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Properties of Equilibrium Asset Prices Under Alternative Learning Schemes*

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Abstract

This paper characterizes equilibrium asset prices under adaptive, rational and Bayesian learning schemes in a model where dividends evolve on a binomial lattice. The properties of equilibrium stock and bond prices under learning are shown to differ significantly compared with prices under full information rational expectations. Learning causes the discount factor and risk-neutral probability measure to become path-dependent and introduces serial correlation and volatility clustering in stock returns. We also derive conditions under which the expected value and volatility of stock prices will be higher under learning than under full information. Finally, we derive restrictions on prior beliefs under which Bayesian and rational learning lead to identical prices and show how the results can be generalized to more complex settings where dividends follow either multi-state i.i.d. distributions or multi-state Markov chains.

1. Introduction

Recent studies have recognized the importance of explicitly incorporating learning effects in equilibrium asset pricing models.¹ Learning introduces a link between state variables and agents' beliefs which the standard assumption of full information rational expectations ignores. This link creates rich dynamics in the mapping from state variables to agents' decisions and thus affects market outcomes such as prices and returns. However, although many alternative learning schemes have appeared in the literature (e.g., adaptive boundedly rational, Bayesian or rational), little is known about their properties when applied to equilibrium asset pricing problems. In fact, the majority of the literature on asset pricing under learning has been developed in a partial equilibrium setting while general equilibrium effects have not received

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¹Brennan and Xia (2001), Bullard and Duffy (2001), Timmermann (1993, 1996, 2001), and Veronesi (1999) are among the contributions on the topic.

nearly as much attention.²

In this paper we show that equilibrium stock and bond prices strongly depend on the nature of the underlying learning process. Our analysis proceeds in the context of one of the cornerstones of modern finance, namely the binomial lattice model proposed by Cox, Ross and Rubinstein (1979). This model is the discrete time equivalent of the geometric Brownian motion process underlying the Black-Scholes model and has thus been used extensively in finance (see, e.g., Stapleton and Subrahmanyam (1984)). While in the classical finance literature asset prices are assumed to follow a binomial lattice, we assume instead that dividends follow a binomial lattice with unknown probability of an up move, π . In equilibrium, asset prices are determined endogenously as a function of the evolution in agents' beliefs and in dividends.

Existing studies can usefully be separated according to whether they use boundedly rational (adaptive) or fully rational learning schemes and whether agents use Bayesian or non-Bayesian approaches. Along these lines we compare three learning models, namely Bayesian, rational and adaptive schemes. Bayesian agents view π as a random variable and start with a set of prior beliefs on the probability distribution of π that are updated through Bayes' rule as new dividend information arrives. Under the two other learning schemes, π is viewed as non-random. The adaptive learning model ignores changes in future parameter estimates, $\hat{\pi}_{t+k}$, viewed from the present (time t), conditioning instead on the current estimate, $\hat{\pi}_t$.³ In contrast, the forward-looking, 'rational' learning scheme accounts for future updates in $\hat{\pi}_t$, acknowledging that although π is constant, the estimator, $\hat{\pi}_{t+k}$, is a random variable that is correlated with future dividends. Under rational learning, asset prices reflect not only the most recent estimate of the parameters, but also all possible future values that the estimator may take. Current equilibrium prices thus reflect all possible future probability distributions of the parameter estimates.⁴

Unfortunately, the need to consider all possible sequences of (path-dependent) predictive distributions normally makes rational learning models difficult to handle. At best, multi-step predictive densities can be approximated numerically. The binomial setup provides an ideal vehicle for addressing these concerns. It generates closed-form expressions for the predictive distribution of future payoffs that allow us to provide analytical results on the properties of asset prices under learning. Furthermore, the results are easy to interpret. Agents update their parameter estimates each period noting which state occurred, and intuition in terms of 'good' news (the up-state) and 'bad' news (the down-state) applies.

We find that the properties of equilibrium prices on learning paths differ strikingly from the full informa-

²For instance, Lakner (1995) investigates consumption and portfolio choice in a finite horizon model in which agents have power utility. While asset prices are observable, their drift and the price shocks are not. As a special case, Lakner studies the case where agents learn by recursive application of Bayes' rule and derives the optimal portfolio policy using martingale methods.

³Most of the early literature on asset pricing implication of learning adopted the adaptive, least-squares learning approach, see e.g. Timmermann (1993), Barsky and De Long (1993), and Barucci (2000). Sargent (1993) contains a number of applications of boundedly rational learning schemes to finance. Evans and Honkapohja (1995), Kuan and White (1994), and Marcet and Sargent (1989) proved convergence for parametric least-squares estimators while Chen and White (1998) considered nonparametric estimators that approximate unknown equilibrium relationships with flexible functions.

⁴Uncertainty about the future mappings from state variables to decisions is thus explicitly incorporated in agents' expected utility maximization problem. Boundedly rational learning rules do not incorporate the effects of future learning on current asset prices and give agents incentives to engage in trading to exploit future learning effects (Townsend (1978, pp. 485-486)).

tion rational expectations case. For instance, under learning, perceived dividends follow a non-stationary distribution and the mapping from realized dividends to equilibrium stock prices also becomes time-varying, even though the true (but unknown) dividend process follows a stationary, homogenous Markov chain. This means that the risk-neutral probability distribution becomes path dependent. We also show that agents' probability beliefs under rational learning form a mean-preserving spread relative to the adaptive learning scheme that ignores the effect of future updating in beliefs. More specifically, the limiting distribution of asset payoffs under rational learning is no longer log-normal but follows a beta-binomial distribution whose parameters reflect agents' current beliefs.

We establish precise links between equilibrium asset prices under the three learning schemes. We show that asset prices under the rational and Bayesian learning schemes are identical provided the Bayesian agents have beta priors. Under different priors, the Bayesian learning equilibrium may not be fully rational, so rationality effectively imposes constraints on the structure of the priors which must reflect the underlying model as they do in the beta-binomial case. Likewise, asset prices under adaptive learning arise as a special case of Bayesian learning when agents have degenerate priors that put full weight on the current probability estimate.

Some papers have considered the equilibrium effects of recursive filtering of hidden state variables. Brennan and Xia (2001) and Veronesi (1999, 2004) develop continuous time models where the dividend drift is unobservable and a filtered estimate is used by a representative agent. In Brennan and Xia's model there are two lognormally distributed state variables, dividends and non-capital income. Veronesi's papers focus on the dividends process but assume that the drift may switch between two values. Timmermann (1993, 1996) studies the equilibrium effect of adaptive least squares learning on asset prices when agents are risk neutral. Lewellen and Shanken (2002) propose a simple overlapping generations model for a risk-averse, Bayesian agent who is learning about the unknown mean of dividends. These papers show that parameter uncertainty can lead to predictability and excess volatility in equity returns. However, this literature has not considered the same array of learning schemes that we entertain here so uncertainty remains as to the characterization and ranking of the effects produced by different assumptions on how investors learn.

A related literature investigates the properties of Bayesian learning schemes under experimentation. In the context of Bayesian learning, Wieland (2000a, 2000b) finds important differences between myopic (adaptive) learning schemes—which minimize (maximize) a Bayesian loss (reward) function conditional on the most recent estimate of the posterior density of some unknown parameters—and fully optimal (rational) schemes where expectations are taken with respect to future beliefs that change over time. Consistent with these papers, we find that rational learning schemes lead to very different equilibrium outcomes (asset prices) relative to myopic-adaptive schemes. Moreover, we also find that Bayesian learning in itself is neither necessary nor sufficient for full rationality of decisions on a learning path.⁵

Brandt, Zeng, and Zhang (2004) perform an exercise related to ours that — using numerical methods applied to a version of the Lucas (1978) asset pricing model — compares the properties of equity risk premia

⁵Wieland also makes it clear that while adaptive policies can easily be characterized in closed-form, rational ones normally require numerical methods. In this paper, we manage to characterize closed form solutions for asset prices by assuming—consistent with a number of papers in finance—that dividends follow a binomial lattice.

under Bayesian learning and under various suboptimal learning rules (e.g. over- and under-confidence, c.f. Abel (2002)). While Bayesian learning does affect risk premia, the suboptimal rules are associated with stronger effects. While the objective of comparing the effects of alternative learning rules on equilibrium stock prices is common to this and our paper, there are clearly many important differences. First, by focusing on the binomial lattice we can derive many analytical results that do not require simulations or approximations. Second, we study learning in the form of parameter estimation uncertainty while Brandt et al. (2004) investigate a filtering problem for a Markov switching model. Finally, while Brandt et al. (2004) compare Bayesian learning to rules from the behavioral finance literature, we limit ourselves to learning schemes that are not commonly construed as the outcome of behavioral biases.

The outline of the paper is as follows. The binomial lattice model is introduced in Section 2. Section 3 derives equilibrium asset prices under full information rational expectations. Section 4 introduces the three learning schemes. Section 5 characterizes equilibrium asset prices on the learning paths and provides conditions for the existence of an equilibrium. Section 6 provides insights into the effect of rational learning on the distribution of asset payoffs and uses simulations to quantify the effects of alternative learning schemes on equilibrium asset returns. Section 7 shows that many of our results and insights can be generalized to more general stochastic processes, although the basic intuition is better conveyed by the binomial lattice case. Section 8 concludes and an appendix contains proofs of the main results.

2. The Binomial Lattice Model

Consider an economy with two assets: a single-period, risk-free, zero-coupon bond in zero net supply trading at time t at a price of B_t and earning interest of $f_t = -\ln B_t$. After one period, the bond pays out a single unit of the consumption good. There is also a stock in unit supply trading at a price, S_t . The stock pays out an infinite stream of dividends $\{D_{t+k}\}_{k=0}^{\infty}$ measured in units of the consumption good. Dividends evolve on a binomial lattice with dividend growth rates $g_{t+k} \equiv \frac{D_{t+k}}{D_{t+k-1}} - 1$ driven by a Bernoulli process:

$$g_{t+k} = \begin{cases} g_h & \text{with prob. } \pi \\ g_l & \text{with prob. } 1 - \pi \end{cases} \quad \pi \in (0, 1) \quad (1)$$

We assume that $g_h > g_l > -1$ so dividends are always positive provided $D_0 > 0$, $D_t \in \mathcal{R}_{++}$.⁶

Limiting our attention to an arbitrarily large, but finite, number of periods T , the information set consists of a finite sample space Ω_T comprising all sequences of 0s and 1s of the form

$$\omega_T = \{I_{\{g_1=g_h\}}, I_{\{g_2=g_h\}}, \dots, I_{\{g_T=g_h\}}\}.$$

$I_{\{\cdot\}}$ is a standard zero-one indicator function. Each $\omega_T \in \Omega_T$ captures a possible sequence of dividend growth rates up to time T . The information available to the representative agent at each point in time includes only knowledge of present and past dividend levels. Hence, the economy's information structure $F_T = \{F_t; t = 0, 1, \dots, T\}$ is a filtration composed of an infinite, nested sequence of σ -algebras, $F_{t+1} \supseteq F_t$

⁶ \mathcal{R}_+ is the set of non-negative real numbers; \mathcal{R}_{++} the set of positive real numbers.

