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Production and Inventory Behavior of Capital*

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Abstract

This paper provides a dynamic optimization model of durable good inventories to study the interactions between investment demand and production of capital goods. There are three major findings: First, capital suppliers' inventory behavior makes investment demand more volatile in equilibrium; Second, equilibrium price of capital is characterized by downward stickiness; Third, the responses of the capital market to interest rate and other environmental changes are asymmetric. All are the results of equilibrium interactions between demand and supply.

JEL Classification: E22, E23, E32.

Keywords: Investment, Capital Theory, Inventory, Durable Goods, Production Volatility, Sticky Price.

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1 Introduction

Inventory investment as a component of aggregate spending accounts for less than one percent of GDP, yet the drop in inventory investment accounts for 87 percent of the drop in GDP during the average postwar recession (Blinder and Maccini, 1991). Among inventories, durable good inventories are the most volatile – nearly five times as volatile as non-durable good inventories in terms of variance (see, e.g., Blinder 1986, table 1). Hence, understanding the production and inventory behavior of the durable goods industry is essential for understanding the business cycle.

This paper focuses on one particular type of durable good: capital. In the U.S., about half of the output produced by the durable-goods sector is sold to producers as capital equipment. Unfortunately, the literature on firms' optimal behavior of production and inventory investment with regard to capital goods is remarkably thin. Most of the literature on capital deals with capital demand (i.e., investment), instead of supply.[1] This may be attributable to the fact that there are no models available for dealing with durable good inventories in general and capital good inventories in particular. The difficulty involved is that, on the one hand, capital is a durable good, and durability is a user's measure, not a producer's measure, hence modeling the production and inventory behavior of capital requires consideration for capacity demand from the view point of capital buyers; and on the other hand, production and inventory accumulation of capital goods is a supply-side problem, hence requiring simultaneous handling of upstream firms which produce, store, and sell capital equipment to downstream firms. The traditional (S,s) approach for inventories, for example, is inadequate for this task. It would assume, for example, that there exists a fixe cost of ordering capital goods, hence firms would have the incentive to order more capital equipment than needed in a (S,s) style, in order to reduce the average fixed cost of capital purchases.[2] This demand-side approach is quite limited for understanding capital good inventories because few firms would order excess capital

[1]The most influential paper on this subject is Tobin's (1969) q theory. For the more recent literature, see Abel, Dixit, Eberly and Pindyck (1996), Able and Eberly (1994), Hayashi (1982), and Lucas and Prescott (1971), among many others.

[2]For the recent literature on the (S,s) inventory model, see, e.g., Caballero and Engel (1999), Fisher and Hornstein (2000), and Kahn and Thomas (2002).

equipment simply because of fixed costs of ordering or delivery, especially considering that most of fixed costs of capital investment are either variable fixed costs or disproportionately small relative to the price of capital. Even if firms do order excess capital in order to reduce the average fixed costs of purchases, the excess capital installed is treated as excess capacity instead of as inventories in accounting books.[3]

According to textbook theories, national savings are the chief source of domestic investment. Yet in reality how savings are translated into investment is a subtle issue. If investment demand is defined as demand for financial capital, then it is rather easy to imagine how household savings (the supply of funds) provide the source of investment. But if investment demand is defined as demand for tangible capital goods (i.e., machineries), than how aggregate savings end up meeting investment demand is not that simple. For one thing, capital goods must be produced, and production of capital goods takes time. Thus, national savings have to come from production determined in the past. Since only productive capital (or finished capital goods which are ready for use) are purchased by firms, the time-to-built factor is on the supply side, not on the demand side. For this reason, the demand for capital may not be satisfied unless the suppliers of capital can anticipate this demand many periods in advance. This time dimension in the supply side of capital is hidden behind the national income accounting. The issue is further complicated by inventories. In national income accounting, inventories are treated as part of aggregate demand. But in reality inventories may be related more closely to the supply side than to the demand side. For example, to enhance the flexibility of supply and to avoid opportunity costs of losing sales, capital suppliers may have incentives to accumulate inventories of capital goods by producing above the expected demand. Such inventory behavior would certainty affect the supply capacity of capital and hence national savings. Thus, while it is easy to determine how an increase in the interest rate affects investment demand from capital buyers (at least according to the textbook theory), it is not clear how this should affect the production and inventory behavior of capital (i.e., the supply of capital). A simple textbook style,

[3]The literature on the lumpiness of investment behavior deals with volatility of capital from the demand side. This literature has left out the issue of capital supply with respect to capital goods production and its associated inventory behavior. See for example, Thomas (2002) and Kahn and Thomas (2002) and the reference therein.

upward-sloping saving curve is clearly inadequate and may be highly misleading in drawing conclusions about the determination of equilibrium investment.

This paper takes a first step towards addressing the supply-side issues of capital by providing a canonical model for the production and inventory behavior of capital. In the model buyers order capital goods from suppliers to produce output, and suppliers produce and sell capital goods to the buyers. The production of capital takes at least one period of time, hence production plans need to be committed before demand is known.[4] Due to uncertainty in investment demand from the buyers (either due to productivity shocks or demand shocks from downstream firms), the suppliers may incur inventories of capital goods produced when demand for capital turns low. The supplier, however, has the option to sell inventories at lower price in order to reduce the cost of holding inventories; or to accumulate inventories, anticipating higher demand in the next period.[5] Optimal production and inventory investment decisions as well as equilibrium price of capital are characterized in a perfectly competitive environment where both buyers and suppliers of capital goods are price takers. Comparative statics are conducted to study the effects of changes in interest rate, capital depreciation rate, demand uncertainty, etc., on the supply behavior of capital.

