Measuring Interest Rates as Determined by Thrift and Productivity

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Measuring interest rates as determined by thrift and productivity

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ABSTRACT

This paper investigates the behavior of short-term real and nominal rates of interest by combining consumption-based and production-based models into a single general equilibrium framework. Based on the theoretical *nonlinear* relationships that link interest rates to both the *marginal rates of substitution* and *transformation* in a monetary production economy, we develop an estimation and simulation procedure to generate historical time series of interest rates. We find that the predictions of interest rates based on a general equilibrium theory are partially consistent with US data.

Keywords: Interest rate; Marginal rate of substitution; Marginal rate of transformation; General equilibrium; Asset pricing

JEL Classification: **C22** Time series models ; **E43** Determination of interest rate

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1. Introduction

This paper attempts to explain the dynamic behavior of interest rates in the US economy using an equilibrium theory. We investigate the behavior of interest rates from two directions of the economy: *thrift* and *productivity*. The first direction comes from a relationship between interest rates and (intertemporal) marginal rates of substitution (MRS). It is thrift that links the supply of funds to interest rates. The second one comes from a link between interest rates and marginal rates of transformation (MRT) that depends on the marginal productivity of capital. The equilibrium rate is determined at a level that balances thrift and productivity.

These two directions, although they have been examined *separately* in the asset pricing literature, have not been examined simultaneously using a single general equilibrium framework. Most existing asset pricing models use a consumption-based approach and focus only on the MRS in an endowment economy (see Kocherakota, 1996). There has been little attention paid to the dynamics of interest rates from the point of view of the MRT using a production-based approach. The important exceptions are studies by Cochraine (1991, 1996), who uses the production-based approach. This work, however, is based on partial equilibrium models.[1] The MRT aspect of interest rates is interesting to consider because it reflects very different mechanisms of interest rate dynamics from those implied by the MRS. By examining these two directions simultaneously in a single general equilibrium framework,[2] we may also be able to identify any inconsistency between the implications of MRS and MRT in a general equilibrium theory. In this paper, we use a limited participation model for a monetary production economy to examine interest rate behavior from the viewpoint of consumers (MRS) and producers (MRT) at the same time.

In a monetary economy, interest rates depend not only on the fundamentals of thrift and productivity but also on monetary factors, which operate together in combined forms. What we observe is the nominal interest rate that can be decomposed into the real interest rate and the

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[1] Using a production (or investment)-based model, Cochrane (1991, 1996) explains the market return with an investment return, inferred from investment data with an adjustment cost production function. The investment return is constructed from a regression analysis, i.e., regressing the investment return on its motivating factors implied by the producer's first-order conditions.

[2] Den Haan (1995) considers a simple production economy that can generate persistence in the interest rate and the slope of the term structure. However, he looks at only one direction of the production economy and does not examine the historical movement of interest rates.

expected inflation. By linking up with consumption side and production side of the economy, we explain the behavior of interest rates by the fundamentals that are entangled with the inflation process. The nominal interest rates will eventually rise by the same amount as any increase in fully anticipated inflation through the so-called Fisher effect. The natural rate of interest reflects the equality between saving and investment, being determined by both thrift and productivity. The market interest rate is affected by the excess demand for the loanable funds that are provided by the banking system, as implied by the liquidity effect.[3]

This paper sheds light on the question of whether or not the general equilibrium framework utilized extensively in the business cycle literature can predict the historical movement of interest rates. It is well known that interest rates are leading indicators of business cycles (e.g., Bernanke and Blinder, 1992). However, most existing studies that attempt to explain the behavior of interest rates based on a general equilibrium framework have focused on the mean and variance (e.g., Weil, 1989; Cecchetti et al., 1993; Abel, 1994; Den Haan, 1995) or covariation with inflation (Giovannini and Labadie, 1991), not on the dynamic properties of interest rates, which would have important implications for investment and output fluctuations. This paper also provides some insights on the risk-free rate puzzle by looking at the production side of the economy.

We start our investigation by deriving the exact theoretical relationships that link interest rates to the MRS on the one hand and to the MRT on the other, using a monetary general equilibrium model featuring the liquidity effect (e.g., Lucas, 1990; Christiano, 1991; Fuerst, 1992; Dow, 1995). We then calibrate the nominal and real interest rates using the actual time series on the fundamentals (such as the consumption growth, inflation, and marginal product of capital) to investigate whether the calibrated MRS and MRT explain the historical movements of the rate of return on the three-month Treasury bill.[4] Since these relationships are highly nonlinear and they involve conditional expectations, we use a simulation method to calibrate theoretical interest rates.

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[3] Saving equals the supply of loanable funds by households and investment equals the demand for loanable funds by firms. If the banking system just intermediates household saving without generating any net injection of loanable funds on its own, the economy will adjust so that the interest rate is driven to its natural level.

[4] Although MRS = MRT in equilibrium, they may nevertheless give very different predictions of calibrated interest rates as they embody different sets of fundamentals and thus contain different information sets about the actual economy , in addition to Euler equation errors.

We examine the consistency between the model prediction and actual data in terms of autocorrelation and spectral density functions and further investigate whether our model can predict the historical movements of the data. We find that, for the real rate, MRT explains the dynamics of the data remarkably well and outperforms MRS. We also find that, for the nominal rate, both MRT and MRS have explanatory power at the business cycle frequency but not at higher frequencies.

The results suggest that our attempt to explain the interest rate behavior by jointly examining consumers and producers' viewpoints is worthwhile and that the dynamic properties of interest rates can be explained to a substantial extent by the fundamentals of thrift and productivity. Nonetheless, there still remains a long way to go for the general equilibrium theory to explain the short-term (high frequency) dynamics of nominal interest rates.

The remainder of this paper is organized as follows. Section 2 describes the relationships linking interest rates to MRS and MRT and derives the stochastic process of endogenous variables from a general equilibrium model. Section 3 develops estimation and simulation methods for calibrating interest rates series. Section 4 presents our empirical results with discussion after a brief description of the data construction. Section 5 concludes.

2. Explaining Interest Rates as Determined by Thrift and Productivity

We use a standard monetary general equilibrium model based on Lucas (1990), Christiano (1991), Fuerst (1992), and Dow (1995). We assume a closed economy since we examine the interest rate behavior in the US economy, which is large enough to abstract influences from abroad. In a growing monetary production economy with portfolio rigidity and capital accumulation, cash-in-advance constraints (CIA) are imposed on all types of transactions, and households choose portfolios before the current state is known. The economy is lumped into families consisting of multiple members, one of which is the financial intermediary that creates the newly injected cash in the financial market.[5] The model is driven by monetary shocks that call for the liquidity and Fisher effects on interest rates.[6]

2.1. The Model

The population consists of many identical families. The number of members of a family, n_t, grows with growth factor η, while the number of families is fixed. The representative family (household) has preferences over uncertain consumption and leisure streams given by

$$E_0\left\{\sum_{t=0}^{\infty}\beta^t\eta^t\{U(c_t)+\ln(1-l_t)\}\right\},\quad \beta>0, \tag{2.1}$$

where E_0 is the expectations operator conditional on period 0, β is the subjective discount factor, c_t is *per capita* consumption, l_t is work effort (leisure endowment is normalized to one), and $U(c_t)=(c_t^{1-\gamma}-1)/(1-\gamma)$ with the relative risk aversion coefficient of γ.

A multiple-member household consists of a shopper, a worker, a firm and a financial intermediary (bank). First, each firm has access to a production technology. With a constant depreciation rate $\delta\in(0,1)$, the law of motion for the *per capita* capital stock is

$$\eta k_{t+1}=(1-\delta)k_t+i_t, \tag{2.2}$$

where i_t is investment. Let the labor in efficiency units grow at the rate $\upsilon-1$. Then the effective labor unit in period t is given by $h_t^E=\upsilon^t h_t$, where h_t is the labor input. The firm produces output, q_t, by means of a Cobb-Douglas function:[7]

$$q_t=f(k_t,h_t)=\upsilon^{(1-\alpha)t}k_t^{\alpha}h_t^{1-\alpha},\alpha\in(0,1).$$

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[5] As in Lucas (1990), the effect of monetary injections are *symmetric across* families, but *asymmetric within* the family since only firms are forced to absorb monetary shocks introduced via the financial intermediary. For the literature on the non-neutrality generated from the asymmetry, see Fuerst (1992) and Christiano (1991).

[6] The money supply change has the first-round loanable funds effect through changes in the excess supply of loanable funds (Friedman and Schwartz, 1982, Ch 10). The liquidity effect represents that an exogenous increase in the money supply forces down the interest rate through agents' portfolio adjustments. Since these two effects cannot be separated from each other in the actual data, we shall refer to them simply as the liquidity effect.

[7] Our model has a balanced-growth equilibrium. The log of output has a trend component and preferences are restricted so that technological progress has no long–run effect on labor supply. For specifications of balanced-growth models, see King et al. (1988) and Cooley and Prescott (1995). This paper, for simplicity, abstracts from productivity shocks. The incorporation of stochastic productivity shocks does not affect our main results.

Second, each household consists of a worker/shopper pair: workers sell their labor services to firms, and shoppers purchase goods from firms. Third, money is introduced via CIA constraints on all transactions. Fourth, a household starting with J_t dollars chooses to deposit D_t of these balances in the financial intermediary before the state of the world for period t is revealed. After this deposit is made, the family separates and the worker travels to the labor market, while both the intermediary and firm travel to the credit market. Fifth, after separation, the state determines the monetary injection (X_t) that is given to each financial intermediary. The representative financial intermediary has $D_t + X_t$ dollars to lend out. The firm borrows B_t at the nominal interest rate R_t from the intermediary. The firm then hires workers at the wage W_t and sells the current product in the goods market at the price P_t.

