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Random Matching and Money in the Neoclassical Growth Model: Some Analytical Results*

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Abstract

I use the monetary version of the neoclassical growth model developed by Aruoba, Waller and Wright (2008) to study the properties of the model when there is exogenous growth. I first consider the planner's problem, then the equilibrium outcome in a monetary economy. I do so by first using proportional bargaining to determine the terms of trade and then consider competitive price taking. I obtain closed form solutions for the balanced growth path of all variables in all cases. I then derive closed form solutions for the transition paths under the assumption of full depreciation and, in the monetary economy, a non-stationary interest rate policy.

JEL Codes: E20, E40,O42
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1 Introduction

The effect of inflation on economic growth is a classic issue in monetary economics. Early contributions by Tobin (1965) and Sidrauski (1967a,1967b) gave us insights as to how inflation could deter (or stimulate) economic growth. The RBC literature revived the neoclassical growth model and made it the workhorse of modern macroeconomics. This gave rise to a renewed interest in studying the effects of inflation on growth with notable work being done by Cooley and Hansen (1989), Gomme (1993) and Ireland (1994). In all of these models, money is 'forced' into the neoclassical growth model, via the assumption of cash-in-advance. Thus, while the real side of these models has well-understood microfoundations, the monetary side does not.

During this same time period, tremendous progress was made understanding the microfoundations of money. Starting with the seminal work of Kiyotaki and Wright (1989, 1993), search theoretic models of money provided deep insights on the role of money as a medium of exchange. These models aided us in understanding how the value of money is affected by information frictions, matching frictions and pricing protocols such as bargaining – features that are absent from the standard neoclassical growth model. As a result, substantial work has been done trying to integrate modern monetary theory with mainstream macroeconomics so we would have a better understanding of how inflation affects capital accumulation and growth. Research along these lines has been done by Shi (1999), Aruoba and Wright (2003), Menner (2006) Aruoba, Waller and Wright (2008), Aruoba and Chugh (2008) and Berentsen, Rojas-Breu and Shi (2009).

My objective here is to contribute to this growing literature. I do so by providing analytical results on steady-state growth and transitional dynamics in the Aruoba, Waller and Wright (AWW) model of money and capital. Whereas AWW focuses mainly on the quantitative aspects of inflation on capital accumulation and growth, in this paper I focus on analytical properties of the model.

The AWW framework combines a monetary search sector with the neoclassical growth model. However, the AWW paper does not have growth nor does it address the conditions needed for balanced growth. Thus, in this paper, I add exogenous labor enhancing technological change to the AWW model and determine the necessary conditions for balanced growth. I then obtain closed form analytical solutions for the steady state capital to labor ratio for: 1) the planner allocation, 2) the monetary equilibrium with proportional bargaining and 3) the monetary equilibrium with price taking. I then study the transition dynamics of the model under the assumption of full depreciation of capital. For the planner allocation, the saving rate is constant, the capital-labor ratio converges monotonically to its steady state value and hours are constant along the transition path. For the monetary economy, given a constant interest rate policy, this is not the case – hours vary along the transition path which makes the saving rate vary as well. I then consider a particular non-stationary policy for the nominal interest rate. Under this policy, the nominal interest adjusts to the growth rate of real wages – if wage growth is excessively high, the nominal interest is below its steady state value. This policy keeps hours worked constant and equal to its steady-state value along the transition path. This is consistent with the planner's desired behavior for hours along the transition path. With this policy, I am able to obtain closed form solutions for the transition paths under both pricing mechanisms. These solutions involve a constant saving rate, constant hours along the transition path and monotone convergence of the capital-labor ratio to its steady state value.

2 Environment

The environment is essentially that of AWW. A $[0, 1]$ continuum of agents live forever in discrete time. Following LW, trade occurs in two separate subperiods. In the first subperiod trade occurs in a decentralized fashion, DM for short, while in subperiod 2, trade occurs in a perfect competitive centralized market, denoted the CM. In the DM, there is a double coincidence problem and private trading histories are private information, i.e., agents are anonymous.

As in AWW, there are two assets available to households, capital and money. Capital is assumed to be non-portable in the DM so buyers must search for sellers. So capital cannot be used as a medium of exchange and claims to such capital can be costlessly counterfeited just as IOU's can be counterfeited. Thus, money has a role even when capital is a storable factor of production.

In the CM there is a general good produced using labor H and capital K that can be used for consumption or investment. Production occurs according to the aggregate production function where $Y_t = F(K_t, Z_t H_t)$ where F is the technology and Z_t is a labor/effort augmenting technology factor that evolves according to the process $Z_t = (1 + \mu) Z_{t-1}$. We also have $Y_t/Z_t = F(K_t/Z_t, H_t)$. Capital is assumed to depreciate at rate $0 \leq \delta \leq 1$.

In the DM, each period with probability σ an agent can consume but not produce, while with the symmetric probablity he can produce but not consume. With probability $1 - 2\sigma$ he is a *nontrader* – he neither produces nor consumes and gets a utility payoff of zero. Due to symmetry in the measure of buyers and sellers, I assume that there is a matching technology that randomly assigns one buyer to one seller. Sellers in the DM can produce output q_t using their own effort e and capital k using a the CRTS technology $f(k_t, Z_t e_t)$. Sellers produce where their capital is located so they have access to their capital, even though buyers do not. We then have $q_t/Z_t = f(k_t/Z_t, e_t)$.

Instantaneous utility for everyone in the CM is $U(x) - Ah$, where x is consumption and h labor. Preferences are separable in consumption and leisure. In the DM, with probability σ you are a buyer and enjoy utility $u(q)$, and with probability σ you are a seller and get disutility $\ell(e)$, where q is consumption and e labor. The utility functions u and U have the usual monotonicity and curvature properties and $u(0) = 0$. Solving $q_t/Z_t = f(k_t Z_t, e_t)$ for $e_t = f^{-1}(q_t/Z_t, k_t/Z_t)$, we get the utility cost of producing q given k – $\ell(e) = \ell[f^{-1}(q_t/Z_t, k_t/Z_t)] \equiv c(q_t/Z_t, k_t/Z_t)$. Monotonicity and convexity imply this latter function has the properties $c_q, c_{qq} > 0$, $c_k < 0$ and $c_{kk} > 0$, and $c_{qk} < 0$ since $f_k f_{ee} < f_e f_{ek}$ holds when k is a normal input. Agents discount across periods at rate $\beta = (1+\rho)^{-1}$ where ρ is the time rate of discount.

