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High Equity Premia and Crash Fears. Rational Foundations*

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Abstract

We show that when in Lucas trees model the process for dividends is described by a lattice tree subject to infrequent but observable structural breaks, in equilibrium recursive rational learning may inflate the equity risk premium and reduce the risk-free interest rate for low levels of risk aversion. The key condition for these results to obtain is the presence of sufficient initial pessimism. The relevance of these findings is magnified by the fact that under full information our artificial economy cannot generate asset returns matching the empirical evidence for any positive relative risk aversion.

JEL codes: G12, D83.

1. **Introduction**

Since Mehra and Prescott (1985, MP), we know that a Lucas-style economy with power, time-additive expected utility, complete markets, no frictions, and in which a representative agent forms rational expectations on the only source of risk (consumption) cannot reproduce the historical mean equity risk premium. This impasse is labeled the *equity premium puzzle*. In practice, unreasonably high levels of relative risk aversion must be used to get a sensible risk premium. Moreover, in MP's framework high risk aversion implies an implausibly low elasticity of intertemporal substitution that forces the real riskless rate to levels in excess of historical averages, the *risk-free rate puzzle*. Following MP (1985) a large effort has been made to explain these puzzles. Many papers have focused on the role of power, time-additive, expected utility preferences constraining the elasticity of intertemporal substitution to be the inverse of the coefficient of

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relative risk-aversion. The assumption of market completeness has been removed, showing that the non-insurable uncertainty in individual consumption increases the equity premium. A literature has assessed the importance of borrowing constraints and transaction costs.

Less attention has been given to the assumption of full-information rational expectations as a way to close the model and impose *some* consistency on the mechanism by which beliefs are formed. Some Authors have approached the puzzles leveraging on events unique to the US history, particularly the Great Depression. Rietz (1988) shows that if agents are characterized by biases in beliefs reflecting catastrophic scenarios not present in the historical data, a high equity premium is generated under reasonable risk aversion. However the required crash state must be truly catastrophic. Danthine and Donaldson (1999) explore the same concept, showing that Peso problems have more dramatic effects in artificial samples in which crashes are not actually present. Cecchetti, Lam, and Mark (2000) study the effects of belief distortions on asset prices and show that some pessimism relative to the maximum-likelihood estimates generates plausible moments. However, the origin of such pessimistic fears is unclear. This literature therefore relies on deviations between realized and subjectively perceived beliefs, often in arbitrary ways. On the opposite, in this paper we try to build a model in which rational pessimism and crash fears may arise as a consequence of the application of simple but optimal estimation methods.

When agents lack full information on some parameters characterizing the environment, their subjective beliefs may rationally deviate from the empirical distribution of the state variables without the need to postulate in an *ad-hoc* fashion that markets agree on some disaster state. A few papers have studied asset pricing under recursive learning (e.g. Barsky and De Long (1993), and Timmermann (1993, 2001)). However, the implications for the equity premium are not pursued. An exception is Brennan and Xia (2001): In a continuous-time general equilibrium setting a representative agent recursively estimates the unobservable drift of dividends. Using a risk aversion coefficient of 15 and a rate of time preference of -10%, they derive an equity premium of 6 percent and a risk-free rate of 2.5 percent. However, Brennan and Xia admit (p. 266) that learning has only second-order effects on the equity premium.[1] We show that first-order effects can be obtained using a plausible degree of curvature of the utility function.

Our paper offers two contributions. First, it removes the assumption lying at the core of the research on the equity premium puzzle that agents form full-information (FI) rational expectations. We focus on a restrictive rational learning mechanism that implies that prices reflect all possible, future perceived distributions of the parameters' estimates (future learning). Second, we prove that under the assumptions of MP (1985), when dividends follow a binomial lattice process, in equilibrium recursive rational learning

[1]MP (1985) argue that a γ in excess of 10 is unreasonable. According to D. Lucas (1994, p. 335) any solution that "(...) does not explain the premium for [a coefficient of relative risk aversion less or equal to] 2.5 is (...) likely to be widely viewed as a resolution that depends on a high degree of risk aversion." Hansen and Singleton (1983) estimate γ to be around 0.8.

may inflate the equity risk premium and reduce the risk-free interest rate for low levels of risk aversion. The key condition for these results to obtain is the presence of sufficient initial pessimism.

Rational learning introduces a new source of risk that supplements the scarce contribution coming from real consumption growth: the instability of the pricing relation $S_t^{RL} = \Psi_t^{RL} D_t$, where S is the asset price and D the dividend. Our model follows the traditional literature in that the price of consumption risk is determined by the coefficient of relative risk aversion (γ). It turns out that Ψ_t^{RL} is the more reactive to changes in beliefs along a learning path the lower γ is. In this sense a small degree of relative risk aversion is able to balance the trade-off between the price of consumption risk and the quantity of risk induced by the variability of the pricing kernel Ψ_t^{RL} in ways that offer (quantitatively) useful insights.

Section 2 introduces the model. Section 3 characterizes equilibrium asset prices under full-information rational expectations. This is a version of MP, specialized to the case of an i.i.d. binomial tree. Section 4 characterizes the rational learning scheme and derives expressions for asset prices. Section 5 discusses the implications for the equity premium and the risk-free rate of the two assumptions on belief formation. Section 6 provides additional comments and concludes.

2. **The Model**

The model is a version of the infinite horizon endowment economy studied by MP (1985). There are two assets: a one-period, risk-free, zero coupon bond in endogenous zero supply, yielding an interest rate of r_t (hence $B_t = \frac{1}{1+r_t}$ where B is the bond price), and a stock index with price S_t in exogenous net unit supply. The stock pays an infinite stream of real dividends $\{D_t\}_{t=1}^{\infty}$. These dividends are perishable; they cannot be reinvested and they must be consumed when they are received. The real growth rate of dividends $g_t \equiv \frac{D_t}{D_{t-1}} - 1$ follows a two-state Bernoulli process with probability parameter π. At each date the real growth rate is either high (g_h, a business cycle expansion) or low (g_l with $-1 < g_l < 0 < g_h$, recession). When confronted with smooth processes such as US consumption growth, a zero first-order autocorrelation is realistic.[2] g_h, g_l, and π may be subject to structural breaks.[3] For simplicity, assume that breaks are *observable.* Events of the magnitude of the Great Depression and the world energy crises are likely to be rapidly recognized. Thus, between today and T and conditioning on no breaks, the rate of growth of dividends follows a $(T-t)$-steps binomial process by which the growth rate in each period

[2] Abel (2002) and Barsky and De Long (1993) stress that to a first approximation dividends follow a random walk. This assumption is not completely at odds with the AR(1) model estimated in Section 2.1 as over the period 1930-1999 the recursive estimates of the autoregressive coefficient ϕ are always below 0.2 in absolute value and never statistically significant.

[3] Guidolin (2004) finds that the null of no breaks is unequivocally rejected in correspondence to the mid 1930s, the Great Depression. When the analysis is repeated conditioning on the occurrence of a first structural break during the 1930s (i.e. for a shorter 1938-1999 period), a second break is located in correspondence to the early 1980s.

$[t, t+1]$ is:

$$g_t = \begin{cases} g_h & \text{with prob. } \pi \\ g_l & \text{with prob. } 1-\pi \end{cases} \qquad \forall t \geq 1, \quad \pi \in (0,1) \tag{1}$$

and the rates of growth are independent. We assume perfect capital markets: unlimited short sales, perfect liquidity, no taxes, no transaction fees, no bid-ask spreads, and no borrowing constraints. There exists a representative agent who has constant relative risk aversion preferences

$$u(c_t) = \begin{cases} \frac{C_t^{1-\gamma}-1}{1-\gamma} & \gamma \neq 1 \\ \ln C_t & \gamma = 1 \end{cases}, \tag{2}$$

where C_t is real consumption. The agent maximizes the discounted value (at a rate $\rho > 0$) of the infinite stream of expected future (instantaneous) utilities derived from the consumption of real dividends.

3. Asset Pricing Under Full Information Rational Expectations

A representative agent knows the stochastic process of dividends – the binomial structure and $\{g_h, g_l, \pi_t\}$ – and forms rational expectations. Assume that breaks occur with such a low frequency to be safely disregarded by agents when they form expectations on future cash flows.[4] The representative agent solves:

$$\max_{\{C_{t+k}, w^s_{t+k}, w^b_{t+k}\}_{k=0}^{\infty}} E\left[\sum_{k=0}^{\infty} \beta^k u(C_{t+k}) | \mathcal{F}_t\right]$$
$$s.t.\ C_{t+k} + w^s_{t+k} S_{t+k} + w^b_{t+k} B_{t+k} = w^s_{t+k-1}(S_{t+k} + D_{t+k}) + w^b_{t+k-1}, \tag{3}$$

where $\beta = \frac{1}{1+\rho}$, and w^s_{t+k}, w^b_{t+k} represent the number of shares of stocks and bonds in the agent's portfolio as of period $t+k$. $E[\cdot|\mathcal{F}_t] \equiv E_t[\cdot]$ denotes the conditional expectation operator measurable with respect to $\mathcal{F}_t$, the information set. Standard dynamic programming methods yield the following Euler equations:

$$S_t = E\left[Q_{t+1}(S_{t+1} + D_{t+1}) \mid \mathcal{F}_t\right] \tag{4}$$
$$B_t = E\left[Q_{t+1} \mid \mathcal{F}_t\right], \tag{5}$$

where $Q_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$ is the pricing kernel (stochastic discount factor) defined as the product of the subjective discount factor and the intertemporal marginal rate of substitution in consumption. In equilibrium dividends are the only source of endowment and consumption, so $C_{t+k} = D_{t+k}\ \forall k \geq 0$.

