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Capital Misallocation and Aggregate Factor Productivity*

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Abstract

We propose a sectoral–shift theory of aggregate factor productivity for a class of economies with AK technologies, limited loan enforcement, a constant production possibilities frontier, and finitely many sectors producing the same good. Both the growth rate and total factor productivity in these economies respond to random and persistent endogenous fluctuations in the sectoral distribution of physical capital which, in turn, responds to persistent and reversible exogenous shifts in relative sector productivities. Surplus capital from less productive sectors is lent to more productive ones in the form of secured collateral loans, as in Kiyotaki–Moore (1997), and also as unsecured reputational loans suggested in Bulow–Rogoff (1989). Endogenous debt limits slow down capital reallocation, preventing the equalization of risk–adjusted equity yields across sectors. Economy–wide factor productivity and the aggregate growth rate are both negatively correlated with the dispersion of sectoral rates of return, sectoral TFP and sectoral growth rates. If sector productivities follow a symmetric two–state Markov process, many of our economies converge to a limit cycle alternating between mild expansions and abrupt contractions. We also find highly periodic and volatile limit cycles in economies with small amounts of collateral.

JEL classification: D90, E32, O47
Keywords: TFP, misallocation, sectoral shocks, collateral, reputation

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1 Introduction

National income accounting exercises conducted by Klenow and Rodriguez-Clare (1997), Hall and Jones (1999) and Chari, Kehoe, and McGrattan (2007) unanimously conclude that total factor productivity is of cardinal importance for both long-run growth and business cycles, including economic depressions. Klenow and Rodriguez-Clare, for example, find that at least 50% of the variation in output per worker in a sample of over 40 countries is attributable to differences in TFP. There is less agreement about the source of productivity differentials. Suggestions range widely from differences in broadly defined social infrastructure advocated by Hall and Jones, to technology adoption barriers proposed by Parente and Prescott (1999), to the labor market frictions studied in Lagos (2006).

This paper is a theoretical investigation of how credit market frictions limit capital mobility and slow down the movement of resources from temporarily less to temporarily more productive sectors, and more generally, from temporarily low to temporarily high valuations. We are pushed in this direction by much evidence connecting poor economic performance with capital misallocation. Chari et al. (2007) find that financial frictions, defined as "efficiency wedges" which distort the allocation of intermediate inputs among firms,[1] account for 60-80% of the US output drop in the 1929-1933 depression and the 1979-1982 recession, and also for 73% of the variance in detrended U.S. output from 1959 to 2004. Eisfeldt and Rampini (2006) point out that capital reallocation among U.S. firms–defined as sales and acquisition of property, plant and equipment–makes up nearly 25% of total investment on average. Finally, there are strong indications that macroeconomic volatility is connected with the dispersion of both sectoral productivities and sectoral rates of return on capital.[2]

[1] Intermediate inputs are misallocated because they are purchased on credit by producers who face different borrowing costs. Interestingly, Chari et al. find very little explanatory power in "investment wedges", that is, in time–varying distortions to the economy–wide cost of capital.

[2] See Eisfeldt and Rampini (2006) on the countercyclical dispersion of capital productivity among firms and on the sectoral dispersion of TFP; Loungani, Rush, and Tave (1990) and Brainard and Cutler (1993) on the countercyclical dispersion of stock market returns across sectors; Diebold and Yilmaz (2008) on the correlation between the volatilities of stock market returns and GDP growth

Lilien (1982) was an early advocate of the importance of sectoral shocks for overall economic activity in an empirical study that connected the aggregate unemployment rate with the cross-sectional dispersion of sectoral employment. More recently, Phelan and Trejos (2000) argue in a model with labor–search frictions that sectoral reallocations are quantitatively important for business cycle dynamics. This paper puts Lilien's idea to work in a growth model with *financial frictions* and looks at the consequences for productivity and capital accumulation. It is this emphasis on sectoral shocks and productivity which separates our work from earlier literature on financial frictions as a cause of macroeconomic volatility.[3]

We describe sectoral shifts as idiosyncratic technology shocks in a class of simple economies populated by identical infinitely-lived households and consisting of finitely many sectors that produce the same consumption good. Capital is the only input in production which means that we focus on the misallocation of investment and ignore potentially larger problems stemming from imperfectly functioning labor markets. Sectoral technologies are assumed to be AK with random idiosyncratic productivities and a constant aggregate production possibility frontier, that is, a fixed value for the maximal idiosyncratic productivity. We ignore declining and expanding industries, assuming instead that all sectoral shocks are temporary and reversible.

An ideal economy of this type without any financial frictions would exploit its unchanging aggregate production possiblities to the fullest by moving all physical capital instantly to the most productive sector, and delivering to its population a constantly growing stream of aggregate output and individual consumption. In what follows, surplus capital from less productive sectors is in the form of collateral loans, as suggested by Kiyotaki and Moore (1997), secured by an exogenous fraction $\lambda \in [0, 1]$ of the borrower's total resources, and also in the form unsecured reputational loans, as in Bulow and Rogoff (1989), which punish defaulters with perpetual exclusion from future borrowing. Both types of loans require endogenous debt limits which rule out default when asset markets are complete. When these limits bind,

in a sample of 40 countries.

[3]Prominent examples are Kiyotaki and Moore (1997) on collateral constraints as a propagator and amplifier of aggregate technology shock; Matsuyama (2007) on the interplay between borrower net wealth, debt limits and investment; and Aghion, Banerjee, and Piketty (1999) on how restricted participation in credit markets contributes to macroeconomic volatility.

they slow down capital reallocation and prevent rates of return on capital from reaching equality across all sectors.

Generous debt limits are typical outcomes for economies with abundant collateral, that is, a relatively high value of λ, and with a shared belief among debtors that substantial lines of unsecured credit will continue to be available in the future. Any deterioration in the amount of collateral or in the expected flow of future unsecured credit will tighten *current* debt limits and impede the flow of capital from lower to higher valuations.

Low capital mobility tends to spell trouble in the class of economies studied in this paper. At one extreme, high values of the collateral parameter λ guarantee perfect capital mobility which, in turn, ensures a unique, rapid and socially desirable balanced growth path. At the other extreme, complete financial autarky, that is, a combination of no collateral and no unsecured lending, leads to macroeconomic disaster. The unique outcome in this case is slow, inefficient and highly volatile growth with strong history dependence.

In between the two extremes of perfect mobility and no mobility lie the relevant cases of modest collateral and unsecured lending. Secured lending by itself tends to generate periodic limit cycles as the distribution of equity shifts in response to persistent changes in relative factor productivities. TFP and the aggregate growth rate do well when most equity is "in the right hands" of highly productive firms, less well when most equity is owned by low–productivity firms. Lower values of the collateral parameter λ typically reduce the average growth rate, increase its dispersion and raise the periodicity of the limit cycle.

Cyclical behavior need not occur if unsecured lending is available to productive firms. Socially desirable balanced growth paths are still open to economies with modest collateral and substantial lines of unsecured credit. The downside of this situation is that unsecured lending is a fragile multiple equilibrium based on the borrower's "reputation", that is, the willingness to repay a current loan. This willingness, in turn, rests on the borrower's faith that unsecured credit will continue to be available in the future. Any event that shakes this faith (more precisely, any event that reduces expected future debt limits for all borrowers) will gradually reduce reputational borrowing until it vanishes altogether.

The rest of the paper is organized as follows. The next section gives a dramatized preview of results by contrasting an economy of perfect capital mobility with the the worst-case scenario of financial autarky or zero capital mobility. Section 3 describes a more general class of economies with financial frictions. Stationary Markov equilibria are defined in Section 4 and described in Section 5 for economies with secured loans only. Section 6 looks at the dynamics of lending when both secured and unsecured loans are traded. Section 7 presents some numerical examples connecting macroeconomic aggregates with collateral availability and sectoral shocks. Extensions are discussed in Section 8 and conclusions are summarized in Section 9.

2 An Example: Financial Autarky

As a first step towards understanding the importance of financial markets for aggregate factor productivity, consider a simple stochastic AK economy which is a special case of the more general model discussed in subsequent sections. There are two sectors $i = 1, 2$, each represented by an infinitely–lived producer with logarithmic utility of consumption and common discount factor β. Both sectors produce the same consumption/investment good by means of linear technologies with time–varying capital productivity. There are two aggregate states of the world ($s = 1, 2$) with transition probabilities

$$\pi(s_+|s) = \begin{cases} \pi \in [0,1] & \text{if} \quad s_+ = s \ , \\ 1 - \pi & \text{if} \quad s_+ \neq s \ . \end{cases}$$

The representative producer in sector i has access to a linear technology transforming capital input into output with sectoral productivity

$$A^i_s = \begin{cases} A > 0 & \text{if} \quad i = s \ , \\ zA & \text{if} \quad i \neq s \ , \end{cases}$$

with $0 < z < 1$. Because of logarithmic utility and linear returns to savings, each producer's saving rate is constant at β. In the absence of financial frictions, less productive households lend out all their savings to the most productive ones. Hence output growth is flat at βA.

At the other extreme to frictionless financial markets stands the scenario of financial autarky. Here output growth depends critically on the distribution of wealth between sectors. We denote by $x \in [0,1]$ the wealth share of the productive sector, by (Y, K) the vector of current aggregate output and capital, and by (Y_+, K_+) the future value of that vector.

Aggregate output and future capital satisfy

$$Y = AK[x + z(1-x)], \quad K_+ = \beta Y. \tag{1}$$

The future value of the wealth share x_+ equals the ratio of the efficient producer's capital tomorrow divided by the future value of aggregate capital K_+. This yields the following stochastic law of motion for the wealth share

$$x_+ = \begin{cases} f(x;z) & \equiv \quad x/[x + z(1-x)] & \text{w. prob. } \pi \ , \\ 1 - f(x;z) & = \quad z(1-x)/[x + z(1-x)] & \text{w. prob. } 1-\pi \ . \end{cases} \tag{2}$$

Equation (2) is graphed in Figure 1. From equation (1), we conclude that aggregate factor productivity:

- includes a correction $x + z(1-x) < 1$ due to financial frictions;
- is lower when sectoral productivities are more dispersed, that is, for small values of z; and
- fluctuates in response to changes in the distribution of wealth between potential "borrowers" and "lenders", that is, between more productive and less productive sectors.

It is easy to check that the aggregate growth rate

$$Y_+/Y = \beta A[z + (1-z)x_+]$$

fluctuates when the wealth distribution changes, even though the aggregate production possibilities frontier is stationary. In a cross section of economies indexed on the value of z, the growth rate would be positively correlated with z and, therefore, negatively correlated with the dispersion of sectoral TFP's.

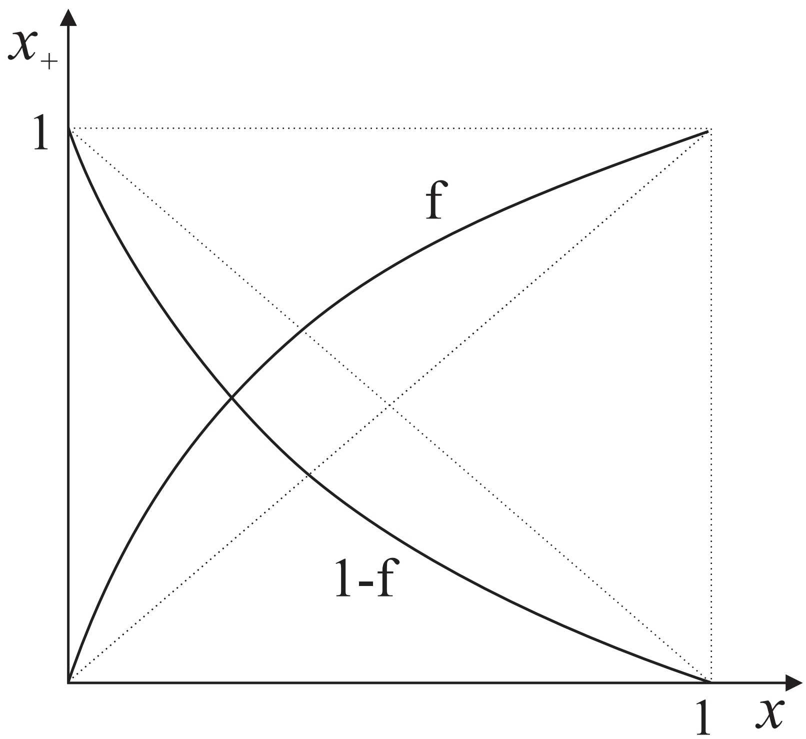

Figure 1: Transition maps for the wealth share under financial autarky.

