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An Analytical Approach to Buffer-Stock Saving under Borrowing Constraints*

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Abstract

The profession has been longing for closed-form solutions to consumption functions under uncertainty and borrowing constraints. This paper proposes an analytical approach to solving general-equilibrium buffer-stock saving models with both idiosyncratic and aggregate uncertainties as well as liquidity constraints. It is shown analytically that an individual's optimal consumption plan follows the rule of thumb: Consumption is proportional to a target level of wealth, with the marginal propensity to consume dependent on the state of the macroeconomy. I apply this method to address two long-standing puzzles in general equilibrium: the "excess smoothness" and "excess sensitivity" of consumption with respect to income changes. Some of my findings sharply contradict the conventional wisdom.

Keywords: Buffer Stock Saving, Borrowing Constraints, Consumption Puzzles, Excess Smoothness, Excess Sensitivity, Permanent Income Hypothesis.

JEL Codes: D91, E21.

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1 Introduction

Postwar aggregate U.S. data show that lagged output growth significantly predicts consumption growth, and the standard deviation of consumption growth is only about half of that of GDP growth. However, the canonical optimizing consumption model with a quadratic utility function, constant interest rate, and stochastic labor income predicts that consumption growth is independent of lagged income changes and should be more volatile than labor-income growth if aggregate income growth has positive serial correlation (as the data suggest it does). Thus, aggregate consumption growth has been described as exhibiting two puzzles: it is both "excessively sensitive" to lagged (or predictable) income changes (Flavin, 1981) and "excessively smooth" relative to current income growth (Deaton, 1987; and Campbell and Deaton, 1989).

Although the quadratic utility function is too stylized, its implications are more general (see, e.g., Hall, 1978). Hence, this special model has served as the modern-day reincarnation of Friedman's (1957) permanent income hypothesis (PIH) that consumption is determined by permanent income rather than by current income. Despite its intuitive appeal, the simple PIH model has encountered some empirical anomalies (such as the aforementioned two puzzles), which has triggered the growth of a vast literature to seek for resolutions.[1]

Chief modification of the canonical PIH model to resolve the two aforementioned puzzles is the buffer-stock saving model (e.g., Deaton, 1991; Carroll, 1992), which modifies the simple PIH model to allow for precautionary saving motives (due to a positive third derivative of the utility function), impatience (due to heavy discounting of the future), and borrowing constraints (due to imperfect financial markets). However, the modified model inherits two crucial features of the canonical PIH model: constant interest rate and exogenous labor income. Such a partial-equilibrium framework has served as the workhorse of modern dynamic consumption theory.[2]

Can buffer-stock saving quantitatively explain the smoothness of aggregate consumption and its correlation with lagged aggregate income? According to a recent study by Ludvigson and Michaelides (2001), the answer is "no". Ludvigson and Michaelides (2001) use a calibrated heterogeneous-agent buffer-stock model to show that aggregate consumption growth is as volatile as aggregate income and it is not predictable by lagged income growth. Even assuming that consumption could not react immediately to current changes in income growth because of information frictions, the

[1] An incomplete list includes Campbell and Deaton (1989), Campbell and Mankiw (1989, 1990), Carroll (1992, 1994, 1997), Carroll and Kimball (1996, 2001), Christiano, Eichenbaum, and Marshall (1991), Deaton (1987, 1991, 1992), Ermini (1993), Flavin (1981, 1985), Gali (1991), Hayashi (1987), Ludvigson and Michaelides (2001), Michaelides (2002), Pischke (1995), Sommer (2007), Quah (1990), and Zeldes (1989a, 1989b).

[2] See, e.g., the literature servey by Carroll (2001).

volatility of consumption growth would still be about 90% of that of income growth, far above the estimated value of 0.48 in the U.S. data. Hence, their analysis suggests that modern consumption theories are inadequate and incapable of explaining the two anomalies of aggregate consumption behaviors.

Ironically, the real-business-cycle (RBC) literature has long predicted that aggregate consumption and its growth rate are very smooth relative to aggregate income and output growth under aggregate technology-level shocks (see, e.g., Kydland and Prescott, 1982; King, Plosser, and Rebelo, 1988). The RBC literature argues that such predictions are precisely what the PIH would imply. Yet despite the popularity of the RBC literature, virtually no research has set out to formally investigate whether standard general-equilibrium RBC models can explain the two well-established consumption puzzles quantitatively.[3] It is well known that under shocks to the level of technology, RBC models generate very smooth consumption both at the level and at the growth rate. However, it is much less clear what may happen when shocks originate from the growth rate of technology (which may be serially correlated) instead from the level. This provides the first motivation of this paper: to investigate whether stochastic changes in the growth rate of technology can lead to the two consumption puzzles in general equilibrium. The results also serve as a reference point for my analysis of heterogeneous-agent buffer-stock models.

But there is a reason for the buffer-stock literature to maintain the assumptions of constant interest rate and exogenous labor income even when the focus of the analysis is on the relationship between aggregate consumption and aggregate income: computational costs. Despite these extreme assumptions, the standard buffer-stock literature still has to rely on numerical computational methods to solve for individuals' decision rules of consumption and savings before aggregation (see, e.g., Deaton, 1991; and the literature survey by Carroll, 2001). The computational difficulty is compounded in general-equilibrium models when the interest rate and labor income are endogenous and time varying (see, e.g., Krusell and Smith, 1998). This computational challenge provides the second motivation of this paper: to provide an analytically tractable method to solve a heterogeneous-agent buffer-stock model in general equilibrium. Analytical tractability is a great virtue because it makes the model's economic mechanisms transparent and comparative statistics easy to conduct.[4]

My analysis yields three major findings: (i) The excess-smoothness and excess-sensitivity puzzles are exaggerated by the consumption literature. In general equilibrium with endogenous interest rate and labor income (in spite of serially correlated shocks to the growth rate of technology), aggre-

[3] The only exception I am aware of is Michener (1984). Michener clearly pointed out that the apparent failure of the PIH has to do with the assumption of a constant interest rate. However, Michener used a RBC model with 100 percent depreciation of capital and considered only the excess sensitivity puzzle. As will become clear in the next section, this special model cannot explain the excess smoothness puzzle. Also, Michener did not consider technology-growth shocks.

[4] As noted by Carroll and Kimball (2001, p.1), "A drawback to numerical solutions ... is that often it is difficult to determine why results come out the way they do." In the case of no borrowing constraints and an extremely simplified form of labor-income uncertainty, tractable models are obtained by Toche (2005) and Carroll and Toche (2009).

gate consumption growth is significantly less volatile than output growth even without borrowing constraints; namely, its standard deviation is only about 70% of that of output growth, not more volatile than output as predicted by partial-equilibrium consumption models. In addition, general-equilibrium theory predicts that current-period consumption growth should be positively correlated with lagged income growth when income growth is serially correlated (as it is in the data). However, discrepancies between theory and data still exist and are surprisingly robust to parameter values. Hence, although the puzzles are exaggerated, a general-equilibrium model with endogenous interest rate and labor supply cannot completely desolve them with empirically plausible parameter values. (ii) Borrowing constraints can significantly reduce the volatility of consumption growth relative to income growth if the degree of heterogeneity (or consumption inequality) is sufficiently large; but it does not improve the predictive power of lagged income for future consumption growth. This is in contrast to the findings of Ludvigson and Michaelides (2001) based on partial-equilibrium analysis where they argue that borrowing constraints are not very effective in both reducing consumption-growth volatility and raising consumption sensitivity to lagged income. (iii). Habit formation is very effective in both reducing the relative volatility of consumption growth and increasing the sensitivity of consumption to lagged income growth. However, habit formation tends to "over-kill" the sensitivity puzzle: with mild degrees of habit formation the model predicts that the correlation between consumption growth and lagged income growth is twice as strong as it is in the data. In other words, under habit formation the data exhibit "excess *insensitivity*" rather than "excess sensitivity" of consumption towards predictable income changes. This is in contrast to the findings of the literature (e.g., Michaelides, 2002). All of the above results are obtained analytically in this paper without resorting to complicated numerical computation methods such as that in Krusell and Smith (1998).

Two simplifying strategies allow me to solve a heterogeneous-agent general-equilibrium buffer-stock model analytically. First, the idiosyncratic shocks are *i.i.d.*, orthogonal to aggregate uncertainty, and come from preferences rather than from labor income. But the analytical tractability carries through and the results remain similar if the idiosyncratic uncertainty is from wealth-income instead.[5] Second, and more importantly, the utility function is linear for leisure and labor-supply decisions are made before observing the idiosyncratic shocks. These simplifying strategies make the expected marginal utility of an individual's consumption and the cutoff value for target wealth independent of idiosyncratic shocks. With these properties, closed-form decision rules for individuals' consumption and saving plans can be obtained. After aggregating individual decision rules by the law of large numbers, the aggregate variables form a system of non-linear dynamic equations as in a representative-agent model. Hence, traditional solution methods available in the RBC liter-

[5] See the Appendix.

ature can be applied to solving the model's equilibrium saddle-path, given the distribution of the idiosyncratic shocks. The impulse response functions to aggregate shocks and second (or higher) moments of the model can then be computed analytically following the RBC literature (e.g., the method of King, Plosser, and Rebelo, 1988).[6]

These simplifying strategies have some costs,[7] but the payoff is significant: They not only make the model analytically tractable with closed-form solutions for individuals' decision rules (despite a time-varying interest rate), but also reduce the computational costs down to the level of solving a representative-agent RBC model. In addition, the mechanisms of buffer-stock saving become transparent and we obtain all the essential insights of the buffer-stock saving theory, such as that the accumulation of wealth follows a target strategy (Deaton, 1991) and that individuals opt to save excessively so as to be well-insured against uncertainty (Aiyagari, 1994; Carroll, 1997; Krusell and Smith, 1998).

The reason general-equilibrium models generate smoother consumption growth than income growth is that technology growth increases the marginal product of capital, which drives up the real interest rate and hence the marginal propensity to save, rather than raising the marginal propensity to consume as a canonical PIH model would predict under a constant interest rate. Borrowing constraints can further reduce consumption-growth volatility not because they prevent consumption from adjusting freely when the constraints bind, but rather because they raise the precautionary saving motive and enhance the buffer-stock role of savings. Consequently, consumption becomes less sensitive to shocks. For the same reason, buffer-stock saving does not increase the sensitivity of current consumption growth towards changes in lagged income because under precautionary-saving motives, agents opt to save excessively so that they are very well self-insured against uncertainty. As a result, consumption is not any more sensitive to past income than it would be without borrowing constraints.

The rest of the paper is organized as follows. Section 2 presents a standard frictionless RBC model as a control model (reference point). It is shown that under serially correlated shocks to aggregate technology growth (i.e., to the source of permanent income), the model exhibits both the "excess smoothness" puzzle and the "excess sensitivity" puzzle, albeit to a significantly less degree than claimed in the consumption literature. It is also shown that endogenous labor supply is irrelevant for these results; hence, a constant interest rate assumed in the consumption literature is crucial for causing the discrepancies. Section 3 introduces borrowing constraints and uninsurable idiosyncratic risks into the control model and shows how to solve analytically for individuals'

[6] If the depreciation rate of capital stock is 100 percent, then closed-form solutions for the aggregate variables can also be obtained for the general-equilibrium buffer-stock model by pencil and paper.

[7] One of the costs is that the elasticity of labor supply is not a free parameter. Another is that the wealth distribution is degenerate under idiosyncratic preference shocks. However, the distribution of consumption and savings are not degenerate. The degenerate wealth distribution can be avoided by considering wealth shocks (see the Appendix).

optimal consumption and saving plans as functions of the aggregate variables. Impulse responses to technology-growth shocks and second moments of the model's growth rates are also computed analytically using the log-linearization method. Section 4 analyzes the effects of habit formation in a heterogeneous-agent buffer-stock model. Section 5 concludes the paper. In the Appendix, I also illustrate how to solve a general-equilibrium buffer-stock model analytically when the idiosyncratic shocks do not originate from preferences but from wealth-income. This also serves as a robustness check to the results in Section 3.

2 The Control Model

The control model is a standard and perhaps the simplest version of the real-business-cycle (RBC) model of Kydland and Prescott (1982). There are two sources of uncertainty in the model: shocks to the level of technology and to the growth rate of technology. Therefore, the model is not stationary in the level but stationary in the growth rate. To solve the model, we first transform the economy into one without growth by a proper normalization and derive decision rules around the steady state. We then uncover the growth dynamics of the original model around its long-run balanced growth path by an inverse transformation.

There is a unit mass of continuum of identical households who, taking as given the market real interest rate and real wage, choose sequences of consumption (C), savings (S), and labor supply (N) to maximize expected life-time utility, $E_0 \sum_{t=0}^{\infty} \beta^t \{\log C_t - aN_t\}$, subject to the budget constraint $C_t + S_{t+1} \leq (1+r_t)S_t + W_t N_t$, where r is the real interest rate and W the real wage. The population is constant over time. Leisure enters the utility linearly to reflect indivisible labor (Hansen, 1985; Rogerson, 1988). The linearity simplifies the analysis of our heterogeneous-agent buffer-stock model in the next section. Without loss of generality, assume $a = 1$.

