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Inventory Accelerator in General Equilibrium*

Peng-fei Wang
Hong Kong Univ. of Sci. & Tech.
pfwang@ust.hk

Yi Wen
Federal Reserve Bank of St. Louis
and
Tsinghua University
yi.wen@stls.frb.org

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Abstract

We develop a general-equilibrium model of inventories with explicit micro-foundations by embedding the production-cost-smoothing motive (e.g., Eichenbaum, AER 1989) into an otherwise standard DSGE model. We show that firms facing idiosyncratic cost shocks have incentives to bunch production and smooth sales by carrying inventories. The optimal inventory target of a firm is derived explicitly. The model is broadly consistent with many of the observed stylized facts of aggregate inventory fluctuations, such as the procyclical inventory investment and the countercyclical inventory-sales ratio. In addition, the model yields novel predictions for the role of inventories in macroeconomic stability: Inventories may not only greatly amplify but also propagate the business cycle. That is, the incentive to accumulate inventories under the cost-smoothing motive can give rise to hump-shaped output dynamics and significantly higher volatility of GDP. Such predictions are in sharp contrast to the implications of the recent general-equilibrium inventory literature (e.g., Khan and Thomas, 2007; and Wen, 2008), which shows that inventory investment induced by traditional mechanisms (e.g., the stockout-avoidance motive and the (S,s) rule) does not increase the variance of aggregate output.

Keywords: Inventories, Accelerator, Business Cycle, Target-Adjustment Model.

JEL codes: E13, E20, E32.

*Correspondence: Yi Wen, Research Department, Federal Reserve Bank of St. Louis, St. Louis, MO, 63144. Phone: 314-444-8559. Fax: 314-444-8731. Email: yi.wen@stls.frb.org.

1 Introduction

"Relative to its importance in business fluctuations, inventory investment must be the most under-researched aspect of macroeconomic activity." (Blinder, 1981, p444). Blinder's assessment probably remains true today despite his and others' best efforts in developing a well established and empirically validated theory of inventory investment for the past quarter century.[1] The general-equilibrium inventory literature with rigorous microfoundations is extremely thin,[2] although the empirical evidence continue to remind us of Blinder's (1990) famous claim that "business cycles are, to a surprisingly large degree, inventory cycles."[3] In particular, the "great moderation" of the U.S. economy since the 1980s seems closely associated with a significant reduction in inventory volatility and inventory-sales ratio (see, e.g., Kahn, McConnell and Perez-Quiros, 2002).

Overwhelming empirical evidence indicate that the variance of production is larger than that of sales and inventory investment is procyclical in a wide variety of sectors and subsectors, and also in the entire economy. Because of this, conventional wisdom views inventories as a destabilizing force to the economy and key to understanding the business cycle.[4]

However, the conventional wisdom is based on a partial equilibrium argument: *Given* sales, procyclical inventory investment implies a higher variance of production; hence, output is more variable than it would be if inventories did not exist or were not procyclical. Such an argument ignores the possible general-equilibrium effects of inventories on sales through prices. Indeed, a recently emerged general-equilibrium inventory literature challenges Blinder's view that inventories are key to understanding economic fluctuations. Khan and Thomas (2007) and Wen (2008) develop DSGE models in which inventories are rigorously introduced through firms' optimization behavior either via the (S,s) policy or the stockout-avoidance motive, and show that procyclical inventory investment does not increase the volatility of output.[5] This is so because in general equilibrium inventories may stabilize sales as much as (or even more than) they destabilize production. For example, firms' intentional accumulation of inventories in a boom attenuates the rise of sales and their purposeful decumulation of inventories in a recession mitigates the fall in demand. Consequently, eliminating inventories from the economy does not necessarily decrease the variance of

[1] An incomplete list of important early works include Blanchard (1983), Blinder (1981, 1986a, 1986b), Blinder and Maccini (1991), Eichenbaum (1989), Kahn (1987), Ramey (1991), West (1986), and many others.

[2] Exceptions include Fisher and Hornstein (2000), Khan and Thomas (2007a, 2007b), Kryvtsov and Midrigany (2008) and Wen (2008).

[3] Inventory investment accounts for less than 1% of GDP, but its movement accounts for more than 60% of the variations in GDP (see, e.g., Blinder, 1981 and 1986a; Blinder and Maccini, 1991). Using updated data, Romer (2001, p170, Table 4.2) shows that declines in inventory investment still accounts for more than 40% of the drop in GDP for post-war U.S. recessions.

[4] See the previously cited literature for reference.

[5] In particular, Wen (2008) shows that inventories *reduce* the variance of GDP.

GDP, contradicting the argument of Kahn, McConnell and Perez-Quiros (2002) that the reduction of inventories is a possible cause of the "great moderation".

While the general-equilibrium analyses of Khan and Thomas (2007) and Wen (2008) are provocative, there are other theoretical possibilities linking inventories to output volatility and motives inducing firms to hold inventories. For example, firms may use inventories to smooth sales when the costs of production are uncertain. That is, profit-maximizing firms (facing cost shocks) may opt to "bunch" production by producing more than sales and carrying the excess supply as inventories when costs are low, and using inventories to meet demand when costs are high. This is the production-cost-smoothing motive emphasized by Eichenbaum (1989). There are ample examples of such an optimal inventory behavior in reality. For example, the great volatility of oil prices in the world market increases uncertainty in production costs and may therefore induce firms to hold excessively large amount of both finished and intermediate-goods inventories than they would otherwise without such uncertainties. Also, empirical studies carried out by Eichenbaum (1989) and others show that cost shocks are indeed very important for explaining inventory fluctuations at the industry level.[6]

This paper provides a rigorous, microfounded, general-equilibrium model of inventories based on the cost-smoothing motive. Our model is shown able to explain many of the stylized facts of aggregate inventory behavior, such as the procyclical inventory investment and the countercyclical inventory-sales ratio. In sharp contrast to Khan and Thomas (2007) and Wen (2008), which are based either on the (S,s) inventory strategy or the stockout-avoidance policy, our model predicts that procyclical inventory investment may greatly amplify the volatility of aggregate output. More importantly, it may also propagate aggregate shocks by generating hump-shaped output dynamics; suggesting that inventory investment may indeed play a key role in the business cycle. This potential role of inventories in propagating the business cycle was first studied by Metzler (1941) but has not been emphasized by the recent theoretical literature.[7]

[6]Costs shocks as a distinct source of uncertainty driving inventory behavior is also emphasized by Blanchard (1983), Eichenbaum (1984, 1989), Durlauf and Maccini (1995), Ramey (1989), West (1986), among others. However, this literature does not provide an explicit microtheory to explain the existence of inventories. For example, in this literature the existence of an optimal target inventory level is assumed, rather than derived from firms' cost minimization problems.