$\forall t \geq 0$, with F_t representing the first t movements of dividends. For given D_0 this ensures that the process $\{D_t\}_{t=1}^T$ is adapted to F_T . Finally the probability measure is given by

$$P(\omega_T) = \pi^j (1 - \pi)^{T-j}, \quad (2)$$

where $\omega_T \in \Omega_T$ is any state with j high growth and $T - j$ low growth occurrences.

2.1. The investor's optimization problem

For a given dividend process, we follow Lucas (1978) and let asset prices be determined in equilibrium by the representative investor's first order conditions. There is a single representative agent who is a price taker and has an infinite horizon. The consumption good, paid out in the form of dividends, is perishable. Ownership of assets is determined by trading in competitive markets for exchange of the consumption good, stocks and bonds. In equilibrium the representative consumer holds the existing (unit) supply of the stock and consumes all of the dividends paid out by the stock (c.f. Lucas (1978, p. 1430)).

Consumer preferences over random consumption sequences are represented by the functional

$$E \left[\sum_{k=0}^{\infty} \beta^k u(C_{t+k}) | F_t \right], \quad (3)$$

where $u(\cdot): \mathbb{R}_+ \rightarrow \mathbb{R}$ is a continuously differentiable, strictly increasing, and strictly concave Von-Neumann Morgenstern utility function, and C_t is real consumption at time t . $\beta = \frac{1}{1+\rho}$, $1 > \rho > 0$ is the subjective rate of impatience, and $E[\cdot | F_t] \equiv E_t[\cdot]$ denotes the conditional expectation operator. We assume that the functional (3) is bounded (although $u(\cdot)$ may be unbounded) but later derive conditions under which this holds. No assumptions are made on the mapping between the probability distribution used to calculate $E[\cdot | F_t]$ and the information set F_t . In particular, this mapping is allowed to change over time as a function of past state variables.

Following common practice in the literature, much of our analysis assumes that the representative investor has constant relative risk aversion preferences:

$$u(C_t) = \begin{cases} \frac{C_t^{1-\gamma}-1}{1-\gamma} & \gamma \neq 1, \gamma \geq 0 \\ \ln C_t & \gamma = 1 \end{cases}. \quad (4)$$

This function is continuously differentiable, strictly increasing, and strictly concave as assumed.

The agent's holdings of stocks and bonds in period t , w_t^s and w_t^b , solve the following problem: Given a pricing function mapping dividends into asset prices, the state of the economy, D_t , and initial asset holdings w_{t-1}^s and w_{t-1}^b , the agent selects a sequence of consumption plans and (end-of-period) asset holdings, $\{C_{t+k}, w_{t+k}^s, w_{t+k}^b\}_{k=0}^{\infty}$, to maximize the discounted value of the infinite stream of expected future utilities (3) subject to the sequence of constraints:

$$\begin{aligned} C_{t+k} + w_{t+k}^s S_{t+k} + w_{t+k}^b B_{t+k} &\leq w_{t+k-1}^s (S_{t+k} + D_{t+k}) + w_{t+k-1}^b \\ C_{t+k} &\geq 0 \quad \forall k \geq 0. \end{aligned} \quad (5)$$

Here $\{S_{t+k}, B_{t+k}\}_{k=0}^{\infty}$ are asset prices consistent with the assumed pricing function. In equilibrium the consumer holds exactly one unit of the stock and no zero-coupon bonds. Without loss of generality we can thus introduce a bound $\bar{w} > 1$ on the asset holdings, $|w_{t+k}^s| \leq \bar{w}$ and $|w_{t+k}^b| \leq \bar{w}$ ($\forall k \geq 0$); so that (5) defines a compact set. None of these bounds will be binding in equilibrium. Our goal is to characterize asset prices in a particular class of equilibria (Prescott and Mehra (1980)):

Definition 1 (Stationary Recursive Competitive Equilibrium). *A stationary competitive equilibrium is defined by:*

- (i) *A stationary pricing function $q: \mathbb{R}_{++} \rightarrow \mathbb{R}_+^2 - q(D_t) \equiv [S_t(D_t) \ B_t(D_t)]'$ – from the current state of the economy, D_t , to asset prices.*
- (ii) *A continuous value function $V: \mathbb{R}_{++} \times \mathbb{R}^2 \rightarrow \mathbb{R}$, $V(D_t, w_{t-1}^s, w_{t-1}^b)$.*
- (iii) *A consumption-portfolio choice policy $\hat{w}^C: \mathbb{R}_{++} \times \mathbb{R}^2 \rightarrow \mathbb{R}_+^3$ maximizing*

$$u(C_t) + \beta E \left[V(D_{t+1}, w_t^s, w_t^b) | F_t \right]$$

subject to the constraints (5). This maps the state variables relevant to the agent into optimizing consumption and portfolio decisions, $\hat{w}^C(D_t) \equiv [\hat{C}_t \ \hat{w}_t^s \ \hat{w}_t^b]'$. $\hat{w}^C(D_t)$ takes the pricing function $q(D_t)$ as given.

- (iv) *$\hat{C}_t = D_t$, $\hat{w}_t^s = 1$ and $\hat{w}_t^b = 0$, i.e. markets clear.*

Given any continuous, strictly positive pricing function $q: \mathbb{R}_{++} \rightarrow \mathbb{R}_{++}^2$, Prescott and Mehra (1980) show the existence of a unique, bounded, measurable, and continuous function $V(D_t, w_{t-1}^s, w_{t-1}^b)$ under our continuity and boundedness assumptions on $u(\cdot)$ provided that the conditional c.d.f. of D_t is continuous, and the feasible set of consumption and investment choices is compact and continuous in the state variables. This value function satisfies the Bellman optimality equation

$$\begin{aligned} V(D_t, w_{t-1}^s, w_{t-1}^b) &= \max_{C_t, w_t^s, w_t^b} \left\{ u(C_t) + \beta E \left[V(D_{t+1}, w_t^s, w_t^b) | F_t \right] \right\} \\ C_t + w_t^s S_t + w_t^b B_t &\leq w_{t-1}^s (S_t + D_t) + w_{t-1}^b \\ C_t \geq 0 \quad w_t^s &\leq \bar{w} \quad w_t^b \leq \bar{w} \end{aligned} \tag{6}$$

$V(\cdot)$ generally depends on the assumed pricing function q . Moreover, when $u(\cdot)$ is concave and the budget constraint convex, $V(D_t, w_{t-1}^s, w_{t-1}^b)$ may also be shown to be strictly increasing and (weakly) concave. Using continuity and concavity of the value function, existence of a recursive competitive equilibrium follows from standard contraction mapping arguments.⁷

⁷It is easy to show that boundedness of the value function follows directly by imposing the condition $\rho > \pi g_h + (1 - \pi)g_l$.

3. Asset Prices under Full Information

To derive the properties of the stationary pricing function q , we use property (iii) of the definition of a competitive equilibrium. Under full information rational expectations, the investor's optimization program implies the first-order conditions

$$\begin{aligned} u'(\hat{C}_t)S_t &= \beta E \left[\frac{\partial V(D_{t+1}, \hat{w}_t^s, \hat{w}_t^b)}{\partial w_t^s} \middle| F_t \right] \\ u'(\hat{C}_t)B_t &= \beta E \left[\frac{\partial V(D_{t+1}, \hat{w}_t^s, \hat{w}_t^b)}{\partial w_t^b} \middle| F_t \right]. \end{aligned}$$

>From $V(D_t, w_{t-1}^s, w_{t-1}^b) = u(w_{t-1}^s(S_t + D_t) + w_{t-1}^b - \hat{w}_t^s S_t - \hat{w}_t^b B_t) + \beta E [V(D_{t+1}, \hat{w}_t^s, \hat{w}_t^b) | F_t]$ subject to (6), the envelope condition implies

$$\begin{aligned} \left. \frac{\partial V(D_t, w_{t-1}^s, w_{t-1}^b)}{\partial w_{t-1}^s} \right|_{\hat{w}^C(D_t)} &= u'(\hat{C}_t)(S_t + D_t) \\ \left. \frac{\partial V(D_t, w_{t-1}^s, w_{t-1}^b)}{\partial w_{t-1}^b} \right|_{\hat{w}^C(D_t)} &= u'(\hat{C}_t). \end{aligned} \quad (7)$$

Substituting (7) into the first order conditions, we obtain stock and bond prices

$$S_t = E [Q_{t+1}(S_{t+1} + D_{t+1}) | F_t], \quad (8)$$

$$B_t = E [Q_{t+1} | F_t]. \quad (9)$$

$Q_{t+1} = \beta u'(\hat{C}_{t+1})/u'(\hat{C}_t)$ is the representative agent's marginal rate of substitution between consumption in period $(t+1)$ and period t . Every pricing function (8)-(9) reflects the representative agent's preferences through Q_{t+1} . Under CRRA preferences Q_{t+1} reduces to $\beta \frac{u'(C_{t+1})}{u'(C_t)} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma}$ or, from the equilibrium condition, $C_{t+k} = D_{t+k}$, $Q_{t+1} = \beta (1 + g_{t+1})^{-\gamma}$. (8)-(9) and $Q_{t+1} = \beta (1 + g_{t+1})^{-\gamma}$ thus constitute the stationary equilibrium pricing function in Definition 1. Lucas (1978, p. 1435) shows that these pricing functions are unique (from uniqueness of the stationary recursive equilibrium).