Furthermore, we assume that the firm sells $(1-\delta)k_t$ existing net capital and purchases ηk_{t+1} future capital stock by the end of period t (i.e., selling and repurchasing the existing capital stock). The firm borrows from the bank the amount to purchase investment goods ($\eta k_{t+1} - (1-\delta)k_t$) and to pay wages. The firm pays the rental cost for the existing capital stock during the period of producing and selling of its product.

Let M_t be the *economy-wide per capita* money stock. The economy-wide per capita money stock, M_t, follows the law of motion given by

$$\eta M_{t+1} = M_t(1+x_t), \tag{2.3}$$

where x_t, the per capita money stock growth, follows a stochastic process that will be specified later. We measure all nominal variables in period t relative to the start-of-period *aggregate* money stock per family, $n_t M_t$. Then $m_t = J_t / (n_t M_t)$ denotes a household's per capita money holdings relative to the economy-wide per capita money holdings. Similarly, define $d_t = D_t / (n_t M_t)$, $b_t = B_t / (n_t M_t)$, $w_t = W_t / P_t$, $p_t = P_t / (n_t M_t)$, and $x_t = X_t / (n_t M_t)$.

Let $V(m,k,\kappa)$ denote the maximized objective function for the representative household that begins a period with m cash balance, k capital stock, and κ, economy-wide per capita capital stock. Now the dynamic optimization motivates the Bellman equation:

$$V(m_t,k_t,\kappa_t) = \max_{\{d_t\}} E_{t-1}\left[\max_{\{c_t,h_t,b_t,l_t,k_{t+1}\}} E_t\{U(c_t) + W(1-l_t) + \beta\eta V(m_{t+1},k_{t+1},\kappa_{t+1})\}\right] \tag{2.4}$$

subject to:

$$m_t - d_t \geq p_t c_t, \tag{2.5}$$

$$b_t \geq p_t(\eta k_{t+1} - (1-\delta)k_t) + w_t h_t, \tag{2.6}$$

$$\begin{aligned} m_{t+1} = \{ & m_t + d_t R_t + w_t l_t - p_t c_t + x_t(1+R_t) \\ & + [p_t f(k_t, h_t) - w_t h_t - p_t(\eta k_{t+1} - (1-\delta)k_t) - b_t R_t]\}/(1+x_t). \end{aligned} \tag{2.7}$$

CIA constraints (2.5)–(2.6) apply to the shopper and to the firm, respectively. The law of motion for money balance is given by Eq. (2.7). The cash balance that a family will have at the start of period $t+1$ is contributed by the worker/shopper ($m_t + d_t R_t + w_t h_t - p_t c_t$), the intermediary ($x_t(1+R_t)$), and the firm [$p_t f(k_t, h_t) - w_t h_t - p_t(\eta k_{t+1} - (1-\delta)k_t) - b_t R_t$].

2.2. Marginal Rate of Substitution and Marginal Rate of Transformation

On the supply side of loans, the interest rate is linked to the (intertemporal) marginal rates of substitution (MRS) in consumption between periods t and $t+1$. From the equilibrium conditions, we have the relationship that links interest rates to the MRS (see Appendix A):

$$1 + R_t = [\Lambda_t + U'(c_t)/P_t]/[\beta\eta E_t\{U'(c_{t+1})/P_{t+1}\}], \tag{2.8}$$

where $\Lambda_t = (\lambda_{2t} - \lambda_{1t})/(n_t M_t)$ measuring the liquidity effect, λ_1 is the liquidity cost incurred by not holding cash, and λ_2 summarizes the borrowing cost of the firm (λ_1 and λ_2 denote the Lagrange multipliers associated with Eqs. (2.5) and (2.6), respectively). The banks decide the bank rate and provide signals to the differentiated groups of the economy via the interest rate. Without an accurate forecast, the value of money will not be equally valued in the goods and financial markets, i.e., $\lambda_{1t} \neq \lambda_{2t}$. Thus, the liquidity effect arises from monetary injections through the banks and the forecast errors attributed to an information structure wherein households make portfolio choices before all state information is available.

One can rewrite Eq. (2.8) as the Fisherian decomposition of the nominal rate into the real rate and expected inflation treating the unobservable liquidity effect as a stochastic error:

$$1 + R_t = U'(c_t) \Big/ \left[\beta\eta E_t\{U'(c_{t+1}) \cdot \frac{P_t}{P_{t+1}}\}\right] + \xi_t, \tag{2.9}$$

where ξ_t is a stochastic error with zero mean due to the liquidity effect.

On the other hand, the demand for loans depends on the evaluated return from investment in physical capital and the opportunity cost of borrowing in the financial market. From the equilibrium conditions, we obtain the following relationship (see Appendix A):

$$(1+R_t)p_t V_m(k_{t+1})/(1+x_t) = E_t\{(\eta + f_{k,t+1} - \delta)p_{t+1}\beta\eta V_m(k_{t+2})/(1+x_{t+1})\}, \qquad (2.10)$$

where $V_m(k_{t+1}) = E_t\{U'(c_{t+1})/p_{t+1}\}$. This represents the condition that the evaluated return from current investment equals the cost of the borrowed cash balance. Suppose a firm borrows P_t dollars from the bank at the cost of interest R_t per dollar. If the firm uses borrowed money for production, it will have, at time $t+1$, the expected cash flow of $E_t[(\eta + f_{k,t+1} - \delta)P_{t+1}]$ that can be used for future purposes with the per dollar gain summarized by $\beta\eta V_m(k_{t+2})/(1+x_{t+1})$. Note that it is not until the start of period $t+2$ that the firm can use this return for future purposes.[8] Then equilibrium conditions require that the expected value of the marginal productivity of capital net of depreciation that can be used for future purposes at the start of period $t+2$ should be equal to the value of borrowed funds that can be used for future purposes at the start of period $t+1$.

Using $p_t = P_t/(n_t M_t)$, $n_{t+1} = \eta n_t$, and Eq. (2.3), we can rewrite Eq. (2.10) to obtain the relation that relates the interest rate to the marginal rate of transformation (MRT):

$$1+R_t = \beta\eta E_t\left\{(\eta + f_{k,t+1} - \delta)\frac{P_{t+1}}{P_{t+2}}U'(c_{t+2})\right\} \Big/ E_t\left\{\frac{P_t}{P_{t+1}}U'(c_{t+1})\right\}. \qquad (2.11)$$

Returns from alternative uses of cash relate the interest rate to the thrift factor (MRS) and productivity factor (MRT). The equilibrium rate will be determined at the level that satisfies the condition MRS$=$MRT, which corresponds to the *Keynes-Ramsey* rule under certainty.[9]

3. Methodology of Generating Interest Rate Series

[8] For the return on the investment to be liquidated, two periods are required: one for the gestation period of capital and the other for selling the product and distributing the return.

[9] The rate of return on deposit (R_t) is taken to be the risk-free, nominal interest rate. The introduction of a nominal discount bond in zero net supply does not affect conditions (2.9) and (2.11), and renders R_t equal to the rate of return on the (default-free) bond. Hence R_t is considered to be the market interest rate.

According to Eqs. (2.9) and (2.11), the nominal interest rate can be explained in two ways. One pertains to the consumption-based approach that links the interest rate to the thrift factor. Another pertains to the production-based approach that links the interest rate to the productivity factor. In this section, we test whether these approaches are consistent with US data. Namely, we calibrate interest rates according to Eqs. (2.9) and (2.11) using the actual data on the consumption growth, inflation, and marginal product of capital, and compare the calibrated series with the actual series. The (ex-ante) real rate is then determined by deflating the nominal rate by the expected inflation factor.

Although the theoretical relationships involve nonlinear expectations, the conditional normality of the consumption growth, inflation, and output-capital ratio in logarithm, consistent with our theoretical model, helps reduce them to linear expectations involving only the conditional mean and covariance of the observables.[10] We then use a vector autoregression (VAR) model to estimate the conditional mean and covariance of the observables. To ensure full consistency between theory and estimation, we impose the VAR structure implied by our theoretical model on the data in estimation. Namely, we estimate the constrained vector autoregressive moving average (CVARMA) model that is implied by the autoregressive moving average (ARMA) structure of the control variables. The CVARMA generates forecasts for the consumption growth, inflation, and capital-output ratio, from which we compute conditional expectations and a covariance matrix of forecast errors for relevant variables. Finally, predictions are made about interest rates by simulating the theoretical relationships after replacing conditional expectations and covariance with their estimates.