[7]A virtue of our approach is that our model is analytically tractable, in contrast to the (S,s) general equilibrium inventory models (e.g., see Fisher and Hornstein, 2000; and Khan and Thomas, 2007). Our strategy to make the model analytically tractable is inspired by the general-equilibrium approach of Wen (2008). Wen (2008) uses the Lagrangian method to derive a firm's inventory decision rules as functions of an endogenous cutoff value for idiosyncratic shocks, which determines the firm's probability of inventory stockout. The optimal value of the endogenous cutoff variable is in turn determined by a dynamic Euler equation for the shadow value of inventories, which equates the marginal cost of inventory to its expected rates of return (which is similar to a Belman's equation). Because each firm takes the aggregate environment (variables) as given, the Belman equation can be explicitly solved to obtain the optimal cutoff value as a function of the distribution of idiosyncratic shocks. Closed-form expressions for firms' decision rules of production, sales, inventory investment, and inventory-sales ratio can all be obtained. After aggregating firms' decision rules by the law of large numbers, the conventional log-linear approximation method can then be applied to solve the model's saddle-path aggregate dynamics.

2 The Model

This is a model of output inventories. The framework can be easily extended to the case of input inventories.[8] There are two types of goods in the economy, final goods and intermediate goods. The final goods sector is perfectly competitive and are produced according to a CES function using a continuum of intermediate goods,

$$Z_t = \left[\int_0^1 y_t(i)^{\frac{\sigma-1}{\sigma}} di \right]^{\frac{\sigma}{\sigma-1}}. \tag{1}$$

Given prices of intermediate goods, $p_t(i)$, the inverse demand function of intermediate good i is given by

$$p(i) = y(i)^{-\frac{1}{\sigma}} Z^{\frac{1}{\sigma}}, \tag{2}$$

where the final good price has been normalized to one. Each type of intermediate good i is supplied by a monopolist firm, which produces output according to

$$x_t(i) = A_t \varepsilon_t(i) k_t(i)^{\alpha} n_t(i)^{1-\alpha}, \tag{3}$$

where $\varepsilon(i)$ is the inverse of an idiosyncratic cost shock to firm i, A is an aggregate TFP shock, $k(i)$ is capital and $n(i)$ is labor. The factor markets are competitive so intermediate goods firms take the real rental rate of capital (r) and the real wage (w) as given. The factor demand functions are given by $k_t(i) = \alpha \frac{x_t(i)}{r_t+\delta_k} \phi_t(i)$ and $n_t(i) = (1-\alpha)\frac{x_t(i)}{w_t}\phi_t(i)$, where $\phi(i)$ denotes the marginal cost of firm i and $r+\delta_k$ the user's cost of capital. These factor demand function imply $\frac{1}{A_t}\left(\frac{r+\delta_k}{\alpha}\right)^a \left(\frac{w}{1-\alpha}\right)^{1-\alpha} = \varepsilon(i)\phi(i)$. We can define $\Phi_t \equiv \frac{1}{A_t}\left(\frac{r+\delta_k}{\alpha}\right)^a \left(\frac{w}{1-\alpha}\right)^{1-\alpha}$ as the aggregate marginal cost and it satisfies $\phi_t(i) = \frac{\Phi_t}{\varepsilon_t(i)}$.

Defining $s_t(i)$ as the stock of inventories that firm i decides to hold in period t, the firm's program is to maximize the expected sum of future profits by solving

$$\max E_t \sum_{j=0}^{\infty} \beta^t \Lambda_t \left\{ p_t(i) y_t(i) - \frac{\Phi_t}{\varepsilon_t(i)} x_t(i) \right\} \tag{4}$$

subject to

$$y_t(i) + s_t(i) = x_t(i) + (1-\delta_s) s_{t-1}(i) \tag{5}$$

$$s_t(i) \geq 0 \tag{6}$$

[8] That is, firms can smooth the cost of inputs by bunching orders when the costs are stochastic.

$$p_t(i)y_t(i) \geq \frac{\Phi_t}{\varepsilon_t(i)} x_t(i) \tag{7}$$

where (5) is the resource constraint, (6) is a nonnegativity constraint on inventory stock, (7) is a non-negativity constraint on profit, Λ_t is the marginal utility of consumption of the household and δ_s is the depreciation rate of inventories.

Because the revenue function, $p(i)y(i) = y(i)^{1-\frac{1}{\sigma}} Z^{\frac{1}{\sigma}}$, is concave in sales and the cost function is linear in production (due to constant returns to scale), firms have incentives to smooth both sales and production cost by accumulating inventories, which can maximize average profits and reduce costs through intertemporal substitution of production activities. For example, when $\varepsilon(i)$ is large (marginal cost is low), firm i can produce more than sales (up to the point of a zero profit) and use inventories to substitute for future production when the next-period marginal cost maybe high. On the other hand, when $\varepsilon(i)$ is small (marginal cost is high), the firm can use inventories to satisfy sales without raising production costs.