There is also a unit mass of continuum of identical firms producing output according to the constant-returns-to-scale technology, $Y_t = A_t K_t^{\alpha} (Z_t N_t)^{1-\alpha}$, where A_t denotes a stationary process of shocks to total factor productivity (TFP) and Z_t a non-stationary process of labor-augmenting technology. Labor augmenting technology grows over time according to $Z_t = (1 + g_t) Z_{t-1}$, where g_t is a stochastic growth rate with mean $\bar{g} \geq 0$. When $\bar{g} = 0$, the dynamic effects of g_t and A_t are identical if g_t is *i.i.d.* and A_t is a random walk without drift. The laws of motion for the two driving processes are given by

$$\log A_t = \rho_A \log A_{t-1} + \varepsilon_{At} \tag{1}$$

$$g_t - \bar{g} = \rho_g (g_{t-1} - \bar{g}) + \varepsilon_{gt}. \tag{2}$$

The capital stock is accumulated according to $K_{t+1} = (1 - \delta) K_t + I_t$, where δ is the depreciation rate and I investment. Firms behave competitively; with perfect capital markets the factor prices are thus determined by marginal products: $W_t = (1-\alpha)\frac{Y_t}{N_t}$ and $r_t + \delta = \alpha \frac{Y_t}{K_t}$.

In the absence of uncertainty, the model has a unique constant balanced growth path, along which the variables $\{C_t, K_{t+1}, S_{t+1}, W_t\}$ all grow at the rate $1+\bar{g}$. Hence, we can transform the model into a stationary one by scaling it down by Z^{-1}. In order for the transformed capital stock to remain as a state variable that does not respond to changes in Z in period t, we scale the model by Z_{t-1}^{-1}, instead of Z_t^{-1}. Using lower-case letters to denote the transformed variables, $x_t \equiv \frac{X_t}{Z_{t-1}}$, a representative household's problem becomes

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \{\log c_t - aN_t\}$$

subject to

$$c_t + (1+g_t)s_{t+1} \leq (1+r_t)s_t + w_t N_t. \tag{3}$$

The production function becomes

$$y_t = A_t k_t^{\alpha} N_t^{1-\alpha} (1+g_t)^{1-\alpha}. \tag{4}$$

The law of motion for capital becomes $(1+g_t)k_{t+1} = (1-\delta)k_t + i_t$. Factor prices become $w_t = (1-\alpha)\frac{y_t}{N_t}$ and $r_t + \delta = \alpha \frac{y_t}{k_t}$. The transformed model has a unique steady state where all variables are constant in the absence of aggregate uncertainty.

In general equilibrium, capital market clearing implies $s_t = k_t$; hence, the household budget constraint becomes

$$c_t + (1+g_t)k_{t+1} - (1-\delta)k_t = y_t, \tag{5}$$

which is also the aggregate resource constraint. The optimal consumption level must satisfy the Euler equation,

$$(1+g_t)u_c'(t) = \beta E_t (1+r_{t+1}) u_c'(t+1), \tag{6}$$

where u_c' is the marginal utility of consumption.

In the steady state, the Euler equation and the budget constraint imply the optimal capital-output ratio and consumption-output ratio,

$$\frac{k}{y} = \frac{\beta\alpha}{1+\bar{g}-\beta(1-\delta)} \tag{7}$$

$$\frac{c}{y} = 1 - (\bar{g}+\delta)\frac{\beta\alpha}{1+\bar{g}-\beta(1-\delta)}, \tag{8}$$

respectively. The saving rate is simply $(\bar{g}+\delta)\frac{k}{y}$, which is the modified golden rule. The real interest rate is given by $1+r = \frac{1+\bar{g}}{\beta}$.

Notice that the steady-state saving rate positively depends on the growth rate: $\frac{\partial}{\partial \bar{g}}\left[(\bar{g}+\delta)\frac{k}{y}\right] = \frac{(1-\beta)(1-\delta)}{[1+\bar{g}-\beta(1-\delta)]^2} > 0$. This implication contradicts the conventional wisdom based on simple PIH models. The conventional wisdom argues that forward-looking consumers should save less in a fast-growing economy because they know they will be richer in the future than they are today. Such an argument is based implicitly on the assumption that the real interest rate (or the marginal product of capital) is constant and not affected by growth. However, productivity growth raises the marginal product of capital and hence the demand for loanable funds. Therefore, in a standard general-equilibrium growth model with endogenous interest rate, faster growth induces higher saving. This important theoretical implication was applied by Chen, Imrohoroglu, and Imrohoroglu (2006) to explain Japan's high saving rate.

Also notice that the steady-state saving rate is independent of the form of the utility functions. That is, even if the utility function is quadratic as in the canonical PIH model, the steady-state saving rate is still given by equation (7) in general equilibrium. This also implies that the growth dynamics of the model around the steady state are the same regardless of the utility function, as long as the elasticity of the marginal utility of consumption, $\frac{\partial u_c'}{\partial c}\frac{c}{u_c'}$, is calibrated to the same value. On the other hand, the Euler equation (6) indicates that the marginal utility of consumption follows a random walk if the interest rate is constant or exogenous. Therefore, the "excess smoothness" puzzle and the "excess sensitivity" puzzle are not special features of the quadratic utility function adopted in the canonical PIH model, but the consequence of the exogenously fixed interest rate. This point is very important for understanding the results below.

With the log utility function in consumption, it is well known that when $\delta = 1$ this model has closed-form solutions for dynamic equilibrium, which are characterized by the linear rules:

$$c_t = (1-\beta\alpha)\, y_t \tag{9}$$

$$(1+g_t)\, k_{t+1} = \beta\alpha y_t. \tag{10}$$

Notice that these closed-form decision rules are obtained regardless of the labor-supply decisions. With a linear function in leisure, we have the first-order condition,

$$c_t = w_t = (1-\alpha)\frac{y_t}{N_t}, \tag{11}$$

so optimal labor supply is given by

$$N_t = \frac{1-\alpha}{1-\beta\alpha} \in (0,1). \tag{12}$$

These decision rules will serve as a reference point for the buffer-stock saving model in the next section.

When $\delta \neq 1$, the dynamic equilibrium of the transformed model can be solved by standard methods such as the log-linearization method of King, Plosser, and Rebelo (1988). Using circumflex to denote percentage deviations of a variable from its steady state value, $\hat{x}_t \equiv \log x_t - \log \bar{x}$, the decision rules of the model can be characterized by the following system of linear equations with a state-space representation:

$$\hat{\mathbf{q}}_{t+1} = M\hat{\mathbf{q}}_t + R\boldsymbol{\epsilon}_{t+1} \tag{13}$$

$$\hat{\mathbf{x}}_t = H\hat{\mathbf{q}}_t, \tag{14}$$

where $\hat{\mathbf{q}} = \begin{bmatrix} \hat{k} & \hat{A} & \hat{g} \end{bmatrix}'$ is a vector of the state variables, $\boldsymbol{\epsilon} = \begin{bmatrix} \varepsilon_A & \varepsilon_g \end{bmatrix}'$ is a vector of innovations, and $\hat{\mathbf{x}}$ is a vector of endogenous variables under interest, such as consumption, labor, investment, output, and so on. Notice that $\log(1+g_t) - \log(1+\bar{g}) \approx g_t - \bar{g} \equiv \hat{g}_t$.

Using Δ to denote the first-difference operator, the growth rate of $\hat{\mathbf{x}}$ is given by $\Delta\hat{\mathbf{x}}$; the system in (13) and (14) can then be expressed in a new state-space form in growth rates:

$$\Delta\hat{\mathbf{q}}_{t+1} = M\Delta\hat{\mathbf{q}}_t + \tilde{R}\begin{pmatrix} \boldsymbol{\epsilon}_{t+1} \\ \boldsymbol{\epsilon}_t \end{pmatrix} \tag{15}$$

$$\begin{pmatrix} \boldsymbol{\epsilon}_{t+1} \\ \boldsymbol{\epsilon}_t \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \mathbf{1} & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{\epsilon}_t \\ \boldsymbol{\epsilon}_{t-1} \end{pmatrix} + \begin{pmatrix} \mathbf{1} \\ 0 \end{pmatrix}\boldsymbol{\epsilon}_{t+1} \tag{16}$$

$$\Delta\hat{\mathbf{x}}_t = H\Delta\hat{\mathbf{q}}_t, \tag{17}$$

where $\mathbf{1}$ is a 2×2 identity matrix and $\tilde{R}$ is an accordingly adjusted matrix. By the definition $x_t \equiv \frac{X_t}{Z_{t-1}}$, the growth rate of a transformed variable ($\Delta\hat{x}_t$) and its counterpart of the original untransformed variable ($\Delta\hat{X}_t$) satisfy the relationship $\Delta\hat{x}_t = \Delta\hat{X}_t - \hat{g}_{t-1} - \bar{g}$. Thus, the de-meaned growth rate of the untransformed variables are given by

$$\Delta\hat{X}_t - \bar{g} = \Delta\hat{x}_t + \hat{g}_{t-1}. \tag{18}$$

Hence, the solution to the transformed model as in system (15)-(17) implies a solution to the growth rates of the corresponding untransformed model through the inverse transform (18). The impulse responses of the growth rates of the untransformed variables and their second moments can thus be easily obtained analytically as shown in the technical appendix of King, Plosser, and Rebelo (1988).[8]

Following the standard RBC literature, we set the time period to be a quarter, the time discounting factor $\beta = 0.98$,[9] capital depreciation rate $\delta = 0.025$, capital's income share $\alpha = 0.4$, the

[8] The law of motion in (16) is useful because it makes the forecasting errors in the state-space presentation *i.i.d.* instead of moving-average processes.

[9] We choose $\beta = 0.98$ rather than $\beta = 0.99$ so as to compare the control model with the buffer-stock model with impatience. The predictions with $\beta = 0.99$ are very similar and are also reported.

average quarterly technology growth rate $\bar{g} = 0.01$, and the persistence of shocks $\rho_g = 0.23$ and $\rho_A = 0.9$.[10] Since the dynamics of the model under transitory TFP shocks are well known in the literature, we focus on the effects of shocks to the growth rate of technology, g_t. Figure 1 shows the impulse responses of the growth rates of aggregate output ($\Delta\hat{Y}_t$), consumption ($\Delta\hat{C}_t$), investment ($\Delta\hat{I}_t$), and labor ($\Delta\hat{N}_t$) to a 1 percent increase in g_t. The dashed lines in each window represent the unit impulse of $\hat{g}_t$.

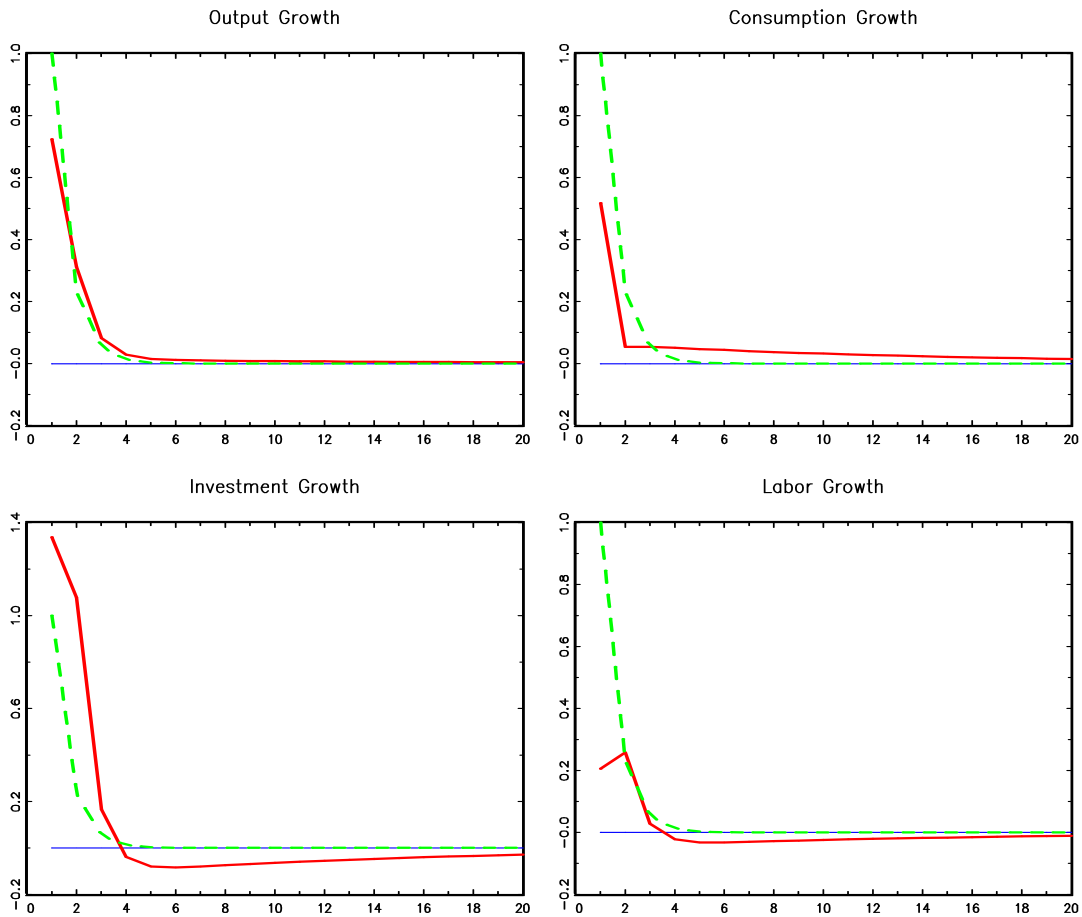

Figure 1. Impulse Responses to Technology Growth (Control Model).

