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Speculative Bubbles and Financial Crisis*

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Abstract

Why are asset prices so much more volatile and so often detached from their fundamental values? Why does the bursting of financial bubbles depress the real economy? This paper addresses these questions by constructing an infinite-horizon heterogeneous agent general equilibrium model with speculative bubbles. We characterize conditions under which storable goods, regardless of their intrinsic values, can carry bubbles and agents are willing to invest in such bubbles despite their positive probability of bursting. We show that perceived changes in the bubbles' probability to burst can generate boom-bust cycles and produce asset price movements that are many times more volatile than the economy's fundamentals, as in the data.

Keywords: Asset Price Volatility, Boom-Bust Cycles, Financial Crisis, Speculative Bubbles, Sunspots, Tulip Mania.

JEL Codes: E21, E22, E32, E44, E63.

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1 Introduction

The current financial crisis caused by the burst of the U.S. housing bubble is not new. History has too often witnessed the rise and collapse of nationwide asset bubbles. Each time, an entire economy cheered for a bubble's birth and then mourned its death. The first recorded nationwide bubble is the "Tulip mania"—a period in Dutch history during which contract prices for tulip bulbs reached extraordinarily high levels and then suddenly collapsed. At the peak of the tulip mania in February 1637, tulip contracts sold for more than 10 times the annual income of a skilled craftsman, which is above the value of a furnished luxury house in seventeenth-century Amsterdam.[1] Figure 1 shows the tulip price index during the 1636-37 period.[2]

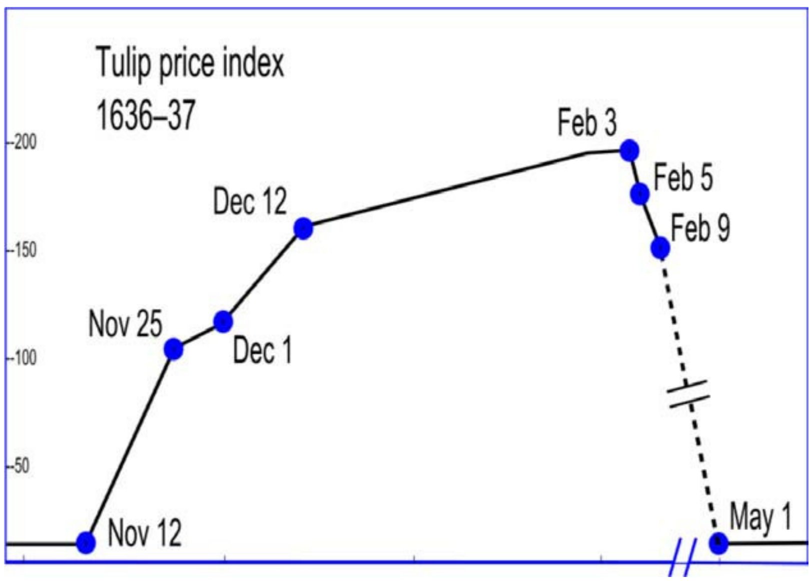

Figure 1. The Tulip mania bubble.

According to Mackay (1841, p. 107), during the tulip mania, people sold their other possessions to speculate in the tulip market:

> ... [T]he population, even to its lowest dregs, embarked in the tulip trade.... Many individuals grew suddenly rich. A golden bait hung temptingly out before the people, and, one after the other, they rushed to the tulip marts, like flies around a honey-pot. Every one imagined that the passion for tulips would last for ever, and that the wealthy from every part of the world would send to Holland, and pay whatever prices were asked for them. The riches of Europe would be concentrated on the shores of the Zuyder Zee, and poverty banished from the favoured clime of Holland. Nobles, citizens, farmers, mechanics, seamen, footmen, maidservants, even chimney-sweeps and old clotheswomen, dabbled in tulips.

People were purchasing tulips at higher and higher prices, intending to resell them for a profit. However, such a scheme could not last because tulip prices were growing faster than income. Sooner or later traders would no longer be able to find new buyers willing to pay increasingly inflated prices. As this realization set

[1] See, for example, Mackay (1841).
[2] Source: Wikipedia (http://en.wikipedia.org/wiki/Tulip_mania) and Thompson (2007).

in, the demand for tulips collapsed and prices plummeted. The Dutch economy went into a deep recession in 1637.

Although historians and economists continue to debate whether the tulip mania was indeed a bubble caused by what Mackay termed "Extraordinary Popular Delusions and the Madness of Crowds" (see, e.g., Dash, 1999; Garber, 1989, 1990; and Thompson, 2007), this paper shows that genuine bubbles with prices far exceeding the bubbles' fundamental values and with movements similar to Figure 1 can be constructed in an infinite-horizon dynamic stochastic general equilibrium (DSGE) model. In the model, infinitely lived agents are willing to invest in bubbles even though they may burst at any moment. The reason is that with incomplete financial markets and borrowing constraints, bubbles provide liquidity and help diversify idiosyncratic risks by serving as stores of value. We show that the burst of such bubbles can generate recessions, and the perceived changes in the probability of the bubbles' burst can cause asset price movements many times more volatile than aggregate output.

People invest in bubbles for many reasons. The idea that infinitely lived rational agents are willing to hold bubbles with no intrinsic values to self-insure against idiosyncratic income risks can be traced back at least to Bewley (1980).[3] This idea is more clearly articulated recently in general equilibrium models by Kiyotaki and Moore (2008) and Kocherlakota (2009), where heterogeneous firms use intrinsically worthless assets to improve resource allocation and investment efficiency when financial markets are incomplete.[4] This paper extends this literature to study asset price volatility and bubbles that may grow on assets with intrinsic values. This extension is not trivial because sunspot equilibrium may disappear in the Kiyotaki-Moore-Kocherlakota model once the object supporting the bubble (e.g., land) is allowed to have fundamental values. More importantly, casual observation suggests that more often the bubbles are likely to exist in goods with fundamental values, such as antiques, bottles of wines, paintings, flower bulbs, rare stamps, houses, land, and so on.

We use a DSGE model to characterize conditions for the existence of rational bubbles that grow on goods with fundamental values. We show that any inelastically supplied storable goods,[5] regardless of their intrinsic values, can support bubbles with the following features: (i) the market price of the goods exceeds their fundamental values and (ii) the market values can collapse to fundamental values with positive probability (namely, the fundamental value is itself a possible equilibrium).[6]

The basic structure of our model closely resembles that of Kiyotaki and Moore (2008) and Kocherlakota (2009) wherein firms, instead of households, invest in bubbles; however, the analysis easily can be extended to households. The main differences between our model and the literature include the following:

1. In addition to characterizing general equilibrium conditions for bubbles to develop on objects with fundamental values, in our model the probability of capital investment is endogenously determined by firms rather than exogenously fixed. That is, firms optimally choose whether to invest in fixed capital

[3] It can be traced further back to Samuelson's (1958) overlapping generations model of money. For a recent extension of Bewley's model to a DSGE model with multiple assets; see Wen (2009).

[4] The related literature also includes Angeletos (2007), Araujo, Pascoa, and Torres-Martinez (2005), Caballero and Krishnamurthy (2006), Farhi and Tirole (2008), Hellwig and Lorenzoni (2009), Kocherlakota (1992), Santos and Woodford (1997), Scheinkman and Weiss (1986), Tirole (1985), and Woodford (1986), among others. These literature reports focus on asset price bubbles and financial market frictions and differ from the indeterminacy literature of Benhabib and Farmer (1994) and Wang and Wen (2007, 2008). For the earlier literature on sunspots, see Cass and Shell (1983) and Azariadis (1981).

[5] Goods can be producible yet at the same time inelastically supplied. For example, antiques and bottles of wine are produced goods, but their dates of production make them unique and nonsubstitutable by newly produced ones.

[6] If the fundamental value is not an equilibrium, then bubbles will never burst and thus it may be argued that bubbles do not exist (because it is difficult to know empirically whether there is a bubble if it never bursts). Also, the existence of multiple fundamental equilibria does not imply bubbles because the asset values never exceed fundamentals in a fundamental equilibrium.

each period. Hence, in equilibrium the number of firms that are investing can respond to aggregate shocks and monetary policy. This extensive margin is missing from the literature.

2. We introduce multiple assets in the model. Our multiple asset approach allows us to construct stochastic sunspot equilibrium and conduct impulse response analyses and time-series simulations.

3. We focus on asset price volatility and calibrate our model to match the second moments of the U.S. data.

4. We provide an analytically tractable method to solve the dynamic paths of our model (without resorting to numerical computational techniques as in Krusell and Smith, 1998) despite a continuum of heterogeneous agents with irreversible investment and borrowing constraints.[7]

The rest of the paper is organized as follows. Section 2 presents a basic model and characterizes conditions under which bubbles can grow on goods with intrinsic values. Section 3 introduces sunspot shocks to a version of the basic model (by allowing the perceived probability of bubbles to burst to be stochastic) and calibrates the model to match the U.S. business cycles and asset price volatility. Section 4 concludes the paper.

2 The Basic Model

2.1 Firms

There is a continuum of competitive firms indexed by $i \in [0,1]$. Each firm maximizes discounted dividends, $E_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t d_t(i)$, where d denotes dividend, Λ the representative household's marginal utility that firms take as given, and $\beta \in (0,1)$ the time-discounting factor. The production technology of firm i is denoted by

$$y(i) = Ak(i)^{\alpha} n(i)^{1-\alpha}, \qquad \alpha \in (0,1), \tag{1}$$

where A is an index of aggregate total factor productivity (TFP), $k(i)$ capital stock, and $n(i)$ employment. The capital stock is accumulated according to the law of motion:

$$k_{t+1}(i) = (1-\delta)k_t(i) + \frac{i_t(i)}{\varepsilon_t(i)}, \tag{2}$$

where investment is irreversible ($i(i) \geq 0$) and is subject to an idiosyncratic rate of return (cost) shock, $\varepsilon_t(i)$, with support, $[\underline{\varepsilon}, \bar{\varepsilon}] \in R^+$, and the cumulative distribution function, $F(\varepsilon)$.

