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Kalman filtering with truncated normal state variables for Bayesian estimation of macroeconomic models

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Abstract

A pair of simple modifications—in the forecast error and forecast error variance—to the Kalman filter recursions makes possible the filtering of models in which one or more state variables is truncated normal and latent. Such recursions are broadly applicable to macroeconometric models, such as vector autoregressions and estimated dynamic stochastic general equilibrium models, that have one or more probit-type equation.

JEL classifications: **C32, C35, E37**

Key words: **Kalman filter, truncated normal, probit model, macroeconometric models**

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1 Introduction

Multivariate models that contain a probit-type equation are emerging in empirical macroeconomics, such as the Qual VAR of Dueker (2005)—a vector autoregression that includes an equation that determines a qualitative variable. The probit-type equation could be one that includes information on business cycle turning points, or it could be a monetary policy equation that uses data on the central bank's interest rate target that changes by discrete increments. Other macroeconomic phenomena also lead to qualitative variables that evolve dynamically, such as currency pegs and central bank exchange-rate interventions. Dynamic probits aim to capture both the limited-dependent nature and serial correlation inherent in qualitative time-series variables.[1]

Until now, however, multivariate macroeconometric models that contain a probit-type equation have relied on single-move sampling techniques, even though multi-move sampling dominates in terms of efficiency and convergence to the joint posterior distribution. In single-move Markov Chain Monte Carlo (MCMC) sampling of a vector, z, the density used to draw an element t for iteration i depends on the previous iteration's draws:

$$\pi(z_t^{(i)} \mid \{z^{(i)r}\}_{r<t}, \{z_s^{(i-1)}\}_{s>t}, \text{Data, parameters}).$$

In multi-move sampling, in contrast, the entire vector of z is drawn without reference to

[1]The term "dynamic probit" is also used in some contexts to refer to a model in which the lagged qualitative variable appears as a covariate; see Horowitz (1992). In this case, the presence of the lagged qualitative variable adds persistence but not dynamics in the sense of time-varying expected durations to the qualitative regimes. We refer here to dynamic probits as those for which the latent variable governing the qualitative variable is autoregressive and, thereby, displays dynamic behavior. Eichengreen et al. (1985) is the first example of an autoregressive dynamic probit.

the last iteration's draws:

$$\pi(\{z^{(i)_t}\}_{t=1}^{T} \mid \text{Data, parameters}).$$

One hallmark feature of MCMC estimation of probit-type models is the sampling of truncated normal latent variables; see Albert and Chib (1993). At the same time, the Kalman filter has seen great use in Bayesian estimation as a tool to derive multi-move sampling distributions, as pioneered by Carter and Kohn (1994), Fruhwirth-Schnatter (1994), Shephard (1994) and De Jong and Shephard (1995). This paper demonstrates that multi-move sampling of truncated normal variables is possible if one modifies the Kalman filter recursions for the case where a state variable is truncated normal. To the author's knowledge, this specific modification of the Kalman filter recursions for truncated state variables has not appeared in the literature to date. With such a modified Kalman filter, it is possible to extend multi-move MCMC sampling to the Qual VAR model of Dueker (2005) and also to extend estimation of dynamic stochastic general equilibrium (DSGE) models to include probit-type features, such as the central bank's discrete target interest rate. Dueker and Nelson (2006) use a Qual VAR to conduct counterfactual business cycle filtering of macroeconomic data. Because many linearized DSGE models take the form of a vector autoregression, subject to coefficient restrictions, the the form of the model we present here covers a wide range of macroeconometric models.

2 State-space model and truncation

Thus, consider a macroeconometric model written as a p-order vector autoregression, where z is the latent variable that lies behind the qualitative data, z^q, and observed macroeconomic time seris, X:

$$Y_t = c_Y + \sum_{i=1}^{p} \Phi^{(i)} Y_{t-i} + \epsilon_t, \tag{1}$$

where $Y_t = (X_t, z_t)'$ is $k \times 1$ and

$$\Phi^{(i)} = \begin{pmatrix} \Phi^{(i)}_{XX} & \Phi^{(i)}_{Xz} \\ \Phi^{(i)}_{zX} & \Phi^{(i)}_{zz} \end{pmatrix}.$$

A standard state-space representation of the VAR from eq. (1) has the following state equation[2] :

$$\begin{pmatrix} Y_t \\ Y_{t-1} \\ \vdots \\ Y_{t-p+1} \end{pmatrix} = \begin{pmatrix} c_Y \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} \Phi^{(1)} & \Phi^{(2)} & \cdots & \Phi^{(p)} \\ I & 0 & \cdots & 0 \\ 0 & \ddots & & \vdots \\ 0 & \cdots & I & 0 \end{pmatrix} \begin{pmatrix} Y_{t-1} \\ Y_{t-2} \\ \vdots \\ Y_{t-p} \end{pmatrix} + \begin{pmatrix} \epsilon_{Y,t} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

