sector to the intermediate goods sector is no longer sufficient to offset the positive effect of inventory investment on GDP volatility when IAC exist.

Alternatively, we can understand the results from a social planner's perspective. For the social planner, inventories have a similar role to capital in smoothing consumption. With IAC, capital's role in smoothing consumption is hindered. Hence, compared with the control model with IAC, the planner opts to accumulate inventories in addition to capital to facilitate consumption smoothing, thus increasing the volatility of GDP. This is illustrated by impulse responses in Figure 1.

The dashed lines in Figure 1 represent the control model and the solid lines the inventory model. The figure shows that without inventories, (i) final sales (dashed line in 2nd window) are highly persistent and hump-shaped because of IAC, and (ii) consumption (dashed line in 3rd window) is highly volatile on impact because capital is not able to smooth consumption under IAC. With inventories, however, the figure shows that (i) consumption is significantly smoothed over time by inventories so it becomes significantly less volatile than in the control model, (ii) final sales also become less volatile as a result of consumption smoothing (while the volatility of investment does not change significantly), and (iii) because of procyclical inventory investment, however, GDP (1st window) becomes far more volatile in the inventory model than in the control model.

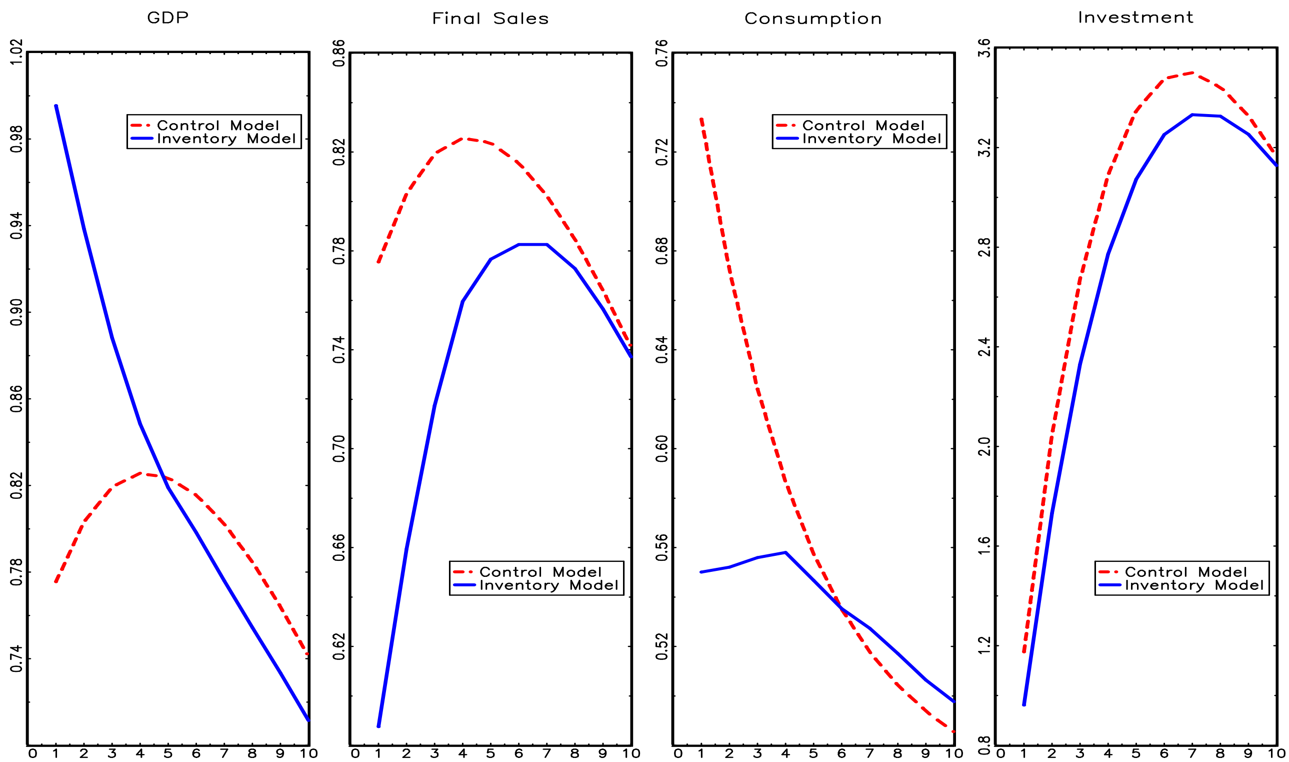

Figure 1. Impulse Responses to a Technology Shock.

Sensitivity analysis. Table 10 provides the results of sensitivity analysis for different values of the adjustment cost parameter ($\varphi''(1)$). It shows that as the adjustment costs increase, the contribution of inventories to the volatility of GDP rises and the variance of capital investment decreases. In the meantime, the sign of the correlation to GDP remains robust for both inventory investment and the inventory-to-sales ratio.

