ECONOMIC RESEARCH
FEDERAL RESERVE BANK OF ST. LOUIS

WORKING PAPER SERIES

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| **Authors** | Tarek Coury, and Yi Wen |
| **Working Paper Number** | 2007-029B |
| **Revision Date** | March 2009 |
| **Citable Link** | https://doi.org/10.20955/wp.2007.029 |
| **Suggested Citation** | Coury, T., Wen, Y., 2009; Global Indeterminacy in Locally Determinate RBC Models, Federal Reserve Bank of St. Louis Working Paper 2007-029. URL https://doi.org/10.20955/wp.2007.029 |

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| **Published In** | International Journal of Economic Theory |
| **Publisher Link** | https://doi.org/10.1111/j.1742-7363.2008.00102.x |

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# Global Indeterminacy in Locally Determinate RBC Models*

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January 28, 2008

**Abstract**

We investigate the global dynamics of Real Business Cycle (RBC) models with production externalities. We confirm that purely local analysis does not tell the full story. With externalities smaller than required for local indeterminacy, local analysis shows the steady state to be a saddle, implying a unique equilibrium. But global analysis reveals the steady state is surrounded by stable deterministic cycles. Our analysis suggests that indeterminacy is more pervasive than previously thought, and the results strengthen the view that caution should be exercised when linearized versions of this class of RBC models are used in applied work.

***Keywords:*** Global Indeterminacy; Real Business Cycles; Sunspots; Chaos; Limit Cycles.

***JEL Classification:*** C62, E13, E32.

*We thank Jess Benhabib, Karl Shell, and participants at the Cornell/Penn Macro Conference, the Meeting of the European Economic Association for comments. We also wish to thank Roger Farmer and an anonymous referee for suggestions. We are however solely responsible for all remaining errors. Correspondence: Yi Wen, Research Department, Federal Reserve Bank of St. Louis, St. Louis, MO, 63144. Phone: 314-444-8559. Fax: 314-444-8731. Email: yi.wen@stls.frb.org.

†Part of this work was conducted while the author was a graduate student at Cornell University and University Lecturer at Cambridge and Oxford. This work was completed while the author was a research fellow at the Dubai School of Government and a visiting scholar at Harvard's Kennedy School of Government, both whose support are gratefully acknowledged.

‡Part of this work was conducted while this author was an Assistant Professor at Cornell University.

# 1 Introduction

The work of Benhabib and Farmer (1994) has triggered a great amount of research and renewed interest in studying expectations-driven fluctuations. The major reason behind this is that it makes quantitative analysis of the business-cycle effects of sunspots possible within the popular framework of Kydland and Prescott (1982). Along with this fast growing literature of indeterminacy, there has also been a growing concern that sunspots equilibria in this class of RBC models may not be robust to parameter calibration and model perturbations. For example, the required degree of returns-to-scale and the elasticity of labor supply for inducing indeterminacy in this class of models may be unrealistically large; and indeterminacy may no longer be possible once realistic adjustment costs of capital or labor are taken into account. Both the indeterminacy literature and the criticisms raised against it, however, are based almost exclusively on local analysis. The point of this paper is to show the danger of drawing conclusions based solely on local analysis. In particular, it is shown that models can exhibit global indeterminacy even when they are locally determinate and that global dynamics can be dramatically different from local dynamics.

In this paper we focus on the Benhabib-Farmer RBC model featuring increasing returns to scale due to production externalities. But we believe our analyses have broader implications for other types of models, such as the New Keynesian monetary models with Taylor rules.[1] By analyzing the global dynamics of this class of RBC models, we confirm that purely local analysis can be misleading. For example, with sufficiently small externalities, local analysis shows the steady state to be a saddle, implying a unique equilibrium featuring monotonic transitional dynamics; but global analysis reveals instead that the steady state is surrounded by stable period-$2n$ cycles.