Denoting $\{\lambda(i), \pi(i), \mu_t(i)\}$ as the Lagrangian multipliers of constraints (5)-(7), respectively, the first-order conditions of $\{x(i), y(i), s(i)\}$ are given by

$$\frac{\Phi_t}{\varepsilon_t(i)} \left(1 + \mu_t(i)\right) = \lambda_t(i) \tag{8}$$

$$\left(\frac{\sigma - 1}{\sigma}\right) y_t(i)^{-\frac{1}{\sigma}} Z_t^{\frac{1}{\sigma}} \left(1 + \mu_t(i)\right) = \lambda_t(i). \tag{9}$$

$$\lambda_t(i) = \beta(1 - \delta_s) E_t \frac{\Lambda_{t+1}}{\Lambda_t} \lambda_{t+1}(i) + \pi_t(i) \tag{10}$$

Equations (8) and (9) imply $\frac{\Phi_t}{\varepsilon_t(i)} = \left(\frac{\sigma-1}{\sigma}\right) y_t(i)^{-\frac{1}{\sigma}} Z_t^{\frac{1}{\sigma}}$, which determines the optimal amount of sales, $y_t(i) = \left(\frac{\sigma-1}{\sigma}\right)^{\sigma} Z_t \left(\frac{\varepsilon_t(i)}{\Phi_t}\right)^{\sigma}$. Consequently, the monopoly price is a markup over the marginal cost,

$$p_t(i) = \frac{\sigma}{\sigma - 1} \frac{\Phi_t}{\varepsilon_t(i)}. \tag{11}$$

Notice that, in the absence of idiosyncratic uncertainty, the incentives to hold inventories are diminished because equation (10) implies $\pi_t = \lambda_t - \beta(1 - \delta_s)\lambda_{t+1}$, which is greater than 0 in the steady state. That is, aggregate shocks do not induce inventory investment near the steady state.

Decision rules for inventories. Consider two possibilities:

Case A. $\varepsilon_t(i) \geq \varepsilon_t^*$. Suppose $s_t(i) > 0, \pi_t(i) = 0$. In such a case, (10) implies $\lambda_t(i) = \beta(1 -$

$\delta_s)E_t \frac{\Lambda_{t+1}}{\Lambda_t}\lambda_{t+1}(i)$, and equation (8) implies

$$\frac{\Phi_t}{\varepsilon_t(i)} \leq \beta(1-\delta_s)E_t \frac{\Lambda_{t+1}}{\Lambda_t}\lambda_{t+1}(i), \tag{12}$$

because $\mu_t(i) \geq 0$. This implies $\varepsilon_t(i) \geq \frac{\Phi_t}{\tilde{\beta}(1-\delta_s)E_t \frac{\Lambda_{t+1}}{\Lambda_t}\lambda_{t+1}(i)} \equiv \varepsilon_t^*$, which defines the cutoff value ε_t^* and the relationship

$$\beta(1-\delta_s)E_t \frac{\Lambda_{t+1}}{\Lambda_t}\lambda_{t+1}(i) \equiv \frac{\Phi_t}{\varepsilon_t^*}. \tag{13}$$

Equation (8) then further implies $1+\mu_t(i) = \frac{\varepsilon_t(i)}{\varepsilon_t^*}$. Hence, we conclude that $\mu_t(i) > 0$ if $\varepsilon_t(i) > \varepsilon_t^*$. In such a case, the non-negative profit constraint binds, $p(i)y(i) = \frac{\Phi}{\varepsilon(i)}x(i)$, which together with (11) implies

$$x(i) = \frac{\sigma}{\sigma-1}y_t(i) > y_t(i), \tag{14}$$

suggesting that inventory investment is strictly positive. That is, in the case of a large enough idiosyncratic productivity shock (or small enough cost shock), the firm produces more than sales and opt to hold the excess supply as inventories.

Case B: $\varepsilon_t(i) < \varepsilon_t^*$. Suppose $p(i)y(i) > \frac{\phi}{\varepsilon(i)}x(i), \mu_t(i) = 0$. Then (8) and (13) imply $\frac{\Phi_t}{\varepsilon_t(i)} = \lambda_t(i) = \frac{\Phi_t}{\varepsilon_t^*} + \pi_t(i)$, which implies $\pi_t(i) = \frac{\Phi_t}{\varepsilon_t(i)} - \frac{\Phi_t}{\varepsilon_t^*} > 0$. Hence, we have $s_t(i) = 0$. In such a case, the firm opts to stockout and the resource identity implies $x_t(i) = y_t(i) - (1-\delta_s)s_{t-1}(i)$.

The decision rules of the firm can thus be summarized by the following policy functions:

$$y_t(i) = \left(\frac{\sigma-1}{\sigma}\right)^{\sigma} Z_t \left(\frac{\varepsilon_t(i)}{\Phi_t}\right)^{\sigma} \tag{15}$$

$$x_t(i) = \begin{cases} \frac{\sigma}{\sigma-1}y_t(i) & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ y_t(i) - (1-\delta_s)s_{t-1}(i) & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases} \tag{16}$$

$$s_t(i) = \begin{cases} \frac{1}{\sigma-1}y_t(i) + (1-\delta_s)s_{t-1}(i) & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ 0 & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases} \tag{17}$$

Since the shadow value of inventory satisfies

$$\lambda_t(i) = \begin{cases} \frac{\Phi_t}{\varepsilon_t^*} & \text{if } \varepsilon_t(i) \geq \varepsilon_t^* \\ \frac{\Phi_t}{\varepsilon_t(i)} & \text{if } \varepsilon_t(i) < \varepsilon_t^* \end{cases}, \tag{18}$$

equation (13) becomes

$$\frac{\Phi_t}{\varepsilon_t^*} = \beta(1-\delta_s)E_t\frac{\Lambda_{t+1}}{\Lambda_t}\left[\int_{\varepsilon_{t+1}(i)<\varepsilon_{t+1}^*}\frac{\Phi_{t+1}}{\varepsilon_{t+1}(i)}dF(\varepsilon)+\int_{\varepsilon_{t+1}(i)\geq\varepsilon_{t+1}^*}\frac{\Phi_{t+1}}{\varepsilon_{t+1}^*}dF(\varepsilon)\right], \tag{19}$$