From the viewpoint of economic theory, the most interesting feature of the law of motion in equation (2) is what it says about the stochastic process of macroeconomic aggregates (factor productivity, the distribution of wealth and the rate of growth) in an economy with no credit market and no capital mobility. All of these stochastic processes turn out to be complex, as one may guess from Figure 1; they visit a countable infinity of states and attain no ergodic distribution. The following result is proved in the Appendix.

Proposition 1: Let $\pi \in (0,1)$.

(a) For any initial value $x_0 \in (0,1)$, the wealth share variable x_t is a Markov chain on the countably infinite asymptotic set

$$\left\{ f(x_0; z^n), f(1-x_0; z^n) \middle| n \in \mathbb{Z} \right\} ,$$

where $\mathbb{Z} = \{n = 0, \pm 1, \pm 2, \dots\}$ is the set of all integers.

(b) The dynamics of aggregate growth and total factor productivity are not ergodic.

This result stands in stark contrast to the equilibrium outcome of the corresponding economy with perfect capital mobility where factor productivity and aggregate output growth are constant. We turn now to a more general environment with active asset markets.

3 The environment

Consider a growth model in discrete time $t = 0, 1, 2, \dots$ with a finite number of agent types (sectors) indexed $i \in \mathcal{I} = \{1, 2, \dots, I\}$ and productivity states $s \in \mathcal{S} = \{1, 2, \dots, S\}$. Each sector comprises a continuum of agents with equal size. All agents produce the same good which is available for consumption and investment purposes. Their common preferences over consumption streams are represented by an additively separable expected utility function

$$E_0(1-\beta)\sum_{t=0}^{\infty}\beta^t \ln[c(s^t)] \ ,$$

where $s^t = (s_t, \dots, s_0) \in \mathcal{S}^{1+t}$ is the state history in period t, and the initial state s_0 is given. The productivity state follows a Markov process with transition probability from s to s_+ equal to $\pi(s_+|s)$. In state s an agent of type i can convert capital into gross output ("resources") with linear technology $y = A_s^i k$. Resources y include current output and undepreciated capital which can be costlessly converted into the single consumption/investment good in the next period. In particular, capital investment is not producer–specific. The simplification that all agents produce the same good isolates the impact of sectoral shocks on capital reallocation while abstracting from relative price effects.

We assume that the economy's production possibility frontier is constant at $A \equiv \max_{i\in\mathcal{I}} A_s^i$ for all $s \in \mathcal{S}$. Though we do not need to impose that agent types are in some way symmetric, it simplifies the exposition to assume that every agent has access to the technological frontier sometimes and that there is always a unique most productive sector:

(A1) Every agent operates the technological frontier sometimes; that is, for each i there exists s such that $A_s^i = A$.

(A2) Not more than one agent type operates the technological frontier; that is, for each s there is exactly one i such that $A_s^i = A$.

(A3) No state is trivial; that is, every $s \in \mathcal{S}$ is in the support of the unique invariant state distribution.

Throughout this paper we focus on stationary Markov equilibria where all endogenous variables depend only on the current state vector of the economy, denoted $\sigma \equiv (\mathbf{x}, s) \in \Sigma \equiv [0,1]^I \times \mathcal{S}$, where $\mathbf{x} = (x^1, \ldots, x^I)$ is the distribution of wealth shares across agent types.

Each period, the less productive agents lend out capital to the more productive agents at the gross interest rate $R(\sigma)$ that prevails in the credit market. An exogenous fraction $\lambda \in [0,1]$ of each agent's resources is pledgeable collateral which can be seized by creditors in the event of default. The value of λ is constant and common for all producers; it depends on technological factors like the collaterizability of income and wealth, as well as on creditor rights and other aspects of economic institutions.[4] Timing within each period is as follows. *First* the productivity state is realized; *second* the credit market opens and agents decide about consumption, investment, borrowing and lending; *third*, agents produce, borrowers redeem their debt, and everyone carries their wealth into the next period.

Borrowers may choose to default at the end of the period. Any agent who does so loses the collateral share of his resources to creditors and is banned from any unsecured borrowing in all future periods. A defaulting agent is still allowed to lend, however, and also retains full access to secured loans. Since no uncertainty is resolved during debt contracts (that is, borrowing and debt redemption happen within the same period), there exist default–deterring debt limits, defined similarly as in the pure–exchange model of Alvarez and Jermann (2000). These limits are the highest values of debt that will prevent default. In the absence of collateral

[4] If resources are split into output and undepreciated capital according to $y = Ak = \tilde{A}k + (1-\delta)k$, a more general expression for collateral would be $\lambda_0 \tilde{A}k + \lambda_1(1-\delta)k$. Our simplifying assumption is that collaterizability of output and capital is the same, $\lambda_0 = \lambda_1$.

($\lambda = 0$), our enforcement mechanism resembles the one discussed by Bulow and Rogoff (1989) and Hellwig and Lorenzoni (2008) who consider unsecured loans and assume that defaulters are denied excess to future loans but are still allowed to accumulate assets. With $\lambda > 0$, secured borrowing is feasible and sometimes, but not always, borrowing limits go beyond an agent's collateral capacity and sustain a higher flow of credit. Borrowing above one's collateral is an *unsecured loan* founded on a producer's desire to maintain a record of solvency and of continued access to future unsecured loans.

We denote the endogenous constraint on borrower i's debt–equity ratio by $\theta^i(\sigma)$. Whenever the cost of capital $R(\sigma)$ is strictly below borrower i's marginal product A_s^i, this producer will borrow up to his debt limit, and the leveraged equity return will be $\tilde{R}^i(\sigma) = A_s^i + \theta^i(\sigma)[A_s^i - R(\sigma)]$. On the other hand, if agent i's productivity is below or equal to the capital yield $R(\sigma)$, this agent's equity return is simply $\tilde{R}^i(\sigma) = R(\sigma)$. A defaulting agent, who has only access to secured (collateral) loans, faces a maximal debt–equity ratio $\theta_c^i(\sigma) = \lambda A_s^i/[R(\sigma) - \lambda A_s^i]$, and his equity return is $\tilde{R}_c^i(\sigma) = A_s^i R(\sigma)(1-\lambda)/[R(\sigma) - \lambda A_s^i]$ when $R(\sigma) < A_s^i$, and $\tilde{R}_c^i(\sigma) = R(\sigma)$ otherwise.

It is worth noting that intra–period credit is the only traded asset in this economy. If agents were to trade insurance or contingent claims against next period's productivity state, these security markets would not open. This immediately follows from the observation that every agent's marginal utility of wealth ω is proportional to $1/\omega$, regardless of the agent's productivity state, so all agents' security demands are proportional to their wealth. That in turn implies that trade of insurance securities must be zero in equilibrium. We also do not consider a stock market distinct from the loan market. In particular, all shares in other agents' technologies are equivalent to loans and are subject to default.

The assumption of logarithmic utility implies that all agents consume a fraction $1-\beta$ of wealth, and that the expected utility of a productive borrower with end–of–period wealth ω can be expressed in the form $\ln(\omega) + V^i(\sigma)$, where $V^i(\sigma)$ is end–of–period utility of agent i with unit wealth when the current state is $\sigma = (\mathbf{x}, s)$.[5] Similarly,

[5]These assertions follow from the observation that agent i's flow budget constraint takes the form $\omega^i = \tilde{R}^i(\sigma)(\omega_-^i - c^i)$ where c^i is consumption and ω^i (ω_-^i) is agent i's wealth at the end of the current period (the previous period, respectively).

if agent i had defaulted in this or in some earlier period, his utility is expressed as $\ln(\omega) + V_c^i(\sigma)$ if end–of–period wealth is ω and $V_c^i(\sigma)$ denotes end–of–period utility of a unit–wealth agent of type i who has access to secured loans only.

4 Stationary Markov equilibrium

A stationary Markov equilibrium is a list of functions

$$\Big[\theta^i(\sigma), R(\sigma), \tilde{R}^i(\sigma), \tilde{R}^i_c(\sigma), v^i(\sigma), X^i(\sigma)\Big]_{i\in\mathcal{I},\sigma\in\Sigma} . \tag{3}$$

The first four objects on that list are respectively the debt–equity limits on solvent agents, the cost of capital, and the equity returns for solvent and bankrupt agents. The functions $v^i(\sigma) = V^i(\sigma) - V_c^i(\sigma)$ define the "penalty of default" for agent i, that is, the difference between the continuation utilities from solvency and default. Finally, the maps $x^i_+ = X^i(\sigma) : \Sigma \to [0,1]$ connect this period's state vector $(\mathbf{x}, s)$ with next period's wealth share for every agent i. Let $\mathbf{X} = (X^i)_{i\in\mathcal{I}} : \Sigma \to [0,1]^I$ be the collection of these maps.

In equilibrium, debt limits are the largest values that will deter default when any borrower with equity E is indifferent between solvency and default:

$$\ln\Big[\tilde{R}^i(\sigma)E\Big] + V^i(\sigma) = \ln\Big[(1-\lambda)A^i_s[1+\theta^i(\sigma)]E\Big] + V_c^i(\sigma) .$$

Here, the right–hand side is expected utility of the defaulting agent i who leaves the default period with unpledged wealth $(1-\lambda)A^i_s[1+\theta^i(\sigma)]E$. This equality is conveniently equivalent to

$$\theta^i(\sigma) = \frac{(e^{v^i(\sigma)} - 1 + \lambda)A^i_s}{(1-\lambda)A^i_s - e^{v^i(\sigma)}[A^i_s - R(\sigma)]} . \tag{4}$$

Equation (4) shows that the maximum default–free debt–equity ratio is increasing in the penalty of default $v^i(\sigma)$ and in the collateral share λ. Debt–equity ratios are also decreasing in the interest rate, and equation (4) gives a lower bound on the equilibrium interest rate: the debt-equity ratio of borrower i tends to infinity when $R(\sigma)$ approaches $A^i_s[1-(1-\lambda)e^{-v^i(\sigma)}]$ from above. Intuitively, when the interest rate is low, some borrowers never opt for default. Thus their demand for loans becomes infinite which cannot be compatible with credit–market equilibrium.

Equation (4) shows that $\theta^i(\sigma)$ is larger than the secured borrowing limit $\theta_c^i(\sigma) = \lambda A_s^i/[R(\sigma) - \lambda A_s^i]$ for all positive default penalties $v^i(\sigma) > 0$; it reduces to $\theta_c^i(\sigma)$ if $v^i(\sigma) = 0$. In the following, we refer to an equilibrium with $v^i(\sigma) = 0$ for all $i \in \mathcal{I}$ and $\sigma \in \Sigma$ as one of *secured borrowing*; an equilibrium where $v^i(\sigma) > 0$ for at least some $i \in \mathcal{I}$ and $\sigma \in \Sigma$ has *secured and unsecured borrowing*: here debt–equity limits are based on collateral *and* reputation.