The money stock is given by M_t and evolves according to the process $M_t = \gamma M_{t-1}$. Agents receive a lump-sum transfer of cash, τM, in the CM. In an earlier version of the paper, I included exogenously determined government spending and taxes; they are excluded here to minimize clutter and focus on how trading frictions and bargaining affect the steady-state allocation and dynamics. For notational simplicity, period $t+1$ is denoted $+1$, and so. Agents discount between the CM and DM at rate β but not between the DM and CM.

3 Planner Allocation

Consider the planner's problem in this economy where agents are treated symmetrically and the planner can dictate quantities traded. The planner's problem is

$$\begin{aligned} J(K) &= \max_{q,X,H,K_{+1}} \left[\sigma u(q) - \sigma c\left(\frac{q}{Z}, \frac{K}{Z}\right) + U(X) - AH + \beta J(K_{+1}) \right] \\ s.t.\ X &= F(K, ZH) + (1-\delta)K - K_{+1} \end{aligned} \tag{1}$$

Eliminating X and differentiating, the first order conditions are

$$\begin{aligned} q: \quad & u'(q) = c_q\left(\tfrac{q}{Z}, \tfrac{K}{Z}\right)\tfrac{1}{Z} \\ H: \quad & A = U'(X) F_H(K, ZH) Z \\ K_{+1}: \quad & U'(X) = \beta J'(K_{+1}) \end{aligned} \tag{2}$$

Also, using $J'(K) = U'(X)[F_K(K, ZH) + 1 - \delta] - \sigma c_k\left(\frac{q}{Z}, \frac{k}{Z}\right)\frac{1}{Z}$, we have

$$U'(X) = \beta U'(X_{+1})[F_K(K_{+1}, Z_{+1}H_{+1}) + 1 - \delta] - \beta\sigma c_k\left(\frac{q_{+1}}{Z_{+1}}, \frac{K_{+1}}{Z_{+1}}\right)\frac{1}{Z_{+1}}. \tag{3}$$

So the equilibrium allocation solves

$$\begin{aligned}
u'(q) &= c_q\left(\frac{q}{Z},\frac{K}{Z}\right)\frac{1}{Z} && (4)\\
A &= U'(X)F_H(K/Z,H)Z && (5)\\
U'(X) &= \beta U'(X_{+1})[F_K(K_{+1}/Z_{+1},H_{+1})+1-\delta]-\beta\sigma c_k\left(\frac{q_{+1}}{Z_{+1}},\frac{K_{+1}}{Z_{+1}}\right)\frac{1}{Z_{+1}} && (6)\\
X &= ZF(K/Z,H)+(1-\delta)K-K_{+1} && (7)
\end{aligned}$$

Two comments are in order. First, if $\sigma=0$, then $q=0$ and the model collapses to the standard neoclassical growth model. Second, if capital is not productive in the DM, then the model dichotomizes as in Aruoba and Wright (2003) – the steady evolution of K,X,H and Y can be determined independently using (5)-(7) while (4) detemines q/Z.

Consider the following functional forms:

$$\begin{aligned}
F(K,ZH) &= K^{\alpha}(ZH)^{1-\alpha} \qquad 0<\alpha<1\\
U(X) &= B\frac{X^{1-\varepsilon}-1}{1-\varepsilon} \qquad \varepsilon>0 \ \text{ or } U(X)=B\ln X \ \text{ for } \varepsilon=1\\
u(q) &= \frac{(q+b)^{1-\eta}-b^{1-\eta}}{1-\eta} \qquad 0<\eta<1 \ \text{ or } u(q)=\ln\left(\frac{q+b}{b}\right) \ \text{ for } \eta=1\\
c\left(\frac{q}{Z},\frac{k}{Z}\right) &= \left(\frac{q}{Z}\right)^{\psi}\left(\frac{k}{Z}\right)^{1-\psi} \qquad \psi\geq 1\\
&\Rightarrow \ c_q\left(\frac{q}{Z},\frac{k}{Z}\right)\frac{1}{Z}=\frac{\psi}{Z}\left(\frac{q}{k}\right)^{\psi-1}\\
&\Rightarrow \ c_k\left(\frac{q}{Z},\frac{k}{Z}\right)\frac{1}{Z}=-\frac{(\psi-1)}{Z}\left(\frac{q}{k}\right)^{\psi}
\end{aligned}$$

Note that without bargaining, we do not need $u(q)$ to go through the origin which occurs when $b>0$. So set $b=0$ and use $u(q)=\frac{q^{1-\eta}-1}{1-\eta}$. Hence, (4)-(7) become

$$\begin{aligned}
X^{\varepsilon} &= \frac{(1-\alpha)B}{A}\left(\frac{K}{ZH}\right)^{\alpha}Z && (8)\\
q^{-\eta} &= \frac{\psi}{Z}\left(\frac{K}{q}\right)^{1-\psi} && (9)\\
\left(\frac{X_{+1}}{X}\right)^{\varepsilon} &= \beta\left[\alpha\left(\frac{K_{+1}}{Z_{+1}H_{+1}}\right)^{\alpha-1}+1-\delta\right]+\beta\sigma(\psi-1)\frac{X_{+1}^{\varepsilon}}{BZ_{+1}}\left(\frac{q_{+1}}{K_{+1}}\right)^{\psi} && (10)\\
\frac{X}{ZH} &= \left(\frac{K}{ZH}\right)^{\alpha}+(1-\delta)\frac{K}{ZH}-\frac{Z_{+1}H_{+1}}{ZH}\frac{K_{+1}}{Z_{+1}H_{+1}} && (11)
\end{aligned}$$

3.1 Steady State

Conjecture a steady state with balanced growth and constant aggregate hours $H_{+1} = H$ for all t. This implies we have a constant value of capital per efficiency labor unit, $\hat{K} = K/ZH$, and all real variables grow at the rate $1 + \mu$.