Table 10. Sensitivity Analysis (Adjustment Costs)

$\varphi''(1)$	0.5	1.3	2.1	2.9	3.8	4.6
$\left(\frac{\text{GDP volatility with inventory}}{\text{GDP volatility without inventory}}\right)$	1.085	1.126	1.148	1.163	1.174	1.181
$\left(\frac{\text{Investment volatility}}{\text{GDP volatility}}\right)$	4.278	3.323	2.812	2.469	2.190	2.000
$Corr$ (Inventory Investment, GDP)	0.677	0.716	0.741	0.758	0.771	0.780
$Corr$ (Inventory/Sales, GDP)	-0.866	-0.687	-0.600	-0.553	-0.525	-0.510

6 Conclusion

This paper shows that in the general equilibrium (S,s) inventory model of KT, inventories can be significantly destabilizing to the economy if capacity utilization (or IAC) is taken into consideration. This finding is consistent with conventional wisdom on the role of inventories in the business cycle, even though the conventional belief is based on partial-equilibrium arguments. Although the KT model is only a special case of (S,s) inventory management and focuses only on intermediate goods inventories, we believe that our findings are robust to a wide class of general equilibrium (S,s) inventory models driven by technology shocks. In all such models, the increased inventory demand triggered by cheaper prices (or lower ordering costs) under positive technology shocks can be met by increased capacity utilization alone in the inventory-goods producing sector without necessarily relying on labor reallocations across sectors, thus facilitating positive inventory accumulation without necessarily reducing final sales.

However, as illustrated by Wang and Wen (2009) and Wen (2011), different incentives for inventory demands can have dramatically different implications for the (de)stabilizing role of inventories. A similar point is also made by Chang, Hornstein, and Sarte (2009). Therefore, conclusions drawn from the (S,s)-type inventory models do not generalize to other types of inventory models. For example, adding capacity utilization and IAC into the general-equilibrium stockout-avoidance inventory model of Wen (2011) does not make inventories investment destabilizing to the economy, in contrast to the (S,s) inventory model studied in this paper. The reason is that variable capacity utilization and capital adjustment costs cannot undo the effects of a procyclical shadow value of inventories in a stockout-avoidance model where firms hold inventories primarily to meet unexpected demand shocks instead of reducing order/production costs. This procyclical shadow value of inventories arises from an endogenously determined procyclical probability of stockout. It is key to rendering inventories stabilizing because it makes final goods more expensive in a boom and less costly in a recession under aggregate demand shocks, dampening the variance of aggregate demand over the business cycle regardless of capacity utilization or IAC. Therefore, our results in this paper reinforce the argument of Wang and Wen (2009) that whether inventories are (de)stabilizing or not depends not only on model structures and sources of aggregate shocks but also on the motives for holding inventories. In the end, which inventory models can better characterize inventory behavior is an empirical question.

Appendix I. Proof of Proposition 1

Proof. We solve the firm's problem in several steps.

1. We solve the firm's labor demand by

$$\theta_n m_t^{\theta_m} n_t^{\theta_n - 1} = W_t, \tag{48}$$

which yields

$$n_t = \left(\frac{\theta_n}{W_t}\right)^{\frac{1}{1-\theta_n}} m_t^{\frac{\theta_m}{1-\theta_n}}. \tag{49}$$

Substituting this solution into the profit function gives

$$m_t^{\theta_m} n_t^{\theta_n} - W_t n_t \equiv R_t m_t^{\theta}, \tag{50}$$

where $\theta = \frac{\theta_m}{1-\theta_n}$ and

$$R_t = (1-\theta_n)\left(\frac{\theta_n}{W_t}\right)^{\frac{\theta_n}{1-\theta_n}}. \tag{51}$$

2. Define $V_t^a(s_t)$ as the value function of an active firm that places an order in period t (excluding the fixed order cost) and $V_t^a(s_t) - \varepsilon_t W_t$ as the firm's value function including the fixed order cost. Define $V_t^n(s_t)$ as the value function of an inactive firm that decides not to order intermediate goods in period t. With these notations, the final goods producer's problem in equation (16) becomes

$$V_t(s_t, \varepsilon_t) = \max\left\{V_t^a(s_t) - W_t\varepsilon_t, V_t^n(s_t)\right\}. \tag{52}$$

Define $\bar{V}_t(s_t) = \int V_t(s_t, \varepsilon)\, dF(\varepsilon)$ as the average (expected) value of a firm with inventory stock s_t. So by definition we can write the Bellman equation for $V_t^a(s_t)$ as

$$V_t^a(s_t) = \max_{x_t, s_{t+1}} R_t(s_t + x_t - s_{t+1})^{\theta} - \sigma s_{t+1} - P_t x_t + \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\bar{V}_t(s_{t+1}). \tag{53}$$