With aggregate capital K, agent i's equity is $x^i K$. The *supply of credit* comes from all agents with productivity $A_s^i \leq R(\sigma)$, and because agents with $A_s^i = R(\sigma)$ are indifferent between lending and borrowing at market rate $R(\sigma)$, the aggregate supply of credit per unit of aggregate capital is a step function, expressed as the correspondence

$$\mathrm{CS}(\sigma) = \left[\sum_{i:A_s^i < R(\sigma)} x^i \ , \ \sum_{i:A_s^i \leq R(\sigma)} x^i \right] .$$

Similarly, the demand for credit per unit of capital is the correspondence

$$\mathrm{CD}(\sigma) = \left[\sum_{i:A_s^i > R(\sigma)} \theta^i(\sigma) x^i \ , \ \sum_{i:A_s^i \geq R(\sigma)} \theta^i(\sigma) x^i \right] ,$$

and the credit market is in equilibrium if

$$\mathrm{CS}(\sigma) \cap \mathrm{CD}(\sigma) \neq \emptyset \ . \tag{5}$$

As we saw earlier, for any interest yield $R(\sigma)$, the equity return of agent i is

$$\tilde{R}^i(\sigma) = \max \left\{ A_s^i + \theta^i(\sigma) \Big[A_s^i - R(\sigma) \Big], R(\sigma) \right\} , \tag{6}$$

while the equity return of a producer with access to secured borrowing is

$$\tilde{R}_c^i(\sigma) = \max \left\{ \frac{A_s^i R(\sigma)(1-\lambda)}{R(\sigma) - \lambda A_s^i}, R(\sigma) \right\} . \tag{7}$$

Agent i's wealth share changes from x^i to

$$x_+^i = X^i(\sigma) = \frac{\tilde{R}^i(\sigma) x^i}{\sum_{j \in \mathcal{I}} \tilde{R}^j(\sigma) x^j} \quad , \ \sigma = (x^1, \ldots, x^I, s) \ . \tag{8}$$

To understand this expression, suppose that total wealth is one unit today; then agent i's wealth next period is β times the numerator of (8) while total wealth is β times the denominator of (8).

Expected utilities satisfy recursive equations

$$V^i(\mathbf{x},s) = (1-\beta)\ln(1-\beta)+\beta\sum_{s_+\in\mathcal{S}}\pi(s_+|s)\left\{\ln\left[\beta\tilde{R}^i[\mathbf{X}(\mathbf{x},s),s_+]\right]+V^i\left[\mathbf{X}(\mathbf{x},s),s_+\right]\right\}. \tag{9}$$

Note again that V^i denotes expected utility of solvent agent i with unit wealth. In the current period, this agent consumes $c = 1-\beta$, and so the first term on the right–hand side is the utility of current consumption; the other terms are discounted future payoffs. In the next period, the distribution of wealth changes from $\mathbf{x}$ to $\mathbf{x}_+ = \mathbf{X}(\mathbf{x},s)$ and the productivity state changes from s to s_+ with probability $\pi(s_+|s)$; the agent saves a fraction β of his unit wealth, ending the period with wealth $\omega_+ = \beta\tilde{R}^i(\mathbf{x}_+,s_+)$ and utility $\ln(\omega_+) + V^i[\mathbf{x}_+,s_+]$.

For an agent who has opted for default in some earlier period, the recursive equation in V_c^i is nearly identical to (9); all that changes is that the equity returns $\tilde{R}^i$ are replaced by the defaulter's lower returns $\tilde{R}_c^i$. By subtracting those equations from (9), we obtain recursive equations in the default penalties $v^i(\sigma) = V^i(\sigma) - V_c^i(\sigma)$:

$$v^i(\mathbf{x},s) = \beta\sum_{s_+\in\mathcal{S}}\pi(s_+|s)\left\{\ln\frac{\tilde{R}^i[\mathbf{X}(\mathbf{x},s),s_+]}{\tilde{R}_c^i[\mathbf{X}(\mathbf{x},s),s_+]} + v^i\left[\mathbf{X}(\mathbf{x},s),s_+\right]\right\}. \tag{10}$$

Definition: *A stationary Markov equilibrium is a list of functions specified in (3) which satisfies equations (4)–(8) and (10) for all* $\sigma = (\mathbf{x},s)\in\Sigma$ *and* $i\in\mathcal{I}$.

In a stationary Markov equilibrium, the state vector σ is also a sufficient statistic for the growth rate that connects aggregate current resources Y with last period's resources Y_-. In particular, current aggregate capital K equals saving βY_-, and current resources are the sum of resources across all agent types:

$$\begin{aligned} Y &= K\sum_{i\in\mathcal{I}} x^i\tilde{R}^i(\sigma) \\ &= \beta Y_-\Big\{R(\sigma) + \sum_{i:A_s^i>R(\sigma)}[A_s^i - R(\sigma)]x^i[1+\theta^i(\sigma)]\Big\}. \end{aligned}$$

The growth factor is

$$\frac{Y}{Y_-} = \beta\Big\{R(\sigma) + \sum_{i:A_s^i>R(\sigma)}[A_s^i - R(\sigma)]x^i[1+\theta^i(\sigma)]\Big\} \le \beta A .$$

This expression has an upper bound βA achieved when no capital is misallocated.

Before we analyze stationary Markov equilibria in detail for some special cases, we state two general results. One of them says that an equilibrium with no unsecured borrowing always exists. In particular,

Proposition 2: *There exists a unique equilibrium in which all borrowing is secured.*

This result generalizes earlier findings by Bulow and Rogoff (1989) and Kehoe and Levine (1993) who showed that financial autarky is an equilibrium in economies where all borrowing is unsecured. Indeed, it is easy to check that $v^i(\sigma) = 0$ together with $\tilde{R}^i(\sigma) = \tilde{R}^i_c(\sigma)$ and $\theta^i(\sigma) = \theta^i_c(\sigma)$ satisfy all equilibrium equations except market clearing for any given interest rate $R(\sigma)$. Existence and uniqueness of the market–clearing interest rate is proven in the appendix.

What is the intuition for the equilibrium without unsecured borrowing? If there are no unsecured loans, there is no penalty of default, and therefore no borrower is permitted to borrow in excess of collateral. And conversely, when debt–equity limits just reflect collateral constraints, a good credit record is worthless because there is no default penalty. Section 5 characterizes the secured borrowing equilibrium completely for a symmetric economy with two agent types and two states.

Our second result says that a first–best allocation can only be an equilibrium if there is enough collateral. Specifically, $\lambda \geq (I-1)/I$ is a necessary and sufficient condition to support the first best with secured borrowing at the symmetric initial wealth distribution $x^i = 1/I$, $i \in \mathcal{I}$. Here returns are equalized, $\tilde{R}^i = A$, and the secured borrowing constraint is large enough to shift all capital to the most productive sector in every state.

Can the first best also be supported by secured *and* unsecured borrowing when $\lambda < (I-1)/I$? Put differently, is there a first–best equilibrium where unsecured borrowing exists? In line with earlier results by Bulow and Rogoff (1989) and Hellwig and Lorenzoni (2008), the answer to this question is no. Unsecured borrowing cannot support first–best allocations. The intuition for this result is as follows. In a first best allocation, the capital cost (the interest rate) equals the capital return of borrowers. Hence there is no leverage gain, so that access to loans has no value for borrowers.

In turn, every borrower would default on an unsecured loan, no matter how small it is. These findings are summed up in

Proposition 3: *When $\lambda \geq (I-1)/I$, the secured–borrowing equilibrium gives rise to a first best allocation for some initial distribution of wealth. Conversely, when $\lambda < (I-1)/I$, no first best allocation can be an equilibrium with secured and unsecured borrowing.*

The inequality $\lambda \geq (I-1)/I$ is stringent, requiring collateral to be a large proportion of available resources, that is, gross national product plus undepreciated capital. In spite of Proposition 3, we will see in Section 6 that unsecured borrowing can still sometimes support production–efficient allocations, particularly in economies with very patient agents and large productivity differences between sectors.

To explore equilibrium with binding constraints in more detail, we focus for the remainder of this paper on the symmetric two–agent, two–state special case of our general environment. In particular, $A_s^i = A$ if $i = s$, and $A_s^i = zA$ if $i \neq s$, for $i \in \{1, 2\}$ and $s \in \{1, 2\}$, where $z < 1$ is a measure of the productivity differential. Both types are equally likely to operate the frontier technology, where π is the probability that any state $s = 1, 2$ does not change from one period to the next. In this symmetric economy, stationary Markov equilibria are also symmetric. Therefore, the only relevant state variable is the share of wealth owned by the borrowing agents (short "borrower wealth"), to be denoted $x \in [0, 1]$. The wealth distribution is thus $(x, 1-x)$ if $s = 1$ and $(1-x, x)$ if $s = 2$. Current rates of return and debt limits depend on borrower wealth x alone, and the productivity state s matters only for the transitional dynamics of borrower wealth.

5 Secured borrowing

In the equilibrium with secured borrowing the debt–equity ratio is $\theta = \lambda A/[R - \lambda A]$ and the market–clearing loan yield can be readily obtained as

$$R(x) = \begin{cases} zA & \text{if } x \leq 1 - \frac{\lambda}{z} \ , \\ \frac{\lambda A}{1-x} & \text{if } x \in [1 - \frac{\lambda}{z}, 1 - \lambda] \ , \\ A & \text{if } x \geq 1 - \lambda \ . \end{cases}$$

When borrower wealth is below $1 - \lambda/z$, credit demand is so low that the equilibrium interest rate makes unproductive lenders indifferent between production and lending. The economy is *production inefficient* because it misallocates its capital stock. When borrower wealth exceeds this threshold, all capital flows to the more productive agents and the economy becomes production efficient. For $x < 1 - \lambda$, borrowers are still debt constrained and enjoy a higher equity yield than do lenders. Consumption growth rates are higher for borrowers which makes the economy *consumption inefficient*. Full efficiency in period t is attained only when borrower wealth exceeds $1 - \lambda$. In what follows, we assume throughout that $\lambda < z$ so that production inefficiency remains a possibility.

The transitional dynamics of borrower wealth is described by two maps. Next period's borrower wealth is $x_+ = X_0(x)$ when the productivity state is unaltered and it is $x_+ = X_1(x) = 1 - X_0(x)$ when the productivity state changes. Using the above expressions for $R(x)$, $\theta(x) = \lambda A/[R(x) - \lambda A]$, and the borrowers' equity return $\tilde{R}(x) = A + \theta(x)[A - R(x)]$, we obtain

$$X_0(x) = \frac{\tilde{R}(x)x}{\tilde{R}(x)x + R(x)(1-x)} = \begin{cases} \frac{(1-\lambda)x}{(1-z)x + z - \lambda} & , \ x \leq 1 - \frac{\lambda}{z} \ , \\ 1 - \lambda & , \ x \in [1 - \frac{\lambda}{z}, 1 - \lambda] \ , \\ x & , \ x \geq 1 - \lambda \ . \end{cases}$$

Figure 2 shows the two maps X_0 and X_1 in three generic situations. It becomes evident from these graphs that the stochastic dynamics of borrower wealth must settle down on the bounded interval $[\lambda, 1 - \lambda]$ or $[1 - \lambda, \lambda]$. Moreover, the asymptotic dynamics must be a stochastic cycle with finite support. The precise statement, which is proved in the Appendix, is

Proposition 4: *In the equilibrium with secured borrowing and for any* $\pi \in (0,1)$ *and* $\lambda > 0$*, the dynamics of wealth* x_t *enters a finite stochastic cycle* $(x_n)_{n=1}^N$*, with probability one as* $t \to \infty$*. The cycle has the following features.*

(a) *Economies with ample collateral* $\lambda \geq 1/2$ *converge to a cycle with two states* $x_2 = 1 - x_1 \in [1-\lambda, \lambda]$*. Production is efficient, debt constraints do not bind, and aggregate output and individual consumption grow at the constant rate* βA*.*

(b) *Economies with medium collateral* $\lambda \in [z/(1+z), 1/2)$ *also converge to a cycle with two states and* $x_1 = \lambda < x_2 = 1-\lambda$*. Production is again efficient and aggregate growth is constant at* βA*. However, individual consumption and wealth growth rates are volatile, and borrowers are constrained in a fraction* $1-\pi$ *of periods. Specifically, agent* i*'s consumption growth in state* s_t *is* βA *if* $s_t = s_{t-1}$*,* $\beta A \lambda/(1-\lambda)$ *if* $i \neq s_t \neq s_{t-1}$*, and* $\beta A(1-\lambda)/\lambda$ *if* $i = s_t \neq s_{t-1}$*.*

(c) *Economies with small collateral* $\lambda < \frac{z}{1+z}$ *converge to a cycle with generically* $N = 2m$ *states, with* $m \geq 2$*. In* $2m-3$ *of these states, aggregate growth is lower than* βA*. Cycles are typically asymmetric with booms lasting longer than recessions. The number of states is a weakly decreasing function of* λ*, and* $m \to \infty$ *as* $\lambda \to 0$*.*

Figure 2 illustrates the three possibilities stated in the proposition. In (a), the typical first–best equilibrium is a cycle where borrower wealth fluctuates between two states which means that every agent's wealth share is constant. Any initial wealth distribution must enter such a cycle with probability one in finitely many periods. In (b), the stochastic cycle again has only two states, but now one of them has constrained borrowers; no capital is misallocated and production is efficient in all periods. And graph (c) shows an example of a cycle with six states, with no misallocation of capital in three of them, and some misallocation in the other three. The red lines indicate the possible transitions between these states.