[4] Our empirical analysis has isolated only 2 breaks in a 110 years long time series, a frequency of 1.8%. Timmermann (2001) uses a monthly probability of 0.3%. Guidolin (2004) calculates by simulation equilibrium prices when future, random breaks are taken into account and concludes that closed-form solutions provide a good approximation.

It is straightforward to prove that the full information rational expectations (FI) stock price, S_t^{FI}, is

$$S_t^{FI} = \lim_{T\to\infty} E_t \left[\sum_{j=1}^{T} \left(\beta^j \prod_{i=1}^{j} (D_{t+i}/D_{t+i-1})^{1-\gamma} \right) \right] \cdot D_t. \tag{6}$$

The linear homogenous form of the pricing function $S_t^{FI} = \Psi_t^{FI} D_t$ is a direct implication of expected utility maximization, where Ψ_t^{FI} denotes the pricing kernel, i.e. the price-dividend ratio, see Abel (2002, p. 1079) and Brennan and Xia (2001, p. 258), a time-varying function of π_t. Assuming

$$\rho > \pi_t g_h^* + (1-\pi_t) g_l^*, \tag{7}$$

where $g_l^* \equiv (1+g_l)^{1-\gamma} - 1$ and $g_h^* \equiv (1+g_h)^{1-\gamma} - 1$, Guidolin and Timmermann (2003a,b) prove that

$$S_t^{FI} = \Psi_t^{FI} D_t = \frac{1 + g_l^* + \pi_t (g_h^* - g_l^*)}{\rho - g_l^* - \pi_t (g_h^* - g_l^*)} D_t, \tag{8}$$

while the positive equilibrium risk-free rate, $r_t^{f,FI}$, is

$$r_t^{f,FI} = \frac{1}{B^{FI}} - 1 = \frac{1+\rho}{(1+g_l)^{-\gamma} + \pi_t \left[(1+g_h)^{-\gamma} - (1+g_l)^{-\gamma} \right]} - 1. \tag{9}$$

Condition (7) ensures not only $\Psi_t^{FI} > 0$ but also existence of the equilibrium.[5]

4. **Asset Prices on a Learning Path**

Suppose the agent is on a learning path: he knows that dividends follow a lattice $\{g_h, g_l, \pi_t\}$. He also knows g_h and g_l. However, π_t is unknown and the agent estimates it using all the available information since its last change (break), time τ_b. The agent recursively employs the simple estimator:

$$\widehat{\pi}_t^{\tau_b} = \frac{n_0^{\tau_b} + \sum_{j=\tau_b+1}^{t} I_{\{g_j = g_h\}}}{N_0^{\tau_b} + t - \tau_b + 1} = \frac{n_0^{\tau_b} + n_t^{\tau_b}}{N_0^{\tau_b} + N_t^{\tau_b}} \quad t > \tau_b, \tag{10}$$

where $I_{\{g_{t+j}=g_h\}}$ takes value 1 when at step/time j of the binomial tree dividends grow at a high rate, and zero otherwise. $n_t^{\tau_b}$ denotes the number of high growth states recorded between $\tau_b + 1$ and time t, while $N_t^{\tau_b}$ is the total number of dividend movements recorded over $[\tau_b + 1, t]$. After a break, investors are assumed to start out with beliefs synthesized by $\{n_0^{\tau_b}, N_0^{\tau_b}\}$.[6] $\widehat{\pi}_0^{\tau_b} = n_0^{\tau_b}/N_0^{\tau_b}$ reflects a starting belief agents hold on the probability of a good state, with $1/N_0^{\tau_b}$ the associated degree of precision.

Agents are on a *rational* learning (RL) path, see Guidolin and Timmermann (2003b), i.e. they take into account that their beliefs on π will be updated for $t' > t$ and incorporate the effects of future learning

[5] As shown in Guidolin and Timmernann (2003b), this condition is necessary and sufficient for existence.

[6] If breaks are observable, agents know when to restart their learning and discard previous information. If agents were uncertain on the breaks, econometric methods to estimate the breaks should be used. We do not pursue this extension.

in their current beliefs.[7] This framework stresses that agents perceive their own future beliefs as random variables measurable with respect to the sequence of future information sets. In the binomial tree model the RL compounded probability distribution can be fully characterized. Guidolin and Timmermann (2003a,b) prove that the distribution for the (gross) dividend growth rate between t and $t+T$ is given by

$$P^{RL}\left\{(1+g_h)^i(1+g_l)^{T-i} \mid n_t^{\tau_b}, N_t^{\tau_b}\right\} = \binom{T}{i}\left[\prod_{k=0}^{T-1}(N_t^{\tau_b}+k)\right]^{-1} \times \times\left[\prod_{k=0}^{i-1}(n_t^{\tau_b}+n_0^{\tau_b}+k)\prod_{k=0}^{T-i-1}(N_0^{\tau_b}+N_t^{\tau_b}-n_0^{\tau_b}-n_t^{\tau_b}+k)\right] \quad (11)$$

$i=0,...,T$, where $\binom{T}{i}=\frac{T!}{(j-i)!\,i!}$ is the permutation operator for $T\geq i$, and $\prod_{k=0}^{-1}(\cdot)=1$. The updated probability distribution of dividends for period $t+T$ only depends on the number of up-states occurring between periods t and $t+T-1$ and is independent of the specific path followed on the binomial lattice.

Guidolin and Timmermann (2003a,b) solve the consumption problem and apply standard methods to show that the following Euler equations characterize an internal optimum:

$$\begin{aligned} B^{RL} &= \widehat{E}_t^{\tau_b}\left[Q_{t+1}\right] \\ S_t^{RL} &= \widehat{E}_t^{\tau_b}\left\{Q_{t+1}\left[D_{t+1}+\widehat{E}_{t+1}^{\tau_b}\left(Q_{t+2}\left(D_{t+2}+\widehat{E}_{t+2}(...\widehat{E}_{t+T-1}^{\tau_b}(Q_{t+T}(D_{t+T}+S_{t+T}))...\right)\right)\right]\right\} \\ &= \widehat{E}_t^{\tau_b}(Q_{t+1}D_{t+1})+\widehat{E}_t^{\tau_b}[Q_{t+1}\widehat{E}_{t+1}^{\tau_b}(Q_{t+2}D_{t+2})]+...+\widehat{E}_t^{\tau_b}\{Q_{t+1}\widehat{E}_{t+1}^{\tau_b}[Q_{t+2}...\widehat{E}_{t+T-1}^{\tau_b}(Q_{t+T}D_{t+T})...]\}. \end{aligned} \quad (12)$$

$\widehat{E}_t^{\tau_b}[\cdot]\equiv E[\cdot|\widehat{\pi}_t^{\tau_b}, N_t^{\tau_b}]$ defines an expectation conditional on the information available at time t, in particular the current estimate of π_t after the last break in $\tau_b<t$. Since the sequence of conditional expectations at the nodes $t+1$, $t+2$, ..., $t+T$ depends on the future states $\widehat{\pi}_{t+1}^{\tau_b}$, $\widehat{\pi}_{t+2}^{\tau_b}$, ..., $\widehat{\pi}_{t+T-1}^{\tau_b}$, the law of iterated expectations can no longer be used as the distributions with respect to which future expectations are taken should also discount future information. Once this fact is recognized, it is straightforward to prove that if a transversality condition holds and $\rho>\max\{g_l^*, g_h^*\}$, the stock price under rational learning, S_t^{RL}, is

$$S_t^{RL}=\Psi_t^{RL}D_t=\left\{\sum_{i=1}^{\infty}\beta^i\sum_{j=0}^{i}(1+g_h^*)^j(1+g_l^*)^{i-j}P^{RL}\{(1+g_h^*)^j(1+g_l^*)^{i-j}|n_t^{\tau_b}, N_t^{\tau_b}\}\right\}\cdot D_t \quad (13)$$

where the RL probabilities are given by (11). The equilibrium risk-free rate is

$$r_t^{f,RL}(\widehat{\pi}_t^{\tau_b})=\frac{1+\rho}{(1+g_l)^{-\gamma}+\widehat{\pi}_t^{\tau_b}\left[(1+g_h)^{-\gamma}-(1+g_l)^{-\gamma}\right]}-1. \quad (14)$$

Clearly, the pricing kernel is no longer a constant, depending on the cumulated knowledge on π, through $n_t^{\tau_b}$ and $N_t^{\tau_b}$. In this sense, dividend changes between time t and $t+1$ acquire a self-enforcing nature:

[7] Most of the papers in the asset pricing literature use adaptive, least-squares learning schemes that do not allow prices to reflect the effects of future learning. For instance, Barsky and De Long (1993) realize that their model "(...) does not allow prices today to be influenced by investors' knowledge that they will be revising their estimate (...) in the future" (p. 299).

news of a certain sign will cause not only a stock price change through the linear pricing relationship $S_t^{RL} = \Psi_t^{RL} D_t$, but also through the revision of the pricing kernel, Ψ_t^{RL}. Moreover, while under FI the risk-free rate was a constant, on a learning path it changes as a function of the 'new' state variables $n_t^{\tau_b}$ and $N_t^{\tau_b}$.