Using (8) and (11) yields

$$K = \left[(1-\alpha) BA^{-1}\right]^{1/\varepsilon} \frac{\hat{K}^{1-\alpha+\alpha/\varepsilon}}{1-(\delta+\mu)\hat{K}^{1-\alpha}} Z^{1/\varepsilon}$$

where $K > 0$ if $[1/(\delta+\mu)]^{\frac{1}{1-\alpha}} > \hat{K}$. This implies that K grows at gross rate $(1+\mu)^{1/\varepsilon}$. With constant hours and Z growing at rate $1+\mu$ we need $\varepsilon = 1$ or log utility in the CM. This is standard in the neoclassical growth model when preferences are separable over consumption and leisure. So impose this. Steady-state hours and consumption are then giving by

$$\begin{aligned} H &= \frac{(1-\alpha) BA^{-1}}{1-(\delta+\mu)\hat{K}^{1-\alpha}} \\ X &= (1-\alpha) BA^{-1}\hat{K}^{\alpha} Z \end{aligned}$$

From (9) we obtain

$$\begin{aligned} q &= \left\{ \frac{1}{\psi}\left[(1-\alpha) BA^{-1}\right]^{\psi-1} \left[\frac{\hat{K}}{1-(\delta+\mu)\hat{K}^{1-\alpha}}\right]^{\psi-1} Z^{\psi} \right\}^{\frac{1}{\psi+\eta-1}} \\ \frac{q_{+1}}{q} &= (1+\mu)^{\frac{\psi}{\psi+\eta-1}} \\ \frac{q_{+1}}{K_{+1}} &= \left\{ \frac{1}{\psi}\left[\frac{1}{(1-\alpha) BA^{-1}}\right]^{\eta} \left[\frac{1-(\delta+\mu)\hat{K}^{1-\alpha}}{\hat{K}}\right]^{\eta} Z^{1-\eta} \right\}^{\frac{1}{\psi+\eta-1}} \end{aligned}$$

The growth rate of q equals $1+\mu$ when $\eta = 1$ which also makes q_{+1}/K_{+1} constant in steady state. Hence, we need log preferences in the DM to have balanced growth in DM production. Impose this from here on. Note that $dq/d\hat{K} > 0$.

Using (8), (10) and (11) with $\varepsilon = \eta = 1$, we obtain the planner's choice of $\hat{K}$ and H:

$$\hat{K}_p = \left\{ \frac{\alpha\beta + \sigma\beta\left(\frac{\psi-1}{\psi B}\right)}{1+\mu-\beta(1-\delta)+(\delta+\mu)\sigma\beta\left(\frac{\psi-1}{\psi B}\right)} \right\}^{\frac{1}{1-\alpha}} \tag{12}$$

$$H_p = (1-\alpha)BA^{-1}\left[\frac{1+\mu-\beta(1-\delta)+(\delta+\mu)\sigma\beta\left(\frac{\psi-1}{\psi B}\right)}{1+\mu-\beta(1-\delta)-(\delta+\mu)\alpha\beta}\right] \tag{13}$$

$$\hat{q}_p \equiv \frac{q}{Z} = \left(\frac{1}{\psi}\right)^{\frac{1}{\psi}}\left[(1-\alpha)BA^{-1}\right]^{\frac{\psi-1}{\psi}}\left[\frac{\hat{K}_p}{1-(\delta+\mu)\hat{K}_p^{1-\alpha}}\right]^{\frac{\psi-1}{\psi}} \tag{14}$$

So we have a balanced growth path with K, X and q all growing at gross rate $1+\mu$. For $\sigma > 0$ and $\psi > 1$, capital has additional value for producing in the DM so the steady-state capital per efficiency unit of labor is higher than in the standard neoclassical growth model. Steady state hours worked in the CM are also higher.

3.2 Dynamics

To obtain some analytical results on the transitional dynamics, let $\delta = 1$. From (9)

$$\frac{q}{K} = \left(\frac{Z}{\psi K}\right)^{1/\psi}$$

while (8) and (11) yield

$$K_{+1} = \left[1 - \frac{(1-\alpha)BA^{-1}}{H}\right]K^{\alpha}(ZH)^{1-\alpha}. \tag{15}$$

We can then write the Euler equation as

$$K_{+1} = \frac{1}{H}\left[\alpha\beta\frac{H_{+1}}{H} + \sigma\beta\left(\frac{\psi-1}{\psi}\right)(1-\alpha)A^{-1}\right]K^{\alpha}Z^{1-\alpha}H^{1-\alpha} \tag{16}$$

Conjecture that hours are constant for all t along the transition path. Combining (15) and (16) gives us the planner's choice of hours

$$H_p = \frac{(1-\alpha)BA^{-1}}{1-\beta\alpha}\left[1+\sigma\beta\left(\frac{\psi-1}{\psi B}\right)\right].$$

With full depreciation, the planner keeps hours at the steady state value. For $\sigma > 0$ and $\psi > 1$, hours are higher along the transition path than in the standard neoclassical growth model. It also implies that investment (CM consumption) is a higher (lower) fraction of output with transitional dynamics

$$
\begin{aligned}
K_{+1} &= \left[\frac{\alpha\beta + \sigma\beta \left(\frac{\psi - 1}{\psi B} \right)}{1 + \sigma\beta \left(\frac{\psi - 1}{\psi B} \right)} \right] K^{\alpha} \left(ZH \right)^{1-\alpha} \\
X &= \left[\frac{1 - \alpha\beta}{1 + \sigma\beta \left(\frac{\psi - 1}{\psi B} \right)} \right] K^{\alpha} \left(ZH \right)^{1-\alpha}
\end{aligned}
$$

and the transition path for $\hat{K}$ is given by

$$
\hat{K}_{+1} = \frac{1}{1 + \mu} \left[\frac{\alpha\beta + \sigma\beta \left(\frac{\psi - 1}{\psi B} \right)}{1 + \sigma\beta \left(\frac{\psi - 1}{\psi B} \right)} \right] \hat{K}^{\alpha}
$$

If $\sigma = 0$ then we have the standard transition path for capital in the Solow model . Thus, the additional productivity of capital in the DM not only generates a higher steady-state capital stock per efficiency unit of labor, but also a higher rate of investment and more rapid growth in the transition to the steady-state.

4 Monetary Economy

In the monetary economy, firms hire labor and capital to produce output which is sold in the CM at the monetary price p. Goods and input markets are perfectly competitive. Profit maximization implies $r_t = F_K(K_t/Z_t, H_t)$ and $w_t = F(K_t/Z_t, H_t)Z_t$, where r is the rental rate, and w is the real wage. Constant returns implies equilibrium profits are 0. Firms do not operate in the DM but agents can use their capital and effort to produce output.