Our analysis is particularly relevant in light of the recent literature of estimating and testing for indeterminacy. Lubik and Schorfheide (2004) use a business-cycle model that allows for indeterminacy to conduct likelihood-based estimation of the effects of monetary policies based on the difference in the propagation mechanism between determinate and indeterminate models. They argue that U.S. monetary policy in the post-1982 period is consistent with determinacy whereas the policy in the pre-Volker period is not. Their estimation pro-

[1] In our working paper version (Coury and Wen 2000), we have also examined other versions of the Benhabib-Farmer model, such as the model of Wen (1998) featuring capacity utilization and the model of Weder (1998) featuring durable consumption goods. We showed that these models all exhibit global indeterminacy while the steady state appears to be a saddle.

cedure, however, is based entirely on local analysis. Thus, the conclusions they draw from their analysis may not be robust if the region of the parameter space they consider to be determinate is actually indeterminate in the global sense.

Our analysis reinforces the concerns expressed by Benhabib and Eusepi (2004) regarding the danger of drawing conclusions based solely on local analysis. Our paper is related to the work of Christiano and Harrison (1999). Rather than focusing on local dynamics, Christiano and Harrison conduct global analysis and show that chaos and regime switching sunspot equilibria can arise in a standard RBC model with increasing returns to scale. They also study the implications of this model for stabilization in relation to government tax policies. However, their analysis is conducted under the condition that the degree of aggregate returns to scale is large enough to trigger local indeterminacy. They do not study whether global indeterminacy continues to exist when the model lies in the determinate region judged by local analysis.[2]

The significance of the present work is that we conduct global analysis under the condition of local determinacy and without policy distortions. Linearized versions of this class of models are now routinely used and calibrated in the empirical literature to explain macroeconomic fluctuations.[3] Since global indeterminacy exists in this class of empirically plausible RBC models even in parameter regions where the equilibrium appears to be locally unique, caution must be exercised when linearized versions of these models are applied to empirical analysis. In practice, for example, people who do not believe in sunspots or indeterminacy may choose the level of the externality small enough so that the model's steady state appears to be a saddle, hence carrying out investigations assuming that the equilibrium is unique, while indeterminacy and sunspot equilibria may exist globally in the model.[4]

*Related Literature.* The broader literature is impossible to survey here because it is so vast. We only mention some of the papers that are most closely related to our analysis in this paper. Benhabib and Farmer (1994) and Farmer and Guo (1994) discuss indeterminacy and sunspot equilibria in a standard one-sector RBC model with production externalities

[2] Another related work is Guo and Lansing (2002). They examine the relationship between government tax policy and endogenous fluctuations in the Benhabib-Farmer (1994) model. They show that when the model is locally indeterminate due to a sufficiently large degree of externalities, the introduction of distortionary taxes into the model can lead to interesting global dynamics. In particular, they find parameter regions where the model is locally determinate but globally indeterminate. However, their finding is restricted to the condition that increasing returns to scale are sufficiently large to trigger local indeterminacy in the absence of policy distortions. Hence, in this regard their model is similar to Christiano and Harrison (1999).

[3] See, e.g., Benhabib and Farmer (1996), Benhabib and Wen (2004), Barinci and Cheron (2001), Farmer and Guo (1994), Guo and Sturzenegger (1998), Harrison and Weder (2002), Perli (1998), Schmitt-Grohe (2000), Weder (1998), Wen (1998a, b) and Xiao (2004), among many others.

[4] A similar point has also been made by Guo and Lansing (2002) in a different context regarding the design of stabilizing government polices.