Assume that in the beginning of time ($t = 0$) there exists one unit of divisible good endowed from nature and equally distributed among the firms. The good can be paid to households (firm owners) as dividends and yield marginal utility, f. Hence, f is the fundamental value of the good.[8] (We call the good "tulips" throughout the paper.) Also assume that households do not have the technology to store tulips but firms do, and there exists a fixed storage cost, $\zeta \geq 0$, per unit per period. Obviously, firms will never want to sell tulips if their market price, (q), is less than f. The question is: Do firms have incentives to hold and invest in tulips when $q > f$? In other words, can $q > f$ be supported as a competitive (bubble) equilibrium in the economy other than the fundamental equilibrium, $q^* = f$? Intuitively, because tulips are storable for firms,

[7] Our method follows that of Wang and Wen (2009). As far as we know, the existing literature—except Wang and Wen (2009)—has not shown how to solve discrete-time models with irreversible investment and borrowing constraints analytically.

[8] For simplicity, assume that the good cannot be used as a factor of production.

they thus allow a firm to self-insure against idiosyncratic shocks by serving as a store of value (i.e., liquidity). For example, if the cost shock ε is large (or the rate of return to capital investment is low), firms may opt to invest in tulips to have liquidity available in the future when the next-period costs of capital investment may be low. On the other hand, if the rate of return to capital investment is high (ε is small), firms may opt to liquidate (sell) tulips in hand and make more income available by purchasing fixed capital and expanding production capacity. Such behavior is rational despite the fact that tulip bubbles have a positive probability to burst.

To characterize the conditions for the existence of a bubble equilibrium, consider a firm's maximization problem, which is to decide whether and how much to invest in tulips to maximize the present value of expected future dividends. The firm's resource constraint is

$$d_t(i) + i_t(i) + (q_t + \zeta)\, h_{t+1}(i) + w_t n_t(i) \leq A_t k_t(i)^{\alpha} n_t(i)^{1-\alpha} + q_t h_t(i), \tag{3}$$

where w is the real wage, h_{t+1} the quantity (or shares) of tulips purchased in the beginning of period t as a store of value, and ζh_{t+1} the total fixed storage costs paid for storing tulips within period t. In addition, we impose the following constraints: $d_t(i) \geq 0$ and $h_{t+1}(i) \geq 0$. That is, firms can neither pay negative dividends nor hold negative amounts of tulips. These assumptions imply that firms are financially constrained and the asset markets are incomplete. Such constraints plus investment irreversibility give rise to speculative (precautionary) motives for investing in tulip bubbles.

The following steps simplify our analysis. Using the firm's optimal labor demand schedule,

$$(1-\alpha) A k(i)^{\alpha} n(i)^{-\alpha} = w, \tag{4}$$

we can express labor demand as a linear function of the capital stock, $k(i)$,

$$n(i) = \left[\frac{(1-\alpha)A}{w}\right]^{\frac{1}{\alpha}} k(i). \tag{5}$$

Accordingly, output $y(i)$ is also a linear function of $k(i)$,

$$y(i) = A\left[\frac{(1-\alpha)A}{w}\right]^{\frac{1-\alpha}{\alpha}} k(i). \tag{6}$$

These linear relations imply that aggregate output and employment may depend only on the aggregate capital stock. Thus, we do not need to track the distribution of $k(i)$ to study aggregate dynamics. Defining $R \equiv \alpha A\left[\frac{(1-\alpha)A}{w}\right]^{\frac{1-\alpha}{\alpha}}$, the firm's net revenue is given by

$$y(i) - wn(i) = Rk(i), \tag{7}$$

which is also linear in the capital stock.

Using the definition of R, the firm's problem is to solve

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t \left[R_t k_t(i) - i_t(i) + q_t h_t(i) - (q_t + \zeta)\, h_{t+1}(i)\right], \tag{8}$$

subject to the following constraints:

$$d_t(i) \geq 0 \tag{9}$$

$$h_{t+1}(i) \geq 0 \tag{10}$$

$$i_t(i) \geq 0 \tag{11}$$

$$k_{t+1}(i) = (1-\delta)\, k_t(i) + \frac{i_t(i)}{\varepsilon_t(i)}. \tag{12}$$

Let $\{\mu(i), \phi_t(i), \lambda(i), \pi(i)\}$ denote the Lagrangian multipliers of constraints (9) through (12), respectively; the first-order conditions for $\{i_t(i), k_{t+1}(i), h_{t+1}(i)\}$ are given, respectively, by

$$1 + \mu_t(i) = \frac{\lambda_t(i)}{\varepsilon_t(i)} + \pi_t(i) \tag{13}$$

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ [1 + \mu_{t+1}(i)] R_{t+1} + (1-\delta)\lambda_{t+1}(i) \right\} \tag{14}$$

$$[1 + \mu_t(i)]\,(q_t + \zeta) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ q_{t+1} \left[1 + \mu_{t+1}(i)\right] \right\} + \phi_t(i), \tag{15}$$

plus the complementary slackness conditions,

$$\pi_t(i) i_t(i) = 0 \tag{16}$$

$$\phi_t(i) h_{t+1}(i) = 0 \tag{17}$$

$$[1 + \mu_t(i)]\,[R_t k_t(i) - i_t(i) + q_t h_t(i) - (q_t + \zeta)\, h_{t+1}(i)] = 0. \tag{18}$$

Notice that equation (14) implies that the value of $\lambda_t(i)$ is the same across firms because $\varepsilon(i)$ is i.i.d. and is orthogonal to aggregate shocks.

2.2 Decision Rules

There are two possible outcomes for the circulation of tulips and their liquidation value in the economy. First, tulips may not be traded among firms and their liquidation value is simply $q = f$. Second, tulips may be traded among firms and their market price is $q \geq f$.[9] In the first possible outcome, each firm does not expect other firms to invest in tulips and therefore has no incentives to deviate by holding tulips as a store of value. This may happen, for example, if the liquidation value, f , is low relative to storage costs so that tulips are not an efficient store of value. In the second possible outcome, each firm expects a positive measure of other firms willing to hold tulips each period and the liquidation value, q, is sufficiently high; thus, it is willing to invest in tulips as well. Which outcome prevails in general equilibrium depends on the parameter space, as the following analysis shows.

The decision rules at the firm level are characterized by a cutoff strategy. Consider two possibilities:

Case A: $\varepsilon_t(i) \leq \varepsilon_t^*$. In this case, the cost of capital investment is low. Suppose $i_t(i) > 0$; accordingly we have $\pi_t(i) = 0$. Equations (13) and (14) imply

$$\varepsilon_t(i)\,[1 + \mu_t(i)] = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ [1 + \mu_{t+1}(i)] R_{t+1} + (1-\delta)\lambda_{t+1}(i) \right\}. \tag{19}$$

[9] Notice that in any case, $q_t < f$ can never be an equilibrium outcome, because in this case the demand for tulips will rise and consequently q_t will increase.

Given that $\mu_t(i) \geq 0$, we must have $\varepsilon_t(i) \leq \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{[1+\mu_{t+1}(i)]R_{t+1} + (1-\delta)\lambda_{t+1}(i)\right\}$, which defines the cutoff value, ε_t^*,

$$\varepsilon_t^* \equiv \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{[1+\mu_{t+1}(i)]R_{t+1} + (1-\delta)\lambda_{t+1}(i)\right\}. \tag{20}$$

Equation (13) then becomes

$$1 + \mu_t(i) = \frac{\varepsilon_t^*}{\varepsilon_t(i)}. \tag{21}$$

Hence, whenever $\varepsilon_t(i) < \varepsilon_t^*$, we must have $\mu_t(i) = \frac{\varepsilon_t^*}{\varepsilon_t(i)} - 1 > 0$ and $d_t(i) = 0$. Equation (15) becomes

$$\frac{\varepsilon_t^*}{\varepsilon_t(i)} (q_t + \zeta) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{q_{t+1}\left[1 + \mu_{t+1}(i)\right]\right\} + \phi_t(i). \tag{22}$$

Defining ϕ_t^* as the cutoff value of $\phi_t(i)$ for firms with $\varepsilon_t(i) = \varepsilon_t^*$, equation (22) implies

$$q_t + \zeta = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{q_{t+1}\left[1 + \mu_{t+1}(i)\right]\right\} + \phi_t^*. \tag{23}$$

Given that $\phi_t^* \geq 0$, the fact that $\phi_t(i) > \phi_t^*$ under Case A yields

$$\phi_t(i) > 0. \tag{24}$$

That is, for any $\varepsilon_t(i) < \varepsilon_t^*$, we must have

$$h_{t+1}(i) = 0 \tag{25}$$

and

$$i_t(i) = R_t k_t(i) + q_t h_t(i). \tag{26}$$

This suggests that firms opt to liquidate all financial assets to maximize investment in fixed capital when the cost of fixed investment is low.