Let $W(m, k, Z)$ and $V(m, k, Z)$ be the value functions of agents in the CM and DM respectively when holding m units of money, k units of capital given the aggregate state Z. Begining with the CM, we have

$$
\begin{aligned}
W(m, k, Z) &= \max_{x, h, m_{+1}, k_{+1}} \left\{ U(x) - Ah + \beta V(m_{+1}, k_{+1}, Z_{+1}) \right\} \\
\text{s.t. } x &= wh + (1 + r - \delta)\, k - k_{+1} + \tau M + \frac{m - m_{+1}}{p}.
\end{aligned}
$$

Eliminating h using the budget equation, we have the first order conditions

$$
\begin{aligned}
x &: \ U'(x) = \frac{A}{w} \\
m_{+1} &: \ \frac{A}{pw} = \beta V_m\left(m_{+1}, k_{+1}, Z_{+1}\right) \\
k_{+1} &: \ \frac{A}{w} = \beta V_k\left(m_{+1}, k_{+1}, Z_{+1}\right).
\end{aligned} \tag{17}
$$

and the envelope conditions,

$$W_m(m,k,Z) = \frac{A}{pw} \tag{18}$$

$$W_k(m,k,Z) = \frac{A\left(1+r-\delta\right)}{w}. \tag{19}$$

In the DM market, we have

$$V(m,k,Z) = \sigma V_b(m,k,Z) + \sigma V_s(m,k,Z) + (1-2\sigma)W(m,k,Z) \tag{20}$$

with

$$V_b(m,k,Z) = u(q_b) + W\left(m - d_b, k, Z\right) \tag{21}$$

$$V_s(m,k,Z) = -c\left(\frac{q_s}{Z}, \frac{k}{Z}\right) + W\left(m + d_s, k, Z\right), \tag{22}$$

where q_b and d_b are the quantities of goods acquired and money spent by buyers in the DM, while q_s and d_s are the quantities of goods produced and money earned by sellers.

Using (18), we have

$$V(m,k,Z) = \sigma\left[u(q_b) - d_b\frac{A}{pw} - c\left(\frac{q_s}{Z}, \frac{k}{Z}\right) + d_s\frac{A}{pw}\right] + W(m,k,Z).$$

Differentiating yields

$$
\begin{aligned}
V_m(m,k,Z) &= \sigma\left[u'\frac{\partial q_b}{\partial m} - \frac{A}{pw}\frac{\partial d_b}{\partial m}\right] + \sigma\left[-c_q\frac{\partial q_s}{\partial m} + \frac{A}{pw}\frac{\partial d_s}{\partial m}\right] + \frac{A}{pw} \\
V_k(m,k,Z) &= \sigma\left[-\frac{c_q}{Z}\frac{\partial q_s}{\partial k} - \frac{c_k}{Z} + \frac{A\left(1+r-\delta\right)}{w}\frac{\partial d_s}{\partial k}\right] \\
&\quad +\sigma\left[u'\frac{\partial q_b}{\partial k} - \frac{A\left(1+r-\delta\right)}{w}\frac{\partial d_b}{\partial k}\right] + \frac{A\left(1+r-\delta\right)}{w}.
\end{aligned}
$$

In order to solve (17), we must evaluate these derivatives. To do that we need to describe how the terms of trade are determined in the DM. One possibility is pricing taking. Another is bargaining.

4.1 Proportional Bargaining

Suppose agents are randomly matched in a bilateral fashion in the D with each buyer being randomly paired with a seller. In the search theoretic models of money, bargaining has traditionally been used to determine the terms of trade in bilateral trades, with Nash bargaining being the standard. However, as Aruoba, Rocheteau and Waller (2007) emphasize, in the LW framework, Nash bargaining generates non-monotonic surpluses for buyers. Thus inefficiencies occurring under the Friedman rule are due to this property of the bargaining solution rather than a holdup problem as suggested by LW.

To avoid this problem, I will consider proportional bargaining as the way in which terms of trade are determined. Under proportional bargaining, the buyer's gains from trade is a fixed share, θ, of the trade surplus:

$$u(q) - \frac{A}{pw}d = \theta\left[u(q) - c\left(\frac{q}{Z}, \frac{k}{Z}\right)\right].$$

Imposing $d = m$ we have

$$\frac{A}{pw}m = (1-\theta)\,u(q) + \theta c\left(\frac{q}{Z}, \frac{k}{Z}\right)$$

and

$$\begin{aligned}
\frac{\partial q}{\partial m} &= \frac{1}{(1-\theta)\,u'(q) + \theta c_q\left(\frac{q}{Z}, \frac{k}{Z}\right)\frac{1}{Z}}\frac{A}{pw} > 0 \\
\frac{\partial q}{\partial k} &= \frac{-\theta c_k\left(\frac{q}{Z}, \frac{k}{Z}\right)\frac{1}{Z}}{(1-\theta)\,u'(q) + \theta c_q\left(\frac{q}{Z}, \frac{k}{Z}\right)\frac{1}{Z}} > 0.
\end{aligned}$$

We have

$$\begin{aligned}
V_m(m,k,Z) &= \sigma\frac{A}{pw}\frac{\theta\left[u'(q) - c_q\left(\frac{q}{Z}, \frac{k}{Z}\right)\frac{1}{Z}\right]}{(1-\theta)\,u'(q) + \theta c_q\left(\frac{q}{Z}, \frac{k}{Z}\right)\frac{1}{Z}} + \frac{A}{pw} \\
V_k(m,k,Z) &= -\sigma c_k\left(\frac{q}{Z}, \frac{k}{Z}\right)\frac{1}{Z}\frac{(1-\theta)\,u'(q)}{(1-\theta)\,u'(q) + \theta c_q\left(\frac{q}{Z}, \frac{k}{Z}\right)\frac{1}{Z}} + \frac{A\,(1+r-\delta)}{w}
\end{aligned}$$