The value function for an inactive firm (with $x_t = 0$) can be written as

$$V_t^n(s_t) = \max_{s_{t+1}} R_t(s_t - s_{t+1})^{\theta} - \sigma s_{t+1} + \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\bar{V}_t(s_{t+1}). \tag{54}$$

3. Obviously, $V_t^a(s_t) \geq V_t^n(s_t)$, since $x_t = 0$ is always a possible solution for the problem defined in (53). Comparing $V_t^a(s_t) - W_t\varepsilon_t$ and $V_t^n(s_t)$ for any given inventory level s_t, it is easy to see that there exists a cutoff value for the fixed cost, ε_t^*, such that

$$V_t^a(s_t) - W_t\varepsilon_t^* = V_t^n(s_t). \tag{55}$$

The above equation defines the cutoff as an implicit function of the firm's inventory stock s_t. So we can denote $\varepsilon_t^* = \varepsilon_t^*(s_t)$. A firm will place an order $(x_t > 0)$ if and only if $\varepsilon_t \leq \varepsilon_t^*(s_t)$.

4. For a firm that decides to place an order, the first-order condition with respect to x_t is

$$\theta R_t m_t^{\theta-1} = P_t, \tag{56}$$

which solves for the optimal input level for an active firm, $m_{0t} = \left(\frac{P_t}{\theta R_t}\right)^{\frac{1}{\theta-1}}$. Note that the solution is independent of the existing inventory stock and the fixed cost shock; i.e., it is state independent. By equation (49), the optimal labor demand is also independent of $\{s_t, \varepsilon_t\}$. We denote these state-independent variables as m_{0t} and n_{0t}. The first-order condition with respect to inventory holding s_{t+1} is

$$\theta R_t m_{0t}^{\theta-1} + \sigma = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \frac{\partial \overline{V}_{t+1}(s_{t+1})}{\partial s_{t+1}}. \tag{57}$$

Combining the previous two equations, we have

$$P_t + \sigma = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \frac{\partial \overline{V}_{t+1}(s_{t+1})}{\partial s_{t+1}}. \tag{58}$$

This implies that the optimal level of inventories for an active firm, s_{t+1}, is also state-independent (i.e., it depends only on the aggregate variables and not on the firm's history). That is, all firms that decide to place an order in period t will replenish their inventory stocks to the same level regardless of their individual histories. We denote $s_{1,t+1}$ as the optimal level of inventory stock carried over to period $t+1$ by active firms.

5. We now turn to inactive firms which do not place orders in period t (i.e., $\varepsilon_t > \varepsilon_t^*$). The first-order condition for s_{t+1} in the problem (54) is given by

$$\theta R_t (s_t - s_{t+1})^{\theta-1} + \sigma = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \frac{\partial \overline{V}_{t+1}(s_{t+1})}{\partial s_{t+1}} + \pi_t, \tag{59}$$

where π_t is a Lagrangian multiplier associated with the non-negative constraint on s_{t+1}. Notice that in this case $m_t = s_t - s_{t+1}$ because $x_t = 0$. The above equation defines the decision rules for intermediate goods input $m_t = m_t(s_t)$ and inventory holdings $s_{t+1} = s_t - m(s_t)$. By equation (49), labor demand can be written as $n_t = n_t(s_t)$. The decision rules at the firm level are summarized by equations (24)-(26).

■

Appendix II. Proof of Proposition 2

Proof. First of all, by definition we have $s_{J+1,t+1} = 0$. Also, for vintage-j firms that do not order in period t, we have $m_{j,t} = s_{j,t} - s_{j+1,t+1}$ for $j = 1, 2, ..., J$. These give us $J+1$ equations that correspond to equations (34) and (36) in Proposition 2.

To prove equation (30), consider the value function of an active firm with vintage j:

$$V_t^a(s_{j,t}) = \max_{m_{0,t},\ s_{1,t+1}} \left\{ R_t m_{0,t}^{\theta} - \sigma s_{1,t+1} - P_t\left(m_{0,t} + s_{1,t+1} - s_{j,t}\right) + \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \bar{V}_{t+1}\left(s_{1,t+1}\right) \right\} \tag{60}$$

where $\bar{V}_{t+1}(s_{1,t+1})$ is the expected value function with respect to idiosyncratic shock ε evaluated at $s_{1,t+1}$. Since the term $P_t s_{j,t}$ on the right-hand side (RHS) does not affect the optimal choices (because $s_{j,t}$ is predetermined), we can define a new value function (for active firms) that is independent of j:

$$V_t^a = \max_{m_{0,t}, s_{1,t+1}} \left\{ R_t m_{0,t}^{\theta} - \sigma s_{1,t+1} - P_t\left(m_{0,t} + s_{1,t+1}\right) + \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \bar{V}_{t+1}\left(s_{1,t+1}\right) \right\}. \tag{61}$$