which determines the endogenous cutoff value ε_t^* and therefore the optimal target inventory level in the model.[9] The left-hand side of the above equation is the shadow value (opportunity cost) of holding inventory when the firm's productivity is high. The right-hand side is the expected rates of return by carrying one unit of inventory to the next period: in the case of low productivity ($\varepsilon(i) < \varepsilon^*$), the firm opts to stockout ($s(i) = 0$) by keeping production low and the shadow value of inventory is $\frac{\Phi_{t+1}}{\varepsilon_{t+1}(i)}$; in the case of high productivity ($\varepsilon(i) \geq \varepsilon^*$), the firm opts to carry the inventory forward and the shadow value is again $\frac{\Phi_{t+1}}{\varepsilon_{t+1}^*}$. Since the probability of stockout is determined by the cutoff value ε^*, the firm chooses ε^* so that the marginal cost of holding inventory in period t equals the expected next-period marginal gains. In other words, the above equation is the Belman equation for determining the inventory target by dynamic programming.

This Belman equation has two implications. First, the value of inventory is higher when the firm stockout (i.e., when $\varepsilon(i) \leq \varepsilon^*$). Second, the shadow price of inventory is downward sticky: given the aggregate component of the marginal cost Φ, the shadow value of inventory is constant (independent of $\varepsilon(i)$) when the firm's marginal cost is low (i.e., $\varepsilon(i)^{-1} \leq \frac{1}{\varepsilon^*}$) and it increases with the firm's marginal cost when the cost is high (i.e., $\varepsilon(i)^{-1} > \frac{1}{\varepsilon^*}$).

The decision rule for production (16) states that production is proportionally larger than sales when the cost of production is below the cutoff value ($\varepsilon > \varepsilon^*$), and it is less than sales when cost is high ($\varepsilon < \varepsilon^*$). Such a decision rule confirms our earlier intuition that firms opt to bunch production and use inventories to smooth sales and maximize the average profits.

The decision rule for inventory accumulation (17) states that inventory investment, $s_t(i) - (1-\delta_s)s_{t-1}(i)$, is procyclical with sales when $\varepsilon \geq \varepsilon^*$,[10] suggesting that on average (or at the aggregate level), inventory investment is procyclical and production is more volatile than sales. This is consistent with the stylized empirical fact. However, despite the procyclical inventory investment, the inventory stock–to-sales ratio in the model is countercyclical. That is, a one-percent increase in sales corresponds to a less than one-percent increase in the inventory stock.[11] Such a prediction is also consistent with the empirical fact emphasized by Bils and Kahn (2000).

[9] The probability of stockout in the model is given by $F(\varepsilon_t^*)$. Firms choose a target inventory level to determine the optimal probability of stockout under cost shocks.

[10] Its contemporaneous correlation with sales is zero when $\varepsilon < \varepsilon^*$.

[11] This is easier to see if $\sigma > 2$. On the one hand, the inventory stock cannot be negative when sales are low; and on the other hand, it increases less proportionally to sales when sales are high.

Bils and Kahn (2000) argues that the countercyclical inventory-sales ratio implies procyclical marginal costs and countercyclical markup. According to them, given procyclical inventory investment, firms may nonetheless choose to accumulate less inventories in a boom because the marginal cost of production is procyclical. This explanation is based on a stockout-avoidance motive for holding inventories. Countering the argument of Bils and Kahn (2007), in our model, the aggregate marginal cost (Φ) is constant and the firm level marginal cost ($\frac{\Phi}{\varepsilon(i)}$) is countercyclical; yet, the model predicts a countercyclical inventory-sales ratio and procyclical inventory investment.

3 General Equilibrium

3.1 Aggregation

By the law of large numbers, the final output equation, $Z = \left[\int y(i)^{\frac{\sigma-1}{\sigma}} di\right]^{\frac{\sigma}{\sigma-1}}$, implies that the marginal cost is constant,

$$\Phi = \left(\frac{\sigma - 1}{\sigma}\right) \left[\int \varepsilon^{\sigma-1} dF(\varepsilon)\right]^{\frac{1}{\sigma-1}}, \tag{20}$$

where $F(\varepsilon)$ denotes the cumulative distribution function (CDF) of the random variable ε. Define $Y \equiv \int_0^1 y(i)di$, $K \equiv \int_0^1 k(i)di$, $N \equiv \int_0^1 n(i)di$, $X \equiv \int_0^1 x(i)di$ and $S \equiv \int_0^1 s(i)$. The level of aggregate sales is given by

$$Y_t = PZ_t, \tag{21}$$

where $P \equiv \left[\int \varepsilon(i)^{\sigma} dF(\varepsilon)\right] \left[\int \varepsilon(i)^{\sigma-1} dF(\varepsilon)\right]^{\frac{\sigma}{1-\sigma}}$ can be interpreted as an aggregate measure of the relative price of the final good in terms of intermediate goods. Note $P = 1$ if $\varepsilon(i)$ is constant cross firms. Using the firm-level production decision rule, the level of aggregate production is given by

$$X_t = Y_t \frac{\left[\int_{\varepsilon(i)<\varepsilon^*} \varepsilon_t(i)^{\sigma} dF(\varepsilon) + \frac{\sigma}{\sigma-1} \int_{\varepsilon(i)\geq\varepsilon^*} \varepsilon_t(i)^{\sigma} dF(\varepsilon)\right]}{\int \varepsilon_t(i)^{\sigma} dF(\varepsilon)} - (1 - \delta_s) S_{t-1} F(\varepsilon_t^*), \tag{22}$$

and the aggregate stock of inventories is given by

$$S_t = \frac{1}{\sigma - 1} Y_t \frac{\int_{\varepsilon(i)\geq\varepsilon^*} \varepsilon_t(i)^{\sigma} dF(\varepsilon)}{\int \varepsilon_t(i)^{\sigma} dF(\varepsilon)} + (1 - \delta_s) S_{t-1} \left[1 - F(\varepsilon_t^*)\right]. \tag{23}$$

