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Hayashi Meets Kiyotaki and Moore: A Theory of Capital Adjustment Costs*

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Abstract

Firm-level investment is lumpy and volatile but aggregate investment is much smoother and highly serially correlated. These different patterns of investment behavior have been viewed as indicating convex adjustment costs at the aggregate level but non-convex adjustment costs at the firm level. This paper shows that financial frictions in the form of collateralized borrowing at the firm level (Kiyotaki and Moore, 1997) can give rise to convex adjustment costs at the aggregate level yet at the same time generate lumpiness in plant-level investment. In particular, our model can (i) derive aggregate capital adjustment cost functions identical to those assumed by Hayashi (1982) and (ii) explain the weak empirical relationship between Tobin's Q and plant-level investment.

Keywords: Adjustment Costs, Collateral, Borrowing Constraints, Tobin's Q, Investment.

JEL Codes: E22, E62, G31.

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1 Introduction

It is well known that firm-level investment behaves quite differently from aggregate investment. In particular, firm-level investment is lumpy, whereas aggregate investment is much smoother and highly serially correlated (see, e.g., Caballero, 1999). Such a sharp difference in investment dynamics at the plant and aggregate levels has often motivated researchers to adopt inconsistent assumptions in explaining investment dynamics: assuming convex adjustment costs in aggregate models and non-convex adjustment costs in micro models. Econometric studies typically find that convex capital adjustment costs (CAC) are consistent with aggregate investment data but not with firm-level data (e.g., Bloom, 2009).

However, CAC are a widely adopted assumption in dynamic macroeconomic models and have a long tradition in the history of investment theory.[1] This assumption is often needed because a theoretical model without CAC would imply (i) the elasticity of capital supply is the same in both the short run and the long run; that is, the equilibrium capital stock can be reached instantaneously because of the possibility of an infinite speed of the investment rate; and (ii) the relative price of the investment and consumption goods is a constant independent of the relative outputs of the two goods.

Such implications not only are inconsistent with data but also create theoretical difficulties in determining the optimal rate of investment in partial equilibrium models of the firm, which motivated the early investment literature to adopt CAC (e.g., Lucas, 1967; Gould, 1968). In addition, theory requires CAC to rationalize investment decisions as a function of firm value and replacement costs of capital (Tobin, 1969; Lucas and Prescott, 1971; Abel, 1979, 1983; and Hayashi, 1982).

CAC also play an important role in contemporary dynamic stochastic general equilibrium (DSGE) models. For example, (i) they help open-economy models to explain the saving-investment correlations and the home bias puzzle (e.g., Baxter and Crucini 1993); (ii) they are essential to explaining the equity premium puzzle in production economies with capital (e.g., Jermann, 1998; Boldrin, Christiano, and Fisher, 2001); (iii) they rationalize large welfare costs of the business cycle (e.g., Barlevy 2004); and (iv) they are key to supporting news shocks as a credible driving force of the business cycle (e.g., Beaudry and Portier, 2007;

[1] For the early literature that assumes CAC, see Gould (1968), Lucas (1967, 1969), Uzawa (1969), Lucas and Prescott (1971), among others. For a literature survey on investment theory, see Caballero (1999).

Jaimovich and Rebelo, 2009).

However, despite the popularity and apparent "necessity" of CAC in macro models, few microfoundations have been provided in the literature to rationalize CAC, especially the properties imposed on the functional forms of CAC (Hayashi, 1982). This lack of microfoundations unavoidably invites criticisms, such as the following:

(i) Empirical analysis based on firm-level data does not find convex adjustment costs important in explaining firm-level investment behavior.[2]

(ii) Firm-level investment is lumpy with very little serial correlation, which is inconsistent with convex adjustment costs which smooth out investment over time.[3]

(iii) CAC imply that Tobin's Q should be a sufficient statistic to explain firm-level investment, but firms' investments are more sensitive to cash flows than to Tobin's Q.[4]

The goal of this paper is to reconcile the apparent inconsistencies between micro and macro behaviors of investment. In particular, we show that financial frictions in the form of collateralized borrowing at the firm level can simultaneously explain convex adjustment costs at the aggregate level and lumpy investment at the firm level if firms are subject to idiosyncratic shocks. A particular advantage of our approach is that the model is analytically tractable with closed-form solutions.

2 CAC and Related Literature

The typical CAC in macro models take the following functional form (Hayashi, 1982):

$$K_{t+1} = (1-\delta) K_t + \psi \left(\frac{I_t}{K_t} \right) K_t, \tag{1}$$

where the function $\psi(\cdot)$ is increasing, concave, and homogeneous of degree zero; K_t denotes the existing capital stock; and I_t denotes total investment expenditure as part of a firm's cash flow (CF): $CF = F(K,N) - WN - PI$, where P is the relative price of investment goods. In a one-good economy, $P = 1$. This type of CAC function $\psi(\cdot)$ implies diminishing returns

[2] See e.g, Cooper and Haltiwanger (2006) and Bloom (2009).

[3] See, e.g., Caballero, Engel and Haltiwanger (1995), Cooper, Haltiwanger and Power (1999), Doms and Dunne (1998), and Power (1994). In the data, as one moves from the plant level to more aggregated levels, such as business establishments, firms, and industries, the lumpiness of investment gradually weakens. However, even at the firm level, investment still appears to be very lumpy, much lumpier than industry-level investment (see, e.g., Doms and Dunne, 1998, p.422). Although most empirical literature used plant-level data to document lumpy investment, in our model we assume that firms and plants are equivalent entities and use these terms synonymously (i.e., each firm has only one plant).

[4] See, e.g., Hassett and Hubbard (1997) and Caballero (1999).

to investment in capital formation—part of the investment spending is lost and does not become productive capital. Under this type of adjustment cost function, the average Tobin's Q is the same as the marginal Q, which greatly facilitates empirical studies of investment behaviors (Hayashi, 1982).

This form of adjustment costs in equation (1) is equivalent to an alternative formulation of CAC that is also popular in the investment literature. This alternative formulation maintains the neoclassical law of motion for capital, $K_{t+1} = (1-\delta) K_t + \tilde{I}_t$, but redefines a firm's cash flow as

$$F(K_t, N_t) - W_t N_t - P_t C(\tilde{I}_t / K_t) K_t, \tag{2}$$

where the function $C(\cdot)$ denotes total real costs associated with investment expenditure $\tilde{I}_t$ measured in capital units and satisfies the properties $C'(\cdot) > 0$ and $C''(\cdot) > 0$ (see, e.g., Abel, 1982, 1983).

These two forms of adjustment costs formulated in equations (1) and (2) are equivalent, since by redefining $I_t/K_t = C(\tilde{I}_t/K_t)$, we have $\tilde{I}_t/K_t = C^{-1}(I_t/K_t) = \psi(I_t/K_t)$. There are other formulations of CAC, but this paper focuses on the more standard form defined in equation (1).

Why does aggregate capital accumulation exhibit convex adjustment costs? At least three plausible explanations are offered in the literature: (i) Installing new capital takes time and involves sunk costs, delivery lags, and learning (e.g., Cooper and Haltiwanger, 2006). (ii) Capital is firm specific, which makes investment irreversible or partially irreversible (i.e., it comes with resale costs). Irreversibility imposes costs in adjusting the capital stock downward. (iii) Firms are borrowing constrained; hence, they are not able to increase capital at an infinite speed. Borrowing constraints impose costs in adjusting capital upward.

Two questions naturally arise: Suppose these frictions are explicitly modeled in firms' optimization decisions; (i) would they necessarily give rise to the form of CAC in equation (1)? (ii) If so, do they have the same policy implications as those implied by equation (1)? (iii) Are these frictions consistent with the lumpiness of firm level investment?

These questions are answered in this paper. We show the following:

(i) If firms' investment projects are subject to idiosyncratic risk (that affects the project's rate of returns) and firms face borrowing constraints with borrowing limit proportional to firms' collateral (capital stock), then the aggregate economy exhibits CAC that are identical in functional form to equation (1).

(ii) Irreversible investment—an important assumption in the investment literature to rationalize convex adjustment costs[5]—is unnecessary for deriving the aggregate CAC function but imposes more structures on the CAC function. In particular, if investment is completely irreversible and the distribution of investment-specific shocks follows the Pareto distribution, then the implied aggregate CAC function takes the popular Cobb-Douglas form:

$$K_{t+1} = (1 - \delta) K_t + b I_t^{\theta} K_t^{1-\theta}, \tag{3}$$

where $\theta \in (0, 1)$ is a parameter that depends on the borrowing constraints and distribution of firm-specific shocks.

(iii) A microfounded CAC model with financial constraints is consistent with the following empirical facts: (a) Firm-level investment is lumpy and (b) firm-level investment has little serial correlation and is insensitive to Tobin's Q.[6]

This paper relates to the work of Carlstrom and Fuerst (1997), who show that the particular type of borrowing constraints studied by Bernanke and Gertler (1989) can imply aggregate CAC. The specific financial frictions studied by Bernanke and Gertler (1989) are private information for investment returns and agency costs associated with costly state verification. However, these types of borrowing constraints do not imply a CAC function identical to that in equation (1) because the implied CAC function under agency costs is not homogeneous of degree zero and does not have the property that the marginal Q equals the average Q. Hence, our paper differs from this literature in at least three important aspects. First, the financial friction we consider is based on costly contract enforcement and collateralized borrowing as in the works of Kiyotaki and Moore (1997) and Jermann and Quadrini (2010).[7] More specifically, in the models of Bernanke and Gertler (1989) and Carlstrom and Fuerst (1997), firms rent capital from entrepreneur households who transform consumption goods into capital by borrowing from unproductive households. In contrast, capital rental markets do not exist in our model and firms must finance fixed investment through external funds with borrowing limits depending on the firm's collateral value. Thus, we can characterize the relationship between the marginal Q and average Q of a firm, following closely the

[5] See, e.g., Abel and Eberly (1994, 1996), Pindyck (1991), Dixit (1992), and Dixit and Pindyck (1994).

[6] The theoretical literature on lumpy investment typically assumes fixed investment costs, which are not assumed in this paper. Important examples include Veracierto (2002), Thomas (2002), Khan and Thomas (2003, 2008), Gourio and Kashyap (2007), Bachmann et al. (2008), among others.

[7] The literature on financial constraints and contract enforceability is vast. A selection of works closely related to those of Kiyotaki and Moore (1997) and Jermann and Quadrini (2010) includes those of Albuquerque and Hopenhayn (2004), Cooley, Marimon, and Quadrini (2004), Jermann and Quadrini (2010), Iacoviello (2005), and Liu and Wang (2010), among many others.

tradition of Tobin (1969) and Hayashi (1982). Second, in an agency-cost model, investment is not lumpy because the entrepreneurs always undertake investment in equilibrium. This feature is inconsistent with data. In contrast, we attempt to quantitatively match the lumpiness of firm-level investment and the correlation between the investment rate and Tobin's Q.

Our work also relates to Lorenzoni and Walentin (2007) and the associated literature that uses simulated data from theoretical models with financial frictions to investigate the quantitative relationship between Tobin's Q and investment (e.g., Gomes, 2001; and others). Lorenzoni and Walentin (2007) show that financial constraints can substantially weaken the correlation between Q and investment, relative to a frictionless benchmark (e.g., Hayashi, 1982). While our model can also explain the weak relationship between Q and investment, our approach differs from that of Lorenzoni and Walentin (2007) in one important aspect: They assume CAC in firms' investment technologies, whereas we do not need this assumption. Consequently, their model cannot explain the lumpiness of firm-level investment. Our paper also differs from theirs in the main focus of the analysis: We try to rationalize and derive CAC from microfoundations.

Thomas (2002) uses a model with non-convex adjustment costs to generate lumpy investment at the firm level and shows that such lumpiness can be unrelated to the volatility of aggregate investment in general equilibrium. Our analysis differs from hers. We show instead that borrowing constraints can simultaneously explain the lumpiness of firm-level investment and the sluggishness of aggregate investment. Consistent with Thomas (2002), however, our results suggest that there can be no causal relations between investment volatility at the firm level and that at the aggregate level. This implication holds in our model regardless of general equilibrium.