An implication of this analysis is that the dispersion of output growth is magnified when the collateral share falls. This is again evident from Figure 2(c): the support of the invariant cycle becomes the larger, the smaller the value of λ is. Lower values of collateral not only reduce the trend growth rate, they also lead to a more complex

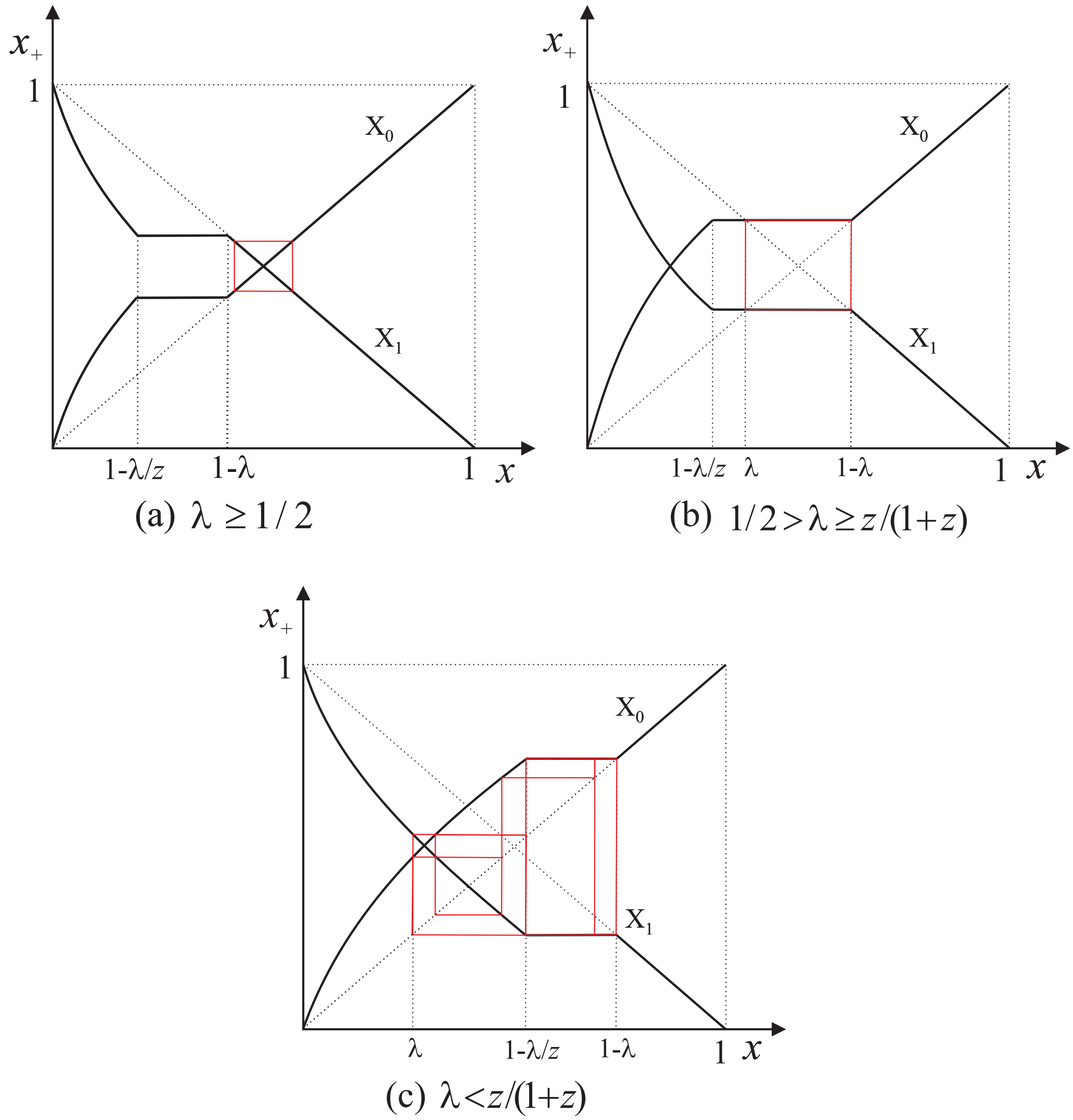

Figure 2: Asymptotic cycles with secured borrowing.

and dispersed distribution of growth rates around trend. In fact, in the limit where $\lambda \to 0$ the economy does not even have a finite limit cycle but is instead described by a stochastic process on a countably infinite state space, as discussed earlier in Proposition 1.

6 Secured and unsecured borrowing

Equilibria with unsecured borrowing are not easy to describe analytically in any degree of generality. Nonetheless, it is possible to derive a few insightful results for some special cases where the asymptotic wealth dynamics settles down to a finite state space. One such case is the deterministic economy ($\pi = 0$), the other is an economy permitting simple production–efficient stochastic cycles with two states. We explore these simpler equilibria in this section.

The deterministic economy admits a steady state with binding constraints where borrower wealth is stable at some x. The wealth share of either type thus periodically alternates between x and $1 - x$. This is in stark contrast to the stochastic economy where equilibria are typically cyclical and the only possible steady state is a first best outcome with unconstrained borrowers achievable only if $\lambda \geq 1/2$. For $\lambda < 1/2$, borrower wealth must fluctuate permanently in a stochastic economy.

One obvious steady state in the deterministic model is the one without unsecured borrowing. Although Proposition 4 requires $\pi > 0$, it is straightforward to extend the result to the deterministic case as follows. The deterministic economy has a unique steady state x with secured borrowing which is (i) first best when $\lambda \geq 1/2$; (ii) production efficient and consumption inefficient when $z/(1+z) \leq \lambda < 1/2$; and (iii) production inefficient when $\lambda < z/(1+z)$. In Figure 2 these steady states are at the intersection of the 45^o line with the map $X_1(.)$.

For a deterministic economy with secured and unsecured borrowing, we prove the following result in the Appendix.[6]

Proposition 5: *Let $\pi = 0$. Then there is a threshold value $\hat{\lambda} \leq \dfrac{z - \beta^2}{1 - \beta^2}$ such that*

(a) *If $\beta \leq z$, there is one steady state with secured borrowing and no steady state with secured and unsecured borrowing.*

(b) *If $\beta > z$ and $\lambda \in [\hat{\lambda}, \dfrac{\beta}{1+\beta})$, there is a steady state with secured and unsecured borrowing which coexists with the steady state with secured borrowing.*

[6]The Appendix gives additional results on stochastic cycles in economies with efficient production and inefficient consumption; see Proposition 7.

(c) *If* $\beta > z > \beta^2$ *and* $\lambda \in (\hat{\lambda}, \dfrac{z-\beta^2}{1-\beta^2})$, *there is a third steady state with secured and unsecured borrowing which coexists with the two other steady states of (b).*

To interpret these results, the inequality $\beta \leq z$ simply says that the gains from credit market participation are not high enough to support an equilibrium with unsecured borrowing. Part (a) extends the well–known result of Kehoe and Levine (1993) that intertemporal financial autarky or, in our setting, secured borrowing is the only equilibrium when agents are too impatient or when income fluctuations are too small. Conversely, says part (b), when $\beta > z$ unsecured borrowing is feasible but now collateral may not exceed the threshold $\beta/(1+\beta)$. If the collateral value is larger than that number, the gain from borrowing above collateral is too small to prevent borrowers from defaulting. Put differently, secured borrowing alone supports efficient allocations with low leverage, so that extended credit limits add very little value. Part (c) establishes a strong form of equilibrium multiplicity. In these situations, one steady state turns out to be production efficient whereas the other two steady states are production inefficient. The explanation for equilibrium multiplicity is a *dynamic complementarity* in the endogenous borrowing limits. Borrowers' expectations of *future* credit market conditions affect their incentives to default today, and this in turn takes an impact on their *current* borrowing limits. If future constraints are tight, the payoff from solvency is modest; agents place a low value on the strategy of participating in credit markets, and their default penalty is low. In this case, current default–deterring debt limits must be low. Conversely, expectations of loose constraints in the future make participation more valuable, lessen default incentives and ease current constraints.

When agents are sufficiently patient, so that $z \leq \beta^2$, the assumptions in part (c) are not valid. Then the deterministic economy has a unique steady state with secured and unsecured borrowing, coexisting with the inferior steady state without unsecured borrowing. Figure 3 shows how steady–state loan yields vary with the collateral parameter λ when $z \leq \beta^2$.

The deterministic economy also permits an analysis of the local dynamics around the steady states. Whenever there exists a unique steady state with unsecured borrowing, it is locally determinate, whereas the secured–borrowing steady state is

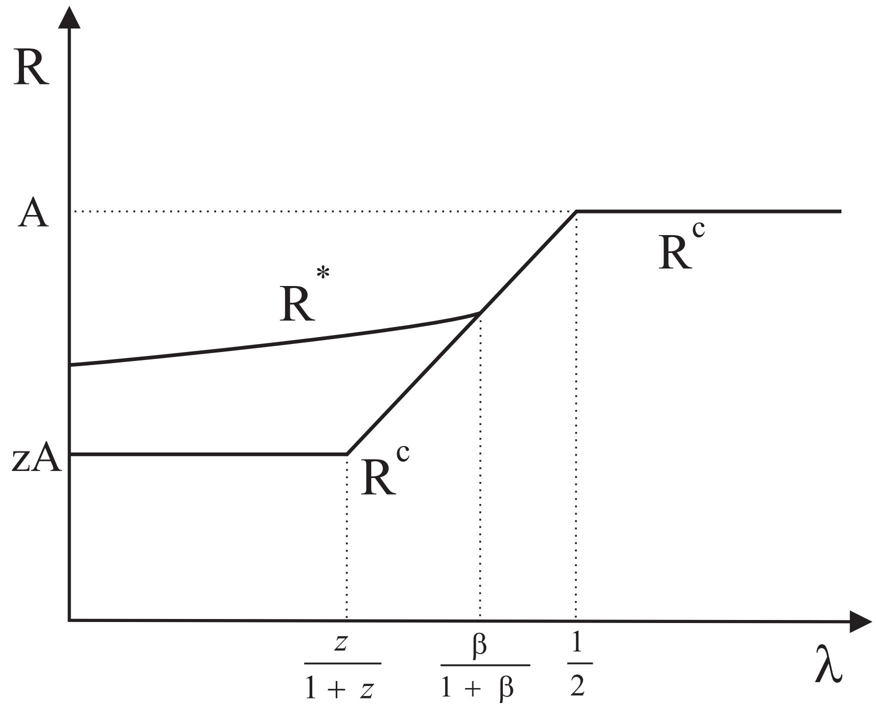

Figure 3: Steady state loan yields with secured borrowing (R^c) and with secured and unsecured borrowing (R^*) when $z < \beta^2$.

locally indeterminate. In particular, there is a continuum of dynamic equilibria in which the value of unsecured loans vanishes asymptotically. Again, this result is explained by the dynamic complementarity between endogenous credit constraints which may trigger a self–fulfilling collapse of the market for unsecured loans: when market participants expect credit constraints to tighten rapidly, the value of reputation would shrink over time until only secured loans are traded. Such equilibria are mathematically similar to hyperinflationary equilibria in overlapping–generation models of fiat money but do not require the existence of an unproductive financial asset.