(i.e., the model of Baxter and King 1991). Since these first-generation indeterminate RBC models require implausibly large degrees of externalities to generate indeterminacy (see, e.g., Schmitt-Grohe 1997), thereby casting doubt on their empirical relevance, subsequent work by Benhabib and Farmer (1998), Benhabib and Nishimura (1997), Benhabib, Meng and Nishimura (2000), Bennett and Farmer (2000), Harrison (2001), Perli (1998), Weder (1998 and 2000) and Wen (1998a), among many others, made efforts to reduce the degree of externalities required for inducing local indeterminacy.[5] This line of research discovers that factors such as additional sectors of production, durable consumption goods, non-separable utility functions, or variable capacity utilization can all help by reducing the required externalities for local indeterminacy to a degree that is empirically plausible. However, Wen (1998b), Kim (2003), and Herrendorf and Valentinyi (2003), among others, show that if adjustment costs are present, indeterminacy may no longer be possible in this class of models regardless of the degree of externalities or returns to scale. Most work in this literature, however, is based on local analysis. The global properties of this class of calibrated RBC models remain largely unknown. For the broader literature on sunspots, see Shell (1977, 1987), Cass and Shell (1983), Shell and Smith (1992), Azariadis (1981), Azariadis and Guesnerie (1986), and Woodford (1986a, 1986b, 1991). For global analysis of indeterminacy and nonlinear dynamics in dynamic optimization models, see Benhabib and Nishimura (1979), Benhabib and Day (1982), Grandmont, Pintus, and Vilder (1998), Michener and Ravikumar (1998), Pintus, Sands, and Vilder (2000), Majumdar, Mitra and Nishimura (2000), Mitra (2001), Mitra and Nishimura (2001a, 2001b), among many others.

# 2 The Benhabib-Farmer Model

A representative agent chooses sequences of consumption $\{c_t\}_{t=0}^{\infty}$, hours to work $\{n_t\}_{t=0}^{\infty}$, and the stock of capital $\{k_{t+1}\}_{t=0}^{\infty}$ to solve

$$\max \sum_{t=0}^{\infty} \beta^t \left\{ \log(c_t) - a \frac{n_t^{1+\gamma}}{1+\gamma} \right\} \tag{1}$$

subject to

$$c_t + k_{t+1} - (1-\delta)k_t \leq X_t k_t^{\alpha} n_t^{1-\alpha}, \tag{2}$$

[5] Schmitt-Grohe and Uribe (1997) proposed an indeterminate model without increasing returns to scale in the production technology. Their model features distortionary taxes and balanced government budget. Indeterminacy arises if the steady-state tax rate is larger than capital's share of aggregate income. Wen (2001) showed that this model is similar to the Benhabib-Farmer (1994) model in reduced form.

where $\delta$ is the rate of depreciation for capital and $X_t = \left(\bar{k}_t^{\alpha}\bar{n}_t^{1-\alpha}\right)^{\eta}$ $(\eta \geq 0)$ represents production externalities taken as parametric by individual agents. In equilibrium, the first order optimality conditions are given by:

$$an_t^{\gamma} = (1-\alpha)\frac{1}{c_t}k_t^{\alpha(1+\eta)}n_t^{(1-\alpha)(1+\eta)-1} \tag{3}$$

$$\frac{1}{c_t} = \beta\frac{1}{c_{t+1}}\left[\alpha k_{t+1}^{\alpha(1+\eta)-1}n_{t+1}^{(1-\alpha)(1+\eta)} + 1 - \delta\right] \tag{4}$$

$$c_t + k_{t+1} - (1-\delta)k_t = k_t^{\alpha(1+\eta)}n_t^{(1-\alpha)(1+\eta)}, \tag{5}$$

plus a transversality condition. Equation (3) is the labor market equilibrium condition, equation (4) is the intertemporal Euler equation for consumption and saving, and equation (5) is the aggregate resource constraint.

# 3 Local Dynamics

Before analyzing the global dynamics of the model, it is illustrative and useful to investigate first the local dynamics of the model. Log-linearizing the first-order conditions (3)-(5) around the steady state and substituting out labor $n$ using equation (3), we can obtain a two-variable linear system in $\{k_t, c_t\}$:

$$\begin{pmatrix} \hat{k}_{t+1} \\ \hat{c}_{t+1} \end{pmatrix} = M \begin{pmatrix} \hat{k}_t \\ \hat{c}_t \end{pmatrix}, \tag{6}$$

where circumflex denotes percentage deviation from steady state. In order to have a unique equilibrium, the existing literature argues that steady state must be a saddle, i.e., one of the eigenvalues of $M$ must lie outside the unit circle, so that the current consumption level $c_t$ can be solved forward as a function of the state $(k_t)$. If both eigenvalues of $M$ lie inside the unit circle, the steady state becomes a sink. Hence a continuum of equilibria exists because any initial value of consumption is consistent with equilibrium. Benhabib and Farmer (1994) show that, with the externality parameter $\eta$ exceeding a critical value $\eta^*$, the steady state becomes a sink.