A selected set of second moments predicted by the model and its U.S. counterpart are reported in Table 1, where the model's predictions under different parameter values are also reported as a comparative statistic analysis.[11] The first column reports the standard deviation of the growth rates of the major variables in the model relative to that of output $(\frac{\sigma_x}{\sigma_y})$, where $x = \{c, i, n\}$ represents

[10] $\rho_g = 0.23$ is the value used by Ludvigson and Michaelides (2009) and is also consistent with the U.S. data based on the growth rate of the Solow residual. The qualitative results of this paper do not hinge on this particular value.

[11] The U.S. data are quarterly real GDP, real fixed non-residential investment, real consumption of nondurables and services, and total private nonfarm employment. The sample is 1947:1-2009:1. The data source is BEA.

the variables in consideration; the second column reports the contemporaneous correlations of the growth rates of these variables with output growth ($cor(x_t, y_t)$); the third column reports the correlations of the current growth rates of the variables with lagged output growth ($cor(x_t, y_{t-1})$), which is also the measure of sensitivity we adopt in this paper; and the last column reports the first-order autocorrelations of growth rates ($cor(x_t, x_{t-1})$).

Based on the impulse responses in Figure 1 and the statistics reported in Table 1, the model's predictions for growth dynamics match those of the U.S. data qualitatively well in many dimensions. First, the model is able to enhance the serial correlation of the exogenous shocks to technology growth. For example, the first-order autocorrelation of output growth is 0.4, more than 70% stronger than that of the driving process. Investment growth and labor growth are even more strongly autocorrelated (with autocorrelation of 0.55 and 0.58, respectively). This is in sharp contrast to the case of technology-level shocks, in which case standard RBC models are not able to enhance the persistence of shocks (see, e.g., Cogley and Nason, 1995; and Rotemberg and Woodford, 1996).

Second, the model also predicts that investment growth and labor growth both lag output and consumption growth by about one quarter. The kink in the impulse responses of investment becomes even more visible if we set capital's income share $\alpha = 0.3$. One of the criticisms of the RBC theory is its inability to explain such a lead-lag relationship among the growth rates of aggregate output, consumption, investment, and labor (see, e.g., Cochrane, 1994; Wen, 2007). But it is now clear that such criticisms are ill based because the literature has mainly considered only technology-level shocks instead of technology-growth shocks when addressing these issues. Because the model generates delayed responses in investment growth and labor growth, lagged income growth predicts these variables as well as in the data.

Table 1. Predicted Second Moments (Control Model)

	σ_c/σ_y	$cor(c_t, y_{t-1})$
U.S. Data	**0.51**	**0.28**
Model 1 ($\alpha = 0.4$)	**0.68**	**0.15**
Model 2 ($\alpha = 0.2$)	**0.81**	**0.13**
Model 3 ($\alpha = 0.6$)	**0.59**	**0.21**
Model 4 ($\beta = 0.9$)	**0.67**	**0.31**
Model 5 ($\gamma = 10^{10}$)	**0.72**	**0.14**
Model 6 ($\delta = 1$)	**1.00**	**0.58**

Third and most importantly, the model predicts two of the most well-known stylized facts of the business cycle: consumption growth is less volatile but investment growth is more volatile than output growth; and these growth rates comove over the business cycle. Since this prediction is related to the two aforementioned consumption puzzles, we will postpone the discussions to later paragraphs.

A surprising and dramatic failure of this simple RBC model is its prediction of the volatility of labor relative to output. The standard deviation of employment growth relative to output growth is 0.77 in the data but 0.44 in the model. This is puzzling given that the RBC literature (e.g., Hansen, 1988) argues that indivisible labor helps to significantly raise labor's volatility to match that of output. The problem is again rooted in the source of shocks. Under technology-level shocks, labor is volatile when it is indivisible; but under technology-growth shocks, labor is no longer volatile because the incentive for supplying labor is reduced when the changes in income are permanent (growth shocks means permanent changes in the income level). Hence, the labor market anomaly of the RBC theory cannot be resolved by indivisible labor, contradicting the argument of Hansen (1988).

Let's now turn to the two consumption puzzles. For the U.S. economy (first row in Table 1), the relative volatility of consumption growth to GDP growth is 0.51 and the correlation between consumption growth and lagged output growth is 0.28. The model predicts that consumption growth is about 0.7 times volatile as output growth and its correlation with lagged output growth is 0.15 (see Model 1 in Table 1). Thus, the model performs much better than the canonical PIH model in explaining consumption dynamics in the data. However, the excess smoothness and the excess sensitivity puzzles remain and they are surprisingly robust to parameter values, as the following discussions show.

There are two features that differentiate the general-equilibrium model from the simple PIH model: endogenous interest rate and elastic labor supply.[12] However, an elastic labor supply is not important in generating the difference between the two models. For example, let the period-utility function be replaced by $u(c, N) = \log C_t - aN_t^{1+\gamma}$ and let the elasticity of labor supply ($\frac{1}{\gamma}$) approach zero so that labor becomes constant (e.g., let $\gamma = 10^{10}$); then the standard deviation of consumption growth is still about 72% of that of income growth and the sensitivity measure remains essentially unchanged (see Model 5 in Table 1). Therefore, the fundamental reason for consumption growth to be less volatile than income growth in the RBC model is mainly because of the endogenous interest rate. The interest rate is procyclical under technology shocks; thus, a higher growth rate of productivity will induce a higher saving rate, dampening its impact on consumption growth. Alternatively, the interest rate is the price of current consumption in terms of future consumption; although individuals take the price as given, their collective action in raising current consumption in responding to a higher permanent income will increase the interest rate,

[12] As noted earlier, the specific form of the utility function is irrelevant for dynamics around the steady state.

which in turn will discourage current consumption. Such an important interest-rate channel is missing in partial-equilibrium models. However, endogenizing the interest rate does not completely eliminate the two puzzles identified by the consumption literature. This discussion also makes it clear that changing the elasticity of the labor supply in the model does not help resolve these puzzles.

For the same reason (i.e., the general-equilibrium effects), the volatility of consumption growth is sensitive to the capital's income-share parameter α, which affects the rate of returns to savings via the marginal product of capital. When α is small, the interest rate (or return to investment) is low; hence, the marginal propensity to consume is high and consumption is more volatile (see, e.g., Model 2 in Table 1 where the ratio of the standard deviation of consumption growth to income growth is 0.81 when $\alpha = 0.2$). The opposite happens when α is large. A larger value of α not only reduces the volatility of consumption but also increases the sensitivity of consumption to lagged income. This is so because more savings make consumption growth more sustainable and thus more persistent, rendering it more predictable by history. For example, when α increases from 0.4 to 0.6 (Model 3 in Table 1), the sensitivity measure $cor(c_t, y_{t-1})$ increases from 0.15 to 0.21 while the autocorrelation of consumption growth also increases from 0.17 to 0.29. This appears to be consistent with the analysis of Campbell and Deaton (1989) that the "excess smoothness" puzzle and the "excess sensitivity puzzle" are intrinsically related and they reflect the two sides of the same coin. That is, a smoother consumption path implies a higher serial correlation, hence a greater sensitivity to history (such as lagged income). Thus, it *appears* that increasing capital's share may resolve the two puzzles. Unfortunately, it cannot. Model 3 in Table 1 indicates that even if $\alpha = 0.6$, which implies a capital's share far larger than that in the U.S. economy, the relative volatility measure is still 0.59 and the sensitivity measure is still 0.21, significantly different from the data (0.51 and 0.28, respectively).

As the consumer becomes less patient (i.e., as β decreases), the relative volatility of consumption growth declines and the sensitivity increases. However, assuming a low value of β does not eliminate the excess smoothness puzzle. For example, even when $\beta = 0.9$, which implies that the steady-state capital to output ratio is only one quarter of that in the U.S. economy, the relative standard deviation of consumption growth is still 0.67 (only slightly below the benchmark of 0.68), although the sensitivity measure rises to 0.31 (matching the data well).

The predictions under a large depreciation rate of capital are perhaps the most counter-intuitive in light of Campbell and Deaton's (1989) analysis (e.g., Model 6 in Table 1). When $\delta = 1$, consumption growth is as volatile as income growth yet its correlation with lagged income growth becomes even stronger – instead of weaker, as Campbell and Deaton's (1989) analysis would indicate. This suggests that the excess smoothness puzzle and the excess sensitivity puzzle may not necessarily go hand in hand.

When $\delta = 1$, both investment and consumption in the model becomes completely proportional to output, as in equations (9) and (10). This implies that capital is no longer a stock variable but a flow variable. This has two consequences. First, since the capital stock is permanent income and since consumption follows permanent income, when the capital stock becomes volatile, so does consumption. Second, shocks to technology growth can be very effectively propagated over time through savings and capital accumulation when $\delta = 1$: a 1% increase in output can translate to a 1% increase in the next-period capital stock, which in turn can translate into an α% increase in future output.[13] This implies that the autocorrelation of the growth rates of all variables in the model (except labor) is strong and is given by $\alpha = 0.4$ even in the absence of any serial correlations in the shock process. This implies that both the volatility and the autocorrelation of consumption growth are very large: consumption growth is now just as volatile as income growth and the sensitivity measure is 0.58, an extremely high value. This result contradicts the belief of Campbell and Deaton (1989) about the link between smoothness and sensitivity because their belief is based on just one sample and they hold a partial-equilibrium perspective. In general equilibrium, permanent income is essentially the capital stock and consumption tracks the capital stock closely. So when δ is close to zero, permanent income (the capital stock) is extremely smooth relative to current income (output) because it is the sum of past investment, whereas output is directly affected not only by labor but most importantly by technology shocks. Hence, consumption as a function of permanent income is not only far soother than output, but also far less predictable by output. The opposite happens when δ becomes large.

To sum up, the above analysis shows that the excess smoothness and excess sensitivity puzzles are exaggerated by the existing literature. In general equilibrium, where labor income and, especially, the real interest rate are endogenous, consumption growth is not as volatile as predicted by the simple PIH model and it is correlated with past income growth, albeit not as strongly as in the data. Thus, the anomalies are less severe than claimed. Nonetheless, the two puzzles remain and are quite robust to parameter values. In the next section we investigate whether borrowing constraints can help resolve them.

3 Buffer-Stock Saving with Borrowing Constraints

In this model households are indexed by $i \in [0, 1]$ and are each subject to idiosyncratic preference shocks $\theta_t(i)$ in every period. These shocks are orthogonal to aggregate shocks, the support of θ is $[\theta_L, \theta_H]$ with $\theta_H > \theta_L > 0$, and the cumulative distribution function of the shocks, $F(\theta)$, is common

[13] Whereas in the case of $\delta = 0.025$, a 1% increase in investment can translate only into 0.025% increase in the next-period capital stock because the changes in a flow variable has little impact on a stock variable.

to all households. Different draws of the preference shocks imply different optimal consumption and saving plans for a household. Hence, consumers are heterogeneous. A key assumption is that households must choose labor supply in the beginning of each period before observing their idiosyncratic preference shocks in that period. This assumption together with the linear leisure function imply that wealth distribution across households is degenerate. This dramatically simplifies the computation of general equilibrium because we do not have to keep track of the distribution of wealth of each individual in the state space. However, the distributions of consumption and savings are not degenerate; hence, individual consumption and aggregate consumption are not the same in the model. All aggregate shocks are realized in the beginning of each period before all decisions are made in that period. The model thus contains the control model as a special case when the distribution of $\theta(i)$ becomes degenerate.[14]

Applying the same transformation as in the control mode, household i's problem is to solve

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \theta_t(i) \log c_t(i) - N_t(i) \right\}$$

subject to

$$c_t(i) + (1 + g_t) s_{t+1}(i) \leq (1 + r_t) s_t(i) + w_t N_t(i) \tag{19}$$

$$s_{t+1}(i) \geq 0, \tag{20}$$

where the second inequality is a simple form of borrowing (or liquidity) constraint.[15] Denoting $\{\lambda(i), \pi(i)\}$ as the Lagrangian multipliers for constraints (19) and (20), respectively, the first-order conditions for $\{c(i), n(i), s(i)\}$ are given, respectively, by

$$\frac{\theta(i)}{c(i)} = \lambda(i) \tag{21}$$

$$1 = w_t E_t^i \lambda(i) \tag{22}$$

$$(1 + g_t) \lambda_t(i) = \beta E_t (1 + r_{t+1}) \lambda_{t+1}(i) + \pi_t(i), \tag{23}$$

where the expectation operator E^i denotes expectations conditional on the information set of time t excluding $\theta_t(i)$. Hence, equation (22) reflects the fact that labor supply $n_t(i)$ must be made before the idiosyncratic taste shocks (and hence the value of $\lambda_t(i)$) are realized. By the law of iterated

[14] This technique of obtaining analytical solutions in heterogeneous-agent models with borrowing constraints is based on my other works on inventory theory and money demand theory (Wen, 2008, 2009a, and 2009b). Similar techniques are also applied by Wang and Wen (2009) to models with heterogeneous firms. The following analysis closely follows this literature.