Case B: $\varepsilon_t(i) > \varepsilon_t^*$. In this case, the cost of investing in fixed capital is high. Suppose $d_t(i) > 0$ and $\mu_t(i) = 0$. Then equations (13) and (14) and the definition of the cutoff ε^* imply $\pi_t(i) = 1 - \frac{\varepsilon_t^*}{\varepsilon_t(i)} > 0$. Hence, we have $i_t(i) = 0$. In such a case, firms opt not to invest in fixed capital and instead pay shareholders a positive dividend. Given that $\mu_t(i) = 0$, equation (15) implies $\phi_t(i) = \phi_t^* \geq 0$. That is, the Lagrangian multiplier $\phi(i)$ is the same across firms under Case B because ϕ_t^* is independent of i. However, depending on the liquidation value of tulips in the next period, there are two possible choices (outcomes) for tulip investment under Case B: (B1) $\int_0^1 h_{t+1}(i)di > 0$ and (B2) $\int_0^1 h_{t+1}(i)di = 0$. The first outcome (B1) implies a positive aggregate demand for tulips (i.e., tulips are held as a store of value in the economy) because firms expect other firms to accept tulips in the future and the liquidation value is high enough to cover storage costs, so we must have $\phi_t^* = 0$. The second outcome (B2) implies that tulips are not traded and all existing tulips are consumed (i.e., paid to households as dividends); hence, we must have $\phi_t^* \geq 0$ and $h_{t+1}(i) = 0$ for all i. Under outcome (B2), we must also have $q_t = f$.

Thus, whether a positive demand exists for tulips under Case B depends on firms' expectation of the liquidation value of tulips in the future (i.e., on whether tulips are traded in the next period). Denoting

$$\Omega_{t+1} \equiv \int_0^1 h_{t+1}(i)di \tag{27}$$

as the aggregate demand of tulips in period t, the two possible outcomes under Case B imply the equilibrium complementary slackness condition,

$$\Omega_{t+1}\phi_t^* = 0. \tag{28}$$

Combining Cases A and B, the decision rule for capital investment is given by

$$i_t(i) = \begin{cases} R_t k_t(i) + q_t h_t(i) & \text{if } \varepsilon_t(i) \leq \varepsilon_t^* \\ \\ 0 & \text{if } \varepsilon_t(i) > \varepsilon_t^* \end{cases}. \tag{29}$$

The rate of returns to tulips depends on the expected marginal value of liquidity (cash flow), which is greater than 1 because of the option of waiting. This option value is denoted by

$$Q(\varepsilon_t^*) \equiv E\left[1 + \mu(i)\right] = \int \max\left\{1, \frac{\varepsilon^*}{\varepsilon(i)}\right\} dF(\varepsilon) > 1. \tag{30}$$

When the cost of capital investment is low (Case A), one tulip yields $\frac{\varepsilon^*}{\varepsilon(i)} > 1$ units of new capital through investment by liquidating the tulip asset. When the cost is high (**C**ase B), firms can opt to hold on to the liquid asset and the rate of return is simply 1.

Using equations (22) and (23), the value of the Lagrangian multiplier for the nonnegativity constraint (10) is determined by

$$\phi_t(i) = \begin{cases} \left(\frac{\varepsilon^*}{\varepsilon(i)} - 1\right)(q_t + \zeta) & \text{if } \varepsilon(i) \leq \varepsilon^* \\ \\ \phi_t^* & \text{if } \varepsilon(i) > \varepsilon^* \end{cases}. \tag{31}$$

This suggests that the cross-firm average shadow value of relaxing the borrowing constraint (10) by purchasing one additional tulip is

$$\begin{aligned} \int_0^1 \phi(i) di &= (q+\zeta) \int_{\varepsilon \leq \varepsilon^*} \left(\frac{\varepsilon^*}{\varepsilon} - 1\right) dF(\varepsilon) + \phi_t^* \left[1 - F\right] \\ &= (q+\zeta)(Q-1) + \phi_t^* \left[1 - F\right], \end{aligned} \tag{32}$$

which is independent of i but positively related to the tulip's price, q. Based on this, integrating equation (15) over i and rearranging yields

$$q_t + \zeta = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} q_{t+1} Q_{t+1} + \phi_t^* \left[1 - F\right]. \tag{33}$$

Equation (33) has several implications (proofs are given in the next section) for the equilibrium price of a tulip:

1. If $\zeta = 0$ and $f = 0$, then $q_t = 0$ and $\phi_t^* \geq 0$ for all t is a fundamental equilibrium, and $q > 0$ and $\phi_t^* = 0$ is a possible bubble equilibrium. This is the case analyzed by Kiyotaki and Moore (2008) and Kocherlakota (2009).

2. If $\zeta = 0$ and $f > 0$, then there can exist at most one equilibrium with $q_t \geq f$ and $\phi_t^* = 0$; hence, the type of sunspot equilibrium discussed by Kocherlakota (2009) is not possible. That is, if $q > f$ is an equilibrium, then $q = f$ is not an equilibrium and vice versa.

3. If $\zeta > 0$ and $f \geq 0$, then multiple equilibria are possible; in particular, $q = f$ and $\phi_t^* \geq 0$ is a

fundamental equilibrium and $q > f$ and $\phi_t^* = 0$ is a possible bubble equilibrium, depending on the parameter values.

2.3 Aggregation

The aggregate variables are defined as $N_t = \int_0^1 n_t(i)di$, $I_t = \int_0^1 i_t(i)di$, $K_t = \int_0^1 k_t(i)di$, and $Y_t = \int_0^1 y_t(i)di$. Given that $k_t(i)$ is a state variable, by the factor demand functions of firms we have $N_t = \left[\frac{(1-\alpha)A_t}{w_t}\right]^{\frac{1}{\alpha}} K_t$ and $Y_t = A_t \left[\frac{(1-\alpha)A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_t$. These two equations imply that aggregate output can be written as a simple function of aggregate labor and capital, $Y_t = A_t K_t^\alpha N_t^{1-\alpha}$. Hence, the real wage is $w_t = (1-\alpha)\frac{Y_t}{N_t}$ and R_t is denoted as $R_t = \alpha \frac{Y_t}{K_t}$, which turns out to be the aggregate marginal product of capital. Equation (14) can be written as

$$\varepsilon_t^* = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \varepsilon_{t+1}^* \left[\alpha \frac{Y_{t+1}}{K_{t+1}} \frac{Q(\varepsilon_{t+1}^*)}{\varepsilon_{t+1}^*} + 1 - \delta\right]. \tag{34}$$

This equation determines the endogenous cutoff value, ε_t^*, and therefore the optimal level of capital investment and production at the firm level. The left-hand side of the equation is the marginal cost of installing one additional unit of capital, whereas the right-hand side is the expected rate of returns to capital. Also, the effective aggregate investment is given by $\int_0^1 \frac{i(i)}{\varepsilon(i)} di = I_t \frac{\int_{\varepsilon \leq \varepsilon^*} \varepsilon^{-1} dF}{F(\varepsilon_t^*)}$, where the coefficient $\frac{\int_{\varepsilon \leq \varepsilon^*} \varepsilon^{-1} dF}{F(\varepsilon_t^*)}$ measures the marginal efficiency of aggregate investment.

By the law of large numbers, the aggregate capital investment is given by

$$I_t = [\alpha Y_t + q_t \Omega_t] F(\varepsilon_t^*). \tag{35}$$

Notice that the tulips affect aggregate capital accumulation through two channels. First, they directly increase all firms' cash flows through the liquidation value, q_t. Second, they influence the cutoff value, ε^*, thus affecting the number of active firms (that make fixed investments) along the extensive margin and, consequently, the marginal efficiency of aggregate investment. The last channel plays a critical role in our model's dynamics but is absent in the models of Kiyotaki and Moore (2008) and Kocherlakota (2009).

2.4 General Equilibrium

To close the model, we add a standard representative household that solves

$$\max \sum_{t=0}^{\infty} \beta^t \left(\log(C_t) - A_n \frac{N_t^{1+\gamma_n}}{1+\gamma_n}\right)$$

subject to

$$C_t \leq w_t N_t + D_t, \tag{36}$$

where

$$D_t = \int_0^1 d_t(i)di = R_t K_t - I_t + f(\Omega_t - \Omega_{t+1}) - \zeta \Omega_{t+1}, \tag{37}$$

where the term $f(\Omega_t - \Omega_{t+1})$ implies that $(\Omega_t - \Omega_{t+1})$ tulips are retired from circulation and each is transformed into f units of consumption goods, and the term $\zeta \Omega_{t+1}$ implies that Ω_{t+1} tulips are carried into the next period and each incurs a storage cost ζ. Notice that if a tulip never retires from circulation, then $\Omega_t = 1$ for all $t \geq 0$; and if tulips are never wanted by firms as a store of value, then $\Omega_0 = 1$ and $\Omega_t = 0$ for all

$t > 0$. The first-order conditions for the household are summarized by $\frac{1}{C_t} w_t = A_n N_t^{\gamma_n}$, and the household's resource constraint implies $C_t = (1-\alpha) Y_t + D_t = Y_t - I_t + f(\Omega_t - \Omega_{t+1}) - \zeta\Omega_{t+1}$.