However, as will be shown shortly, there can still exist multiple equilibrium paths even though the steady state is locally a saddle. That is, the model can still be indeterminate globally even if $\eta < \eta^*$).[6] To demonstrate the possible topological changes of this model as

[6] Externalities are still crucial for global indeterminacy.

we vary the externality parameter $\eta$, table 1 tabulates the eigenvalues of the linear system.[7] We calibrate the remaining structural parameters according to the existing RBC literature, namely, we set $\beta = 0.99$, $\gamma = 0$, $\alpha = 0.3$, and $\delta = 0.025$ for a quarterly model. The existence of global indeterminacy is not sensitive to the particular parameter values. For a wide region of the parameter space, global indeterminacy will arise if the externality parameter $\eta$ is larger than a critical value. These critical values, however, is smaller than the value required for local indeterminacy. In other words, the steady state remains a saddle when global indeterminacy appears.

Table 1.

| $\eta$ | $\{\lambda_1, \lambda_2\}$ | | modulus |
|---|---|---|---|
| 0 | 0.9250 | 1.0920 | |
| 0.4 | 0.9124 | 1.3917 | |
| 0.4800 | 0.9046 | 18627 | |
| 0.4801 | 0.9046 | -278.75 | |
| 0.4934 | 0.9026 | -1.0004 | 1.0 |
| 0.4935 | 0.9025 | -0.9852 | |
| 0.506 | 0.9003 | -0.0011 | |
| 0.507 | 0.9001 | 0.0384 | |
| 0.58 | 0.8515 | 0.8324 | |
| 0.581 | $0.8433 \pm 0.0089i$ | | 0.8434 |
| 1 | $0.9533 \pm 0.0382i$ | | 0.9540 |

Table 1 shows that the steady state goes through noticeable topological changes as the externality parameter $\eta$ increases from zero. For example, for $\eta$ slightly greater than 0.48, the largest eigenvalue changes sign from positive infinity to negative infinity. At about $\eta = 0.4934$, the steady state changes from a saddle to a sink and the largest eigenvalue (in absolute value) passes through 1. Hence, the model admits a flip bifurcation. Whether that flip bifurcation is supercritical or not remains to be examined in the next section by global analyses. For $\eta \geq 0.4934$, the steady state is locally a sink and the model becomes locally indeterminate. Complex eigenvalues emerge when $\eta$ increases still further. Computation shows that the eigenvalues become real again if we increase the externality beyond $\eta = 1$.

[7] In table 1, we report the value of $\eta$ in the first column, the corresponding eigenvalues in the second and third columns. In the last column we report the modulus of the largest eigenvalue when they cross the unit circle or when they become complex numbers.

# 4 Global Dynamics

We first prove that the flip bifurcation identified in the local analyses is *supercritical*. This means that, within a small open neighborhood of $\eta^*$, and as $\eta$ becomes larger, the period-1 steady state goes from being a sink to being a saddle surrounded by an attracting period-2 cycle: trajectories starting in a neighborhood of the period-1 steady state would end up converging to this cycle (if the cycle were *repelling*, then the bifurcation would be called *subcritical*.) We then use numerical methods to investigate the global behavior of the model when parameters are sufficiently far away from the bifurcation point.