It is easy to check that these aggregate relations satisfy the aggregate resource identity,

$$Y_t + S_t = (1 - \delta_s) S_{t-1} + X_t. \tag{24}$$

The factor demand functions imply $(r_t + \delta_k)K_t = \alpha\Phi M_t$ and $w_t N_t = (1-\alpha)\,\Phi M_t$, where $M_t \equiv \int_0^1 \frac{x(i)}{\varepsilon(i)} di$. Since $\Phi \equiv \frac{1}{A}\left(\frac{r+\delta_k}{\alpha}\right)^{\alpha}\left(\frac{w}{1-\alpha}\right)^{1-\alpha}$, these aggregate factor demand functions imply the aggregate production function,

$$M_t = A_t K_t^{\alpha} N_t^{1-\alpha}. \tag{25}$$

Using the definition of M_t and the firm-level production decision rule, we have

$$M_t = Y_t \frac{\left[\int_{\varepsilon<\varepsilon_t^*} \varepsilon^{\sigma-1} dF(\varepsilon) + \frac{\sigma}{\sigma-1}\int_{\varepsilon\geq\varepsilon_t^*} \varepsilon^{\sigma-1} dF(\varepsilon)\right]}{\int \varepsilon^{\sigma} dF(\varepsilon)} - (1-\delta_s)S_{t-1}\int_{\varepsilon<\varepsilon_t^*} \varepsilon^{-1} dF(\varepsilon). \tag{26}$$

Notice that when $\varepsilon(i)$ is constant across firms, we have $M_t = \frac{X_t}{\varepsilon}$ by comparing (26) and (22).

Notice that Equation (23) is the familiar stock-adjustment model of inventories widely used in the empirical literature (see, e.g., Blinder 1986b). However, the crucial difference here is that the speed of adjustment in our model is time-varying and depends in general equilibrium on other aggregate variables via the optimal cutoff variable ε_t^*.

3.2 Household

To close the model we add a representative household. The household's role is to supply labor and accumulate capital. The household derives utility from consumption (C_t) and disutility of working (N_t) by solving

$$\max E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\theta} - 1}{1-\theta} - a\frac{N_t^{1+\gamma}}{1+\gamma} \right) \tag{27}$$

subject to

$$C_t + K_{t+1} \leq (1+r_t)K_t + w_t N_t + \Pi_t \tag{28}$$

where Π denotes aggregate profit income distributed from firms. Denoting Λ_t as the Lagrangian multiplier for the budget constraint, the first order conditions with respective to $\{C_t, N_t, K_{t+1}\}$ are given, respectively, by

$$C_t^{-\theta} = \Lambda_t \tag{29}$$

$$aN_t^{\gamma} = w_t \Lambda_t \tag{30}$$

$$\Lambda_t = \beta E_t (1+r_{t+1})\Lambda_{t+1}. \tag{31}$$

The aggregate profit income is given by

$$\Pi = \int_0^1 \left[y(i)^{1-\frac{1}{\sigma}} Z^{\frac{1}{\sigma}} - \frac{\Phi}{\varepsilon(i)} x(i) \right] di = Z_t - w_t N_t - (r_t + \delta_k)K_t \tag{32}$$

This implies that in equilibrium the household's budget constraint is given by

$$C_t + K_{t+1} - (1-\delta_k)K_t = Z_t, \tag{33}$$

which is also the equilibrium market-clearing condition for the final good.

The aggregate variables that need to be determined are $\{C_t, K_{t+1}, N_t, X_t, M_t, Y_t, S_t, \varepsilon_t^*\}$. Define the functions $g_1(\varepsilon^*) \equiv \int_{\varepsilon<\varepsilon^*} \varepsilon^{\sigma-1} dF(\varepsilon)$, $g_2(\varepsilon^*) \equiv \int_{\varepsilon\geq\varepsilon^*} \varepsilon^{\sigma-1} dF(\varepsilon)$, $h(\varepsilon^*) \equiv \int_{\varepsilon\geq\varepsilon^*} \varepsilon^{\sigma} dF(\varepsilon)$, and $q(\varepsilon^*) \equiv \int_{\varepsilon<\varepsilon^*} \varepsilon^{-1} dF(\varepsilon)$. The system of equations that solve these variables are given by

$$\frac{1}{\varepsilon_t^*} = \beta(1-\delta_s)E_t \frac{C_{t+1}^{-\theta}}{C_t^{-\theta}} \left[q(\varepsilon_{t+1}^*) + \frac{1-F(\varepsilon_{t+1}^*)}{\varepsilon_{t+1}^*} \right], \tag{34}$$

$$M_t = Y_t \frac{\left[g_1(\varepsilon_t^*) + \frac{\sigma}{\sigma-1} g_2(\varepsilon_t^*) \right]}{\int \varepsilon^\sigma dF(\varepsilon)} - (1-\delta_s)S_{t-1} q(\varepsilon^*). \tag{35}$$

$$S_t = \frac{1}{\sigma-1} Y_t \frac{h(\varepsilon_t^*)}{\int \varepsilon^\sigma dF(\varepsilon)} + (1-\delta_s)S_{t-1}\left[1 - F(\varepsilon_t^*)\right]. \tag{36}$$

$$Y_t + S_t = (1-\delta_s)S_{t-1} + X_t. \tag{37}$$

$$aN_t^\gamma = (1-\alpha)\,\Phi \frac{M_t}{N_t} C_t^{-\theta} \tag{38}$$

$$C_t^{-\theta} = \beta E_t \left(\alpha\Phi \frac{M_{t+1}}{K_{t+1}} + 1 - \delta_k \right) C_{t+1}^{-\theta}. \tag{39}$$