It is found that a competitive capital supplier's optimal behavior is characterized by an inventory target policy that specifies the optimal level of production based on expected investment demand from capital buyers. Such inventory holding behavior of the capital supplier can dramatically change the dynamics of equilibrium investment demand. Without inventories, the demand for capital is met completely by capital production. Due to time-to-built, production plans are determined by past information about expected future demand. Thus *ex post* the investment demand cannot be re-adjusted to reflect news about the current productivity of capital, leading to less volatile investment demand. With inventories, however, the supply of capital becomes effectively perfectly elastic up to the point of a stockout, enabling capital buyers to re-adjust investment

[4]This reflects the important concept of time-to-built (see Kydland and Prescott, 1982).

[5]Thee are two types of capital, equipments and structures. Since structures are much less divisible and hence far more costly both in terms of price and inventory storage, they are mostly produced according to orders. Hence inventories of structures are less common than inventories of equipments. However, according to the U.S. housing data (houses are a form of structures), the suppliers often start housing construction before the orders come in, suggesting that there are also inventories in structures.

demand according to new information about the returns to capital. Hence, investment demand becomes more volatile in equilibrium. It is also shown that the responses of the capital market to policy changes are asymmetric, due to the capital suppliers' production and inventory behavior. For example, an increase in the interest rate has an effect on equilibrium investment only when the market is thick, despite that investment demand is always a function of the interest rate. Another interesting implication of the model is that price of capital appears to be downward sticky and upward flexible. This is also a consequence of the inventory behavior of the capital suppliers.[6]

The rest of the paper is organized as follows. The model is described in Section 2. Closed-form policies for optimal demand, supply, inventory investment and equilibrium price of capital are derived and characterized in Section 3. Comparative statics are carried out in Section 4. Finally, section 5 concludes the paper.

2 The Model

Downstream Firms: A representative buyer purchases capital goods as capacity investment and produces output according to the production technology,

$$f(k_t, \theta_t),$$

where k represents capital stock, θ is an $i.i.d$ random variable representing shocks to the firm's demand or productivity, and $f()$ satisfies

$$f'_k > 0, f''_{kk} < 0, f''_{k\theta} > 0;$$

where the last assumption indicates that θ shifts the capital demand curve upwards. The market price of new capital (cost of investment) is λ_t which the firm takes as given. Assume full capacity utilization, the firm chooses sequences of either the capital stock, $\{k_{t+j}\}_{j=0}^{\infty}$, or the rate of investment, $\{I_{t+j}\}_{j=0}^{\infty}$, to maximize the discounted expected profit,

$$\max E_t \sum_{j=0}^{\infty} \beta^j \left[f(k_{t+j}, \theta_{t+j}) - \lambda_{t+j} I_{t+j} \right]$$

[6]The downward sticky price behavior has also been studied by Reagan (1982) in a monopolistic competition model with non-durable goods. The stockout-avoidance motive for holding inventories in the model is similar to that studied by Abel (1985), Reagan (1982), and Kahn (1987).

subject to

$$k_{t+j} = I_{t+j} + (1-\delta)k_{t+j-1};$$

where $\beta \in (0,1)$ is the inverse of interest rate (discounting factor) and δ is the depreciation rate of capital.

Upstream Firms: A representative supplier produces capital goods (y_t) according to a linear production technology. This implies that the cost function is linear in output, ay_t, where a is a positive constant. Assume that there is a one period production lag between the commitment of input and the availability of output for sale (i.e., the firm must make production plans one period in advance before demand for capital in period t is known), hence the total output (capital goods) available for sale in period t is the existing stock of inventories carried from last period (s_{t-1}) plus the current output (y_t) that was committed last period, $s_{t-1} + y_t$. This assumption of production lags reflects the important concept of time-to-built (e.g., see Kydland and Prescott, 1982). Without loss of generality the depreciation rate for inventories is assumed to be zero and there is no other costs for holding inventories except the cost associated with time discounting, β. The firm takes expected output price (λ_t) and expected investment demand from buyers (I_t) as given and chooses sequences of production plans (y_t) and inventory investment ($s_t - s_{t-1}$) to maximizes's discounted sum of expected profits,

$$\max_{\{y_{t+j}\}} E_{t-1}\left\{ \max_{\{s_{t+j}\}} \left\{ E_t \sum_{j=0}^{\infty} \beta^j \left[\lambda_{t+j} I_{t+j} - a y_{t+j}\right] \right\} \right\}$$

subject to

$$I_{t+j} + s_{t+j} = s_{t+j-1} + y_{t+j},$$

and

$$s_{t+j} \geq 0,$$
$$y_{t+j} \geq 0,$$

where the expectation operators, $\{E_{t-1}, E_t\}$, indicate the relevant information sets when decisions are made.