To sum up, the equilibrium paths of the model, $\{C_t, I_t, N_t, Y_t, K_{t+1}, q_t, \varepsilon_t^*, \Omega_{t+1}\}$, are fully characterized by the following system of eight nonlinear difference equations:

$$Y_t = A_t {K_t}^{\alpha} N_t^{1-\alpha} \tag{38}$$

$$C_t + I_t = Y_t + f(\Omega_t - \Omega_{t+1}) - \zeta\Omega_{t+1} \tag{39}$$

$$(1-\alpha)\frac{Y_t}{C_t} = A_n N_t^{1+\gamma_n} \tag{40}$$

$$\frac{q_t + \zeta}{C_t} = \beta E_t \left\{ \frac{q_{t+1}}{C_{t+1}} Q_{t+1} \right\} + \frac{\phi_t^* \left[1 - F(\varepsilon_t^*)\right]}{C_t} \tag{41}$$

$$I_t = \left[\alpha Y_t + q_t \Omega_t\right] F(\varepsilon^*) \tag{42}$$

$$\frac{\varepsilon_t^*}{C_t} = \beta E_t \left\{ \frac{\varepsilon_{t+1}^*}{C_{t+1}} \left[\alpha \frac{Y_{t+1}}{K_{t+1}} \frac{Q_{t+1}}{\varepsilon_{t+1}^*} + 1 - \delta \right] \right\} \tag{43}$$

$$K_{t+1} = (1-\delta) K_t + I_t \frac{\int_{\varepsilon \leq \varepsilon^*} \varepsilon^{-1} dF}{F(\varepsilon_t^*)} \tag{44}$$

$$\Omega_{t+1} \phi_t^* = 0. \tag{45}$$

Two steady states are possible in the model. In one steady state, tulips are never consumed and their market price is greater than their fundamental value—namely, $q_t > f$, $\Omega_t = 1$, and $\phi^* = 0$. In the other steady state, the market price equals the fundamental value and tulips are not circulated among firms—namely, $q = f$, $\Omega_t = 0$, and $\phi^* \geq 0$. We are now ready to characterize conditions under which particular steady state(s) may arise in general equilibrium.

Steady State A: $q \geq f$, $\Omega = 1$, and $\phi^* = 0$. In this steady state, equation (39) and equations (41) through (44) become

$$C + I = Y - \zeta \tag{46}$$

$$q + \zeta = \beta q Q(\varepsilon^*) \tag{47}$$

$$I = (\alpha Y + q) F(\varepsilon^*) \tag{48}$$

$$1 - \beta(1-\delta) = \beta\alpha \frac{Y}{K} \frac{Q(\varepsilon^*)}{\varepsilon^*} \tag{49}$$

$$\delta K = I \frac{\int_{\varepsilon \leq \varepsilon^*} \varepsilon^{-1} dF}{F(\varepsilon^*)}. \tag{50}$$

Equations (49) and (50) solve for the capital-to-output ratio ($\frac{K}{Y}$) and the saving rate ($\frac{I}{Y}$) given the cutoff ε^*. Equation (46) then determines the consumption-to-output ratio. Equation (40) and the production function then determine the levels of aggregate output and employment and hence the levels of consumption and investment. Equations (47) and (48) then jointly determine the cutoff (ε^*) and the asset price (q). Notice equation (47) suggests $Q(\varepsilon^*) > 1$; hence, an interior solution for the cutoff $\varepsilon^* \in [\underline{\varepsilon}, \bar{\varepsilon}]$ exists provided that the storage cost ζ is not too high.[10]

Steady State B: $q = f$, $\Omega = 0$, and $\phi^* \geq 0$. In this steady state, no firm will invest in tulip. Denoting

[10] If ζ is too high, then equation (47) becomes an inequality, $q + \zeta > \beta q Q(\bar{\varepsilon})$; hence, no firm will hold tulips.

X_b as the value of a variable X in steady state B, the first-order conditions (39) and (41) through (44) become

$$C_b + I_b = Y_b \tag{51}$$

$$f + \zeta = \beta f Q(\varepsilon_b^*) + \phi^* \left[1 - F(\varepsilon_b^*)\right] \tag{52}$$

$$I_b = \alpha Y_b F(\varepsilon_b^*) \tag{53}$$

$$1 - \beta(1-\delta) = \beta\alpha \frac{Y_b}{K_b} \frac{Q(\varepsilon_b^*)}{\varepsilon_b^*} \tag{54}$$

$$\delta K_b = I_b \frac{\int_{\varepsilon \le \varepsilon_b^*} \varepsilon^{-1} dF}{F(\varepsilon_b^*)}. \tag{55}$$

Equations (53) through (55) imply

$$1 - \beta(1-\delta) = \beta\delta \frac{Q(\varepsilon_b^*)}{\varepsilon_b^* \int_{\varepsilon \le \varepsilon_b^*} \varepsilon^{-1} dF}, \tag{56}$$

which solves for the cutoff ε_b^*. Given ε_b^*, we can then solve for $\{Y_b, C_b, K_b, N_b, \phi^*\}$. Notice that to ensure the condition $\phi^* \ge 0$ holds, equation (52) implies $\zeta \ge f\left[\beta Q(\varepsilon_b^*) - 1\right]$; that is, the storage cost must be large enough to have a no-bubble equilibrium. This suggests that when $\zeta = 0$ and $f > 0$, steady state B may not be possible (see Proposition 1 below).

We call steady state A a bubble equilibrium and steady state B a fundamental equilibrium. The following propositions characterize the nature of the equilibria and conditions for each equilibrium to arise.

Proposition 1 *There are fewer firms investing in fixed capital in steady state A than in steady state B, and the marginal efficiency of aggregate investment is higher in a bubble equilibrium than in a no-bubble equilibrium. As a result, the aggregate capital stock-to-output ratio is higher in the bubble equilibrium than in the no-bubble equilibrium.*

Proof. *In steady state B, by equation (56), we have*

$$1 - \beta(1-\delta) = \beta\delta \frac{Q(\varepsilon_b^*)}{\varepsilon_b^* \int_{\varepsilon \le \varepsilon_b^*} \varepsilon^{-1} dF} = \beta\delta \left[1 + \frac{1 - F(\varepsilon_b^*)}{\varepsilon_b^* \int_{\varepsilon \le \varepsilon_b^*} \varepsilon^{-1} dF}\right]; \tag{57}$$

whereas in steady state A, we have $\frac{I}{Y} > \alpha F(\varepsilon^*)$ by equation (48); hence, equations (49) and (50) imply

$$1 - \beta(1-\delta) = \beta\delta\alpha \frac{Y}{I} \frac{Q(\varepsilon^*) F(\varepsilon^*)}{\varepsilon^* \int_{\varepsilon \le \varepsilon^*} \varepsilon^{-1} dF} < \beta\delta \left[1 + \frac{1 - F(\varepsilon^*)}{\varepsilon^* \int_{\varepsilon \le \varepsilon^*} \varepsilon^{-1} dF}\right]. \tag{58}$$

*Equations **(57)** and **(58)** then imply*

$$\frac{1 - F(\varepsilon^*)}{\varepsilon^* \int_{\varepsilon \le \varepsilon^*} \varepsilon^{-1} dF} > \frac{1 - F(\varepsilon_b^*)}{\varepsilon_b^* \int_{\varepsilon \le \varepsilon_b^*} \varepsilon^{-1} dF},$$

*or $\varepsilon^* < \varepsilon_b^*$. That is, there are fewer firms investing in fixed capital in the bubble equilibrium because the optimal cutoff ε^* is lower. The marginal efficiency of aggregate investment is given by $\frac{\int_{\varepsilon \le \varepsilon^*} \varepsilon^{-1} dF}{F(\varepsilon^*)}$ in equatio**n** (44), which is decreasing in ε^*. Also, the capital-**to**-output ratio is decreasing in the cutoff by equations (49) and (54)**;** thus, we have $\frac{K}{Y} > \frac{K_b}{Y_b}$.* ■

Proposition 2 *(i) If $\zeta = 0$ and $f = 0$, then $q_t = 0$ and $\phi^* \geq 0$ for all t is a fundamental equilibrium, and $q > 0$ and $\phi^* = 0$ is a possible bubble equilibrium. (ii) If $\zeta = 0$ and $f > 0$, then there exists at most one equilibrium with $q \geq f$ and $\phi^* = 0$. (iii) If $\zeta > 0$ and $f \geq 0$, then $q = f$ and $\phi^* \geq 0$ is **a f**undamental equilibrium; and $q > f$ and $\phi^* = 0$ is a possible bubble equilibrium.*

Proof. We prove the proposition case by case.

(i) Suppose $\zeta = 0$ and $f = 0$. Then equation (52) is clearly satisfied if $\phi_t^* = 0$. In such a case, $\Omega = 0$ and $q = 0$ is an equilibrium because no firm has an incentive to deviate by holding tulips when the liquidation value of a tulip is zero. Hence, a fundamental equilibrium with $q = f = 0$ exists. To prove the bubble equilibrium, suppose $q > 0$. Equation (47) implies $Q(\varepsilon^*) = \frac{1}{\beta}$, which solves for the cutoff value ε^* as an interior point in the support $\varepsilon \in [\underline{\varepsilon}, \bar{\varepsilon}]$ because $\frac{1}{\beta} > 1$, provided the upper bound of the support $\bar{\varepsilon}$ is large enough. Given this, we have $0 < F(\varepsilon^*) < 1$. Equation (49) implies the capital-to-output ratio,

$$\frac{K}{Y} = \frac{\alpha}{1 - \beta(1-\delta)} \frac{1}{\varepsilon^*}. \tag{59}$$

This ratio and equation (50) give the household's saving rate,

$$\frac{I}{Y} = \frac{\alpha\delta}{1 - \beta(1-\delta)} \frac{F}{Q - 1 + F}, \tag{60}$$

where $Q - 1 + F = \varepsilon^* \int_{\varepsilon \leq \varepsilon^*} \varepsilon^{-1} dF$. Equation (48) then implies the asset value-to-output ratio as a function of the saving rate,

$$\frac{q}{Y} = \left[\frac{I}{Y} \frac{1}{F(\varepsilon^*)} - \alpha\right]. \tag{61}$$

To ensure $q > 0$ (i.e., $\frac{q}{Y} > 0$), we must have $\frac{I}{Y} > \alpha F$**,** which implies the following restriction on the paramete**rs:**

$$\delta > \left(\beta^{-1} - 1 + F\right)\left(1 - \beta\left(1 - \delta\right)\right). \tag{62}$$

That is, if the household is sufficiently patient (i.e., β close to 1**),** then firms have incentives to hold bubbles with $q > 0$; hence, $\Omega = 1$ and $\phi^* = 0$. (Note that when β is close to 1**,** $Q(\varepsilon^*)$ is also close to 1**;** hence, ε^* is close to its lower bound $\underline{\varepsilon}$ and $F(\underline{\varepsilon}) = 0$.)