$$C_t + K_{t+1} - (1-\delta_k)K_t = \frac{1}{P} Y_t, \tag{40}$$

$$M_t = A_t K_t^\alpha N_t^{1-\alpha}. \tag{41}$$

4 Impulse Responses

4.1 Definition of GDP

Because of imperfect competition and inventories, care must be taken when measuring GDP in the model economy. By the value added approach, GDP in the model economy is the sum of the final good sector's value added and the intermediate goods sector's value added. The final good sector's value added is its revenue minus the costs of intermediate goods used up in the production, $Z_t - \int_0^1 p_t(i) y_t(i) di$, which is zero under perfect competition; whereas the intermediate goods sector's value added is $\int_0^1 p_t(i) x_t(i) di$. Hence, $GDP = \int_0^1 p_t(i) x_t(i) di$. Alternatively, by the expenditure approach, GDP is the sum of Z (aggregate consumption plus business investment) and inventory investment (by market values), $GDP = Z_t + \int_0^1 p_t(i)[s_t(i) - (1-\delta_s)s_{t-1}(i)]di$. Since inventory investment equals production minus sales, by the zero profit condition we have $GDP = \int_0^1 p_t(i)x_t(i)di$. Hence, both approaches give the same results. Substituting the relationship $p_t(i) = \frac{\sigma}{\sigma-1}\frac{\Phi_t}{\varepsilon_t(i)}$ into GDP gives

$$GDP_t = \frac{\sigma}{\sigma-1}\Phi \int_0^1 \frac{x_t(i)}{\varepsilon_t(i)}di = \frac{\sigma\Phi}{\sigma-1}M_t. \tag{42}$$

Notice that in the absence of idiosyncratic uncertainty, we have $GDP = Z = AK^{\alpha}N^{1-\alpha}$ because firms will no longer have incentives to hold inventories.

4.2 Calibration

Assume $\varepsilon(i)$ is drawn from a Pareto distribution, $F(\varepsilon) = 1 - \left(\frac{1}{\varepsilon}\right)^{\eta}$, with the shape parameter $\eta > 0$ and the support $\varepsilon \in (1, \infty]$. We further assume $\eta > \sigma$ so as to make the following integrations meaningful. With this assumption, the functions $\{g_1(\varepsilon^*), g_2(\varepsilon^*), h_1(\varepsilon^*), h_2(\varepsilon^*), q(\varepsilon^*)\}$ are given by $g_1(\varepsilon^*) = \frac{\eta}{\eta+1-\sigma}\left[1-(\varepsilon^*)^{\sigma-\eta-1}\right]$, $g_2(\varepsilon^*) = \frac{\eta}{\eta+1-\sigma}(\varepsilon^*)^{\sigma-\eta-1}$, $h(\varepsilon^*) = \frac{\eta}{\eta-\sigma}(\varepsilon^*)^{\sigma-\eta}$, $q(\varepsilon^*) = \frac{\eta}{\eta+1}\left[1-(\varepsilon^*)^{-1-\eta}\right]$. Similarly, we have the relative price $P = \left[\frac{\eta}{\eta-\sigma}\right]\left[\frac{\eta}{\eta+1-\sigma}\right]^{\frac{\sigma}{1-\sigma}}$, and the aggregate marginal cost $\Phi = \left(\frac{\sigma-1}{\sigma}\right)\left[\frac{\eta}{\eta+1-\sigma}\right]^{\frac{1}{\sigma-1}}$.

The time period is one quarter. To illustrate the dynamic properties of the model, we set $\beta = 0.99$, $\alpha = 0.35$, $\delta_k = 0.035$, $\gamma = 0$ (indivisible labor), $\theta = 1$ (log utility in consumption), $\delta_s = 0.005$, $\sigma = 4$, $\eta = 4.35$. The parameters $\{\sigma, \eta, \beta, \delta_s\}$ jointly determine the inventory-to-sales ratio in the steady-state. At a quarterly frequency, if $\beta = 0.99$ and $\delta_s = 0.005$, a realistic inventory-to-sales ratio ($\frac{S}{Y} \simeq 1$) in the steady-state requires small η to generate enough variance in firms' cost shocks (ε) so that they have strong incentives to hold inventories. For this purpose, we set $\eta = 4.5$ and and $\sigma = 4$, hence the constraint $\eta > \sigma$ is satisfied. The implied steady-state inventory-to-sales ratio is $\frac{S}{Y} = 1.01$ and the stock-to-sales ratio $\frac{(1-\delta_s)S+X}{Y} \simeq 2$. The aggregate technology shock is assumed to follow an $AR(1)$ process, $\hat{A}_t = \rho\hat{A}_{t-1} + \xi_t$ with $\rho = 0.9$. The parameter values of the model are summarized in Table 1.

Table 1. Parameter Values

α	β	δ_k	δ_s	θ	γ	σ	η	ρ
0.35	0.99	0.035	0.005	1.0	0.0	4.0	4.35	0.90

The impulse responses of GDP, consumption, labor, investment, inventory investment, and the inventory-sales ratio ($\frac{S_t}{Y_t}$) to a one-standard-deviation aggregate technology shock are graphed in Figure 1 (solid lines). Under aggregate technology shocks, aggregate inventory investment is strongly procyclical and far more volatile than GDP. However, the inventory-sales ratio is countercyclical, suggesting that the inventory stock fails to track sales one-for-one despite the procyclical changes in inventory investment. Hence, the model is able to explain the stylized facts of inventory behavior emphasized by the empirical literature (e.g., Bils and Kahn, 2000). In addition, consumption is less volatile and investment is more volatile than GDP, as in a standard RBC model.