The rest of the paper is organized as follows. Section 3 presents a benchmark model with a simple form of borrowing constraints and shows how to derive equation (1) from the model. Section 4 studies a model with endogenous borrowing limits and their policy implications. Section 5 provides a rationalization for the special forms of borrowing constraints using limited contract enforceability. Section 6 conducts quantitative simulations of our microfounded model and examines the model's predictions for the lumpiness of firm-level investment and its correlation with Tobin's Q. Section 7 discusses the robustness of the results. Section 8 concludes the paper.

3 The Benchmark Model

3.1 Firms

There is a continuum of competitive firms indexed by $i \in [0,1]$. Firm i's objective is to maximize its discounted dividends,

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} D_t(i), \tag{4}$$

where $D_t(i)$ represents firm i's dividend in period t and Λ_t the marginal utility of a representative household. The production function has constant returns to scale and is given by

$$Y_t(i) = F(K_t(i), A_t N_t(i)), \tag{5}$$

where A_t represents aggregate labor-augmenting technology which can be either deterministic or stochastic, and $N_t(i)$ and $K_t(i)$ are firm-level employment and capital, respectively. Each firm accumulates capital according to the law of motion,

$$K_{t+1}(i) = (1-\delta)K_t(i) + \varepsilon_t(i) I_t(i), \tag{6}$$

where $I_t(i)$ denotes investment expenditure and $\varepsilon_t(i) \in \mathbf{R}^+$ is an idiosyncratic shock to the marginal efficiency of investment, which has the probability density function $\phi(\varepsilon)$ and cumulative density function $\Phi(\varepsilon)$. For simplicity, assume that this shock is orthogonal to any aggregate shocks. A firm's dividend in period t is hence given by $D_t(i) = Y_t(i) - P_t I_t(i) - W_t N_t(i)$, where $P_t = 1$ denotes the relative price of investment goods and W_t the competitive real wage.

What is the meaning of $\varepsilon_t(i)$? There are at least two interpretations. First, as it is modeled here, $\varepsilon_t(i)$ is a shock to the rate of returns to investment. A higher realization of $\varepsilon_t(i)$ implies that the same amount of investment expenditure leads to more finished capital goods. In a world with time-to-build (Kydland and Prescott, 1982), it takes time and additional efforts for invested resources to become productive capital. So the efficiency shock $\varepsilon_t(i)$ captures any idiosyncratic factors involved in the process between the time of investment spending and the time of project completion.

Second, the results are identical if we assume that the dividend is given by $D_t(i) = Y_t(i) - \varepsilon_t(i) I_t(i) - W_t N_t(i)$ and the law of motion of capital is given by $K_{t+1}(i) = (1-\delta) K_t(i) + I_t(i)$. In this alternative setting, $\varepsilon_t(i)$ measures the cost (or its inverse) of investment. So $\varepsilon_t(i)$

captures idiosyncratic costs associated with the ordering and installation of capital for a firm. This interpretation of $\varepsilon_t(i)$ directly relates to the micro adjustment cost literature pertaining to equation (2).

In addition, investment-specific cost shock $\varepsilon_t(i)$ allows us to handle the case with occasional binding financial constraints. In the original Kiyotaki and Moore (1997) model, the constraint is assumed to be always binding even when desired investment is low. In reality, however, this may not be the case.

Denote $n_t(i) \equiv N_t(i)/K_t(i)$ as the labor-to-capital ratio and $f(\cdot) \equiv F(1, \cdot)$ as the output-to-capital ratio. Given the real wage, the firm's optimal labor demand is determined by the equation $f_n(A_t n_t(i)) A_t = W_t$. Note that the labor demand function implies that all firms choose the same labor-to-capital ratio, namely, $n_t(i) = n(w_t, A_t)$ for all i. Firm i's operating profits can then be expressed as $R_t K_t(i) = \max_{N_t(i)} \{Y_t(i) - W_t N_t(i)\}$, where

$$R_t \equiv f(\cdot) - w_t n_t \tag{7}$$

is independent of i and the capital stock. Hence, a firm's operating profit is proportional to its capital stock. The dividend is then given by $D_t(i) = R_t K_t(i) - I_t(i)$.

We make the following additional assumptions:

(i) A firm's investment is financed by credit and is subject to the borrowing constraint:

$$I_t(i) \leq \theta K_t(i), \tag{8}$$

where $\theta > 0$ is a constant. This borrowing constraint specifies that total investment cannot exceed an amount proportional to the existing capital stock. We defer discussions about the justifications of such a form of borrowing constraints to a later section.

(ii) Firm-level investment may be partially irreversible:

$$K_{t+1}(i) \geq \rho (1 - \delta) K_t(i), \tag{9}$$

where the parameter $\rho \in [0, 1]$ indicates the degree of irreversibility. For example, if $\rho = 1$, then investment is completely irreversible and equation (9) becomes $I_t(i) \geq 0$. At the other extreme, if $\rho = 0$, then investment is completely reversible and equation (9) becomes $K_{t+1}(i) \geq 0$. Hence, the restriction in equation (9) encompasses both reversible and irreversible investment as special cases. Equation (9) can also be rewritten as

$$I_t(i) \geq -\frac{\tilde{\rho}}{\varepsilon_t(i)} K_t(i), \tag{10}$$

where $\tilde{\rho} \equiv (1-\rho)(1-\delta)$. Since our general results hold for $\rho = 0$, irreversible investment is not essential for our analysis.

With the definition in equation (7), a firm's maximization problem can be rewritten as

$$\max_{\{i_t\}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} \left(R_t K_t(i) - I_t(i)\right) \tag{11}$$

subject to equations (6), (8), and (10).

Denote $\{\lambda_t(i), \mu_t(i), \pi_t(i)\}$ as the Lagrangian multipliers of constraints (6), (8), and (10), respectively. The firm's first-order conditions for $\{I_t(i), K_{t+1}(i)\}$ are given, respectively, by

$$1 = \varepsilon_t(i)\lambda_t(i) + \pi_t(i) - \mu_t(i), \tag{12}$$

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1-\delta)\lambda_{t+1}(i) + \theta\mu_{t+1}(i) + \tilde{\rho}\frac{\pi_{t+1}(i)}{\varepsilon_{t+1}(i)} \right\}, \tag{13}$$

plus the complementarity slackness conditions, $\pi_t(i)\left[I_t(i) + \rho\frac{K_t(i)}{\varepsilon_t(i)}\right] = 0$ and $\mu_t(i)[\theta_t K_t(i) - I_t(i)] = 0$. It is obvious that when $\varepsilon_t(i)$ is i.i.d, the Lagrangian multipliers $\{\lambda_t(i),\, \pi_t(i),\, \mu_t(i)\}$ depend only on aggregate states $\mathbf{S}_t$ and $\varepsilon_t(i)$, which implies that the expected values of the Lagrangian multipliers are independent of i; namely, $E_t\lambda_{t+1}(i) = \bar{\lambda}_{t+1}$, $E_t\mu_{t+1}(i) = \bar{\mu}_{t+1}$, and $E_t\pi_{t+1}(i) = \bar{\pi}_{t+1}$. So equation (13) can be rewritten as

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1-\delta)\bar{\lambda}_{t+1} + \theta\bar{\mu}_{t+1} + \tilde{\rho}\int \frac{\pi_{t+1}(i)}{\varepsilon_{t+1}(i)} d\Phi(\varepsilon) \right\}, \tag{14}$$

which shows that $\lambda_t(i) = \lambda(\mathbf{S}_t) \equiv \lambda_t$ is also independent of i. Since the marginal cost of investment is 1 and the marginal value of newly installed capital stock is λ_t, the market-based measure of Tobin's Q is given by $Q_t = \lambda_t$, which is independent of i.

3.2 Investment Decision Rules

We use a guess-and-verify strategy to derive closed-form decision rules at the firm level. The decision rules are characterized by a cutoff strategy where the cutoff (ε_t^*) pertains to the realization of investment-specific shocks and is defined by the opportunity cost of installing one unit of capital:

$$\lambda_t = \frac{1}{\varepsilon_t^*}. \tag{15}$$

Consider the following possible cases:

Case A: $\varepsilon_t(i) > \varepsilon_t^*$. In this case, the marginal efficiency of investment is high. Since the return to investment is high, firms opt to undertake investment up to the borrowing limit, $I_t(i) = \theta_t K_t(i)$, so the constraint (10) does not bind. Hence, we have $\pi_t(i) = 0$. Equation (12) implies $\mu_t(i) = \frac{\varepsilon_t(i)}{\varepsilon_t^*} + \pi_t(i) - 1 = \frac{\varepsilon_t(i)}{\varepsilon_t^*} - 1 > 0$.

Case B: $\varepsilon_t(i) < \varepsilon_t^*$. In this case, the marginal efficiency of investment is low. Given this, firms opt to make minimum investment, which means $I_t(i) = -\tilde{\rho}\frac{K_t(i)}{\varepsilon_t(i)}$. So the constraint (8) does not bind and we have $\mu_t(i) = 0$. Equation (12) implies $\pi_t(i) = 1 + \mu_t(i) - \frac{\varepsilon_t(i)}{\varepsilon_t^*} = 1 - \frac{\varepsilon_t(i)}{\varepsilon_t^*} > 0$.

Case C: $\varepsilon_t(i) = \varepsilon_t^*$. By equation (12), $\mu_t(i) = \frac{\varepsilon_t(i)}{\varepsilon_t^*} + \pi_t(i) - 1 = \pi_t(i)$. Suppose $\{\mu_t(i), \pi_t(i)\} > 0$; by the slackness conditions we have $I_t(i) = -\tilde{\rho}\frac{K_t(i)}{\varepsilon_t(i)}$ and $I_t(i) = \theta K_t(i)$, which is a contradiction. Hence, it must be true that $\mu_t(i) = \pi_t(i) = 0$. In this marginal case, equation (12) implies $\lambda_t = \frac{1}{\varepsilon_t^*}$, which confirms that the cutoff is indeed given by equation (15). Without loss of generality, we assume that in this marginal case a firm undertakes maximum investment.

Notice that from an individual firm's own perspective, Tobin's Q is measured by $q_t(i) \equiv \frac{\varepsilon_t(i)}{\varepsilon_t^*}$. A firm will undertake positive investment if $q(i) \geq 1$, otherwise the firm disinvest or remains inactive. However, because markets are incomplete and the idiosyncratic shocks are not observable (or insured) through markets, the market-based measure of Tobin's Q is $\frac{1}{\varepsilon_t^*}$, which is independent of $\varepsilon_t(i)$.

Based on the above analysis, the Lagrangian multipliers satisfy $\mu_t(i) = \max\{q_t(i) - 1, 0\}$ and $\pi_t(i) = \max\{1 - q_t(i), 0\}$. The firm's decision rules for investment and capital accumulation are thus given by

$$I_t(i) = \begin{cases} \theta K_t(i) & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ -\tilde{\rho}\frac{K_t(i)}{\varepsilon_t(i)} & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases} \tag{16}$$

$$\frac{1}{\varepsilon_t^*} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\left\{R_{t+1} + \frac{(1-\delta)}{\varepsilon_{t+1}^*} + O(\varepsilon_{t+1}^*)\right\}, \tag{17}$$

where the implicit function $O(\cdot)$ in equation (17) is defined by

$$O(\varepsilon_{t+1}^*) \equiv E_t\left[\theta\mu_{t+1}(i) + \tilde{\rho}\frac{\pi_{t+1}(i)}{\varepsilon_{t+1}(i)}\right] \tag{18}$$
$$= \theta\int_{\varepsilon_{t+1}(i)\geq\varepsilon_{t+1}^*}\frac{\varepsilon_{t+1}(i)-\varepsilon_{t+1}^*}{\varepsilon_{t+1}^*}d\Phi(\varepsilon) + \tilde{\rho}\int_{\varepsilon_{t+1}(i)<\varepsilon_{t+1}^*}\left(\frac{1}{\varepsilon_{t+1}(i)} - \frac{1}{\varepsilon_{t+1}^*}\right)d\Phi(\varepsilon).$$

The investment function (16) indicates that firm-level investment is lumpy with little serial correlation. Each firm in any period has only probability $1 - \Phi(\varepsilon_t^*)$ of undertaking positive investment and probability $\Phi(\varepsilon_t^*)$ of remaining inactive (or disinvesting). These probabilities are determined by aggregate economic conditions that influence the cutoff (ε_t^*) and are independent of each firm's investment history (which is highly idiosyncratic). Also, such lumpiness is independent of the value of $\tilde{\rho}$—namely, the lumpiness does not hinge on irreversibility.