Competitive Equilibrium: A competitive equilibrium is a set of decision rules for capital sales (I_t), capital production (y_t), inventory holdings (s_t) and the price of capital (λ_t) such that the following first order conditions hold:

$$f'_k(k_t, \theta_t) = \lambda_t - \beta(1-\delta)E_t\lambda_{t+1} \tag{1}$$

$$a = E_{t-1}\lambda_t \tag{2}$$

$$\lambda_t = \beta E_t \lambda_{t+1} + \pi_t \tag{3}$$

$$[k_t - (1-\delta)k_{t-1}] + s_t = s_{t-1} + y_t \tag{4}$$

$$\pi_t s_t = 0 \tag{5}$$

where equation (1) determines the buyer's optimal demand for capital, equation (2) determines the supplier's optimal production of capital, equation (3) determines the supplier's optimal inventory holdings, equation (4) is the capital goods market clearing condition, and equation (5) is the Kuhn-Tucker condition for the nonnegativity constraint on the supplier's inventories (hence π is the complementarity slackness multiplier).[7]

Equation (1) shows that the optimal demand for capital decreases when δ increases (i.e., when the durability of goods decreases), holding capital prices constant. This is the familiar user's cost effect of durability on demand. Equation (2) shows that the optimal supply of capital goods is chosen to the point such that the marginal cost of production (a) equals the expected value of capital in the goods market (λ_t). Equation (3) shows that the optimal level of inventories held by the supplier is determined by the point where the cost of increasing one extra unit of inventory holding, which is the opportunity cost for not selling the good (λ_t), equals the discounted expected benefit of having one more unit of inventory available for sale next period (λ_{t+1}) plus the benefit of relaxing the slackness constraint by one unit (π_t), which is zero if the constraint does not bind. Since capital is durable, there is an intertemporal substitution effect of durability on future demand of capital, which can be seen from the equation,

$$I_t = k_t - (1-\delta)k_{t-1},$$

where purchase of the capital stock last period reduces the current investment demand for capital. The more durable is the good, the larger such effect is.

[7]Given that investment demand is always positive (since $f'_k > 0$), the nonnegativity constraint on production will never bind. Hence the constraint, $y \geq 0$, is ignored.

3 Optimal Supply of Capital

The only source of uncertainty in the model stems from the capital buyer, θ. A high θ implies a high demand for capital and a low θ implies a low demand for capital. Since production of capital takes one period of time, the supplier needs to forecast future demand of capital and decides the optimal level of inventory to hold. To characterize equilibrium decision rules of the suppliers, consider two possibilities: a thin market where the realized value of the shock (θ) or the associated investment demand for capital is below "normal", or a thick market where θ is above "normal" such that the nonnegativity constraint on inventory binds.

Case A: If θ is below normal, suggesting that the investment demand for capital is low, the nonnegativity constraint on the supplier's inventories does not bind. Hence $\pi_t = 0$ and $s_t \geq 0$. Equations (2) and (3) imply that the competitive price of capital is constant,

$$\lambda_t = \beta a.$$

Thus equation (1) implies

$$f'_k(k_t, \theta_t) = \beta\delta a,$$

which gives the optimal capital demand under case A as an increasing function of θ,

$$k_t = k^*(\theta_t), \quad \text{where } \frac{\partial k^*(\theta)}{\partial \theta} > 0.$$

The market clearing condition (4) then implies

$$s_t = y_t + s_{t-1} + (1-\delta)k_{t-1} - k^*(\theta_t).$$

The threshold value for θ is determined by the constraint, $s_t \geq 0$, which implies

$$k_t^*(\theta_t) \leq y_t + s_{t-1} + (1-\delta)k_{t-1}, \tag{6}$$

or

$$\begin{aligned} \theta_t &\leq (k^*)^{-1}\left(y_t + s_{t-1} + (1-\delta)k_{t-1}\right) \\ &\equiv z(y_t), \end{aligned} \tag{7}$$

where $z(y)$ denotes the optimal cutoff point for θ such that there is a stockout if $\theta > z$. Namely, z_t is defined as

$$k^*(z_t) \equiv y_t + s_{t-1} + (1-\delta)k_{t-1}. \tag{8}$$

Since $k^*(\theta)$ is a monotonically increasing function, we have

$$\frac{\partial k^*(z)}{\partial z} > 0 \text{ and } \frac{\partial z(y)}{\partial y} > 0. \tag{8$'$}$$

Case B: If investment demand is above normal due to a large shock on the buyer's productivity or output demand, then the supplier's nonnegativity constraint on inventories binds. Hence $\pi_t > 0$ and $s_t = 0$. The market-clearing condition (4) implies that the investment demand is met with the supplier's entire existing stock of capital goods,

$$k_t - (1-\delta)k_{t-1} = y_t + s_{t-1}. \tag{9}$$

Clearly, the probabilities of case A and case B depend on the production level committed last period, y_t. To determine the optimal production policy, we can utilize equation (2). Denote $\phi()$ as the probability density function of θ with support $[A, B]$, then equation (2) can be expanded as

$$\begin{aligned} a &= E_{t-1}\lambda_t \\ &= \int_A^{z(y_t)} \beta a \phi(\theta) d\theta + \int_{z(y_t)}^B \left[f_k'(k_t, \theta_t) + \beta(1-\delta)a \right] \phi(\theta) d\theta \end{aligned} \tag{10}$$

where the cutoff point that determines the probability of stocking out, $z(y)$, is defined in (7) and (8).

The interpretation of (10) is straightforward. The expected value of equilibrium capital price, λ, is a probability distribution of two cases: $\lambda = \beta a$ if the realized shock to the buyer is small so that there is no stockout for the supplier ($\pi = 0$); or $\lambda = f_k'(k, \theta) + \beta(1-\delta)a$ if the realized shock to the buyer is large so that there is a stockout for the supplier ($\pi > 0$). In the latter case the optimal level of capital demand (k_t) is given by (9). In other words, the left-hand side of (10) is the cost of producing one extra unit of capital goods today, a; but the marginal benefit of having one extra unit of capital goods available next period is given by the right-hand side of (10) with two possibilities. First, in the event of no stockout due to a low demand, the firm gets to save on the marginal cost of production by postponing production for one period. The present value of this term is βa and this event happens with probability $\int_A^{z(y)} \phi(\theta)d\theta$. Second, in the event of a stockout due to a high demand, the firm can sell the product at the competitive market price, λ_t, which equals the marginal product of capital plus the present market value of the nondepreciated good, $f'_k(k,\theta) + \beta(1-\delta)a$, where k is determined by (9) under case B. This event happens with probability $\int_{z(y)}^{B} \phi(\theta)d\theta$.