3.1. ARMA Structure of the Theoretical Model

To derive the stochastic processes of variables on which the conditional expectations are based, consider the log-linearized equilibrium conditions of the model. The information set at the beginning of period t includes the stock of capital, k_t, and the exogenous money growth shock, x_t. Since the decision on the deposit, d_t, has to be made before the shock is realized, it is a function of k_t and x_{t-1},

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[10] This approach to calibrate a time series is in line with previous studies such as Cochrane (1991) and Watson (1993). Assuming the log-normality of the driving shocks of the economy, the Euler equations are approximated as log-linear functions for endogenous variables (e.g., Campbell 1994), which are expressed as a CVARMA model in this paper.

abstracting from a constant term for simplicity. As a result, the state space of the model contains innovations realized in both time period t and period t-1. The state transition equation for k_{t+1} has the form: $\ln k_{t+1} = \phi_k \ln k_t + \phi_{kx1} x_t + \phi_{kx2} x_{t-1}$. The decision rule for a control variable has the form: $\ln z_t = \phi_{zk} \ln k_t + \phi_{zx1} x_t + \phi_{zx2} x_{t-1}$, where z_t represents any control variable in the system. To derive the ARMA structure for the controls, we substitute out the state $\ln k_t$ in the control equation using the state transition equation to get

$$\ln z_t = \phi_k \ln z_{t-1} + \phi_{zx1} x_t + (\phi_{zx2} + \phi_{zk}\phi_{kx1} - \phi_k \phi_{zx1}) x_{t-1} + (\phi_{zk}\phi_{kx2} - \phi_k \phi_{zx2}) x_{t-2}\,. \tag{3.1}$$

The monetary shock is assumed to follow a first-order autoregressive, AR(1), process as in Christiano (1991) and Christiano and Eichenbaum (1992):

$$x_t = \rho x_{t-1} + \varepsilon_t\,. \tag{3.2}$$

Then the equation for z_t becomes an ARMA(2,2) given by

$$\begin{aligned} \ln z_t = {} & (\rho + \phi_k) \ln z_{t-1} - \rho\phi_k \ln z_{t-2} + \phi_{zx1}\varepsilon_t + (\phi_{zx2} + \phi_{zk}\phi_{kx1} - \phi_k\phi_{zx1})\varepsilon_{t-1} \\ & + (\phi_{zk}\phi_{kx2} - \phi_k\phi_{zx2})\varepsilon_{t-2}\,. \end{aligned} \tag{3.3}$$

Notice that all of the control variables share the same autoregressive coefficients but different moving average coefficients, and that the dynamic structure of the exogenous shock (reflected by ρ) affects only the autoregressive coefficients, not the moving average coefficients.[11] Since x is eliminated in Eq. (3.3), we can avoid measuring the (exogenous) money supply.

For the purpose of our current analysis, we form a VAR for the growth rate of consumption, the inflation rate, and the output-capital ratio. Appendix C shows that the stochastic processes of these variables can be expressed in a CVARMA (2,3) model.

3.2. MRS-based and MRT-based Interest Rates

[11] If the monetary shock follows a higher order autoregressive process, the ARMA structure in Eq. (3.3) will be affected. We find, however, that this consideration does not affect our empirical results qualitatively.

The assumption of conditional normality about the fundamentals helps reduce nonlinear expectations in the theoretical MRS and MRT to the linear ones involving only conditional mean and variance of the observables. Consider the stochastic process given by

$$y_{i,t+1} \mid \Omega_t \sim N(E_t(y_{i,t+1}), V_t(y_{i,t+1})), \tag{3.4}$$

where y_{it} is a time series from the variable set $\{\Delta \ln c_t, \Delta \ln P_t, \ln(q_t / k_t)\}$, Ω_t is the information set available at time t, and $E_t(\cdot)$ and variance $V_t(\cdot)$ are the conditional mean and variance, respectively. Note that $E_t\{\exp(y_{i,t+1})\} = \exp\{E_t(y_{i,t+1}) + 0.5V_t(y_{i,t+1})\}$ from the conditional lognomality. The log-linearized equilibrium relationships described in Section 3.1 and Appendix C imply that the consumption growth, inflation and the logarithm of the output-capital ratio conditional on the information set are jointly normally distributed.

To calibrate the MRS-based nominal rate that relies on the consumption side, we rewrite Eq. (2.9) using Eqs. (2.1) and (3.4) as:

$$\begin{aligned} 1 + R_t^{MRS} &= [\beta\eta\, E_t\{\exp(-\gamma\Delta \ln c_{t+1} - \Delta \ln P_{t+1})\}]^{-1} + \xi_t \\ &= (\beta\eta)^{-1} \exp\{\gamma E_t(\Delta \ln c_{t+1}) + E_t(\Delta \ln P_{t+1}) - 0.5\gamma^2 V_t(\Delta \ln c_{t+1}) \\ &\qquad - 0.5V_t(\Delta \ln P_{t+1}) - \gamma Cov_t(\Delta \ln c_{t+1}, \Delta \ln P_{t+1})\} + \xi_t. \end{aligned} \tag{3.5}$$

To calibrate the MRT-based nominal rate that relies on the production side, we can express Eq. (2.11) using Eqs. (2.1) and (3.4) as:

$$1 + R_t^{MRT} = \frac{\beta E_t\{\exp(\ln NMPK_{t+1} - \gamma\Delta \ln c_{t+2} - \gamma\Delta \ln c_{t+1} - \Delta \ln P_{t+2})\}}{E_t\{\exp(-\gamma\Delta \ln c_{t+1} - \Delta \ln P_{t+1})\}}, \tag{3.6}$$

where $NMPK_{t+1} \equiv (\eta + f_{k,t+1} - \delta)$. The calibrated ex-ante real rate is given by:

$$1 + rr_t^* = (1 + R_t) E_t(\frac{1}{1 + \pi_{t+1}}) = (1 + R_t) \exp\{-E_t(\Delta \ln P_{t+1}) + 0.5V_t(\Delta \ln P_{t+1})\}, \tag{3.7}$$

where R_t is either R_t^{MRS} or R_t^{MRT} defined by Eqs. (3.5) or (3.6). Hence, using condition MRS = MRT from Eqs. (3.5)–(3.6) and taking historical means, we have

$$\beta\eta = \left[\frac{1}{T} \sum_{t=1}^{T} \frac{1}{E_t \{\exp\{\ln\ NMPK_{t+1} - \gamma\Delta \ln c_{t+2} - \gamma\Delta \ln c_{t+1} - \Delta \ln P_{t+2}\}} \right]^{\frac{1}{2}}. \tag{3.8}$$

All variables in Eqs. (3.5)–(3.8) can be replaced by their estimates from the CVARMA model. Appendix D provides the steps for calibrating the MRS-based and MRT-based interest rate series.

3.3. Risk-Free Rate Puzzle and Keynes-Ramsey Rule

Eqs. (3.5) and (3.6) show that the MRS-based real rate is an increasing function of γ and that the MRT-based real rate is a decreasing function of γ. This suggests that if we look only at the consumption-based MRS alone, then a high value of γ needs to be assumed in order to sustain a higher risk-free rate. On the hand, if we look only at the production-based MRT alone, then a low value of γ needs to be assumed in order to sustain a higher risk-free rate. However, the steady-state equilibrium condition, MRS=MRT, implies a unique value for γ. This means that, in a general equilibrium model, we do not have the freedom of assigning arbitrary value for γ once other structural parameters (such as β) of the model are specified. The situation is depicted in Fig. 1.

As Mehra and Prescott (1985) suggest, to explain a high equity premium using a conventional consumption-based approach, a very high value of γ (e.g., above 10) should be assumed. Such a high value is not plausible empirically, as implied by findings from micro data (see, e.g., Prescott, 1986). This mirrors the risk-free rate puzzle that the return on the default-free asset is too low to be explained by a consumption-based model (Weil, 1989).[12]

On the other hand, a production-based approach can resolve the risk-free rate puzzle with only a low value for γ. Intuitively, the firm, as a member of the family, discounts future returns to a greater extent if the household is more risk averse. Thus, the less risk averse the household is, the higher the borrowing costs (future returns) is required by the firm, implying a negative slope of the schedule for MRT-based real rates. This implies that the risk-free rate puzzle does not exist from the viewpoint of a firm as a family member.[13]

[12] The existing literature suggests that the risk-free rate puzzle can be resolved by incorporating preference modifications (generalized expected utility or habit formation) and incomplete markets in the consumption-based approach (see, for the listing of related studies, Kocherlakota 1996).

[13] The fact that the MRT-based rate is a decreasing function of γ suggests that the risk free rate puzzle may be solved by shedding light on the production side, whereas a consumption-based model should relax standard assumptions through preference modifications or incorporation of incomplete markets to solve the puzzle.

Here, we argue that it is theoretically inconsistent to assume an arbitrary value of γ in a general equilibrium model in order to resolve the risk-free puzzle when we account for both the consumption and production sides. Namely, the risk aversion parameter is not a free parameter in general equilibrium.

Specifically, Fig. 2A depicts the relation between β and γ implied by Eq. (3.8), on the basis of actual data that will be described in Section 4.1. It shows the schedule of β as an increasing function of γ (the grid for γ is 0.1). Once β is given, then γ is determined. For example, when $\beta = 0.9935$, we have $\gamma = 0.5$. Fig. 2B shows how the MRS-based and MRT-based real rates in our model determine γ, given a specific value of the discount factor, $\beta = 0.9935$. The two blades of scissors, MRS and MRT, determine the value of γ, given β. For an alternative value of β, the two blades cross at a different location and pick up a different value of γ.

4. Empirical Investigations

4.1. Data

Most time series of the US economy that we use are taken from *Citibase* for the post-Korean War period 1954:1−1992:4. We use the rate of return on the three-month Treasury bill as the nominal interest rate. All the variables except interest rates are seasonally adjusted.