The measurement equation for the Qual VAR is simply

$$X_t = \begin{pmatrix} I_{k-1} & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} Y_t \\ Y_{t-1} \\ \vdots \\ Y_{t-p+1} \end{pmatrix} \tag{2}$$

To achieve multi-move sampling, it is necessary to derive the conditional distribution of the entire vector of the latent variable conditional on the parameter vector, Θ, the observed qualitative data, z^q, and the other macroeconomic data, X_t:

$$\pi(\{z_t\}_{t=1}^T \mid \Theta, \{z_t^q\}_{t=1}^T, \{X_t\}_{t=1}^T).$$

We can summarize the above state-space model (suppressing the constants) as:

[2]Macroeconometric models have made great use of the state-space form to take account of latent state variables, e.g., Kuttner (1994).

$$
\begin{aligned}
X_t &= HS_t \\
S_t &= FS_{t-1} + \epsilon_t
\end{aligned} \tag{3}
$$

The number of rows in the state vector is k and F_k denotes row k of F.

Without loss of generality, assume that z is the last element in the state vector S. Because we want to allow for the possiblity that the covariance matrix of ϵ_t has non-zero off-diagonal elements, let

$$\epsilon_t = W\eta_t,$$

where W is an upper-triangular Cholesky decomposition of the covariance matrix Q of ϵ_t. The truncation from having z_t^q in category j implies that z_t lies in the range (d_j, d_{j+1}), and this truncation of z will change the mean and variance of the last element of η from the unconditional values of zero and one, respectively.

The mean of ϵ, conditional on trunction, becomes

$$E[\epsilon_{t+1} \mid z_{t+1}^q] = W \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a \end{pmatrix},$$

where we denote $\alpha_{j+1} = W^{-1}[z_{t+1} - F_k S_{t|t} + d_{j+1}]$, $\alpha_j = W^{-1}[z_{t+1} - F_k S_{t|t} + d_j]$ such that

$$a = \frac{\phi(\alpha_{j+1}) - \phi(\alpha_j)}{\Phi(\alpha_{j+1}) - \Phi(\alpha_j)}.$$

and the variance of ϵ, conditional on truncation, becomes

$$\tilde{Q} = W \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \ddots & 0 & 0 \\ 0 & \cdots & 1 & 0 \\ 0 & \cdots & 0 & v \end{pmatrix} W',$$

where

$$v = 1 - a^2 - \frac{\alpha_{j+1}\phi(\alpha_{j+1}) - \alpha_j\phi(\alpha_j)}{\Phi(\alpha_{j+1}) - \Phi(\alpha_j)}.$$

3 Kalman filter recursions

The Kalman filter with the truncated state variable proceeeds as follows:

1. Data forecast error (DFE) conditional on z^q:

$$y_{t+1} - y_{t+1|t} = y_{t+1} - HFS_{t|t} - HE[\epsilon_{t+1} \mid z^q_{t+1}]$$

2. State forecast variance:

$$P_{t+1|t} = FP_{t|t}F' + \tilde{Q}$$

3. Data forecast error variance:

$$\text{Var}_t(DFE_{t+1}) = HP_{t+1|t}H'$$

The equations to update the state and the precision matrix are based on

$$S_{t+1|t+1} = S_{t+1|t} + E[(S_{t+1} - S_{t+1|t})DFE_{t+1}] \times [HP_{t+1}H']^{-1}DFE_{t+1}.$$

Nevertheless, because

$$E[(S_{t+1} - S_{t+1|t})DFE_{t+1}] = P_{t+1|t}H',$$

the Kalman filter update equations take the usual form, although the truncation information alters the data forecast error (DFE) and forecast variance inputs:

$$S_{t+1|t+1} = S_{t+1|t} + P_{t+1|t}H'[HP_{t+1|t}H']^{-1}DFE_{t+1} \tag{4}$$

$$P_{t+1|t+1} = P_{t+1|t} - P_{t+1|t}H'[HP_{t+1|t}H']^{-1}HP_{t+1|t} \tag{5}$$

Thus, these modifications to the Kalman filter recursions show that the Kalman filter remains a useful tool to calculate conditional densities in the case where one or more of the state variables is truncated normal. The same logic would apply if an observation equation variable were truncated normal.

After the modified Kalman filtering, the usual (no modification necessary) Kalman smoothing equations can be applied in order to draw values of z backwards from the end of the sample, with the net result being a draw from the density $\pi(\{z\} \mid \{z^q\}, \{X\}, \text{parameters})$. See Fruwirth-Schnatter (1994) for a discussion of backwards sampling from Kalman filtered and smoothed conditional densities.

4 Conclusion

A few simple modifications of the Kalman filter recursions make the approach applicable to the case where one or more state variables is truncated normal. No approximation of the Kalman filter is required, in contrast to the extended Kalman filter and its first-order Taylor approximations. This extension of the Kalman filter makes it possible to do multi-move sampling for Bayesian estimation of vector autoregressions and dynamic stochastic general equilibrium macroeconomic models that involve a probit-type equation that makes use of qualitative or discrete-valued data.

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