Clearly, the probability of stocking out in period t, $\int_{z(y)}^{B} \phi(\theta)d\theta$, is determined by the level of production (y) committed in period $t-1$ plus the existing inventory stock, s_{t-1}. The larger is y, the more inventory the firm has (i.e., the larger $z(y)$ is), hence the smaller the probability of stocking out. Since holding inventories is costly due to time discounting, and stocking out is also costly due to loss of opportunities for sale, the level of production is determined to the point where the expected marginal revenue ($E_{t-1}\lambda_t$) equals marginal cost (a).

Proposition 1 *An optimal cutoff point (which is also the optimal inventory target of the supplier), $z(y) \in [A, B]$, exists and is unique. This optimal target inventory level is also constant, $z(y) = \bar{z}$. Furthermore, $\bar{z}$ positively depends on the variance of θ.*

Proof. Rewrite (10) as (by substituting out k_t using equation 9):

$$
\begin{aligned}
a &= \int_A^{z(y_t)} \beta a \phi(\theta)d\theta + \int_{z(y_t)}^{B} \left[f'_k(k_t,\theta_t) + \beta(1-\delta)a\right]\phi(\theta)d\theta \\
&= \int_A^{z(y_t)} \beta a \phi(\theta)d\theta + \int_{z(y_t)}^{B} \left[f'_k\left((y_t + s_{t-1} + (1-\delta)k_{t-1}),\theta_t\right) + \beta(1-\delta)a\right]\phi(\theta)d\theta \\
&= \int_A^{z(y_t)} \beta a \phi(\theta)d\theta + \int_{z(y_t)}^{B} \left[f'_k\left(k^*(z_t),\theta_t\right) + \beta(1-\delta)a\right]\phi(\theta)d\theta,
\end{aligned}
$$

where the last equality utilized the definition of $z(y)$. The above equation can be simplified (after rearranging terms) to:

$$(1-\beta)a = \int_{z(y_t)}^{B} \left[f_k'(k^*(z_t),\theta_t) - \beta\delta a\right]\phi(\theta)d\theta \tag{11}$$

$$\equiv \int_{z_t}^{B} g(z_t,\theta_t)\phi(\theta)d\theta.$$

Notice that $k^*(z)$ is an increasing function of z (see equation $8'$), hence f_k' is a decreasing function of z. Thus, $g_z' = f_{kk}''\frac{\partial k^*(z)}{\partial z} < 0$. Since $g > 0$ (by equation 1, $f_k' > \beta\delta a$ under case B)[8], then clearly the right-hand side of (11) is monotonically decreasing in z :

$$\frac{\partial \int_{z_t}^{B} g(z_t,\theta_t)\phi(\theta)d\theta}{\partial z} = -g(z,z)f(z) + \int_{z}^{B} g_z'\phi(\theta)d\theta < 0.$$

It is easy to see that the minimum of the right-hand side of (11) is zero when $z = B$ and the maximum is greater than $(1-\beta)\,a$ when $z = A$ (since $f_k'(k^*(A),\theta_t)$ can be made arbitrarily large as $A \to -\infty$ by assuming that f_k' is sufficiently diminishing in k). Hence a unique solution for z_t exists. Furthermore, since θ is $i.i.d$, the right-hand side of (11) after integration is an implicit function in the form, $G(z_t,\Omega) = 0$, where Ω is a set of constant parameters. Hence, z_t must be a constant, $z_t = \bar{z}$, which solves $G(\bar{z},\Omega) = 0$ or

$$(1-\beta)a = \int_{\bar{z}}^{B} g(\bar{z},\theta_t)\phi(\theta)d\theta. \tag{12}$$

Denote the mean of θ as $\bar{\theta} \equiv E(\theta_t)$, and notice that in the steady state (i.e., in the absence of uncertainty) $s_t = 0$ for all t, hence the optimal cut-off point $\bar{z} \geq \bar{\theta}$, because under uncertainty (off the steady state) the optimal level of inventories cannot be less than that in the steady state (which corresponds to $\bar{\theta}$) due to the positive probability of a stockout. Now, consider an increase in the variance of θ that preserves the mean. A mean-preserving spread increases the weight of the tail of the distribution, hence the right hand side of (12) increases, indicating that $\bar{z}$ must also increase in order to maintain (12) since the right hand side is decreasing in $\bar{z}$.■

[8] $E_t\lambda_{t+1} = a$ by equation (2).