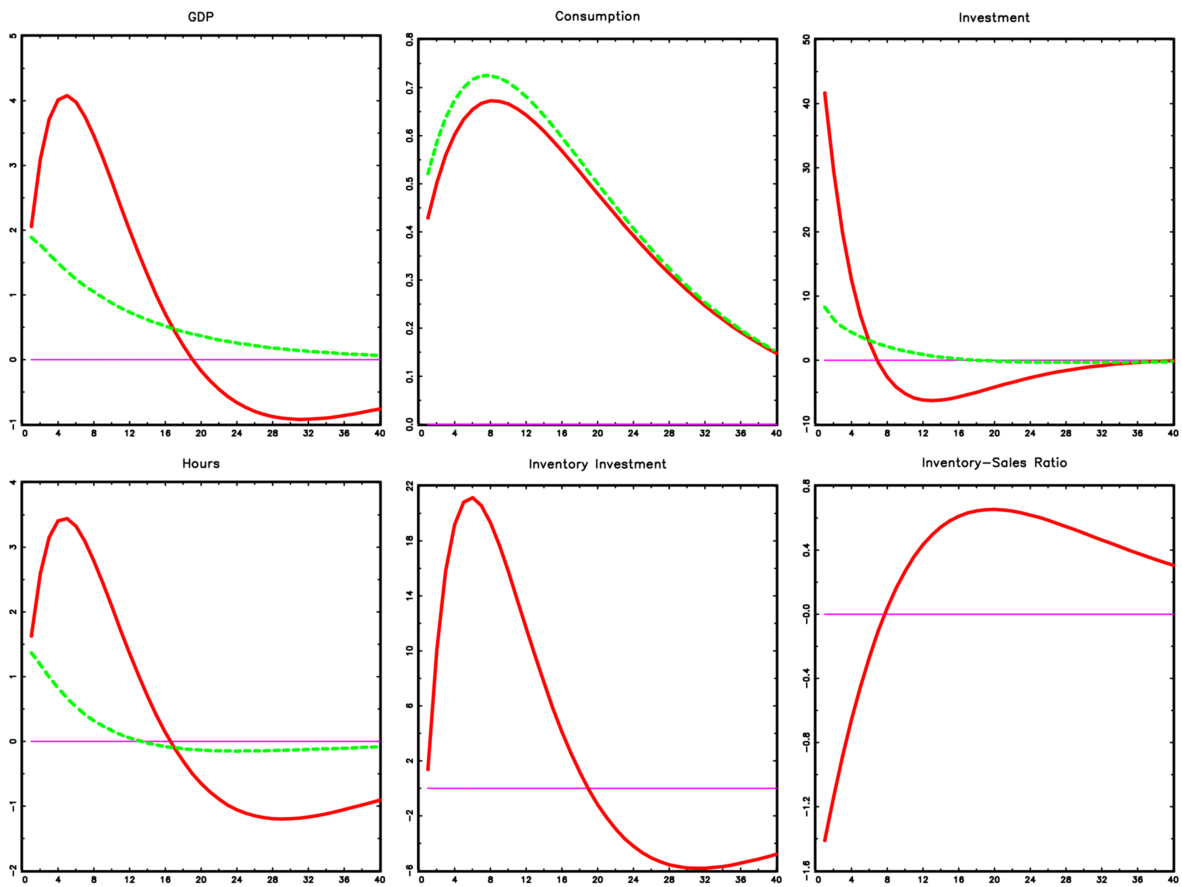

Figure 1. Impulse Responses to Technology Shock.

Most notable of Figure 1 are the hump-shaped impulse responses of GDP and labor to a technology shock. This dynamic pattern of aggregate output is a defining feature of the business cycle that has been emphasized by the literature as a litmus test for quantitative business cycle models (see, e.g., Cogley and Nason, 1995). The fact that our inventory model is able to generate such a hump-shaped output dynamics is striking. The general-equilibrium inventory models of Khan and Thomas (2007) and Wen (2008) are not able to deliver this result. As the variance of the idiosyncratic shocks increases from our benchmark value of $\eta = 4.35$, the hump-shaped output dynamics are gradually dampened. Without inventories, our model is a standard Dixit-Stiglitz RBC model and it is not able to generate hump-shaped dynamics.

As a comparison, the dashed lines in Figure 1 show the responses of our control model which has no inventories.[12] It is evident that inventories destabilize the economy: aggregate output and labor are about 2.5 times more variable with inventories than without; and similarly, aggregate investment is 4 times more volatile with inventories than without. However, aggregate consumption is slightly less volatile with inventories than without, indicating that inventories help smoothing consumption by encouraging capital investment.

Under a positive aggregate technology shock, the marginal product of capital and labor both increase, giving rise to the initial boom in the economy. This is true either with or without inventories. However, inventories facilitate sales – not only firms with $\varepsilon_t(i)$ above the cutoff ε_t^* will want to produce and sell more intermediate goods to the final-good sector but those below the cutoff can also sell more because of previously accumulated inventories. Thus, aggregate investment increases more than it would if inventories did not exist. Firms, knowing that the technology shock is highly persistent, are thus able to accumulate capital at a faster rate, further stimulating employment and inventory accumulation in the subsequent periods. This results in a cumulative process of expansion.[13] Sooner or later the technology shock will die out so that the engine of inventory accumulation will lose steam. Once the economy starts to contract, sales fall, firms opt to decumulate inventories so that the relative price of intermediate good rises to prevent revenue from declining sharply. In particular, because intermediate-goods prices are much higher when firms run out of inventories, imperfectly competitive firms opt to reduce production at a faster speed than it would in the controlled model economy. This causes the economy to over-shoot its steady state from above, giving rise to a boom-bust like propagation mechanism.

Therefore, procyclical inventory investment and countercyclical inventory prices not only amplify aggregate shocks but also propagate them over time. Conforming to the conventional wisdom of Blinder (1981), inventories are destabilizing if their existence is induced by cost shocks and motivated by production-cost-smoothing (Eichenbaum, 1989).

Another way to understanding why inventories enhance the propagation mechanism in this model is looking at the optimal inventory behavior specified by Equation (23). Wen (2008) shows that under a stockout-avoidance motive, the aggregate inventory stock (S_t) is proportional to aggregate sales (Y_t) with the coefficient of proportionality depending on the cutoff variable ε^*. However, here in our model it is more than proportional to sales because it also positively depends on the last-period inventory stock (S_{t-1}). Given sales, the more the inventories in the last period, the larger the inventory stock firms want to hold today. This generates a strongly autocorrelated

[12] That is, each intermediate-good firm i maximizes $p(i)y(i) - \frac{\Phi}{\varepsilon(i)}x(i)$ subject to $y(i) = x(i)$ and $p(i) = Z^{\frac{1}{\sigma}}y(i)^{-\frac{1}{\sigma}}$.