**Proposition 1** *For an open set of parameter values, close to those of the real business cycle model, the flip bifurcation in the Benhabib and Farmer model is supercritical.*

**Proof.** See the appendix.■

As the system undergoes the flip bifurcation, the steady state becomes saddle-path stable. In local analysis, this is associated with a unique equilibrium trajectory that is consistent with initial conditions. Because the flip bifurcation is supercritical, there is in fact a continuum of equilibria consistent with a given initial condition (in the case of a subcritical bifurcation, it would remain true that there is only one equilibrium path associated with a given initial condition.)

An analytical bifurcation analysis can reveal only the global behavior of the system in a neighborhood of the parameter values for which a bifurcation occurs. Outside of this neighborhood, the system may undergo further changes in the global topology of the phase space that can be revealed by numerical methods. The above proposition therefore does not guarantee that the period-2 cycle will persist or that a period-doubling bifurcation to chaos will take place when $\eta$ differs from the critical value $\eta^* = 0.4934$. However, it does guarantee that there exists an open interval around the critical value such that for smaller values of $\eta < \eta^*$, there exists a period-2 attracting cycle around the saddle steady state; and for larger values of $\eta > \eta^*$, the system is globally a sink.

In order to understand the global dynamics of the model for parameters sufficiently far away from the bifurcation point, we rearrange the model's first order conditions as follows:

$$an^{\gamma} = (1-\alpha)\frac{1}{c_t}k_t^{\alpha(1+\eta)}n_t^{(1-\alpha)(1+\eta)-1} \tag{7}$$

$$k_{t+1} = k_t^{\alpha(1+\eta)}n_t^{(1-\alpha)(1+\eta)} + (1-\delta)k_t - c_t \tag{8}$$

$$\frac{1}{c_t} = \beta \frac{1}{c_{t+1}} \left[ \alpha k_{t+1}^{\alpha(1+\eta)-1} n_{t+1}^{(1-\alpha)(1+\eta)} + 1 - \delta \right]. \tag{9}$$

To simulate global dynamics, we disturb the steady state by an arbitrary amount. This is our initial value for the vector $(c_t, k_t)$, which also implies the initial value for $n_t$ (by equation (7)). Given these initial values, we note that equation (8) defines $k_{t+1}$ as a function of $(c_t, k_t)$. Replacing this function into equation (9), we obtain an implicit function in $c_{t+1}$. We then solve this equation numerically for $c_{t+1}$ using the Newton method. We then have a new set of initial values $(c_{t+1}, k_{t+1})$ that we can iterate using the same method.[8]

Starting from the critical value, $\eta^*$, at which the flip bifurcation takes place, it is found that if we decrease $\eta$ slightly, the system becomes locally a saddle but is globally surrounded by a stable period two cycle. As $\eta$ decreases slightly further, the system undergoes a cascade of period-doubling flip bifurcations. A simulation of period doubling is shown in figure 1.[9] For values of $\eta$ larger than its critical value of the flip bifurcation $\eta^*$, the steady state is a global sink. This is a consequence of the fact that the flip bifurcation is supercritical as is proved in proposition 1.

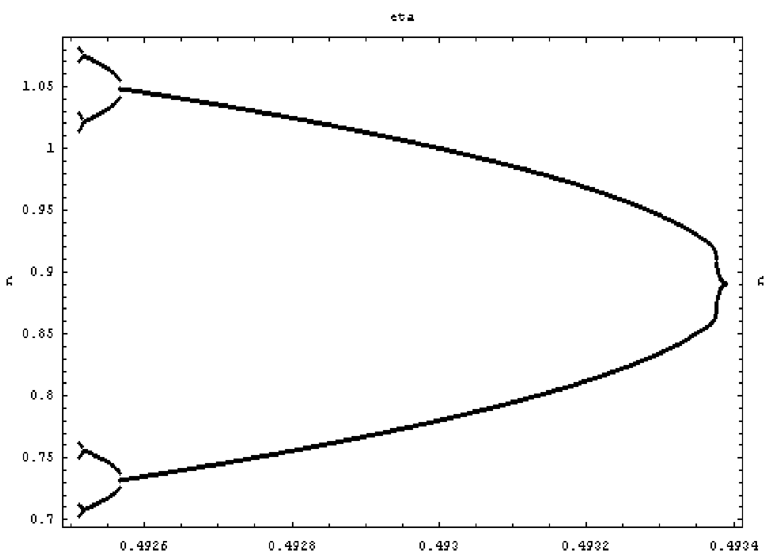


Figure 1. Period-Doubling Global Dynamics.