Proposition 2 *The equilibrium decision rules for demand, supply, inventory investment and market price of capital are given by*

$$k_t = \begin{cases} k^*(\theta_t)\,, \textit{if } \theta_t \le \bar{z} \\ k^*(\bar{z})\ \ , \textit{if } \theta_t > \bar{z} \end{cases}$$

$$I_t = \begin{cases} k^*(\theta_t) - (1-\delta)k^*(\theta_{t-1})\,, \textit{if } \theta_t \le \bar{z} \ \&\ \theta_{t-1} \le \bar{z} \\ k^*(\theta_t) - (1-\delta)k^*(\bar{z}) \qquad, \textit{if } \theta_t \le \bar{z} \ \&\ \theta_{t-1} > \bar{z} \\ k^*(\bar{z}) - (1-\delta)k^*(\theta_{t-1})\ \ , \textit{if } \theta_t > \bar{z} \ \&\ \theta_{t-1} \le \bar{z} \\ \delta k^*(\bar{z}) \qquad\qquad\qquad\qquad\quad, \textit{if } \theta_t > \bar{z} \ \&\ \theta_{t-1} > \bar{z} \end{cases}$$

$$y_t = \begin{cases} \delta k^*(\theta_{t-1})\,, \textit{if } \theta_{t-1} \le \bar{z} \\ \delta k^*(\bar{z}) \qquad, \textit{if } \theta_{t-1} > \bar{z} \end{cases}$$

$$s_t = \begin{cases} k^*(\bar{z}) - k^*(\theta_t)\,, \textit{if } \theta_t \le \bar{z} \\ 0 \qquad\qquad\qquad\ , \textit{if } \theta_t > \bar{z} \end{cases}$$

$$\lambda_t = \begin{cases} \beta a \qquad\qquad\qquad\qquad\qquad\quad\ , \textit{if } \theta_t \le \bar{z} \\ [f_k'(k^*(\bar{z}), \theta_t) + \beta(1-\delta)a]\,, \textit{if } \theta_t > \bar{z} \end{cases}$$

where the constant $\bar{z}$ is the optimal inventory target set by the supplier of capital goods.

Proof. By proposition (1) and equation (8), the optimal production policy is given by

$$y_t = k^*(\bar{z}) - s_{t-1} - (1-\delta)k_{t-1}.$$

Substituting this into the values of inventory (s_t) discussed above under case A and case B respectively gives

$$s_t = \begin{cases} k^*(\bar{z}) - k^*(\theta_t) \text{ if } \theta_t \le \bar{z} \\ 0 \qquad\qquad\qquad\ \text{if } \theta_t > \bar{z} \end{cases}.$$

Similarly, we have

$$k_t = \begin{cases} k^*(\theta_t) \text{ if } \theta_t \le \bar{z} \\ k^*(\bar{z})\ \text{ if } \theta_t > \bar{z} \end{cases}.$$

Shifting the time subscribe backward by one period for s_t and k_t and then substituting them into the production policy give

$$y_t = \begin{cases} \delta k^*(\theta_{t-1}) \text{ if } \theta_{t-1} \le \bar{z} \\ \delta k^*(\bar{z}) \qquad \text{if } \theta_{t-1} > \bar{z} \end{cases}.$$

The other decision rules follow in straightforward fashion.■

These decision rules show that because of the existence of inventories of capital, the variances of investment demand is increased. Without inventories, investment demand is equal to a pre-determined level of production, hence the optimal demand of capital is determined by

$$E_{t-1}f'(k_t, \theta_t) = (1 - \beta(1 - \delta))a,$$

suggesting that investment demand is not responsive to new information about returns to capital as measured by the shock process θ. With inventories, however, the optimal demand of capital is determined by (utilizing equations 1-3):

$$f'(k_t, \theta_t) = \delta\lambda_t + (1 - \delta)\pi_t,$$

which suggests a higher elasticity of capital with respect to news (θ): In the case investment demand is low, inventories can be used to absorb the excess supply; and in the case investment demand is high, inventories can be used to fullfil the excess demand until a stockout happens. Thus, with probability $P = \Pr[\theta \leq \bar{z}]$, we have $\pi_t = 0$ and $\lambda_t = \beta a$, implying that k_t is perfectly correlated with θ_t. An interesting consequence of this is that the competitive market price of capital, λ_t, has the property described by Reagan (1982). Namely, it is downward sticky when demand is low (i.e., $\lambda_t = \beta a$), because firms opt to hold inventories rather than to sell them at a price below marginal cost, speculating that demand may be stronger in the future. Such rational behavior attenuates downward pressure on price. When realized demand is high, on the other hand, the firm draws down its inventories until a stockout occurs and price rises to clear the market ($\lambda_t = [f'_k(k^*(\bar{z}), \theta) - \beta\delta a] + \beta a > \beta a$ and in this case λ_t is an increasing function of θ).

4 Comparative Statics

4.1 A Change in the Interest Rate

Proposition 3 *The target inventory level is decreasing in the interest rate:* $\frac{\partial \bar{z}}{\partial r} < 0$.

Proof. The interest rate is the inverse of β: $r = \frac{1}{\beta}$. A decrease in the interest rate is the same as an increase in β. According to equation (11),

$$(1-\beta)a = \int_{\bar{z}}^{B} \left[f_k' \left(k^*(\bar{z}), \theta_t \right) - \beta\delta a \right] \phi(\theta) d\theta,$$

which can also be expressed as

$$a - \beta a \left[1 - \delta \left(1 - \Phi(\bar{z}) \right) \right] = \int_{\bar{z}}^{B} f_k' \left(k^*(\bar{z}), \theta_t \right) \phi(\theta) d\theta, \tag{13}$$

where $\Phi()$ denotes the cumulative density function of θ. Notice that $k^*(z)$ is increasing in z (see equation ($8'$)), hence f_k' is decreasing in $\bar{z}$ and the right-hand side of (13) is also decreasing in $\bar{z}$. Given that $\Phi(\bar{z}) < 1$, the left-hand side of (13) decreases as β increases. Hence when β increases, $\bar{z}$ must increase to balance the equation.■

This proposition says that a higher interest rate implies a lower target inventory level. The intuition is that a higher interest rate implies not only a higher cost to the user for capital (thus a lower investment demand), but also a higher opportunity cost for holding inventories (i.e., a higher discounting of the future), hence the target inventory level falls.