5. **Implications for the Equity Premium**

5.1. *Full information rational expectations*

In the FI case, the mapping simply consists of a relationship between preferences $[\rho\ \gamma]'$ and equilibrium asset returns. It is straightforward to derive the expression for the equity premium:

$$E[r_t^{p,FI}] = (1+\rho)\left\{\frac{1+g_l+\pi_t(g_h-g_l)}{\pi_t(1+g_h^*)+(1-\pi_t)(1+g_l^*)} - \frac{1}{\pi_t(1+g_h)^{-\gamma}+(1-\pi_t)(1+g_l)^{-\gamma}}\right\}. \qquad (15)$$

Since dividends are i.i.d., there is no difference between the time t conditional and unconditional equity premium. Moreover, besides $\{g_h, g_l, \pi_t\}$, (15) depends on $[\rho\ \gamma]'$ only. To stress this dependency, we write $E[r_t^{p,FI}(\rho,\gamma)]$ and similarly $r_t^{f,FI}(\rho,\gamma)$ and $E[r_t^{e,FI}(\rho,\gamma)]$. It is then natural to ask what is the shape of $E[r_t^{p,FI}(\rho,\gamma)]$ and equilibrium expected returns as $[\rho\ \gamma]'$ vary.

Recalling condition (7), it turns out that investigating the behavior of equilibrium asset returns and of the equity premium as $[\rho\ \gamma]'$ changes implies two distinct sets of issues. Since given ρ and $g_l < 0$ a too high or too low γ may violate (7), a first crucial task is to characterize the interval $(\underline{\gamma}, \bar{\gamma}) \subseteq [0, +\infty)$ over which the FI equilibrium exists, such that (15) and other quantities of interest are actually defined. To this purpose we introduce the following notation. Define

$$K_t^e(\gamma) \equiv \pi_t g_h^* + (1-\pi_t) g_l^*,$$

the denominator (minus one) of the stock return implied by (8), and $K_t^f(\gamma) \equiv \pi_t[(1+g_h)^{-\gamma}-1] + (1-\pi_t)[(1+g_l)^{-\gamma}-1]$, the denominator (minus one) of (9). Notice that $K_t^f(\gamma) = K_t^e(\gamma+1)$ and

$$E[r_t^{e,FI}(\rho,\gamma)] = (1+\rho)\frac{1+g_l+\pi_t(g_h-g_l)}{1+K_t^e(\gamma)} - 1 \quad r_t^{f,FI}(\rho,\gamma) = \frac{1+\rho}{1+K_t^f(\gamma)} - 1.$$

(7) can then be re-written as $1+\rho > 1+K_t^e(\gamma)$. Since

$$K_t^e(0) = \pi_t g_h + (1-\pi_t) g_l = E[g_t] \quad \lim_{\gamma\to\infty}\sup K_t^e(\gamma) = \infty,$$

(7) fails as $\gamma \to \infty$ or, equivalently, one can find a $\bar{\gamma}(\rho) < \infty$ such that only for $\gamma < \bar{\gamma}(\rho)$ the equilibrium exists. As for $\underline{\gamma}(\rho)$, if $\rho > E[g_t]$ then $\underline{\gamma}(\rho) = 0$; otherwise if one can find a $\underline{\gamma}(\rho) > 0$ such that $1+\rho = 1+K_t^e(\underline{\gamma}(\rho))$, the equilibrium exists only for $\gamma > \underline{\gamma}(\rho)$. Naturally, it makes sense to study the behavior of

the equity premium and of asset returns as a function of $[\rho\ \gamma]'$ only when such returns (and differentials thereof) are defined in equilibrium, i.e. for $\gamma \in (\underline{\gamma}(\rho),\ \bar{\gamma}(\rho))$. Interestingly, for $\rho > 0$ (as assumed) the equilibrium always exists since $1+\rho > 1+K_t^e(1) = 1$ holds. Thus either $(\underline{\gamma}(\rho),\ \bar{\gamma}(\rho))$ is an interval or it reduces to the singleton $\{1\}$. Since $\bar{\gamma}(\rho) < \infty$, it follows that in this lattice model one cannot always generate a realistic equity premium by simply increasing γ and exploiting the agent's risk aversion.

Assuming the equilibrium exists on an interval $(\underline{\gamma}(\rho),\ \bar{\gamma}(\rho))$, it becomes then interesting to investigate the behavior of expected asset returns and the equity premium as γ changes. The following result characterizes the basic properties of asset returns in this artificial economy. Let, whenever defined, γ^e, γ^f, and $\gamma^{\max}$ be the maximum points on $(\underline{\gamma}(\rho),\ \bar{\gamma}(\rho))$ of the risk-free rate, expected stock returns, and the equity risk premium (as functions of γ), respectively.

Proposition 1. *Assume the full-information rational expectations equilibrium exists on* $(\underline{\gamma}(\rho),\ \bar{\gamma}(\rho))$.

(a) $E[r_t^{e,FI}(\rho,\gamma)]$ *may be either monotone decreasing in* γ *or it may be at first increasing over* $(\underline{\gamma}(\rho),\ \gamma^e)$ *and then decreasing over* $(\gamma^e,\ \bar{\gamma}(\rho))$. *There exists a* ρ^* *such that for all* $\rho \geq \rho^*$ $E[r_t^{e,FI}(\rho,\gamma)]$ *admits a global maximum at* γ^e.

(b) $r_t^{f,FI}(\rho,\gamma)$ *may be either monotone decreasing in* γ *or it may be at first increasing over* $(\underline{\gamma}(\rho),\ \gamma^f)$ *and then decreasing over* $(\gamma^f,\ \bar{\gamma}(\rho))$. *If and only if* $\rho \geq \rho^*$ *such that* $\gamma^e > 1$, $r_t^{f,FI}(\rho,\gamma)$ *admits a global maximum at* γ^f. *Furthermore,* $\gamma^f < \gamma^e$.

(c) $E[r_t^{p,FI}(\rho,\gamma)]$ *may be either monotone increasing in* γ *or it may be at first increasing over* $(\underline{\gamma}(\rho),\ \gamma^{\max}(\rho))$ *and then decreasing over* $(\gamma^{\max},\ \bar{\gamma}(\rho))$. *There exists a* ρ^{**} *such that for all* $\rho \geq \rho^{**}$ $E[r_t^{p,FI}(\rho,\gamma)]$ *admits a global maximum at* $\gamma^{\max}$.

Proof: See Appendix.

Parts (a) and (b) have a clear interpretation: since under power utility the intertemporal rate of substitution is the inverse of γ, eventually equilibrium asset returns must decline as γ increases and the rate of intertemporal substitution vanishes. The intuition is that a low intertemporal rate of substitution gives strong incentives to the agent to smooth consumption by demanding tradeable assets; given exogenous supply, strong demand increases equilibrium prices and lowers expected rates of return. Logically, this effect is weakened when an agent strongly discounts future consumption streams (high ρ): in this case, despite the strong consumption-smoothing motive, future portfolio payoffs receive less weight than current consumption. In fact, if ρ is sufficiently high, there exists a range of values in $(\underline{\gamma}(\rho),\ \bar{\gamma}(\rho))$ over which expected asset returns increase in γ. In the case of stocks, this derives from the need to be compensated for risk; in the case of bonds from that fact that when $\gamma^e > 1$, parameters are such that over $(\underline{\gamma}(\rho),\ \gamma^f)$ the agent would like to short-sell bonds to invest more in stocks and this drives interest rates up. Interestingly, the phenomenon is stronger for bonds in the sense that there is a range (γ^f, γ^e) over which the risk-free rate declines in γ while expected stock returns increase: Investors use at first bonds for consumption smoothing purposes while risk averse investors keep requiring higher stock returns as γ increases.