Proposition 6: *Let $z < \beta^2$ and $\lambda < \beta/(1+\beta)$.*

(a) *The steady state with secured borrowing (θ^c, x^c) is locally indeterminate. Par-*

ticularly, there is a continuum of equilibria (θ_t, x_t) such that $\theta_t \to \theta^c$ and x_t converges to a cycle with period two around x^c.

(b) *The steady state with secured and unsecured borrowing is locally determinate.*

The stochastic economy cannot have a steady state unless it is the first best, and cycles with unsecured borrowing are too complex to describe analytically. Numerically we find that the qualitative features of the transition maps for borrower wealth are much like the ones shown in Figure 2, with the only difference that the cycles do not have finite support, as they do in Figure 2(c).

However, there are still situations where the economy has a stochastic cycle of order two which is production efficient, like the one shown in Figure 2(b). Paralleling Proposition 5, what is necessary for such cycles is that z is small relative to β: agents must be patient enough and their productivity must fluctuate sufficiently so that exclusion from unsecured borrowing is a severe enough punishment. In fact, production–efficient cycles may even exist when there is no collateral at all. The Appendix gives more details about production–efficient cycles.

7 Numerical examples

For a fuller description of stochastic cycles, especially ones that exhibit some misallocation of capital, we use value function iteration to isolate stationary Markov equilibria with reputational borrowing. Specifically, for arbitrary initial default penalties for agent 1 $v_0^1(.,1) > 0$ and $v_0^1(.,2) > 0$,[7] we calculate constraints and interest rates for all x using the equilibrium conditions (4)–(7) and the wealth iteration maps (8). The results are then substituted in the right–hand side of (10) to calculate new default penalties $v_1^1(.,1)$ and $v_1^1(.,2)$, and so on. Our previous results on equilibrium multiplicity imply that this map cannot be a global contraction, so one cannot expect a definitive proof that equilibrium exists. We find, however, that these iterations converge fast, and we are able to identify the theoretical equilibria in the special

[7] Because of symmetry, default penalties for agent 2 are simply $v^2(x,1) = v^1(x,2)$ and $v^2(x,2) = v^1(x,1)$.

cases analyzed in previous sections. We conjecture that the iteration procedure generally converges to the determinate equilibrium whenever there is equilibrium multiplicity. In the deterministic economy $\pi = 0$, for example, we know that the secured–borrowing equilibrium is determinate whenever no other equilibrium exists, and indeterminate otherwise (Proposition 6). Numerically we find indeed that value function iteration converges to the determinate equilibrium.

For a plausible benchmark parameterization, we study how aggregate growth and volatility depend on the model parameters and how they qualitatively correlate with the sectoral dispersions of equity returns and total factor productivities. It is not the purpose of this exercise to conduct a full–blown quantitative analysis; that would require a much more detailed model incorporating labor and some other features discussed in the next section. We fix the three parameters $A = 1.08$, $\beta = 0.96$ and $\pi = 0.9$ so that annual growth in the first–best economy is at 3.7% and sectoral productivity shifts are rather persistent with a mean duration of 10 years. We then explore how the features of the economy change when we vary the key parameters λ and z.

Figure 4 shows the result of the parameter variation as λ goes from zero to 1/2 when z is fixed at 0.85. From Proposition 4 follows that the economy is production efficient (so aggregate growth is constant at βA) when $\lambda \geq z/(1+z) \approx 0.46$ and this outcome is achieved by secured borrowing which is the unique equilibrium. As the collateral share λ falls below that value, the growth rate decreases and becomes more volatile. When λ falls below 0.41, secured borrowing ceases to be the unique equilibrium. Now another equilibrium with both secured and unsecured borrowing emerges which has higher and more stable output growth than the pure–collateral equilibrium. For lower values of λ, the differences between these equilibria are substantial. For instance at $\lambda = 0.2$, the collapse of unsecured lending would trigger a fall in output growth from about 3.5% to less than 0.5%, and the standard deviation of output growth would rise from just 0.5% to over 4%. A further notable observation is that a reduction in the collateral share λ has opposite effects in equilibria with and without unsecured borrowing. At the pure–collateral equilibrium, a lower value of λ clearly leads to lower and more volatile growth since less capital is shifted to the most productive sectors. However, when we add unsecured borrowing, less collateral makes the potential exclusion from unsecured loans more harmful and thereby *relaxes* credit limits; hence growth increases slightly and becomes less volatile.

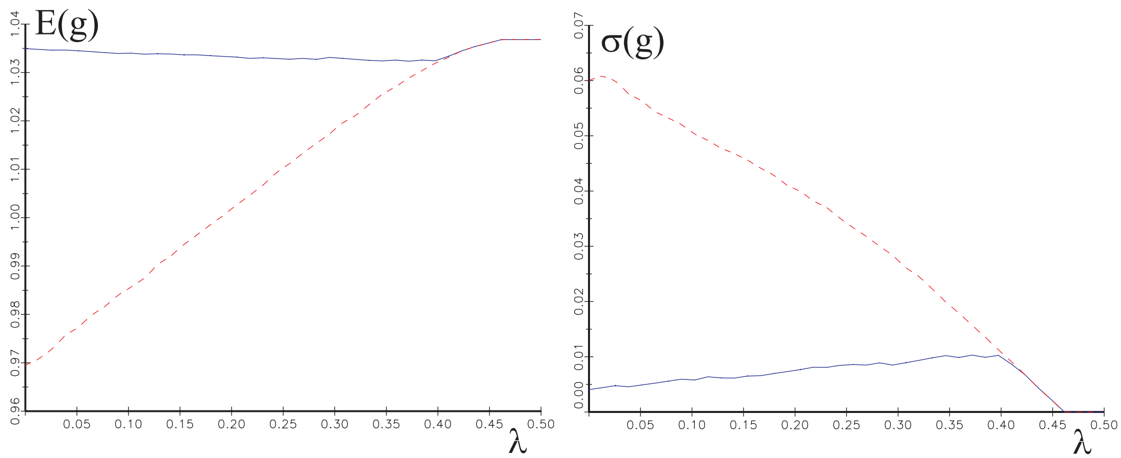

Figure 4: Mean and standard deviation of growth as λ varies from 0 to 0.5 where $A = 1.08$, $\beta = 0.96$, $\pi = 0.9$ and $z = 0.85$. The secured–borrowing equilibrium is shown by the red (dashed) curve, and the equilibrium with secured and unsecured borrowing is blue (solid). $E(g)$ and $\sigma(g)$ are calculated as sample averages for a time series of 50,000 periods.

Figure 5 shows the result of a similar simulation when λ is fixed at 0.2 and z varies from 0.6 to 1.0. Clearly, when z is close to one, the economy is almost first best and growth is constant at $\beta A \approx 1.037$. For values of z above 0.94, there is a unique equilibrium with secured borrowing; a second equilibrium with unsecured borrowing emerges for a larger productivity spread between the two sectors. Interestingly, unsecured borrowing permits a more efficient allocation of capital when the productivity difference between the two sectors is larger. Indeed, the equilibrium with secured and unsecured borrowing achieves a production efficient allocation of resources for values of z below 0.67.

At the benchmark parameter values with $\lambda = 0.2$ and $z = 0.85$ we also calculate equity return dispersion as the spread between sectoral equity returns, measured by the weighted standard deviation between $\tilde{R}_t^1$ and $\tilde{R}_t^2$ where weights are the corresponding wealth shares. At the "good" equilibrium with secured and unsecured

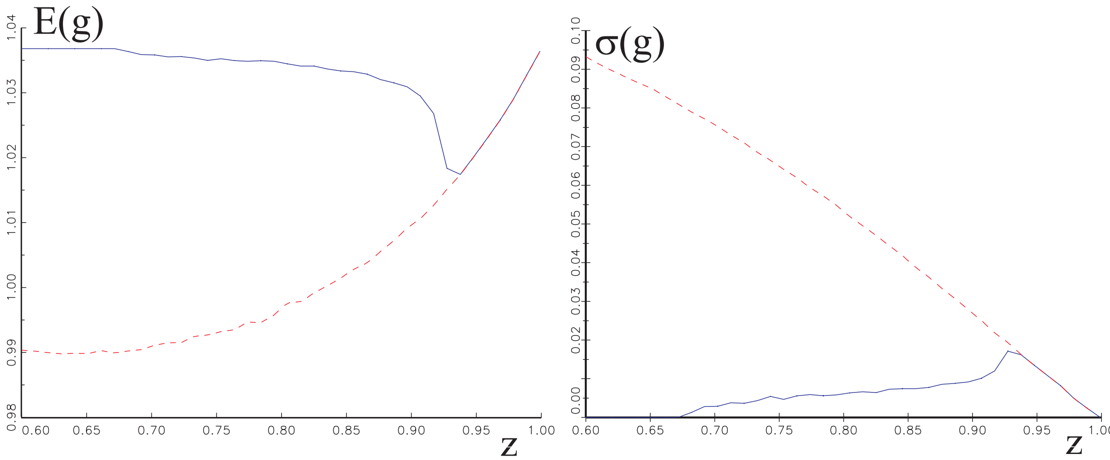

Figure 5: Mean and standard deviation of growth as z varies from 0.6 to 1 where $A = 1.08$, $\beta = 0.96$, $\pi = 0.9$ and $\lambda = 0.2$. The secured–borrowing equilibrium is shown by the red (dashed) curve, and the equilibrium with secured and unsecured borrowing is blue (solid). $E(g)$ and $\sigma(g)$ are calculated as sample averages for a time series of 50,000 periods.

borrowing, we find that this measure averages around 5.4% and has a standard deviation of 7%, the same order of magnitude as the stock–market dispersion indices reported in Loungani, Rush, and Tave (1990) (Fig. 1). The correlation coefficient between equity–return dispersion and g_t is -0.80, which is in line with the evidence listed in the introduction. Growth is low when capital is misallocated in which case the dispersion between sectoral equity returns is large. Figure 6 shows time series for the growth rate of aggregate resources and for the dispersion of equity returns for a simulation of 50 periods at the benchmark parameter values.

In accordance with results of Eisfeldt and Rampini (2006) (Table 3), our model further produces countercyclicality of TFP dispersion across sectors. Using the standard deviation of sectoral factor productivities A and zA, weighted by their relative output shares, as a measure of TFP dispersion, we find a mean dispersion of 2.5%, a standard deviation of 4.1% and a contemporaneous correlation with growth of -0.96.