[15] My analytical method is not limited to log utility. The only problem is how to deal with balanced growth when the leisure function is linear.

expectations and the orthogonality assumption of aggregate and idiosyncratic shocks, equation (23) can be written as

$$(1+g_t)\,\lambda_t(i) = \beta E_t\,(1+r_{t+1})\,\frac{1}{w_{t+1}} + \pi_t(i), \tag{24}$$

where $\frac{1}{w}$ is the marginal utility of consumption in terms of labor.

The decision rules for an individual's consumption and savings are characterized by a cutoff strategy, taking as given the aggregate environment (such as interest rate and real wage). Consider two possible cases:

Case A. $\theta_t(i) \leq \theta_t^*$. In this case the urge to consume is low. It is hence optimal to save so as to prevent possible liquidity constraints in the future. So $s_{t+1}(i) \geq 0$, $\pi_t(i) = 0$ and the shadow value of good $\lambda_t(i) = \beta E_t \frac{1+r_{t+1}}{(1+g_t)w_{t+1}}$. Equation (21) implies that consumption is given by $c(i) = \theta(i)\left[\beta E_t \frac{1+r_{t+1}}{(1+g_t)w_{t+1}}\right]^{-1}$. Defining

$$x(i) \equiv (1+r_t)s_t(i) + wn_t(i) \tag{25}$$

as the wealth (cash in hand) of household i, the budget identity (19) then implies $(1+g_t)\,s_{t+1}(i) = x_t(i) - \theta(i)\left[\beta E_t \frac{1+r_{t+1}}{(1+g_t)w_{t+1}}\right]^{-1}$. The requirement $s_{t+1}(i) \geq 0$ then implies

$$\theta(i) \leq \left[\beta E_t \frac{1+r_{t+1}}{(1+g_t)\,w_{t+1}}\right] x_t(i) \equiv \theta_t^*, \tag{26}$$

which defines the cutoff θ^*. Notice that the cutoff is independent of i because wealth $x(i)$ is determined before the realization of $\theta_t(i)$ and all households face the same distribution of idiosyncratic shocks. This property simplifies the computation of the general equilibrium of the model tremendously.

Case B. $\theta_t(i) > \theta_t^*$. In this case the urge to consume is high. It is then optimal not to save, so $s_{t+1}(i) = 0$ and $\pi_t(i) > 0$. By the resource constraint (19), we have $c_t(i) = x_t(i)$, which by equation (26) implies $c(i) = \theta_t^*\left[\beta E_t \frac{1+r_{t+1}}{(1+g_t)w_{t+1}}\right]^{-1}$. Equation (21) then implies that the shadow value is given by $\lambda_t(i) = \frac{\theta_t(i)}{\theta_t^*}\left[\beta E_t \frac{1+r_{t+1}}{(1+g_t)w_{t+1}}\right]$. Since $\theta(i) > \theta^*$, equation (24) implies $\pi_t(i) = \left[\beta E_t \frac{1+r_{t+1}}{w_{t+1}}\right]\left[\frac{\theta(i)}{\theta^*} - 1\right] > 0$. Notice that the shadow value of goods (the marginal utility of consumption), $\lambda(i)$, is higher under case B than under case A because of the binding borrowing constraint.

The above analyses imply that the expected shadow value of goods, $E^i\lambda(i)$, and hence the optimal cutoff value θ^*, is determined by the following asset-pricing equation for savings based on

(22):

$$\frac{1}{w_t} = \left[\beta E_t \frac{1 + r_{t+1}}{(1 + g_t)\, w_{t+1}}\right] R(\theta_t^*), \tag{27}$$

where

$$R(\theta_t^*) \equiv \left[\int_{\theta(i)\leq\theta^*} dF(\theta) + \int_{\theta(i)>\theta^*} \frac{\theta(i)}{\theta^*} dF(\theta)\right] \tag{28}$$

measures the extra rate of return to savings due to the liquidity value of the buffer stock (i.e., a liquidity premium). Equation (27) can be compared with equation (6). The left-hand side of equation (27) is the utility cost of saving one more unit of liquidity. The right-hand side is the expected gains of such an investment, which takes two possible values. The first is simply the discounted next-period marginal utility of investment ($\beta E_t \frac{1+r_{t+1}}{(1+g_t)w_{t+1}}$) in the case of low consumption demand ($\theta(i) \leq \theta^*$), which has probability $\int_{\theta(i)\leq\theta^*} dF(\theta)$. The second is the effective rate of return to investment adjusted by the marginal utility of consumption ($\frac{\theta_t(i)}{\theta_t^*}\left(\beta E_t \frac{1+r_{t+1}}{(1+g_t)w_{t+1}}\right)$) in the case of high demand ($\theta(i) > \theta^*$), which has probability $\int_{\theta(i)>\theta^*} dF(\theta)$. The optimal cutoff θ^* is chosen so that the marginal cost equals the expected marginal gains. Hence, savings play the role of a buffer stock and the rate of return to liquidity is determined by the real interest rate plus a liquidity premium, $(1 + r)\, R(\theta^*)$, rather than just by $1 + r$. Notice that $R(\theta^*) > 1$ as long as θ^* lies in the interior of the support $[\theta_L, \theta_H]$.

An alternative interpretation of $R > 1$ is that a saving decision is to exercise an option; the option value of one dollar exceeds one because it provides liquidity in the case of the urge to consume. The optimal level of the buffer stock is always such that the probability of stockout (being liquidity constrained) is strictly positive ($\int_{\theta(i)>\theta^*} dF(\theta) > 0$), so that the option value always exceeds one.

The cutoff strategy implies that the optimal level of wealth (cash in hand) in period t is determined by a "target" policy given by $x_t(i) = \theta_t^* \left[\beta E_t \frac{1+r_{t+1}}{(1+g_t)w_{t+1}}\right]^{-1}$, which specifies that wealth (total past savings plus labor income) is set to a target level that is independent of $\theta(i)$ but depends on the distribution of $\theta(i)$. Such a target policy was also derived implicitly by Deaton (1991) under idiosyncratic labor-income shocks in a partial-equilibrium buffer-stock model with a constant interest rate and inelastic labor supply. This target policy here implies that individual labor supply will always adjust so that the wealth level meets its target (recall $x(i) = (1 + r)s(i) + wn(i)$).

Utilizing equation (27), the decision rules of household i are summarized by

$$x_t(i) = w_t R(\theta_t^*)\theta_t^* \tag{29}$$

$$c_t(i) = w_t R(\theta_t^*) \times \min\{\theta(i), \theta_t^*\} \tag{30}$$

$$(1+g_t)s_{t+1}(i) = w_t R(\theta_t^*) \times \max\left\{\theta_t^* - \theta(i), 0\right\}. \tag{31}$$

Notice that $c_t(i) + (1+g_t)s_{t+1}(i) = x_t(i)$. Because of the Leontief functional form, an individual's consumption function is very concave (as noted by Deaton, 1991). These decision rules imply that consumption increases one-for-one with wealth for $\theta^* < \theta(i)$: $c(i) = x(i)$. Beyond the point $\theta^* \geq \theta(i)$, the marginal propensity to consume out of wealth is reduced and becomes less than one: $c(i) = \frac{\theta(i)}{\theta^*}x(i)$. Since different individuals have different $\theta(i)$, their turning points are also different. More importantly, aggregate shocks will affect the distribution of consumption and savings across households by affecting the cutoff θ_t^*.

Aggregation. Denoting $c \equiv \int c(i)di$, $s \equiv \int s(i)di$, $N \equiv \int N(i)di$, and $x \equiv \int x(i)di$ and integrating the household decision rules over i by the law of large numbers, the aggregate variables are given by

$$(1+r_t)s_t + w_t N_t = w_t R(\theta_t^*)\theta^* \tag{32}$$

$$c_t = \frac{D(\theta_t^*)}{\theta_t^*}\left[(1+r_t)s_t + w_t N_t\right] \tag{33}$$

$$(1+g_t)s_{t+1} = \frac{H(\theta_t^*)}{\theta_t^*}\left[(1+r_t)s_t + w_t N_t\right], \tag{34}$$

where

$$D(\theta^*) \equiv \int_{\theta(i)\leq\theta^*} \theta(i)dF(\theta) + \int_{\theta(i)>\theta^*} \theta^* dF(\theta) > 0 \tag{35}$$

$$H(\theta^*) \equiv \int_{\theta(i)\leq\theta^*} \left[\theta^* - \theta(i)\right] dF(\theta) > 0 \tag{36}$$

and the functions satisfy $D(\theta^*) + H(\theta^*) = \theta^*$, $\frac{\partial D}{\partial \theta^*} = 1 - F > 0$, and $\frac{\partial H}{\partial \theta^*} = F > 0$.

Partial-Equilibrium Analysis. Aggregate consumption and savings are related to aggregate wealth according to the following relationships:

$$c_t = \frac{D(\theta_t^*)}{\theta_t^*}x_t \tag{37}$$

$$(1+g_t)s_{t+1} = \left[1 - \frac{D(\theta_t^*)}{\theta_t^*}\right]x_t, \tag{38}$$

where $\frac{D(\theta_t^*)}{\theta_t^*} < 1$ is the aggregate marginal propensity to consume (MPC). Aggregate MPC is less than one because only a fraction of households have MPC equal to one and the rest have MPC less than one due to a binding borrowing constraint. Notice that $\frac{\partial MPC}{\partial \theta^*} = \frac{(1-F)\theta^* - D}{\theta^{*2}} < 0$ because

$(1-F)\,\theta^* = D - \int_{\theta(i)\leq\theta^*} \theta(i)dF(\theta)$ according to (35). This suggests that a rise in the cutoff θ^* will lower the marginal propensity to consume and increase the marginal propensity to save.

Suppose the economy is in a steady state, which is defined as the situation without aggregate uncertainty. Hence, the cutoff θ^* is determined by the relation (27),

$$\beta(1+r)R(\theta^*) = 1 + \bar{g}. \tag{39}$$

Because $\frac{\partial R}{\partial \theta} < 0$, the cutoff θ^* positively depends on the interest rate. That is, a higher interest rate implies a lower propensity to consume and a stronger saving motive. For simplicity, assume $\bar{g} = 0$. Equation (38) implies that the wealth level is given by

$$x = (1+r)\,s + wn = \frac{wn}{1-(1+r)\,(1-MPC)}.$$

Further assume that $r \approx 0$, then the wealth level is approximately given by $\frac{wn}{MPC}$; hence, consumption is given by $c \approx wn$. That is, consumption is approximately as volatile as labor income in a partial-equilibrium buffer-stock model. This result is independent of the degree (distribution) of heterogeneity and explains the findings of Ludvigson and Michaelides (2001) that borrowing constraints do not help resolve the excessive smoothness puzzle. However, if the interest rate is endogenous, then the implications are entirely different, as we show next.

General-Equilibrium Analysis. Under perfect competition, factor prices are determined by marginal products, $r_t + \delta_t = \alpha \frac{y_t}{k_t}$ and $w_t = (1-\alpha)\frac{y_t}{N_t}$. Market clearing implies $s_{t+1} = k_{t+1}$ and $\int N_t(i) = N_t$. The constant-returns-to-scale property of the production function implies $x_t = y_t + (1-\delta)\,k_t$. The aggregate household resource constraint implies the aggregate goods-market clearing condition,

$$c_t + (1+g_t)\,k_{t+1} - (1-\delta)k_t = A_t k_t^{\alpha} N_t^{1-\alpha}\,(1+g_t)^{1-\alpha}\,. \tag{40}$$

A general equilibrium is defined as the sequence $\{c_t, y_t, N_t, k_{t+1}, w_t, r_t, \theta_t^*\}$, such that all households maximize utility subject to their resource and borrowing constraints, firms maximize profits, all markets clear, the law of large numbers holds, and the set of standard transversality conditions are satisfied.[16] The equations needed to solve for the general equilibrium are (27), (33), (34), (40), and the factor price equations given by firms' first-order conditions with respect to $\{k, N\}$. The aggregate model has a unique steady state. The aggregate dynamics of the model can be solved by log-linearizing the aggregate model around the steady state and then applying the method outlined in the previous control-model section to find the stationary saddle paths of the growth rates.

[16]For example, a transversality condition in this model is $\lim_{t\to\infty} \beta^t \frac{k_{t+1}}{w_t} = 0$, where $\frac{1}{w}$ is the shadow value of capital (marginal utility of consumption).