Part (c) is more puzzling and implies that $-$ provided ρ is sufficiently high $-$ the equity premium is declining in γ over $(\gamma^{\max}, \bar{\gamma}(\rho))$, contrary to standard intuition that more risk-averse investors are rewarded higher risk premia. Over selected sub-intervals of $(\underline{\gamma}(\rho), \bar{\gamma}(\rho))$ this result is a mechanical consequence of (a) - (b) for the case in which ρ is sufficiently high and $\gamma^e \in (\underline{\gamma}(\rho), \bar{\gamma}(\rho))$. If $\gamma^e \leq 1$, then there exists a region $(\underline{\gamma}(\rho), \gamma^e)$ over which expected stock returns increase in γ while the risk-free rate declines: over that region the equity premium must be increasing. A similar effect takes place when $\gamma^e > 1$: there exists an interval (γ^f, γ^e) over which expected stock returns increase in γ while the risk-free rate declines. This depends on the fact that bond prices seem to be affected by the progressive decline in the elasticity of substitution 'before' stocks are. Finally, over $(\underline{\gamma}(\rho), \gamma^f)$ (assuming γ^f exists) both expected stock returns and the risk-free rate are increasing, but it turns out that $E[r_t^{e,FI}(\rho,\gamma)]$ increases at a faster rate than $r_t^{f,FI}(\rho,\gamma)$; on the opposite, in $(\gamma^e, \bar{\gamma}(\rho))$ the equity premium must eventually be decreasing in γ as $E[r_t^{e,FI}(\rho,\gamma)]$ decreases faster than the risk-free rate. In both cases, however, the idea is still that a declining intertemporal rate of substitution seems to hit bond prices at a comparatively stronger rate than stock prices. Given the inverse relationship between asset prices and expected returns, we obtain (c).

The proposition offers direct implications for the possibility to produce under FI plausible equity premia and interest rates. The naive notion that a high γ offers a way to resolve the puzzles does not apply. From (c), we can only hope that $\bar{\gamma}(\rho)$ is high enough to span an interval that includes a γ (either $\gamma^{\max}$ or $\bar{\gamma}(\rho)$) such that a risk premium as high as in the data obtains.

5.2. *Rational Learning*

Under RL, preferences affect equilibrium prices in ways that depend on the state of beliefs. From (13), $S_t^{RL} = \Psi^{RL}(n_t^{\tau_b}, N_t^{\tau_b}; \rho, \gamma) D_t$. Therefore — assuming the absence of a structural break between t and $t+1$ — the excess stock return over the interval $[t, t+1]$ is:

$$r_{t+1}^{p,RL}(n_t^{\tau_b}, N_t^{\tau_b}, n_{t+1}^{\tau_b}; \rho, \gamma) = \frac{(1+g_{t+1})}{\Psi^{RL}(n_t^{\tau_b}, N_t^{\tau_b})} + \frac{\Psi^{RL}(n_{t+1}^{\tau_b}, N_t^{\tau_b}+1)}{\Psi^{RL}(n_t^{\tau_b}, N_t^{\tau_b}\gamma)}(1+g_{t+1}) - 1 - r_t^{f,RL}(n_t^{\tau_b}, N_t^{\tau_b}) \tag{16}$$

On a learning path, *realized* excess equity returns depend on both the agent's initial beliefs $(n_t^{\tau_b}, N_t^{\tau_b})$ as well as on the change between t and $t+1$ (through the ratio $\Psi^{RL}(n_{t+1}^{\tau_b}, N_t^{\tau_b}+1)/\Psi^{RL}(n_t^{\tau_b}, N_t^{\tau_b})$).

A different concept is the equity premium *expected* at time t, conditional on the information on the process for dividends then available $(n_t^{\tau_b}, N_t^{\tau_b})$:

$$\hat{E}_t^{\tau_b}\left[r_{t+1}^{pRL,}(\rho,\gamma)\right] = \frac{\hat{E}_t^{\tau_b}[(1+g_{t+1})]}{\Psi^{RL}(n_t^{\tau_b}, N_t^{\tau_b};\rho,\gamma)} + \frac{\hat{E}_t^{\tau_b}[(1+g_{t+1})\Psi^{RL}(n_{t+1}^{\tau_b}, N_{t+1}^{\tau_b};\rho,\gamma)]}{\Psi^{RL}(n_t^{\tau_b}, N_t^{\tau_b};\rho,\gamma)} - \frac{1+\rho}{\hat{E}_t^{\tau_b}[(1+g_{t+1})^{-\gamma}]}. \tag{17}$$

Notice that (17) represents a *subjective* notion of the equity premium as $\hat{E}_t^{\tau_b}[\cdot]$ is a subjective expectation operator that depends on the agents' information set $(n_t^{\tau_b}, N_t^{\tau_b})$. Importantly,

$$E\left[r_{t+1}^{p,RL}(\rho,\gamma)\right] \neq E\left\{\hat{E}_t^{\tau_b}\left[r_{t+1}^{p,RL}(\rho,\gamma)\right]\right\},$$

i.e. the *objective* unconditional equity premium (left-hand side) on a rational learning path differs from the *objectively expected subjective* equity premium (right-hand side). This difference raises an important issue. Traditionally, the literature has discussed the circumstances under which a model generates a stationary distribution for excess returns matching sample moments. In particular, the equity premium $E[r_{t+1}^p(\rho,\gamma)]$ is identified as a long-run sample mean of excess equity returns. While $E[r_{t+1}^{p,RL}(\rho,\gamma)]$ can be quantified by simulating prices in (16) and (14) and taking averages over simulation trials, this quantity is in general different from the estimate of $E\left\{\hat{E}_t^{\tau_b}\left[r_{t+1}^{p,RL}(\rho,\gamma)\right]\right\}$ that can be similarly obtained by simulation. Although the average (expected) subjective equity premium may be interesting in itself (see Abel (2000)), it is clear that the only quantity that can be directly compared to the data is $E[r_{t+1}^{p,RL}(\rho,\gamma)]$.

This section reports results for the objective *conditional* expectation of the equity premium under RL; Section 7 uses simulations from a calibrated version of our model to produce results on the objective *unconditional* premium to be compared with the empirical evidence. Guidolin and Timmermann (2003b) show that Ψ_t^{RL} is an increasing and convex function of $n_t^{\tau_b}$ when $\gamma < 1$. Additionally, define the functions $\epsilon^*(n_0^{\tau_b} + n_t^{\tau_b}, N_0^{\tau_b} + N_t^{\tau_b})$ and $\delta^*(\pi_t, n_0^{\tau_b} + n_t^{\tau_b}, N_0^{\tau_b} + N_t^{\tau_b})$ as:

$$\epsilon^* \quad : \quad \Psi^{RL}(n_0^{\tau_b} + n_t^{\tau_b} + \epsilon^*, N_0^{\tau_b} + N_t^{\tau_b} + 1) = \Psi^{RL}(n_0^{\tau_b} + n_t^{\tau_b}, N_0^{\tau_b} + N_t^{\tau_b}) \tag{18}$$

$$\begin{aligned} \delta^* \quad : \quad & \pi_t \Psi^{RL}(n_0^{\tau_b} + n_t^{\tau_b} + 1 + \delta^*, N_0^{\tau_b} + N_t^{\tau_b} + 1) + (1-\pi_t)\,\Psi^{RL}(n_0^{\tau_b} + n_t^{\tau_b} + \delta^*, N_0^{\tau_b} + N_t^{\tau_b}+1) = \\ = \quad & \frac{n_0^{\tau_b}+n_t^{\tau_b}}{N_0^{\tau_b}+N_t^{\tau_b}}\Psi^{RL}(n_0^{\tau_b}+n_t^{\tau_b}+1+\delta^*, N_0^{\tau_b}+N_t^{\tau_b}+1) + \left(1\text{-}\frac{n_0^{\tau_b}+n_t^{\tau_b}}{N_0^{\tau_b}+N_t^{\tau_b}}\right)\Psi^{RL}(n_0^{\tau_b}+n_t^{\tau_b}+\delta^*, N_0^{\tau_b}+N_t^{\tau_b}+1). \end{aligned} \tag{19}$$

$\epsilon^* > 0$ increases $n_{t+1}^{\tau_b}$ over $n_t^{\tau_b}$ enough to compensate the adverse effect of the updating of $N_{t+1}^{\tau_b}$ to $N_t^{\tau_b}+1$; δ^* fills the gap between the objective $E_t[\Psi_{t+1}^{RL}]$ (calculated using the true but unknown π_t) and the subjective $\hat{E}_t^{\tau_b}[\Psi_{t+1}^{RL}]$ (calculated using $\hat{\pi}_t$). It is straightforward to show that when beliefs are pessimistic, $\widehat{\pi}_t^{\tau_b} < \pi_t$, $\delta^* > 0$; furthermore, $\delta^* \geq \epsilon^*$ means that the effects on the RL kernel of an increase in $N_0^{\tau_b} + N_t^{\tau_b}$ are weaker than those caused by pessimism. Using these results we can state the following:

Proposition 2. *When $\gamma < 1$ and for any given π_t, there exists a pair $[n_0^{\tau_b} + n_t^{\tau_b},\ N_0^{\tau_b} + N_t^{\tau_b}]'$ defining* *pessimistic beliefs* ($\widehat{\pi}_t^{\tau_b} \equiv (n_0^{\tau_b} + n_t^{\tau_b})/(N_0^{\tau_b} + N_t^{\tau_b}) < \pi_t$) *and implying* $\delta^* \geq \epsilon^*$ *such that*

$$\begin{aligned} r_t^{f,RL}(\rho,\gamma) &< r_t^{f,FI}(\rho,\gamma) \\ E_t[r_{t+1}^{p,RL}(\rho,\gamma)] &> E[r_{t+1}^{p,FI}(\rho,\gamma)]. \end{aligned}$$

Proof: See Appendix.