In the full information case where the parameters of the dividend process $\{\pi, g_h, g_l\}$ are known to agents, an explicit solution for asset prices is obtained using the method of undetermined coefficients:

$$S_t^{FI} = \lim_{T \rightarrow \infty} E_t \left[\sum_{s=1}^T \left(\beta^s \prod_{i=1}^s (D_{t+i}/D_{t+i-1})^{1-\gamma} \right) \right] D_t \equiv \Psi^{FI}(\pi) D_t. \quad (10)$$

The linear homogenous form of the equilibrium pricing function $S^{FI}(F_t, \pi) = \Psi^{FI}(\pi) D_t$ is a direct implication of expected utility maximization.⁸ Ψ^{FI} denotes the constant pricing kernel. The solution to (10) can conveniently be stated in terms of the transformed parameters $g_l^* = (1 + g_l)^{1-\gamma} - 1$ and $g_h^* = (1 + g_h)^{1-\gamma} - 1$:

Proposition 1. *Suppose that $\rho > \pi g_h^* + (1 - \pi) g_l^*$ and that the transversality condition $\lim_{T \rightarrow \infty} E_t[(\prod_{k=1}^T Q_{t+k}) S_{t+T}] = 0$ holds. Then the full information rational expectations (FI) stock price, S_t^{FI} , is given by*

$$S_t^{FI}(D_t) = \Psi^{FI} D_t = \frac{1 + g_l^* + \pi(g_h^* - g_l^*)}{\rho - g_l^* - \pi(g_h^* - g_l^*)} D_t. \quad (11)$$

⁸Mehra and Prescott (1985, p.152) note that linear homogeneity of the pricing function in dividends is a general property of constant relative risk aversion preferences.

The full information bond price, B_t^{FI} , is

$$B^{FI} = \frac{(1 + g_l)^{-\gamma} + \pi [(1 + g_h)^{-\gamma} - (1 + g_l)^{-\gamma}]}{1 + \rho} > 0. \quad (12)$$

The proof is given in Appendix A. Since the stock price is homogeneous of degree one in dividends, S_t^{FI} follows the same binomial lattice $\{g_h, g_l, \pi\}$ as dividends.⁹

Notice that while Cox et al. (1979) take the process for the underlying price as exogenous, we derive the underlying stock price in an equilibrium model. This result can also be related to Stapleton and Subrahmanyam (1984) who value options in a general equilibrium model where markets are incomplete and the stock price evolves on a lattice. In contrast to Stapleton and Subrahmanyam, our model assumes that markets are complete, but –as in their paper–the exogenous lattice process applies to dividends. Obviously, in both cases preferences affect the equilibrium stock price. Moreover, while in Cox et al. (1979, p. 232) stock prices follow an exogenous binomial lattice process so that the restriction $g_l < f < g_h$ is necessary and sufficient to rule out the existence of arbitrage opportunities, no such restrictions are required here. The condition

$$\rho > \pi g_h^* + (1 - \pi) g_l^* \quad (13)$$

ensures that $\Psi^{FI} > 0$ and is also necessary and sufficient for convergence of the infinite sum $\sum_{s=1}^{\infty} E_t[(\prod_{i=1}^s Q_{t+i}) D_{t+s}]$ and therefore existence of the equilibrium.

4. Learning Models

Suppose now that agents do not know the true value of the parameter π and instead use the available sample information on current and past dividends F_t to estimate $\pi \in \Pi \equiv [0, 1]$. Since the seminal papers by Blume et al. (1982) and Bray and Kreps (1987) boundedly rational and fully rational learning schemes have been considered in the literature. In *boundedly rational* or *adaptive learning* models agents treat their time- t perception $\hat{\pi}_t$ of the unknown π as if it were the true parameter value. $\hat{\pi}_t$ is recursively updated over time, as new information arrives. Changes in the perceptions $\hat{\pi}_t$ induce non-stationarities in the equilibrium relationship, but such (ex-post) time-variation is ignored in agents' decisions.¹⁰ In contrast, a *rational learning* scheme assumes that agents account for the effect of learning on the equilibrium mapping between payoff-relevant state variables and prices (c.f. Townsend (1978)). Since recursive updating induces changes in $\hat{\pi}_{t+k}$ ($k \geq 0$), future estimates of π are treated as random variables. Rational learning requires consistent updating of beliefs on π and therefore that Bayes rule be recursively applied.

Under Bayesian learning, agents view π as a random variable. A Bayesian learning scheme is neither necessary nor sufficient for rational learning.¹¹ Our paper shows that an explicit Bayesian set-up is not

⁹As in Abel (1988), when $\gamma < 1$, prices will increase as a function of the proportion of high growth states (π), while the opposite result holds for $\gamma > 1$. Under logarithmic utility ($\gamma = 1$) the asset price is independent of π .

¹⁰Examples include Bray (1982), Bray and Savin (1986), and Marcet and Sargent (1989).

¹¹Indeed, Blume and Easley (1982) presented “ (...) a boundedly rational version of Bayesian learning.” (p. 341). Their agents use a Bayesian decision-theoretic set-up but fail to recognize that the true relationship between prices and states of nature is intrinsically non-stationary.

necessary to a rational learning model. On the other hand, Bayesian learning *per se* is not sufficient to obtain full rationality of the learning process. This is similar to Wieland's (2000a) notion of myopic Bayesian decision policies under experimentation. In our binomial lattice set up we derive restrictions on the priors that ensure that such an equivalence obtains. Although the classical applications of Townsend (1978) and Frydman (1982) were based on a Bayesian framework, the essence of a rational scheme lies in agents' awareness of being on a recursive learning path (see Bray and Kreps (1987, p. 604) for further discussion). Perhaps surprisingly, it turns out that Bayesian and rational learning give identical equilibrium prices under a particular choice of the prior in the Bayesian learning scheme.¹²

4.1. Adaptive Learning

First consider the learning scheme of a 'frequentist' agent that views π as non-random but estimates this parameter recursively through the maximum likelihood rule:

$$\hat{\pi}_t = \frac{n_t}{N_t}. \quad (14)$$

Here n_t denotes the number of high growth states recorded up to time t , while N_t is the total number of periods. This maximum likelihood estimator, $\hat{\pi}_t$, is a function of the dividend process and is thus itself a random variable. The key question is whether the agent accounts for parameter estimation uncertainty and its effect on future asset prices. To show the significance of this point, consider the agent's forecast of D_{t+T} . If the agent does not know π but conditions on the most recent estimate, $\hat{\pi}_t$, the forecast becomes the simple projection:

$$P^{AL} \left\{ D_{t+T}^j = (1 + g_h)^j (1 + g_l)^{T-j} D_t | F_t, \hat{\pi}_t \right\} = \binom{T}{j} \hat{\pi}_t^j (1 - \hat{\pi}_t)^{T-j} \quad j = 0, 1, \dots, T.$$

Obviously, this scheme is fundamentally misspecified since optimal decisions in the future will be based on different estimates $\{\hat{\pi}_{t+k}\}_{k=1}^T$. Equivalently, this learning scheme relies on an application of the certainty equivalence principle even though the required conditions (e.g. a quadratic objective function) do not apply, c.f. Gennotte (1986).

4.2. Rational Learning

We start by adapting the following generic definition from Bray and Kreps (1987, p. 606):

Definition 2 (Rational Learning). *In a rational learning model, the sequence of agents' optimizing choices and equilibrium market outcomes are derived:*

- (i) *conditional on agents' use of all available data relevant to make inferences on $\pi \in \Pi$;*
- (ii) *from a statistical framework of belief formation (and updating) concerning $\pi \in \Pi$ that is consistent with the resulting sequence of equilibrium market outcomes;*

¹²Blume and Easley (1982, p. 341) put it this way: "Fully rational learning would require each trader to take into account the effect of his learning (...) on equilibrium prices."

- (iii) *from recursive Bayesian updating of beliefs using a correctly specified likelihood function for the data, i.e. incorporating the equilibrium mapping from $\pi \in \Pi$ to market outcomes.*

We note that (i) is a simple efficiency requirement that all available information is used to make optimal decisions. (ii) requires that the sequence of equilibrium market outcomes reflects the recursive belief updating process. It also requires beliefs to account for the mapping from beliefs to market outcomes. Rational learning thus represents a fixed point in the space of learning strategies (Townsend (1978)). Indeed (iii) requires the likelihood function for the data to be consistent with the equilibrium mapping from $\pi \in \Pi$ to market outcomes.

In the context of our model rational learning implies that agents recursively update their estimate of π , $\hat{\pi}_t$, using Bayes rule although they need not view π as a random variable.¹³ They account for estimation uncertainty in future estimates, $\hat{\pi}_{t+k}$, and conditions (i) - (iii) in definition 2 apply so agents forecast

$$D_{t+T}^{RL} = \hat{E} \left\{ \dots \hat{E} \left[\hat{E} (D_{t+T} | F_{t+T-1}, \hat{\pi}_{t+T-1}) | F_{t+T-2}, \hat{\pi}_{t+T-2} \right] \dots | F_t, \hat{\pi}_t \right\}. \quad (15)$$

$\hat{E}(\cdot | F_{t+k}, \hat{\pi}_{t+k})$ conditions on future estimates of π , again assuming that agents use Bayes' rule to update the parameter estimates. Under rational learning the entire sequence $\{\hat{\pi}_t, \hat{\pi}_{t+1}, \dots, \hat{\pi}_{t+T-1}\}$ enters the forecasting problem and future probability beliefs are recognized to be random (Wieland (2000, p. 507)).

4.3. Bayesian Learning

Bayesian agents perceive π as a random variable and have prior distributions, $p(\pi)$, over the values that π can assume. These prior beliefs are recursively updated into posterior beliefs by applying Bayes' rule. The estimator for π is chosen to minimize the agent's loss function which we next derive.¹⁴ Define the optimal value of the infinite stream of expected future utilities derived from consumption of real dividends and investment under the true but unknown π :

$$\left\{ \hat{C}_{t+k}(\pi), \hat{w}_{t+k}^s(\pi), \hat{w}_{t+k}^b(\pi) \right\}_{k=0}^{\infty} \equiv \hat{a}(\pi) \equiv \arg \max_{\{C_{t+k}, w_{t+k}^s, w_{t+k}^b\}_{k=0}^{\infty}} E \left[\sum_{k=0}^{\infty} \beta^k u(\hat{C}_{t+k}(\pi)) | F_t, \pi \right] \quad (16)$$

$$s.t. \hat{C}_{t+k}(\pi) + \hat{w}_{t+k}^s(\pi) S_{t+k} + \hat{w}_{t+k}^b(\pi) B_{t+k} = \hat{w}_{t+k-1}^s(\pi) (S_{t+k} + D_{t+k}) + \hat{w}_{t+k-1}^b(\pi).$$

Then the loss incurred under the consumption-investment plan $a \equiv \{C_{t+k}, w_{t+k}^s, w_{t+k}^b\}$ is

$$L(\pi, a) \equiv E \left[\sum_{k=0}^{\infty} \beta^k u(\hat{C}_{t+k}(\pi)) | F_t, \pi \right] - E \left[\sum_{k=0}^{\infty} \beta^k u(C_{t+k}) | F_t, \pi \right] \geq 0$$

$$s.t. C_{t+k} + w_{t+k}^s S_{t+k} + w_{t+k}^b B_{t+k} = w_{t+k-1}^s (S_{t+k} + D_{t+k}) + w_{t+k-1}^b.$$

This loss function is defined over $\Pi \times \mathbb{R}_+ \times \mathbb{R}^2$ and has values in \mathbb{R}_+ . For a given $\pi \in \Pi$, minimization of $L(\pi, a)$ is equivalent to maximization of $E \left[\sum_{k=0}^{\infty} \beta^k u(C_{t+k}) | F_t, \pi \right]$. Since $L(\pi, a)$ is a function of the

¹³This is a subtle distinction: a rational learner always perceives $\hat{\pi}_{t+k}$ ($k \geq 1$) as random though π itself is treated as a random variable only in the Bayesian case.