Notice that $O(\cdot)$ is the option value of one unit of installed capital[8]: If the firm receives a favorable shock in the next period, one unit of installed capital can expand the firm's borrowing capacity by θ units and each additional unit of installed capital can bring a net profit of $q(i) - 1$ ($= \frac{\varepsilon(i)-\varepsilon^*}{\varepsilon^*}$) units. This case occurs with probability $\int_{\varepsilon\geq\varepsilon_{t+1}^*} d\Phi(\varepsilon)$. In the case of an unfavorable shock, the firm can withdraw $\tilde{\rho} \geq 0$ of investment units and each unit of saving can be transformed into $\frac{1}{\varepsilon}$ units of consumption goods. By doing so, the firm can increase net profit by $\frac{1-q}{\varepsilon}$ ($= \frac{1}{\varepsilon(i)} - \frac{1}{\varepsilon^*}$) units.

Hence, equation (17) implies that the optimal level of investment is determined at the point where the marginal cost ($\frac{1}{\varepsilon_t^*}$) equals the marginal benefits (= the marginal product of capital + the value of nondepreciated capital + the option value of capital). Because the optimal level of investment depends on the expected returns, which in turn depend on the probability weights of the different cases considered above (i.e., the cutoff ε_t^*), equation (17) states that a firm chooses the optimal cutoff ε_t^* (as an implicit function of aggregate economic conditions) so that the marginal cost of investment equals the expected marginal gains.

Equation (17) also shows that the optimal cutoff ε_t^* is independent of i, so it is the same across all firms. More specifically, the optimal cutoff is independent of a firm's investment

[8] Note that we used the orthogonality condition between idiosyncratic shocks and aggregate uncertainty to derive equation (17).

rate and existing capital stock. This property allows us to characterize aggregate investment dynamics in a tractable manner without needing to use numerical methods (as in the work of Krusell and Smith, 1998).

3.3 Properties of Aggregate Investment Function

Integrating the firm-level decision rules by the law of large numbers, the aggregate investment, aggregate capital stock, and the optimal cutoff are determined jointly by equation (17) and the following two equations:

$$\frac{I_t}{K_t} = \theta\left[1 - \Phi(\varepsilon_t^*)\right] - \tilde{\rho}\int_{\varepsilon<\varepsilon_t^*}\frac{1}{\varepsilon}d\Phi(\varepsilon); \tag{19}$$

$$K_{t+1} = (1-\delta)K_t + \theta K_t\int_{\varepsilon\geq\varepsilon_t^*}\varepsilon d\Phi(\varepsilon) - \tilde{\rho}K_t\int_{\varepsilon<\varepsilon_t^*}d\Phi(\varepsilon), \tag{20}$$

where equation (19) is derived from equation (16) and equation (20) from equation (6). It can be confirmed by the eigenvalue method that this three-equation dynamic system has a unique saddle-path steady state. Hence, given the stochastic process of $\{R_t, \Lambda_t\}$, the equilibrium path of $\{I_t, K_{t+1}, \varepsilon_t^*\}$ is uniquely determined.

Equation (19) suggests that the aggregate investment rate is fully determined by ε_t^*. Given the parameters θ and $\tilde{\rho}$, this equation also defines the cutoff as an implicit function of the investment rate: $\varepsilon_t^* = \varepsilon^*(\frac{I_t}{K_t})$. Therefore, equation (20) can be written as

$$K_{t+1} = (1-\delta)K_t + \varphi(\frac{I_t}{K_t})K_t, \tag{21}$$

where

$$\varphi(\frac{I_t}{K_t}) \equiv \theta\int_{\varepsilon\geq\varepsilon^*(\frac{I_t}{K_t})}\varepsilon d\Phi(\varepsilon) - \tilde{\rho}\int_{\varepsilon<\varepsilon^*(\frac{I_t}{K_t})}d\Phi(\varepsilon) \tag{22}$$

is an implicit function of the aggregate investment rate.

Proposition 1 *The implicit function $\varphi(\cdot)$ is increasing, strictly concave, and homogeneous of degree zero in $\{I_t, K_t\}$.*

Proof. See Appendix I. ■

3.4 Equivalence between CAC and Borrowing Constraints

If we define the market value of one unit of newly installed capital (or Tobin's Q) of a firm as

$$Q_t \equiv \lambda_t = \frac{1}{\varepsilon_t^*} \tag{23}$$

and the aggregate investment rate as $i_t \equiv \frac{I_t}{K_t}$, using equation (19), we can simplify the implicit function $O(\varepsilon_t^*)$ in equation (18) to

$$\begin{aligned} O(\varepsilon_t^*) &= \frac{1}{\varepsilon_t^*}\left[\theta \int_{\varepsilon \geq \varepsilon_t^*} \varepsilon d\Phi(\varepsilon) - \tilde{\rho} \int_{\varepsilon < \varepsilon_t^*} d\Phi(\varepsilon)\right] - \left\{\theta[1-\Phi(\varepsilon_t^*)] - \tilde{\rho}\int_{\varepsilon<\varepsilon_t^*} \frac{1}{\varepsilon} d\Phi(\varepsilon)\right\} \\ &= Q_t \varphi(i_t) - i_t. \end{aligned} \tag{24}$$

Therefore, using the defined functions for $\{\varphi(\cdot), \varphi'(\cdot), O(\cdot), Q\}$, the system of equations that solves for the aggregate investment rate (i_t), the capital stock (K_{t+1}), and Q_t is given by

$$Q_t \varphi'(i_t) = 1, \tag{25}$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\left\{R_{t+1} + Q_{t+1}(1-\delta) + Q_{t+1}\varphi(i_{t+1}) - i_{t+1}\right\}, \tag{26}$$

$$K_{t+1} = (1-\delta)K_t + \varphi(i_t)K_t. \tag{27}$$

Now consider a standard representative-agent macro model with CAC (e.g., Hayashi, 1982), where a representative firm solves (taking as given the marginal product of capital R_t)

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0}\left(R_t K_t - I_t\right) \tag{28}$$

subject to

$$K_{t+1} = (1-\delta)K_t + \psi\left(\frac{I_t}{K_t}\right)K_t. \tag{29}$$

Defining Q_t (Tobin's Q) as the Lagrangian multiplier for the constraint (29) and $i_t \equiv \frac{I_t}{K_t}$ as the investment rate, the first-order conditions for $\{I_t, K_{t+1}\}$ are given, respectively, by

$$Q_t \psi'(i_t) = 1 \tag{30}$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\{R_{t+1} + (1-\delta)Q_{t+1} + Q_{t+1}\left[\psi(i_{t+1}) - \psi'(i_{t+1})i_{t+1}\right]\}. \tag{31}$$

Using equation (30), we can rewrite equation (31) as

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\{R_{t+1} + (1-\delta)Q_{t+1} + Q_{t+1}\psi(i_{t+1}) - i_{t+1}\}. \tag{32}$$

Notice that the system of equations (29), (30), and (32) is identical to the system of equations (25)-(27) because the two CAC functions, $\varphi(\cdot)$ and $\psi(\cdot)$, have the same properties. Since our microfounded model is equivalent to the aggregate CAC model regardless of the value of $\tilde{\rho}$, the equivalence result is established without relying on the assumption of irreversible investment. The key assumption instead is the collateralized borrowing constraint (8).

The equivalence result holds regardless of the exogenous driving processes of $\{R_t, W_t, \Lambda_t\}$. That is, the two models are identical not only in the steady state but also along any transitional dynamic path. For example, the impulse responses of the two models are completely identical under either aggregate technology shocks that affect $\{R_t, W_t\}$ or aggregate demand shocks that affect Λ_t.

Example To further illustrate the equivalence result, suppose the distribution of ε is Pareto with support $[1, \infty)$ and shape parameter $\eta > 1$—namely, $\Phi(\varepsilon) = 1 - \varepsilon^{-\eta}$; and assume that investment is completely irreversible, i.e., $\tilde{\rho} = 0$. With these assumptions, equations (19) and (20) become, respectively, $\frac{I_t}{K_t} = \theta\varepsilon_t^{*-\eta} = \theta Q_t^{\eta}$ and $K_{t+1} = (1-\delta)K_t + \frac{\eta\theta}{\eta-1}\varepsilon_t^{*1-\eta}K_t$, where the adjustment cost function $\frac{\eta\theta^{\frac{1}{\eta}}}{\eta-1}\left(\frac{I_t}{K_t}\right)^{\frac{\eta-1}{\eta}} = \varphi(\frac{I_t}{K_t})$ is homogeneous of degree zero and satisfies $\varphi'(\frac{I_t}{K_t})Q_t = 1$. Substituting for ε_t^*, equation (32) then becomes $Q_t = E_t\beta_{t+1}\left\{R_{t+1} + Q_{t+1}(1-\delta) + \frac{1}{\eta-1}\frac{I_{t+1}}{K_{t+1}}\right\}$, and the law of capital accumulation becomes

$$K_{t+1} = (1-\delta)K_t + \frac{\eta\theta^{\frac{1}{\eta}}}{\eta-1} I_t^{\frac{\eta-1}{\eta}} K_t^{\frac{1}{\eta}}, \tag{33}$$

which has the familiar Cobb-Douglas form (equation (3)) commonly assumed in the macro literature.

3.5 Intuition of the Equivalence

In representative-agent models, CAC imply that the aggregate investment rate is sluggish in responding to macroeconomic environmental changes because of diminishing returns to investment in capital formation. In other words, because $\psi(\cdot)$ is concave, aggregate investment responds to a higher future capital productivity (R_{t+1}) less elastically than it would otherwise. As a result, the optimal capital stock can be reached only through multiple periods of investment at a finite speed instead of through a single-period investment at an infinite speed.

In our heterogeneous-agent model, firm-level investment is lumpy because a firm undertakes either a large amount of positive investment ("active") or a large amount of negative investment ("inactive"), depending on the idiosyncratic shock to the rate of return to investment in a particular period. However, despite the lumpiness of firm-level investment, aggregate investment is sluggish. Aggregate investment in our model has two margins: an intensive margin that depends on each firm's maximum investment level (θ) and an extensive margin that depends on the number of active firms (ε_t^*) in each period. Equation (19) shows that the aggregate investment rate depends on θ (the intensive margin) and the proportion of active firms, $1-\Phi(\varepsilon^*) = \Pr[\varepsilon \geq \varepsilon^*]$ (the extensive margin, assuming $\tilde{\rho} = 0$ for a moment). However, the extensive margin is determined by the optimal cutoff ε_t^*, which behaves sluggishly because by equation (17) the inverse of the cutoff ε_t^*—Tobin's Q—is a slow-moving (weighted) average of expected future marginal products ($R_{t+j}, j = 1, 2, ...$) as well as the option values of capital ($O_{t+j}, j = 1, 2, ...$). Hence, when R_{t+1} changes, the optimal level of aggregate capital stock cannot be reached through a single-period aggregate investment because the extensive margin (Tobin's Q) adjusts slowly over time (since an increase in R_{t+1} has only a small impact on the cutoff).

Aggregate investment (both in our microfounded model and in the representative CAC model) depends fully and positively on Tobin's Q because Tobin's Q contains all information about the marginal costs and benefits of investment—a higher value of capital is required for a higher investment rate when the marginal cost of investment is increasing. However, it is well known that this Q-theory of investment has not fared well empirically. Variables such as firms' cash flows are always found to be significant in explaining firm-level investment other than the average Q (see, e.g., Hassett and Hubbard, 1997).

Our approach provides an explanation for this apparent failure of the Q-theory. In our model, firm-level investment is driven by idiosyncratic cost (or efficiency) shocks $\varepsilon(i)$ while the market-based measure of Q does not capture (reveal) such information. Hence, firms' cash positions and net worth will appear to be more important than the market value of Tobin's Q in determining the rate of investment. On the other hand, without borrowing constraints, only the most productive firm—the most efficient firm with the highest draw of $\varepsilon(i)$—will undertake investment in each period; the model then degenerates to a one-firm (or representative-firm) model in which Q is a sufficient statistic for determining a firm's investment. Hence, both idiosyncratic shocks and borrowing constraints are important in rendering firm-level investment insensitive to Q.