Remark 1 *The equilibrium decision rules show that the economy's response to changes in the interest rate is asymmetric. For example, output level is sensitive to the interest rate only when the market is thick (under high demand). In particular, production decreases as the interest rate increases if $\theta > \bar{z}$. Similarly, a change in the interest rate affects the demand for capital only when the market is thick. When the market is thin (low demand), a change in the interest rate has no effect on demand and production of capital.*

Remark 2 *The variance of production decreases when the interest rate increases (β decreases), since the resulting lower value of the inventory target ($\bar{z}$) leads to a higher probability of stockouts, increasing the likelihood of a thick market. Given that production is constant when $\theta > \bar{z}$, the variance of output decreases:*

$$\sigma_y^2 = P\delta^2 var(k^*(\theta_{t-1})),$$

where $P \equiv \Pr[\theta \leq \bar{z}]$ falls due to a lower $\bar{z}$. The same is also true for the volatility of the capital stock. Hence, a high interest rate period corresponds to *a less volatile capital market. This is because the adjustment of the inventory target (due to a change in the interest rate) changes the probability distribution of thin and thick market, hence the volatilities of equilibrium demand and supply change accordingly.*

Remark 3 *Capital price is more sensitive to an interest rate change when the market is thin than when it is thick. This can be seen in the derivative of the price of capital with respect to* β*:*

$$\frac{\partial \lambda}{\partial \beta} = \begin{cases} a & , \textit{if } \theta_t \leq \bar{z} \\ f''_{kk} \frac{\partial k^*(\bar{z})}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial \beta} + (1-\delta)a & , \textit{if } \theta_t > \bar{z} \end{cases}$$

where $f''_{kk} \frac{\partial k^*(\bar{z})}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial \beta} + (1-\delta)a < a$ *since* $f''_{kk} < 0$ *and* $\frac{\partial k^*(\bar{z})}{\partial \bar{z}} \frac{\partial \bar{z}}{\partial \beta} > 0$*. Also, the volatility of capital price increases as interest rate rises. This implication stems also from the fact that the inventory target level decreases with the interest rate, hence the non-negativity constraint on inventories is easier to bind under high interest rate than under low interest rate, raising the probability of a thick market.*

4.2 A Change in the Rate of Capital Depreciation

Capital depreciation can occur for three reasons: First, capital simply wears out through use. Second, capital breaks down due to accidents or poor quality. Third, capital becomes less productive through technological obsolescence. In all cases, the productivity of capital decreases through depreciation. Although on surface, the rate of capital depreciation appears to be exogenous to firms, but in reality it is often firms' profit maximization considerations that determine whether old capital should retire or not. In addition, government tax policies may also induce firms to retire existing capital at an earlier or latter stage.

A change in the rate of capital depreciation, whether due to natural or economic reasons, can have important effects on the production and inventory behavior of capital, since it affects the demand for capital. There are two opposite effects regarding how the rate of depreciation affects capital demand. The first pertains to a user's-cost effect and the second pertains to an intertemporal substitution effect. A lower depreciation rate on the one hand increases capital demand due to a lower user's cost, but on the other hand it decreases the expected future capital demand since capital is replaced less frequently when it lasts longer.

Consequently, depending on the relative strength of the two effects, demand and supply of capital may be positively or negatively affected by a change in the depreciation rate. The following proposition shows that depreciation rate can have an unambiguous effect on the relative volatility of production and sales of capital.

Proposition 4 *The volatility of production relative to that of sales increases as the depreciation rate increases.*

Proof. Denote $P \equiv \Pr[\theta \leq \bar{z}]$ and denote σ_k^2 as the variance of capital. Then the variance of production and sales (investment demand) are given respectively by

$$\sigma_y^2 = P\delta^2\sigma_k^2$$

$$\sigma_I^2 = P^2\left[1 + (1-\delta)^2\right]\sigma_k^2 + P(1-P)\left[1 + (1-\delta)^2\right]\sigma_k^2 = P\left[1 + (1-\delta)^2\right]\sigma_k^2$$

Hence the variance ratio of production to sales is given by

$$\frac{\sigma_y^2}{\sigma_I^2} = \frac{\delta^2}{1 + (1-\delta)^2}$$

which is strictly less than one and strictly increasing in δ.■

The intuition behind this proposition is as follows. As plans for current production cannot be altered, any rise in current sales must be satisfied entirely by a reduction in inventories. On its own, this implies a one-for-one rise in the production committed for the next period to replenish the depleted inventory stock. However, if goods are durable, increased purchase in the current period raises buyers' stock of goods available for subsequent periods, reducing the anticipated increase in future sales, and hence the response in production as well. This proposition suggests that, everything else equal, durability *per se* is not the source of the more volatile production in the durable goods sector in comparison to the non-durable goods sector.