An equilibrium allocation solves

$$U'(X) = \frac{A}{F_H(K, ZH)} \tag{23}$$

$$\frac{A}{pw} = \beta \frac{A}{p_{+1}w_{+1}} \left[\frac{\sigma\theta \left[u'(q_{+1}) - c_q\left(\frac{q_{+1}}{Z_{+1}}, \frac{k_{+1}}{Z_{+1}}\right)\frac{1}{Z_{+1}} \right]}{(1-\theta)\, u'(q_{+1}) + \theta c_q\left(\frac{q_{+1}}{Z_{+1}}, \frac{k_{+1}}{Z_{+1}}\right)\frac{1}{Z_{+1}}} + 1 \right] \tag{24}$$

$$\begin{aligned} U'(X) &= \beta U'(X_{+1})\left[1 + F_K(K_{+1}, Z_{+1}H_{+1}) - \delta\right] \\ &\quad -\beta\sigma c_k\left(\frac{q_{+1}}{Z_{+1}}, \frac{k_{+1}}{Z_{+1}}\right)\frac{1}{Z_{+1}} \frac{(1-\theta)\, u'(q_{+1})}{(1-\theta)\, u'(q_{+1}) + \theta c_q\left(\frac{q_{+1}}{Z_{+1}}, \frac{k_{+1}}{Z_{+1}}\right)\frac{1}{Z_{+1}}} \end{aligned} \tag{25}$$

$$X = F(K, ZH) + (1-\delta)K - K_{+1}. \tag{26}$$

4.1.1 Steady State

Along the balanced growth path, hours are constant and X, K_{+1} and q grow at a rate $1+\mu$. Conjecture that real balances M/p also grow at the rate $1+\mu$ implying

$$1+\tau = (1+\pi)(1+\mu).$$

It then follows that the nominal interest satisfies

$$1+i = (1+\pi)(1+\mu)(1+\rho).$$

Using the functional forms above, conjecture there is a constant value of $\hat{K} = K/ZH$ along the balanced growth path. Then (23) and (26) yield

$$\begin{aligned} X &= (1-\alpha)\, BA^{-1}\hat{K}^{\alpha} Z \\ K &= (1-\alpha)\, BA^{-1}\left[\frac{\hat{K}}{1-(\delta+\mu)\,\hat{K}^{1-\alpha}}\right] Z \\ H &= \frac{(1-\alpha)\, BA^{-1}}{1-(\delta+\mu)\,\hat{K}^{1-\alpha}}. \end{aligned}$$

With $\hat{K} = K/ZH$ and letting $b \to 0$, (24) yields

$$i = \frac{\sigma\theta\left[u'(q_{+1}) - c_q\left(\frac{q_{+1}}{Z_{+1}}, \frac{K_{+1}}{Z_{+1}}\right)\frac{1}{Z_{+1}}\right]}{(1-\theta)\, u'(q_{+1}) + \theta c_q\left(\frac{q_{+1}}{Z_{+1}}, \frac{K_{+1}}{Z_{+1}}\right)\frac{1}{Z_{+1}}}$$

Note that if $i = 0$, then for any $0 < \theta \leq 1$ we have

$$u'(q_{+1}) = c_q \left(\frac{q_{+1}}{Z_{+1}}, \frac{K_{+1}}{Z_{+1}} \right) \frac{1}{Z_{+1}}$$

which is the efficient quantity *given* the current capital stock K_{+1}. This is consistent with the results in Aruoba, Rocheteau and Waller (2007) – under the Friedman rule, proportional bargaining generates the efficient quantity of goods traded in the DM even though buyers do not get the entire trade surplus. In short, there is no holdup problem on *buyers* at the Friedman rule. Note, even though q_{+1} is efficient, it is not equal to the planner's choice of q_{+1} unless K_{+1} is the same as the planner choice. As we show below, this is not the case due to the hold-up problem on capital discussed in AWW.

Solving for q_{+1} yields

$$q_{+1} = \left[\frac{\sigma\theta - (1-\theta)\, i}{\theta\,(i+\sigma)} \right]^{\frac{1}{\psi+\eta-1}} \left[\frac{(1-\alpha)\, B\hat{K}A^{-1}}{1-(\delta+\mu)\,\hat{K}^{1-\alpha}} \right]^{\frac{\psi-1}{\psi+\eta-1}} Z_{+1}^{\frac{\psi}{\psi+\eta-1}}$$

Again, q grows at $1+\mu$ when $\eta = 1$, i.e., utility is log in the DM. Also note that for $q_{+1} > 0$, we need

$$\frac{\sigma\theta}{1-\theta} > i$$

For a given value of θ, $q_{+1} = 0$ at a finite inflation rate. In short the monetary equilibrium collapses. In what follows, I assume this condition holds.

The steady state has

$$\begin{aligned}
\hat{K}_b &= \left[\frac{\alpha\beta + (1-\theta)^2 \beta \left(\frac{\psi-1}{\psi B\theta}\right)\left(\frac{\sigma\theta}{1-\theta} - i\right)}{1+\mu-\beta(1-\delta) + (1-\theta)^2(\delta+\mu)\beta\left(\frac{\psi-1}{\psi B\theta}\right)\left(\frac{\sigma\theta}{1-\theta} - i\right)} \right]^{\frac{1}{1-\alpha}} \\
H_b &= \frac{(1-\alpha)\, BA^{-1}\left[1+\mu-\beta(1-\delta) + (\delta+\mu)(1-\theta)^2\beta\left(\frac{\psi-1}{\psi B\theta}\right)\left(\frac{\sigma\theta}{1-\theta} - i\right)\right]}{1+\mu-\beta(1-\delta)-(\delta+\mu)\,\alpha\beta} \\
\hat{q}_b &\equiv \left[\frac{\sigma\theta - (1-\theta)\, i}{\theta\,(i+\sigma)} \right]^{\frac{1}{\psi}} \left[\frac{(1-\alpha)\, BA^{-1}\hat{K}_b}{1-(\delta+\mu)\,\hat{K}_b^{1-\alpha}} \right]^{\frac{\psi-1}{\psi}}
\end{aligned}$$

Note that even if the FR holds $i = 0$, we do not replicate the planner allocation since $1-\theta \leq 1$. The reason is that $1-\theta$ appears in front of the second term of the numerator and denominator on the RHS. This is capturing the holdup problem on capital. Thus, while the FR eliminates the holdup problem on money, there is still a holdup problem on capital.