At the aggregate level, however, total investment depends positively and fully on Q for the following reasons: Since a firm's investment is constrained by the firm's capital stock, only the most efficient firms will undertake positive investment and the rest will remain inactive in each period. Thus, an increase in the aggregate stock of capital requires a greater proportion of active firms. This is possible in equilibrium only if the market value of capital (Q) increases (or the cutoff ε^* decreases) so that more firms (including the less efficient ones) will find investment profitable. In other words, the less efficient firms raise the aggregate marginal cost of investment, hence calling for a higher Q to balance it in equilibrium. Therefore, aggregate investment is closely related to Q. This explains why empirical work based on micro data tends to find firms' cash flows more important than Q in determining the rate of firm investment in the short run, but aggregate data and long-run analysis tend to find Q important and significant in determining aggregate investment (see, e.g., Caballero, 1999; Cooper and Haltiwanger, 2006).

So if we were to use both our model and the standard CAC model to generate artificial data for firms' investment rate and run regressions between this variable and Tobin's Q for the two models, the results would indicate that the regression's $R^2 = 1$ in the aggregate CAC model and is < 1 in our model.[9]

4 Endogenous Borrowing Constraints

In the previous benchmark model, the borrowing limit is assumed to be a fixed proportion of the existing capital stock. In general, firms' borrowing limits may depend on the value of the collateral (Kiyotaki and Moore, 1997). That is, the parameter θ may be endogenous. To take this into account, consider the following borrowing constraint with endogenous credit limits:

$$I_t(i) \leq \theta Q_t(i) K_t(i), \tag{34}$$

[9] For quantitative results from such regression analyses, see Table 3 in our working paper (Wang and Wen, 2010).

where $Q_t(i)$ denotes the market value of firm i's existing capital stock and $\theta > 0$ is a parameter.[10]

Proposition 2 *Assuming $\tilde{\rho} = 0$ for simplicity (without loss of generality), a firm's investment decision rule is given by*

$$I_t(i) = \begin{cases} \theta Q_t K_t(i) & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ 0 & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases}; \tag{35}$$

the market value Q_t is determined by

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1-\delta) Q_{t+1} + \theta Q_{t+1} \int_{\varepsilon_{t+1}(i) \geq \varepsilon_{t+1}^*} \left[\frac{\varepsilon_{t+1}(i)}{\varepsilon_{t+1}^*} - 1 \right] d\Phi(\varepsilon) \right\}, \tag{36}$$

or, equivalently, by $Q_t = \frac{1}{\varepsilon_t^*} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} v_{t+1}(i)$; and the firm's private value $v_t(i)$—the value function of a firm per unit of capital—is determined by

$$v_t(i) = \begin{cases} R_t + (1-\delta) Q_t + \theta Q_t [Q_t \varepsilon_t(i) - 1] & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ R_t + (1-\delta) Q_t & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases}. \tag{37}$$

Proof. See Appendix II. ■

The investment decision rule has the same form as that in the benchmark model (except here the parameter θ in the benchmark model is replaced by θQ_t). Note that the private value of a firm is proportional to its capital stock (i.e., $v_t(i)$ is independent of $K_t(i)$) and the market value of a firm, $Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} v_{t+1}(i)$, is the same across all firms, as in the benchmark model.[11]

By the law of large numbers, the aggregate investment is given by

$$I_t = \theta K_t Q_t [1 - \Phi(\varepsilon_t^*)]. \tag{38}$$

[10] For example, if the non-depreciated capital stock is fully collateralized, then $\theta = 1 - \delta$.

[11] As mentioned earlier, because markets are incomplete in the model, idiosyncratic shocks to a firm's investment return are uninsured. Hence, $q_t(i) = \frac{\varepsilon_t(i)}{\varepsilon_t^*}$ cannot be used by the market to determine a firm's Q. This is why the market-based measure of Tobin's Q is $\frac{1}{\varepsilon_t^*}$ instead of $\frac{\varepsilon(i)}{\varepsilon^*}$.

Since $Q_t = \frac{1}{\varepsilon_t^*}$, the above equation defines the implicit function $Q_t = Q\left(I_t/K_t\right)$. We can use this implicit equation to rewrite equation (36) as

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1-\delta)Q_{t+1} + \theta Q_{t+1}^2 \int_{\varepsilon \geq \frac{1}{Q(I_{t+1}/K_{t+1})}} \varepsilon d\Phi(\varepsilon) - \frac{I_{t+1}}{K_{t+1}} \right\}. \tag{39}$$

Similarly, the aggregate law of motion for capital accumulation is given by

$$K_{t+1} = (1-\delta)K_t + \Psi(\frac{I_t}{K_t})K_t, \tag{40}$$

where

$$\Psi(\frac{I_t}{K_t}) \equiv \theta Q(I_t/K_t) \int_{\varepsilon \geq \frac{1}{Q(I_t/K_t)}} \varepsilon d\Phi(\varepsilon). \tag{41}$$

Proposition 3 *For any probability density function $\phi(\varepsilon)$ that satisfies $\phi'(\varepsilon) \leq 0$, the implicit function $\Psi(\cdot)$ in equation (40) is increasing, concave, and homogeneous of degree zero in $\{I_t, K_t\}$.*

Proof. See Appendix III. ■

Example Many standard distributions, such as the Pareto distribution, the exponential distribution, and the uniform distribution, satisfy the property $\phi'(\varepsilon) \leq 0$. As an example, consider the Pareto distribution, $\Phi(\varepsilon) = 1 - \varepsilon^{-\eta}$ with $\eta > 1$. Equation (38) becomes $\frac{I_t}{K_t} = \theta \varepsilon_t^{*-\eta-1}$, and the capital accumulation equation becomes $K_{t+1} = (1-\delta)K_t + \theta \frac{\eta}{\eta-1} \varepsilon_t^{*-\eta}$. Combining these two equations implies

$$K_{t+1} = (1-\delta)K_t + \varphi_0 K_t^{\frac{1}{\eta+1}} I_t^{\frac{\eta}{\eta+1}} \tag{42}$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left[R_{t+1} + (1-\delta)Q_{t+1} + \frac{1}{\eta-1} \frac{I_{t+1}}{K_{t+1}} \right] \tag{43}$$

$$\frac{I_t}{K_t} = \theta Q_t^{\eta+1}, \tag{44}$$

where $\varphi_0 \equiv \frac{\eta}{\eta-1} \theta^{\frac{1}{\eta+1}} > 0$. Thus, similar to the benchmark model, we obtain a reduced-form Cobb-Douglas CAC function $\Psi(I_t/K_t) = \varphi_0 \left(\frac{I_t}{K_t}\right)^{\frac{\eta}{\eta+1}}$ with irreversible investment and the Pareto distribution, Therefore, borrowing constraints at the firm level can fully rationalize the CAC function in equation (1), regardless of whether the borrowing limits are endogenous or exogenous. In other words, the specific form of aggregate CAC assumed by Hayashi (1982) and others in the existing literature can be derived from microfoundations with financial frictions that hinder firms' ability to borrow. However, there are subtle but important differences between the exogenous borrowing limit model and the endogenous borrowing limit model, as shown below.

4.1 Caveats on Equivalence

With endogenous borrowing limits, the equivalence between the microfounded heterogeneous-firm model and the representative-agent CAC model holds only with respect to equation (1). Unlike the benchmark model, however, the equilibrium in the endogenous borrowing limit model and that in the aggregate CAC model are not completely equivalent because the trajectories of investment and capital stock in the endogenous borrowing limit model are no longer identical to those implied by the aggregate CAC model. That is, even though the two models share the same law of motion for aggregate capital accumulation as in equation (29), the first-order conditions in equations (30) and (32) (derived in the representative-firm model) no longer hold in the microfounded model with endogenous borrowing limits.

The source of the discrepancy stems from the endogeneity of the borrowing constraints in equation (34), where the market value of capital, $Q(I_t/K_t)$, is positively affected by the rate of aggregate investment. Hence, the more investment a firm makes, the higher its value, and thus the more creditworthy it becomes. However, this type of credit externality is not internalized by firms because Q_t is a market price taken as given by individual firms. As a result, the microfounded model appears to have an insufficient investment level relative to the counterpart representative-agent CAC model.

The following proposition shows that the credit externality in the endogenous borrowing limit model is equivalent to a form of aggregate "investment externality" in the conventional CAC model, where the source of the aggregate investment externality is a social rate of return to the average investment rate that individual firms take as given.

Proposition 4 *The heterogeneous-firm model with an endogenous borrowing limit is observationally equivalent to the following representative-firm CAC model with investment externalities:*

$$K_{t+1} = (1-\delta)K_t + \tilde{\Psi}(\bar{\imath}_t, i_t)K_t, \tag{45}$$

where $\bar{\imath}_t \equiv \frac{\bar{I}_t}{\bar{K}_t}$ denotes the average investment-to-capital ratio in the economy that the representative firm takes as given, and the CAC function $\tilde{\Psi}(\cdot,\cdot)$ is increasing and concave in $\{\bar{\imath}_t, i_t\}$ and satisfies the decomposition: $\tilde{\Psi}(\bar{\imath}_t, i_t) = \theta Q(\bar{\imath}_t)\varphi(i_t)$, where the function $\varphi(\cdot)$ satisfies

$$\varphi(i_t) = \int_{\varepsilon \geq \varepsilon^*(i_t)} \varepsilon d\Phi(\varepsilon), \tag{46}$$

which is also increasing and concave.

Proof. *See Appendix IV.* ■

Example As an example, consider the microfounded model with Pareto distribution. The model's equilibrium is characterized by equations (42), (43), and (44). Now consider a representative-firm CAC model with investment externality $\left(\frac{\bar{I}_t}{\bar{K}_t}\right)^a$:

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} [R_t K_t - I_t] \tag{47}$$

subject to

$$K_{t+1} = (1-\delta)K_t + \varphi_0 \left(\frac{\bar{I}_t}{\bar{K}_t}\right)^a K_t^{1-b} I_t^b, \tag{48}$$

where $\varphi_0 = \frac{\eta}{\eta-1}\theta^{\frac{1}{\eta+1}}$, $a = \frac{1}{\eta(\eta+1)}$, $b = \frac{\eta-1}{\eta}$, and $\frac{\bar{I}_t}{\bar{K}_t}$ denotes the average investment rate in the economy that the representative firm takes as given. Denoting Q_t as the Lagrangian multiplier for the constraint, the first-order conditions with respect to I_t and K_{t+1} are given, respectively, by

$$Q_t \varphi_0 b \left(\frac{\bar{I}_t}{\bar{K}_t}\right)^a K_t^{1-b} I_t^{b-1} = 1 \tag{49}$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1-\delta)Q_{t+1} + Q_{t+1}\varphi_0(1-b) \left(\frac{\bar{I}_t}{\bar{K}_t}\right)^a K_{t+1}^{-b} I_{t+1}^b \right\}. \tag{50}$$

Imposing the equilibrium condition, $\frac{\bar{I}_t}{\bar{K}_t} = \frac{I_t}{K_t}$, and plugging in the values of $\{\varphi_0, a, b\}$, equation (48) becomes

$$K_{t+1} = (1-\delta)K_t + \frac{\eta\theta^{\frac{1}{\eta+1}}}{\eta-1} K_t^{\frac{1}{\eta+1}} I_t^{\frac{\eta}{\eta+1}}; \tag{51}$$

equation (49) becomes

$$\frac{I_t}{K_t} = \theta Q_t^{1+\eta}; \tag{52}$$

and equation (50) becomes

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1-\delta) Q_{t+1} + \frac{1}{\eta - 1} \frac{I_{t+1}}{K_{t+1}} \right\}. \tag{53}$$

The three equations are identical to equations (42) through (44) in the microfounded model.

4.2 Policy Implications

The previous analysis suggests that if CAC is not a form of technology but a consequence of firm-level borrowing constraints, then the policy implications of an aggregate CAC model and those of a microfounded model may be different. As an example of the different policy implications of the two models, we have the following proposition:

Proposition 5 *The optimal rate of capital tax in the representative-agent CAC model is zero in the steady state, while that in the endogenous credit limit model is negative.*

Proof. See Appendix V. ■

The intuition behind this proposition is straightforward. The endogenous credit limit model features a positive credit externality in firms' investment. Because firms consider the borrowing limit as exogenous when in fact it is endogenously determined by the market equilibrium, the competitive equilibrium features suboptimal investment and leads to insufficient capital stock. Alternatively, since the model is equivalent to a representative-agent CAC model with positive investment externalities, the investment level determined by a representative firm in a competitive equilibrium is suboptimal. Therefore, the adoption of a negative capital tax rate to encourage more investment improves social welfare.