*Robustness.* Our findings are not limited to the Benhabib-Farmer model. For example, in our working paper version (Coury and Wen 2000), we have also examined the models of ∈

[8] This method is also used by Guo and Lansing (2002).

[9] Figure 1 was obtaind by computing 1000 iterations of an orbit starting near the steady state, given a value of $\eta$. The last 100 iterates are kept. This is done for 1000 values of $\eta$. The resulting picture is projected in the $\{\eta, n\}$ space. The figure was generated using Mathematica 4.2 under Windows XP. The algorithm used to compute the zero of implicit functions is Newton's method.

and Weder (1998), where the required degree of externalities for inducing local indeterminacy is much smaller, due to the presence of capacity utilization or durable goods consumption, so that the implied aggregate returns to scale are almost constant. We show that both models have global indeterminacy in parameter regions where the steady state appears to be locally a saddle. Since the analysis and the analytical proof for the stability of bifurcation are very similar, the results are not presented here in order to conserve space. Interested readers are referred to our working paper, Coury and Wen (2000).[10]

# 5 Conclusion

In this paper we use the Benhabib-Farmer (1994) model to show that externalities may lead to an even richer set of dynamics than previously known once one departs from the standard local analysis. Global analysis reveals that endogenous business cycles driven by sunspots can arise even in parameter regions where the steady state appears to be locally unique and determinate. Our analysis indicates that indeterminacy may be more pervasive than previously thought, and the findings strengthen the view that caution should be exercised when linearized versions of this class of RBC models are used in applied work. While the literature emphasizes that externalities-driven RBC models have a strong propagation mechanism when evaluated in the parameter region of local indeterminacy[11], it is possible to construct stationary sunspot equilibria around the deterministic period-$2n$ cycles in a way similar to Christiano and Harrison (1999). The empirical investigation of such a model could be an interesting topic for future research.

# 6 Appendix

**Proposition 1** The flip bifurcation in the Benhabib-Farmer model is supercritical for an open set of parameter values close to those of the real business cycle model.

**Proof.** Let $x_t = (c_t, k_t)$. Consider the system $x_{t+1} = f(x_t) = (f_1(x_t), f_2(x_t))$, where parameters are such that one of the eigenvalues of the system is $-1$. Let $x^*$ be the unique steady

[10] Other RBC models with distortionary taxes and balanced budget rules, such as Schmitt-Grohè and Uribe (1997), are likely to display the same kind of global indeterminacy. Indeed, Wen (2001) shows that such a model is similar to the Benhabib-Farmer model in its reduced-form dynamics.

[11] See, for example, Benhabib and Wen (2004). Their model does a good job in matching selected second moments of the US data for combinations of parameters that give complex eigenvalues. Such results are obtained for values of the externality parameter that are away from the bifurcation point. Complex eigenvalues improve the model's performance by inducing smooth oscillatory behavior.

state of the system. Identify this system with its truncated Taylor series expansion:

$$x_{t+1} = A(x_t - x^*) + F(x_t - x^*), \tag{10}$$

where $A$ is the $2 \times 2$ matrix of first-order partial derivatives evaluated at the steady state and where $F(x) = \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x) + O(\|x\|^4)$, in which $B \equiv (B_1, B_2), C \equiv (C_1, C_2)$ are $2 \times 1$ vectors. The terms in these vectors are defined as follows:

$$B_i(x,y) = \sum_{j,k=1}^{2} \frac{\partial^2 f_i(\xi)}{\partial \xi_j \partial \xi_k} |_{\xi = x^*} \, x_j y_k, \quad C_i(x,y) = \sum_{j,k,l=1}^{2} \frac{\partial^3 f_i(\xi)}{\partial \xi_j \partial \xi_k \partial \xi_l} |_{\xi = x^*} \, x_j y_k z_l \tag{11}$$