(ii) Suppose $\zeta = 0$ and $f > 0$. In this case, $q < f$ is clearly not an equilibrium because the demand for tulips will increase to infinity. So let $q \geq f$. First, steady state B with $q = f$, $\Omega = 0$, and $\phi^* \geq 0$ is not an equilibrium. To see this, suppose firm i deviates from this equilibrium and decides to hold tulips as a store of value. This makes firm i's position better because tulips always have liquidation value $f > 0$ in any period and there is no storage cost; hence, tulips help diversify idiosyncratic risk in capital investment and the demand for tulips will rise. Therefore, all firms have incentives to deviate and q may rise above f. Alternatively, by Proposition 1, we have $Q(\varepsilon_b^*) > Q(\varepsilon^*) = \beta^{-1}$; then by equation (52), we must have $\phi^* = f(1 - \beta Q(\varepsilon_b^*))/[1 - F(\varepsilon_b^*)] < 0$, which is contrary to the requirement $\phi^* \geq 0$. Thus, steady state B can never be an equilibrium and we only need to consider steady state A with $q \geq f$ as a possible equilibrium. In this case, following similar steps in case (i), equation (61) implies that $q \geq f$ is equivalent to the following condition:

$$\frac{\alpha\delta}{1 - \beta(1-\delta)} \frac{1}{\beta^{-1} - 1 + F} - \alpha \geq \frac{f}{Y}. \tag{63}$$

There exists a unique equilibrium whenever this condition is satisfied. For example, condition (63) is satisfied when $\beta \to 1$.

(iii) Suppose $\zeta > 0$ and $f \geq 0$. In this case, $q = f$ and $\phi^* \geq 0$ is an equilibrium if ζ is sufficiently large, because firms do not have incentives to deviate from the fundamental equilibrium by investing in tulips if the storage cost is too high. Now consider $q > f$. Equation (47) implies $\beta Q(\varepsilon^*) = \frac{q+\zeta}{q} \equiv \kappa > 1$. Substituting this definition into equation (49) gives $\frac{K}{Y} = \frac{\alpha}{1-\beta(1-\delta)} \frac{\kappa}{\varepsilon^*}$. Equation (50) gives

$$\frac{I}{Y} = \frac{\alpha\delta}{1-\beta(1-\delta)} \frac{\kappa F}{\beta^{-1}\kappa - 1 + F}. \tag{64}$$

Because equation (48) implies (61), the requirement $q > f$ then implies

$$\frac{\alpha\delta}{1-\beta(1-\delta)} \frac{\kappa}{\beta^{-1}\kappa - 1 + F} - \alpha > \frac{f}{Y}. \tag{65}$$

This condition is easier to satisfy, for example, when $\beta \to 1$ and Y is large enough (e.g., with a large value of TFP). ■

Case (ii) in Proposition 2 states that when $\zeta = 0$ and $f > 0$, if the model's structural parameters are such that $q > f$ is a possible equilibrium, then $q = f$ cannot be an equilibrium. In other words, bubbles will never burst. This suggests that sunspot equilibrium does not exist in the models of Kiyotaki and Moore (2008) and Kocherlakota (2009) if land has intrinsic values with zero or small storage costs. The intuition is that firms will always benefit from using tulips as a store of value to diversify idiosyncratic risks if the liquidation value of tulips is strictly positive ($f > 0$) and the inventory-carrying costs are small.

Also notice that the left-hand side of condition (63) approaches infinity as $\beta \to 1$ (because in this case $Q(\varepsilon^*) \to 1$ and $F(\varepsilon^*) \to 0$); hence, assets with any positive intrinsic values will always carry bubbles as long as agents are sufficiently patient. However, given β, the larger the fundamental value of an asset, the more difficult it is for bubbles to develop because when f is too high, the benefit of using tulips as a store of value does not outweigh the marginal utility of consumption.

Similarly, case (iii) in Proposition 2 (i.e., equation 65) states that the bubble-to-fundamental value ratio, $\frac{q}{f}$, can be made arbitrarily large if β is sufficiently close to 1 and the economy is sufficiently productive (i.e., the output level is sufficiently high due to a high TFP). Conversely, bubbles do not grow on an object if its fundamental value is too high and the economy is unproductive. Case (iii) also indicates that multiple equilibria are possible when $f > 0$ if and only if the storage cost ζ is strictly positive but not too large. A mild storage cost discourages an individual firm to deviate from the no-bubble equilibrium but cannot prevent all firms from deviating simultaneously because the equilibrium bubble price $q > f$, which makes the liquidation value of the bubble asset sufficiently attractive for investing.

3 Systemic Risk and Asset Price Volatility

This section applies a version of the basic model to explain asset price volatility in the U.S. economy by allowing for the possibility for bubbles to burst (as in Kocherlakota, 2009). We introduce multiple assets and stochastic sunspot shocks to affect the probability of bubbles to burst.[11] Although the steps for deriving equilibrium conditions are similar and analogous to those in the basic mode, we detail most of the equations for the sake of completeness and self-containedness.

Assume there is a continuum of types of "tulips" indexed by a spectrum of colors $j \in \mathbf{R}^+$. For simplicity, tulips are assumed to (i) be perfectly storable with no storage costs ($\zeta = 0$), (ii) differ only in their colors

[11] The idea of multiple bubble assets is akin to that in Kareken and Wallace (1981).

(types), and (iii) have no intrinsic values ($f = 0$).[12] Thus, according to Proposition 2, each type of tulip asset can be a bubble with the following property: Its equilibrium price is zero if no firms in the economy expect other firms to invest in it, and the price is strictly positive if all agents expect others to hold it.

In each period a constant measure z of new colors of tulips is born (issued).[13] The supply of each color (variety) of tulips is normalized to 1; hence, each tulip has a unique color. Also, all tulips have the same probability, $\mathbf{p}_t$, to lose their values in each period regardless of color. This assumption captures the concept of systemic risk. The newborn tulips are distributed equally to all agents (firms) as endowments, and issuing (producing) new tulips does not cost any social resources. Let q_t^j denote the price of a tulip (with color) j and $h_{t+1}^j(i)$ the quantity of the tulip j demanded by firm $i \in [0,1]$. The aggregate number (stock) of tulips evolves over time according to the law of motion:

$$\Omega_{t+1} = (1 - \mathbf{p}_t)\Omega_t + z, \tag{66}$$

where Ω is the measure of the stock of tulips in the entire economy. The market clearing condition for each tulip with color j is

$$\int_{i=0}^{1} h_{t+1}^j(i)di = 1. \tag{67}$$

As in the basic model, firms have the same constant returns to scale production technologies and are hit by idiosyncratic cost shocks to the marginal efficiency of investment $\varepsilon(i)$. A firm's problem is to determine a portfolio of tulips to maximize discounted future dividends. Its resource constraint is

$$d_t(i) + i_t(i) + \int_{j\in\Omega_{t+1}} q_t^j h_{t+1}^j(i)dj + w_t n_t(i) \leq A_t k_t(i)^\alpha n_t(i)^{1-\alpha} + \int_{j\in\Omega_t} q_t^j h_t^j(i)\mathbf{1}_t^j dj + \int_{j\in z} q_t^j dj, \tag{68}$$

where w is the real wage, Ω is the set of available colors of tulips, and the index variable $\mathbf{1}_t^j$ satisfies

$$\mathbf{1}_t^j = \begin{cases} 1 & \text{with prob. } 1 - \mathbf{p}_t \\ & \\ 0 & \text{with prob. } \mathbf{p}_t \end{cases}. \tag{69}$$

Namely, each tulip bought in period $t-1$ may lose its value completely with probability $\mathbf{p}_t$ in the beginning of period t. As in the basic model, we impose the following constraints: $i_t(i) \geq 0$, $d_t(i) \geq 0$, and $h_{t+1}^j(i) \geq 0$ for all $j \in \Omega$.