In the special case of $\delta = 1$, the model has closed-form solutions for aggregate dynamics. In this special case, we have $x_t = y_t$ and

$$c_t = \frac{D(\theta_t^*)}{\theta_t^*} y_t \tag{41}$$

$$(1 + g_t)\, k_{t+1} = \frac{H(\theta_t^*)}{\theta_t^*} y_t \tag{42}$$

$$N_t = (1 - \alpha)\, R(\theta_t^*)\theta_t^*. \tag{43}$$

Since $w = (1 - \alpha)\, \frac{y}{N}$, utilizing equation (27), we get

$$H(\theta_t^*) = \beta\alpha E_t R(\theta_{t+1}^*)\theta_{t+1}^*, \tag{44}$$

which suggests that $\theta_t^* = \theta^*$ for all t (i.e., a constant) is a solution and labor supply is thus fixed over time. Once the distribution of $\theta_t(i)$ is given, the constant θ^* can then be solved by equation (44) and we then have $\frac{H(\theta^*)}{\theta^*} = \beta\alpha R(\theta^*)$. Substituting this into (41)-(43) gives $c_t = [1 - \beta\alpha R(\theta^*)]\, y_t$ and $(1 + g_t)\, k_{t+1} = [\beta\alpha R(\theta^*)]\, y_t$, which are comparable to (9) and (10) in the control model and differ only by the liquidity premium R.[17] Clearly, regardless of the distribution of θ, borrowing constraints do not matter for the model's aggregate dynamics if $\delta = 1$. In such a case, aggregate consumption will always be as volatile as aggregate income because the marginal propensity to consume is constant. This special case clearly does not match the U.S. data.

Steady State. The system of equations determining the model's steady state include

$$1 + \bar{g} = \beta(1 + r)R(\theta^*) \tag{45}$$

$$c = wR(\theta^*)D(\theta^*) \tag{46}$$

$$(1 + \bar{g})\, k = wR(\theta^*)H(\theta^*) \tag{47}$$

$$c + (\bar{g} + \delta)\, k = y, \tag{48}$$

where $w = (1 - \alpha)\, \frac{y}{N}$, $r + \delta = \alpha\frac{y}{k}$, and $y = k^\alpha N^{1-\alpha}\, (1 + \bar{g})^{1-\alpha}$. This system of seven equations uniquely solves for the seven endogenous variables $\{c, k, N, y, w, r, \theta^*\}$ in the steady state.

Notice that, as long as the probability of a binding borrowing constraint is strictly positive (i.e., $1 - F(\theta^*) > 0$), or the fraction of borrowing constrained population is not zero, then we must have $R(\theta^*) > 1$. In this case equation (45) implies that the real interest rate is less than the golden-rule rate implied by equation (6) in the control model. That is, precautionary motives under borrowing constraints induce households to over save, resulting in dynamic inefficiency. This confirms the

[17] When the variance of the distribution for θ is degenerate, we have $R(\theta^*) = D(\theta^*) = 1$ and $\theta^* = [1 - \beta\alpha]^{-1}$, so the buffer-stock model reduces completely to the representative-agent RBC model.

findings of Aiyagari (1994). The distance, $|R(\theta^*) - 1|$, can thus be used as a measure of dynamic inefficiency. The model becomes dynamically efficient when $R(\theta^*) = 1$, which is then identical to an RBC model without borrowing constraints. As will be shown shortly, when the variance of $\theta_t(i)$ approaches zero, we must have $R(\theta^*)$ approach one; so borrowing constraints will cease to bind in the limit and the model reduces to the control model.

In the steady state, equation (45) implies that the output-capital ratio must satisfy $(1 + \bar{g}) = \beta\left(1 - \delta + \alpha \frac{y}{k}\right) R(\theta^*)$. Equations (46) and (47) imply the consumption-capital ratio, $\frac{c}{k} = (1 + \bar{g})\frac{D}{H}$. Substituting this consumption-capital ratio into the resource constraint (48) gives another equation for the output-capital ratio: $(1 + \bar{g})\frac{D}{H} + \bar{g} + \delta = \frac{y}{k}$. Putting these two restrictions for output-capital ratio together gives the following implicit equation to uniquely solve for the cutoff value:

$$\frac{1 + \bar{g}}{R(\theta^*)} = \beta\left[1 - \delta + \alpha\left(\bar{g} + \delta + (1 + \bar{g})\frac{D(\theta^*)}{H(\theta^*)}\right)\right]. \tag{49}$$

Because $\frac{\partial R(\theta^*)}{\partial \theta^*} < 0$, the left-hand side ($LHS$) increases monotonically with θ^* and has its maximum equal to $LHS(\theta_H) = 1 + \bar{g}$ and minimum equal to $LHS(\theta_L) = (1 + \bar{g})\frac{\theta_L}{E\theta} < 1 + \bar{g}$, where $E\theta$ is the mean. On the other hand, because $\frac{\partial(D/H)}{\partial\theta^*} = \frac{(1-F)H - FD}{H^2} = \frac{H - F\theta^*}{H^2} = -\left[\int_{\theta\leq\theta^*}\theta(i)dF\right]/H^2 < 0$, the right-hand side ($RHS$) decreases with θ^* with its maximum equal to infinity at $\theta^* = \theta_L$ because $D(\theta_L) = \theta_L$ and $H(\theta_L) = 0$, and with its minimum given by $\beta\left[1 - \delta + \alpha\left(\bar{g} + \delta + (1 + \bar{g})\frac{E\theta}{\theta_H - E\theta}\right)\right]$. Hence, as long as

$$1 + \bar{g} > \beta\left[1 - \delta + \alpha\left(\bar{g} + \delta + (1 + \bar{g})\frac{E\theta}{\theta_H - E\theta}\right)\right], \tag{50}$$

a unique interior solution for θ^* exists. Condition (50) is satisfied if agents are sufficiently impatient (i.e., with β small enough) and the distribution of θ is not degenerate (i.e., $\theta_H > E\theta$).

With the cutoff value θ^* determined, the capital-output ratio and consumption-output ratio are then given by

$$\frac{k}{y} = \frac{\beta\alpha R(\theta^*)}{1 + \bar{g} - \beta(1 - \delta)R(\theta^*)} \tag{51}$$

$$\frac{c}{y} = 1 - (\bar{g} + \delta)\frac{\beta\alpha R(\theta^*)}{1 + \bar{g} - \beta(1 - \delta)R(\theta^*)}, \tag{52}$$

respectively, which differ from those in the control model (7 and 8) by the liquidity premium $R(\theta^*) > 1$. These ratios become identical to those in the control model when the borrowing constraint no longer binds (i.e., $R(\theta^*) = 1$ when $\Pr[\theta(i) > \theta^*] = 0$).

Calibration and Impulse Responses. To facilitate quantitative analysis, we assume the idiosyncratic shocks $\theta(i)$ follow the Pareto distribution, $F(\theta) = 1 - \theta^{-\sigma}$, with $\sigma > 1$ and the support $\theta \in (1, \infty)$. With the Pareto distribution, we have

$$R(\theta_t^*) = 1 + \frac{1}{\sigma - 1}\theta^{*-\sigma} \tag{53}$$

$$D(\theta^*) = \frac{\sigma}{\sigma - 1} - \frac{1}{\sigma - 1}\theta^{*1-\sigma} \tag{54}$$

$$H(\theta^*) = \theta^* - \frac{\sigma}{\sigma - 1} + \frac{1}{\sigma - 1}\theta^{*1-\sigma}. \tag{55}$$

As in the control model, we set the time period to be a quarter of a year, and $\beta = 0.98$, $\delta = 0.025$, and $\alpha = 0.4$. We choose a degree of heterogeneity by setting the shape parameter $\sigma = 1.5$ as our benchmark value.[18] The impulse responses of the model to a 1% increase in the growth rate of labor-augmenting technology g_t, with persistence $\rho_g = 0.23$, are shown in Figure 2 (where the dashed lines represent the impulses of g_t). The figure shows that the impulse responses of the heterogeneous-agent buffer-stock model are qualitatively similar to those in the representative-agent control model. Quantitatively, however, there are important differences.

Under the calibrated parameter values, the steady-state capital-output ratio is 8.657 in the buffer-stock model and is 7.193 in the control model. Hence, the saving rate has increased by about 0.20% because of borrowing constraints. The probability for the borrowing constraint to bind is $1 - F(\theta^*) = \theta^{*-\sigma} = 0.005$, or half of 1%. This is not surprising given the analysis of Krusell and Smith (1998). That is, rational individuals take into consideration the borrowing constraints and opt to save aggressively so as to reduce the probability of binding constraints. With σ close to 1 (which is the well-known Zipf distribution), say $\sigma = 1.05$, the precautionary saving motive becomes even stronger because the degree of uncertainty is much greater. As a result, the probability of a binding borrowing constraint is further reduced to 0.0018, and the steady-state capital-output ratio is now about 20, nearly three times higher than that in the control model. This is an extraordinary amount of savings and indicates how borrowing constraints affect people's saving behaviors under uncertainty.

[18] The variance of the Pareto distribution is a decreasing function of σ. The empirical literature based on distributions of income and wealth typically finds $\sigma \in (1.1, 3.5)$ or centered around $1.5 \sim 2.5$ (see, e.g., Wolff, 1996; Fermi, 1998; Levy and Levy, 2003; Clementi and Gallegati, 2005; and Nirei and Souma, 2007). Hence, $\sigma = 1.5$ is within the empirical estimates. However, other values of σ will also be studied.

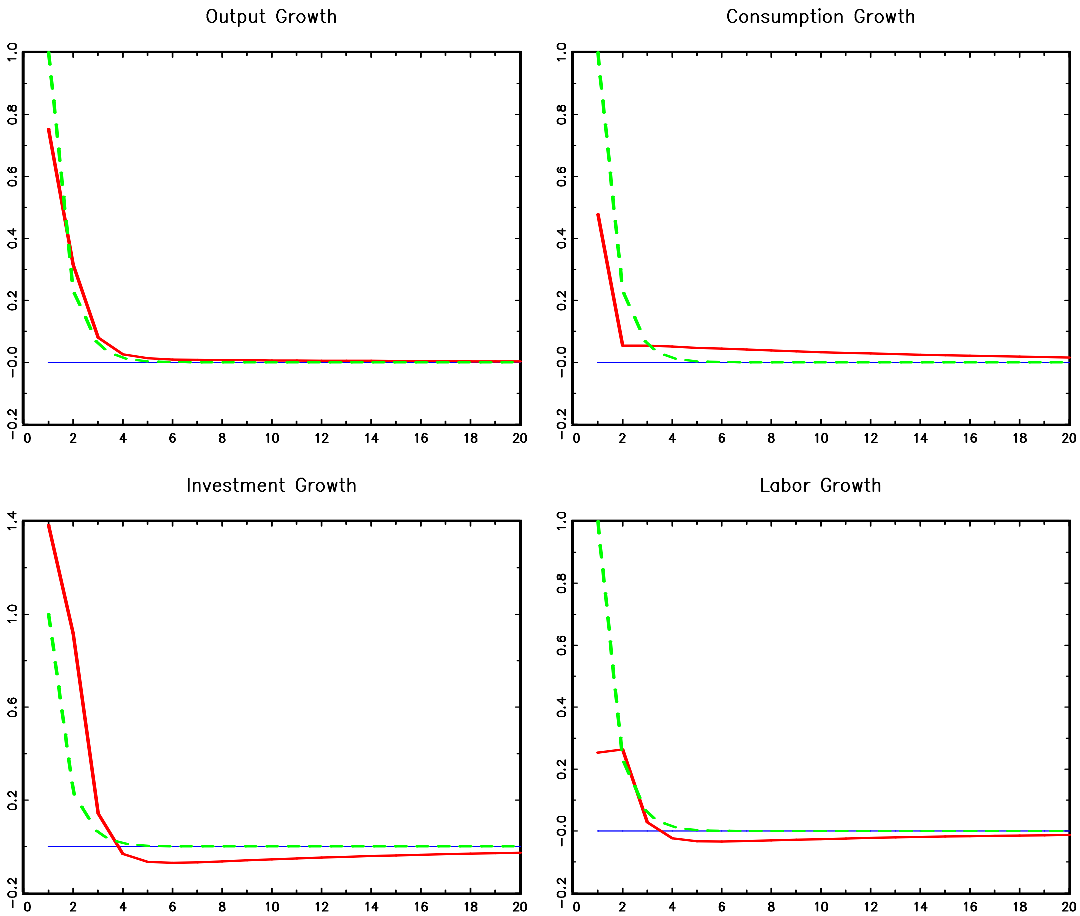

Figure 2. Impulse Responses to Technology Growth (Buffer-Stock Model).

Table 2 reports the predicted second moments of the buffer-stock model under the calibrated parameter values (Model 1 in Table 2). With borrowing constraints, the relative volatility of consumption growth has declined to 0.61, a more than 10% reduction compared with a value of 0.68 in the control model. The sensitivity measure has also increased from 0.15 to 0.16, albeit an insignificant amount. As in the control model, increasing capital's income share (α) will increase both the smoothness and sensitivity of consumption (Model 3 in Table 2); but the discrepancies between the model and data cannot be completely eliminated.