That is, V_t^a equals $V_t^a(s_{j,t})$ evaluated at $s_{j,t} = 0$. Now $V_t^a(s_{j,t})$ can be rewritten as

$$V_t^a(s_{jt}) \equiv V_{j,t}^a = V_t^a + P_t s_{jt}. \tag{62}$$

According to equation (55), the value function of inactive firms can be rewritten as

$$V_{j,t}^n = V_t^a + P_t s_{j,t} - W_t \varepsilon_{j,t}^*. \tag{63}$$

For the maximization problem in equation (61), the first-order condition with respect to $m_{0,t}$ and $s_{1,t+1}$ are given, respectively, by

$$\theta R_t m_{0t}^{\theta-1} = P_t \tag{64}$$

$$P_t + \sigma = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \frac{\partial \bar{V}_{t+1}\left(s_{1,t+1}\right)}{\partial s_{1,t+1}}, \tag{65}$$

where equation (64) corresponds to equation (35) in Proposition 2. Now we need to determine the derivative, $\frac{\partial \bar{V}_t(s_{j,t})}{\partial s_{j,t}}$. Notice that by equations (52) and (55), the expected value function $\bar{V}_t\left(s_{j,t}\right)$ is given by

$$\bar{V}_t\left(s_{j,t}\right) = F(\varepsilon_{j,t}^*) V_t^a\left(s_{j,t}\right) + \left[1 - F(\varepsilon_{j,t}^*)\right] V_t^n\left(s_{j,t}\right) - W_t \int\limits_{\varepsilon < \varepsilon_{j,t}^*} \varepsilon dF\left(\varepsilon\right). \tag{66}$$

Thus,

$$\frac{\partial \bar{V}_t\left(s_{j,t}\right)}{\partial s_{j,t}} = F\left(\varepsilon_{j,t}^*\right) \frac{\partial V_t^a\left(s_{j,t}\right)}{\partial s_{j,t}} + \left[1 - F\left(\varepsilon_{j,t}^*\right)\right] \frac{\partial V_t^n\left(s_{j,t}\right)}{\partial s_{j,t}} + \left[V_t^a\left(s_{j,t}\right) - \varepsilon_{j,t}^* - V_t^n\left(s_{j,t}\right)\right] f\left(\varepsilon_{j,t}^*\right) \frac{\partial \varepsilon_{j,t}^*}{\partial s_{j,t}} W_t. \tag{67}$$

By equation (55), the last term is zero, so we have

$$\frac{\partial \bar{V}_t\left(s_{jt}\right)}{\partial s_{jt}} = F\left(\varepsilon_{jt}^*\right) \frac{\partial V_t^a\left(s_{jt}\right)}{\partial s_{jt}} + \left[1 - F\left(\varepsilon_{jt}^*\right)\right] \frac{\partial V_t^n\left(s_{jt}\right)}{\partial s_{jt}}. \tag{68}$$

The task of computing $\frac{\partial \bar{V}_t(s_{j,t})}{\partial s_{j,t}}$ now reduces to calculating the partial derivatives $\frac{\partial V_t^a(s_{jt})}{\partial s_{jt}}$ and $\frac{\partial V_t^n(s_{jt})}{\partial s_{jt}}$. According to equation (62), we immediately have:[10]

$$\frac{\partial V_{j,t}^a}{\partial s_{j,t}} = P_t. \tag{69}$$

To obtain $\frac{\partial V_t^n(s_{jt})}{\partial s_{jt}}$ in equation (68), we need to consider the value function of the inactive firms of vintage j. For $j = 1, 2, ...J$, we have

$$V_t^n\left(s_{j,t}\right) = \max_{m_{j,t},\ s_{j+1,t+1}} \{R_t m_{j,t}^\theta - \sigma s_{j+1,t+1} + \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \bar{V}_{t+1}\left(s_{j+1,t+1}\right)\}, \tag{70}$$

where $m_{j,t} = s_{j,t} - s_{j+1,t+1}$. The first-order condition with respect to $s_{j+1,t+1}$ $(j = 1, 2, ..., J)$ is given by

$$\theta R_t m_{jt}^{\theta-1} + \sigma = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \frac{\partial \bar{V}_{t+1}\left(s_{j+1,t+1}\right)}{\partial s_{j+1,t+1}}. \tag{71}$$

[10]This equation can also be obtained by applying the envelop theorem to equation (60)

By the envelop theory we have

$$\frac{\partial V_t^n(s_{j,t})}{\partial s_{j,t}} = \theta R_t m_{j,t}^{\theta-1}. \tag{72}$$

Now, putting (69) and (72) into equation (68) gives

$$\frac{\partial \bar{V}_t(s_{j,t})}{\partial s_{j,t}} = F\left(\varepsilon_{j,t}^*\right) P_t + \left[1 - F\left(\varepsilon_{j,t}^*\right)\right] \theta R_t m_{j,t}^{\theta-1}. \tag{73}$$