Using the same definition of R as in the basic model, the firm's problem is to solve

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \Lambda_t \left[R_t k_t(i) - i_t(i) + \int_{j\in\Omega_t} q_t^j h_t^j(i)\mathbf{1}_t^j dj + \int_{j\in z} q_t^j dj - \int_{j\in\Omega_{t+1}} q_t^j h_{t+1}^j(i)dj \right], \tag{70}$$

subject to

$$d_t(i) \geq 0 \tag{71}$$

$$h_{t+1}^j(i) \geq 0 \quad \text{for all } j \tag{72}$$

$$i_t(i) \geq 0 \tag{73}$$

[12] These assumptions reduce the number of parameters and simplify our calibration analysis.
[13] We can also allow z to be stochastic.

$$k_{t+1}(i) = (1-\delta)\, k_t(i) + \frac{i_t(i)}{\varepsilon_t(i)}. \tag{74}$$

Let $\left\{\mu(i), \phi_t^j(i), \lambda(i), \pi(i)\right\}$ denote the Lagrangian multipliers of constraints (71) through (73), respectively, the first-order conditio**ns** for $\left\{i_t(i), k_{t+1}(i), h_{t+1}^j(i)\right\}$ are similar to those in the basic model and denoted, respectively, by

$$1 + \mu_t(i) = \frac{\lambda_t(i)}{\varepsilon_t(i)} + \pi_t(i) \tag{75}$$

$$\lambda_t(i) = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{[1+\mu_{t+1}(i)]R_{t+1} + (1-\delta)\lambda_{t+1}(i)\right\} \tag{76}$$

$$[1+\mu_t(i)]\, q_t^j = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ q_{t+1}^j \mathbf{1}_{t+1}^j \left[1+\mu_{t+1}(i)\right]\right\} + \phi_t^j(i), \tag{77}$$

plus the complementary slackness conditions,

$$\pi_t(i) i_t(i) = 0 \tag{78}$$

$$\phi_t^j(i) h_{t+1}^j(i) = 0 \quad \text{for all } j \tag{79}$$

$$[1+\mu_t(i)] \left[R_t k_t(i) - i_t(i) + \int_{j\in\Omega_t} q_t^j h_t^j(i) \mathbf{1}_t^j dj + \int_{j \in z} q_t^j dj - \int_{j\in\Omega_{t+1}} q_t^j h_{t+1}^j(i) dj \right] = 0. \tag{80}$$

3.1 Decision Rules

As in the basic model, the decision rules at the firm level are characterized by a cutoff strategy. The following steps are analogous to those in the basic model. Consider two possibilities:

Case A: $\varepsilon_t(i) \le \varepsilon_t^*$. In this case, the cost of capital investment is low. Suppose $i_t(i) > 0$; accordingly we have $\pi_t(i) = 0$. Equations (75) and (76) imply

$$\varepsilon_t(i)\left[1+\mu_t(i)\right] = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{[1+\mu_{t+1}(i)]R_{t+1} + (1-\delta)\lambda_{t+1}(i)\right\}. \tag{81}$$

Given that $\mu_t(i) \ge 0$, we must have $\varepsilon_t(i) \le \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t}\left\{[1+\mu_{t+1}(i)]R_{t+1} + (1-\delta)\lambda_{t+1}(i)\right\}$, which defines the cutoff valu**e,** ε_t^*:

$$\varepsilon_t^* \equiv \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{[1+\mu_{t+1}(i)]R_{t+1} + (1-\delta)\lambda_{t+1}(i)\right\}. \tag{82}$$

Equation (75) then becomes $1+\mu_t(i) = \frac{\varepsilon_t^*}{\varepsilon_t(i)}$. Hence, whenever $\varepsilon_t(i) < \varepsilon_t^*$, we must have $\mu_t(i) = \frac{\varepsilon_t^*}{\varepsilon_t(i)} - 1 > 0$ and $d_t(i) = 0$. Equation (77) becomes

$$\frac{\varepsilon_t^*}{\varepsilon_t(i)} q_t^j = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ q_{t+1}^j \mathbf{1}_{t+1}^j \left[1+\mu_{t+1}(i)\right]\right\} + \phi_t^j(i). \tag{83}$$

Defin**ing** ϕ_t^* as the cutoff value of $\phi_t^j(i)$ for firms with $\varepsilon_t(i) = \varepsilon_t^*$, equation **(84) im**plies

$$q_t^j = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} \left\{ q_{t+1}^j \mathbf{1}_{t+1}^j \left[1+\mu_{t+1}(i)\right]\right\} + \phi_t^* \tag{84}$$

Given that $\phi_t^* \ge 0$, the fact that $\phi_t^j(i) > \phi_t^*$ under **C**ase A yields $\phi_t^j(i) > 0$. That is, for any $\varepsilon_t(i) < \varepsilon_t^*$, we must have $h_{t+1}^j(i) = 0$ for all $j \in \Omega_{t+1}$ and $i_t(i) = R_t k_t(i) + \int_{j\in\Omega_t} q_t^j h_t^j(i) \mathbf{1}_t^j dj + \int_{j\in z} q_t^j dj$. This suggests that firms opt to liquidate all tulip assets to maximize investment in fixed capital when the cost of fixed investment is low.

Case B: $\varepsilon_t(i) > \varepsilon_t^*$. In this case, the cost of investing in fixed capital is high. Suppose $d_t(i) > 0$ and $\mu_t(i) = 0$. Then equations (75) and (76) and the definition of the cutoff ε^* imply $\pi_t(i) = 1 - \frac{\varepsilon_t^*}{\varepsilon_t(i)} > 0$. Hence, we have $i_t(i) = 0$. In such a case, firms opt not to invest in fixed capital and, instead, pay the shareholders a positive dividend. Because the market clearing condition for each tulip j is $\int h_{t+1}^j(i)di = 1$, we must also have $h_{t+1}^j(i) > 0$ and $\phi_t^j(i) = 0$ for all $j \in \Omega_{t+1}$ under Case B. Thus, equations (77) and (84) then imply the cutoff $\phi_t^* = 0$.

Combining these two cases, the decision rule for capital investment is given by

$$i_t(i) = \begin{cases} R_t k_t(i) + \int_{j \in \Omega_t} q_t^j h_t^j(i) \mathbf{1}_t^j dj + \int_{j \in z} q_t^j dj & \text{if } \varepsilon_t(i) \leq \varepsilon_t^* \\ 0 & \text{if } \varepsilon_t(i) > \varepsilon_t^* \end{cases}. \tag{85}$$

The option value of liquidity is again defined by $Q \equiv E\left[1 + \mu(i)\right] = \int \max\left\{1, \frac{\varepsilon^*}{\varepsilon(i)}\right\} dF(\varepsilon)$. Using equations (83) and (84), the Lagrangian multiplier for the no**nn**egativity constraint (72) is given by

$$\phi_t^j(i) = \begin{cases} \left(\frac{\varepsilon^*}{\varepsilon(i)} - 1\right) q_t^j & \text{if } \varepsilon(i) \leq \varepsilon^* \\ 0 & \text{if } \varepsilon(i) > \varepsilon^* \end{cases}, \tag{86}$$

and the average shadow value of ϕ^j is $\int \phi^j(i)di = q^j \int_{\varepsilon \leq \varepsilon^*} \left(\frac{\varepsilon^*}{\varepsilon} - 1\right) dF(\varepsilon) = q^j (Q - 1)$, which is independent of i but proportional to the tulip's price**,** q^j.

Integrating equation (77) over i and rearranging gives

$$q_t^j = \beta E_t \frac{\Lambda_{t+1}}{\Lambda_t} q_{t+1}^j \mathbf{1}_{t+1}^j Q_{t+1}, \tag{87}$$

where the right-hand side is the expected rate of return to tulip j. This equation shows that if $\mathbf{p}_t = 1$ (i.e., $\mathbf{1}_t^j = 0$ with probability 1), then tulip j's equilibrium price is given by $q_t^j = 0$ for all t because the demand for such an asset is zero when it has no market value in the next period. More importantly, even if $\mathbf{p} < 1$ (e.g., $\mathbf{p} = 0$), $q_t^j = 0$ for all t is still an equilibrium because no firms will hold tulip j if they do not expect others to hold it. In the next section, we define restrictions on the value of $\mathbf{p}$ so that $q_t^j > 0$ constitutes a bubble equilibrium.

3.2 Aggregation and General Equilibrium

As in the basic model, we have at the aggregate level $N_t = \left[\frac{(1-\alpha)A_t}{w_t}\right]^{\frac{1}{\alpha}} K_t$, $Y_t = A_t \left[\frac{(1-\alpha)A_t}{w_t}\right]^{\frac{1-\alpha}{\alpha}} K_t$, $Y_t = A_t K_t^\alpha N_t^{1-\alpha}$, $w_t = (1-\alpha)\frac{Y_t}{N_t}$, and $R_t = \alpha \frac{Y_t}{K_t}$. Consider a symmetric equilibrium[14] where

$$q_t^j \mathbf{1}_t^j = \begin{cases} q_t & \text{with prob. } 1 - \mathbf{p}_t \\ 0 & \text{with prob. } \mathbf{p}_t \end{cases}, \tag{88}$$

[14] A symmetric equilibrium exists because of arbitrage across tulips of different colors.

where q_t is the price of tulips of all colors. Define $x_t(i) \equiv \int_{j\in\Omega_t} q_t^j h_t^j(i)\mathbf{1}_t^j dj + \int_{j\in z} q_t^j dj$ and $X_t \equiv \int_0^1 x(i)di$. By the law of large numbers, we have $X_t = (1-\mathbf{p}_t)\,\Omega_t q_t + zq_t$. Hence, aggregate capital investment is given by $I_t = (\alpha Y_t + [(1-\mathbf{p}_t)\,\Omega_t + z]\,q_t)\,F(\varepsilon_t^*)$.