4.3 Rationalizing Collateral Constraints through Limited Contract Enforceability

So far the collateral constraints have been imposed on firms in an *ad hoc* fashion. This subsection is intended to rationalize to the assumptions we have made. The discussions below follow that of Jermann and Quadrini (2010).

Suppose that investment needs to be paid in advance, i.e., before production. Assuming that these payments could be financed with intra-period loans that do not incur interests, it is more convenient for the firm to finance the payments with debt than to carry cash over from the previous period. However, after taking out the intra-period loan and before making the investment, the firm could renegotiate the loan as pointed out by Kiyotaki and Moore (1997). In case of default or liquidation the lender would recover a fraction of the existing capital stock, $\theta K_t(i)$, which can be converted into consumption goods. By assuming that the firm has all the bargaining power, the lender will be willing to lend up to $\theta K_t(i)$. By further assuming that firms cannot raise dividends at the beginning of the period, we obtain the desired constraint (8) in the benchmark model.

On the other hand, if we assume that the lender takes over the firm in the case of default, the value recovered by the lender would be proportional to the firm's value, $\theta Q_t K_t(i) = \theta \left[\beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \int v_{t+1}(i)\, d\Phi(\varepsilon)\right] K_t(i)$, where $v_t(i)$ is the firm's private value as defined in equation (37). So the constraint (8) would be replaced by equation (34).

5 Why is Investment Lumpy at the Firm Level?

This section solves for a general-equilibrium version of our microfounded investment model and uses simulated data from the model to investigate the lumpiness in firm-level investment. Because a firm's investment rate depends on the firm's value and other macroeconomic variables such as the real wage, a general-equilibrium model is required.

A representative consumer (i.e., the owner of firms) solves

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \{\log C_t - a_L N_t\} \tag{54}$$

subject to

$$C_t \leq W_t N_t + \Pi_t, \tag{55}$$

where Π_t denotes the lump-sum profit income from all firms. Notice that, for simplicity, the household does not save. Introducing an equity market where households can buy firms' shares would give identical results. Denoting Λ_t as the Lagrangian multiplier of the household's budget constraint, the first-order conditions of the representative household are given by

$$\Lambda_t = \frac{1}{C_t}, \tag{56}$$

$$\frac{(1-\alpha)Y_t}{N_t}\frac{1}{C_t} = a_L. \tag{57}$$

The firm's problem is identical to that in the previous section with endogenous borrowing limits. The firm's decision rules are again given by equations (35) through (37), and the following relationships hold: $Y_t = A_t K_t^{\alpha} N_t^{1-\alpha}$, $W_t = (1-\alpha)\frac{Y_t}{N_t}$, and $R_t = \alpha \frac{Y_t}{K_t}$. Under the assumption of Pareto distribution, the competitive general equilibrium of the aggregate economy is characterized by these three relationships, plus equations (42), (43), (44), (56), and (57). This system of eight equations determines the equilibrium path of $\{C_t, N_t, Y_t, I_t, K_{t+1}, Q_t, W_t, R_t\}$. The equilibrium cutoff is determined by $\varepsilon_t^* = Q_t^{-1}$. The model has a unique saddle path near the steady state as can be easily confirmed by the eigenvalue method. We solve the model by log-linearizing around the steady state under the assumption that the aggregate productivity (A_t) evolves according to the law of motion,

$$\log A_t = \rho \log A_{t-1} + \sigma \xi_t, \tag{58}$$

where ξ_t is i.i.d. with the standard deviation normalized to 1.

Calibration. We calibrate the model at quarterly frequency by setting the time discounting factor $\beta = 0.99$, the capital's income share $\alpha = 0.3$, the persistence of technology shock $\rho = 0.95$, and the standard deviation of innovation $\sigma = 0.0072$ (as in the standard RBC literature). Since a_L does not enter the model's log-linear dynamic system, we choose a_L such that $N = 1$ in the deterministic steady state. The other three parameters—the depreciation rate of capital δ, the borrowing limit θ, and the Pareto distribution parameter η—are chosen so that the model matches the distribution of firm-level investment. The parameter values are summarized in Table 1.

Table 1. Parameter Values

β	α	a_L	ρ	σ	δ	θ	η
0.99	0.3	1.097	0.95	0.0072	0.032	0.08	2.4

We follow Cooper and Haltiwanger (2006) by defining $i_t(i) = \frac{K_{t+1}(i)-(1-\delta)K_t(i)}{K_t(i)}$ as a firm's investment rate. The annual investment rate in the model is calculated by simulation and time aggregation. We simulate 200,000 quarters of data. We first use a general-equilibrium model to obtain the cutoff ε_t^*. We then make 200,000 independent draws of $\varepsilon_t(i)$ for a typical firm by normalizing its initial capital stock. Finally, we calculate the annual investment rate for $\tau = 1, 2, ...50,000$ by

$$i_\tau^A = \frac{K_{4\tau} - (1-\delta)^4 K_{4*(\tau-1)}}{K_{4*(\tau-1)}}. \tag{59}$$

(More details of the simulation procedure can be found in Appendix VI). The statistics for the annualized investment rate i_t^A are reported in Table 2, where the empirical counterpart are based on statistics reported by Cooper and Haltiwanger (2006, p. 615, Table 1).

Table 2. Summary Statistics for Annualized Investment Rate

Investment Rate	$i^A \leq 0$	$0 < i^A \leq 20\%$	$i^A > 20\%$	$E[i^A]$	$std[i^A]$	$\frac{E[i^A \mid i^A \geq 0.2]}{E[i^A]}$	$\rho(i_t^A, i_{t-1}^A)$
Model (%)	17.2	62.6	20.2	12.7	31.7	57.1	-0.0056
Data (%)	18.5	62.9	18.6	12.2	33.7	50.0	0.058

The table shows that our microfounded model matches the basic features of plant-level investment dynamics reported by Cooper and Haltiwanger (2006). For example, our model predicts that in any given year, about 17% of firms are inactive (making zero or negative investment), about 20% of firms undertake big investment projects (with values exceeding 20 percent of the existing capital stock), and the average investment rate is 12.7% a year. These predictions are extremely close to data. The standard deviation of the investment rate is 32% in the model, whereas it is 34% in the data. (iii) Firm-level investment is not serially correlated. The model predicts an autocorrelation of -0.0056, while this value is 0.058 in the data (the last column in the table). Therefore, our model performs quite well in explaining the lumpiness and lack of serial correlations in firms' investment behavior.

6 Robustness Analyses

6.1 A More General Form of Financial Structure

The financing constraints in the previous sections may appear restrictive—that is, all investment must be financed by credit. This means that firms cannot accumulate financial assets and are "forced" to distribute all their profits as dividends in each period. It would be important to see whether (and how) the results of the analysis carry through to more flexible specifications of firm financial structures (e.g., those considered by Gomes, Yaron, and Zhang, 2006). Although it is beyond the scope of this paper to consider a general form of firm financial structure, this section demonstrates the robustness of our results by considering a slightly more general form of financial constraints where firms can finance investment by both credit and savings.

Suppose that firms have the option of not distributing all profits as dividends and that they can borrow from each other's past savings to finance investment in addition to using bank credit. We can model this additional source of finance as an internal loan market where firms issue one-period bonds backed up by past savings. Denote $B_{t-1}(i)$ as the savings of firm i. Notice that if $B_{t-1}(i) < 0$, then firm i lends a portion of its previous-period profits to other firms through the internal loan market. The rate of return (interest rate) is R_{bt-1}.

At the beginning of each period before production, the internal loan market opens and firms use both outside credit and their previous-period savings to finance the current-period investment. The objective function of firm i is to maximize the discounted dividends:

$$\max_{\{I_t(i),B_t(i)\}} E_0 \sum_{t=0}^{\infty} \beta^t \frac{\Lambda_t}{\Lambda_0} \left[R_t K_t(i) - I_t(i) - B_t(i) + B_{t-1}(i) R_{bt-1}\right] \tag{60}$$

subject to

$$K_{t+1}(i) = (1-\delta)K_t(i) + \varepsilon_t(i) I_t(i) \tag{61}$$

$$I_t(i) \geq -\frac{\tilde{\rho}}{\varepsilon_t(i)} K_t(i) \tag{62}$$

$$I_t(i) \leq \theta K_t(i) + B_{t-1}(i) R_{bt-1}, \tag{63}$$

where equation (63) indicates that a firm's investment can be financed by both outside credit (limited by collateral $\theta K_t(i)$) and past savings.

Proposition 6 *Changing the financial structure by allowing an internal loan market does not change our results.*

Proof. See appendix VII. ■

6.2 Decreasing Returns and Non-Constrained Firms

In our model all firms that make positive investment are subject to financing constraints. This stems from the homogeneity of degree one of the production function, which implies that it is always optimal for a firm to expand its investment level. However, in reality financing constraints may apply only to some firms (for example, low-productivity firms or small businesses). In the models presented above, even a firm that has experienced several positive shocks to the productivity of its investment and has accumulated a sizeable capital stock is financially constrained. So a more realistic model with heterogenous firms should allow some firms to be unconstrained. The question is: Would the results be preserved in a framework in which some firms are not borrowing constrained?

The answer is yes. If we assumed a technology with decreasing returns (as in Thomas, JPE 2002), so that a firm's optimal capital stock is finite, then some firms can have a debt capacity larger than their optimal capital stock (i.e. the financing constraint is not binding). Even though the model is no longer analytically tractable because of the decreasing returns to scale technology, our results should continue to hold. The reason is that as long as a positive fraction of firms are financially constrained and such constraints are sometimes binding in equilibrium, the aggregate investment should appear to be more sluggish compared with that in a model without financial constraints, indicating increasing marginal costs or convex adjustment costs. Firm-level investment will remain lumpy because there are no adjustment costs at the firm level and the fraction of firms undertaking investment (positive or negative) is strictly positive. That is, borrowing constraints at the firm level will manifest as convex adjustment costs at the aggregate level regardless of the returns to scale, as long as some firms are borrowing constrained in equilibrium.

This point can also be illustrated using a tractable model with constant returns but with firm-level capital adjustment costs. Because of the adjustment costs, each firm has an optimal level of investment. So a firm will increase investment to its borrowing limit if it receives a good shock, but will keep investment at the optimal level if it receives a bad shock. Thus in the model there is always a positive fraction of firms operating at optimal investment level yet without being financially constrained. We can show that imposing financial constraints in this model leads to an aggregate CAC function that is more convex than the one originally assumed for firms. Hence, borrowing constraints can lead to (or enhance) convex adjustment costs at the aggregate level even if some firms are not financially constrained. The details of the analysis are provided in Appendix VIII (available only upon request).

7 Conclusion

This paper has addressed a long-standing inconsistency problem in investment theory: The assumption of convex adjustment costs in aggregate models and the assumption of non-convex adjustment costs in micro models. The former assumption is consistent with aggregate investment behavior but inconsistent with firm-level data. The latter assumption is consistent with micro evidence but not with aggregate data. Therefore, it is difficult to view either types of adjustment costs as a pure form of technology. This paper has shown that borrowing constraints based on limited contractual enforcement can rationalize CAC at the aggregate level and at the same time generate lumpy investment at the firm level.

The intuition is simple. In the typical CAC models, the marginal cost increases continuously as the investment increases. This assumption can rationalize the sluggishness of aggregate investment but is inconsistent with firm-level lumpy investment. In this paper, however, the marginal cost at the firm level is instead constant (at zero) until it reaches a borrowing limit, and then goes to infinity above this level. Thus, firm-level investment can be lumpy while aggregate investment can appear sluggish.

Our model can also explain the empirical puzzle of why Tobin's Q is not a sufficient statistic to explain firm-level investment in disaggregated data. The reason is that in our model, Tobin's Q is an aggregate statistic while firm-level investment depends crucially on firm-specific shocks which are not captured by the market value of Q. We have also shown that if convex adjustment costs are no longer assumed to be part of the aggregate technology but are derived instead from market frictions and interactions, then aggregate CAC are not necessarily policy invariant.