We construct the quarterly series of the marginal productivity of capital by calibrating a Cobb-Douglas function under the CRS assumption on production technology. The marginal productivity of capital net of depreciation is measured as:

$$NMPK_t = \alpha(q_t / k_t) - \delta \,, \tag{4.1}$$

where q_t is the real GNP per capita and k_t is the net capital stock per capita in the business sector (see Appendix B).[14] Parameter α is set at 0.296, which is calibrated as the capital income share in the business sector following Cooley and Prescott (1995).[15] Also, we estimate that

[14] The use of a broad measure of the capital stock that includes the household capital (consumer durables) and government capital provided qualitatively similar results for a variety of tasks performed in this paper.
[15] We also estimated the production function and capital share equation simultaneously under the restriction of CRS by the full information maximum likelihood method. This exercise provided a similar estimate for α

$\delta = 0.049$ and $\eta = 1.0036$. The sample average of $NMPK_t$ is 12 percent per year. The predicted $NMPK_t$ is obtained after replacing the actual with the predicted *q/y* ratio in Eq. (4.1).[16] The ex-ante real interest rate from the data, rr_t^e, is measured as:

$$1 + rr_t^e \equiv (1 + R_t) E_t \left(\frac{1}{1 + \pi_{t+1}} \right) = (1 + R_t) \cdot \exp\{-E_t(\Delta \ln P_{t+1}) + 0.5 V_t(\Delta \ln P_{t+1})\}, \qquad (4.2)$$

where R_t is the three-month Treasury bill rate.

To obtain conditional expectation of variables, the CVARMA(2,3) model given by Eq. (c.4) in Appendix C is estimated by a nonlinear seemingly uncorrelated regression (SUR) method to reflect that we impose parameter restrictions on the system in which error terms are correlated across equations. The variable set is: $y_t = (\Delta \ln c_t, \Delta \ln P_t, \ln(q_t / k_t))'$, where c_t is the *per capita* real consumption expenditure on nondurables and services and P_t is the consumer price index. The constraints on the parameters are imposed as implied by the theory. The estimated result of CVARMA is summarized in Table 1. As shown by R^2, equations for inflation and output-capital ratio fit the data quite well, whereas the consumption growth equation shows a low value of 0.15. The restriction implied by Eq. (c.4) is not rejected at the 5 percent level. Although the parameter restriction can be rather strong as implied by its p-value of 0.04, we impose this restriction to be consistent with the theory.

4.2. Matching Moments

The mean and standard deviation of the calibrated series based on the MRS and MRT for the 1954:3−1992:4 period are computed by applying the simulation method to Eqs. (3.5)–(3.7). The ex-ante real rate of interest from the actual data is constructed using the expected inflation computed from the CVARMA. The mean and standard deviation of ex-ante real rates (per annum) is 1.41 percent and 2.49, respectively. We consider the selected values of the risk aversion parameter, $\gamma = \{0.1, 0.3, 0.5, 0.7, 1.0, 1.5, 2.0, 5.0\}$ and the corresponding time preference parameter, β, determined by the Keynes-Ramsey rule.

[16] The *NMPK* shows a downward trend. Business cycle models, however, suggest a stationary process without trends for the *NMPK*. We thus linearly detrend the estimated *NMPK* in calibrating the MRT-based interest rate.

Table 2 reports the mean and standard deviation of calibrated real and nominal rates with the combination of β and γ implied by Eq. (3.8). We obtain notable implications from matching moments. First, in terms of matching means, there is an upward bias in the MRS-based real rate, rr^{MRS}, reflecting the risk-free rate puzzle. Second, strong similarities are found between rr^{MRT} and the actual data. A higher *NMPK* is required for assuring positive values of rr^{MRT} as consumers are more risk averse and more impatient. This suggests how the 'high' marginal productivity of capital (12 percent) reconciles the 'low' risk-free rate. The marginal productivity premium over the risk-free rate is due to the *time to receive cash flow* from physical investments as well as the risk aversion in a growing economy. Third, since the variability of rr^{MRS} sharply rises with a higher γ, matching second moments enables us to exclude the cases of γ below 0.5 and above 2. Also, too much variability is involved in all series if $\gamma > 1.5$ while there is too little variability in R^{MRT} and rr^{MRS} if γ <0.3.

Finally, we set $\gamma = 0.5$ and $\beta = 0.9935$, with which the mean and standard deviation of interest rates are explained reasonably well by the calibrated series. Specifically, both real and nominal rates have an upward bias in the mean by about 0.5 percentage point, much moderating the risk-free rate puzzle. The upward bias in the MRT-based series arises too, as a result of imposing the Keynes-Ramsey rule. In terms of variability, rr^{MRT} matches well the actual data whereas rr^{MRS} has a downward bias of 1.48. R^{MRT} is somewhat less variable while R^{MRS} is a bit more variable than the actual data. We henceforth use these parameter values in assessing the closeness between the calibrated series and actual data.

4.3. Prediction

We first take a look at how the calibrated series explains the historical movements of the actual data. Fig. 3 shows the calibrated series (lines with symbols) along with the actual series for ex-ante real and nominal rates of interest. Real rates are depicted in Windows A and C. The MRT-based series (Window A) predicts well the actual data for most of the period. The MRS-based series (Window C) predicts the actual series quite well before 1980 but does not after 1980, and it is less volatile than the actual series. Apparently, the drastic rise in the mean of real rates the early 1980s is not explained by both of calibrated series. Nominal rates are depicted in Windows B and D. Both calibrated series move around the actual data rather closely before 1980. They, however, deviate much from the actual data in the first half of the 1980s: in particular, the MRS-based series sometimes violates the non-negativity of the nominal rate due to negative expected inflation rates.

Now we assess the similarity between the actual data and calibrated series in two dimensions. For this purpose, we use not only the levels of series but also detrended series with different filters since the levels of interest rates may not be covariance stationary. Namely, we also use the first-differenced series; series detrended by the Hodrick-Prescott (HP, 1997) filter; and the series detrended by the band-pass (BP) filter for the 6-32 quarter business cycle frequency as described in Baxter and King (1995), with the caveat that the use of the HP filter may cause spurious cycles (e.g., King and Rebelo, 1993).

In the first dimension, we estimate and compare autocorrelation and spectral density functions of the actual and calibrated series. To save space, we display figures only for the cases with the first difference and BP filter. Fig. 4 depicts estimated autocorrelation functions for the actual data (solid lines), MRT-based series (lines with symbols), MRS-based series (dotted lines). For the real rate, both MRS and MRT show close similarities in the oscillation of the autocorrelation function to the actual data, with MRT displaying better matches under both the first-difference and BP filters. For the nominal rate, the MRS and MRT perform reasonably well under the BP filter, but rather poorly under the first-difference filter. This implies that the model explains the nominal rate better at lower frequencies because of the Fed's interest rate smoothing.

Fig. 5 displays estimated spectra. The spectra are normalized by variances, hence the area underneath each spectral density function is unity. The height of the spectrum at a given frequency indicates the contribution to the total variance from fluctuations at that frequency.[17] For the real rate, both the MRS and MRT show remarkable predictive power on the data regardless of the filter used, with the exception that the MRS fails to generate enough fluctuations at the very low frequency (see Window C). This is consistent with Fig. 4C where the MRS has a weak autocorrelation at lags longer than 8. For the nominal rate, neither the MRS nor the MRT can explain the dynamics of the data under the first-difference filter (Window B): they show too much power at the high frequency

[17] Cycles per quarter equal to frequency/2π, where the frequency ranges from 0 to π. The period of the cycle is the inverse of cycles per quarter.

interval to explain the actual data's high power at lower frequencies only. This is perhaps because the model does not account for the Fed's interest rate smoothing. Under the BP filter (Window D), on the other hand, the MRS performs very well in explaining the data while the MRT does not.[18] This is a reversed situation compared to the case of the real rate, indicating that the productivity-based approach better explains the real rate movement while the consumption-based approach better explains the nominal rate movement (at the business cycle frequency).

Table 3 provides summary statistics of the relative mean square approximation errors (RMSAEs) based on the spectra with different filters. The RMSAEs are suggested by Watson (1993) as a measure of fit and are similar to $1–R^2$ in a standard regression: the smaller the better (see also Yi, 1998). The second panel reports the statistics for each series. First, the MRT-based real rate achieves a very good fit, regardless of the choice of filter (e.g., the RMSAEs for rr^{MRT} in column 2 are less than 0.05). The MRS-based real rate also achieves a good fit (as indicated by RMSAEs for rr^{MRS} in column 4), although, overall, it is not as good as the MRT-based real rate. Second, the nominal rate is not well explained by the model as the real rate. At the business cycle frequency, however, the fit is quite good. The MRS-based series has rather lower RMSAEs than the MRT-based series.

In the second dimension, we assess the similarity of two time series more directly. Table 4 shows the cross-correlation coefficients between the actual and calibrated series. For the real rate, the MRT-based series outperform the MRS-based series. Regardless of the choice of filter, the correlation for rr^{MRT} is quite high, ranging between 0.42–0.66. The correlation for rr^{MRS} ranges between 0.24–0.39. On the other hand, the model tends to predict better the nominal rate than the real rate when the calibration is based on MRS, whereas the converse is true when it is based on MRT. Correlations for R^{MRT} and R^{MRS} are reasonably high in levels (0.510 and 0.632, respectively) and under the BP filter (0.565 and 570, respectively). As shown in Fig. 6, the model largely fits well the actual data at the business cycle frequency, although the model does not predict smooth interest

[18] In levels, we found the astonishing similarity between the spectra of rr^{MRT} and the actual data but less similarity between the spectra of rr^{MRS} and the actual data, and found too much power of the nominal rate at low frequencies to be explained by the calibrated series. With the use of the HP filter, the calibrated series showed strong similarities to the real rate, whereas neither of the two calibrated series captures the high persistence of the nominal rate.

rates in the 1960s, due to the Fed's emphasis on financial market stability,[19] and a drift in real and nominal rates in the early 1980s. Remarkably, much of the short-term dynamics of nominal rates is not explained by our model prediction: under the first-difference and HP filters, correlations for nominal rates are too low for the model to explain nominal rates.