The household remains the same as in the basic model except the aggregate dividend is now given by $D_t = \int_0^1 d_t(i)di = R_t K_t - I_t$ because new tulips are constantly being born so that $\Omega_{t+1} = (1-\mathbf{p}_t)\Omega_t + z$. The first-order conditions of the household are the same as before; however, the household's resource constraint now becomes $C_t = (1-\alpha)Y_t + D_t = Y_t - I_t$. The equilibrium paths of the model, $\{C, I, N, Y, K', q, \varepsilon^*, \Omega'\}$, are fully characterized by the following system of eight nonlinear difference equations:

$$Y_t = A_t {K_t}^{\alpha} N_t^{1-\alpha} \tag{89}$$

$$C_t + I_t = Y_t \tag{90}$$

$$(1-\alpha)\frac{Y_t}{C_t} = A_n N_t^{1+\gamma_n} \tag{91}$$

$$\Omega_{t+1} = (1-\mathbf{p}_t)\Omega_t + z \tag{92}$$

$$\frac{q_t}{C_t} = \beta E_t \left\{ \frac{q_{t+1}}{C_{t+1}}(1-\mathbf{p}_{t+1})Q_{t+1} \right\} \tag{93}$$

$$I_t = (\alpha Y_t + [(1-\mathbf{p}_t)\,\Omega_t + z]\,q_t)\,F(\varepsilon^*) \tag{94}$$

$$\frac{\varepsilon_t^*}{C_t} = \beta E_t \left\{ \frac{\varepsilon_{t+1}^*}{C_{t+1}} \left[\alpha \frac{Y_{t+1}}{K_{t+1}} \frac{Q_{t+1}}{\varepsilon_{t+1}^*} + 1 - \delta \right] \right\} \tag{95}$$

$$K_{t+1} = (1-\delta)\,K_t + I_t \frac{\int_{\varepsilon \le \varepsilon^*} \varepsilon^{-1} dF}{F(\varepsilon_t^*)}. \tag{96}$$

The model has a unique bubble steady state and the equilibrium dynamics of the model are solved by log-linearizing the above system of equations around the steady state.

3.3 Stationary Sunspot Equilibria

We call tulip j a bubble if $q_t^j > 0$. When the bubble bursts, we have $q_t^j = 0$. By arbitrage, after a bubble bursts, its value must become zero permanently, otherwise people may opt to hold it indefinitely based on speculation. In each period there are fraction $\mathbf{p}_t$ of the bubbles that burst and a measure of z new bubbles that are born. Changes in $\mathbf{p}_t$ are driven by sunspots (i.e., the mood of the population), which can follow any stochastic processes. In what follows, we focus on stationary sunspot equilibria with positive and bounded asset prices ($q_t > 0$ for all t).[15]

The sunspot equilibrium condition, $q > 0$, puts some restrictions on the values of $\mathbf{p}$. Given that $q > 0$, equation (93) implies $1 = \beta(1-\mathbf{p})Q(\varepsilon^*)$ in the steady state. Equation (95) implies $Q(\varepsilon^*) = \frac{1-\beta(1-\delta)}{\beta R}\varepsilon^*$. Together we have

$$1 - \mathbf{p} = \frac{R}{1-\beta\,(1-\delta)} \frac{1}{\varepsilon^*}. \tag{97}$$

This equation determines the cutoff value $\varepsilon^*(\mathbf{p})$ as a function of $\mathbf{p}$. An interior solution for the cutoff requires

[15] A nonstationary explosive bubble occurs when the deterministic growth rate of q_t is greater than zero. In this paper, we assume there is no deterministic growth in asset prices in a bubble equilibrium; namely, $\frac{q_t^j}{q_{t-1}^j} = 1$ in the steady state. Relaxing this assumption is straightforward. If asset prices have deterministic growth rates $r^j \ge 0$, then by arbitrage we must have the expected rate of return, $(1-\alpha^j)(1+r^j)$, equalized across tulips, where α^j is the probability for tulip j to burst.

$\varepsilon^*(\mathbf{p}) \in (\underline{\varepsilon}, \bar{\varepsilon})$, where $\underline{\varepsilon}, \bar{\varepsilon} > 0$ are the lower and upper bounds of the support for $\varepsilon_t(i)$, respectively. Hence, we must have

$$\frac{R}{1-\beta(1-\delta)}\frac{1}{\underline{\varepsilon}} > (1-\mathbf{p}) > \frac{R}{1-\beta(1-\delta)}\frac{1}{\bar{\varepsilon}}. \tag{98}$$

Notice that $\frac{R}{1-\beta(1-\delta)} = \sum_{j=0}^{\infty} \beta^j (1-\delta)^j \alpha \frac{Y}{K}$ is the present value of the marginal products of capital and $\frac{1}{\varepsilon}$ is the marginal efficiency of investment. Hence, the conditions in equation (98) state that the real expected rate of return to tulips (i.e., the survival probability of a speculative bubble) must be comparable to that of capital investment (Tobin's q) **t**o induce people to hold both capital and bubbles.

The wider the support of the idiosyncratic shocks, the larger the permissible region for the value of $\mathbf{p}$. This simply restates the finding that idiosyncratic uncertainty in the expected rate of returns to capital investment (or Tobin's q) is the fundamental reason for people to invest in bubbles. When such idiosyncratic assessments of risks converge (e.g., $\underline{\varepsilon} = \bar{\varepsilon}$), it is virtually impossible for bubbles to arise (i.e., the measure of sunspot equilibria becomes zero).

In the steady state, equations (93) through (96) become

$$1 = \beta(1-\mathbf{p})Q(\varepsilon^*) \tag{99}$$

$$I = (\alpha Y + \Omega q) F(\varepsilon^*) \tag{100}$$

$$1 - \beta(1-\delta) = \beta\alpha\frac{Y}{K}\frac{Q(\varepsilon^*)}{\varepsilon^*} \tag{101}$$

$$\delta K = I\frac{\int_{\varepsilon \le \varepsilon^*} \varepsilon^{-1} dF}{F(\varepsilon^*)}. \tag{102}$$

Equation (99) solves for the cutoff value ε^*. Equation (101) implies the capital-to-output ratio, $\frac{K}{Y} = \frac{\beta\alpha}{1-\beta(1-\delta)}\frac{Q}{\varepsilon^*}$. Equations (102) and (101) give the household's saving rate, $\frac{I}{Y} = \frac{\alpha\beta\delta}{1-\beta(1-\delta)}\frac{FQ}{Q-1+F}$, where $Q - 1 + F = \varepsilon^* \int_{\varepsilon \le \varepsilon^*} \varepsilon^{-1} dF$. Equation (100) implies the asset-to-output ratio as a function of the saving rate, $\frac{\Omega q}{Y} = \frac{1}{F(\varepsilon^*)}\frac{I}{Y} - \alpha$. As in the basic model, to ensure $\frac{\Omega q}{Y} > 0$, we must have $\frac{I}{Y} > \alpha F$, which implies $\delta > \left(\beta^{-1} - 1 + F\right)(1 - \beta(1-\delta))$.

3.4 Calibration and Impulse Responses

Assume that $\varepsilon(i)$ follows the Pareto distribution, $F(\varepsilon) = 1 - \varepsilon^{-\theta}$, with the shape parameter $\theta = 1.5$ and the support $(1, \infty)$.[16] With this distribution, we have $Q = \frac{\theta}{1+\theta}\varepsilon^* + \frac{1}{1+\theta}\varepsilon^{*-\theta}$. We normalize the steady-state values $z = 1$ and $A = 1$ **a**nd calibrate the structural parameters of the model as follows: The time period is a quarter, the capital's income share $\alpha = 0.4$, the time-discounting factor $\beta = 0.99$, the capital depreciation rate $\delta = 0.025$, and the inverse elasticity of labor supply $\gamma_n = 0$ (indivisible labor).

The driving processes of the model are assumed to follow AR(1) processes,

$$\ln \mathbf{p}_t = \rho_{\mathbf{p}} \ln \mathbf{p}_{t-1} + (1 - \rho_{\mathbf{p}}) \ln \bar{\mathbf{p}} + \varepsilon_{\mathbf{p}t} \tag{103}$$

$$\ln A_t = \rho_A \ln A_{t-1} + \varepsilon_{At}, \tag{104}$$

where the steady-state probability for bubbles to burst is set to $\bar{\mathbf{p}} = 0.1$. To show the potential of the model in explaining asset price volatility, Figure 2 plots the impulse responses of the output and tulip price to a

[16] The results are not sensitive to the values of θ. For example, with $\theta = 3$ we obtain similar results.

1 percent decrease in productivity ε_{At} and compares them with a 10 percent increase in the probability for bubbles to burst. A positive shock to $\varepsilon_{\mathbf{p}}$ is akin to a financial crisis because it implies a higher systemic financial risk. We set $\rho_{\mathbf{p}} = \rho_A = 0.9$ in the impulse responses.