[13] Notice that firms have incentives to hold an infinite amount of inventories when the idiosyncratic marginal cost is low because each firm takes the aggregate marginal cost as external. However, this is not possible because of the zero profit constraint. Hence, firms opt to gradually increase the speed of inventory investment along with the rising capital stock (production capacity), which relaxes the profit constraint over time.

inventory behavior, which helps propagating the business cycle.

To quantify our results, Table 2 reports some selected moments implied by the inventory model (Model A) and the U.S. economy.[14] First notice that the model can generate very volatile and procyclical inventory investment, with a volatility more than 5 times that of GDP and a correlation with GDP about 0.98. Second, despite this, the inventory-sales ratio is countercyclical. Its correlation with GDP is -0.41. These statistics are qualitatively consistent with the data.

The benchmark value of $\sigma = 4$ in our model implies a vary large markup, around 30%. However, with more realistic values of σ, we can still get similar hump-shaped impulse responses as in Figure 1, as long as η is sufficiently close to σ. For example, the lower panel of Table 2 (Model B) reports statistics implied by the model when $\sigma = 10$ (implying 10% markup) and $\eta = 10.12$. The results are very similar.[15]

Table 2. Selected Inventory Statistics

Variable (x)		ΔS	S/Y
	σ_x/σ_{gdp}	21.6	0.89
U.S. Data	$cor(x_t, GDP_t)$	0.42	-0.71
Model A	σ_x/σ_{gdp}	5.3	0.30
	$cor(x_t, GDP_t)$	0.98	-0.41
Model B	σ_x/σ_{gdp}	7.7	0.39
	$cor(x_t, GDP_t)$	0.96	-0.36

To quantify the destabilizing effects of inventories on GDP, we compare similar models with different inventory-sales ratios (controlled by different values of η). As η increases, the variance of the idiosyncratic shocks decreases, reducing firms' incentives for holding inventories rapidly. The results are reported in Table 3 (Model A). It shows that the standard deviation of GDP decreases rapidly as η rises. As a robustness analysis, we also report the results for the case where $\sigma = 10$ in the lower panel of Table 3 (Model B).[16] The results are similar. In both cases, for a small enough value of η satisfying the constraint $\eta > \sigma$, the volatility of GDP may become several times larger than that of the controlled model without inventories, suggesting that inventories are potentially a great destabilizing force to the economy for a wide range of parameter values. Notice that the difference, $(\eta - \sigma)$, instead of σ or η alone, is crucial for determining the variance of GDP when inventories exist. The reason is this: under the Pareto distribution of the cost shocks, all variables in the model follow Pareto distribution, so that the two parameters $\{\sigma, \eta\}$ (especially, $\eta - \sigma$) jointly

[14] The U.S. statistics are taken from Wen (2008, Table 2).

[15] However, larger values of $\{\sigma, \eta\}$ reduce the steady-state inventory-sales ratio in the model. See Table 3 for more information and robustness analysis.

[16] Notice that the values of $\{\sigma, \eta\}$ do not matter for the volatility of GDP in the controlled model without inventories.

determine the variance of inventories and output, as is evident from the expressions of the functions $\{g_1(\varepsilon^*), g_2(\varepsilon^*), h(\varepsilon^*), q(\varepsilon^*)\}$. If other distribution functions are assumed, the effects of σ and η can be disentangled; but the results are no longer analytically tractable.

Table 3. The Effect of Inventory on Aggregate Volatility

Model A ($\sigma = 4$)	Inventory-Sales Ratio	*std* of GDP	Relative *std*
$\eta = 4.35$	1.01	12.4	2. 58
$\eta = 4.50$	1.00	7.5	1.56
$\eta = 5.00$	0.90	4.9	1. 0
No Inventory	0	4.8	1
Model B ($\sigma = 10$)	Inventory-Sales Ratio	*std* of GDP	Relative *std*
$\eta = 10.12$	0.24	9.24	1. 93
$\eta = 10.15$	0.23	5.45	1.14
$\eta = 12.0$	0.22	4.81	1.0
No Inventory	0	4.8	1

5 Conclusion

This paper provides a general-equilibrium model of inventories with microfoundations. We argue that idiosyncratic cost shocks can induce firms to bunch production and smooth sales by holding inventories. The intertemporal substitution of production activities at the firm level is based on a production-cost-smoothing motive emphasized by Eichenbaum (1989) and others in the empirical literature. However, our model rationalizes the *ad hoc* cost functions and their target-adjustment forms assumed in this empirical literature. The predictions of our general-equilibrium model is consistent with the stylized facts of aggregate inventory behavior, such as the procyclical inventory investment and countercyclical inventory-sales ratio. Most importantly, our analysis reveals that inventory investment motivated by sales-smoothing not only amplify aggregate shocks but also propagate them. This finding confirms a long-standing conjecture in the history of economic thought (e.g., Metzler, 1941) that inventories can serve as an accelerator of the business cycle.

In light of the provocative findings of Khan and Thomas (2007) and Wen (2008), our analysis suggests that whether inventories are stabilizing or destabilizing to the aggregate economy depends on the source of uncertainties at firm level. If idiosyncratic marginal-cost shocks dominate idiosyncratic demand shocks, for example, then the sales-smoothing motive studied in this paper is more important than the stockout-avoidance motive studied by Wen (2008); hence, inventories are destabilizing; otherwise, inventories are stabilizing. In other words, in contrast to the partial-equilibrium tradition of Blinder and others in the earlier literature, the insight learned from general-equilibrium analysis (Khan and Thomas, 2007; Wen, 2008; and this paper) is that the destabilizing nature of inventories does not hinge on whether inventory investment is procyclical (or whether production is more variable than sales), but on the motives for holding inventories.

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