¹⁴Blume and Easley (1984) use a similar decision theoretic approach to study convergence of rational learning in general equilibrium. Implicitly, they identify their rationality requirement with a Bayesian approach.

conditional expectation under F_t , the optimal consumption-investment plan at time t is also a function of F_t . The investor chooses a decision rule mapping all possible information sets in F_t into an optimal action, $\delta(F_t) : F_t \rightarrow \mathfrak{R}_+ \times \mathfrak{R}^2$. The loss function, $L(\pi, \delta(F_t))$, can then be written as a mapping from $\Pi \times \Delta$ to \mathfrak{R}_+ , where Δ is the space of decision rules.

The Bayesian agent's objective is to choose $\delta(F_t)$ to minimize the Bayes risk:

$$\min_{\delta(F_t) \in \Delta} \Lambda(\pi, \delta(F_t)) \equiv \min_{\delta(F_t) \in \Delta} E_p \{ E_{F_t|\pi} [L(\pi, \delta(F_t))] \}, \quad (17)$$

where $E_p[\cdot]$ denotes the expectation over the prior $p(\pi)$ and $E_{F_t|\pi}[\cdot]$ is the expectation over the sample distribution parameterized by $\pi \in \Pi$. We refer to this solution as the Bayes rule, $\delta^p(F_t)$.

D_t is a sufficient statistic for the realization of dividends on the sample path, F_t , so the objective simplifies as follows:

$$\begin{aligned} \Lambda(\pi, \delta(F_t)) &\equiv E_p \{ E_{D_t|\pi} [L(\pi, \delta(D_t))] \} \\ &= \int_{\Pi} \int_{D_t} L(\pi, \delta(D_t)) p(D_t|\pi) p(\pi) dD_t d\pi \\ &= \int_{\Pi} \int_{D_t} L(\pi, \delta(D_t)) p(\pi|D_t) m(D_t) dD_t d\pi. \end{aligned}$$

$p(\pi|D_t)$ is the posterior distribution of π given D_t , and $m(D_t)$ is the marginal distribution of D_t .

(17) is a functional optimization problem with control $\delta(D_t) \in \Delta$. Solving this problem is usually a daunting task but is made easier by the following lemma (c.f. Brown and Purves (1973))

Lemma 1. *Solving (17) is equivalent to finding the elements of the $(t+1) \times 1$ vector $a \equiv [a_0 \ a_1 \ \dots \ a_t]'$ that maximize*

$$\Lambda^j(\pi, a_j) = \int_{\Pi} L(\pi, a_j) p(\pi | (1+g_l)^{t-j} (1+g_h)^j D_0) d\pi \quad j = 0, 1, \dots, t.$$

so $\hat{a}_j = \hat{\delta}^p((1+g_l)^{t-j} (1+g_h)^j D_0)$ where $\delta^p \equiv [\delta((1+g_l)^t D_0) \ \delta((1+g_l)^{t-1} (1+g_h) D_0) \ \dots \ \delta((1+g_h)^t D_0)]'$.

In practice, the \hat{a}_j s simply solve consumption-investment problems conditional on the posterior distribution of π . Therefore in what follows we simply concentrate on solving the program:

$$\min_{a_j} \Lambda^j(\pi, a_j) \equiv \min_{a_j} E_{\pi|D_t} [L(\pi, a_j)] \quad j = 0, 1, \dots, t.$$

$E_{\pi|D_t}[\cdot]$ is the expectation taken over the Bayesian posterior distribution of π . This distribution must be characterized before asset prices can be derived and requires specifying the agent's priors.

We follow conventional practice in the Bayesian literature and assume that the agent has a beta prior with parameters n_0 and N_0 , $\pi \sim \text{beta}(n_0, N_0 - n_0)$, $n_0, N_0 - n_0 > 0$. The most natural interpretation of this prior is that the agent has pre-sample information with n_0 of N_0 realizations being 'up'.¹⁵ This prior, when combined with the Bernoulli dividend process gives a standard setup, c.f. Zellner (1971, page 39) and ensures that the posterior $p(\pi|D_t)$ is also a beta distribution:

¹⁵The requirements $n_0, N_0 - n_0 > 0$ imply that at least one period of high growth and one period of low growth must have been observed. When $n_0 = N_0 - n_0 = 1$ the beta prior becomes uniform.

Lemma 2. *A prior $\pi \sim \text{beta}(n_0, N_0 - n_0)$ implies the following posterior on the binomial lattice:*

$$\pi|(1 + g_l)^{t-j}(1 + g_h)^j D_0 \sim \text{beta}(n_0 + j, N_0 - n_0 + t - j) \quad j = 0, 1, \dots, t.$$

The $\text{beta}(c, d)$ posterior for π incorporates sample information in a simple way: n_0 is updated to $n_0 + j$ by adding the number of high growth realizations between time 0 and t , while $N_0 - n_0$ is updated to $N_0 - n_0 + (t - j)$, where $(t - j)$ is the number of low growth realizations.

5. Equilibrium Asset Prices under Learning

This section derives equilibrium asset prices under the adaptive, rational and Bayesian learning schemes. As in Brandt et al. (2004) we use a common asset pricing model and vary the learning rules followed by the representative agent.

5.1. Asset Prices under Adaptive Learning

Suppose that although agents recursively update their current probability estimate, they do not take into account the effect of future revisions in $\hat{\pi}_t$ when computing the current stock price, S_t . Uncertainty about π is entirely disregarded. Conditional on agents' current estimate of the proportion of up-states ($\hat{\pi}_t$), the asset price under such an adaptive (AL) learning scheme is¹⁶

$$\begin{aligned} S_t^{AL}(D_t) &= \sum_{s=1}^{\infty} E \left[\left(\prod_{i=1}^s Q_{t+i} \right) D_{t+s} | F_t, \hat{\pi}_t \right] \\ &= D_t \sum_{s=1}^{\infty} E \left[\prod_{i=1}^s Q_{t+i} \frac{D_{t+i}}{D_{t+i-1}} | F_t, \hat{\pi}_t \right] \\ &= D_t \sum_{s=1}^{\infty} (1 + \rho)^{-s} \prod_{i=1}^s \left[\hat{\pi}_t (1 + g_h)^{1-\gamma} + (1 - \hat{\pi}_t) (1 + g_l)^{1-\gamma} \right] \\ &= \frac{1 + g_l^* + \hat{\pi}_t (g_h^* - g_l^*)}{\rho - g_l^* - \hat{\pi}_t (g_h^* - g_l^*)} D_t \end{aligned} \quad (18)$$

As in all adaptive learning models, this formula is generally misspecified (c.f. Bray and Kreps (1987, pp. 599-600)). Only in the limit as $t \rightarrow \infty$, does it follow from the Mann-Wald theorem that $\hat{\pi}_t$ converges to the true π and S_t^{AL}/S_t^{FI} converges to 1.

5.2. Asset Prices under Rational Learning

Under rational learning (RL), agents account for the state dependence of future estimates ($\hat{\pi}_{t+k}$):

$$S_t^{RL} = \sum_{s=1}^{\infty} \frac{D_t}{(1 + \rho)^s} \hat{E}_t \left[\left(\frac{D_{t+1}}{D_t} \right)^{1-\gamma} \hat{E}_{t+1} \left(\frac{D_{t+2}}{D_{t+1}} \right)^{1-\gamma} \dots \left[\hat{E}_{t+s-1} \left[\left(\frac{D_{t+s}}{D_{t+s-1}} \right)^{1-\gamma} \right] \dots \right] \right]. \quad (19)$$

¹⁶For this to be well-defined we assume that $\rho > \hat{\pi}_t g_h^* + (1 - \hat{\pi}_t) g_l^*$ for all t . Given γ and ρ , this may impose restrictions on the values taken by $\hat{\pi}_t$ on a learning path.

$\hat{E}_t[\cdot]$ is the expectation operator conditional on the period- t estimate, $\hat{\pi}_t$. Future probability beliefs, $\hat{E}_{t+1}, \hat{E}_{t+2}, \dots$ do not disappear from this expression and have to be accounted for. Since the sequence of conditional expectations at the nodes $t+1, t+2, \dots, t+T$ implied by the Euler condition under RL depends on the future states $\{\hat{\pi}_{t+1}^{\tau_b}, \hat{\pi}_{t+2}^{\tau_b}, \dots\}$, the law of iterated expectations can no longer be applied to reduce (19) since the distributions over which future expectations are computed depend on future information.

On the binomial lattice, the probability distribution under rational learning, $\hat{E}_t[\dots \hat{E}_{t+s-1}[\cdot] \dots]$, can be fully characterized. We prove in Appendix A that the compound probability distribution for the number of up-states occurring between period t and $t+s$, is given by

$$P^{RL} \left\{ \frac{D_{t+s}}{D_t} = (1+g_h)^i (1+g_l)^{s-i} \mid \hat{\pi}_t, N_t \right\} = \binom{s}{i} \frac{\prod_{k=0}^{i-1} (n_t + k) \prod_{k=0}^{s-i-1} (N_t - n_t + k)}{\prod_{k=0}^{s-1} (N_t + k)}, \quad i = 0, \dots, s \quad (20)$$

where $\binom{s}{i} = \frac{s!}{(s-i)!i!}$ and $\prod_{k=0}^{-1} (\cdot) = 1$. The updated probability distribution of dividends for period $t+s$ only depends on the number of up-states between periods t and $t+s-1$ and is independent of the specific path followed on the lattice. Using this result, the equilibrium stock price under rational learning can be derived by summing the probability-weighted product of the marginal rate of substitution and dividends across the number of up-states (i) and time (s):

$$S_t^{RL} = \lim_{T \rightarrow \infty} \left\{ \sum_{s=1}^T \frac{D_t}{(1+\rho)^s} \sum_{i=0}^s \binom{s}{i} (1+g_h^*)^i (1+g_l^*)^{s-i} \frac{\prod_{k=0}^{i-1} (n_t + k) \prod_{k=0}^{s-i-1} (N_t - n_t + k)}{\prod_{k=0}^{s-1} (N_t + k)} \right\}.$$

These findings lead directly to the stock price under rational learning (RL):