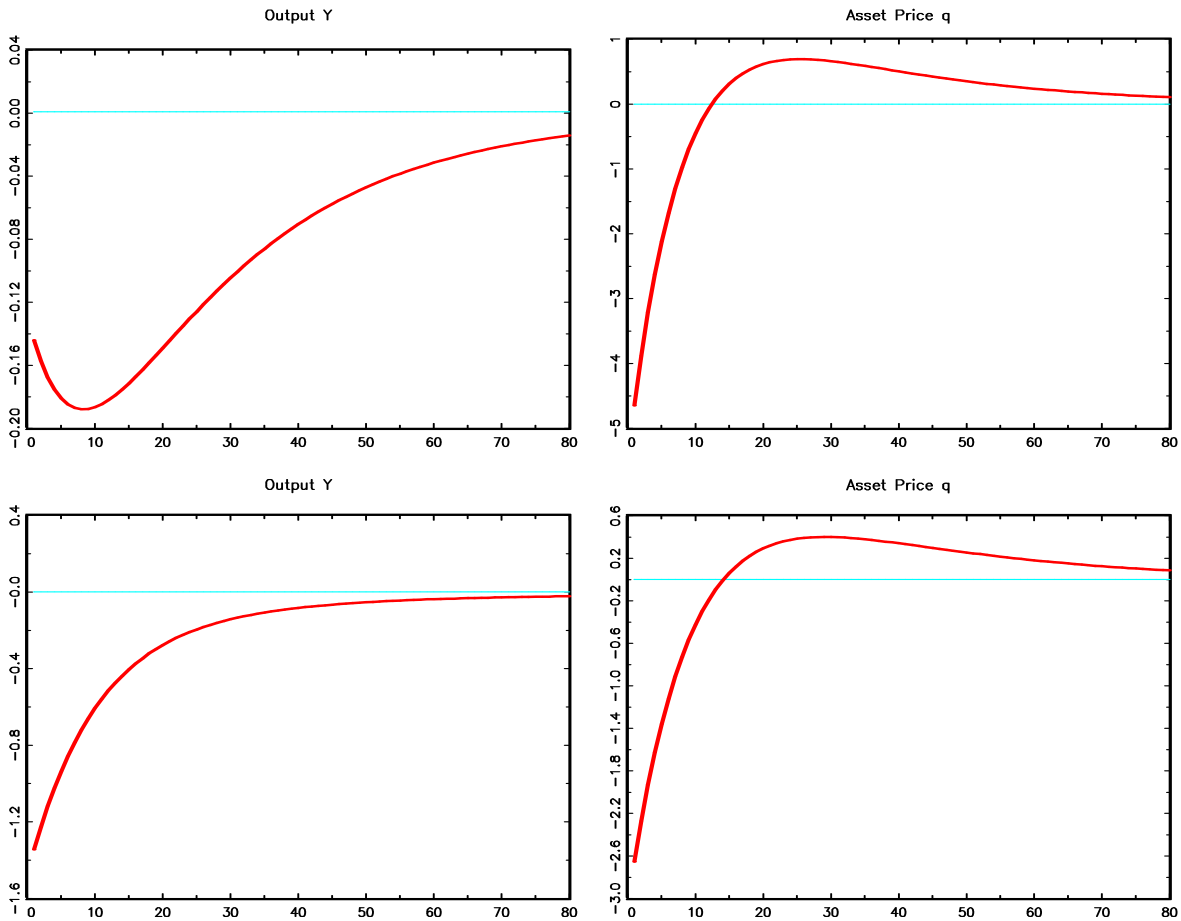

Figure 2. Impulse reponses to probably shock (top windows) and productivity shock (bottom windows).

The top windows in Figure 2 show that an increase in the bubbles' probability to burst generates a recession in aggregate output (upper-left window) and a dramatic drop in asset prices (upper-right window). When the perceived rate of return to tulips decreases (or the financial risk increases), agents rationally decrease their demand for tulips, leading to a sharp fall in asset price. Because tulip assets provide liquidity for firms, "panic" sales of tulips reduce firms' net worth and working capital, leading to a U-shaped decline in output, employment, and capital investment. Such a hump-shaped output dynamics suggest that asset price movements lead the business cycle and provide a key litmus test of business cycle models (see Cogley and Nason, 1995). Our model passes this test with flying colors in this dimension. More importantly, asset prices are far more volatile than the fundamentals. For example, the initial drop in asset prices is more than 32 times larger than that in output, resembling a typical stock market crash. In contrast, an adverse shock to aggregate productivity generates a fall in asset prices (lower-right window) that is only twice as large as the fall in output (lower-left window). Hence, to understand the excessive asset market volatilities, shocks to the probability of the burst of asset bubbles (i.e., systemic financial risk) are essential.

To test whether the model has the potential to match the U.S. time-series data quantitatively, we calibrate the two driving processes $\{A_t, \mathbf{p}_t\}$ using the Solow residuals and the S&P 500 price index (normalized by the GDP deflator) from the U.S. economy (1947:1–2009:1), respectively. For each time series, we apply the Hodrick-Prescott filter on the logged series and estimate an univariate AR(1) model to obtain the coefficients $\{\rho_{\mathbf{p}}, \rho_A\}$. The covariance matrix of the two residuals from the AR(1) models is then used to calibrate $\{\varepsilon_{\mathbf{p}t}, \varepsilon_{At}\}$. The results are as follows: $\rho_{\mathbf{p}} = 0.85$, $\rho_A = 0.75$, $\sigma_{\varepsilon_{\mathbf{p}}} = 0.056$, $\sigma_{\varepsilon_A} = 0.00625$, and $corr(\varepsilon_{\mathbf{p}}, \varepsilon_A) = -0.3$. A negative correlation suggests that it is less likely for bubbles to burst when productivity is high. The data show that the innovations in stock prices are about 9 times more volatile than the innovations in productivity. Based on these values, we reset $\bar{\mathbf{p}} = 0.15$ so that the implied relative volatility of asset price to output is in line with the data. We generate $10,000$ observations from the model and estimate the model's moments. Given the large sample size, the standard errors are small and they are thus not reported. Table 1 reports the predicted second moments of the model and their counterparts in the data.[17]

Table 1. Selected Second Moments

	std σ_x					relative std $\frac{\sigma_x}{\sigma_y}$				$corr(x,y)$				$corr(x_t, x_{t-1})$				
x	y	c	i	n	q	c	i	n	q	c	i	n	q	y	c	i	n	q
Data	.017	.010	.050	.016	.099	.60	2.98	.98	5.93	.78	.77	.83	.46	.83	.81	.84	.90	.80
Model	.012	.009	.025	.005	.068	.74	2.12	.42	5.76	.93	.92	.75	.35	.79	.83	.80	.83	.80

According to the table, the model's predictions are broadly consistent with the U.S. data. For example, in terms of standard deviations, the model is able to explain about 70 percent of output fluctuations and 70 percent of stock price movements in the data. In terms of relative volatilities with respect to output, the model predicts that consumption is about 25 percent less volatile, investment about 2 times more volatile, and asset price nearly 6 times more volatile than output; these predictions are broadly consistent with the U.S. economy. In the data, the correlation between stock prices and output at the business cycle frequencies is 0.46; this value is 0.35 in the model, qualitatively matching the data. The model also can generate strong autocorrelations in output, consumption, investment, labor, and asset prices that are very close to the data. The gap between model and data is most significant in the relative volatility of employment to output; the model significantly underestimated the volatility of employment even with indivisible labor.

We can simulate a tulip bubble using the model. For example, assuming the time period to be a quarter and letting the probability of bubble to burst follow a moving average process,

$$\mathbf{p}_t = \bar{\mathbf{p}} + \sum_{j=0}^{T-1} \alpha_j \varepsilon_{t-j},$$

where ε is zero mean i.i.d. innovations. Suppose $T = 8$, $\bar{\mathbf{p}} = 0.1$, and the probability weight vector $\boldsymbol{\alpha} = \frac{1}{100}[1, 2, 1, 1, 0, -1, -1, -2]$. The simulated tulip bubble is graphed in Figure 3. The larger the value of $\bar{\mathbf{p}}$, the larger the bubble will be. The vector $\boldsymbol{\alpha}$ has zero mean an**d** determines the shape of the bubble. The intuition behind the values of $\boldsymbol{\alpha}$ is as follows: Because agents are forward looking, they react to good financial news by buying tulips now when they perceive that the probability of the bubbles to **burst will** be lower several periods from now. Thus, tulip prices would increase immediately. To prevent a big jump in the current tulip prices, there must be enough bad news today so that investors are cautious in entering the tulip market. This is why $\boldsymbol{\alpha}$ takes positive values initially so that the bubble only grows slowly and gradually.

[17] The data are quarterly and include real GDP (y), nondurable goods consumption (c), total fixed investment (i), total private employment by establishment survey, and the S&P 500 price index normalized by the GDP deflator. The sample covers the period 1947:1–2009:1.

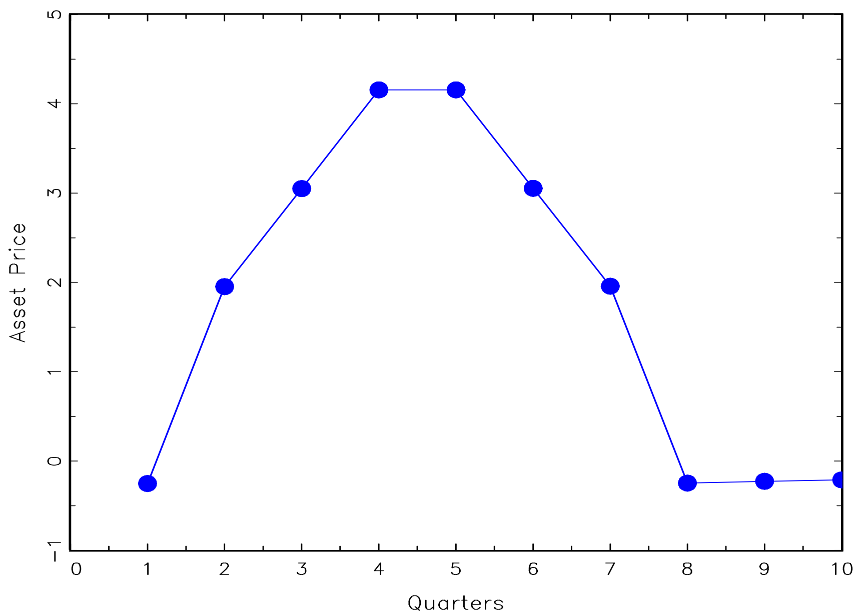

Figure 3. Simulated tulip bubble.

4 Conclusion

This paper provides an infinite-horizon DSGE model with heterogeneous agents to explain asset price volatility. It characterizes conditions under which bubbles with market values exceeding their fundamental values may arise. It is shown that rational agents are willing to invest in such bubbles despite their positive probability to burst and that changes in the perceived systemic risk in the asset market can trigger boom-bust cycles and asset price collapse. Calibration exercises confirm that the model has the potential to quantitatively explain the U.S. business cycle and asset price volatility. As potential research topics, it would be interesting to consider welfare analysis and optimal policies in a bubble economy as in Kiyotaki and Moore (2008) and Kocherlakota (2009) and to consider bubbles with nonstationary prices. We leave these issues to future research.

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