4.1.2 Dynamics

To obtain analytical results, again assume $\delta = 1$. We then have

$$\frac{q_{+1}}{Z_{+1}} = \left\{ \frac{1}{\theta\psi} \left[\frac{1 - \theta + \sigma\theta - (1+i)(1-\theta)\left(\hat{K}^{\alpha}_{+1}/\hat{K}^{\alpha}\right)}{(1+i)\left(\hat{K}^{\alpha}_{+1}/\hat{K}^{\alpha}\right) - 1 + \sigma} \right] \right\}^{1/\psi} \left(\frac{K_{+1}}{Z_{+1}}\right)^{\frac{\psi-1}{\psi}}$$

and

$$\begin{aligned} \hat{K}_{+1} &= \frac{1}{1+\mu} \left\{ \alpha\beta + \frac{(1-\alpha)}{AH_{+1}} \beta \left(\frac{\psi-1}{\psi}\right) \left(\frac{1-\theta}{\theta}\right) \left[1 - \theta(1-\sigma) - (1+i)(1-\theta)\frac{\hat{K}^{\alpha}_{+1}}{\hat{K}^{\alpha}} \right] \right\} \hat{K}^{\alpha} \\ \hat{K}_{+1} &= \frac{1}{1+\mu} \frac{H}{H_{+1}} \left[1 - \frac{(1-\alpha)B}{AH} \right] \hat{K}^{\alpha} \end{aligned}$$

These two equations can be combined to obtain a non-linear equation for H_{+1} as a function of H and $\hat{K}$.

As with price taking consider a non-stationary interest rate policy given by

$$1 + i = \frac{1 - \theta(1-\sigma) - \lambda}{1-\theta} (1+\mu) \frac{w}{w_{+1}}$$

where λ is a constant and satisfies $\sigma\theta \geq \lambda$. When wages grow at the balanced path growth rate, we have $i = \frac{\sigma\theta - \lambda}{1-\theta} \geq 0$ with the Friedman rule corresponding to $\sigma\theta = \lambda$. As I will show shortly, this policy has the effect of keeping hours worked in the CM constant along the transition path, just as the planner would choose. One way of thinking about this policy is that it aims at employment stability.

It then follows that the transition paths for $\hat{K}_{+1}$ and H_{+1} are given by:

$$\begin{aligned} \hat{K}_{+1} &= \frac{1}{1+\mu} \left[\alpha\beta + \frac{(1-\alpha)}{AH_{+1}} \beta \left(\frac{\psi-1}{\psi}\right) \left(\frac{1-\theta}{\theta}\right) \lambda \right] \hat{K}^{\alpha} \\ \hat{K}_{+1} &= \frac{1}{1+\mu} \frac{H}{H_{+1}} \left[1 - \frac{(1-\alpha)B}{AH} \right] \hat{K}^{\alpha}. \end{aligned}$$

Conjecture that hours are constant along the transition path. Then we have

$$
\begin{aligned}
\hat{K}_{+1} &= \frac{1}{1+\mu}\left[\frac{\alpha\beta+\beta\left(\frac{\psi-1}{\psi B}\right)\left(\frac{1-\theta}{\theta}\right)\lambda}{1+\beta\left(\frac{\psi-1}{\psi B}\right)\left(\frac{1-\theta}{\theta}\right)\lambda}\right]\hat{K}^{\alpha} \\
H &= \frac{(1-\alpha)\,BA^{-1}\left[1+\beta\left(\frac{\psi-1}{\psi B}\right)\left(\frac{1-\theta}{\theta}\right)\lambda\right]}{1-\alpha\beta} \\
\frac{q}{Z} &= \left[\frac{\lambda(1-\theta)}{\theta\psi(\sigma-\lambda)}\right]^{1/\psi}\left(\hat{K}H\right)^{\frac{\psi-1}{\psi}}
\end{aligned}
$$

Under this policy the transition path for $\hat{K}_{+1}$ is monotone. Note that even at the Friedman rule $\lambda = \sigma\theta$ the transition paths do not mimic the planner allocation due to the hold-up problem on capital. Thus, the holdup problem on capital leads to a lower steady state $\hat{K}$, lower investment along the transition path and thus a lower growth rate of the economy for $\hat{K} < \hat{K}_b$.

4.2 Price taking

As shown in AWW, price taking eliminates the holdup problems on both buyers and sellers. This leaves the time cost of holding money as the only remaining friction. In this section, I consider price taking in order to see how the model behaves in the absence of holdup problems. Assume that agents trade anonymously in a competitive market in the DM and take the market price $\tilde{p}$ parametrically. The buyer's problem is

$$
\begin{aligned}
V_b(m,k,Z) &= \max_{q_b,d}\; u(q_b) + W(m-d,k,Z) \\
\text{s.t. } \tilde{p}q_b &= d \text{ and } d \le m
\end{aligned}
$$

while the seller's problem becomes

$$
V_s(m,k,Z) = \max_{q_s} -c\left(\frac{q_s}{Z},\frac{k}{Z}\right) + W\left[m+\tilde{p}q,k,Z\right].
$$

It is easy to show that the buyer's constraint $d \le m$ is binding in equilibrium, and so $q = M/\tilde{p}$. The seller's choice satisfies $c_q\left(\frac{q_s}{Z},\frac{k}{Z}\right)\frac{1}{Z} = \tilde{p}A/pw$. In equilibrium we have

$$
\begin{aligned}
\frac{\partial q_s}{\partial m} &= \frac{\partial d_s}{\partial m} = \frac{\partial q_b}{\partial k} = \frac{\partial d_b}{\partial k} = \frac{\partial d_s}{\partial k} = 0 \\
\frac{\partial q_b}{\partial m} &= \frac{1}{\tilde{p}},\; \frac{\partial d_b}{\partial m} = 1
\end{aligned}
$$

So,

$$
\begin{aligned}
V_m(m,k,Z) &= \sigma\frac{u'(q)}{\tilde{p}} + (1-\sigma)\frac{A}{pw} \\
V_k(m,k,Z) &= -\sigma c_k\left(\frac{q_s}{Z},\frac{k}{Z}\right)\frac{1}{Z} + \frac{A\,(1+r-\delta)}{w}.
\end{aligned}
$$

We now have

$$
\begin{aligned}
\frac{A}{pw} &= \beta\frac{A}{w_{+1}p_{+1}}\left[\sigma\frac{u'(q_{+1})}{c_q\left(\frac{q_{+1}}{Z_{+1}},\frac{K_{+1}}{Z_{+1}}\right)\frac{1}{Z_{+1}}}+1-\sigma\right] && (27) \\
U'(X) &= \beta U'(X_{+1})\left[F_K(K_{+1},H_{+1})+1-\delta\right] - \sigma\beta c_k\left(\frac{q_{+1}}{Z_{+1}},\frac{K_{+1}}{Z_{+1}}\right)\frac{1}{Z_{+1}} && (28)
\end{aligned}
$$

$$
\begin{aligned}
U'(X) &= \frac{A}{ZF_H(K,ZH)} && (29) \\
X + K_{+1} &= F(K,ZH) + (1-\delta)K. && (30)
\end{aligned}
$$

A monetary equilibrium is a sequence of quantities $\{X, K_{+1}, H, q\}$ solving (27)-(30) given an initial capital stock K_0 and money stock M_0.