Appendix I. Proof of Proposition 1

Proof. Denoting $i_t \equiv \frac{I_t}{K_t}$ and taking derivative of the function $\varphi(\cdot)$ in equation (22) with respect to i_t gives

$$\varphi'(i_t) = [-\theta\varepsilon_t^*\phi(\varepsilon_t^*) - \tilde{\rho}\phi(\varepsilon_t^*)]\frac{\partial\varepsilon_t^*}{\partial i_t}, \tag{64}$$

where $\phi(\varepsilon)$ denotes the PDF of ε. Differentiating equation (19) with respect to $i_t \equiv \frac{I_t}{K_t}$, we have

$$\frac{\partial i_t}{\partial\varepsilon_t^*} = -\theta\phi(\varepsilon_t^*) - \tilde{\rho}\frac{1}{\varepsilon_t^*}\phi(\varepsilon_t^*). \tag{65}$$

The above two equations together imply

$$\varphi'(i_t) = \frac{\theta\varepsilon_t^*\phi(\varepsilon_t^*) + \tilde{\rho}\phi(\varepsilon_t^*)}{\theta\phi(\varepsilon_t^*) + \tilde{\rho}\frac{1}{\varepsilon_t^*}\phi(\varepsilon_t^*)} = \varepsilon_t^* > 0. \tag{66}$$

Differentiating this equation again with respect to i_t and using equation (65) gives

$$\varphi''(i_t) = \frac{\partial\varepsilon_t^*}{\partial i_t} = \frac{1}{-\theta\phi(\varepsilon_t^*) - \tilde{\rho}\frac{1}{\varepsilon_t^*}\phi(\varepsilon_t^*)} < 0. \tag{67}$$

Therefore, the function $\varphi(i_t)$ is increasing and strictly concave in i_t. Since $\varphi(i_t)$ depends only on the investment-to-capital ratio, it is homogeneous of degree zero in $\{I_t, K_t\}$. ■

Appendix II. Proof of Proposition 2

Proof. Denote $V[K_t(i), \varepsilon_t(i)]$ as the value function of firm i with capital stock $K_t(i)$. Based on the analysis of Hayashi (1982), we conjecture that a firm's value is linearly homogeneous in its capital stock because of constant returns to scale production technology:

$$V[K_t(i), \varepsilon_t(i)] = v[\varepsilon_t(i)]K_t(i) \equiv v_t(i)K_t(i). \tag{68}$$

We verify later that this conjecture is correct. Define $\bar{v}_t \equiv Ev_t(i) = \int v_t(\varepsilon)d\Phi(\varepsilon)$ as the average value of the firm across states and $i_t(i) \equiv \frac{I_t(i)}{K_t(i)}$ as the firm's investment rate. Firm i solves the following dynamic programming problem:

$$v_t(i)K_t(i) = \max_{K_{t+1}(i), I_t(i)} \left\{R_tK_t(i) - I_t(i) + \beta E_t\left[\frac{\Lambda_{t+1}}{\Lambda_t}v_{t+1}(i)K_{t+1}(i)\right]\right\} \tag{69}$$

subject to

$$K_{t+1}(i) = (1-\delta)K_t(i) + \varepsilon_t(i)I_t(i) \tag{70}$$

$$I_t(i) \geq -\frac{\tilde{\rho}}{\varepsilon_t(i)}K_t(i) \tag{71}$$

and the borrowing constraint in equation (34). To simplify the analysis, assume $\tilde{\rho} = 0$.

Denote $\{\lambda_t(i), \pi_t(i), \mu_t(i)\}$ as the Lagrangian multipliers of constraints (70), (71), and (34), respectively. The firm's first-order conditions for $\{I_t(i), K_{t+1}(i)\}$ are given, respectively, by $1 = \varepsilon_t(i)\lambda_t(i) + \pi_t(i) - \mu_t(i)$ and

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \bar{v}_{t+1}. \tag{72}$$

The envelope condition is given by $v_t(i) = R_t + (1-\delta)\,\lambda_t(i) + \theta Q_t(i)\mu_t(i)$. Substituting this expression into equation (72) gives

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\left\{R_{t+1} + (1-\delta)\lambda_{t+1}(i) + \theta Q_{t+1}(i)\mu_{t+1}(i)\right\}. \tag{73}$$

Hence, all first-order conditions are the same as those in the benchmark model except here $\theta = (1-\delta)\,Q_t(i)$. Therefore, following the same steps of analysis as in the benchmark model (Section 2.2) by considering different cases for the possible values of the Lagrangian multipliers, it can be easily shown that the Lagrangian multipliers are given by $\mu_t(i) = \max\{q_t(i) - 1, 0\}$, $\pi_t(i) = \max\{1 - q_t(i), 0\}$, where $q_t(i) = \frac{\varepsilon_t(i)}{\varepsilon_t^*}$; the firm's optimal decision rules for investment and capital accumulation are given by equations (35) and (36); and the firm's value function is given by equation (37). Clearly, since R_t and Q_t are independent of $K_t(i)$, equation (37) implies that the value of a firm is proportional to its capital stock: $V[\varepsilon_t(i), K_t(i)] = v_t(i)K_t(i)$. This confirms our initial conjecture. ■

Appendix III. Proof of Proposition 3

Proof. Denote $i_t \equiv \frac{I_t}{K_t}$, then

$$\frac{\partial \Psi}{\partial i_t} = (1-\delta)\frac{\partial Q}{\partial i_t}\int_{\varepsilon \geq \frac{1}{Q(i_t)}} \varepsilon d\Phi(\varepsilon) + (1-\delta)\,\phi(\varepsilon_t^*)Q_t^{-2}\frac{\partial Q}{\partial i_t}. \tag{74}$$

Since $Q_t = \frac{1}{\varepsilon_t^*}$, equation (38) implies

$$\frac{\partial i_t}{\partial Q_t} = (1-\delta)\left[1 - \Phi(\varepsilon_t^*)\right] + (1-\delta)\,\varepsilon_t^*\phi(\varepsilon_t^*). \tag{75}$$

The above two equations together imply

$$\Psi'(i_t) = \varepsilon_t^* \left[\frac{\int_{\varepsilon \geq \varepsilon^*} \frac{\varepsilon}{\varepsilon^*} d\Phi(\varepsilon) + \varepsilon_t^* \phi(\varepsilon_t^*)}{[1 - \Phi(\varepsilon_t^*)] + \varepsilon_t^* \phi(\varepsilon_t^*)} \right] > \varepsilon_t^* > 0, \tag{76}$$

where the inequality holds because $\int_{\varepsilon \geq \varepsilon^*} \frac{\varepsilon}{\varepsilon^*} d\Phi(\varepsilon) > 1 - \Phi(\varepsilon^*)$ and the support of ε is in the positive region of the real line.

Integrating by parts and rearranging, the first term in the numerator of $\Psi'(i)$ can be written as $\int_{\varepsilon \geq \varepsilon_t^*} \varepsilon d\Phi(\varepsilon) = \varepsilon_t^*[1 - \Phi(\varepsilon^*)] + \int_{\varepsilon \geq \varepsilon_t^*} [1 - \Phi(\varepsilon)]\, d\varepsilon$. Thus,

$$\begin{aligned} \Psi'(i_t) &= \frac{\varepsilon_t^*[1 - \Phi(\varepsilon^*)] + \varepsilon_t^{*2} \phi(\varepsilon_t^*) + \int_{\varepsilon \geq \varepsilon_t^*} [1 - \Phi(\varepsilon)]\, d\varepsilon}{[1 - \Phi(\varepsilon_t^*)] + \varepsilon_t^* \phi(\varepsilon_t^*)} = \varepsilon_t^* + \frac{\int_{\varepsilon \geq \varepsilon_t^*} [1 - \Phi(\varepsilon)]\, d\varepsilon}{[1 - \Phi(\varepsilon_t^*)] + \varepsilon_t^* \phi(\varepsilon_t^*)} \\ &\equiv f(\varepsilon^*). \end{aligned} \tag{77}$$

Notice that

$$\begin{aligned} f'(\varepsilon_t^*) &= 1 + \frac{-[1 - \Phi(\varepsilon_t^*)] \{\varepsilon_t^* \phi(\varepsilon_t^*) + [1 - \Phi(\varepsilon_t^*)]\} - \varepsilon_t^* \phi'(\varepsilon_t^*) \int_{\varepsilon \geq \varepsilon_t^*} [1 - \Phi(\varepsilon)]\, d\varepsilon}{\{\varepsilon_t^* \phi(\varepsilon_t^*) + [1 - \Phi(\varepsilon_t^*)]\}^2} \\ &= 1 - \frac{[1 - \Phi(\varepsilon_t^*)]}{\varepsilon_t^* \phi(\varepsilon_t^*) + [1 - \Phi(\varepsilon_t^*)]} - \frac{\varepsilon_t^* \phi'(\varepsilon_t^*) \int_{\varepsilon \geq \varepsilon_t^*} [1 - \Phi(\varepsilon)]\, d\varepsilon}{\{\varepsilon_t^* \phi(\varepsilon_t^*) + [1 - \Phi(\varepsilon_t^*)]\}^2} \\ &= \frac{\varepsilon_t^* \phi(\varepsilon_t^*)}{\varepsilon_t^* \phi(\varepsilon_t^*) + [1 - \Phi(\varepsilon_t^*)]} - \frac{\varepsilon_t^* \phi'(\varepsilon_t^*) \int_{\varepsilon \geq \varepsilon_t^*} [1 - \Phi(\varepsilon)]\, d\varepsilon}{\{\varepsilon_t^* \phi(\varepsilon_t^*) + [1 - \Phi(\varepsilon_t^*)]\}^2}. \end{aligned} \tag{78}$$

Clearly, as long as $\phi'(\varepsilon_t^*) \leq 0$, we have

$$f'(\varepsilon_t^*) \geq 0 \tag{79}$$

and

$$\Psi''(i_t) = f'(\varepsilon_t^*) \frac{\partial \varepsilon_t^*}{\partial i_t} \leq 0 \tag{80}$$

since $\frac{\partial \varepsilon_t^*}{\partial i_t} < 0$ by equation (38). Therefore, $\Psi(\cdot)$ is increasing and concave. In addition, it is clear that $\Psi(\cdot)$ depends only on the investment-to-capital ratio i_t, so it is homogeneous of degree zero in $\{I, K\}$. ■

Appendix IV. Proof of Proposition 4

Proof. Consider a representative firm solving the program in equation (28) subject to equation (45), taking $\bar{\imath}_t$ as given. Denoting Q_t as the Lagrangian multiplier for the constraint and imposing the equilibrium condition $\bar{\imath}_t = i_t$, the first-order conditions for I_t and K_{t+1} are given, respectively, by

$$\theta Q_t^2 \varphi'(i_t) = 1 \tag{81}$$

$$Q_t = \beta E_t \left\{ \frac{\Lambda_{t+1}}{\Lambda_t} R_{t+1} + (1-\delta) Q_{t+1} + \theta Q_{t+1}^2 [\varphi(i_t) - \varphi'(i_{t+1}) i_{t+1}] \right\}. \tag{82}$$