Plugging this equation into (65) and (71), respectively, gives

$$P_t + \sigma = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left[F\left(\varepsilon_{1,t+1}^*\right) P_{t+1} + \left[1 - F\left(\varepsilon_{1,t+1}^*\right)\right] \theta R_{t+1} m_{j+1,t+1}^{\theta-1}\right] \tag{74}$$

$$\theta R_t m_{jt}^{\theta-1} + \sigma = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left[F\left(\varepsilon_{j+1,t+1}^*\right) P_{t+1} + \left[1 - F\left(\varepsilon_{j+1,t+1}^*\right)\right] \theta R_{t+1} m_{j+1,t+1}^{\theta-1}\right]. \tag{75}$$

These two equations, together with equation (64), correspond to the $J+1$ equations in equation (33) in Proposition 2.

The remaining $J+2$ equations are related to V_t^a and the cutoff $\varepsilon_{j,t}^*$ for $j = 1, 2, ...J+1$, which are determined by equation (63). We can use equation (63) to substitute out $V_t^n(s_{j,t})$ in equation (66) to obtain

$$\begin{aligned} \bar{V}_t(s_{j,t}) &= \int\limits_{\varepsilon \le \varepsilon_{jt}^*} \left[V_t^a(s_{jt}) - W_t \varepsilon\right] dF(\varepsilon) + \int\limits_{\varepsilon > \varepsilon_{jt}^*} \left[V_t^a(s_{jt}) - W_t \varepsilon_{jt}^*\right] dF(\varepsilon) \\ &= V_t^a(s_{jt}) - W_t \int \min\left\{\varepsilon, \varepsilon_{jt}^*\right\} dF(\varepsilon) \\ &= V_t^a + P_t s_{jt} - W_t \int \min\left\{\varepsilon, \varepsilon_{jt}^*\right\} dF(\varepsilon), \end{aligned} \tag{76}$$

where the third line comes from equation (62). Substituting the above equation for $\bar{V}_t(s_{j,t})$ into equation (61) under the optimal choices gives equation (30) in Proposition 2. Using the relation (63) and the function $\bar{V}_t(s_{j+1,t+1})$ defined in equation (76) to substitute out $V_{j,t}^n$ and $\bar{V}(s_{j+1,t+1})$ in equation (70) under the optimal choices gives equations (31) and (32) in Proposition 2. These together give us $J+2$ additional equations. The total number of equations is thus $3(J+1)+1$ in Proposition 2. ■

Appendix III. Steps for Solving the Steady State

We solve the steady state of our inventory model in several steps: in steps 1 and 2, we list all the variables and the corresponding equations needed to solve for the variables; in steps 3 and 4, we illustrate how to recursively solve the steady state using the system of equations listed in steps 1 and 2.

Step 1. We first list the equations needed to solve for the steady-state distributions of final goods firms, taking as given the aggregate variables, $\{P, W, R\}$. Assume that the fixed order cost ε follows the power distribution, $F(\varepsilon) = (\frac{\varepsilon}{\bar{\varepsilon}})^{\kappa}$ with support $\varepsilon \in [0, \bar{\varepsilon}]$. The uniform distribution is a special case when $\kappa = 1$. Given the power distribution, we have the relationship

$$\int \min\{\varepsilon, \varepsilon^*\} \, dF(\varepsilon) = \left[1 - \frac{1}{1+\kappa} \left(\frac{\varepsilon^*}{\bar{\varepsilon}}\right)^{\kappa}\right] \varepsilon^*. \tag{77}$$

The distribution of firms can then be solved using the following system of $4(J+1)$ equations implied by those in Proposition 2 and the following relationship:

$$V^a_{j,t} = V^a_t + P s_{j,t}, \quad j = 1, 2, ..., J. \tag{78}$$

First, using the steady-state relationship implied by equation (78), $V^a_j = V^a + P s_j$, we have

$$V^a_j = V^a_1 - P(s_1 - s_j), \tag{79}$$

where V^a_1 is determined by equations (30) and (78) as:

$$V^a_1 = R m^{\theta}_0 - \sigma s_1 - P m_0 + \beta \left[V^a_1 - W \int \min\{\varepsilon, \varepsilon^*_1\} \, dF(\varepsilon)\right]. \tag{80}$$

These $J+1$ equations can be used in determining V^a_j, $j = 1, 2, ..., J+1$.