We now provide implications from the comparison between model predictions and actual data. First, both sides capture dynamic aspects of the real rate, for which MRT outperforms MRS. MRT has information superiority to MRS at high frequencies while MRT's information superiority is much reduced at low frequencies. This is perhaps because calibration based on MRT utilizes information on production technology as well as consumption growth whereas MRS does not include the liquidity effect component, which itself contains information on the production side. Second, the model predicts well the nominal rate at the business cycle frequency but not at high frequencies. This finding implies that the nominal rate can be explained by fundamentals of thrift and productivity to some extent although its fluctuations at high frequencies can be influenced by monetary policy.

4.4. Discussion

Although we have provided a variety of diagnostics for the performance of the model, our main purpose is not to test the model against a range of other models, but to argue that MRS and MRT should be jointly examined in a single framework. By doing so, we attempt to reveal consistency between the implications of MRS and MRT in a general equilibrium model. As argued by Cochrane (1996, p. 573), there are many factors that can 'delink' MRS from interest rates. In our model, the liquidity effect appears through portfolio rigidity in a monetary production economy. The *unobservable* liquidity effect term has been treated as a stochastic disturbance in calibrating the MRS-based interest rates. The MRT-based interest rates, however, utilize information on both production technology and preferences, and thus embodies the liquidity effect. As a result, the wedge between the two series is attributable to the liquidity effect term.[20]

[19] In the 1960s, monetary policy was pursued to maintain the stability of financial markets, focusing on the movements of free reserves to control the bank loan rate or the return on long-term bonds.
[20] The wedge could also be due to misspecification of the model. The existence of the wedge due to other sources that renders MRS different from the conventional formula implies that a production-based asset pricing model gets around the puzzle (see also footnotes 14-15).

A friction is imposed on the production side of the economy since firms are assumed to take a quarter to build the capital stock from investment and another quarter to receive cash flow from output sales.[21] As a result, a compensation for the *time to receive cash flow* and the risk involved in physical investment render the high marginal productivity compatible with the low risk-free rate: note that marginal utility from future consumption is involved in MRT, too. In practice, however, firms face longer gestation lags (e.g., 2-4 quarters) in installing new capital. The incorporation of the time-to-build idea (e.g., Kydland and Prescott, 1982) will involve longer period lags in the interest rate dynamics in conjunction with the inflation dynamics and the term-structure of interest rates.

Our model does not predict an abrupt upward drift in real interest rates in the early 1980s. A conventional hypothesis is that the real interest rate follows a stationary process with a constant mean. The time-series literature on real interest rates (e.g., Nelson and Schwert, 1977; Rose, 1988; Garcia and Perron, 1996), however, has provided evidence against this hypothesis for the post-WWII period. A simple way to relax the hypothesis is to allow for mean drifts. We employ a statistical procedure following Quandt (1958) to identify structural break dates assuming the existence of two breaks. This exercise suggests that two break dates for the ex-ante real rate are 73:2 and 80:3.[22] The break at 73:2, plausibly associated with the first oil shock, is captured only by the MRT-based series. Thus, this break seems to be arising from productivity and inflation factors embodied in our model. The break at 80:3, however, is explained neither by the MRT-based series nor by the MRS-based ones.[23] We attribute the break in the early 1980s to changes in institutional factors. The highest and most volatile interest rates in post-WWII history were preceded by the change in the operating target from the Federal funds rate to nonborrowed reserves in October 1979 and the Depository Institutions Deregulation Act of 1980 that eliminated all deposit interest

[21] A typical observation is that firms hold the one-to-three months' worth of sales (e.g., Bils and Kahn, 1996). The median of the mean of the manufacturing firm data from COMPUSTAT (for 1975-1994) shows the yearly inventory/sales ratio is 0.244 (Choi et al., 1997), implying that firms holds about three months' worth of sales.

[22] Break points, (t_1, t_2), are chosen to minimize the log of the sum of squares residual function: $LSSR = t_1 * \ln(\hat{e}_1^{'}\hat{e}_1) + (t_2 - t_1) * \ln(\hat{e}_2^{'}\hat{e}_2) + (T - t_2) * \ln(\hat{e}_3^{'}\hat{e}_3)$, where $\hat{e}_i$ is the sub-period residual of the regression, $rr_t^e = \overline{rr} + e_t$ with e_t being covariance stationary. Allowing for a Markov switching in the inflation process instead, Garcia and Perron (1996) suggest mean drifts in the ex-ante real rate occurring around the same dates as ours.

[23] The break dates are 73:1 and 81:4 for the MRT-based series, and 68:4 and 82:3 for the MRS-based series.

rate ceilings. Also, fiscal policies and the government debt in the 1980s may have caused high interest rates.[24]

More importantly, the nominal rate shows much smoother movements than the calibrated series in short-time horizons as indicated by the spectra estimates (see also footnote 17) and Windows B and D of Fig. 4. This finding provides an important policy implication. Our model assumes that the monetary growth follows an autoregressive process. However, the Fed's concern with the financial market stability and interest rate smoothing, indeed, has affected the nominal interest rate (Rudebusch, 1995; Choi, 1999). Also, inflation targeting would alter the dynamics of interest rates: e.g., Fuhrer and Moore (1992) suggest that aggressive inflation targeting raises the variability of interest rates relative to that of inflation. Hence, we expect the Fed's monetary policy based on a feedback rule to be able to explain a substantial portion of the discrepancy between the nominal rate and calibrated series.

5. Conclusion

This paper examines the behavior of interest rates by linking the consumption-based and production-based approaches into a general equilibrium framework for a growing monetary economy. We derive two theoretical nonlinear relationships that link interest rates to thrift and to productivity and calibrate historical time series consistent with the theoretical model using the constrained VARMA estimation and simulation methods.

We find that the movement of the real rate can be explained to some extent by thrift and quite well by productivity,[25] which provides insight on the risk-free rate puzzle. The calibrated series based on theoretical relationships, however, fail to explain the abnormal drift in the real rate in the early 1980s, which is presumably due to institutional factors. We also find some similarities between the actual and calibrated series for the nominal rate. Nonetheless, the nominal rate shows an excessive smoothness compared with the calibrated series although these series show close

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[24] Fitoussi and Phelps (1988) suggest that the US fiscal policy has been accompanied by the high interest rate around the world from 1980 onwards.

[25] If the wedge between the MRS and the MRT is indeed due to the liquidity effect, then our approach can also shed light on the nature of the liquidity effect itself. Since this issue is quite involved is beyond the scope of the current project, we leave it to future explorations.

similarities at the business cycle frequency. It is then puzzling why the calibration of the nominal rate by the Lucas-Fuerst type general equilibrium model fails to deliver the smooth movement of the nominal interest rate at high frequencies.

To solve this puzzle, the extension of the model in the following directions, left for future study, will be helpful. First and most importantly, the incorporation of an *endogenous* monetary policy that aims to smooth market rates (and perhaps to target inflation) will contribute to capturing the little variability of nominal rate movements at high frequencies.[26] Second, taking into account the time-to-build idea and elaborating the production process with the capacity utilization idea (e.g., Bils and Cho, 1994) may improve the performance of calibrating the MRT-based interest rate. Furthermore, to understand the puzzle, it may be beneficial to consider the role of government debt (Mankiw, 1987; Evans, 1987) and the sources of friction that may affect the persistence of interest rates such as incomplete asset markets (Telmer, 1993), adjustment costs (Cochrane, 1991, 1996; Cogley and Nason, 1995), and transaction costs (Luttmer, 1996).

[26] Restrictions on the goods market's price adjustment speed will affect the dynamics of inflation. The introduction of sluggish price adjustments may render movements of expected inflation and thus nominal rates smoother.

Appendix A. Derivation of Eqs. (2.8) and (2.11)

Equilibrium conditions of the model include market clearing conditions for the five markets: $m_t = m_{t+1} = 1, h_t = l_t, b_t = d_t + x_t, k_t = \kappa_t$, and $c_t + \eta k_{t+1} - (1-\delta)k_t = f(k_t, h_t)$. Let λ_1 and λ_2 denote the Lagrange multipliers associated with (2.5) and (2.6), respectively. The first-order conditions with respect to indicated variables, evaluated at the equilibrium are:

d_t: $E_{t-1}(\lambda_{1t}) = \beta\eta E_{t-1}[R_t V_m(k_{t+1})/(1+x_t)]$, (a.1)

c_t: $U'(c_t) = p_t[\lambda_{1t} + \beta\eta V_m(k_{t+1})/(1+x_t)]$, (a.2)

l_t: $-1/(1-l_t) = \beta\eta V_m(k_{t+1}) w_t/(1+x_t)$, (a.3)

b_t: $\lambda_{2t} = \beta\eta R_t V_m(k_{t+1})/(1+x_t)$, (a.4)

h_t: $p_t f_{h,t}\, \beta\eta V_m(k_{t+1})/(1+x_{t+1}) = w_t[\lambda_{2t} + \beta\eta V_m(k_{t+1})/(1+x_t)]$, (a.5)

k_{t+1}: $\beta\eta V_k(k_{t+1}) = p_t[\lambda_{2t} + \beta\eta V_m(k_{t+1})/(1+x_t)]$, (a.6)

λ_{1t} : $1 - d_t = p_t c_t$, (a.7)

λ_{2t} : $d_t + x_t = p_t(k_{t+1} - (\eta-\delta)k_t) + w_t h_t$, (a.8)

where $V_m(k_{t+1}) = E_t[U'(c_{t+1})/p_{t+1}]$ and $V_k(k_{t+1}) = \beta\eta E_t[(\eta + f_{k,t+1} - \delta)p_{t+1}V_m(k_{t+2})/(1+x_{t+1})]$. Combining (a.2) and (a.4) yields Eq. (2.8). Combining (a.4) and (a.6) yields Eq. (2.11).