We are able to isolate a combination of risk-aversion and beliefs — low risk aversion and pessimism — for which the conditional risk premium is higher under RL than the (unconditional) risk premium under FI. In principle, the incorporation of learning effects points in the direction of higher equity premia for plausible preferences, provided the economy is characterized 'on average' by some degree of pessimism. Moreover, under the same assumptions, $r_t^{f,RL}(\rho,\gamma) \leq r_t^{f,FI}(\rho,\gamma)$ should make the occurrence of a risk-free rate puzzle unlikely. However, Proposition 2 has only partial relevance for the quantitative assessment of the unconditional risk premium. On one hand, a straightforward corollary to Proposition 2 obtains:

Corollary. *When* $\gamma < 1$ *and for any given* π_t, *there exists a pair* $[n_0^{\tau_b} + n_t^{\tau_b},\ N_0^{\tau_b} + N_t^{\tau_b}]'$ *defining pessimistic beliefs* ($\widehat{\pi}_t^{\tau_b} \equiv (n_0^{\tau_b} + n_t^{\tau_b})/(N_0^{\tau_b} + N_t^{\tau_b}) < \pi_t$) *such that the initial precision* $N_0^{\tau_b}$ *in (10) is sufficiently large, there exists a date* $T > t$ *(function of* $n_0^{\tau_b}, n_t^{\tau_b}$ *and* $N_0^{\tau_b}$*) s.t.*

$$\frac{1}{T}\sum_{j=t}^{T}\sum_{i=1}^{t} E_j\left[r_{j+1}^{p,RL}(n_j^{\tau_b}+i, N_t^{\tau_b}+j;\rho,\gamma)\right] P^{RL}\{(1+g_h)^i(1+g_h)^{t-i}|n_j^{\tau_b}, N_t^{\tau_b}\} > E\left[r_{t+1}^{p,FI}(\rho,\gamma)\right]$$

$$\frac{1}{T}\sum_{j=t}^{T}\sum_{i=1}^{t} r_j^{f,RL}(n_j^{\tau_b}+i, N_t^{\tau_b}+j;\rho,\gamma) P^{RL}\{(1+g_h)^i(1+g_h)^{t-i}|n_j^{\tau_b}, N_t^{\tau_b}\} < r_t^{f,FI}(\rho,\gamma).$$

Proof: When $[n_0^{\tau_b} + n_t^{\tau_b},\ N_0^{\tau_b} + N_t^{\tau_b}]'$ is such that $\widehat{\pi}_t^{\tau_b}$ is sufficiently below π_t and $N_0^{\tau_b}$ is sufficiently large, there is a T such that in (10) $\widehat{\pi}_j^{\tau_b} < \pi_t\ \forall j \in [t,T]$ even if the realizations of the dividend growth rate are uniformly high in $[t,T]$, i.e. for all sample paths that beliefs may follow. Clearly, given $\widehat{\pi}_t^{\tau_b}$ the higher is $N_0^{\tau_b}$, the higher is T. From Proposition 2, $\widehat{\pi}_j^{\tau_b} < \pi_t\ \forall j \in [t,T]$ and $\gamma < 1$ imply $E_j[r_{j+1}^{p,RL}(n_j^{\tau_b}+i, N_t^{\tau_b}+j;\rho,\gamma)] > E[r_{j+1}^{p,FI}(\rho,\gamma)]$ independently of the specific time j beliefs. Therefore, summing over all possible future values of $n_j^{\tau_b}$ (hence $\hat{\pi}_j^{\tau_b}$):

$$\sum_{i=1}^{t} E_j\left[r_{j+1}^{p,RL}(n_j^{\tau_b}+i, N_t^{\tau_b}+j;\rho,\gamma)\right] P^{RL}\{(1+g_h)^i(1+g_h)^{t-i}|n_j^{\tau_b}, N_t^{\tau_b}\} > E\left[r_{t+1}^{p,FI}(\rho,\gamma)\right].$$

The inequality for the premium follows. Identical steps lead to the claim concerning the riskless rate. □

In short, the Corollary states that although nothing can be said about the unconditional equity premium and risk-free rates, provided that the averaging takes place over a sufficiently short sample $[t,T]$, mean excess returns (riskless interest rates) will exceed (be inferior) to the FI unconditional levels independently of the sample path of beliefs over $[t, T]$. Importantly, the type of time averages appearing in the Corollary correspond to the estimates of the equity premium we will consider in the simulations of Section 7.

Contrary to the FI case, under RL it is difficult to characterize the behavior of equilibrium expected returns and of the equity premium as a function of preferences only. In fact, no analog to Proposition 1 can be proven because ρ and γ have effects on asset returns that depend on the state of beliefs, $\widehat{\pi}_t^{\tau_b}$. The most striking property of the equity premium under RL is that $-$ contrary to Proposition 1(c) $-$ for $\gamma < 1$ and pessimistic beliefs, it is possible for the conditional premium to be decreasing in $\gamma < 1$ over some range. This fact implies the possibility of increasing the equity premium generated by the model by *lowering* γ, instead of increasing it. In the light of the vast debate on what upper bound on the coefficient of relative risk aversion is plausible, it is crucial to find explanations for the high historical premium relying on low risk aversion. Our intuition for this result is that as an economy becomes risk-neutral, the intertemporal elasticity of substitution $(1/\gamma)$ becomes larger and $r_t^{f,RL}(\rho, \gamma)$ declines towards ρ while the demand for bonds decreases as the agent reduces her allocation to bonds for consumption smoothing reasons. Even though (14) implies that interest rates rise as the pessimism is eliminated by repeated observation of the fundamental outcomes, such an economy is characterized by modest risk-free rates. On the other hand, over time the agent revises her pessimistic beliefs towards the true, unobserved π_t. Since the RL pricing kernel increases in $n_t^{\tau_b}$, the price-dividend ratio progressively climbs up and generates high stock returns.

6. **Discussion**

RB are beliefs about state variables that are generally incorrect but that are not inconsistent with the unconditional properties of the unknown DGP. Simulations in Kurz (1996, 1997) and Kurz and Motolese (2001) have documented that under parametric restrictions on the distribution of beliefs across agents, high premia may be generated along with low interest rates. Restrictions on beliefs are equivalent to patterns of prevalence and intensity of pessimists vs. optimists. In Kurz and Motolese (2001) a key condition requires that on average the majority of agents are optimistic about the prospects of capital gains while a minority is more intensely pessimistic.

There are obvious similarities between the learning-based and the RB approaches. First, we also study the case in which agents fail to know distribution of state variables. Agents are wrong about the distribution of state variables at all dates, although by standard asymptotic arguments the long-run distribution can eventually be learnt. Departures of beliefs from the unknown true distribution generate fluctuations in asset prices that differ from rational expectations values. Second, we study a stable but non-stationary system characterized by infrequent structural breaks, i.e. we approximate non-stationarity with a sequence of time intervals over which the unknown DGP is invariant, see Kurz's (1997) notion of "environments". Third, also the RB approach to the equity premium puzzle has generated results from restrictions on the structure of beliefs that can be read in terms of pessimism vs. optimism. We do the same, although our restriction is weaker and concerns only initial beliefs after the occurrence of breaks and not the entire sample path. One can see rational learning as a selection device for one set of rational beliefs.

Clearly, the crucial difference consists in the fact that RB theory has focused on the effects of heterogeneity, in the sense that divergent beliefs are not only allowed under RB, but they are essential for the approach to produce interesting pricing implications. On the contrary, our paper adopts the standard fiction of a representative agent and thus concentrates on the effects of dynamics in beliefs. Second, while in our model this dynamics is governed by the realizations of the dividend process, RB theory has employed the additional construct of generating variables (Nielsen (1996)). Although these variables are not essential to RB theory and they could be a function of observed state variables, in Kurz (1997, 1998), and Kurz and Motolese (2001) they are assumed to be independently and identically distributed. Therefore recursive rational learning imposes a further restriction on the mapping between observed variables and belief dynamics. Third, RB theory lists future spot asset prices as part of the state space, that is therefore enlarged to imply *endogenous* uncertainty. In our model uncertainty remains strictly *exogenous*, although it is greatly amplified by learning. Fourth, while the RB literature has simultaneously explained a number of features of financial markets besides the equity premium and the risk-free rate puzzles (e.g. the forward exchange rate puzzle, long-horizon predictability, endogenous stochastic volatility, etc. see Kurz, Jin, and Motolese (2003)), our RL model can only claim partial success on the first two moments of the most primitive asset returns and shows some deficiency when its dynamic (conditional) implications are assessed. We conjecture that only extensions that jointly model recursive learning effects and heterogeneous beliefs about other agents' beliefs might provide a unified framework to understand asset prices.