Proposition 4 deals only with the relative volatility of production to sales. What happens to the absolute variance of production, however, depends on the details of the model, in particular, the specific functional forms of $f(k,\theta)$. This is so not only because a higher value of δ increases the user's cost of capital, lowering the optimal demand for capital and reducing the volatility of k^*, hence $\frac{\partial \sigma_k^2}{\partial \delta} < 0$, but because the optimal inventory target ($\bar{z}$) may also be affected by δ, causing the probability of a stockout to change as δ changes. In other words, the total effect of a change in δ on the volatility of production is given by three terms,

$$\frac{\partial \sigma_y^2}{\partial \delta} = 2\delta P \sigma_k + \delta^2 \frac{\partial \sigma_k^2}{\partial \delta} P + \delta^2 \sigma_k^2 \frac{\partial P}{\partial \delta},$$

where the first term shows a direct positive effect of δ on the volatility of y due to the intertemporal substitution effect of durability on future demand (i.e., a higher δ raises the anticipated future demand for capital), the second term shows a negative effect of δ on the volatility of y due to the user's cost effect (i.e., a higher δ lowers the current demand for capital), and the third term shows the effect of δ on the firm's inventory target policy ($\bar{z}$), which can no longer be unambiguously determined using equation (13) because according to (8) we have $k^*(z) = y_t + s_{t-1} + (1-\delta)k_{t-1}$, thus a larger δ has a negative effect on k^*, hence a positive effect on the marginal product of capital, causing the right hand side of equation (13) to increase. But a higher δ also raises the left hand side of (13), hence it is not clear whether $\bar{z}$ should increase or decrease based on equation (13). Thus, the effect of δ on the volatilities of capital supply cannot be explicitly determined unless further details are given for the demand function of capital, $k^*()$, and the probability distribution function of θ, $\phi()$.

In what follows I give two examples. The first example shows clearly that the absolute volatility of production decreases as the durability of goods increases. The second example shows the possibility that the opposite may be true.

Economy 1: The production function for the buyer is given by the quadratic form,

$$f(k_t, \theta_t) = \theta_t k_t - \frac{1}{2} k_t^2.$$

Proposition 5 *In this economy the equilibrium decision rules for demand, supply, inventory investment and market price of capital are given by*

$$k_t = \begin{cases} \theta_t - \beta\delta a \text{ if } \theta_t \leq \bar{z} \\ \bar{z} - \beta\delta a \text{ if } \theta_t > \bar{z} \end{cases}$$

$$I_t = \begin{cases} \theta_t - (1-\delta)\theta_{t-1} - \beta\delta^2 a & \text{if } \theta_t \leq \bar{z} \text{ \& } \theta_{t-1} \leq \bar{z} \\ \theta_t - (1-\delta)\bar{z} - \beta\delta^2 a & \text{if } \theta_t \leq \bar{z} \text{ \& } \theta_{t-1} > \bar{z} \\ \bar{z} - (1-\delta)\theta_{t-1} - \beta\delta^2 a & \text{if } \theta_t > \bar{z} \text{ \& } \theta_{t-1} \leq \bar{z} \\ \delta\bar{z} - \beta\delta^2 a & \text{if } \theta_t > \bar{z} \text{ \& } \theta_{t-1} > \bar{z} \end{cases}$$

$$y_t = \begin{cases} \delta\theta_{t-1} - \beta\delta^2 a & \text{if } \theta_{t-1} \leq \bar{z} \\ \delta\bar{z} - \beta\delta^2 a & \text{if } \theta_{t-1} > \bar{z} \end{cases}$$

$$s_t = \begin{cases} \bar{z} - \theta_t & \text{if } \theta_t \leq \bar{z} \\ 0 & \text{if } \theta_t > \bar{z} \end{cases}$$

$$\lambda_t = \begin{cases} \beta a & \text{if } \theta_t \leq \bar{z} \\ \theta_t - \bar{z} + \beta a & \text{if } \theta_t > \bar{z} \end{cases}.$$

Proof. The marginal product of capital is given by $\theta_t - k_t$ and the capital demand function $k^*()$ is given by

$$k^*(x) = x - \beta\delta a$$

where $x = \theta$ in case there is no stockout ($\theta \leq \bar{z}$) and $x = \bar{z}$ in case there is a stockout ($\theta > \bar{z}$). Substituting $k^*(x)$ into the decision rules in proposition 2 gives the desired results.■

Proposition 6 *In this economy the inventory target, $\bar{z}$, is independent of δ.*

Proof. Applying the decision rule for k_t, equation (11) now becomes,

$$\begin{aligned}(1-\beta)a &= \int_z^B \left[\theta_t - k^*(z) - \beta\delta a\right]\phi(\theta)d\theta \\ &= \int_{\bar{z}}^B \left[\theta - \bar{z}\right]\phi(\theta)d\theta.\end{aligned}$$

Clearly, $\bar{z}$ is independent of δ.■

Thus, the parameter $P \equiv \Pr\left[\theta \leq \bar{z}\right]$ is also independent of δ. Based on the decision rules, the variances of demand and production can be found as:

$$\sigma_I^2 = P\left[1 + (1-\delta)^2\right]\sigma_\theta^2$$

$$\sigma_y^2 = P\delta^2\sigma_\theta^2$$

Since P is independent of δ, we have

$$\frac{\partial \sigma_I^2}{\partial \delta} < 0 \text{ and } \frac{\partial \sigma_y^2}{\partial \delta} > 0.$$

Namely, despite that the variance of capital demand decreases as the depreciation rate increases (indicating a strong user's cost effect), the variance of production increases nonetheless, indicating that the intertemporal substitution effect dominates the user's cost effect on production.