Appendix B. Measuring the marginal productivity of capital

The *quarterly* series of the capital stock is constructed by using the annual series of capital stocks (Department of Commerce, 1993; Bureau of Economic Analysis, 1993) and related quarterly data. The quarterly data on private fixed investment and the stock of inventories are taken from *Citibase*. The quarterly depreciation rates are generated by using the consumption of fixed capital stock (GCCJQ in *Citibase*) and a proper interpolation of annual depreciation data from the Department of Commerce (1993). The private capital stock includes the fixed reproducible private capital stock and the stock of inventories. The share of capital income in the business sector output, α, is computed following Cooley and Prescott (1995): $\alpha = (\text{Unambiguou s Capital Income} + DEP)/(GNP - \text{Ambiguous Capital Income})$, where GNP is the nominal GNP, and DEP is the nominal consumption of fixed capital. The sample mean (standard errors) of the capital income share and the depreciation rate for the 54:1–92:4 period are computed as α=0.296 (0.012) and δ=0.049 (0.008), respectively. To obtain per capita values, variables are divided by population (PAN17 in *Citibase*). Then the net marginal productivity of capital is given by Eq. (4.1) in the text.

Appendix C. Derivation of the CVARMA

Denote $\Delta = 1 - L$, where L is the lag operator. The consumption growth is given by

$$\begin{aligned}\Delta \ln c_t &= (\rho + \phi_k)\Delta \ln c_{t-1} - \rho\phi_k \Delta \ln c_{t-2} + \phi_{cx1}\varepsilon_t + (\phi_{cx2} + \phi_{ck}\phi_{kx1} - \phi_k\phi_{cx1} - \phi_{cx1})\varepsilon_{t-1} \\ &\quad + (\phi_{ck}\phi_{kx2} - \phi_k\phi_{cx2} - \phi_{cx2} - \phi_{ck}\phi_{kx1} + \phi_k\phi_{cx1})\varepsilon_{t-2} - (\phi_{ck}\phi_{kx2} - \phi_k\phi_{cx2})\varepsilon_{t-3}, \qquad \text{(c.1)}\end{aligned}$$

and the inflation rate is given by

$$\Delta \ln P_t = (\rho + \phi_k)\Delta \ln P_{t-1} - \rho\phi_k \Delta \ln P_{t-2} + \phi_{px1}\varepsilon_t + (\phi_{px2} + \phi_{pk}\phi_{kx1} - \phi_k\phi_{px1} - \phi_{px1})\varepsilon_{t-1}$$
$$+ (\phi_{pk}\phi_{kx2} - \phi_k\phi_{px2} - \phi_{px2} - \phi_{pk}\phi_{kx1} + \phi_k\phi_{px1})\varepsilon_{t-2} - (\phi_{pk}\phi_{kx2} - \phi_k\phi_{px2})\varepsilon_{t-3}\,. \tag{c.2}$$

To derive the output-capital ratio, we note

$$\ln q_t = (\rho + \phi_k)\ln q_{t-1} - \rho\phi_k \ln q_{t-2} + \phi_{yx1}\varepsilon_t + (\phi_{yx2} + \phi_{yk}\phi_{kx1} - \phi_k\phi_{yx1})\varepsilon_{t-1} + (\phi_{zk}\phi_{yx2} - \phi_k\phi_{yx2})\varepsilon_{t-2}\,,$$
$$\ln k_t = (\rho + \phi_k)\ln k_{t-1} - \rho\phi_k \ln k_{t-2} + \phi_{kx1}\varepsilon_t + \phi_{kx2}\varepsilon_{t-1}\,.$$

Hence the output-capital ratio is given by

$$\ln(q_t / k_t) = (\rho + \phi_k)\ln(q_{t-1}/k_{t-1}) - \rho\phi_k \ln(q_{t-2}/k_{t-2}) + (\phi_{yx1} - \phi_{kx1})\varepsilon_t$$
$$+ (\phi_{yx2} + \phi_{yk}\phi_{kx1} - \phi_k\phi_{yx1} - \phi_{kx2})\varepsilon_{t-1} + (\phi_{zk}\phi_{yx2} - \phi_k\phi_{yx2})\varepsilon_{t-2}\,. \tag{c.3}$$

Taken together, we have the following CVARMA form:

$$\begin{pmatrix} \Delta \ln c_t \\ \Delta \ln P_t \\ \ln(q_t/k_t) \end{pmatrix} = (\rho + \phi_k)\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \Delta \ln c_{t-1} \\ \Delta \ln P_{t-1} \\ \ln(q_{t-1}/k_{t-1}) \end{pmatrix} - \rho\phi_k \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \Delta \ln c_{t-2} \\ \Delta \ln P_{t-2} \\ \ln(q_{t-2}/k_{t-2}) \end{pmatrix}$$
$$+ \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{12} & 0 \\ 0 & 0 & \sigma_{13} \end{pmatrix}\varepsilon_t + \begin{pmatrix} \sigma_{21} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{23} \end{pmatrix}\varepsilon_{t-1}$$
$$+ \begin{pmatrix} \sigma_{31} & 0 & 0 \\ 0 & \sigma_{32} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}\varepsilon_{t-2} + \begin{pmatrix} \sigma_{41} & 0 & 0 \\ 0 & \sigma_{42} & 0 \\ 0 & 0 & \sigma_{43} \end{pmatrix}\varepsilon_{t-3}\,, \tag{c.4}$$

where $\sigma_{11} \equiv \phi_{cx1}$, $\sigma_{12} \equiv \phi_{px1}$, $\sigma_{13} \equiv \phi_{yx1} - \phi_{kx1}$, $\sigma_{21} \equiv (\phi_{cx2} + \phi_{ck}\phi_{kx1} - \phi_k\phi_{cx1} - \phi_{cx1})$, $\sigma_{22} = (\phi_{px2} + \phi_{pk}\phi_{kx1} - \phi_k\phi_{px1} - \phi_{px1})$, $\sigma_{23} = (\phi_{yx2} + \phi_{yk}\phi_{kx1} - \phi_k\phi_{yx1} - \phi_{kx2})$, $\sigma_{31} \equiv (\phi_{ck}\phi_{kx2} - \phi_k\phi_{cx2} - \phi_{cx2} - \phi_{ck}\phi_{kx1} + \phi_k\phi_{cx1})$, $\sigma_{32} \equiv (\phi_{pk}\phi_{kx2} - \phi_k\phi_{px2} - \phi_{px2} - \phi_{pk}\phi_{kx1} + \phi_k\phi_{px1})$, $\sigma_{33} \equiv (\phi_{zk}\phi_{yx2} - \phi_k\phi_{yx2})$, $\sigma_{41} \equiv (-\phi_{ck}\phi_{kx2} + \phi_k\phi_{cx2})$, $\sigma_{42} \equiv (-\phi_{pk}\phi_{kx2} + \phi_k\phi_{px2})$, and $\sigma_{43} \equiv 0$. The model imposes the following restrictions: (a) the autoregressive roots must be identical across the three equations; (b) the off-diagonal elements and the coefficient σ_{43} must be zero. No further restrictions are imposed unless the deep parameters of the model are specified. The ARMA structure in Eq. (c.4) are imposed on the estimation of the conditional moments of the observables when calibrating interest rates.

Appendix D. Calibration of interest rate series using the estimated CVARMA

Consider a k-dimensional multiple time series with the sample size T, $\{y_1, \cdots, y_T\}$, generated by a VARMA(p,q) process:

$$y_t = \mu + A_1 y_{t-1} + \cdots + A_p y_{t-p} + u_t + \sigma_1 u_{t-1} + \cdots + \sigma_q u_{t-q}\,,$$

where μ is a $(k\times1)$ vector of intercept terms, the A_is are $k\times k$ coefficient matrices, and u_t is a k-dimensional white noise. With restrictions as given by Eq. (c.4), the off-diagonal elements of the A_is are zero. Define

$$
\begin{array}{llll}
y_t=(y_{1t},y_{2t},\cdots,y_{kt})' & (k\times1), & Y=(y_1,y_2,\cdots,y_T) & (k\times T), \\
B=(\mu,A_1,\cdots,A_p) & k\times(kp+1), & Z_t=(1,y_t,\cdots,y_{t-p+1})' & (kp+1)\times1, \\
Z=(Z_0,Z_1,\ldots,Z_{T-1}) & (kp+1)\times T, & U=(u_1,u_2,\cdots,u_T) & (k\times T).
\end{array}
$$

Then the CVARMA(2,3) model can be rewritten as: $Y=BZ+U$. The one-step-ahead conditional forecast is given by $E_t[y_{t+1}]=\hat{B}Z_{t+1}$, where $\hat{B}$ is the estimate of vector B.

The MRS-based rate, abstracting the unobservable ξ_t from Eq. (3.5), is computed as:

$$1+R_t^{MRS}=(\beta\eta)^{-1}\exp(W_1\hat{y}_{t+1}+0.5W_1\hat{\Sigma}_uW_1'), \tag{d.1}$$

where $y_t=[\Delta\ln c_t,\ \Delta\ln P_t,\ \ln NMPK_t]'$, $W_1=[\gamma\quad 1\quad 0]$, $\hat{y}_{t+1}=E_t(y_{t+1})$, and the estimator of the covariance matrix is defined by $\hat{\Sigma}_u=\hat{U}\hat{U}'/T$ with $\hat{U}=Y-\hat{B}Z$. To compute the MRT-based rate, Eq. (3.6) requires the two-step-ahead forecast given by $E(y_{t+2}\,|\,\Omega_t)\ =E(y_{t+2}\,|\,\hat{y}_{t+1},y_{t,\cdots,}y_1)$, generated by the estimated model after replacing y_{t+1} with $\hat{y}_{t+1}$. Let $\hat{O}_t$ and $\hat{U}_t$ be forecast values and errors of $[\ln NMPK_{t+1},\Delta\ln c_{t+2},\ \ \Delta\ln c_{t+1},\Delta\ln P_{t+2}]'$ conditional on Ω_t, respectively. The numerator in Eq. (3.6) is computed as:

$$\beta\exp(W_2\hat{O}_t+0.5W_2\frac{\hat{U}\hat{U}'}{T}W_2), \tag{d.2}$$

where $W_2=[1\quad-\gamma\quad-\gamma\quad-1]$. Similarly, the denominator of Eq. (3.6) can be computed.