In conclusion, although differences remain, both approaches to the puzzles spring off the principle that the structure and dynamic of agents' beliefs are as important primitive explanations variables of economic phenomena as the usual list of 'suspects', preferences, technology, and frictions/constraints.

This paper shows that there exists an alternative way in which extreme events such as the Great Depression or the oil shocks can generate high equity premia. While previous literature has focused on the induced, permanent biases in the stationary beliefs of investors in an *ad hoc* fashion, we show that if agents are on a recursive learning path, tail events may produce long-lasting effects on equilibrium prices.

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Appendix

Proof of Proposition 1. (a) To establish how expected stock returns change as a function of γ (for given ρ), let's study the behavior of $K_t^e(\gamma)$ in $(\underline{\gamma}, \bar{\gamma})$. Differentiating $K_t^e(\gamma)$ with respect to γ we get a function $k_t^e(\gamma) \equiv dK^e(\gamma)/d\gamma$ defined as:

$$k_t^e(\gamma) = -\pi_t[(\ln(1+g_h))\,(1+g_h)^{1-\gamma}] - (1-\pi_t)[(\ln(1+g_l))\,(1+g_l)^{1-\gamma}].$$

When $g_h > 0 > g_l > -1$, $k_t^e(\gamma)$ is strictly increasing in γ (each of the summands is), and $\lim_{\gamma\to\infty} \sup k_t^e(\gamma) = \infty$. The behavior of $E[r_t^{e,FI}(\rho,\gamma)]$ can be completely characterized by these properties of $k_t^e(\gamma)$ and the behavior of $k_t^e(\gamma)$ at $\gamma = 0$, which depends on ρ. Four cases can be considered:
(i) $k_t^e(0) < 0$ and $1+\rho \leq 1+K_t^e(0) = 1+\pi_t g_h + (1-\pi_t)g_l$. Since $k_t^e(\gamma)$ is continuous in γ, $k_t^e(0) < 0$ and $k_t^e(\gamma) \to \infty$ as $\gamma \to \infty$, there exists a γ^e such that $k_t^e(\gamma^e) = 0$, i.e. $K_t^e(\gamma)$ is U-shaped with a global minimum in γ^e. This implies the existence of $\underline{\gamma}(\rho)$ and $\bar{\gamma}(\rho)$ such that $1+\rho = 1+K_t^e(\underline{\gamma}(\rho)) = 1+K_t^e(\bar{\gamma}(\rho))$ and $\gamma^e \in (\underline{\gamma}(\rho), \bar{\gamma}(\rho))$. Therefore $E[r_t^{e,FI}(\rho,\gamma)]$ is bell-shaped with an interior global maximum at γ^e.
(ii) $k_t^e(0) < 0$ and $1+\rho > 1+K_t^e(0)$. Similarly to case (i), there must exist a γ^e such that $k_t^e(\gamma^e) = 0$, i.e. $K_t^e(\gamma)$ is U-shaped with a global minimum in γ^e. However in this case there is no $\underline{\gamma}(\rho) \geq 0$ such that $1+\rho = 1+K_t^e(\underline{\gamma}(\rho))$ since $1+\rho > 1+K_t^e(0)$. Therefore $\underline{\gamma}(\rho) = 0$. On the other hand, there is a $\bar{\gamma}(\rho) < \infty$ such that $1+\rho = 1+K_t^e(\bar{\gamma}(\rho))$. $E[r_t^{e,FI}(\rho,\gamma)]$ is bell-shaped with an interior global maximum at γ^e.
(iii) $k_t^e(0) \geq 0$ and $1+\rho \leq 1+K_t^e(0)$. Since $k_t^e(\gamma)$ is positive over $[0,\infty)$, $K_t^e(\gamma)$ is a convex and everywhere strictly increasing function. This implies that for no value of $\gamma \neq 1$ (7) is satisfied, as

$1+\rho \leq 1+K_t^e(0) < 1+K_t^e(\gamma)$. Therefore this case is empty and the interval $(\underline{\gamma}(\rho), \bar{\gamma}(\rho))$ collapses to the singleton $\gamma = 1$.

(iv) $k_t^e(0) \geq 0$ and $1+\rho > 1+K_t^e(0)$. Since $k_t^e(\gamma)$ is positive over $[0, \infty)$, $K_t^e(\gamma)$ is a convex and everywhere strictly increasing function. While we let $\underline{\gamma}(\rho) = 0$ for the same reasons in (ii), there is a $\bar{\gamma}(\rho) < \infty$ such that $1+\rho = 1+K_t^e(\bar{\gamma}(\rho))$. In this case, $E[r_t^{e,FI}(\rho,\gamma)]$ is monotone decreasing over $(0, \bar{\gamma}(\rho))$.

In conclusion, when the equilibrium exists on an interval $(\underline{\gamma}(\rho), \bar{\gamma}(\rho))$ and given ρ, $E[r_t^{e,FI}(\rho,\gamma)]$ is either bell-shaped with an interior global maximum γ^e independent of ρ (cases (i) and (ii)) or monotone decreasing in γ (case (iv)). Since as $\rho \to \infty$ only cases (ii) and (iv) become possible, $\limsup_{\rho\to\infty} \underline{\gamma}(\rho) = 0$ and $\limsup_{\rho\to\infty} \bar{\gamma}(\rho) = \infty$, there exists a ρ^* such that for $\rho \geq \rho^*$ $\gamma^e \in (\underline{\gamma}(\rho), \bar{\gamma}(\rho))$ represents the global maximum for expected stock returns.

(b) To establish how expected stock returns change as a function of γ (for given ρ), study the behavior of $K_t^f(\gamma)$ in $(\underline{\gamma}, \bar{\gamma})$. Differentiating $K_t^f(\gamma)$ w.r.t. γ we get a function $k_t^f(\gamma) \equiv dK^f(\gamma)/d\gamma$:

$$k_t^f(\gamma) = -\pi_t[(\ln(1+g_h))(1+g_h)^{-\gamma}] - (1-\pi_t)[(\ln(1+g_l))(1+g_l)^{-\gamma}].$$

Clearly, $K_t^f(0) = 0$, when $g_h > 0 > g_l > -1$ $k_t^f(\gamma)$ is strictly increasing in γ (each of the summands is), and $\lim_{\gamma\to\infty} \sup k_t^e(\gamma) = \infty$. Two cases are then relevant: either $k_t^e(0) \geq 0$ so that $k_t^e(\gamma) > 0$ for all γ's, or $k_t^e(0) < 0$ so the continuity of $k_t^e(\gamma)$ in γ implies that a γ^f can be found such that $k_t^e(\gamma^f) = 0$. In the former case, $K_t^f(\gamma)$ is a convex and monotonic increasing function, implying that $r_t^{f,FI}(\rho,\gamma)$ is monotonically decreasing in γ. In the latter case, $K_t^f(\gamma)$ is U-shaped, implying that $r_t^{f,FI}(\rho,\gamma)$ is bell-shaped with a global maximum over $[0, \infty)$ (independent of ρ). This last result does not imply though that the riskless rate has a global maximum in $(\underline{\gamma}(\rho), \bar{\gamma}(\rho))$. To this purpose, notice that from the definitions of $k_t^e(\gamma)$ and $k_t^f(\gamma)$, it follows that $k_t^f(\gamma-1) = k_t^e(\gamma)$ so that if γ^e exists such that $k_t^e(\gamma^e) = 0$, then $k_t^f(\gamma^e - 1) = k_t^f(\gamma^f) = 0$. Therefore $\gamma^f = \gamma^e - 1$ and $\gamma^e > \gamma^f$. Therefore an interior global minimum for $k_t^f(\gamma)$ (hence an interior global maximum for the risk-free rate) over $(\underline{\gamma}(\rho), \bar{\gamma}(\rho))$ exists if and only if γ^e exists and $\gamma^f = \gamma^e - 1 > 0$ or $\gamma^e > 1$.

In conclusion, when the equilibrium exists in $(\underline{\gamma}(\rho), \bar{\gamma}(\rho))$, $r_t^{f,FI}(\rho,\gamma)$ is either bell-shaped with an interior global maximum γ^f or monotone decreasing in γ. Since it is always possible to find a ρ^* such that for $\rho \geq \rho^*$ $\gamma^e \in (\underline{\gamma}(\rho), \bar{\gamma}(\rho))$, analogously for $\rho \geq \rho^*$ there exists $\gamma^f \in (\underline{\gamma}(\rho), \bar{\gamma}(\rho))$ if and only if $\gamma^e > 1$.