Second, The following $J+1$ equations can be used in determining ε^*_j, $j = 1, 2, ..., J+1$. Equations (31) and (32) imply

$$V^a_j - W\varepsilon^*_j = R m^{\theta}_j - \sigma s_{j+1} + \beta \left[V^a_{j+1} - W \int \min\left\{\varepsilon, \varepsilon^*_{j+1}\right\} dF(\varepsilon)\right], \text{ for } j = 1, ..., J \tag{81}$$

$$V^a_{J+1} - W\varepsilon^*_{J+1} = \beta \left[V^a_{J+1} - W \int \min\left\{\varepsilon, \varepsilon^*_{J+1}\right\} dF(\varepsilon)\right]. \tag{82}$$

Third, from the first-order conditions for inventories, we have additional $J+1$ equations that can be used in determining s_j for $j = 1, 2, ..., J+1$. Specifically, equations (33) and (34) imply

$$\theta R m_j^{\theta-1} + \sigma = \beta \left\{ F\left(\varepsilon_{j+1}^*\right) P + \left[1 - F\left(\varepsilon_{j+1}^*\right)\right] \theta R m_{j+1}^{\theta-1} \right\}, \quad j = 0, 2, ...J-1 \tag{83}$$

$$s_{J+1} = 0. \tag{84}$$

Finally, from the policy functions of input materials, the following $J+1$ equations can be used in determining m_j, $j = 0, 1, 2, ..., J$. Equations (35) and (36) imply

$$\theta R m_0^{\theta-1} = P \tag{85}$$

$$m_j = s_j - s_{j+1}, \;\; j = 1, 2, ...J. \tag{86}$$

Step 2. Now, we solve for the aggregate variables $\{W, R\}$ as a function of the relative price of intermediate goods, P. By the first-order conditions of intermediate goods firms, equations (10) and (13), the real wage W can be expressed as:

$$W = (1-\alpha) P \left(\frac{K}{L}\right)^{\alpha}, \tag{87}$$

where $\frac{K}{L} = \left[\left(\frac{1}{\beta} - 1 + \delta\right) / (\alpha P)\right]^{\frac{1}{\alpha-1}}$. Given W, the steady-state R can be solved using equation (42).

Step 3. We now show how to recursively solve $\left\{\{s_j\}_{j=1}^{J+1}, \{m_j\}_{j=0}^{J}, \{\varepsilon_j^*\}_{j=1}^{J+1}, \{V_j^a\}_{j=1}^{J+1}\right\}$ as functions of P from the system of equations listed above. Equation (85) implies

$$m_0 = \left(\frac{P}{\theta R}\right)^{\frac{1}{\theta-1}}. \tag{88}$$

So given P and $\{s_1, \varepsilon_1^*\}$ for vintage 1 firms, we can compute $\{m_j\}_{j=1}^{J}, \{s_j\}_{j=2}^{J+1}, \{\varepsilon_j^*\}_{j=2}^{J+1}$, and $\{V_j^a\}_{j=1}^{J+1}$ recursively below. Then we will use two additional constraints to obtain (s_1, ε_1^*) at the end.

Equation (80) implies that V_1^a is a function of (s_1, ε_1^*) and P:

$$V_1^a = \frac{R m_0^{\theta} - \sigma s_1 - P m_0 - \beta W \int \min\{\varepsilon, \varepsilon_1^*\} dF(\varepsilon)}{1-\beta}. \tag{89}$$

From the recursive equation (83), we can compute m_1 in terms of m_0 and ε_1^*:

$$m_1 = \left\{ \frac{\theta R m_0^{\theta-1} + \sigma - \beta F\left(\varepsilon_1^*\right) P}{\beta \left[1 - F\left(\varepsilon_1^*\right)\right] \theta R} \right\}^{\frac{1}{\theta-1}}. \tag{90}$$

From equations (86) and (79), s_2 and V_2^a can be updated to

$$s_2 = s_1 - m_1 \tag{91}$$

$$V_2^a = V_1^a - P m_1. \tag{92}$$

Finally, from equation (81), we can solve for the cutoff ε_2^* according to the following equation:

$$\left[1 - \frac{1}{1+\kappa}\left(\frac{\varepsilon_2^*}{\bar{\varepsilon}}\right)^{\kappa}\right]\varepsilon_2^* - \frac{V_1^a - W\varepsilon_1^* - R m_1^{\theta} + \sigma s_2 - \beta V_2^a}{\beta W} = 0. \tag{93}$$

Repeating the above steps will give us $\left\{s_j, \varepsilon_j^*, m_{j-1}, V_j^a\right\}$ for $j = 2, ..., J+1$. That is, by equation (83), we can update m_j. By equation (86), we can compute s_{j+1}. Then we can use equation (79) to compute V_{j+1}^a. Finally, using equation (81), we can obtain ε_{j+1}^*.