4.2.1 Steady state

Along the balanced growth path, hours are constant and X, K_{+1} and q grow at a rate $1+\mu$. Conjecture that real balances M/p also grow at the rate $1+\mu$ implying

$$1+\tau = (1+\pi)(1+\mu).$$

It then follows that the nominal interest satisfies

$$1+i = (1+\pi)(1+\mu)(1+\rho).$$

As with the planner, (28) and (30) yield

$$
\begin{aligned}
X &= (1-\alpha)\,BA^{-1}\hat{K}^{\alpha}Z && (31) \\
K &= \frac{(1-\alpha)\,BA^{-1}\hat{K}}{1-(\delta+\mu)\,\hat{K}^{1-\alpha}}Z && (32) \\
H &= \frac{(1-\alpha)\,BA^{-1}}{1-(\delta+\mu)\,\hat{K}^{1-\alpha}} && (33)
\end{aligned}
$$

From (27) we have

$$\frac{i}{\sigma} = \frac{u'(q_{+1}) - c_q\left(\frac{q_{+1}}{Z_{+1}}, \frac{K_{+1}}{Z_{+1}}\right)\frac{1}{Z_{+1}}}{c_q\left(\frac{q_{+1}}{Z_{+1}}, \frac{K_{+1}}{Z_{+1}}\right)\frac{1}{Z_{+1}}}.$$

So $i = 0$ generates the efficient quantity of goods in the DM. All that remains to determine is whether $i = 0$ generates the planner's choice for K_{+1}. Rewriting this expression we get

$$\frac{q}{Z} = \left[\frac{\sigma}{\psi(i+\sigma)}\right]^{\frac{1}{\psi}}\left[\frac{(1-\alpha)BA^{-1}\hat{K}}{1-(\delta+\mu)\hat{K}^{1-\alpha}}\right]^{\frac{\psi-1}{\psi}}.$$

Using this expression as well as (29) in the Euler equation we obtain the equilibrium values of $\hat{K}$ and H in the monetary economy with price taking:

$$\begin{aligned}
\hat{K}_m &= \left[\frac{\alpha\beta\left(1+\frac{i}{\sigma}\right)+\sigma\beta\left(\frac{\psi-1}{\psi B}\right)}{[1+\mu-\beta(1-\delta)]\left(1+\frac{i}{\sigma}\right)+(\delta+\mu)\sigma\beta\left(\frac{\psi-1}{\psi B}\right)}\right]^{\frac{1}{1-\alpha}} \\
H_m &= (1-\alpha)BA^{-1}\left[\frac{1+\mu-\beta(1-\delta)+(\delta+\mu)\sigma\beta\left(\frac{\psi-1}{\psi B}\right)\frac{\sigma}{i+\sigma}}{1+\mu-\beta(1-\delta)-(\delta+\mu)\alpha\beta}\right].
\end{aligned}$$

Compared to the planner allocation (12) and (13) we have $\hat{K}_m < \hat{K}_p$ and $H_m < H_p$ for any $i > 0$. Furthermore, we have $d\hat{K}_m/di < 0$ and $dH_m/di < 0$. We have $\hat{K}_p = \hat{K}_m$ if $i = 0$ or

$$\pi = \frac{1}{(1+\mu)(1+\rho)} - 1.$$

So at the Friedman rule, deflation must be greater than the time rate of discount – it must also account for growth to the real return to capital.

Finally we have

$$\hat{q}_m = \frac{q_m}{Z} = \left[\frac{\sigma}{\psi(i+\sigma)}\right]^{\frac{1}{\psi}}\left[\frac{(1-\alpha)BA^{-1}\hat{K}_m}{1-(\delta+\mu)\hat{K}_m^{1-\alpha}}\right]^{\psi-1}$$

Since $\hat{K}_m = \hat{K}_p$ at $i = 0$, we have $\hat{q}_m = \hat{q}_p$ at the Friedman rule. Intuitively, inflation acts as a tax on DM consumption which reduces the equilibrium value of q. This in turn lowers the marginal value of capital and agents accumulate less capital and work fewer hours.

4.2.2 Dynamics

Set $\delta = 1$. The Euler equation and intra-temporal condition are given by

$$\hat{K}_{+1} = \frac{1}{1+\mu}\left[\beta\alpha + \sigma^2\beta\left(\frac{\psi-1}{\psi}\right)\frac{1}{\frac{\hat{K}_{+1}^{\alpha}}{\hat{K}^{\alpha}}(1+i)-1+\sigma}\frac{(1-\alpha)}{AH_{+1}}\right]\hat{K}^{\alpha}$$

$$\hat{K}_{+1} = \left[\frac{H-(1-\alpha)BA^{-1}}{1+\mu}\right]\frac{\hat{K}^{\alpha}}{H_{+1}}$$

Combining these two equations gives us a non-linear dynamic equation in H_{+1} in term of H and $\hat{K}^{\alpha}$. So the dynamical system

$$\begin{aligned} & [AH-(1-\alpha)B]\left[\frac{H-(1-\alpha)BA^{-1}}{1+\mu}\right]^{\alpha}\hat{K}^{\alpha}(1+i) \\ = & \left[\sigma^2\beta\left(\frac{\psi-1}{\psi}\right)(1-\alpha)+[AH-(1-\alpha)B](1-\sigma)\right]H_{+1}^{\alpha} \\ & +\alpha\beta AH_{+1}\left[\frac{H-(1-\alpha)BA^{-1}}{1+\mu}\right]^{\alpha}\hat{K}^{\alpha}(1+i)-(1-\sigma)\alpha\beta AH_{+1}^{\alpha+1} \end{aligned}$$

$$\hat{K}_{+1} = \frac{1}{1+\mu}\frac{H}{H_{+1}}\left[1-\frac{(1-\alpha)BA^{-1}}{H}\right]\hat{K}^{\alpha}$$

determines the paths of H_{+1} and $\hat{K}_{+1}$ as a function of current H and $\hat{K}$.