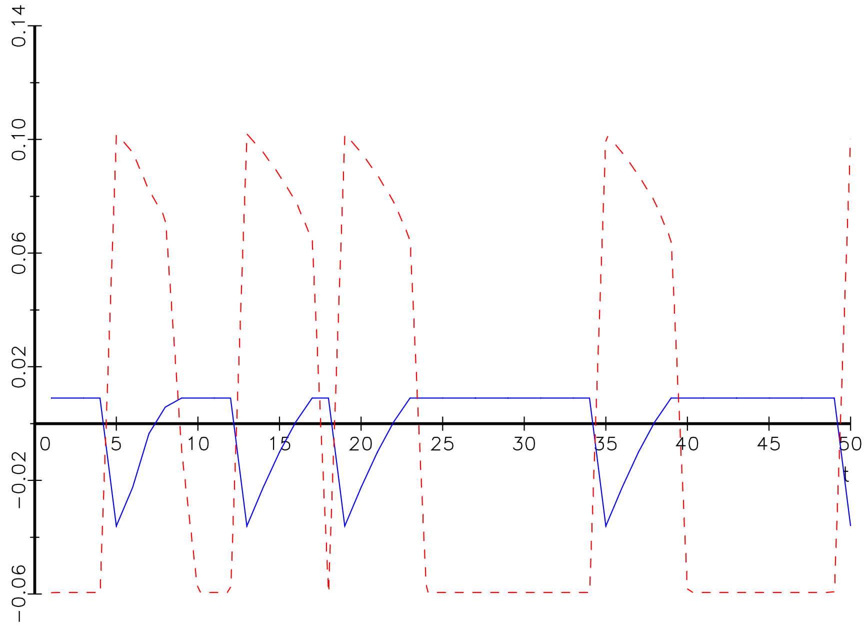

Figure 6: Simulated equilibrium with secured and unsecured borrowing at parameters $A = 1.08$, $\beta = 0.96$, $\pi = 0.9$, $z = 0.85$ and $\lambda = 0.2$: Growth rate (solid, blue) and dispersion of equity returns (dashed, red), both as deviations from their mean.

8 Extensions

8.1 Sector–specific labor

The absence of any input from labor is the biggest abstraction in our model, but its impact is quantitative rather than qualitative. A quick remedy, along the lines suggested by Kiyotaki and Moore (2008) and Kocherlakota (2009), would replace our AK technologies with standard neoclassical constant return ones of the form $Y = A^i_s F(K^i_s, N^i_s)$. Each sector would be populated with one unit of immobile labor. Labor will consume its marginal product and all saving would still be done by capitalists $i \in \mathcal{I}$ who hire labor at a competitive wage w^i_s and face a competitive yield on each unit of capital, that is,

$$R^i_s = \max_{N \geq 0} \left\{ A^i_s F(1, N) - w^i_s N \right\} .$$

Finally, pledgeable collateral will be an exogenous fraction λ of capital income rather than of all income, and long–term growth can be added by endowing sectoral productivity shocks with a common trend. We expect this extension to be relatively straightforward provided that labor is completely immobile.

8.2 Irreversible investment

Unsurprisingly, each of the two sectors is considerably more volatile than the aggregate economy. Because every stochastic cycle must enter some occasional periods of production efficiency, output in any sector falls occasionally to zero. This circumstance prevents a meaningful calculation of sector growth rates. Further, it appears to be a strong abstraction to assume that all gross resources can, in the extreme, move between sectors from one period to the next. To deal with these limitations, it seems a sensible extension to augment the model by some sector specificity (investment irreversibility).[8] For simplicity, suppose that a constant fraction of resources is sector specific and cannot be employed in the other sector, and let $\Psi < 1$ be the share of resources which is usable in both sectors. Then the only change to the model is that the expression for credit supply $\mathrm{CS}(\sigma)$ is multiplied by the factor Ψ, and all other equations remain unchanged. It is important to emphasize, however, that now even the first–best economy involves fluctuations of aggregate output, as resources move sluggishly between sectors when the productive state changes.

In the numerical example, suppose that 90 percent of capital is sector specific ($\Psi = 0.1$). For the benchmark parameter values with $\lambda = 0.2$ and $z = 0.95$, mean growth falls from 2.1% (at $\Psi = 1$) to 1.4% (at $\Psi = .1$) whilst its standard deviation increases from 1.3% to 1.9%. The growth rate of either sector, however, has a standard deviation of about 9.6%, five times larger than the standard deviation of aggregate growth. In line with the evidence discussed in the introduction, the dispersion between sector growth rates is countercyclical; its correlation coefficient with aggregate growth is -0.36.

[8] An alternative variation would be heterogeneity of final goods: if the two goods produced are not perfect substitutes and essential for consumers, sectoral shifts induce relative price changes which prevent that all resources are shifted to one sector.

8.3 Sectoral comovement

Another feature of the cycles characterized in the previous sections is that growth rates in the two sectors are negatively correlated. As an implication, either one sector is pro–cyclical and the other is counter–cyclical or both sectors are acyclical. The evidence however strongly supports comovement of virtually all two–digit industries with aggregate output (see e.g. Christiano and Fitzgerald (1998)). In the previous example, the absence of comovement is an artefact of the two–sector, two–state specification, where business cycles are *exclusively* driven by sectoral productivity shifts between two constant productivities A and zA. When technologies are in some way correlated with the sectoral productivity shifts, comovement follows easily. This is relatively simple when the technology frontier is volatile, but is also possible with a constant frontier. To see this in an extension of the model which retains perfect symmetry between sectors, suppose that the technology parameter z attains one of two values z^c and z^n, depending on whether the sectoral state changes or not. That is, "change of states" has $z_t = z^c$ when $s_t \neq s_{t-1}$, and "no change of states" has $z_t = z^n$ when $s_t = s_{t-1}$. This extension thus has two sectors and four productivity states $(s_t, s_{t-1}) \in \{1, 2\}^2$. In the numerical example with the same benchmark parameter values, we find that there is comovement (i.e. positive correlations between aggregate growth and growth rates of each sector) if $z^c = 0.99 > z^n = 0.95$ (and $\Psi = 0.9$). Alternative mechanisms are possible which can generate comovement for aggregate shocks to A or to the collateral share λ which are correlated with sectoral shifts.

8.4 Other loan enforcement mechanisms

Reputational loans in our model are supported by perpetual exclusion of defaulting agents from all borrowing in excess of collateral. Alternative punishment mechanisms are conceivable and have been explored in the literature, mostly in economic environments without production. A much more powerful enforcement of credit arrangements is obtained when defaulters can be shut out of all intertemporal trade (borrowing and lending) as is the case in the models of Kehoe and Levine (1993) and Alvarez and Jermann (2000). In their pure–exchange models with zero collateral it is well known that first–best allocations can be implemented with unsecured loans provided that all agents share a *common* high discount factor and have sufficiently large income variability (see also Azariadis and Kaas (2007)). Similar results can also be obtained for our model; the only change in the model's equilibrium equations is that the defaulters' equity returns $\tilde{R}^i_c(\mathbf{x}, s)$ in equations (10) must be replaced by the autarkic returns A^i_s. In the two–agent, two–state special case without collateral, it is straightforward to show that there is a first–best equilibrium at the symmetric wealth distribution iff

$$\ln \frac{1}{z} \geq \frac{(1-\beta)(1+\beta-2\beta\pi)}{\beta(1-\pi)} \ln 2 \ .$$

Thus the first best is an equilibrium if the productivity differential is sufficiently large or if agents are sufficiently patient.

On the other hand, one can also think of *weaker* punishment scenarios in our model. For example, defaulting agents may be shut out of unsecured credit for a finite number of periods before they regain unlimited access to unsecured loans. Alternatively, defaulters may sometimes evade punishment and have a positive chance of obtaining unsecured loans. Both mechanisms are rather straightforward extensions of our model. Shortened punishment periods and lower detection probabilities tighten debt limits considerably and thereby contribute to lower growth and higher aggregate TFP volatility.

9 Conclusions

This paper outlines a financial theory of aggregate factor productivity which connects the sectoral allocation of capital with sectoral productivity shocks and credit market frictions. We emphasize frictions arising from insufficient collateral for secured loans and from the limited enforcement of unsecured loans. Both of these lead to endogenous debt limits which slow down the reallocation of surplus capital from less productive to more productive sectors, and prevent the equalization of sectoral productivities and sectoral rates of return.

Our model is consistent with much empirical evidence suggesting that economy-wide factor productivity and economic growth are both negatively correlated with the dispersion of sectoral stock returns, the dispersion of sectoral TFP's, and the dispersion of sectoral growth rates. If we reinterpret "sectors" to be individual firms, then our results are also consistent with recent work by Hsieh and Klenow (2007) who find that industry productivity dispersion is negatively correlated with industry productivity in a panel that includes data from the U.S., China and India. Alternatively, if we index countries by the fraction of collateral assets to total resources, our results are in line with Diebold and Yilmaz (2008) who find that macroeconomic volatility is positively correlated with stock market volatility in a cross section of countries.

A clear example of macroeconomic volatility is the one that, as of this writing, is gripping developed economies throughout the world. Its symptoms are a substantial fall in economic activity and unusually steep reductions in asset prices and the volume of credit. For example, seasonally adjusted weekly averages of all commercial paper issued in the United States fell from \$170bn for all of 2007 to just about \$100bn for the week ending June 12, 2009. The seasonally adjusted stock of all commercial paper outstanding dropped by 20% in the six months following November 30, 2008.

Our model offers a simple explanation for such episodes of rapid disintermediation. We interpret them as transitions from a well—intermediated, socially desirable and fragile state with plenty of unsecured credit to a poorly–intermediated, less desirable but asymptotically stable state in which all loans are collateralized. The impulse for this transition is widespread skepticism about the ability of financial intermediaries to continue the provision of unsecured credit at the volume needed to support the socially desirable equilibrium. An equivalent interpretation of the movement from the good to the bad state is that it is triggered by a "sunspot" variable, similar to the bubble bursting equilibrium considered by Kocherlakota (2009). We conjecture that bubbles on financial assets are equivalent to unbacked private debt in our environment, as they are in the pure–exchange model of Hellwig and Lorenzoni (2008).

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Appendix

Proof of Proposition 1:

(a) Let $g(x;z) \equiv 1 - f(x;z)$ and define (f^n, g^n) to be the n–order iterates of the maps (f,g), where $(f^0(x;z), g^0(x;z)) = (x,x)$ and $(f^1, g^1) = (f,g)$. Then one easily shows

$$\begin{aligned}
f^n(x;z) &= f(x;z^n) \quad \text{for all } n \geq 0 \ , \\
g^2(x;z) &= x \quad , \\
g[f(x;z);z] &= g(x;z^2) \quad , \\
f[g(x;z^n);z] &= g(x;z^{n-1}) \quad \text{for all } n \geq 1 \ ,
\end{aligned}$$

for all $z \in [0,1]$. For any initial value $x_0 \in (0,1)$, we can easily show by induction that x_t is a Markov chain on the asymptotic set

$$\left\{x_0, f(x_0; z^n), f(1-x_0; z^n), 1-x_0, g(x_0; z^n), g(1-x_0; z^n) \text{ for } n \geq 1\right\} .$$

However, $x_0 = f(x_0; z^n)$ for $n = 0$, $1 - x_0 = g(x_0; z^n)$ for $n = 0$. In addition, $g(x; z^n) = f(1-x; z^{-n})$ for all x and $n \geq 1$. Therefore the set of asymptotic states consists of the list in Proposition 1 if we allow n to be any integer.