Once we have finished the above recursive procedure, we still need two more equations to pin down s_1 and ε_1^*. Remember that we still have two additional equations that have not been used yet: equations (82) and (84). By equation (86) at $j = J$ and equation (84), we have

$$s_J(P, \varepsilon_1^*, s_1) = m_J(P, \varepsilon_1^*, s_1), \tag{94}$$

which yields one additional equation. For the other equation, notice that from previous recursive calculations, we have obtained $V_{J+1}^a(P, \varepsilon_1^*, s_1)$ and $\varepsilon_{J+1}^*\left(P, \varepsilon_1^*, s_1\right)$. Plugging them into equation (82) gives

$$V_{J+1}^a - W\varepsilon_{J+1}^* = \beta\left[V_{J+1}^a - W\int \min\left\{\varepsilon, \varepsilon_{J+1}^*\right\} dF\left(\varepsilon\right)\right], \tag{95}$$

which gives the other equation needed for solving $\{\varepsilon_1^*, s_1\}$. Therefore given P, equations (94) and (95) constitute two nonlinear equations that can be used to jointly solve for ε_1^* and s_1.

Once we know the cutoffs, $\varepsilon_1^*, \varepsilon_2^*, ...\varepsilon_{J+1}^*$, the distribution $\{\omega_j\}$ can then be solved by evaluating equations (27) to (29) at steady state.

Step 4. Now we specify the final step to solve for P. According to equations (39) and (40), the total production of intermediate goods, given P, is

$$X = M = \sum_{j=0}^{J-1} m_j \omega_{j+1} + m_J \left[1 - F\left(\varepsilon_J^*\right)\right] \omega_J. \tag{96}$$

Since the Euler equation for capital stock, (10), implies

$$\alpha P \frac{X}{K} = \frac{1}{\beta} - 1 + \delta, \tag{97}$$

we can solve for K as function of P. Since investment equals $\bar{\delta} K$, we can obtain $I = \bar{\delta} K$. Also, from the household optimal condition of consumption (3), we can solve for aggregate consumption using

$$\frac{1}{C} W = \tau. \tag{98}$$

According to equation (41), the aggregate production for final goods can be determined by

$$Y = \sum_{j=0}^{J-1} y_j \omega_{j+1} + y_J \left[1 - F\left(\varepsilon_J^*\right)\right] \omega_J. \tag{99}$$

where $y_j = R m_j^\theta / (1 - \theta_n) - \sigma s_{j+1}$, for $j = 0, ..., J$. Therefore, the final goods market clearing condition implies

$$Y(P) = C(P) + I(P), \tag{100}$$

which can be used to solve for P.■

References

[1] Abel, A., 1985, Inventories, stock-outs and production smoothing, Review of Economic Studies 52(2), 283-293.

[2] Alessandria, G., J. Kaboski, and V. Midrigan, 2010, Inventories, lumpy trade, and large devaluations, American Economic Reivew 100(5), 2304-2339.

[3] Baumol, W. J., 1952, The Transactions Demand for Cash: An Inventory Theoretic Approach, The Quarterly Journal of Economics 66(4), pp. 545-556.

[4] Benhabib, J. and Y. Wen, 2004, Indeterminacy, aggregate demand, and the real business cycle, Journal of Monetary Economics 51(3), 503-30.

[5] Bils, M., 2004, Studying Price Markups from Stockout Behavior, University of Rochester Working Paper.

[6] Bils, M. and J. Kahn, 2000, What inventory behavior tells us about business cycles? American Economic Review 90(3), 458-481.

[7] Blanchard, O., 1983, The production and inventory behavior of the American automobile industry, Journal of Political Economy 91 (no. 3), 365-400.

[8] Blinder, A., 1981, Retail inventory behavior and business fluctuations, Brookings Papers on Economic Activity 12(2), 443-520.

[9] Blinder, A., 1986, Can the production smoothing model of inventory behavior be saved? The Quarterly Journal of Economics 101(3), 431-453.

[10] Blinder, A., 1990, Inventory theory and consumer behavior, University of Michigan Press.

[11] Blinder, A. and L. Maccini, 1991, Taking stock: A critical assessment of recent research on inventories, Journal of Economic Perspectives 5(1), 73-96.

[12] Brown, W. and U. Haegler, 2004, Financing constraints and inventories, European Economic Review 48(5), 1091-1123.

[13] Caballero, R. and E. Engel, 1991, Dynamic (S, s) Economies, Econometrica, 59(6), 1659-1686.

[14] Caplin, A., 1985, The variability of aggregate demand with (S,s) inventory policies, Econometrica 53, 1395-1410.

[15] Caplin, A. and J. Leahy, 1991, State-dependent pricing and the dynamics of output and prices, Quarterly Journal of Economics 106, 683-708.

[16] Chang, Y., A. Hornstein, and P.D. Sarte, 2009, On the employment effects of productivity shocks: The role of inventories, demand elasticity, and sticky prices, Journal of Monetary Economics 56(3), 328-343.

[17] Christiano, L., 1988, Why does inventory investment fluctuate so much? Journal of Monetary Economics 21(2-3), 247-280.