Most notably, when the degree of idiosyncratic uncertainty is further increased (say $\sigma = 1.15$), then the buffer-stock saving model is able to perfectly match the excess smoothness of the data (see Model 5 in Table 2 where the predicted relative volatility of consumption growth is 0.51). However, this has little effect on the excess sensitivity puzzle. On the other hand, a combination of strong borrowing constraints and impatience (i.e., $\sigma = 1.25$ and $\beta = 0.92$ as in Model 6 in Table 2), the model can resolve both puzzles perfectly: the relative volatility of consumption growth is 0.50 and its correlation with lagged income growth is 0.27. However, the cost is that the implied capital-output ratio is too low, about 6.3 (this number is around 10 in the data). We can also show that the model's dynamics converge to those of the control model when borrowing constraints are relaxed by increasing the shape parameter in the Pareto distribution (σ). For example, when $\sigma = 3$ (Model 7 in Table 2), there is virtually no difference between the predictions of the buffer-stock model and those of the representative-agent model (Model 1 in Table 1).

Table 2. Predicted Second Moments (Buffer-Stock Model)

	σ_c/σ_y	$cor(c_t, y_{t-1})$
U.S. Data	**0.51**	**0.28**
Model 1 ($\alpha = 0.4$)	**0.61**	**0.16**
Model 2 ($\alpha = 0.2$)	**0.66**	**0.13**
Model 3 ($\alpha = 0.6$)	**0.56**	**0.21**
Model 4 ($\beta = 0.9$)	**0.58**	**0.31**
Model 5 ($\sigma = 1.15$)	**0.51**	**0.15**
Model 6 ($\sigma = 1.25, \beta = 0.92$)	**0.50**	**0.27**
Model 7 ($\sigma = 3.0$)	**0.68**	**0.15**

The reason that borrowing constraints can significantly increase the smoothness of consumption growth relative to income growth is not mainly because consumers are unable to borrow when income growth rises, but rather because rational consumers have a much stronger incentive to save so as to relax future borrowing constraints.[19] This result differs from that in Ludvigson and Michaelides (2001), where they show that borrowing constraints cannot significantly reduce the volatility of consumption relative to income and are thus not effective in resolving the excess smoothness puzzle.

4 Habit Formation

Michaelides (2002) shows that habit formation is very effective in resolving both the excess smoothness puzzle and the excess sensitivity puzzle.[20] However, Michaelides' analysis is carried out in the traditional partial-equilibrium framework with a constant interest rate. It is therefore interesting to extend his analysis to general equilibrium to see if his results are robust. To make the general-equilibrium model with habit formation analytically tractable, we assume external habit rather than internal habit.

[19] This forward-looking precautionary saving behavior is also noted by Zelds (1989b).

[20] Articles proposing habit formation as a possible resolution to consumption puzzles also include Deaton (1992) and Sommer (2007), among others.

By a similar transformation as in the previous sections, household i's objective function can be written as

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left\{ \theta_t(i) \log \left[c_t(i) - \frac{\tau}{1+g_{t-1}} c_{t-1} \right] - N_t(i) \right\},$$

where $1+g_{t-1} = \frac{Z_{t-1}}{Z_{t-2}}$, subject to (19) and (20). The first-order conditions for $\{n(i), s(i)\}$ are the same as before but that for $c(i)$ is replaced by

$$\frac{\theta(i)}{c(i) - \frac{\tau}{1+g_{t-1}} c_{t-1}} = \lambda(i). \tag{56}$$

Because lagged consumption is taken as parametric by individuals, it does not change the way the decision rules are derived in the previous section. Hence, the decision rules of household i are summarized by

$$c_t(i) = w_t R(\theta_t^*) \times \min \{\theta(i), \theta_t^*\} + \frac{\tau}{1+g_{t-1}} c_{t-1} \tag{57}$$

$$(1+g_t) s_{t+1}(i) = w_t R(\theta_t^*) \times \max \{\theta_t^* - \theta(i), 0\} \tag{58}$$

$$x_t(i) = w_t R(\theta_t^*) \theta_t^* + \frac{\tau}{1+g_{t-1}} c_{t-1}, \tag{59}$$

where the the liquidity premium, $R(\theta^*)$, is the same as in (27) and (28). These decision rules are similar to those in the previous section except the consumption and the target-wealth level both have an additional term, $\frac{\tau}{1+g_{t-1}} c_{t-1}$. This shows that habit formation makes consumption history-dependent and raises the target wealth level by an amount determined by that history. That is, the optimal plan for wealth accumulation is again a target policy as before but with the target level also depending on the average living standard of other households in the economy (c_{t-1}). This suggests a higher saving rate than the case without habit formation.

These decision rules imply that the relationship between consumption and wealth is given by

$$c_t(i) = x_t(i) \left[\frac{\min\{\theta(i), \theta_t^*\}}{\theta^*} \right] + \frac{\tau}{1+g_{t-1}} c_{t-1} \left[\frac{\max\{\theta^* - \theta(i), 0\}}{\theta^*} \right].$$

Thus, habit formation significantly alters the growth dynamics of consumption in the following sense: When $\theta^* \leq \theta(i)$, we have $c_t(i) = x_t(i)$ as in the case without habit; namely, the marginal propensity to consume is one. However, when $\theta^* > \theta(i)$, we have $c_t(i) = x_t(i) \frac{\theta(i)}{\theta^*} + \frac{\tau}{1+g_{t-1}} c_{t-1} \frac{\theta^* - \theta(i)}{\theta^*}$. In this latter case, although the marginal propensity to consume is less than one (because $\frac{\theta(i)}{\theta^*} < 1$), consumption is also raised by the term, $\frac{\tau}{1+g_{t-1}} c_{t-1} \left[\frac{\max\{\theta^* - \theta(i), 0\}}{\theta^*} \right] > 0$, which positively depends on the value of τ. This implies that the stronger the degree of habit formation, the smoother the consumption. Hence, the growth rate of consumption is less volatile with habit formation than without. Also, because habit formation increases the serial correlation in consumption growth, it will also enhance the sensitivity of current-period consumption growth towards changes in lagged income.

The aggregated decision rules are given by

$$c_t = w_t R(\theta_t^*) D(\theta_t^*) + \frac{\tau}{1+g_{t-1}} c_{t-1} \tag{60}$$

$$(1+g_t) s_{t+1} = w_t R(\theta_t^*) H(\theta_t^*) \tag{61}$$

$$x_t = (1+r_t) s_t + w N_t = w_t R(\theta_t^*) \theta^* + \frac{\tau}{1+g_{t-1}} c_{t-1}, \tag{62}$$

where the functions $D(\theta^*)$ and $H(\theta^*)$ are the same as in (35) and (36). In general equilibrium, $s_t = k_t$, $x_t = y_t + (1-\delta) k_t$; hence, aggregate consumption and savings are related to aggregate output according to the following relationships:

$$c_t = \frac{D(\theta_t^*)}{\theta_t^*} (y_t + (1-\delta) k_t) + \left[1 - \frac{D(\theta^*)}{\theta^*}\right] \frac{\tau}{1+g_{t-1}} c_{t-1} \tag{63}$$

$$(1+g_t) k_{t+1} = \left[1 - \frac{D(\theta_t^*)}{\theta_t^*}\right] (y_t + (1-\delta) k_t) - \left[1 - \frac{D(\theta^*)}{\theta^*}\right] \frac{\tau}{1+g_{t-1}} c_{t-1}, \tag{64}$$

where, as before, $\frac{D(\theta_t^*)}{\theta_t^*}$ is the aggregate MPC with $MPC < 1$ and $\frac{\partial MPC}{\partial \theta^*} < 0$.

Steady State. The system of equations determining the steady state of the habit-formation model is the same as in (45)-(48) except the consumption function is replaced by

$$c = \frac{1}{1 - \frac{\tau}{1+\bar{g}}} w R(\theta^*) D(\theta^*). \tag{65}$$

As in the previous section, the output-capital ratio must satisfy $(1+\bar{g}) = \beta \left(1 - \delta + \alpha \frac{y}{k}\right) R(\theta^*)$. Equations (65) and (61) imply the steady-state consumption-capital ratio, $\frac{c}{k} = \frac{(1+\bar{g})}{1-\frac{\tau}{1+\bar{g}}} \frac{D}{H}$. These relationships together with resource constraint (48) give the following implicit equation to solve for the cutoff value that is analogous to (49):

$$\frac{1+\bar{g}}{R(\theta^*)} = \beta \left[1 - \delta + \alpha \left(\bar{g} + \delta + \frac{(1+\bar{g})}{1 - \frac{\tau}{1+\bar{g}}} \frac{D(\theta^*)}{H(\theta^*)}\right)\right]. \tag{66}$$

Because $\frac{\partial(D/H)}{\partial\theta^*} < 0$, the right-hand side implies that, everything else equal, the larger the value of τ, the higher the cutoff θ^*. With the cutoff value θ^* determined, the capital-output ratio and consumption-output ratio are then given by

$$\frac{k}{y} = \frac{\beta\alpha R(\theta^*)}{1+\bar{g}-\beta(1-\delta)R(\theta^*)} \tag{67}$$

$$\frac{c}{y} = 1-(\bar{g}+\delta)\,\frac{\beta\alpha R(\theta^*)}{1+\bar{g}-\beta(1-\delta)R(\theta^*)}, \tag{68}$$

respectively, which differ from those in the previous buffer-stock model only because the values of θ^* in the two models are different. In particular, since θ^* is larger with habit formation, the capital-output ratio is lower (because $\frac{\partial R}{\partial\theta} < 0$). That is, habit formation reduces the rate of saving. The reason can be seen from equation (**??**). A higher degree of habit formation (τ) raises the level of consumption by (i) increasing the relative weight of the habit stock in the consumption function and (ii) decreasing the marginal propensity to consume ($mpc = \frac{D(\theta^*)}{\theta^*}$) out of wealth ($y+(1-\delta)\,k$). This suggests that habit formation does not necessarily enhance the positive link between growth and saving, in sharp contrast to the analysis of Carroll, Overland, and Weil (2000) in a model with constant marginal product of capital.

Calibration and Impulse Responses. As a benchmark value, we set $\tau = 0.4$. The rest of the parameters are set at the same values as in the previous model; that is, $\beta = 0.98$, $\alpha = 0.4$, $\delta = 0.025$, $\sigma = 1.5$, and $\bar{g} = 0.01$. The impulse responses are shown in Figure 3 (where the dashed lines represent the impulses of g_t). The most notable difference in Figure 3 compared with Figure 2 is that consumption growth (top left window) is much smoother than before.

The predicted second moments of the habit-model are reported in Table 3. The effects of habit formation in reducing consumption volatility and enhancing its sensitivity to lagged income is obvious from the table. When $\tau = 0.4$ (Model 1 in Table 3), the standard deviation of consumption relative to output matches the U.S. data almost perfectly. At the same time, the sensitivity measure is also increased remarkably to 0.50. However, habit formation tends to over-kill the sensitivity puzzle: In comparison with the model, the data exhibit excess insensitivity rather than excess sensitivity of consumption to lagged income. When $\tau = 0.2$ (Model 2 in Table 3), the excess insensitivity problem is less severe, but the effect on consumption volatility is weakened.

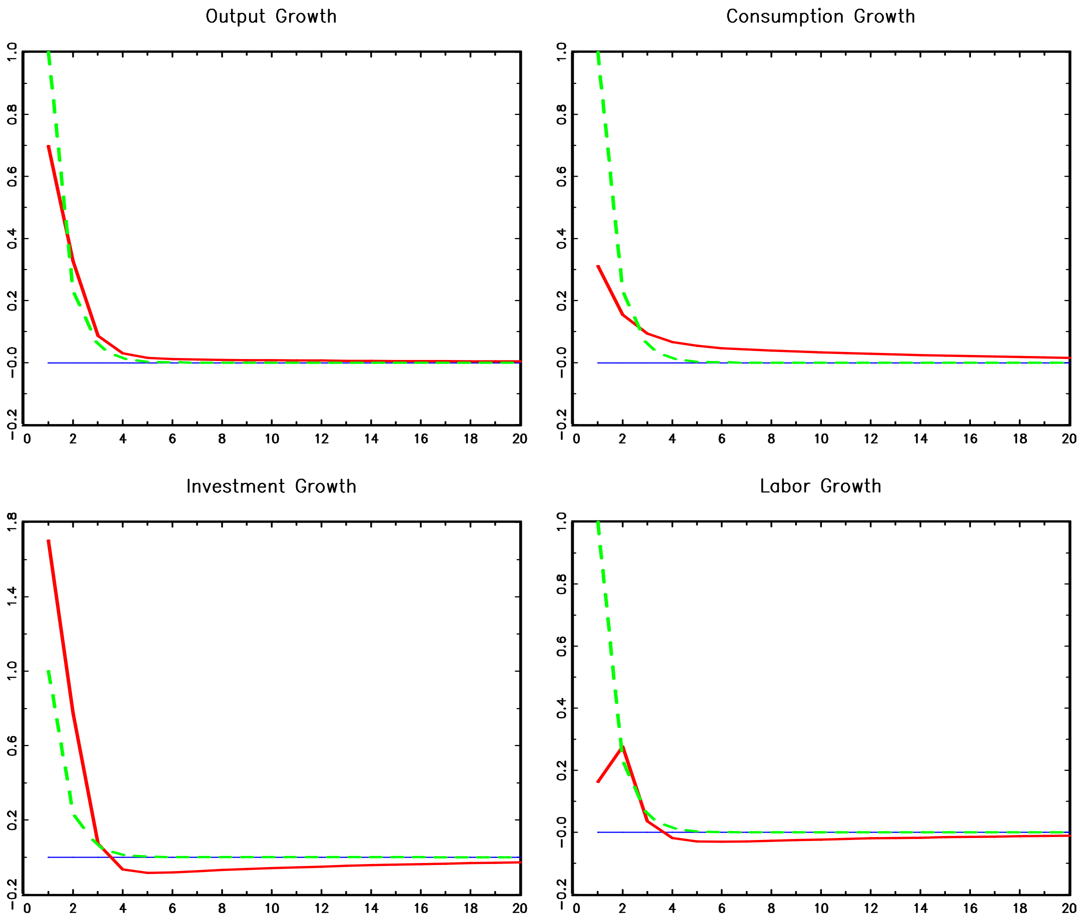

Figure 3. Impulse Responses to Technology Growth (Habit Model).