Economy 2: The production function for the buyer is given by the constant elasticity form,

$$f(k_t, \theta_t) = \frac{\theta_t^{\gamma} k_t^{1-\gamma}}{1-\gamma}, \quad 1 \geq \gamma \geq 0.$$

Proposition 7 *In this economy (economy 2) the equilibrium decision rules for demand, supply, inventory investment and market price of capital are given by*

$$k_t = \begin{cases} \left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \theta_t \textit{ if } \theta_t \leq \bar{z} \\ \left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \bar{z} \textit{ if } \theta_t > \bar{z} \end{cases}$$

$$I_t = \begin{cases} \left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \theta_t - (1-\delta)\left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \theta_{t-1} & \textit{if } \theta_t \leq \bar{z} \,\&\, \theta_{t-1} \leq \bar{z} \\ \left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \theta_t - (1-\delta)\left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \bar{z} & \textit{if } \theta_t \leq \bar{z} \,\&\, \theta_{t-1} > \bar{z} \\ \left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \bar{z} - (1-\delta)\left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \theta_{t-1} & \textit{if } \theta_t > \bar{z} \,\&\, \theta_{t-1} \leq \bar{z} \\ \delta\left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \bar{z} & \textit{if } \theta_t > \bar{z} \,\&\, \theta_{t-1} > \bar{z} \end{cases}$$

$$y_t = \begin{cases} \delta\left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \theta_{t-1} \textit{ if } \theta_{t-1} \leq \bar{z} \\ \delta\left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \bar{z} \quad \textit{if } \theta_{t-1} > \bar{z} \end{cases}$$

$$s_t = \begin{cases} \left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} (\bar{z} - \theta_t) \textit{ if } \theta_t \leq \bar{z} \\ 0 \quad \textit{if } \theta_t > \bar{z} \end{cases}$$

$$\lambda_t = \begin{cases} \beta a & \textit{if } \theta_t \leq \bar{z} \\ \left(\left[\frac{\theta_t}{\bar{z}}\right]^{\gamma} - 1\right)\beta\delta a + \beta a \textit{ if } \theta_t > \bar{z} \end{cases}.$$

Proof. The marginal product of capital is given by $\left(\frac{\theta}{k}\right)^{\gamma}$ and the capital demand function is given by

$$k^*(x) = \left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} x$$

where $x = \theta$ in case there is no stockout $(\theta \leq \bar{z})$ and $x = \bar{z}$ in case there is a stockout $(\theta > \bar{z})$. Substituting $k^*(x)$ into the decision rules in proposition 2 gives the desired results.■

Proposition 8 *In economy 2 the inventory target, $\bar{z}$, positively depends on δ.*

Proof. Applying the decision rule for k_t in this economy to equation (11) gives,

$$\begin{aligned}
(1-\beta)a &= \int_z^B \left[\left(\frac{\theta_t}{k^*(\bar{z})}\right)^{\gamma} - \beta\delta a\right] \phi(\theta) d\theta \\
&= \int_z^B \left[\left(\frac{\theta_t}{\left[\frac{1}{\beta\delta a}\right]^{\frac{1}{\gamma}} \bar{z}}\right)^{\gamma} - \beta\delta a\right] \phi(\theta) d\theta \\
&= \int_z^B \beta\delta a \left[\left(\frac{\theta_t}{z}\right)^{\gamma} - 1\right] \phi(\theta) d\theta,
\end{aligned}$$

which can also be expressed as

$$\frac{(1-\beta)}{\beta\delta} = \int_{\bar{z}}^B \left[\left(\frac{\theta_t}{\bar{z}}\right)^{\gamma} - 1\right] \phi(\theta) d\theta.$$

Since the right hand side is decreasing in $\bar{z}$, thus $\bar{z}$ positively depends on δ.■

Thus, the probability measure, $P \equiv \Pr[\theta \leq \bar{z}]$, also positively depends on δ. Based on the decision rules, the variances of investment demand and production can be found as:

$$\sigma_I^2 = \left[\frac{1}{\delta}\right]^{\frac{2}{\gamma}} \left[\frac{1}{\beta a}\right]^{\frac{2}{\gamma}} \left[1 + (1-\delta)^2\right] P\sigma_\theta^2$$

$$\sigma_y^2 = \left[\frac{1}{\delta}\right]^{\frac{2(1-\gamma)}{\gamma}} \left[\frac{1}{\beta a}\right]^{\frac{2}{\gamma}} P\sigma_\theta^2$$

Clearly, holding P constant, we have

$$\frac{\partial \sigma_I^2}{\partial \delta} < 0 \text{ and } \frac{\partial \sigma_y^2}{\partial \delta} < 0.$$

Hence, as long as P does not increase too fast when δ increases, the volatility of both investment demand and capital production may both decrease as the depreciation rate increases, provided that the inventory target does not move substantially with δ and/or the cumulative density function for θ is sufficiently flat near $\bar{z}$. This situation is certainly a possibility. Nonetheless, the volatility ratio of production to sales is still given by $\frac{\delta^2}{1+(1-\delta)^2} < 1$, hence the relative volatility of production to sales will still be an increasing function of δ, suggesting that the volatility of production increases faster than that of sales as the rate of capital depreciation increases.

5 Conclusion

The demand side of the capital market has been intensively studied by the literature and hence relatively well understood, but the supply side of the capital market has been largely neglected. Since, in equilibrium, demand equals supply, understanding the supply side of the capital market is no less important than understanding the demand side. Capital is a special type of durable good (it is the reproductive force of the economy), and the production of capital takes time (e.g., according to Kydland and Prescott, 1982, the average time period for capital production is about 4 quarters). Thus to understand how investment demand, one of the most volatile economic variables over the business cycle, is satisfied by national savings in equilibrium, an understanding of the production and inventory behavior of capital is essential. This paper showed that production and inventory behavior of capital suppliers can dramatically alter the equilibrium dynamics of the capital market. In particular, due to capital suppliers' strategic production and inventory behavior, equilibrium investment demand becomes more volatile, capital price becomes downward sticky, and the responses of the capital market towards policy shocks become asymmetric. In particular, a change in the interest rate has little effect on the capital market if it is thin. Policy tends to be more effective at influencing equilibrium investment only if the market is thick.

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