Since $\theta Q_{t+1}^2 \varphi'(i_{t+1}) = 1$, equation (82) can be written as

$$Q_t = \beta E_t \left\{ \frac{\Lambda_{t+1}}{\Lambda_t} R_{t+1} + (1-\delta) Q_{t+1} + \theta Q_{t+1}^2 \int_{\varepsilon \geq \frac{1}{Q(i_{t+1})}} \varepsilon d\Phi(\varepsilon) - \frac{I_{t+1}}{K_{t+1}} \right\}, \tag{83}$$

which is identical to equation (39) in the microfounded model. ■

Appendix V. Proof of Proposition 5

Proof. The first part of the proposition—the optimal capital tax rate in the representative-agent CAC model without externalities is zero—is a standard result in the literature. Hence, we need only to prove the second part of the proposition. We add a representative household into the model so that the government's objective function is well defined. We prove the proposition in an environment without aggregate uncertainty. The household's problem is to choose consumption (C_t) and labor supply (N_t) in each period to solve

$$\max \sum_{t=0}^{\infty} \beta^t [u(C_t) - v(N_t)] \tag{84}$$

subject to $C_t \leq w_t N_t + \Pi_t + T_t$, where Π_t denotes aggregate dividends distributed from firms and $T_t = \int \tau_t [Y_t(i) - w_t N_t(i)] di$ is a lump sum transfer from the government based on capital tax revenues collected from all firms, where τ_t is the tax rate for capital income. The first-order conditions of the household can be summarized by

$$u'(C_t) w_t = v'(N_t). \tag{85}$$

On the firm side, we can show that, regardless of capital tax, the endogenous credit limit model is always equivalent to a representative-firm model with investment externality. Hence, based on the equivalence, we need only to prove that the optimal capital tax rate is negative in the representative-firm model with investment externality. For simplicity, we consider the Pareto distribution for firms' idiosyncratic shocks $\varepsilon_t(i)$ (in the microfounded model) and the Cobb-Douglas production function, $Y_t = AK_t^{\alpha}N_t^{1-\alpha}$. Thus, the equivalent CAC function is of the Cobb-Douglas form and a representative firm in the investment externality model must solve

$$\max \sum_{\tau=0}^{\infty} \beta^{\tau} \frac{\Lambda_{t+\tau}}{\Lambda_t} \{(1-\tau_t)(Y_t - w_t N_t) - I_t\} \tag{86}$$

subject to

$$K_{t+1} = (1-\delta)K_t + \varphi_0 \left(\frac{\bar{I}_t}{\bar{K}_t}\right)^a K_t^{1-b} I_t^b, \tag{87}$$

where $a = \frac{1}{\eta(\eta+1)}$, $b = \frac{\eta-1}{\eta}$, and $\frac{\bar{I}_t}{\bar{K}_t}$ denotes the average investment rate in the economy that each firm takes as given. The first-order conditions for $\{I_t, K_{t+1}\}$ in this model are given by

$$Q_t \varphi_0 b \left(\frac{\bar{I}_t}{\bar{K}_t}\right)^a K_t^{1-b} I_t^{b-1} = 1 \tag{88}$$

$$Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{(1-\tau_{t+1})R_{t+1} + (1-\delta)Q_{t+1} + Q_{t+1}\varphi_0(1-b)\left(\frac{\bar{I}_t}{\bar{K}_t}\right)^a K_{t+1}^{-b} I_{t+1}^b\right\}, \tag{89}$$

where $R_t = \alpha \frac{Y_t}{K_t}$ and $w_t = \frac{(1-\alpha)Y_t}{N_t}$. Imposing the equilibrium conditions, $\frac{\bar{I}_t}{\bar{K}_t} = \frac{I_t}{K_t}$ and $\Lambda_t = u'(C_t)$, and plugging in the values of $\{\varphi_0, a, b\}$, and substituting out $\{R_t, w_t, Q_t\}$, the above two first-order conditions become

$$\frac{I_t}{K_t} = \theta Q_t^{1+\eta} \tag{90}$$

$$\theta^{-\frac{1}{\eta+1}} \left(\frac{I_t}{K_t}\right)^{\frac{1}{\eta+1}} = \beta \frac{u'(C_{t+1})}{u'(C_t)} \left[(1-\tau_{t+1})\frac{\alpha Y_{t+1}}{K_{t+1}} + (1-\delta)\theta^{-\frac{1}{\eta+1}} \left(\frac{I_{t+1}}{K_{t+1}}\right)^{\frac{1}{\eta+1}} + \frac{1}{\eta-1}\frac{I_{t+1}}{K_{t+1}}\right] \tag{91}$$

The law of motion for capital accumulation becomes

$$K_{t+1} = (1-\delta)K_t + \frac{\eta \theta^{\frac{1}{\eta+1}}}{\eta-1} K_t^{\frac{1}{\eta+1}} I_t^{\frac{\eta}{\eta+1}}, \tag{92}$$

and the household resource constraint becomes

$$I_t + C_t = Y_t = AK_t^{\alpha} N_t^{1-\alpha}. \tag{93}$$

Notice that equations (85), (91), (92), (93), and the aggregate production function can uniquely solve the competitive equilibrium path of $\{C_t, I_t, Y_t, N_t, K_{t+1}\}$ as a function of the tax rate τ_t in the externality model. The optimal tax policy is to design a sequence of tax rates $\{\tau_t\}_{t=0}^{\infty}$ to solve

$$V(K_0) = \max_{\{\tau_t\}} \sum_{t=0}^{\infty} \beta^t [u(C(\tau_t) - v(N(\tau_t)] \tag{94}$$

subject to equations 85, (91), (92), (93), and the aggregate production function.

Instead of directly solving program (94), we first study the "first best allocation" in the externality model, which pertains to the highest possible utility that a social planner can achieve in the model when the investment externality is fully endogenized. Hence, the first best allocation also pertains to the highest possible utility that the government can achieve using tax policies in program (94).

The first best allocation solves

$$V^*(K_0) = \max_{\{C_t, N_t, I_t, K_{t+1}\}} \sum_{t=0}^{\infty} \beta^t [u(C_t) - v(N_t)] \tag{95}$$

subject to

$$K_{t+1} = (1-\delta)K_t + \frac{\eta\theta^{\frac{1}{\eta+1}}}{\eta - 1} K_t^{\frac{1}{\eta+1}} I_t^{\frac{\eta}{\eta+1}} \tag{96}$$

$$C_t + I_t = AK_t^{\alpha} N_t^{1-\alpha} \tag{97}$$

It is obvious that the lifetime utility defined in program (95) is at least as large as that defined in program (94): $V^*(K_0) \geq V(K_0)$, because the former gives the first best allocation. The first-order conditions for $\{I_t, C_t, K_{t+1}\}$ in program (95) are given, respectively, by

$$Q_t \frac{\eta}{\eta+1} \frac{\eta\theta^{\frac{1}{\eta+1}}}{\eta-1} K_t^{\frac{1}{\eta+1}} I_t^{\frac{\eta}{\eta+1}-1} = 1 \tag{98}$$

$$u'(C_t) \frac{(1-\alpha)Y_t}{N_t} = v'(N_t) \tag{99}$$

$$Q_t = \beta \frac{u'(C_{t+1})}{u'(C_t)} \left[\frac{\alpha Y_{t+1}}{K_{t+1}} + (1-\delta) Q_{t+1} + Q_{t+1} \frac{1}{\eta+1} \frac{\eta \theta^{\frac{1}{\eta+1}}}{\eta - 1} K_t^{\frac{1}{\eta+1}-1} I_t^{\frac{\eta}{\eta+1}} \right]. \tag{100}$$

Equation (98) implies $Q_t = \frac{\eta^2-1}{\eta^2}\theta^{-\frac{1}{\eta+1}} \left(\frac{I_t}{K_t}\right)^{\frac{1}{\eta+1}}$. Using this relationship to substitute out Q, equation (100) becomes

$$\frac{\eta^2-1}{\eta^2}\theta^{-\frac{1}{\eta+1}} \left(\frac{I_t}{K_t}\right)^{\frac{1}{\eta+1}} = \beta \frac{u'(C_{t+1})}{u'(C_t)} \left[\frac{\alpha Y_{t+1}}{K_{t+1}} + (1-\delta) \frac{\eta^2-1}{\eta^2}\theta^{-\frac{1}{\eta+1}} \left(\frac{I_{t+1}}{K_{t+1}}\right)^{\frac{1}{\eta+1}} + \frac{1}{\eta}\frac{I_{t+1}}{K_{t+1}} \right]. \tag{101}$$

Notice that equations (99), (101), (96), (97), and the aggregate production function together uniquely solve for the first best allocation $\{C_t, I_t, Y_t, N_t, K_{t+1}\}$ under program (95). Similarly, equations (85), (91), (92), (93), and the aggregate production function together uniquely solve for the equilibrium path of $\{C(\tau_t), I(\tau_t), Y(\tau_t), N(\tau_t), K(\tau_t)\}$ in a competitive equilibrium with investment externalities. Comparing these two systems of equations, except that equation (101) is different from equation (91), all other equilibrium conditions in the first best allocation are identical to those in a competitive equilibrium in terms of mathematical relationship. In particular, equations (99), (96), and (97) are identical to equations (85), (92), and (93), respectively.

Denote the equilibrium path of the first best allocation as $\{C_t^*, I_t^*, Y_t^*, N_t^*, K_{t+1}^*\}$. By comparing equation (101) under program (95) with equation (91) in the competitive equilibrium, it is obvious that the government can achieve the first best allocation in program (94) by setting the tax rate such that equation (101) and equation (91) are identical, which implies

$$\begin{aligned} & \frac{\eta^2}{\eta^2-1}\frac{\alpha Y_{t+1}^*}{K_{t+1}^*} + (1-\delta)\theta^{-\frac{1}{\eta+1}} \left(\frac{I_{t+1}^*}{K_{t+1}^*}\right)^{\frac{1}{\eta+1}} + \frac{\eta}{\eta^2-1}\frac{I_{t+1}^*}{K_{t+1}^*} \\ = \; & (1-\tau_{t+1})\frac{\alpha Y_{t+1}^*}{K_{t+1}^*} + (1-\delta)\theta^{-\frac{1}{\eta+1}} \left(\frac{I_{t+1}^*}{K_{t+1}^*}\right)^{\frac{1}{\eta+1}} + \frac{1}{\eta-1}\frac{I_{t+1}^*}{K_{t+1}^*}. \end{aligned} \tag{102}$$

Simplification gives

$$\left(\frac{1}{\eta^2-1} + \tau_{t+1}\right)\alpha Y_{t+1}^* = \frac{1}{\eta^2-1} I_{t+1}^*. \tag{103}$$

Since $Q_t^* = \frac{\eta^2-1}{\eta^2}\theta^{-\frac{1}{\eta+1}} \left(\frac{I_t^*}{K_t^*}\right)^{\frac{1}{\eta+1}}$, we have $Q_{t+1}^* \frac{\eta\theta^{\frac{1}{\eta+1}}}{\eta-1} K_{t+1}^{\frac{1}{\eta+1}} I_{t+1}^{\frac{\eta}{\eta+1}} = \frac{\eta+1}{\eta} I_{t+1}^*$. So we can rewrite equation (101) by multiplying both sides by K^*_{t+1} as

$$Q^*_t K^*_{t+1} = \beta \frac{u'(C^*_{t+1})}{u'(C^*_t)} \left[\alpha Y^*_{t+1} - I^*_{t+1} + Q^*_{t+1} K^*_{t+2} \right]. \tag{104}$$

This equation implies that in the steady state we must have $\alpha Y^* > I^*$. Then by equation (103), we must have $\tau < 0$ in the steady state to achieve the first best allocation. ■

Appendix VI. Model Simulation

1. Simulating aggregate variables. We solve the equilibrium path of the aggregate variables by log-linear approximation around the deterministic steady state. The log-linearized variable is defined as

$$\hat{x}_t \equiv \log(X_t) - \log \bar{X}, \tag{105}$$

where $\bar{X}$ indicates the steady-state value. We simulate the aggregate model for $t = 200,000$ periods using the law of motion of aggregate technology in equation (58). Based on the simulated variables, we can use the following transformation to obtain the value of aggregate variables:

$$X_t = \bar{X} \exp(\hat{x}_t). \tag{106}$$

In this way, we obtain the sequences of capital K_t, aggregate investment I_t, Tobin's Q_t, and the cutoff $\varepsilon^*_t = \frac{1}{Q_t}$.

2. Generating firm data. In order to generate firm data, we need to simulate the idiosyncratic shocks, $\varepsilon_t(i)$. A random sample with $200,000$ observations for $\varepsilon(i)$ in each time period t can be generated using inverse transform sampling. Given a random variable U drawn from the uniform distribution on the unit interval $(0, 1)$, the variable

$$\varepsilon = \frac{1}{U^{\frac{1}{\eta}}} \tag{107}$$

is Pareto distributed with the distribution function

$$F(\varepsilon) = 1 - \varepsilon^{-\eta}. \tag{108}$$

Given the sequences of aggregate variables (especially the cutoff ε^*_t), we obtain firm-level investment based on the firm's decision rule,

$$I_t(i) = \begin{cases} Q_t \theta K_t(i) & \text{if } \varepsilon_t(i) \geq \varepsilon^*_t \\ 0 & \text{if } \varepsilon_t(i) < \varepsilon^*_t \end{cases}. \tag{109}$$

We normalize each firm's initial capital stock to the aggregate steady-state capital $\bar{K}$; namely, $K_0(i) = \bar{K}$. We construct the firm-level capital sequence by the law of motion:

$$K_{t+1}(i) = (1-\delta)K_t(i) + \varepsilon_t(i)I_t(i). \tag{110}$$

In each period $t = 0, 1, ..., 200,000$, we track each firm i's capital stock and positive investment level whenever $\varepsilon_t(i) \geq \varepsilon_t^*$.