(b) Follows from the fact that the set of asymptotic states depends on the initial value x_0. □

Proof of Proposition 2:

It remains to show existence and uniqueness of a market–clearing interest rate $R = R(\sigma)$ with secured borrowing for any $\sigma = (\mathbf{x}, s)$. For any $R > \lambda A \geq \lambda A_s^i$, collateral debt limits are $\theta_c^i(R) = \dfrac{\lambda A_s^i}{R - \lambda A_s^i}$. Debt limits are decreasing in the interest rate, and so is the demand for credit $\mathrm{CD}(\sigma)$; it is a downward–sloping function with finitely many upward jumps at $R = A_s^i$, it is zero at $R \geq A$ and it tends to infinity as $R \to \lambda A$. On the other hand, the supply of credit $\mathrm{CS}(\sigma)$ is a weakly increasing step function which is zero at $R = 0$ and finite at $R \geq A$. Because of these features, there exists a unique market–clearing interest rate for any σ. □

Proof of Proposition 3:

A candidate first–best equilibrium has stable wealth shares $\mathbf{x}^* = (x^{i*})_{i \in \mathcal{I}}$ and an interest factor equal to the frontier productivity, $R(\mathbf{x}^*, s) = A$ for all $s \in \mathcal{S}$. With secured borrowing ($v^i = 0$), the debt limits then follow from (4) as $\theta^i(\mathbf{x}^*, s) = \lambda/(1-\lambda)$ for all $i \in \mathcal{I}$ and $s \in \mathcal{S}$. In every productive state s, there is by assumption (A2) a unique agent $i(s)$ using the frontier technology. Because of (A3), no state is trivial, hence credit market equilibrium requires that for any $s \in \mathcal{S}$, agent $i(s)$'s maximum demand for credit does not exhaust credit supply of all other agents:

$$\frac{\lambda}{1-\lambda} x^{i(s)*} \geq \sum_{j \neq i(s)} x^{j*} = 1 - x^{i(s)*} \ , \ \ s \in \mathcal{S} \ ,$$

which is $\lambda \geq 1 - x^{i(s)*}$ for all s. By assumption (A1), every agent has access to the frontier technology in at least one state. Thus the first best is an equilibrium with secured borrowing iff $\lambda \geq 1 - x^{i*}$ for all $i \in \mathcal{I}$. For this to be true *at some distribution of wealth* x^{i*}, it must hold in particular at the symmetric distribution of wealth, $x^{i*} = 1/I$. Therefore, the condition $\lambda \geq (I-1)/I$ is necessary and sufficient for the first best to be an equilibrium with secured borrowing for some distribution of initial wealth.

Now suppose that $\lambda < (I-1)/I$ and suppose that there is a first–best equilibrium with secured and unsecured borrowing at stable wealth distribution $\mathbf{x}^* = (x^{i*})_{i\in\mathcal{I}}$ and interest yields $R(\mathbf{x}^*, s) = A$, $s \in \mathcal{S}$. But then from (6) and (7), $\tilde{R}^i(\mathbf{x}^*, s) = \tilde{R}^i_c(\mathbf{x}, s) = A$ for all $i \in \mathcal{I}$ and $s \in \mathcal{S}$, and from (10) follows that $v^i(\mathbf{x}^*, s) = 0$ for all (i, s). But this in connection with (4) implies again that all borrowing is secured, so debt limits are $\theta^i(\mathbf{x}^*, s) = \lambda/(1-\lambda)$, and the credit market cannot be in equilibrium, as seen above. $\square$

Proof of Proposition 4:

Parts (a)–(b) follow simply from inspection of Figure 2 (a) and (b). To prove (c), it is useful to note the following features of the maps $X_0(x)$ and $X_1(x)$. For any $x \leq 1 - \lambda/z$, it holds that $X_1X_1(x) = x$ and that $X_0X_1(x) = 1 - x$.

Again it is clear from the graph that the minimum and maximum elements of the asymptotic invariant set are $\underline{x} = \lambda$ and $\overline{x} = 1 - \lambda$. Let $\ell \geq 1$ be the unique number such that $X_0^{\ell-1}(\lambda) < 1-\lambda/z$ and $X_0^{\ell}(\lambda) \geq 1-\lambda/z$ and suppose the last inequality is strict (a generic feature). Obviously then, $X_0^k(\lambda)$, $k = 1, \ldots, \ell$ are also elements of the asymptotic invariant set. Further elements are the ℓ wealth states $X_1X_0^k(\lambda)$ for $k = 0, \ldots, \ell-1$, which are all in the interval $(\lambda, 1-\lambda/z]$ and which are generically different from the other elements. Note that $X_1X_0^{\ell}(\lambda) = \lambda$ is not a new element of the asymptotic invariant set. To see that there are no further elements, note that any further iteration from $X_1X_0^k(\lambda)$ can only lead either to $X_1X_1X_0^k(\lambda) = X_0^k(\lambda)$ or to $X_0X_1X_0^k(\lambda) = 1 - X_0^k(\lambda) = X_1X_0^{k-1}(\lambda)$ which are both already elements of the asymptotic invariant set. Hence the asymptotic invariant set comprises λ, $1-\lambda$, $X_0^k(\lambda)$ for $k = 1, \ldots, \ell$ and $X_1X_0^k(\lambda)$ for $k = 0, \ldots, \ell-1$, which are $2\ell + 2 = 2m$ elements with $m = \ell + 1 \geq 2$. Of these, exactly the three $[1-\lambda,\ X_0^{\ell}(\lambda)$ and $X_1(\lambda) = 1-\lambda/z]$ are not smaller than the threshold $1-\lambda/z$ and have growth rates at $\beta A - 1$. All other states have growth rates below $\beta A - 1$. $\square$

Proof of Proposition 5:

Without loss of generality, set $A = 1$ to simplify notation. Consider first a production–efficient steady state x with debt–equity constraint $\theta = (1-x)/x$ and interest rate $R \in (z, 1)$. x is a steady state if $x = X_1(x) = R(1-x)$, and hence $x = R/(1+R)$, $\theta = 1/R$, $\tilde{R} = 1 + \theta(1-R) = 1/R$, and $\tilde{R}_c = R(1-\lambda)/(R-\lambda)$. Let v and w be the default penalties for borrowing and lending agents (of both types) in steady state. From (10) follows that v and w satisfy

$$\begin{aligned} v &= \beta w \ , &\quad (11)\\ w &= \beta\Big\{\ln\Big[\frac{\tilde{R}}{\tilde{R}_c}\Big] + v\Big\} \ . &\quad (12)\end{aligned}$$

Hence,

$$v = \frac{\beta^2}{1-\beta^2}\ln\Big(\frac{\tilde{R}}{\tilde{R}^c}\Big) = \frac{\beta^2}{1-\beta^2}\ln\Big(\frac{R-\lambda}{R^2(1-\lambda)}\Big) \ .$$

On the other hand, (4) implies that

$$v = \ln\Big[(1-\lambda)(1+R)\Big] \ . \quad (13)$$

From these two equations follows that the steady–state interest rate must solve

$$(1-\lambda)^{1/\beta^2} R^2 (1+R)^{(1-\beta^2)/\beta^2} = R - \lambda \ . \quad (14)$$

Moreover, unsecured borrowing requires that $v > 0$ which, from (13), implies that $R > R^c \equiv \lambda/(1-\lambda)$, where R^c is the interest rate with secured borrowing which is always a solution of equation (14). Another solution $R^* > R^c$ exists provided that the slope of the LHS at R^c is smaller than one. This turns out to be the case if and only if $\lambda < \beta/(1+\beta)$. Now R^* indeed constitutes an equilibrium provided that $R^* > z$ and $R^* < 1$. The latter inequality follows if LHS>RHS at $R = 1$. But this turns out to be equivalent to $\lambda < 1/2$ which follows from $\lambda < \beta/(1+\beta)$. The first inequality is true either if $R^c \geq z$ (which is the same as $\lambda \geq z/(1+z)$) or if LHS<RHS at $R = z$. This last inequality is expressed as

$$\frac{(z-\lambda)^{\beta^2}}{1-\lambda} > z^{2\beta^2}(1+z)^{1-\beta^2} \ . \quad (15)$$

This inequality becomes an equality at $\lambda = z/(1+z)$, and the left–hand side is decreasing in $\lambda > \lambda_0 \equiv (z-\beta^2)/(1-\beta^2)$ and increasing in $\lambda < \lambda_0$. When $z \geq \beta$,

$\lambda_0 \geq z/(1+z)$ holds, and hence (15) is violated for any $\lambda \leq z/(1+z)$; hence there is no steady state with unsecured borrowing in this case. When $z < \beta$, there must be a threshold $\tilde{\lambda} < \lambda_0$ where (15) holds with equality. Hence, with $\hat{\lambda} = \max(0, \tilde{\lambda})$, there exists a steady state with unsecured borrowing for any $\lambda \in [\hat{\lambda}, \beta/(1+\beta))$.

Next consider a production–inefficient steady state where $R = z$, $\tilde{R} = 1 + \theta(1-z)$ and $\tilde{R}_c = z(1-\lambda)/(z-\lambda)$, and again let v be the stationary penalty of default for a borrowing agent. Now (11) and (12) can be expressed as

$$v = \frac{\beta^2}{1-\beta^2} \ln \left(\frac{[1+\theta(1-z)](z-\lambda)}{z(1-\lambda)} \right) ,$$

and (4) becomes

$$v = \ln \left[\frac{(1-\lambda)(1+\theta)}{1+\theta(1-z)} \right] . \tag{16}$$

Equating the two yields an equation in the debt–equity ratio,

$$1 + \theta = \Big(1 - \lambda/z\Big)^{\frac{\beta^2}{1-\beta^2}} \left(\frac{1+\theta(1-z)}{1-\lambda} \right)^{\frac{1}{1-\beta^2}} . \tag{17}$$

Here, the debt–equity ratio with secured borrowing $\theta^c = \lambda/(z-\lambda)$ solves this equation. Another solution θ^* corresponds to an equilibrium with unsecured borrowing only if $\theta^* > \theta^c$, and such a solution exists iff the slope of the RHS at θ^c is smaller than one. This is the case iff

$$\lambda < \lambda_0 = \frac{z - \beta^2}{1 - \beta^2} .$$

The solution $\theta^* > \theta^c$ indeed gives rise to a production–inefficient equilibrium at $R = z$ if $\theta^* < (1-x^*)/x^*$ at the stationary borrower share x^* which satisfies

$$x = X_1(x) = \frac{z(1-x)}{[1+\theta^*(1-z)] + z(1-x)} ,$$

or

$$(1-z)(1+\theta^*)x^2 + 2zx - z = 0 .$$

Clearly this quadratic has a unique solution $x^* \in (0,1)$ for any $\theta^* > 0$. Now $\theta^* < (1-x^*)/x^*$ when $x^* < 1/(1+\theta^*)$ which is the case if the quadratic is positive at $x = 1/(1+\theta)$, which in turn is equivalent to $\theta^* < 1/z$. This condition is fulfilled whenever in (17) the LHS is smaller than the RHS at $\theta = 1/z$. But this inequality is equivalent to (15) again. Because the LHS in (15) is increasing in $\lambda \in [0, \lambda_0]$, the inequality is satisfied for all $\lambda \in (\hat{\lambda}, \lambda_0)$. Hence

$$\hat{\lambda} < \lambda < \frac{z - \beta^2}{1 - \beta^2}$$

is a necessary and sufficient condition for a production inefficient steady state equilibrium with secured and unsecured borrowing. $\square$

Proof of Proposition 6:

In the deterministic economy, the dynamic versions of the steady–state equations (11) and (12) can be simplified to

$$v_t = \beta^2 \ln\left(\frac{\tilde{R}_{t+2}}{\tilde{R}^c_{t+2}}\right) + \beta^2 v_{t+2} \ . \tag{18}$$

Suppose first that the economy is production inefficient in all periods. Then, $R_t = z$, $\tilde{R}^c_t = z(1-\lambda)/(z-\lambda)$, and from (4) follows

$$\tilde{R}_t = \frac{(1-\lambda)z}{1 - \lambda - e^{v_t}(1-z)} \ .$$

Substitution into (18) yields

$$v_t = \beta^2 \ln\left(\frac{z-\lambda}{1-\lambda-e^{v_{t+2}}(1-z)}\right) + \beta^2 v_{t+2} \ .$$

Note that v_t is a forward–looking (jump) variable. Hence, the steady state $v = 0$ is locally indeterminate iff $dv_t/(dv_{t+2})|_{v=0} > 1$. But this condition turns out to be the same as

$$\lambda > \frac{z - \beta^2}{1 - \beta^2} \ ,$$

which follows from $z < \beta^2$. Hence, there is an infinity of equilibria with $v_t \to 0$ (and thus $\theta_t \to \theta^c$). In the limit, the dynamics of borrower wealth becomes

$$x_{t+1} = \frac{(1-x_t)(z-\lambda)}{z - \lambda + x_t(1-z)} = X_1(x_t) \ ,$$

which satisfies $x_{t+2} = X_1^2(x_t) = x_t$. Hence, in all these equilibria, wealth converges to a cycle of periodicity two (where one of these "cycles" is the secured–borrowing steady state).