for $i = 1, 2$. Let $q \in R^2$ be the eigenvector associated with the eigenvalue $\mu_1 = -1$ of $A$, which spans the eigenspace $T^c$, the tangent space to the center manifold. Let $p \in R^2$ be the adjoint eigenvector such that $A^T p = \mu_1 p$, normalized so that $\langle p, q \rangle = 1$, where $\langle ., . \rangle$ is the usual real inner product. We have the following lemma in the continuous case (Kuznetsov, 1998, Section 5.2):

**Lemma** Let $T^{su}$ denote an $(n-1)$ dimensional linear eigenspace of $A$ corresponding to all eigenvalues other than 0, which is the space tangent to the stable-unstable manifold. Then $y \in T^{su}$ if and only if $\langle p, y \rangle = 0$.□

Applying this lemma in the discrete case to the matrix $(A - \mu_1 E)$ (where $E$ is the $2 \times 2$ identity matrix), we conclude that $y \in T^{su}$ if and only if $\langle p, y \rangle = 0$, where $T^{su}$ spans the eigenspace corresponding to all eigenvalues other than $-1$.

We can decompose any vector $x \in R^2$ as $x = uq + y$ (where $u = \langle p, x \rangle$, $y = x - \langle p, x \rangle q$, $uq \in T^c$ and $y \in T^{su}$). The mapping (after a shift of coordinates back to $(0,0)$), $\tilde{x} = Ax + F(x)$, can be written in the $(u, y)$ coordinate system as:

$$\begin{aligned} \tilde{u} &= \mu_1 u + \frac{1}{2}\sigma u^2 + u \langle b, y \rangle + \frac{1}{6}\delta u^3 + ... \\ \tilde{y} &= Ay + \frac{1}{2}a u^2 + ... \end{aligned} \tag{12}$$

where

$$\sigma = \langle p, B(q,q) \rangle, \quad \delta = \langle p, C(q,q,q) \rangle, \quad a = B(q,q) - \langle p, B(q,q) \rangle q.$$

In order to reduce the dimension of the above system, we appeal to the Reduction Principle restated here:

**Theorem** (Kuznetsov,1998, p157) Consider the hyperbolic discrete-time dynamical system: $x \mapsto f(x)$. Using an eigenbasis, the system:

$$\begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} Bu + g(u,v) \\ Cv + h(u,v) \end{pmatrix} \text{ is locally topologically equivalent to } \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} Bu + g(u, V(u)) \\ Cv \end{pmatrix}$$

where $B$ has eigenvalues on the unit circle, and all the eigenvalues of $C$ are not on the unit circle and where $V(u)$ is a function defining the local representation of the center manifold $W^c = \{(u,v) : v = V(u)\}$. $\square$

The local approximation of the center manifold has the form $y = V(u) = \frac{1}{2}w_2u^2 + O(u^3)$. Substituting $y$ into the second equation in (12), it can be shown that $w_2 = -(A-E)^{-1}a$. Using $\langle b, y\rangle = \langle p, B(q,y)\rangle$ and substituting it into the first equation in (12) yields (after simplification):

$$\tilde{u} = -u + a(0)u^2 + b(0)u^3 + O(u^4) \equiv h(u,0), \tag{13}$$

where

$$\begin{aligned} a(0) &= \frac{1}{2}\langle p, B(q,q)\rangle \\ b(0) &= \frac{1}{6}\langle p, C(q,q,q)\rangle - \frac{1}{4}(\langle p, B(q,q)\rangle)^2 - \frac{1}{2}\langle p, B(q,(A-E)^{-1}B(q,q))\rangle \end{aligned}$$

It can be shown that equation (13) is topologically equivalent to the normal form:

$$\tilde{\xi} = -\xi + c(0)\xi^3 + O(\xi^4), \tag{14}$$

where $c(0) = a^2(0) + b(0)$. This equivalence is true as long as two nondegeneracy conditions hold (see Theorem 4.3 in Kuznetsov, 1998). These conditions will be checked later. Furthermore, we can show that (14) is locally topologically equivalent to $\tilde{\xi} = -\xi + c(0)\xi^3$ around the steady state (see lemma 4.2 in Kuznetsov, 1998).