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Table 1
Estimated result of CVARMA

Equation	Intercept	$(\rho+\phi_k)$	$(-\rho\phi_k)$	σ_{2j}	σ_{3j}	σ_{4j}	R^2
$\Delta \ln c_t : j$=1	0.0002**	0.670**	0.269	0.465*	0.377*	0.001	0.114
	(0.0001)	(0.229)	(0.221)	(0.236)	(0.184)	(0.080)	
$\Delta \ln P_t : j$=2	0.0007 †	0.670**	0.269	0.046	0.284**	–0.199*	0.746
	(0.0004)	(0.229)	(0.221)	(0.235)	(0.098)	(0.091)	
$\ln(q_t / k_t) : j$=3	–0.036**	0.670**	0.269	–0.520**	–0.214**	–	0.977
	(0.012)	(0.229)	(0.221)	(0.232)	(0.084)	–	
$\chi^2(5)$			11.66 (0.040)				

Notes: The system is estimated by a nonlinear SUR method for the period 1954:3–1992:4 (standard errors in parentheses). Each regression equation includes an intercept. A chi-square test is performed for the parameter restriction in the unconstrained VARMA(3,3), following Gallant and Jorgenson (1979). The test statistic follows a $\chi^2(5)$ distribution (p-value in parenthesis). ** indicates significance at the 1% level, * indicates significance at the 5% level, and † indicates significance at the 10% level.

Table 2
Mean and standard deviation of calibrated interest rates

β	0.9919	0.9927	0.9935	0.9944	0.9956	0.9977	0.9997	1.0122
γ	0.1	0.3	0.5	0.7	1.0	1.5	2.0	5.0
rr^{MRT}	2.04	2.04	2.05	2.05	2.06	2.08	2.11	2.54
	(2.84)	(2.70)	(2.63)	(2.64)	(2.77)	(3.27)	(4.01)	(9.87)
R^{MRT}	6.36	6.36	6.36	6.37	6.38	6.41	6.44	6.90
	(1.24)	(1.45)	(1.75)	(2.10)	(2.67)	(3.68)	(4.73)	(11.23)
rr^{MRS}	2.02	2.03	2.03	2.03	2.03	2.04	2.04	2.07
	(0.20)	(0.60)	(1.01)	(1.41)	(2.02)	(3.02)	(4.03)	(10.05)
R^{MRS}	6.36	6.36	6.36	6.36	6.35	6.35	6.37	6.37
	(3.14)	(2.97)	(2.86)	(2.80)	(2.82)	(3.13)	(3.71)	(8.99)
Actual data		rr^e: 1.41			R: 5.73			
		(2.49)			(2.93)			

Note: Means and standard errors in parentheses are in percentage per annum for the period 54:3–92:4.

Table 3
Watson's test results for the fit of calibrated series: RMSAEs

Variable	rr^{MRT}	R^{MRT}	rr^{MRS}	R^{MRS}
Level	0.010	0.262	0.149	0.053
First difference	0.043	0.307	0.035	0.100
HP filter	0.022	0.445	0.029	0.116
BP filter	0.011	0.079	0.036	0.011

Notes: Figures are the RMSAEs (relative mean square approximation errors). The HP filter denotes the Hodrick-Prescott detrending method, and the BP filter denotes the 6–32 quarter band-pass filter (Baxter and King 1995).

Table 4
Cross-correlation between actual and calibrated interest rates

Period	rr^{MRT}	R^{MRT}	rr^{MRS}	R^{MRS}
Level	0.526	0.510	0.243	0.632
First difference	0.422	–0.231	0.393	0.146
HP filter	0.537	0.180	0.315	0.430
BP filter	0.661	0.565	0.235	0.570

Note: Calibrated rates are based on the assumption of $\beta = 0.9935$ and $\gamma = 0.5$.

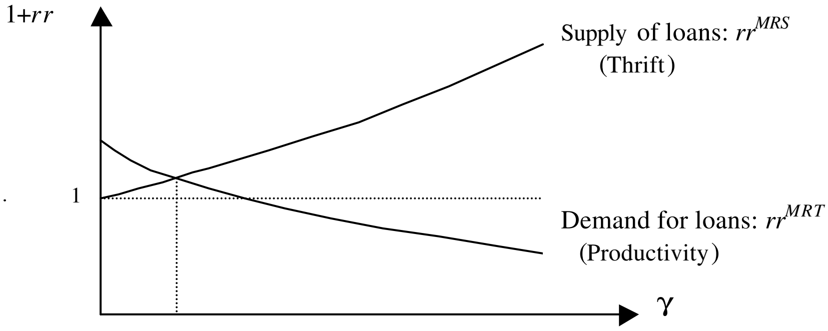

Fig. 1. Interest rates and risk aversion parameter in a growing economy.

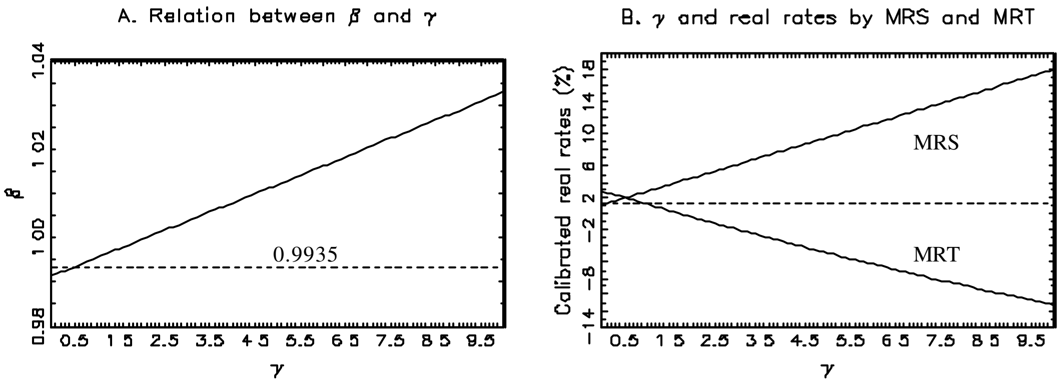

Fig. 2. Tuning the discount factor (β) and risk aversion parameter (γ).

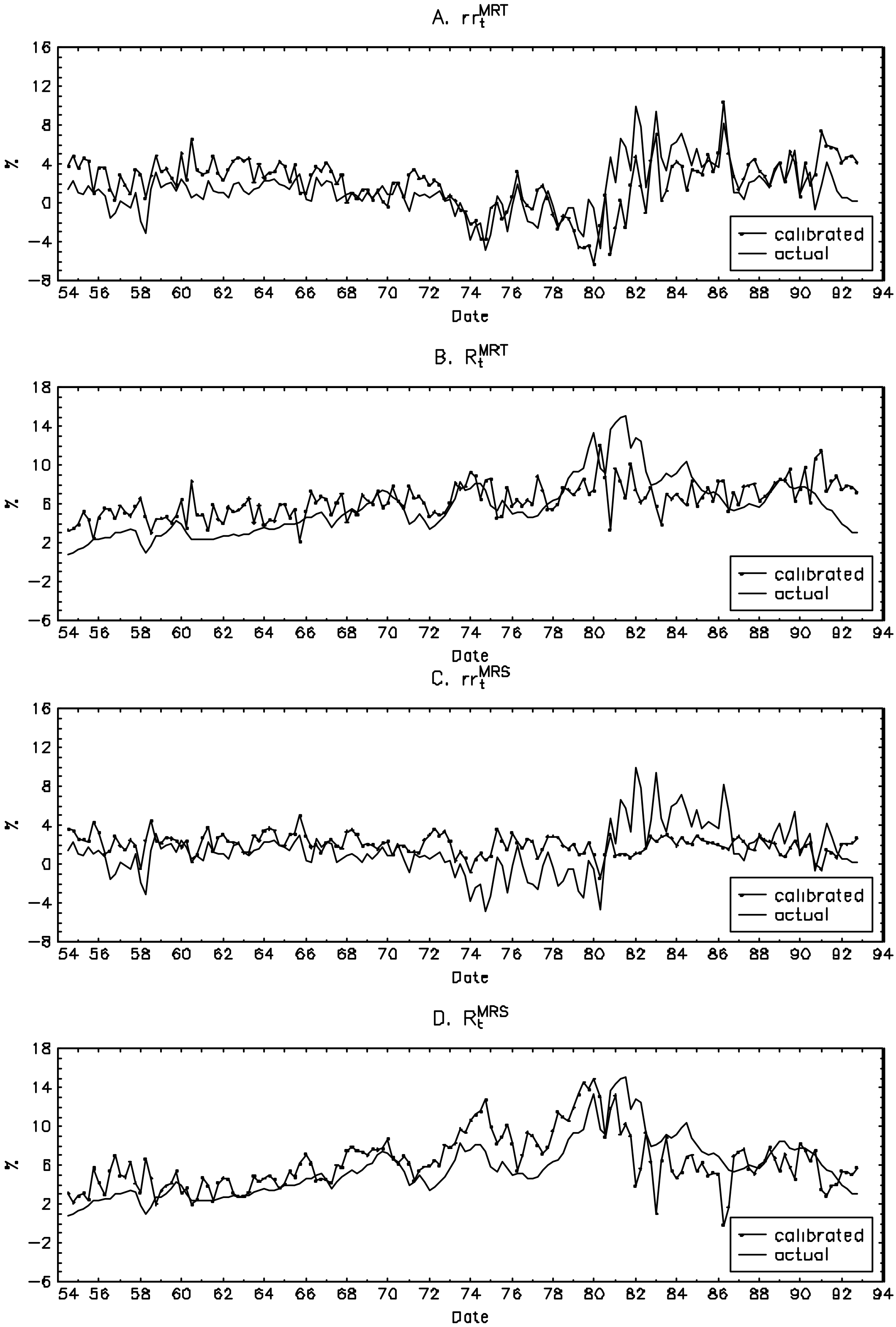

Fig. 3. Calibrated real and nominal interest rates.