Consider a non-stationary monetary policy along the transition path. One such policy is

$$\frac{\hat{K}_{+1}^{\alpha}}{\hat{K}^{\alpha}}(1+i)-1+\sigma = \Omega$$

where $\Omega \geq \sigma$ is some constant. It then follows that current interest rates satisfy

$$1+i = (\Omega+1-\sigma)\frac{\hat{K}^{\alpha}}{\hat{K}_{+1}^{\alpha}}.$$

Manipulate this expression to write it in terms of real wages

$$1+i = (\Omega+1-\sigma)\frac{w}{w_{+1}}(1+\mu)$$

If real wages converge to the balanced growth rate, then this policy rule converges to the value $i = \Omega - \sigma$. If $\Omega = \sigma$, this policy rule generates the Friedman rule along the balanced growth path. As shown above, such a policy keeps hours constant along the transition path, just as the planner would choose.

Under this non-stationary interest rate policy we have

$$\begin{aligned}
\hat{K}_{+1} &= \frac{1}{1+\mu}\left[\frac{\alpha\beta+\sigma\beta\left(\frac{\psi-1}{\psi B}\right)\frac{\sigma}{\Omega}}{1+\sigma\beta\left(\frac{\psi-1}{\psi B}\right)\frac{\sigma}{\Omega}}\right]\hat{K}^{\alpha} \\
H &= \frac{(1-\alpha)BA^{-1}}{1-\alpha\beta}\left[1+\sigma\beta\left(\frac{\psi-1}{\psi B}\right)\frac{\sigma}{\Omega}\right] \\
\frac{q}{Z} &= \left(\frac{\sigma}{\psi\Omega}\right)^{1/\psi}\left(\hat{K}H\right)^{\frac{\psi-1}{\psi}}.
\end{aligned}$$

Under this policy rule, the transition path for $\hat{K}$ is monotone. It mimics the planner's transition path but at a slower growth rate when $\Omega > \sigma$. What does this policy do? It adjusts the interest rate such that the cost of acquiring money in t and $t+1$, i.e., the growth rate of real wages, is unaffected by the transition to the steady state. If real wages are going to grow unusually fast, then it is cheaper to acquire a unit of money in $t+1$ than acquire a unit of money in t and carry it to $t+1$. Hence the demand for money would fall along with its real value. To counter this, the policy above lowers i to improve the value of money when wage growth is excessively high. To see this in more detail, from (27) we have

$$\frac{K_{+1}^{\alpha}Z_{+1}^{1-\alpha}H_{+1}^{-\alpha}}{K^{\alpha}Z^{1-\alpha}H^{-\alpha}}(1+\pi)(1+\rho)-1=\sigma\left[\frac{u'(q_{+1})}{c_q\left(\frac{q_{+1}}{Z_{+1}},\frac{K_{+1}}{Z_{+1}}\right)\frac{1}{Z_{+1}}}-1\right].$$

The RHS is the marginal liquidity value of money. The LHS is the marginal cost of holding money from t to $t+1$. Under the proposed policy, the LHS is constant. Thus, q_{+1} adjusts to equate the marginal liquidity value to this constant cost of holding money along the transition path.

5 Conclusion

This paper contributes to our analytical understanding of the role of matching frictions, bargaining and money on growth dynamics. Whereas AWW focus on numerical analysis, I am able to derive analytical results that provide additional insight for the numerical results obtained in AWW. The benefit of this analysis is that it provides clear and simple intuition for how bargaining, random matching and changes in the nominal interest rate affect the steady state-capital labor ratio, consumption and short-run growth rates of the economy.

References

[1] ARUOBA, S.B. and S. CHUGH (2008), "Money and Optimal Capital Taxation", (Mimeo, University of Maryland).

[2] ARUOBA, S.B., G. ROCHETEAU and C. WALLER (2007), "Bargaining and the Value of Money", *Journal of Monetary Economics* **54**, 2636–2655.

[3] ARUOBA, S.B. and R. WRIGHT (2003), "Search, Money and Capital: A Neoclassical Dichotomy", *Journal of Money, Credit and Banking* **35**, 1086-1105.

[4] ARUOBA, S.B., C. WALLER and R. WRIGHT (2008), "Money and Capital: A Quantitative Analysis", (Mimeo, University of Notre Dame).

[5] BERENTSEN, ROJAS-BREU and SHI (2009). "Liquidity and Economics Growth", (Mimeo, University of Basel).

[6] GOMME, P. (1993), "Money and Growth Revisited: Measuring the Costs of Inflation in an Endogenous Growth Model", *Journal of Monetary Economics* **32**, 51-77.

[7] IRELAND, P. (1994), "Money and Growth: An Alternative Approach", *American Economic Review* **84**, 47-65.

[8] KIYOTAKI, N. and R. WRIGHT (1989), "On Money as a Medium of Exchange", *Journal of Political Economy* **97**, 927-954.

[9] KIYOTAKI, N. and R. WRIGHT (1993), "A Search-Theoretic Approach to Monetary Economics", *American Economic Review* **83**, 63-77.

[10] MENNER, M. (2006), "A Search-Theoretic Monetary Business Cycle with Capital Formation", *Contributions to Macroeconomics,* **6**, Article 11.

[11] SHI S. (1999), "Search, Inflation and Capital Accumulation", *Journal of Monetary Economics* **44**, 81-104.

[12] SIDRAUSKI, M. (1967a), "Rational Choice and Patterns of Growth in a Monetary Economy", *American Economic Review* **57**, 534-544.

[13] SIDRAUSKI, M. (1967b), "Inflation and Economic Growth", *Journal of Political Economy* **75**, 796-810.

[14] TOBIN, J. (1965), "Money and Economic Growth", *Econometrica* **33**, 671-684.