Proposition 2. *Suppose that a transversality condition holds*

$$\lim_{T \rightarrow \infty} \hat{E}_t \left\{ Q_{t+1} \hat{E}_{t+1} \left[Q_{t+2} \dots \hat{E}_{t+T-1} (Q_{t+T} S_{t+T}) \dots \right] \right\} = 0,$$

and that $\rho > \max \{g_l^*, g_h^*\}$. Then the stock price under rational learning, S_t^{RL} , is

$$S_t^{RL}(D_t) = \Psi_t^{RL}(\hat{\pi}_t, N_t) \cdot D_t^j = \left\{ \sum_{s=1}^{\infty} \beta^s \sum_{i=0}^s (1+g_h^*)^i (1+g_l^*)^{s-i} P^{RL}(D_{t+s}^i \mid \hat{\pi}_t, N_t) \right\} \cdot D_t^j \quad (21)$$

where $P^{RL}(D_{t+s}^i = (1+g_h)^i (1+g_l)^{s-i} D_t^j \mid \hat{\pi}_t, N_t)$ is given by

$$P^{RL} \{ D_{t+s}^i \mid \hat{\pi}_t, N_t \} = \binom{s}{i} \frac{\prod_{k=0}^{i-1} (n_t + k) \prod_{k=0}^{s-i-1} (N_t - n_t + k)}{\prod_{k=0}^{s-1} (N_t + k)}. \quad (22)$$

The equilibrium bond price under rational learning is

$$B_t^{RL}(D_t) = \hat{E}_t [\beta(1+g_{t+1})^{-\gamma}] = \frac{(1+g_l)^{-\gamma} + \hat{\pi}_t [(1+g_h)^{-\gamma} - (1+g_l)^{-\gamma}]}{1+\rho}.$$

Proposition 2 has several implications. First the price-dividend ratio is no longer a constant and depends on $\hat{\pi}_t = n_t/N_t$. Dividend shocks between time t and $t+1$ lead to a change in the stock price not only through the linear relationship, $S_t^{RL} = \Psi_t^{RL} D_t$, but also through revisions to the pricing kernel. Second,

while under full information the risk-free rate is constant, on a learning path it changes as a function of the state variables n_t and N_t . High dividend growth raises the risk free rate by raising $\hat{\pi}_{t+1}$ above $\hat{\pi}_t$. Third, as the rational learning stock price is a time-varying multiple of dividends, stock prices no longer follow the dividend lattice $\{g_h, g_l, \pi\}$. On a learning path, a recombining, flexible lattice is needed to capture the stochastic process of equilibrium stock prices. However, the tree is still recombining so that a period of high growth followed by one of low growth leads to the same stock price as a period of low price growth followed by one of high price growth. The period $[t, t+1]$ (gross) capital gain on the stock index is:

$$(1 + g_{t+1}) \frac{\Psi_{t+1}^{RL}}{\Psi_t^{RL}} = \begin{cases} \frac{\Psi^{RL}(\hat{\pi}_{t+1}^u, N_{t+1})}{\Psi^{RL}(\hat{\pi}_t, N_t)} (1 + g_h) & \text{w/prob. } \pi \\ \frac{\Psi^{RL}(\hat{\pi}_{t+1}^d, N_{t+1})}{\Psi^{RL}(\hat{\pi}_t, N_t)} (1 + g_l) & \text{w/prob. } (1 - \pi) \end{cases},$$

where $\hat{\pi}_{t+1}^u \equiv \frac{1}{N_{t+1}} (N_t \hat{\pi}_t + 1)$ and $\hat{\pi}_{t+1}^d \equiv \frac{N_t}{N_{t+1}} \hat{\pi}_t$. The capital gain is time-varying and depends on n_t and N_t , as well as on the realized state between t and $t+1$. Therefore it changes as we move along the lattice for ex-dividend stock prices. The (local) volatility of the stock price – in this set up, the size of the jump in price caused by dividend news – changes over time (as N_t increases deterministically with t) and as a function of the underlying cash index.

The description of equilibrium asset prices on a rational learning path is completed by establishing restrictions on the values taken by $\hat{\pi}_t$ which rule out arbitrage opportunities:

Corollary 1. *Under the conditions stated in Proposition 2, the following inequalities are sufficient for the absence of arbitrage opportunities:*

$$\frac{1 + \Psi^{RL}(\hat{\pi}_{t+1}^d, N_{t+1})}{\Psi^{RL}(\hat{\pi}_t, N_t)} (1 + g_l) < \frac{1 + \rho}{(1 + g_l)^{-\gamma} + \hat{\pi}_t [(1 + g_h)^{-\gamma} - (1 + g_l)^{-\gamma}]} < \frac{1 + \Psi^{RL}(\hat{\pi}_{t+1}^u, N_{t+1})}{\Psi^{RL}(\hat{\pi}_t, N_t)} (1 + g_h). \quad (23)$$

Completeness and absence of arbitrage opportunities are guaranteed by (23), so a unique risk neutral measure exists for our model although it will be a function of the entire sequence of probability beliefs. This can most easily be seen by comparing the value as of period $t+1$ of one dollar in period $t+3$ when a high and a low dividend growth state occur. With high growth followed by low growth,

$$\frac{[\exp(f_{t+1}^{RL}(\hat{\pi}_{t+1}^u))]^{-1} [\exp(f_{t+2}^{RL}(\hat{\pi}_{t+2}^{ud}))]^{-1}}{(1 + g_l)^{-\gamma} + \frac{(n_t+1)}{(N_t+1)} [(1 + g_h)^{-\gamma} - (1 + g_l)^{-\gamma}]} \times \frac{1 + \rho}{(1 + g_l)^{-\gamma} + \frac{(n_t+1)}{(N_t+2)} [(1 + g_h)^{-\gamma} - (1 + g_l)^{-\gamma}]}.$$

where $\hat{\pi}_{t+2}^{ud} = \frac{n_t+1}{N_t+2} = \hat{\pi}_{t+2}^{du}$. Reversing the sequence (low growth followed by high growth), we get

$$\frac{[\exp(f_{t+1}^{RL}(\hat{\pi}_{t+1}^d))]^{-1} [\exp(f_{t+2}^{RL}(\hat{\pi}_{t+2}^{du}))]^{-1}}{(1 + g_l)^{-\gamma} + \frac{n_t}{N_t+1} [(1 + g_h)^{-\gamma} - (1 + g_l)^{-\gamma}]} \times \frac{1 + \rho}{(1 + g_l)^{-\gamma} + \frac{n_t+1}{N_t+2} [(1 + g_h)^{-\gamma} - (1 + g_l)^{-\gamma}]}.$$

Taking the ratio of these two expressions and defining $B = [(1 + g_h)^{-\gamma} - (1 + g_l)^{-\gamma}]$, we have

$$\frac{(1 + g_l)^{-2\gamma} + \frac{n_t(n_t+1)}{(N_t+1)(N_t+2)} B^2 + \frac{n_t(N_t+2) + (n_t+1)(N_t+1)}{(N_t+1)(N_t+2)} (1 + g_l)^{-\gamma} B}{(1 + g_l)^{-2\gamma} + \frac{(n_t+1)^2}{(N_t+1)(N_t+2)} B^2 + \frac{(n_t+1)(2N_t+3)}{(N_t+1)(N_t+2)} (1 + g_l)^{-\gamma} B},$$

which in general differs from unity. For discounting purposes, the exact sequence of dividend realizations thus matters.

5.3. Asset Prices under Bayesian Learning

At time t , the optimal decision rule of a Bayesian agent with prior $\text{beta}(n_0, N_0 - n_0)$ solves

$$\min_{a_j} E_{\pi|D_t} [L(\pi, a_j)],$$

where $\pi|D_t \sim \text{beta}(n_0 + j, N_0 - n_0 + t - j)$. From the definition of $L(\pi, a_j)$ this is equivalent to solving the program:

$$\begin{aligned} & \max_{\{C_{t+k}^j, w_{t+k}^{s,j}, w_{t+k}^{b,j}\}_{k=0}^{\infty}} E_{\pi|D_t^j} \left\{ E \left[\sum_{k=0}^{\infty} \beta^k u(C_{t+k}^j) | D_t^j, \pi \right] \right\} \\ \text{s.t. } & C_{t+k}^j + w_{t+k}^{s,j} S_{t+k} + w_{t+k}^{b,j} B_{t+k} = w_{t+k-1}^{s,j} (S_{t+k} + D_{t+k}) + w_{t+k-1}^{b,j} D_{t+k}^j \\ & D_t^j = (1 + g_l)^{t-j} (1 + g_h)^j D_0, \end{aligned} \quad (24)$$

where $a_j \equiv \{C_{t+k}^j, w_{t+k}^{s,j}, w_{t+k}^{b,j}\}_{k=0}^{\infty}$ is the optimal plan conditional on current dividends $D_t^j = (1 + g_l)^{t-j} (1 + g_h)^j D_0$. (24) is identical to the consumption-portfolio problem in Section 2, the only difference lying in the probability distribution with respect to which the expectation in (24) is taken. Using Lemma 2, asset prices under Bayesian learning can be characterized as follows:

Proposition 3. *Suppose that $\rho > \max\{g_l^*, g_h^*\}$ and that the Bayesian agent has a $\text{beta}(n_0, N_0 - n_0)$ prior density on π . Under the transversality condition $\lim_{T \rightarrow \infty} E_{t,\pi}^j \left[\left(\prod_{k=1}^T Q_{t+k} \right) S_{t+T} \right] = 0, \forall \pi, j$, the Bayesian learning (BL) equilibrium stock price, S_t^{BL} , is given by*

$$\begin{aligned} S_t^{BL}(D_t) &= D_t^j \Psi^{BL} = D_t^j \cdot \left[\frac{\Gamma(N_0 + t)}{\Gamma(j + n_0) \Gamma(t + N_0 - n_0 - j)} \right. \\ &\quad \times \left. \int_0^1 \frac{1 + g_l^* + \pi (g_h^* - g_l^*)}{\rho - g_l^* - \pi (g_h^* - g_l^*)} \pi^{j+n_0-1} (1 - \pi)^{t+N_0-n_0-j-1} d\pi \right], \end{aligned}$$

where j is the number of ‘up’ moves in dividends between period 1 and t . The Bayesian learning bond price, B_t^{BL} , is (recalling that $\hat{\pi}_t \equiv \frac{n_0+j}{N_0+t}$)

$$B_t^{BL}(D_t) = \frac{(1 + g_l)^{-\gamma} + \hat{\pi}_t [(1 + g_h)^{-\gamma} - (1 + g_l)^{-\gamma}]}{1 + \rho} > 0.$$