The main challenge in computing the sign of $c(0)$, which determines the direction of the bifurcation, is figuring out the coefficients of the third-order Taylor series expansion in equation (11). In principle, $c(0)$ can be expressed as a complicated function of all the structural parameters, and the sign of $c(0)$ can then be determined if all the parameters satisfy an implicit relationship: $h(\alpha, \beta, \delta, \eta...) = 0$ which represents all the points where a bifurcation takes place. However, since the system $x_{t+1} = f(x_t)$ is implicit in $x_{t+1}$ and $x_t$ in our model, obtaining analytical Taylor series expansion terms as functions of the parameters is difficult. Instead, we choose specific parameter values at which the bifurcation takes place and compute the numerical values of $c(0)$ to prove the direction of bifurcation.

The dynamical system of the benchmark model corresponding to $x_{t+1} = f(x_t)$ is:

$$\begin{array}{c} \alpha\beta\left[\zeta_2 k_t^{\zeta_3} c_t^{\zeta_4} + (1-\delta)k_t - c_t\right]^{\zeta_3 - 1} c_{t+1}^{\zeta_4 - 1} + \frac{\beta(1-\delta)}{c_{t+1}} - \frac{1}{c_t} = 0, \\ k_{t+1} = k_t^{\zeta_3} c_t^{\zeta_4} + (1-\delta)k_t - c_t; \end{array} \tag{15}$$

where $\zeta_1 = \frac{1}{1+\gamma-(1-\alpha)(1+\eta^*)}, \zeta_4 = -\zeta_1(1-\alpha)(1+\eta), \zeta_2 = (1-\alpha)^{-\zeta_4}$, $\zeta_3 = \alpha(1+\eta)[1+(1+\eta)(1-\alpha)\zeta_1]$. We now appeal to the implicit function theorem to obtain the desired Taylor series expansion for system (15). As an example, we set $\beta = 0.99$, $\gamma = 0$, $\alpha = 0.3$, and $\delta = 0.025$. At these parameter values, the flip bifurcation takes place when the externality parameter $\eta = 0.4934....$ Using these values, we obtain coefficients for the Taylor series, $c_{k,j}^h$, which satisfy $f_h(c,k) = \sum\limits_{k,j} c_{k,j}^h (c-c^*)^k (k-k^*)^j$, $h = 1, 2$ and where the summation is taken over:

$$(k,j) \in \{(0,0), (1,0), (0,1), (1,1), (2,1), (1,2), (0,3), (3,0), (2,0), (0,2)\}.$$

Using these coefficients, the matrix $A$ as well as $(p, q)$ can be computed as:

$$A = \begin{pmatrix} c_{1,0}^1 & c_{0,1}^1 \\ c_{1,0}^2 & c_{0,1}^2 \end{pmatrix} = \begin{pmatrix} 0.082637.. & 0.03137.. \\ 28.2945.. & -0.18008.. \end{pmatrix},$$

$$p = (-14.8781, \quad 0.569283), \; q = (-0.0289657, \quad 0.99958).$$

Using these data, one can compute that $c(0) = 0.557624 > 0$, which implies that the flip bifurcation in the benchmark model is supercritical.

Let's now check the nondegeneracy conditions . $\frac{1}{2}\left(h_{uu}(0,0)\right)^2 + \frac{1}{3}h_{uuu}(0,0) = 2c(0) \neq 0$. The second condition , $h_{u\eta}(0,0) \neq 0$, is true as long as $\left|\mu_1'(\eta^*)\right| \neq 0$ holds. Our computation shows that $\left|\mu_1'(\eta^*)\right| = 153.629$. This completes the proof.■

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