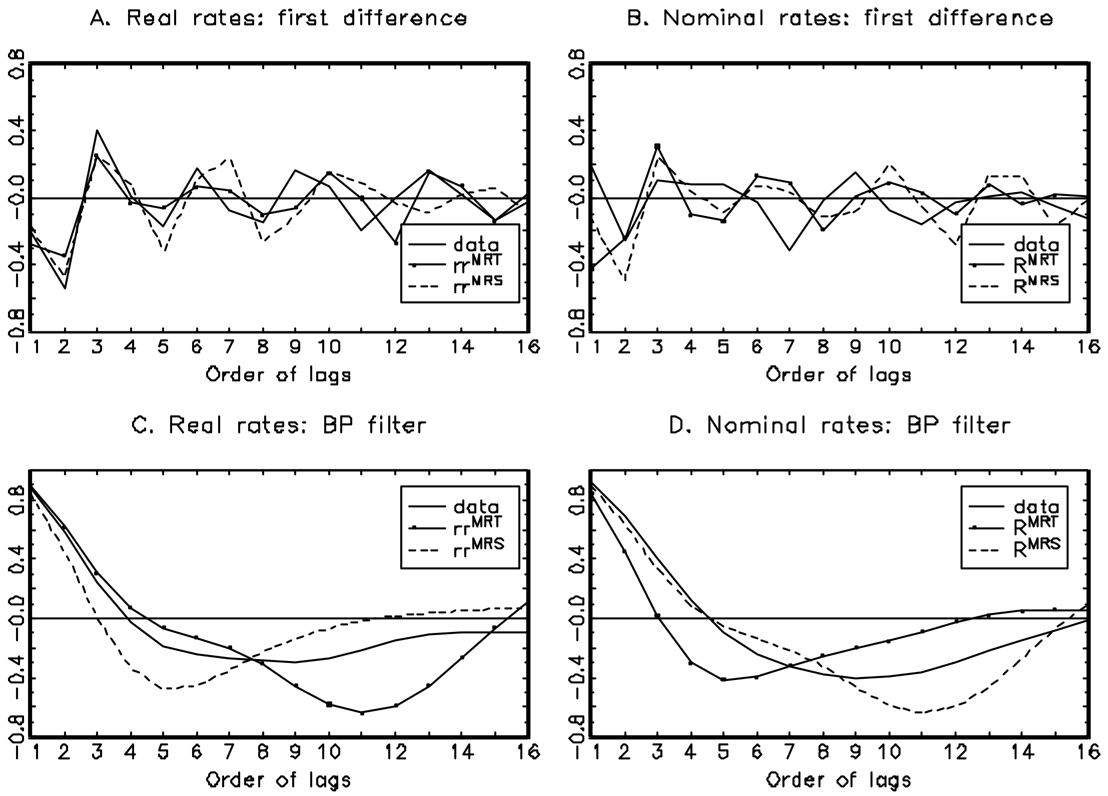

Fig. 4. Autocorrelation functions for interest rates.

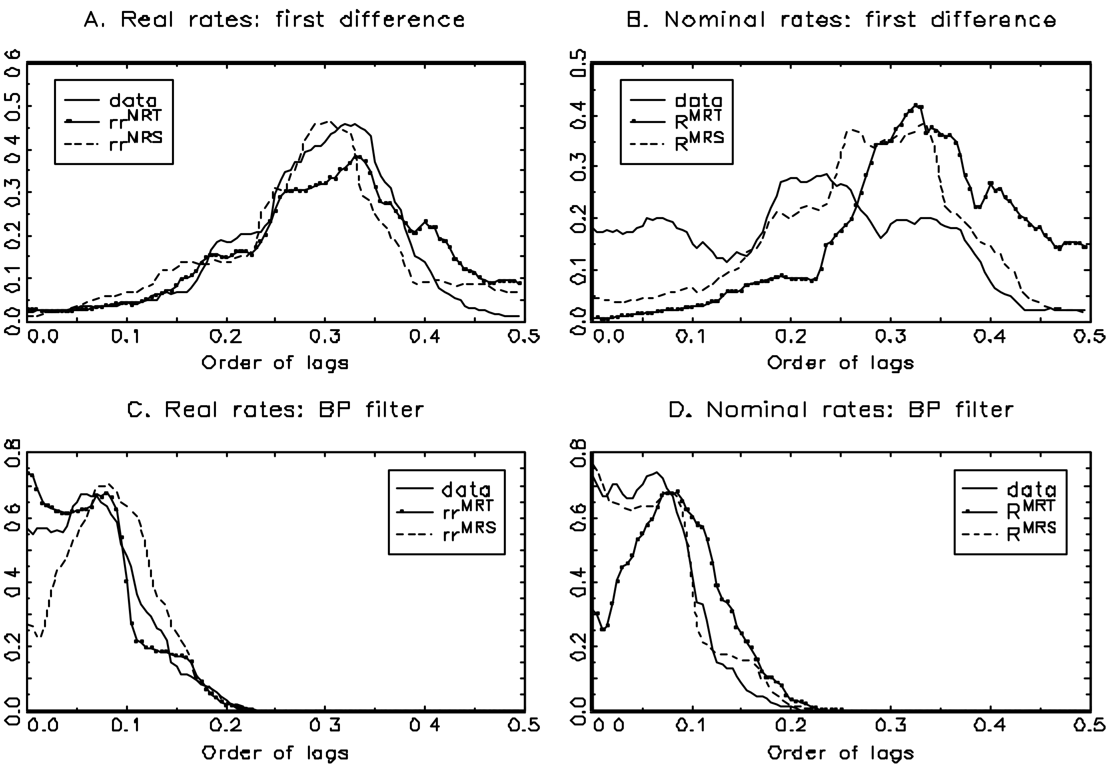

Fig. 5. Spectra for interest rates.

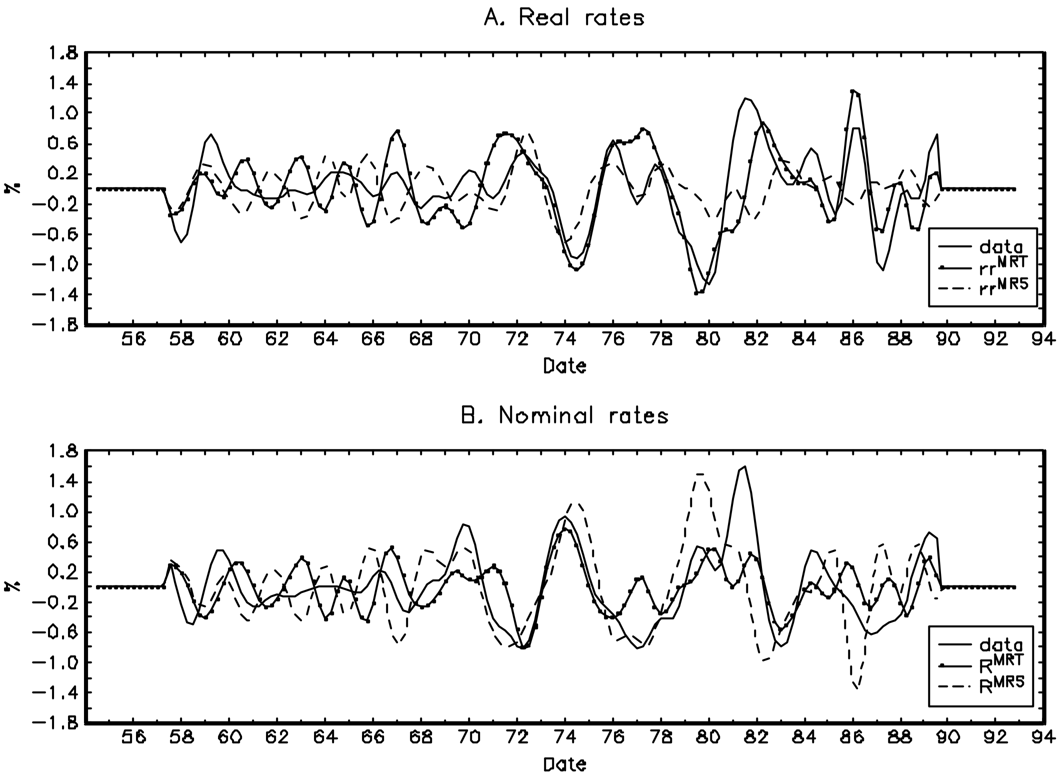

Fig. 6. BP filtered, calibrated real and nominal interest rates