5.4. Relationships Between Asset Prices under Bayesian, Rational, and Adaptive Learning

5.4.1 Bayesian and Rational learning

There is a simple relationship between asset prices under the Bayesian and rational learning schemes. Recall that on a rational learning path, the stock price is

$$S_t^{RL} = D_t^j \cdot \left\{ \sum_{s=1}^{\infty} \beta^s \sum_{i=0}^s (1 + g_h^*)^i (1 + g_l^*)^{s-i} P^{RL} \left(D_{t+s}^i | \hat{\pi}_t^j, N_t \right) \right\},$$

where the expectation is taken under the RL distribution for future dividends (given $\hat{\pi}_t$ and N_t). Under the assumptions of Proposition 3, S_t^{BL} can be written in a similar way based on the Bayesian predictive

density for future dividends, $p(D_{t+i})$, calculated as:

$$p(D_{t+i}) = \int_{\Pi} p(D_{t+i}|\pi) p(\pi|D_t) d\pi \quad i \geq 1.$$

where $p(D_{t+i}|\pi)$ is the predictive distribution of future dividends given π and $p(\pi|D_t)$ is the posterior for π . Using Lemma 2, the following result applies:

Proposition 4. *Suppose that $\rho > \max(g_l^*, g_h^*)$ and that the Bayesian agent has a $\text{beta}(n_0, N_0 - n_0)$ prior density on π . Then the Bayesian learning (BL) equilibrium stock price is given by*

$$S_t^{BL}(D_t) = D_t \cdot \left[\sum_{s=1}^{\infty} \sum_{i=0}^s \beta (1 + g_l^*)^{s-i} (1 + g_h^*)^i P^{BL}(D_{t+s}^i) \right], \quad (25)$$

where $P^{BL}(D_{t+s}^i = (1 + g_l)^{s-i} (1 + g_h)^i D_t^j)$ is given by

$$\binom{s}{i} \frac{\Gamma(N_0 + t)}{\Gamma(n_0 + j) \Gamma(N_0 - n_0 + t - j)} \frac{\Gamma(n_0 + j + i) \Gamma(N_0 + t + s - n_0 - j - i)}{\Gamma(N_0 + t + s)}. \quad (26)$$

Provided the Bayesian agent has a beta prior, $P^{BL}(D_{t+s})$ follows a beta-binomial distribution with parameters $(s, N_0 + t, N_0 + t - n_0 - j)$ and $P^{RL}\{D_{t+s}\} = P^{BL}(D_{t+s})$ so the two asset prices are identical:

Proposition 5. *Suppose that $\rho > \max\{g_l^*, g_h^*\}$. (i) if the Bayesian agent has a $\text{beta}(n_0, N_0 - n_0)$ prior density on π and $N_t = N_0 + t$, $n_t = n_0 + j$. Then the rational learning (RL) equilibrium stock and bond prices are identical to the BL prices:*

$$\begin{aligned} S_t^{BL}(D_t) &= S_t^{RL}(D_t) \\ B_t^{BL}(D_t) &= B_t^{RL}(D_t). \end{aligned}$$

(ii) Under different priors the rational learning and Bayesian learning asset prices will generally differ.

Of course, Bayesian learning induces a beta distribution only if the agent's prior follows a beta distribution. With other, non-conjugate types of priors, different posterior and predictive distributions are obtained and asset prices will be different under the two learning schemes. Two points fundamentally distinguish RL from BL. The first is the role of the prior under the Bayesian learning model. A rational learner need not view π as random but certainly perceives the estimator $\hat{\pi}_{t+k}$ as a random variable. A Bayesian agent instead adopts a prior over π and regards π as random. Future dividend information is used to update the posterior density for π .

The second difference is equally fundamental: while the rational learner is extremely smart and acknowledges the effect of updates in his future beliefs on the current price, the Bayesian agent is myopic and only accounts for his current beliefs - as reflected in the conditional probability distribution for π . It is only to the extent that the posterior distribution is consistent with the data generating process for dividends that the myopic Bayesian learning scheme will be rational. If the Bayesian learner used, say, a truncated normal prior on $[0, 1]$, the resulting prices would not be fully rational. Rationality under a

representative agent's Bayesian learning scheme hence requires restrictions on the prior distribution which must be chosen to be consistent with the structure of the model.¹⁷

These differences are deep and represent very different learning mechanisms, so it is surprising that the two approaches can lead to identical asset prices although of course only under restrictive assumptions.

5.4.2 Bayesian and Adaptive Learning

Asset prices under Bayesian and adaptive learning form a general-to-special relation. Suppose the agent has a degenerate prior

$$p(\pi) = \begin{cases} 1 & \text{if } \pi = t^{-1} \sum_{i=1}^t I_{\{g_i = g_h\}} \\ 0 & \text{otherwise} \end{cases}.$$

Then the BL stock price can be written as

$$\begin{aligned} S_t^{BL} &= D_t \sum_{s=1}^{\infty} \int_0^1 \left[\sum_{i=0}^s \beta^s (1 + g_l^*)^{s-i} (1 + g_h^*)^i \binom{s}{i} (1 - \pi)^{s-i} \pi^i \right] p(\pi|D_t) d\pi \\ &= D_t \sum_{s=1}^{\infty} \sum_{i=0}^s \beta^s (1 + g_l^*)^{s-i} (1 + g_h^*)^i \binom{s}{i} (1 - \hat{\pi}_t)^{s-i} \hat{\pi}_t^i \\ &= D_t \left(\frac{1 + g_l^* + \hat{\pi}_t (g_h^* - g_l^*)}{\rho - g_l^* - \hat{\pi}_t (g_h^* - g_l^*)} \right). \end{aligned}$$

Hence $S_t^{BL}(D_t) = S_t^{AL}(D_t)$ for all dividend levels.¹⁸ Adaptive learning can thus be viewed as a special case of Bayesian learning in which a very particular (infeasible) prior is employed.¹⁹ An adaptive learner effectively conditions on the current estimate of $\hat{\pi}_t$ and is systematically surprised by changes to future values, $\hat{\pi}_{t+k}$. Apart from this special case, Bayesian and adaptive stock prices differ.

While Bayesian learning provides sufficient flexibility to obtain asset prices under adaptive learning as a special case, the same does not occur for finite t under rational learning. To see this, write S_t^{AL} as:²⁰

$$S_t^{AL} = \lim_{T \rightarrow \infty} \left\{ \sum_{s=1}^T \frac{D_t}{(1 + \rho)^s} \sum_{i=0}^s \binom{s}{i} (1 + g_h^*)^i (1 + g_l^*)^{s-i} \cdot \frac{\prod_{k=0}^{i-1} n_t \prod_{k=0}^{s-i-1} (N_t - n_t)}{\prod_{k=0}^{s-1} N_t} \right\}.$$

Clearly for $i = 0, 1, \dots, s$

$$\binom{s}{i} \frac{\prod_{k=0}^{i-1} (n_t + k) \prod_{k=0}^{s-i-1} (N_t - n_t + k)}{\prod_{k=0}^{s-1} (N_t + k)} \neq \binom{s}{i} \frac{\prod_{k=0}^{i-1} n_t \prod_{k=0}^{s-i-1} (N_t - n_t)}{\prod_{k=0}^{s-1} N_t}.$$

RL and AL stock prices can therefore not be identical. Another way to characterize such differences is through the speed of the learning clock used by agents to update their beliefs over time. Rational learning

¹⁷On the other hand, our result does *not* impose restrictions on how n_0 and N_0 are selected (other than $n_0 \neq 0, n_0 \neq N_0$). The proof of Proposition 4 holds irrespective of the exact choice of values for n_0, N_0 .

¹⁸ $B_t^{BL}(D_t) = B_t^{AL}(D_t)$ only when the degenerate prior assigns unit mass to a value of π that coincides with the mean of the beta density.

¹⁹This prior is not feasible as $\hat{\pi}_t$ is a function of future information unknown at time 0. Hence the probability that this prior is used is zero and this case is only useful to illustrate the formal relationship between BL and RL stock prices.

²⁰The equilibrium risk-free rates do coincide, $f_t^{LS} = f_t^{RL}$. In general, rational and adaptive learning schemes imply the same equilibrium asset prices one-period prior to expiration when future learning effects can be disregarded.

is obtained when the learning clock is updated every time new information arrives. If beliefs are never updated (i.e. the updating frequency is infinity and the learning clock has stopped), S_t^{AL} is obtained.

Apart from the special case of a degenerate prior, under a rational or Bayesian learning scheme, agents' probability beliefs form a mean-preserving spread relative to adaptive probability beliefs:

Proposition 6. *The probability weights assigned to the tails of the binomial lattice are higher under rational learning than under adaptive learning. Conversely the adaptive learning scheme puts more weight on the centre of the lattice.*

In fact, the Bayesian and rational learning schemes both adopt a Beta distribution over the unknown value of π . Under rational learning this is obtained as a limiting result as the horizon, T , goes to infinity. To demonstrate this, Figure 1 plots the probability distribution for the proportion of up-states, using different values of the forecast horizon ($T = 2, 10, 100, 1000$), assuming $\hat{\pi}_t = \frac{2}{10}$. The probability mass in the tails is always higher under rational than under adaptive learning.

6. Properties of Asset Prices under Learning

This section explores properties of asset prices and returns under learning. Given the equivalence between equilibrium asset prices under rational and Bayesian learning under the assumptions of Proposition 5, we focus on rational learning results.²¹ Because of its boundedly rational foundations, we disregard adaptive learning.

Properties of asset prices and returns under rational learning crucially depend on the mapping from agents' beliefs, $\hat{\pi}_t$, to stock prices. We first establish conditions for monotonicity and convexity of this mapping:

Proposition 7. *Under the assumptions of Proposition 2, the rational learning price-dividend ratio,*

$$\Psi^{RL}(\hat{\pi}_t, N_t) = \sum_{s=1}^{\infty} \beta^i \sum_{i=0}^s (1 + g_h^*)^i (1 + g_l^*)^{s-i} P_t^{RL}(i, s),$$

is a nondecreasing and convex function of $\hat{\pi}_t = \frac{n_t}{N_t}$ when $\gamma \leq 1$. For $\gamma > 1$ the pricing function is a decreasing and convex function of $\hat{\pi}_t$.

Figure 2 illustrates this result when $\pi = 0.6$. The other parameters are $N_t = 100$, $g_h = +6\%$, $g_l = -4\%$, $\rho = 6\%$. The coefficient of relative risk aversion is either 0.5 or 1.5.