Next consider a production–efficient economy. Here $\theta_t = (1 - x_t)/x_t$, and from (4) follows

$$R_t = \frac{1 - e^{-v_t}(1-\lambda)}{1 - x_t} \ , \ \tilde{R}_t = \frac{1-\lambda}{e^{v_t} x_t} \ , \ \tilde{R}_t^c = \frac{[1 - e^{-v_t}(1-\lambda)](1-\lambda)}{1 - e^{-v_t}(1-\lambda) - \lambda(1-x_t)} \ .$$

Substitution into (18) yields

$$v_t = \beta^2 \ln\left(\frac{1 - e^{-v_{t+2}}(1-\lambda) - \lambda(1 - x_{t+2})}{[e^{v_{t+2}} - 1 + \lambda]x_{t+2}}\right) + \beta^2 v_{t+2} \ . \tag{19}$$

The dynamics of borrower wealth is

$$x_{t+1} = \frac{R_t(1-x_t)}{R_t(1-x_t) + \tilde{R}_t x_t} = 1 - e^{-v_t}(1-\lambda) \ .$$

Substitution into (19) gives

$$v_t = \beta^2 \ln\left(\frac{1 - e^{-v_{t+2}}(1-\lambda) - \lambda(1-\lambda)e^{-v_{t+1}}}{[e^{v_{t+2}} - 1 + \lambda][1 - e^{-v_{t+1}}(1-\lambda)]}\right) + \beta^2 v_{t+2} \ .$$

Using $\varphi_t = e^{v_t}$, this equation is more conveniently expressed as

$$\varphi_t = \left[\frac{\varphi_{t+1}(\varphi_{t+2} - 1 + \lambda) - \lambda(1-\lambda)\varphi_{t+2}}{(\varphi_{t+2} - 1 + \lambda)(\varphi_{t+1} - 1 + \lambda)}\right]^{\beta^2} \ . \tag{20}$$

A steady state is a solution of

$$\varphi^{(1-\beta^2)/\beta^2} = \frac{\varphi - 1 + \lambda^2}{(\varphi - 1 + \lambda)^2} \ . \tag{21}$$

One solution is $\varphi = 1$ which (under appropriate conditions) gives rise to a steady state with secured borrowing. A steady state with secured and unsecured borrowing must be a solution with $\varphi > 1$. Again φ_t is a forward–looking jump variable; hence a steady state is locally determinate if both eigenvalues of the backward dynamics (20) have modulus less than one, and a steady state is locally indeterminate if at least one eigenvalue has modulus larger than one. It is straightforward to calculate the determinant and trace of the Jacobian at the steady state:

$$\begin{aligned} D = -\frac{d\varphi_t}{d\varphi_{t+2}} &= \frac{\beta^2 \lambda (1-\lambda)^2}{(\varphi - 1 + \lambda)(\varphi - 1 + \lambda^2)} \ , \\ T = \frac{d\varphi_t}{d\varphi_{t+1}} &= -\frac{\beta^2(\varphi - 1)(1-\lambda)^2}{(\varphi - 1 + \lambda)(\varphi - 1 + \lambda^2)} \ . \end{aligned}$$

At a steady state with secured borrowing ($\varphi = 1$), $D > 1$ if and only if $\lambda < \beta/(1+\beta)$. Hence, this steady state is indeterminate whenever there is another one with secured and unsecured borrowing. Such a steady state is determinate, provided that $D < 1$ and $D + T > -1$. At $\lambda = 0$, $D = 0$ and $T > -1$ requires that $\varphi > 1 + \beta^2$. Since $\varphi = (\varphi - 1)^{-\beta^2/(1-\beta^2)}$, this inequality is true provided that

$$1 + \beta^2 < (\beta^2)^{-\beta^2/(1-\beta^2)} ,$$

which is true for all $\beta^2 \in (0, 1)$. When λ increases, it can be shown numerically that both $D(\lambda)$ and $D(\lambda) + T(\lambda)$ increase (where $\varphi(\lambda) > 1$ adjusts to solve (21)), regardless of the value of β. Moreover $D(\lambda)$ converges to 1 and $T(\lambda)$ converges to zero when $\lambda \to \beta/(1 + \beta)$ (and $\varphi(\lambda) \to 1$). Therefore $D(\lambda) < 1$ and $D(\lambda) + T(\lambda) > -1$ are satisfied for any $\lambda \in [0, \beta/(1 + \beta))$, and hence the steady state with secured and unsecured borrowing is locally determinate. □

Stochastic equilibrium cycles

Proposition 7: *A production–efficient cycle with secured and unsecured borrowing:*

(a) *Does not exist for any λ if z is large, that is, if*

$$z \geq \frac{\beta(1-\pi)}{1-\beta\pi} .$$

(b) *Exists for intermediate λ and small z, that is, if*

$$\lambda \in \left[\underline{\lambda}, \frac{\beta(1-\pi)}{1+\beta-2\beta\pi}\right) \quad \textit{and} \quad z < \frac{\beta(1-\pi)}{1-\beta\pi} ,$$

for some threshold $\underline{\lambda}$ which equals zero whenever z falls below another threshold $\underline{z} < \beta(1-\pi)/(1-\beta\pi)$.

Part (a) says that production efficient cycles with unsecured borrowing cannot exist if agents are too impatient or if productivity fluctuations are too small. Part (b) says the opposite: sufficiently large productivity fluctuations give rise to a production–efficient equilibrium which also requires collateral to be neither too large nor too small. For example, if collateral is large enough, production efficient (or even first best) allocations are achieved by secured borrowing alone; a good credit reputation is then worthless. And when collateral is too small, any equilibrium with unsecured borrowing must involve some states of production inefficiency. But when z is smaller than threshold $\underline{z}$, unsecured loans can support production–efficient allocations even in the complete absence of collateral.

Proof:

A production–efficient cycle of order two has a support at $x_1 < x_2$ where x_1 applies if the productive state changes ($s_t \neq s_{t-1}$), and x_2 applies if the state stays the same. The corresponding default penalties for borrowers and lenders in these two situations are denoted v_j, w_j, $j = 1, 2$. The cycle has the following features:

- the allocation is first best at $x = x_2$, i.e. $R_2 = A$ and

$$\theta_2 = \frac{e^{v_2} + \lambda - 1}{1 - \lambda} \geq \frac{1 - x_2}{x_2} \ .$$

- the allocation is production–efficient but not first–best at $x = x_1$, so that $\theta_1 = (1 - x_1)/x_1$ and

$$R_1 = A\frac{e^{v_1} + \lambda - 1}{e^{v_1}(1 - x_1)} \in (zA, A) \ .$$

Again simplify notation by setting $A = 1$. Because the economy is first–best at x_2, $X_0(x_2) = x_2$ and $X_1(x_2) = 1 - x_2$. Hence $x_1 = 1 - x_2$. At $x = x_1$, rates of return are

$$R_1 = \frac{e^{v_1} + \lambda - 1}{e^{v_1}(1 - x_1)} < \tilde{R}_1 = \frac{1 - \lambda}{x_1 e^{v_1}} \ .$$

If the productive state is unaltered, borrower wealth must thus increase from x_1 to x_2, which implies

$$X_0(x_1) = \frac{\tilde{R}_1 x_1}{\tilde{R}_1 x_1 + R_1(1 - x_1)} = e^{-v_1}(1 - \lambda) = x_2 = 1 - x_1 \ ,$$

and therefore $x_1 = 1 - e^{-v_1}(1 - \lambda)$.

The recursive equations in default penalties are

$$\begin{aligned}
v_1 &= \beta\pi v_2 + \beta(1-\pi)w_1 \ , \\
w_1 &= \beta\pi w_2 + \beta(1-\pi)\Big[\ln(\tilde{R}_1/\tilde{R}_1^c) + v_1\Big] \ , \\
v_2 &= \beta\pi v_2 + \beta(1-\pi)w_1 \ , \\
w_2 &= \beta\pi w_2 + \beta(1-\pi)\Big[\ln(\tilde{R}_1/\tilde{R}_1^c) + v_1\Big] \ .
\end{aligned}$$

From these follows $v_1 = v_2 = v$, $w_1 = w_2 = w$, and

$$\begin{aligned}
v &= C\ln\Big(\frac{\tilde{R}_1}{\tilde{R}_1^c}\Big) \\
&= C\ln\Big(\frac{e^v + \lambda - 1 - \lambda e^v(1-x_1)}{(e^v+\lambda-1)x_1 e^v}\Big) \\
&= C\ln\Big(\frac{e^v - 1 + \lambda^2}{(e^v+\lambda-1)^2}\Big) \ ,
\end{aligned} \tag{22}$$

with

$$C \equiv \frac{\beta^2(1-\pi)^2}{(1-\beta)(1+\beta-2\pi\beta)} \ .$$

Equation (22) has a solution $v > 0$ (necessary for unsecured borrowing) provided that $\lambda^2 + 2\lambda C - C < 0$, which is equivalent to

$$\lambda < \frac{\beta(1-\pi)}{1+\beta-2\beta\pi} \equiv \lambda_1 \ . \tag{23}$$

To make sure that this is indeed a production–efficient equilibrium, it must be verified that $R_1 \in [z, 1)$ and that $\theta_2 \geq (1-x_2)/x_2$. It is easy to show that the last condition holds with equality. $R_1 < 1$ is fulfilled provided that $e^v < 2(1-\lambda)$. But at $e^v = 2(1-\lambda)$ the RHS of (22) is zero; thus the equilibrium e^v must be smaller than $2(1-\lambda)$. Hence it remains to check that $R_1 \geq z$. This is true iff $e^v \geq (1-\lambda)(1+z)$ which is valid either if $\lambda \geq z/(1+z)$ or if RHS$\geq$LHS in equation (22) at $e^v = (1-\lambda)(1+z)$. Hence, $R_1 \geq z$ iff

$$\lambda \geq \frac{z}{1+z} \quad \text{or} \quad \Phi(\lambda) \equiv \frac{(z-\lambda)^C}{(1-\lambda)^{1+C}} \geq z^{2C}(1+z) \ . \tag{24}$$

The last condition holds with equality at $\lambda = z/(1+z)$ and Φ has a maximum at $\tilde{\lambda} = z(1+C) - C$. It is straightforward to verify that $\underline{\lambda} \geq \lambda_1$ iff $z/(1+z) \geq \lambda_1$ iff $z \geq \beta(1-\pi)/(1-\beta\pi)$. Hence, if z exceeds this threshold, there is no λ satisfying both (23) and (24), and hence there cannot be a production–efficient cycle with secured and unsecured borrowing, which proves part (a). Conversely, when $z < \beta(1-\pi)/(1-\beta\pi)$, there is a unique $\hat{\lambda} < \underline{\lambda}$ such that the second condition in (24) holds with equality. In that case, any $\lambda \in [\hat{\lambda}, \lambda_1)$ gives rise to a production–efficient cycle with secured and unsecured borrowing, which proves part (b). Finally, the lower bound $\hat{\lambda}$ is non–positive provided that the second condition in (24) holds at $\lambda = 0$ which is the same as

$$1 \geq z^C(1+z) \ .$$

This condition is the same as $z \leq \underline{z}$ for another threshold $\underline{z}(\beta)$ which converges to one when $\beta \to 1$ ($C \to \infty$). $\square$