A perfect match on both dimensions can be achieved with the help of a tightened borrowing constraint. For example, when $\sigma = 1.25$ and $\tau = 0.15$ (Model 3 in Table 3), the relative volatility of consumption growth is the same as in the data (0.51) and the sensitivity measure is also close to the data (0.29). In the meantime, the implied capital-output ratio is 10.6, matching the data almost perfectly. This reinforces the previous two basic findings: (i) borrowing constraints are effective in resolving the excess smoothness puzzle but not for the excess sensitivity puzzle; (ii) habit formation is very effective in resolving the excess smoothness puzzle but it generates an excess insensitivity puzzle. Therefore, a proper combination of these two factors can effectively eliminate both puzzles.

The reason that habit formation has a much stronger relative force in raising the sensitivity of consumption than in reducing its volatility is that, as habit level rises, output also become much less volatile through the reduction in labor supply (under the well-known intertemporal substitution effect) and much more serially correlated. Hence, this leads to a significant increase in its power to predict future consumption growth. However, this tends to over-kill the excess sensitivity puzzle: The bottom row in Table 3 (RBC model) shows that in the absence of borrowing constraints, habit formation alone ($\tau = 0.5$) can resolve the excess smoothness puzzle ($\sigma_c/\sigma_y = 0.51$) but creates an excess insensitivity puzzle ($cor(c_t, y_{t-1}) = 0.56$ in the model but 0.28 in the data).

Table 3. Predicted Second Moments (Habit Model)

	σ_c/σ_y	$cor(c_t, y_{t-1})$
U.S. Data	**0.51**	**0.28**
Model 1 ($\tau = 0.4$)	**0.50**	**0.50**
Model 2 ($\tau = 0.2$)	**0.55**	**0.33**
Model 3 ($\sigma = 1.25, \tau = 0.15$)	**0.51**	**0.29**
RBC with Habit ($\tau = 0.5$)	**0.51**	**0.56**

5 Conclusion

This paper provides an analytical approach to inspecting buffer-stock saving behavior in general equilibrium under borrowing constraints. My approach greatly simplifies the analysis and reduces the computational costs. Consequently, the mechanisms of buffer-stock saving become more transparent even with a time-varying interest rate and endogenous labor income. The methodology is applied to addressing two long-standing puzzles in consumption theory: the "excess smoothness" and "excess sensitivity" of consumption growth with respect to income growth. My analysis shows: (i) In contrast to the analysis of Campbell and Deaton (1989), the PIH is not *per se* the root of the puzzles, but the assumption of a constant (or exogenous) interest rate is; consequently, the excess smoothness and excess sensitivity of consumption growth have been exaggerated. (ii) Borrowing constraints are able to resolve the excess-smoothness puzzle if the degree of idiosyncratic uncertainty is strong enough; but it is not able to solve the excess-sensitivity puzzle. (iii) Habit formation is very effective in eliminating the excess-smoothness puzzle but it "over-kills" the excess-sensitivity puzzle. In this regard, habit formation creates an "excess *insensitivity*" puzzle. However, a combination of weak habit formation and strong borrowing constraints can resolve both puzzles simultaneously.

Appendix

This appendix shows that my methodology to solving buffer-stock saving models analytically is neither restricted to idiosyncratic preference shocks nor relies on a degenerate wealth distribution. Here I give an example by considering a multiplicative shock to individuals' wealth-income (or cash in hand), $x(i) = (1 + r_t) s_t(i) + wN_t(i)$. In the model, household i solves

$$\max E \sum_{t=0}^{\infty} \beta^t \{\log c_t(i) - N_t(i)\}$$

subject to

$$c_t(i) + (1 + g_t) s_{t+1}(i) \leq \varepsilon_t(i) [(1 + r_t) s_t(i) + wN_t(i)] \tag{69}$$

$$s_{t+1}(i) \geq 0, \tag{70}$$

where $\varepsilon(i)$ is an idiosyncratic *i.i.d.* shock with support $\varepsilon \in [\varepsilon_L, \varepsilon_H]$ and the cumulative distribution function $F(\varepsilon)$. Denoting $\{\lambda(i), \pi(i)\}$ as the Lagrangian multipliers for constraints (69) and (70), respectively, the first-order conditions for $\{c(i), n(i), s(i)\}$ are given, respectively, by

$$\frac{1}{c(i)} = \lambda(i) \tag{71}$$

$$1 = w_t E_t^i \varepsilon_t(i) \lambda(i) \tag{72}$$

$$(1 + g_t) \lambda_t(i) = \beta E_t (1 + r_{t+1}) \varepsilon_{t+1}(i) \lambda_{t+1}(i) + \pi_t(i), \tag{73}$$

where the expectation operator E^i denotes expectations conditional on the information set of time t excluding $\varepsilon_t(i)$. Hence, equation (72) reflects the fact that labor supply $n_t(i)$ must be made before the idiosyncratic wealth shocks (and hence the value of $\lambda_t(i)$) are realized. By the law of iterated expectations and the orthogonality assumption of aggregate and idiosyncratic shocks, equation (73) can be written (by using 72) as

$$(1 + g_t) \lambda_t(i) = \beta E_t (1 + r_{t+1}) \frac{1}{w_{t+1}} + \pi_t(i). \tag{74}$$

Similar to the previous analysis, the decision rules for an individual's consumption and savings are characterized by a cutoff strategy where the cutoff is defined by ε^*. Consider two possible cases:

Case A. $\varepsilon_t(i) \geq \varepsilon_t^*$. In this case the wealth level is high. It is hence optimal to save so as to prevent possible liquidity constraints in the future when wealth may be low. So $s_{t+1}(i) \geq 0$, $\pi_t(i) = 0$, and the shadow value of good $\lambda_t(i) = \beta E_t \frac{1+r_{t+1}}{(1+g_t)w_{t+1}}$. Equation (71) implies that consumption is given by $c(i) = \left[\beta E_t \frac{1+r_{t+1}}{(1+g_t)w_{t+1}}\right]^{-1}$. Defining

$$x(i) \equiv (1 + r_t) s_t(i) + wn_t(i) \tag{75}$$

as the wealth (cash in hand) of household i in the absence of the idiosyncratic shock, the budget constraint (69) then implies $(1+g_t)\, s_{t+1}(i) = \varepsilon_t(i)x_t(i) - \left[\beta E_t \frac{1+r_{t+1}}{(1+g_t)w_{t+1}}\right]^{-1}$. The requirement $s_{t+1}(i) \geq 0$ then implies

$$\varepsilon_t(i) \geq \frac{1}{x_t(i)} \left[\beta E_t \frac{1+r_{t+1}}{(1+g_t)\, w_{t+1}}\right]^{-1} \equiv \varepsilon_t^*, \tag{76}$$

which defines the cutoff ε^*. Notice that the cutoff is independent of i because wealth $x(i)$ is determined before the realization of $\varepsilon_t(i)$ and all households face the same distribution of idiosyncratic shocks.

Case B. $\varepsilon_t(i) < \varepsilon_t^*$. In this case the wealth level is low. It is then optimal not to save, so $s_{t+1}(i) = 0$ and $\pi_t(i) > 0$. By the resource constraint (69), we have $c_t(i) = \varepsilon_t(i)x_t(i)$, which by equation (76) implies $c(i) = \frac{\varepsilon_t(i)}{\varepsilon_t^*}\left[\beta E_t \frac{1+r_{t+1}}{(1+g_t)w_{t+1}}\right]^{-1}$. Equation (71) then implies that the marginal utility of consumption is given by $\lambda_t(i) = \frac{\varepsilon_t^*}{\varepsilon_t(i)}\left[\beta E_t \frac{1+r_{t+1}}{(1+g_t)w_{t+1}}\right]$. Since $\varepsilon(i) < \varepsilon^*$, equation (74) implies $\pi_t(i) = \left[\beta E_t \frac{1+r_{t+1}}{w_{t+1}}\right]\left[\frac{\varepsilon^*}{\varepsilon} - 1\right] > 0$.

The above analyses imply that the expected shadow value of goods, $E^i\varepsilon(i)\lambda(i)$, and hence the optimal cutoff value ε^*, is determined by the following equation for savings based on (72):

$$\frac{1}{w_t} = \left[\beta E_t \frac{1+r_{t+1}}{(1+g_t)\, w_{t+1}}\right] R(\varepsilon_t^*), \tag{77}$$

where

$$R(\varepsilon_t^*) \equiv \left[\int_{\varepsilon<\varepsilon^*} \varepsilon^* dF(\varepsilon) + \int_{\varepsilon\geq\varepsilon^*} \varepsilon dF(\varepsilon)\right]. \tag{78}$$

Notice that, unlike the case with preference shocks, the value of $R(\varepsilon^*)$ is no longer necessarily greater than one because the option value of liquidity is now measured by $\frac{R(\varepsilon^*)}{\varepsilon^*}$ instead of by $R(\varepsilon^*)$. However, here we have something analogous: $R(\varepsilon^*) > \varepsilon^*$. The cutoff strategy continues to imply that the optimal level of wealth (cash in hand) in period t is determined by a "target" policy given by $x_t(i) = \frac{1}{\varepsilon_t^*}\left[\beta E_t \frac{1+r_{t+1}}{(1+g_t)w_{t+1}}\right]^{-1}$. Thus, labor supply will still adjust so that the wealth level meets its target, as in the previous sections.

Utilizing equation (77), the decision rules of household i are summarized by

$$c_t(i) = w_t R(\varepsilon_t^*) \times \min\left\{\frac{\varepsilon(i)}{\varepsilon^*}, 1\right\} \tag{79}$$

$$(1+g_t)s_{t+1}(i) = w_t R(\varepsilon_t^*) \times \max\left\{\frac{\varepsilon(i)-\varepsilon^*}{\varepsilon^*}, 0\right\} \tag{80}$$

$$x_t(i) = w_t R(\varepsilon_t^*) \varepsilon_t^{*(-1)}. \tag{81}$$

Notice that $c_t(i) + (1 + g_t)s_{t+1}(i) = \varepsilon_t(i)x_t(i)$. Denoting $c \equiv \int c(i)di$, $s \equiv \int s(i)di$, $N \equiv \int N(i)di$, and $x \equiv \int x(i)di$ and integrating the household decision rules over i by the law of large numbers, the aggregate variables are given by

$$c_t = w_t R(\varepsilon_t^*) D(\varepsilon_t^*) \tag{82}$$

$$(1 + g_t)s_{t+1} = w_t R(\varepsilon_t^*) H(\varepsilon_t^*) \tag{83}$$

$$\int \varepsilon(i)x(i)di = \bar{\varepsilon}\left[(1 + r_t)s_t + wN_t\right] = w_t R(\varepsilon_t^*) \frac{\bar{\varepsilon}}{\varepsilon_t^*}, \tag{84}$$

where $\bar{\varepsilon}$ is the mean of $\varepsilon(i)$ and

$$D(\varepsilon^*) \equiv \int_{\varepsilon \leq \varepsilon^*} \frac{\varepsilon}{\varepsilon^*} dF(\varepsilon) + 1 - F(\varepsilon^*) \tag{85}$$

$$H(\varepsilon^*) \equiv \int_{\varepsilon > \varepsilon^*} \frac{\varepsilon}{\varepsilon^*} dF(\varepsilon) - (1 - F(\varepsilon^*)) \tag{86}$$

and these two functions satisfy $D(\varepsilon^*) + H(\varepsilon^*) = \frac{\bar{\varepsilon}}{\varepsilon^*}$, $\varepsilon^* D < \bar{\varepsilon}$, $\frac{\partial D}{\partial \varepsilon^*} < 0$, and $\frac{\partial H}{\partial \varepsilon^*} < 0$. Aggregate consumption and savings are related to aggregate wealth ($\Omega_t \equiv \bar{\varepsilon} x_t$) according to the following relationships:

$$c_t = \frac{\varepsilon_t^*}{\bar{\varepsilon}} D(\varepsilon_t^*) \Omega_t \tag{87}$$

$$(1 + g_t)s_{t+1} = \left[1 - \frac{\varepsilon^*}{\bar{\varepsilon}} D\right] \Omega_t, \tag{88}$$

where $\frac{\varepsilon_t^*}{\bar{\varepsilon}} D(\varepsilon_t^*) < 1$ is the aggregate marginal propensity to consume (MPC).