6.1. Serial Correlation and Volatility Clustering

Even when dividend shocks are i.i.d., equilibrium stock returns under learning will in general be serially correlated and heteroskedastic. Serial correlation in both the level of returns and in squared asset returns

²¹However, Section 6.3 shows that many of the theoretical findings in Sections 6.1-6.2 continue to apply to AL and BL under non-conjugate priors. It is straightforward to prove that $\Psi^{BL}(\hat{\pi}_t, N_t)$ is increasing in $\hat{\pi}_t$ when $\gamma < 1$, decreasing in $\hat{\pi}_t$ when $\gamma > 1$, independent of $\hat{\pi}_t$ when $\gamma = 1$. Therefore, the results of Proposition 8 on serial correlation and volatility clustering follow more generally.

is commonly found in empirical studies, c.f. Lo and MacKinlay (1988) and Bollerslev et. al. (1992). Proposition 8 relates these properties to learning effects. To state the result we define the continuously compounded rate of return on the stock between time t and $t + 1$ as $r_{t+1} \equiv \ln \left(\frac{S_{t+1} + D_{t+1}}{S_t} \right)$.

Proposition 8. *When dividends evolve on a binomial lattice, the rate of return and the squared rate of return will be serially uncorrelated under full information rational expectations. However, in general returns will have a non-zero correlation under learning. Furthermore, if $\text{Cov}[r_{t+s}^{RL}, r_t^{RL}] > 0$ and $E[r_{t+s}^{RL}]$ and $E[r_t^{RL}]$ have the same sign, then $\text{Cov}[(r_{t+s}^{RL})^2, (r_t^{RL})^2] > 0$ for $s \geq 1$.*

The condition that $E[r_{t+s}^{RL}]$ and $E[r_t^{RL}]$ have the same sign is equivalent to requiring that s cannot be too large. Otherwise $\hat{\pi}_{t+s}$ could be so different from $\hat{\pi}_t$ that the expected stock return could change sign. When serial correlation in returns is attributed to learning, this suggests that we can also expect to find volatility clustering in returns.

In the presence of serial correlation in the estimator, $\hat{\pi}_t$, a monotone $\Psi^{RL}(\hat{\pi}_t, N_t)$ is sufficient to produce serial correlation. In contrast, the heteroskedasticity induced by learning relies on the convexity of the RL price-dividend ratio: The larger is $\hat{\pi}_t$, the larger will be the effect on the asset price of a shock to dividends through its effect on $\Psi^{RL}(\hat{\pi}_t, N_t)$.

6.2. Excess Volatility

A number of empirical studies have found stock prices to be excessively volatile compared to a full information rational expectations benchmark (see Shiller (1981) and Leroy and Porter (1981)). Provided agents are less risk averse than under log-utility ($\gamma < 1$), we can show that rational learning generates higher return volatility than under full information:²²

Proposition 9. *When $0 \leq \gamma < 1$ the expected stock price under rational learning will exceed the expected stock price under full information:*

$$E[S_t^{RL}] > E[S_t^{FI}].$$

Furthermore, the variance of stock returns under rational learning exceeds the variance of stock returns under full information rational expectations:

$$\text{Var}(r_t^{RL}) > \text{Var}(r_t^{FI}).$$

When $\gamma = 1$, $S_t^{RL} = S_t^{FI}$, and the moments are identical since the price-dividend ratio is independent of $\hat{\pi}_t$. When $\gamma > 1$ no general ranking can be established. A negative covariance between $\Psi_t^{RL}(\hat{\pi}_t)$ and D_t lowers the expected stock price, but the strict convexity of $\Psi_t^{RL}(\hat{\pi}_t)$ increases the average stock price. Likewise, the negative covariance between $\Psi_t^{RL}(\hat{\pi}_t)$ and D_t leads to lower volatility while the additional price variation induced by the dependence of the price-dividend ratio on $\hat{\pi}_t$ tends to increase it. That learning leads to higher volatility for low levels of risk aversion is encouraging in the light of the general finding that excessive degrees of risk aversion are required to explain movements in asset prices, c.f. Grossman and Shiller (1981).

²²The variance expressions in Proposition 9 consider stock returns since in our model stock prices are nonstationary.

6.3. Simulation Results

To study the quantitative implications of different learning schemes for the properties of equilibrium asset returns, we perform a simulation exercise with parameters calibrated to match real dividend data on Standard & Poors companies, adjusted to account for inflation as measured by the CPI (see Shiller (2000) for data sources). We supplement these data with value-weighted index returns and 3-month T-bill returns from CRSP. The data are aggregated to obtain *real* quarterly series spanning the period 1950:I - 2003:IV, a total of 216 observations.²³ We calibrate the fundamental (dividend) process to $g_h = +1.7\%$, $g_l = -1.5\%$, $\pi = 0.565$. These parameters imply a maximum annualized real growth rate of 7% and a minimum growth rate of -5.9% – both plausible values. The annualized real mean growth rate is 1.2% and its volatility is 3.2%, both set to match the sample moments of our real dividend series for 1950-2003. To avoid biases in our simulation results, we set $\hat{\pi}_0 \equiv n_0/N_0 = 1/2$, where 1950:I is the initial time period (time 0). This implies relatively strong learning in the immediate post-WWII period. We set $\rho = 3.5\%$ per annum and use 5000 independent simulations of the 216-quarter path for real dividends, stock prices, and interest rates and report sample statistics averaged across simulations.

To investigate the importance of the restriction on the prior in Proposition 5 and assess the quantitative implications across RL and BL schemes, Table 1 considers a third case, Bayesian learning when the prior is non-conjugate. In particular, we assume that the Bayesian learner initially has a truncated normal (over $(0, 1)$) prior density on π , with initial parameters $\mu_0 = n_0/N_0$ and $\sigma_0^2 = n_0(N_0 - n_0)/[(N_0 + 1)N_0]$. Since the problem in this case does not admit closed forms for the posterior distribution and must be handled through numerical integration, we proceed to recursively update the posterior and calculate the predictive density of future growth rates.

Table 1 reports sample means and standard deviations for real stock returns, excess stock returns, and real short-term (3-month) interest rates over our sample. The data display all the typical features that are well known in the asset pricing literature: high mean excess returns (5.6% per year), low real interest rates (1.5%), highly volatile excess stock returns (14.9%) and stable interest rates (1.3% annualized volatility).²⁴ There is evidence of both persistence and volatility clustering in stock returns, as shown by the Ljung-Box statistics in the first row of the table. To evaluate this, we follow Lo and MacKinlay (1988) and compute variance ratios – i.e. the ratio between the variance of q -period stock returns and q times the variance of annual returns – for $q = 2, \dots, 5$. For $q = 2$ the variance ratio is approximately one (1.04), but as q grows the ratio declines (e.g. at $q = 5$ the variance ratio is 0.92), indicating the presence of mean-reversion in long-horizon stock returns. Second, we report OLS estimates and R^2 coefficients in long-horizon predictive regressions of the type proposed by Campbell and Shiller (1988):

$$\sum_{j=1}^q (r_{t+j} - r_{t+j}^f) = a(q) + b(q) \cdot \ln(dy_t) + \epsilon_t^q,$$

²³For brevity we do not test for or model the presence of structural breaks in fundamentals and simply start our simulations after WWII. In the presence of an oil shock-related break (as in Timmermann (2001)) in the 1970s, the volatility and risk premia induced by learning may be higher than what is reported here.

²⁴The 5.6% mean excess stock returns is heavily influenced by the inclusion of the 2000-2003 period and is closer to the higher estimate in excess of 6% reported by Mehra and Prescott when based on the sample 1950-1999.

where r_t^f is the short-term real interest rate and dy_t is the dividend yield. Consistent with the literature, we find that both $\hat{b}(q)$ and $R^2(q)$ are monotonically increasing in the horizon, q . For instance, while $\hat{b}(2) = 0.17$, $R^2(2) = 0.08$, $\hat{b}(5) = 0.26$, $R^2(5) = 0.19$. This indicates predictability of long-horizon returns from the log-dividend yield.

Table 1 compares the quantitative properties of equilibrium stock and bond returns under alternative choices of γ and different learning schemes. Since Propositions 7 and 9 focus on the case where $\gamma < 1$, we start by considering $\gamma = 0.5$ and then increase γ to 0.9, and 1.5.²⁵ In the absence of learning (under FI), stock returns are i.i.d. and the lattice model fails to generate a plausible risk premium, serial correlation, volatility clustering and sufficiently volatile stock returns. In addition, the interest rate is too high and counterfactually constant.²⁶ Conversely, when $\gamma = 0.5$, the rational learning model generates very plausible asset prices with a mean excess return of 2.3%, volatility close to the 15% implied by the data, serial correlation patterns that approximately match the data (with variance ratios close to one for $q = 2$ and significantly below one for $q = 5$) and ARCH effects. Furthermore, the dividend yield predicts long-horizon returns with R^2 -values close to those estimated from the data. One aspect of the data that is missed by the RL model is the (average) level of the price-dividend ratio, that is overestimated.²⁷ Finally, Table 1 reveals interesting information on the differences among learning schemes. AL generates price effects that are qualitatively similar, but weaker than RL. This is explained by the fact that treating Ψ^{RL} as a random variable and integrating over the infinite sequence of its future values generates stronger volatility, predictability, and ARCH effects. Both under AL and RL, the resulting price-dividend ratio is too high, however, and the equilibrium short interest rate is both too high and insufficiently volatile (although learning effects lower it). The same qualitative finding applies to BL under non-conjugate priors. In this case learning generates stock returns whose mean is closer to the sample estimate. However, this learning scheme implies too strong predictability in real stock returns, with high dividend yields predicting negative long-run returns, and regression coefficients whose magnitude decline in q -features that are at odds with the empirical evidence.

Panels A and B of Table 1 also illustrate the effect of raising γ . Interestingly—and consistent with Guidolin’s (2005) results for the RL scheme—under learning, stock return volatility, mean returns, and the Ljung-Box statistics capturing serial correlation and volatility clustering follow a U-shape when plotted against γ , with minimum values at $\gamma = 1$, the log-utility case. When $\gamma = 1$, asset prices under FI and under learning coincide.²⁸ Across values of γ and across learning schemes, the case that best matches the

²⁵Propositions 3-5 imply that the equilibrium price exists for $\rho > \max(g_l^*, g_h^*)$. Under our calibration, $0.035/4 = 0.0088 > \max(g_l^*, g_h^*) = g_h^* = (1.017)^{1-\gamma} - 1$ fails for $\gamma < 0.5$; $0.0088 > \max(g_l^*, g_h^*) = g_l^* = (0.985)^{1-\gamma} - 1$ fails for $\gamma > 1.5$. Therefore we restrict our simulations to the range $\gamma \in [0.5, 1.5]$. Notice also that for $\gamma = 1$, Propositions 3-5 imply that the stock price under learning is identical to the price under FI. Hence we do not use the log-utility case in our simulations.