(c) Finally, the equity premium is simply the difference between $E[r_t^{e,FI}(\rho,\gamma)]$ and $r_t^{f,FI}(\rho,\gamma)$. Given γ, (15) makes it evident that the premium is an increasing function of ρ. On the interval $[0, \infty)$ it is clear that $E[r_t^{e,FI}(\rho,0)] = 1+\rho = r_t^{f,FI}(\rho,0)$ so that $E[r_t^{p,FI}(\rho,0)] = 0$, which corresponds to standard intuition, a risk-neutral investor will demand a zero risk premium; also, since $\limsup_{\gamma\to\infty} E[r_t^{e,FI}(\rho,0)] = -1 = \limsup_{\gamma\to\infty} r_t^{f,FI}(\rho,0)$, it follows that $\limsup_{\gamma\to\infty} E[r_t^{p,FI}(\rho,0)] = 0$. However $E[r_t^{p,FI}(\rho,0)]$ is the difference of two continuous functions of γ and is everywhere non-negative since positivity of (15) is guaranteed by:

$$\begin{gathered}
[1+g_l+\pi_t(g_h-g_l)]\left[\pi_t(1+g_h)^{-\gamma}+(1-\pi_t)(1+g_l)^{-\gamma}\right] > \pi_t(1+g_h)^{1-\gamma}+(1-\pi_t)(1+g_l)^{1-\gamma} \\
\Longrightarrow [\pi_t(1+g_h)+(1-\pi_t)(1+g_l)]\left[\pi_t(1+g_h)^{-\gamma}+(1-\pi_t)(1+g_l)^{-\gamma}\right] > \\
> \pi_t(1+g_h)(1+g_h)^{-\gamma}+(1-\pi_t)(1+g_l)(1+g_l)^{-\gamma} > \pi_t(1+g_h)^{1-\gamma}+(1-\pi_t)(1+g_l)^{1-\gamma}.
\end{gathered}$$

Given the assumption of (weak) risk aversion this result is almost obvious. As such there exists a global maximum $\gamma^{\max} \in (0, \infty)$ (independent of ρ), and $E[r_t^{p,FI}(\rho,0)]$ is at first increasing and then decreasing.

The existence of $\gamma^{\max} \in (0, \infty)$ does not imply $\gamma^{\max} \in (\underline{\gamma}(\rho), \bar{\gamma}(\rho))$. However, from the properties of $K_t^e(\gamma)$ in (a), it is clear that $\limsup_{\rho\to\infty} \underline{\gamma}(\rho) = 0$ and $\limsup_{\rho\to\infty} \bar{\gamma}(\rho) = \infty$, so that there exists a ρ^{**} such that for $\rho \geq \rho^{**}$ $\gamma^{\max} \in (\underline{\gamma}(\rho), \bar{\gamma}(\rho))$ represents the global maximum for the FI equity premium. □

Proof of Proposition 2. All the expectations are taken with respect to true probability measure π_t. To simplify notations, define $\hat{n}_t^{\tau_b} \equiv n_0^{\tau_b} + n_t^{\tau_b}$ and $\hat{N}_t^{\tau_b} \equiv N_0^{\tau_b} + N_t^{\tau_b}$ the counters that incorporate initial beliefs, $n_0^{\tau_b}$ and $N_0^{\tau_b}$. The proof exploits a result in Guidolin and Timmermann (2003b), Proposition 4, who show that when $\gamma < 1$, the RL pricing kernel is monotone increasing and convex in $n_t^{\tau_b}$ (given $N_t^{\tau_b}$). Since for $\gamma < 1$ Ψ_t^{RL} is increasing in $\hat{\pi}_t^{\tau_b}$ and pessimism is defined as $\hat{\pi}_t^{\tau_b} < \pi_t$, it follows that $\Psi_t^{RL} < \Psi_t^{FI}$ (see Guidolin and Timmermann (2003b), Proposition 4). Hence:

$$E_t\left[1 + r_{t+1}^{e,RL}(\rho,\gamma)\right] = \frac{E_t[(1+g_{t+1})]}{\Psi^{RL}(\hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b})} + \frac{E_t[(1+g_{t+1})\Psi^{RL}(\hat{n}_{t+1}^{\tau_b}, \hat{N}_t^{\tau_b}+1)]}{\Psi^{RL}(\hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b})} > \frac{E_t[(1+g_{t+1})]}{\Psi_t^{FI}} +$$
$$+ \frac{E_t[(1+g_{t+1})\Psi^{RL}(\hat{n}_{t+1}^{\tau_b}, \hat{N}_t^{\tau_b}+1)]}{\Psi^{RL}(\hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b})} > \frac{E_t[(1+g_{t+1})]}{\Psi_t^{FI}} + \frac{E_t[\Psi^{RL}(\hat{n}_{t+1}^{\tau_b}, \hat{N}_t^{\tau_b}+1)]}{\Psi^{RL}(\hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b})} E_t[(1+g_{t+1})]$$

where the last line derives from an application of the covariance inequality (as $\Psi^{RL}(\hat{n}_{t+1}^{\tau_b}, \hat{N}_t^{\tau_b}+1)$ is an increasing function of g_{t+1}, through $n_{t+1}^{\tau_b}$). As for the term $\frac{E_t[\Psi^{RL}(\hat{n}_{t+1}^{\tau_b}, \hat{N}_t^{\tau_b}+1)]}{\Psi^{RL}(\hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b})}$, notice that:

$$\begin{aligned} E_t[\Psi^{RL}(\hat{n}_{t+1}^{\tau_b}, \hat{N}_t^{\tau_b}+1)] &= \pi_t \Psi^{RL}(\hat{n}_t^{\tau_b}+1, \hat{N}_t^{\tau_b}+1) + (1-\pi_t)\Psi^{RL}(\hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b}+1) \\ &> \hat{\pi}_t^{\tau_b} \Psi^{RL}(\hat{n}_t^{\tau_b}+1, \hat{N}_t^{\tau_b}+1) + (1-\hat{\pi}_t^{\tau_b})\Psi^{RL}(\hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b}+1) \end{aligned}$$

where the last line uses the fact that $\hat{\pi}_t^{\tau_b} < \pi_t$ whereas $\Psi^{RL}(\hat{n}_t^{\tau_b}+1, \hat{N}_t^{\tau_b}+1) > \Psi^{RL}(\hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b}+1)$ follows from the fact that RL pricing kernel is increasing in $\hat{n}_t^{\tau_b}$. Define now ϵ^* and δ^* as in (18) - (19). $\epsilon^*(\hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b})$ is a positive function (Guidolin and Timmermann (2003b) show that under $\gamma < 1$ and given $\hat{n}_t^{\tau_b}/\hat{N}_t^{\tau_b}$, Ψ_{t+1}^{RL} is decreasing in $\hat{N}_t^{\tau_b}$) implicitly defined by

$$\lambda_\epsilon(\hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b}, \epsilon^*; \rho, \gamma, g_h, g_l) \equiv \Psi^{RL}(\hat{n}_t^{\tau_b}+\epsilon^*, \hat{N}_t^{\tau_b}+1) - \Psi^{RL}(\hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b}) = 0,$$

and it is continuous (from continuity of Ψ_{t+1}^{RL} in its arguments). As for δ^*, under the assumption of pessimistic beliefs, it is non-negative and defined by the function

$$\lambda_\delta(\pi_t, \hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b}, \delta^*; \rho, \gamma, g_h, g_l) \equiv \pi_t \Psi^{RL}(\hat{n}_t^{\tau_b}+1, \hat{N}_t^{\tau_b}+1) + (1-\pi_t)\Psi^{RL}(\hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b}+1) +$$
$$- \frac{\hat{n}_t^{\tau_b}}{\hat{N}_t^{\tau_b}} \Psi^{RL}(\hat{n}_t^{\tau_b}+1+\delta^*, \hat{N}_t^{\tau_b}+1) + \left(1 - \frac{\hat{n}_t^{\tau_b}}{\hat{N}_t^{\tau_b}}\right) \Psi^{RL}(\hat{n}_t^{\tau_b}+\delta^*, \hat{N}_t^{\tau_b}+1) = 0.$$

Since for given $\pi_t, \hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b}$, $\lambda(\pi_t, \hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b}, \delta^*)$ is continuous, strictly decreasing and changes sign over $\delta^* \in [0, \infty)$, δ^* is in fact implicitly defined by $\lambda(\pi_t, \hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b}, \delta^*) = 0$ as a function of $\pi_t, \hat{n}_t^{\tau_b}$, and $\hat{N}_t^{\tau_b}$ (besides preferences and the dividend process), $\delta^*(\pi_t, \hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b})$; from continuity of Ψ_{t+1}^{RL} in its arguments, $\lambda(\pi_t, \hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b}, \delta^*)$ is also continuous and as such $\delta^*(\pi_t, \hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b})$ is continuous in its arguments.