In general equilibrium, we have $r_t + \delta_t = \alpha \frac{y_t}{k_t}$ and $w_t = (1 - \alpha) \frac{y_t}{N_t}$. Market clearing implies $s_{t+1} = k_{t+1}$ and $\int N_t(i) = N_t$. The constant-returns-to-scale property of the production function implies $x_t = y_t + (1 - \delta) k_t$. The aggregated household resource constraint implies the aggregate goods-market clearing condition,

$$c_t + (1 + g_t) k_{t+1} - \bar{\varepsilon}(1 - \delta)k_t = \bar{\varepsilon} y_t. \tag{89}$$

Notice the wedge in the aggregate budget identity, $\bar{\varepsilon}$. This wedge exists because of the idiosyncratic weal shock.

In the special case of $\delta = 1$, the model also has closed-form solutions with $x_t = y_t$ and

$$c_t = \left[\frac{\varepsilon_t^*}{\bar{\varepsilon}} D(\varepsilon_t^*)\right] \bar{\varepsilon} y_t \tag{90}$$

$$(1+g_t)\,k_{t+1} = \left[1 - \frac{\varepsilon_t^*}{\bar{\varepsilon}} D(\varepsilon_t^*)\right] \bar{\varepsilon} y_t \tag{91}$$

$$N_t = (1-\alpha)\,\frac{R(\varepsilon_t^*)}{\varepsilon_t^*}. \tag{92}$$

Since $w = (1-\alpha)\,\frac{y}{N}$, utilizing equation (77), we get

$$\left[\frac{\bar{\varepsilon}}{\varepsilon_t^*} - D(\varepsilon_t^*)\right] = \beta\alpha E_t \frac{R(\varepsilon_{t+1}^*)}{\varepsilon_{t+1}^*}, \tag{93}$$

which suggests that $\varepsilon_t^* = \varepsilon^*$ for all t (i.e., a constant) is a solution and labor supply is constant over time. Once the distribution of $\varepsilon_t(i)$ is given, the constant ε^* can then be solved by equation (93) and we then have $\left[1 - \frac{\varepsilon_t^*}{\bar{\varepsilon}} D(\varepsilon_t^*)\right] = \beta\alpha\frac{1}{\bar{\varepsilon}} R(\varepsilon^*)$. Substituting this into equations (90) and (91) gives $c_t = \left[1 - \beta\alpha\bar{\varepsilon}^{-1} R(\theta^*)\right] \bar{\varepsilon} y_t$ and $(1+g_t)\,k_{t+1} = \left[\beta\alpha\bar{\varepsilon}^{-1} R(\theta^*)\right] \bar{\varepsilon} y_t$.[21] Hence, as in the previous buffer-stock models, regardless of the distribution of ε, borrowing constraints do not matter for the model's aggregate dynamics when $\delta = 1$.

The model's steady state and impulse responses can be solved analogously to the previous sections. Since the steps are similar, they are not repeated here. When we assume that the distribution of $\varepsilon(i)$ is given by the power function, $F(\varepsilon) = \left(\frac{\varepsilon(i)}{\varepsilon_{\max}}\right)^{\sigma}$, with support $\varepsilon(i) \in [0, \varepsilon_{\max}]$ and the upper-bound parameter $\varepsilon_{\max} = \frac{(1+\sigma)}{\sigma}$ so that the mean $\bar{\varepsilon} = 1$, the results of this model are then completely identical to those obtained in the previous sections when the variance of wealth shocks is chosen properly to match that of preference shocks in the previous models.[22]

[21] When the variance of the distribution for ε is degenerate, we have $R(\theta^*) = \varepsilon^* = \bar{\varepsilon}$, so this buffer-stock model also reduces completely to the representative-agent RBC model.

[22] A power-law distribution is the inverted Pareto distribution. That is, if ε follows the Pareto distribution, then ε^{-1} follows the power distribution.

References

[1] Aiyagari, R., 1994, Uninsured idiosyncratic risk and aggregate saving, Quarterly Journal of Economics 109(3), 659-684.

[2] Blinder, A. S., and A. Deaton, 1985, "The Time Series Consumption Function Revisited," Brookings Papers of Economic Activity, 2, 465-511.

[3] Campbell, J. and A. Deaton, 1989, Why is Consumption So Smooth? The Review of Economic Studies 56(3), 357-373.

[4] Campbell, J. and M. Gregory, 1989, "Consumption, Income, and Interest Rates: Reinterpreting the Time Series Evidence," in Olivier Jean Blanchard and Stanley Fischer, eds., NBER macroeconomics annual. Cambridge, MA: MIT Press, 185-216.

[5] Campbell, J. and M. Gregory, 1990, "Permanent Income, Current Income, and Consumption." Journal of Business and Economic Statistics, July 1990, 8(3), pp. 265-279.

[6] Carroll, C., 1992, "The Buffer-Stock Theory of Saving: Some Macroeconomic Evidence." Brookings Papers on Economic Activity 23(2), 61-156.

[7] Carroll, C., 1994, "How Does Future Income Affect Current Consumption?" The Quarterly Journal of Economics. 109(1), 111-148.

[8] Carroll, C., 1997, "Buffer-Stock Saving and the Life Cycle/Permanent Income Hypothesis." Quarterly Journal of Economics, 112(1), 1-56.

[9] Carroll, C., 2001, A theory of the consumption function, with and without liquidity constraints, Journal of Economic Perspectives 15(3), 23-45.

[10] Carroll, C. and M. Kimball, 1996, "On the Concavity of the Consumption Function." Econometrica. 64(4), 981-992.

[11] Carroll, C. and M. Kimball, 2001, "Liquidity Constraints and Precautionary Saving," NBER Working Paper No. 8496.

[12] Carroll, C., J. Overland, and D. Weil, 2000, Saving and Growth with Habit Formation, American Economic Review 90(3), 341-355.

[13] Carroll, C. and P. Toche, 2009, A Tractable Model of Buffer Stock Saving, Johns Hopkins University Working Paper.

[14] Chen, K., A. Imrohoroglu, and S. Imrohoroglu, 2006, The Japanese Saving Rate, American Economic Review 96(5), 1850-1858.

[15] Christiano, L., M. Eichenbaum, and M. David, 1991, "The Permanent Income Hypothesis Revisited." Econometrica, March 59(2), 397-423.

[16] Clementi, F. and M. Gallegati, 2005, Pareto's Law of Income Distribution: Evidence for Germany, the United Kingdom, and the United States, in Econophysics of Wealth Distributions. Milan: Springer-Verlag Italia. ed. by Chatterjee, A., Yarlagadda, S., and Chakrabarti B. K.

[17] Cochrane, J., 1994, Shocks, Carnegie–Rochester Conference Series on Public Policy 41(1), 295–364.

[18] Cogley, T., Nason, J., 1995, Output dynamics in real-business-cycle models. American Economic Review 85(3), 492–511.

[19] Deaton, A., 1987, "Life-Cycle Models of Consumption: Is the Evidence Consistent with the Theory?" in Advances in Econometrics. Fifth World Congress, Vol. 2, ed. by Truman F. Bewley. Cambridge, UK: Cambridge University Press, 121-148.

[20] Deaton, A., 1991, "Saving and Liquidity Constraints." Econometrica. 59(5), pp. 1221-1248.

[21] Deaton, A., 1992, Understanding Consumption. Oxford: Oxford University.

[22] Ermini, L., 1993, "Effects of Transitory Consumption and Temporal Aggregation on the Permanent Income Hypothesis." Review of Economics and Statistics 75(4), pp. 736-740.

[23] Fermi, D. (Ed.), QUID (Paris: RTL, 1998).

[24] Flavin, M., 1981, "The Adjustment of Consumption to Changing Expectations about Future Income." Journal of Political Economy 89(5), 974-1009.

[25] Flavin, M., 1985, Excess sensitivity of consumption to current income: liquidity constraints or myopia? Canadian Journal of Economics 18, 117-136.

[26] Friedman, M., 1957, A Theory of the Consumption Function. Princeton: Princeton Univ Press.

[27] Gali, J., 1991, "Budget Constraints and Time Series Evidence on Consumption," American Economic Review, 81, 1238-1253.

[28] Hall, R., 1978, "Stochastic Implications of the Life Cycle-Permanent Income Hypothesis: Theory and Evidence." Journal of Political Econ- omy. 86, pp. 971-87.

[29] Hansen, G., 1985, Indivisible labour and the business cycle. Journal of Monetary Economics 16, 309–325.

[30] Hayashi, F., 1987, "Tests for Liquidity Constraints: A Critical Survey," in Advances in Econometrics, Fifth World Congress, T. Bewley, ed. (Cambridge, UK: Cambridge University Press, 1987).

[31] Hubbard, G., J. Skinner, and S. Zeldes, 1995, "Precautionary Saving and Social Insurance." Journal of Political Economy 103(2), pp. 360-399.

[32] Kimball, M., 1990, "Precautionary Saving in the Small and in the Large." Econometrica 58(1), pp. 53-73.

[33] King, R., Plosser, C., Rebelo, S., 1988, Production, growth and business cycles: I – The basic neoclassical model. Journal of Monetary Economics 21(2), 195-232.

[34] Krusell, P. and A. Smith, 1998, "Income and Wealth Heterogeneity in the Macro- economy." Journal of Political Economy 106(5), pp. 867-96.

[35] Kydland, F. and E. Prescott, 1982. Time to build and aggregate fluctuations. Econometrica 50(6), 1345-1370.

[36] Levy, M. and H. Levy, 2003, Investment talent and the Pareto wealth distribution: Theoretical and empirical analysis, The Review of Economics and Statistics 85(3), 709-725.

[37] Long, J. and C. Plosser, 1983, Real business cycles, Journal of Political Economy 91(1), 39-69.

[38] Ludvigson, S. and A. Michaelides, 2001, "Can Buffer Stock Saving Explain the Consumption Excesses?" The American Economic Review 91(3), 631-647.

[39] Mankiw, G. and M. Shapiro, 1985, Trends, Random Walks, and Tests of the Permanent Income Hypothesis, Journal of Monetary Economics 16(2), pp. 165-74.

[40] Michaelides, A., 2002, Buffer Stock Saving and Habit Formation, London School of Economics, Centre for Economic Policy Research (CEPR) Working Paper.

[41] Michener, R., 1984, Permanent income in general equilibrium, Journal of Monetary Economics 13(3), 297-305.

[42] Pischke, J. S., 1995, "Individual Income, Incom- plete Information and Aggregate Consumption." Econometrica, 63(4), pp. 805-40.

[43] Rogerson, R., 1988, Indivisible labor, lotteries and equilibrium, Journal of Monetary Economics 21(1), 3-16.

[44] Rotemberg, J., Woodford, M., 1996. Real-business-cycle models and the forecastable movements in output, hours, and consumption. American Economic Review 86, 71–89.

[45] Sommer, M., 2007, "Habit Formation and Aggregate Consumption Dynamics," The B.E. Journal of Macroeconomics 7(1) (Advances), Article 21.

[46] Toche, P., 2005, "A Tractable Model of Precautionary Saving in Continuous Time," Economics Letters, 87(2), 267–272.

[47] Quah, D., 1990, "Permanent and Transitory Movements in Labor Income: An Explanation For 'Excess Smoothness' in Consumption." Journal of Political Economy 98(3), pp. 449-475.

[48] Wang, PF. and Y. Wen, 2009, Financial development and economic volatility: An unified explanation, Federal Reserve Bank of St. Louis Working Paper 2009-022C.

[49] Wen, Y., 2007, "Granger Causality and Equilibrium Business Cycle Theory." Federal Reserve Bank of St. Louis Review 89(3), 195-205.

[50] Wen, Y., 2008, Input and output inventory dynamics, Federal Reserve Bank of St. Louis Working Paper 2008-008B.

[51] Wen, Y., 2009a, Liquidity and welfare in a heterogeneous-agent economy, Federal Reserve Bank of St. Louis Working Paper 2009-019B.

[52] Wen, Y., 2009b, When does heterogeneity matter? Federal Reserve Bank of St. Louis Working Paper 2009-024A.

[53] Wolf, E., 1996, Trends in Household Wealth during l989-1992, submitted to the Department of Labor (New York University).

[54] Zeldes, S., 1989a, Optimal Consumption with Stochastic Income: Deviations from Certainty Equivalence, The Quarterly Journal of Economics, Vol. 104(2), 275-298.

[55] Zeldes, S., 1989b, Consumption and Liquidity Constraints: An Empirical Investigation, The Journal of Political Economy 97(2), 305-346.