[18] Christiano, L., M. Eichenbaum and C. Evans, 2005, Nominal rigidities and the dynamic effects of a shock to monetary policy, Journal of Political Economy 113(1), 1-45.

[19] Coen-Pirani, D., 2004, Markups, aggregation, and inventory adjustment, American Economic Review 94(5), 1328-53.

[20] Dotsey, M., R.G. King, and A. Wolman, 1999, State-dependent pricing and the general equilibrium dynamics of money and output, Quarterly Journal of Economics 114(2), 655-690.

[21] Eichenbaum, M., 1989, Some empirical evidence on the production level and production cost smoothing models of inventory investment, American Economic Review 79 (4), 853-64.

[22] Engelhardty, L., 2011, Inventories, Business Cycles, and Variable Capital Utilization, Kent State University Working Paper.

[23] Feldstein, M. and A. Auerbach, 1976, Inventory behavior in durable-goods manufacturing: The target-adjustment model, Brooking Papers on Economic Activity, vol. 2, 351-408.

[24] Fisher, J. and A. Hornstein, 2000, (S,s) inventory policies in general equilibrium, Review of Economic Studies 67(1), 117-145.

[25] Greenwood, J., Z. Hercowitz, and G. Huffman, 1988, Investment, capacity utilization, and the real business cycle, American Economic Review 78, 402-417.

[26] Haltiwanger, J. and L. Maccini, 1988, A Model of Inventory and Layoff Behaviour Under Uncertainty, The Economic Journal 98(392), 731-745.

[27] Hornstein, A., 1998, Inventory investment and the business cycle, Federal Reserve Bank of Richmond Economic Quarterly 84 (Spring), 49-71.

[28] Humphreys, B., L. Maccini and S. Schuh, 2001, Input and output inventories, Journal of Monetary Economics 47(2), 347-375.

[29] Iacoviello, M., F. Schiantarelliz, and S. Schuh, 2007, Input and Output Inventories in General Equilibrium, Working Paper, Boston College.

[30] Kahn, J., 1987, Inventories and the volatility of production, American Economic Review 77(4), 667-679.

[31] Kahn, J., 1992, Why is production more volatile than sales? Theory and evidence on the stockout-avoidance motive for inventory-holding, The Quarterly Journal of Economics 107(2), 481-510.

[32] Kahn, J., M. McConnell and G. Perez-Quiros, 2002, On the causes of the increased stability of the U.S. economy, Economic Policy Review (May), p 183-202.

[33] Khan, A. and J. Thomas, 2007a, Inventories and the business cycle: An equilibrium analysis of (S,s) policies, American Economic Review 97(4), 1165-88.

[34] Khan, A. and J. Thomas, 2007b, Explaining Inventories: A Business Cycle Assessment of the Stockout Avoidance and (S,s) Motives, Macroeconomic Dynamics 11(5), 638-64.

[35] King, R.G. and S. Rebelo, 1999, Resuscitating real business cycles, Handbook of Macroeconomics Vol. 1A., eds. M. Woodford and J. Taylor, Amsterdam; New York and Oxford: Elsevier Science, North-Holand.

[36] Kryvtsov, O. and V. Midrigan, 2008, Inventories, Markups, and Real Rigidities in Menu Cost Models, Working Paper, New York University.

[37] Kydland, F. and Prescott, E., 1982, Time to Build and Aggregate Fluctuations, Econometrica 50(6), p 1345-1370.

[38] Ramey, V., 1989, Inventories as factors of production and economic ‡uctuations, American Economic Review 79(3), 338-354.

[39] Ramey, V., 1991, Nonconvex costs and the behavior of inventories, Journal of Political Economy 99(2), 306-334.

[40] Ramey, V. and K. West, 1999, Inventories, chapter 13 in *Handbook of Macroeconomics*, J. Taylor and M. Woodford (ed.), Vol. 1B, 863-423, North-Holland.

[41] Samuelson, P., 1971, Stochastic Speculative Price, Proceedings of the National Academy of Sciences of the United States of America, vol. 68(2), 335-337.

[42] Wang P. and Y. Wen, 2009, Inventory accelerator in general equilibrium, Federal Reserve Bank of St. Louis Working Paper 2009-010A.

[43] Wang, P. and Y. Wen, 2010, Hayashi meets Kiyotaki and Moore: A theory of capital adjustment costs, Federal Reserve Bank of St. Louis Working Paper 2009-037A.

[44] Wen, Y., 1998, Capacity utilization under increasing returns to scale, Journal of Economic Theory 81(1), 7-36.

[45] Wen, Y., 2005, Understanding the inventory cycle, Journal of Monetary Economics 52(8), 1533-55.

[46] Wen, Y., 2011, Input and output inventory dynamics, American Economic Journal: Macroeconomics (forthcoming).