Given some π_t, we now look for a pair $\hat{\hat{n}}_t^{\tau_b}, \hat{\hat{N}}_t^{\tau_b}$ such that $\hat{\hat{n}}_t^{\tau_b}/\hat{\hat{N}}_t^{\tau_b} > 0$, $\delta^*(\pi_t, \hat{\hat{n}}_t^{\tau_b}, \hat{\hat{N}}_t^{\tau_b}) = \epsilon^*(\hat{\hat{n}}_t^{\tau_b}, \hat{\hat{N}}_t^{\tau_b})$ or $\Delta(\pi_t, \hat{\hat{n}}_t^{\tau_b}, \hat{\hat{N}}_t^{\tau_b}) \equiv \delta^*(\pi_t, \hat{\hat{n}}_t^{\tau_b}, \hat{\hat{N}}_t^{\tau_b}) - \epsilon^*(\hat{\hat{n}}_t^{\tau_b}, \hat{\hat{N}}_t^{\tau_b}) = 0$. Notice that $\Delta(\pi_t, \hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b})$ is continuous because

$\delta^*$and ϵ^* are continuous. Consider first the case of no pessimism, $\hat{n}_t^{\tau_b}/\hat{N}_t^{\tau_b} = \pi_t$: in this case $\epsilon^* > \delta^* = 0$ and $\Delta(\pi_t, \hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b}) < 0$. On the contrary, under extreme pessimism – i.e. $\hat{n}_t^{\tau_b} = 0$ given $\hat{N}_t^{\tau_b} > 0$ – notice that $\Psi^{RL}(0, \hat{N}_t^{\tau_b}) = \sum_{i=1}^{\infty} \beta^i (1+g_l^*)^i = \frac{\beta(1+g_l^*)}{1-\beta(1+g_l^*)} = \Psi^{RL}(0, \hat{N}_t^{\tau_b}+1)$, since $P^{RL}\{D_{t+i} = (1+g_l)^i \mid 0, \hat{N}_t^{\tau_b}\} = 1$. Therefore

$$\begin{aligned} \pi_t \Psi^{RL}(1, \hat{N}_t^{\tau_b}+1) + (1-\pi_t)\Psi^{RL}(0, \hat{N}_t^{\tau_b}+1) &= \pi_t \Psi^{RL}(1, \hat{N}_t^{\tau_b}+1) + (1-\pi_t)\Psi^{RL}(0, \hat{N}_t^{\tau_b}) \\ &= \pi_t \Psi^{RL}(1, \hat{N}_t^{\tau_b}+1) + (1-\pi_t)\Psi^{RL}(\epsilon^*, \hat{N}_t^{\tau_b}+1) \\ &= \Psi^{RL}(\delta^*, \hat{N}_t^{\tau_b}+1) \\ &= \pi_t \Psi^{RL}(\delta^*, \hat{N}_t^{\tau_b}+1) + (1-\pi_t)\Psi^{RL}(\delta^*, \hat{N}_t^{\tau_b}+1) \end{aligned}$$

This can be written as

$$\pi_t \left[\Psi^{RL}(1, \hat{N}_t^{\tau_b}+1) - \Psi^{RL}(\delta^*, \hat{N}_t^{\tau_b}+1)\right] + (1-\pi_t)\left[\Psi^{RL}(\epsilon^*, \hat{N}_t^{\tau_b}+1) - \Psi^{RL}(\delta^*, \hat{N}_t^{\tau_b}+1)\right] = 0.$$

Clearly, the sum is nil insofar as $\Psi^{RL}(\epsilon^*, \hat{N}_t^{\tau_b}+1) - \Psi^{RL}(\delta^*, \hat{N}_t^{\tau_b}+1) < 0$ and $\Psi^{RL}(1, \hat{N}_t^{\tau_b}+1) - \Psi^{RL}(\delta^*, \hat{N}_t^{\tau_b}+1) > 0$. This implies: $\Psi^{RL}(0, \hat{N}_t^{\tau_b}+1) < \Psi^{RL}(\delta^*, \hat{N}_t^{\tau_b}+1)$ and $\delta^* > \epsilon^*$, i.e. $\Delta(\pi_t, \hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b}) > 0$. Therefore, as $\hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b}$ change over the pessimistic range $0 \leq \hat{n}_t^{\tau_b}/\hat{N}_t^{\tau_b} \leq \pi_t$, $\Delta(\pi_t, \hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b})$ switches sign, from negative to positive. From standard results, it then follows that a pair $\widehat{\hat{n}}_t^{\tau_b}, \widehat{\hat{N}}_t^{\tau_b}$ exists such that $\Delta(\pi_t, \widehat{\hat{n}}_t^{\tau_b}, \widehat{\hat{N}}_t^{\tau_b}) = 0$, i.e. $\delta^* = \epsilon^*$ with $\widehat{\hat{n}}_t^{\tau_b}/\widehat{\hat{N}}_t^{\tau_b} > 0$. This shows that the set of pairs $\hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b}$ such that $\delta^* \geq \epsilon^*$ is non-empty. Consider now some beliefs from such set. Then:

$$\begin{aligned} E_t[\Psi^{RL}(\hat{n}_{t+1}^{\tau_b}, \hat{N}_t^{\tau_b}+1)] &= \hat{\pi}_t^{\tau_b}\Psi^{RL}(\hat{n}_t^{\tau_b}+1+\delta^*, \hat{N}_t^{\tau_b}+1) + (1-\hat{\pi}_t^{\tau_b})\Psi^{RL}(\hat{n}_t^{\tau_b}+\delta^*, \hat{N}_t^{\tau_b}+1) \\ &> \Psi^{RL}(\hat{E}_t^{\tau_b}\left[\hat{n}_{t+1}^{\tau_b}\right] + \delta^*, \hat{N}_t^{\tau_b}+1) \\ &= \Psi^{RL}\left(\hat{n}_t^{\tau_b} + \hat{\pi}_t^{\tau_b} + \delta^*, \hat{N}_t^{\tau_b}+1\right) \\ &> \Psi^{RL}\left(\hat{n}_t^{\tau_b} + \delta^*, \hat{N}_t^{\tau_b}+1\right) \geq \Psi^{RL}\left(\hat{n}_t^{\tau_b} + \epsilon^*, \hat{N}_t^{\tau_b}+1\right) = \Psi^{RL}\left(\hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b}\right). \end{aligned}$$

The second inequality follows from an application of Jensen's inequality to the convex Ψ_t^{RL}, the following equality from $\hat{E}_t^{\tau_b}\left[\hat{n}_{t+1}^{\tau_b}\right] = \hat{\pi}_t^{\tau_b}(\hat{n}_t^{\tau_b}+1) + (1\text{-}\hat{\pi}_t^{\tau_b})\hat{n}_t^{\tau_b} = \hat{\pi}_t^{\tau_b} + \hat{n}_t^{\tau_b} > \hat{n}_t^{\tau_b}$, and the second and third inequalities from the fact that Ψ_t^{RL} increases in $\hat{n}_{t+1}^{\tau_b}$ given $\hat{N}_{t+1}^{\tau_b}$, with $\hat{\pi}_t^{\tau_b} > 0$, and $\delta^* \geq \epsilon^*$. Therefore $\frac{E_t[\Psi^{RL}(\hat{n}_{t+1}^{\tau_b}, \hat{N}_t^{\tau_b}+1)]}{\Psi^{RL}(\hat{n}_t^{\tau_b}, \hat{N}_t^{\tau_b})} > 1$. We conclude that:

$$E_t\left[1+r_{t+1}^{e,RL}(\rho,\gamma)\right] > \frac{E_t[(1+g_{t+1})]}{\Psi_t^{FI}} + \frac{E_t[\Psi_{t+1}^{RL}]}{\Psi_t^{RL}} E_t[(1+g_{t+1})] > \left[1+\frac{1}{\Psi_t^{FI}}\right] E_t[(1+g_{t+1})] = E\left[1+r_{t+1}^{e,FI}(\rho,\gamma)\right],$$

the FI expected gross return on stocks. Hence $E_t\left[r_{t+1}^{e,RL}(\rho,\gamma)\right] > E\left[r_{t+1}^{e,FI}(\rho,\gamma)\right]$. On the other hand,

$$r_t^{f,RL}(\rho,\gamma) = \frac{1+\rho}{\hat{\pi}_t^{\tau_b}(1+g_h)^{-\gamma} + (1-\hat{\pi}_t^{\tau_b})(1+g_l)^{-\gamma}} < \frac{1+\rho}{\pi_t(1+g_h)^{-\gamma} + (1-\pi_t)(1+g_l)^{-\gamma}} = r_t^{f,FI}(\rho,\gamma),$$

the FI risk-free rate, since $(1+g_h)^{-\gamma} < (1+g_l)^{-\gamma}$ and $\hat{\pi}_t^{\tau_b} < \pi_t$. Finally, since the equity premium is the difference between expected stock returns and the risk-free rate, and we have shown that $E_t\left[r_{t+1}^{e,RL}(\rho,\gamma)\right] > E\left[r_{t+1}^{e,FI}(\rho,\gamma)\right]$ and $r_t^{f,RL}(\rho,\gamma) < r_t^{f,FI}(\rho,\gamma)$, then $E_t\left[r_{t+1}^{p,RL}(\rho,\gamma)\right] > E\left[r_{t+1}^{p,FI}(\rho,\gamma)\right]$ follows. □