3. Regression analysis. We run two regressions. The first is based on aggregate time series[12]:

$$\frac{K_{t+1} - (1-\delta)K_t}{K_t} = \beta_0 + \beta_1 Q_t. \tag{111}$$

The second is based on firm-level data:

$$\frac{K_{t+1}(i) - (1-\delta)K_t(i)}{K_t(i)} = \beta_0 + \beta_1 Q_t. \tag{112}$$

The adjusted R^2 is almost the same if we use log variables for the aggregate model. For the firm-level data, since $\frac{K_{t+1}(i)-(1-\delta)K_t(i)}{K_t(i)}$ can be zero in some periods, we cannot use log values in the regression.

Appendix VII. Proof of Proposition 6

Proof. Denoting $\{\lambda_t(i), \pi_t(i), \mu_t(i)\}$ as the Lagrangian multipliers of constraints (61), (62), and (63), respectively, the firm's first order conditions for $\{I_t(i), K_{t+1}(i), B_t(i)\}$ are given, respectively, by

$$1 = \varepsilon_t(i)\lambda_t(i) + \pi_t(i) - \mu_t(i), \tag{113}$$

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ R_{t+1} + (1-\delta)\lambda_{t+1}(i) + \theta\mu_{t+1}(i) + \tilde{\rho}\frac{\pi_{t+1}(i)}{\varepsilon_{t+1}(i)} \right\}, \tag{114}$$

$$1 = \beta R_{bt} E_t[1 + \mu_{t+1}(i)], \tag{115}$$

plus the following complementarity slackness conditions:

$$\pi_t(i)\left[I_t(i) + \rho\frac{K_t(i)}{\varepsilon_t(i)}\right] = 0 \tag{116}$$

$$\mu_t(i)[\theta_t K_t(i) + B_{t-1}(i)R_{bt-1} - I_t(i)] = 0. \tag{117}$$

[12] Tobin's Q is a sufficient statistic to determine aggregate investment in both our model and the CAC model.

Following the same analysis and solution method as in the previous sections, we have the following decision rules and equilibrium conditions for each firm:

$$I_t(i) = \begin{cases} \theta K_t(i) + B_{t-1}(i)R_{bt-1} & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ -\tilde{\rho}\frac{K_t(i)}{\varepsilon_t(i)} & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases} \tag{118}$$

$$\frac{1}{\varepsilon_t^*} = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\left\{R_{t+1} + \frac{(1-\delta)}{\varepsilon_{t+1}^*} + O(\varepsilon_{t+1}^*)\right\} \tag{119}$$

$$1 = \beta R_{bt} E_t \frac{\Lambda_{t+1}}{\Lambda_t}\left[1 + \int_{\varepsilon \geq \varepsilon_{t+1}^*} \frac{\varepsilon}{\varepsilon_{t+1}^*} d\Phi(\varepsilon)\right], \tag{120}$$

where the implicit function

$$\begin{aligned} O(\varepsilon_{t+1}^*) &\equiv E_t\left[\theta\mu_{t+1}(i) + \tilde{\rho}\frac{\pi_{t+1}(i)}{\varepsilon_{t+1}(i)}\right] \qquad (121) \\ &= \theta \int_{\varepsilon_{t+1}(i) \geq \varepsilon_{t+1}^*} \frac{\varepsilon_{t+1}(i) - \varepsilon_{t+1}^*}{\varepsilon_{t+1}^*} d\Phi(\varepsilon) + \tilde{\rho}\int_{\varepsilon_{t+1}(i) < \varepsilon_{t+1}^*}\left(\frac{1}{\varepsilon_{t+1}(i)} - \frac{1}{\varepsilon_{t+1}^*}\right) d\Phi(\varepsilon). \end{aligned}$$

Market clearing for the internal loan market implies $\int B_{t-1}(i)di = 0$ for all t. Notice that as long as $\varepsilon_{t+1}^* < \varepsilon_{\max}$, equation (120) implies that $\beta E_t R_{bt}\frac{\Lambda_{t+1}}{\Lambda_t} < 1$, so a representative household (firm owner) will not want to hold the one-period bond issued by firms in the internal loan market. So the one-period bonds will only be traded among firms. The aggregate investment is given by

$$I_t = \theta K_t[1 - \Phi(\varepsilon_t^*)] - \tilde{\rho}K_t \int_{\varepsilon < \varepsilon_t^*} \frac{1}{\varepsilon} d\Phi(\varepsilon), \tag{122}$$

which is identical to equation (19). Hence, the aggregate capital stock evolves according to equation (20). So changing the financial structure by allowing internal financing does not change our results. ■

Appendix VIII. Possibly Non-Binding Borrowing Constraints (Not For Publication)

In this appendix, we show that having some firms operate at the optimal investment level without being financially constrained does not change our results. To allow for some firms to be unconstrained when investing positively at the optimal level, we consider capital adjustment costs at the firm level with the investment technology:

$$K_{t+1}(i) = (1-\delta)K_t(i) + \varepsilon_t(i)K_t^{\gamma}(i)I_t^{1-\gamma}(i), \tag{123}$$

where $0 < \gamma < 1$ indicates convex adjustment costs and $\varepsilon_t(i)$ is i.i.d with Pareto distribution. The financial constraint takes the following form:

$$I_t(i) \leq \theta K_t(i). \tag{124}$$

The analysis is conducted in two steps. First, we show that if $\theta = \infty$, namely there are no financial constraints, the aggregate CAC function is identical to equation (123). Second, we show that if $\theta < \infty$, then the implied aggregate CAC function is more convex than the CAC function in equation (123).

Consider $\theta = \infty$ first. The value function of a firm is given by

$$V_t(\varepsilon_t(i), K_t(i)) = \max\{R_t K_t(i) - I_t(i)\} + \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} V_{t+1}(\varepsilon_{t+1}(i), K_{t+1}(i)) \tag{125}$$

subject to the technological constraint

$$K_{t+1}(i) = (1-\delta)K_t(i) + \varepsilon_t(i)K_t^{\gamma}(i)I_t^{1-\gamma}(i). \tag{126}$$

Following Hayashi (1982) and the solution method in the main text of this paper, assuming $V_t(\varepsilon_t(i), K_t(i)) = v_t(\varepsilon_t(i))K_t(i)$ and defining $Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \int v_{t+1}(\varepsilon)\phi(\varepsilon)d\varepsilon$, the first-order condition with respect to investment is

$$1 = (1-\gamma)Q_t \varepsilon_t(i)K_t^{\gamma}(i)I_t^{-\gamma}(i), \tag{127}$$

or

$$I_t(i) = \left[(1-\gamma)Q_t \varepsilon_t(i)\right]^{\frac{1}{\gamma}} K_t(i). \tag{128}$$

Hence the aggregate investment in the economy is

$$I_t = \left[(1-\gamma)Q_t\right]^{\frac{1}{\gamma}} K_t \int \varepsilon^{\frac{1}{\gamma}}\phi(\varepsilon)d\varepsilon, \tag{129}$$

and the aggregate capital stock follows the law of motion:

$$\begin{aligned} K_{t+1} &= (1-\delta)K_t + \int \varepsilon_t(i)K_t(i)\left[(1-\gamma)Q_t\varepsilon_t(i)\right]^{\frac{1-\gamma}{\gamma}} \\ &= (1-\delta)K_t + K_t\left[(1-\gamma)Q_t\right]^{\frac{1-\gamma}{\gamma}} \int \varepsilon^{\frac{1}{\gamma}}\phi(\varepsilon)d\varepsilon. \end{aligned} \tag{130}$$

Substituting out Q_t with equation (129), the above equation becomes

$$K_{t+1} = (1-\delta)K_t + K_t \left[\frac{I_t}{K_t \int \varepsilon^{\frac{1}{\gamma}} \phi(\varepsilon) d\varepsilon} \right]^{1-\gamma} \int \varepsilon^{\frac{1}{\gamma}} \phi(\varepsilon) d\varepsilon. \tag{131}$$

Rearranging gives

$$K_{t+1} = (1-\delta)K_t + \bar{\varepsilon} K_t^{\gamma} I_t^{1-\gamma}, \tag{132}$$

where the parameter $\bar{\varepsilon} \equiv \left(\int \varepsilon^{\frac{1}{\gamma}} \phi(\varepsilon) d\varepsilon \right)^{\gamma}$. So the aggregate investment technology in the above equation exhibits a CAC function that is identical to the CAC function in equation (123) at the firm level. Equation (129) indicates that the elasticity of investment with respect to Tobin Q is $\frac{1}{\gamma}$:

$$\frac{\partial I_t}{\partial Q_t} \frac{Q_t}{I_t} = \frac{1}{\gamma}. \tag{133}$$

Now consider the case with $\theta < \infty$. The firm's value function satisfies

$$V_t(\varepsilon_t(i), K_t(i)) = \max \{ R_t K_t(i) - I_t(i) \} + \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} V_{t+1}(\varepsilon_{t+1}(i), K_{t+1}(i)) \tag{134}$$

with the constraint

$$K_{t+1}(i) = (1-\delta)K_t(i) + \varepsilon_t(i) K_t^{\gamma}(i) I_t^{1-\gamma}(i) \tag{135}$$

and

$$I_t(i) \leq \theta K_t(i). \tag{136}$$

As before, assuming $V_t(\varepsilon_t(i), K_t(i)) = v_t(\varepsilon_t(i)) K_t(i)$ and defining $Q_t = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \int v_{t+1}(\varepsilon) \phi(\varepsilon) d\varepsilon$, optimal investment can be determined by solving

$$\max \left\{ -I_t(i) + Q_t[(1-\delta)K_t(i) + \varepsilon_t(i) K_t^{\gamma}(i) I_t^{1-\gamma}(i)] \right\} \tag{137}$$

subject to

$$I_t(i) \leq \theta K_t(i). \tag{138}$$

If $I_t(i) < \theta K_t(i)$, the first-order condition for investment is still given by

$$1 = (1-\gamma) Q_t \varepsilon_t(i) K_t^{\gamma}(i) I_t^{-\gamma}(i). \tag{139}$$

This defines a cutoff

$$\frac{\theta^{\gamma}}{(1-\gamma)Q_t} = \varepsilon_t^*, \tag{140}$$

so that equation (139) holds if $\varepsilon_t(i) < \varepsilon_t^*(Q)$, and $I_t(i) = \theta K_t(i)$ if $\varepsilon_t(i) \geq \varepsilon_t^*(Q)$. Hence, the aggregate investment is given by

$$I_t = \theta K_t \int_{\varepsilon_t^*(Q)}^{\infty} d\Phi(\varepsilon) + K_t \int_{\varepsilon_{\min}}^{\varepsilon_t^*(Q)} [(1-\gamma)Q_t\varepsilon]^{\frac{1}{\gamma}}\, d\Phi(\varepsilon), \tag{141}$$

which implies

$$\frac{\partial I_t}{\partial Q_t}\frac{Q_t}{I_t} = \frac{1}{\gamma}\frac{\int_{\varepsilon_{\min}}^{\varepsilon_t^*} [(1-\gamma)Q_t\varepsilon]^{\frac{1}{\gamma}}\, d\Phi(\varepsilon)}{\theta \int_{\varepsilon_t^*}^{\varepsilon_{\max}} d\Phi(\varepsilon) + \int_{\varepsilon_{\min}}^{\varepsilon_t^*} [(1-\gamma)Q_t\varepsilon]^{\frac{1}{\gamma}}\, d\Phi(\varepsilon)}. \tag{142}$$

So as long as $\varepsilon_t^* < \varepsilon_{\max}$, namely, as long as some firms are financially constrained, we have

$$\frac{\partial I_t}{\partial Q_t}\frac{Q_t}{I_1} < \frac{1}{\gamma}, \tag{143}$$

regardless of the distribution of $\varepsilon_t(i)$. Hence, aggregate investment is less responsive to Tobin's Q because financial constraints imply greater convex adjustment cost at the aggregate level. However, when $\theta < \infty$, there does not exist an analytical expression for CAC function analogous to equation (132). But given that $\frac{\partial I_t}{\partial Q_t}\frac{Q_t}{I_1} < \frac{1}{\gamma}$, we can infer that the aggregate CAC is more convex than that in equation (132).

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