²⁶Departures from unit variance ratios, zero Ljung-Box statistics, and zero predictability is explained by small sample effects.

²⁷This explains why the predictability regression coefficients in columns 12 and 14 of the table are much larger under RL than in the data: since RL over-estimates the price-dividend ratios, it also generates too small dividend yields.

²⁸However, mean stock returns as well as short-term interest rates are monotonically increasing in γ . The intuition for why RL and AL mean excess returns increase as γ goes from 1 to 0.5 is that as we approach risk-neutrality, the intertemporal elasticity of substitution ($1/\gamma$) becomes larger and the equilibrium interest rate declines towards ρ , while the convexity of learning pricing kernels makes upward revisions of the estimated $\hat{\pi}$ more important (in terms of pricing effects) than downward revisions, so

data is rational learning and $\gamma = 1/2$.

An estimate of $\gamma = 1/2$ may appear to be well below values required to resolve the equity premium puzzle. Although most available econometric estimates of γ tend to exceed one-half, this value is not inconsistent with estimates reported in studies such as Hansen and Singleton (1982, 1983) or Ferson and Constantinides (1991). Furthermore, the bulk of the econometric evidence comes from models based on a representative agent who is assumed to know the stationary stochastic process for asset payoffs and so the estimates depend on the validity of this assumption. This is important since, in the presence of learning, the representative agent's beliefs are themselves subject to time-variation and the investor perceives a non-stationary process for asset payoffs. It is not clear how methodologies could be developed to account for such non-stationarities, although Table 1 indicates that this may be an important issue. Intuition suggests that, in the presence of learning, γ controls two types of behavior: (i) agents' aversion to consumption risk, as in standard models solved under FI; (ii) the sensitivity of equilibrium asset prices with respect to revisions in the predictive distribution of π . While the first effect is monotonically increasing in γ , the second is monotonically decreasing in γ . This explains the U-shaped pattern in the mapping from γ into the equity premium, return predictability and ARCH effects.²⁹

The ability of learning models to modify standard results on what a plausible value for γ is can be understood in the following simple method of moment framework based on Campbell et al. (1997, pp. 306-307). Under the assumption of homoskedastic and joint normally distributed dividend growth rates (g_t) and asset returns, the unconditional mean of the riskless rate and the equity risk premium are given by:

$$\begin{aligned} E[r_t^f] &= -\ln \beta - \frac{\gamma^2 \text{Var}[g_t]}{2} + \gamma E[g_t] \\ E[r_t - r_t^f] &= \gamma \text{Cov}[r_t, g_t], \end{aligned} \tag{27}$$

where the equity premium expression ignores a Jensen's inequality term. As the frequency of movements on the binomial tree increases, the joint distribution of returns under FI converges to a bivariate Gaussian distribution (see Cox et al. (1979)). Equation (27) can therefore be expected to capture the moments of asset returns with high accuracy under FI, while it should only be viewed as a rough but informative approximation under learning. Using $\beta = (1 + 0.035)^{-1} = 0.966$ on an annualized basis, (27) can be evaluated using the moments for real dividend growth and stock returns implied by the available time series. In our data, we have $E[g_t] = 0.01216$, $\text{Var}[g_t] = 0.00097$, $\text{Cov}[r_t, g_t] = 0.00079$, and $\text{Var}[r_t] = 0.01459$. Small risk aversion coefficients, e.g. $\gamma = 0.5$, generate a plausible risk-free rate but much too low an equity premium, while a value of γ as high as 71 is required to generate the 5.6% equity premium found in our

that mean stock returns increase compared to the FI case. The result is a considerably higher equity premium than under FI, see Guidolin (2005).

²⁹We thank an anonymous referee for drawing our attention to this implication of our results. An expanding literature has obtained results consistent with the notion that many asset pricing phenomena may be explained at relatively low levels of risk aversion. For instance, David (2004) illustrates that although a declining γ reduces the required compensation for consumption risk, it also makes agents more aggressive (based on erroneous but credible models of the economy), increases trading risk, and hence can raise the equity premium.

data—although the resulting riskless rate becomes negative. For intermediate values of γ (in the range 4-10), small positive equity premia (e.g. 0.8% for $\gamma = 10$) and interest rates below 10% per annum can be generated under FI.

We next argue heuristically why assuming that the representative investor is on a learning path may generate estimates of γ smaller than what is typically reported in the literature. Using the simulations underlying the RL results in Table 1, we calculate sample averages of the *perceived* moments $E[g_t]$, $Var[g_t]$, $Cov[r_t, g_t]$, and $Var[r_t]$ under RL. In other words, at each step on our simulation paths, we calculate the predictive density of dividend growth rates and stock returns using (20). We then average these perceptions over time and across simulations, obtaining $\hat{E}^{RL}[g_t] = 0.01216$, $\widehat{Var}^{RL}[g_t] = 0.01025$, while $\widehat{Cov}^{RL}[r_t, g_t]$ and $\widehat{Var}^{RL}[r_t]$ depend on the assumed level of γ . For $\gamma = 0.5$ we have $\widehat{Cov}^{RL}[r_t, g_t] = 0.02464$, $\widehat{Var}^{RL}[r_t] = 0.01593$, so (27) implies an equity premium of 1.3%, a plausible mean stock return of 5.2%, and a short-term interest rate of 3.9%. Although these values do not perfectly match the sample moments, $\gamma = 0.5$ appears to adequately trade off several competing moments. Rational learning effects magnify the perception of the riskiness of the fundamental process and the perception of a high degree of correlation between the real dividend growth rate and stock returns since the price-dividend ratio Ψ^{RL} is increasing in $\hat{\pi}_t$ for $\gamma < 1$.

7. Extensions of the Model

Since the seminal paper by Cox et al. (1979), the binomial lattice has played a key role in the development of finance theory. However, it is of interest to see whether our analysis applies to a more general setup where the dividend growth rate can take more than two values at each point in time or the state transitions are allowed to follow a Markov process. Sections 7.1 - 7.2 show that the result that BL and RL equilibrium asset prices are identical under restrictions on the choice of the prior generalizes both to multi-state and to Markovian processes. Section 7.3 discusses the limitations arising from our assumption of a representative investor and points to ways in which our analysis may carry over to environments with heterogeneous investors.

7.1. Generalization to a Multinomial Lattice

First, we show that relaxing the assumption that g_{t+j} can only take two possible values is not important for our result on the equivalence of BL and RL asset prices under restrictions on the choice of the (conjugate) priors. Suppose that $\forall t \geq 1$, g_t can take $K \geq 2$ possible values, $\{g_1, g_2, \dots, g_K\}$ with probabilities $\{\pi_1, \pi_2, \dots, \pi_K\}$ such that $\sum_{k=1}^K \pi_k = 1$, i.e. the probability distribution is defined over the K simplex. In this case g_t is drawn from a multinomial distribution with K possible outcomes. Furthermore, assume that under BL the investor has a Dirichlet prior with parameters $\{n_{1,0}, n_{2,0}, \dots, n_{K,0}\}$, $n_{k,0} > 0$, $k = 1, \dots, K$, such that $n_{K,0} = N_0 - \sum_{k=1}^{K-1} n_{k,0}$:

$$p\left(\pi_1, \pi_2, \dots, \pi_K | n_{1,0}, n_{2,0}, \dots, N_0 - \sum_{k=1}^{K-1} n_{k,0}\right) = \frac{\Gamma(N_0)}{\prod_{k=1}^K \Gamma(n_{k,0})} \prod_{k=1}^K \pi_k^{n_{k,0}-1}.$$

The Dirichlet prior is simply a vector generalization of the beta distribution. Once again, the most natural interpretation is that the agent has pre-sample information with $n_{k,0}$ realizations of the dividend growth rate, g_k , $k = 1, \dots, K$, where N_0 is the total number of pre-sample observations. Since the Dirichlet prior is conjugate for multinomial distributions, the resulting posterior for $p(\pi_1, \pi_2, \dots, \pi_K | F_t)$ is also Dirichlet with parameters $\{n_{1,0} + X_{1,t}, n_{2,0} + X_{2,t}, \dots, N_0 + t - \sum_{k=1}^{K-1} n_{k,0} - \sum_{k=1}^{K-1} X_{k,t}\}$, where $X_{k,t}$ is a counter that measures the number of realizations of the dividend growth rate that ‘fall’ in the k -th cell, $k = 1, \dots, K$ (see Zellner (1971) for a proof).

If the probabilities $\{\pi_1, \pi_2, \dots, \pi_K\}$ were known, the FI equilibrium asset prices could be found, subject to standard transversality conditions and provided that $\rho > \sum_{k=1}^K \pi_k g_k^*$ (where $g_k^* \equiv (1 + g_k)^{1-\gamma} - 1$), as follows:

$$S_t^{FI} = D_t \sum_{s=1}^{\infty} E_t \left[\prod_{l=1}^s Q_{t+l} \frac{D_{t+l}}{D_{t+l-1}} \right] = D_t \sum_{s=1}^{\infty} (1+\rho)^{-s} \prod_{l=1}^s \sum_{k=1}^K \left[\pi_k (1 + g_k)^{1-\gamma} \right] = \frac{1 + \sum_{k=1}^K \pi_k g_k^*}{\rho - \sum_{k=1}^K \pi_k g_k^*}. \quad (28)$$

Similarly, the equilibrium risk-free rate in this setting is:

$$B^{FI} = E_t [\beta(1 + g_{t+1})^{-\gamma}] = \frac{\sum_{k=1}^K \pi_k (1 + g_k)^{-\gamma}}{1 + \rho}. \quad (29)$$

Clearly, (28)-(29) reduce to (11)-(12) in the special case where $K = 2$ and the dividend growth rate evolves on a binomial lattice. BL asset prices are then obtained as an extension of Proposition 3. Under a similar set of assumptions (including a Dirichlet prior), S_t^{BL} is given by

$$S_t^{BL} = D_t \cdot \frac{\Gamma(N_0)}{\prod_{k=1}^K \Gamma(n_{0,k})} \int_0^1 \int_0^1 \dots \int_0^1 \frac{1 + \sum_{k=1}^K \pi_k g_k^*}{\rho - \sum_{k=1}^K \pi_k g_k^*} \prod_{k=1}^K \pi_k^{n_{k,0} + X_{k,t} - 1} d\pi_1 \cdot d\pi_2 \cdot \dots \cdot d\pi_K,$$

while the bond price is given by

$$B_t^{BL} = \int_0^1 \int_0^1 \dots \int_0^1 \frac{\sum_{k=1}^K \pi_k (1 + g_k)^{-\gamma}}{1 + \rho} p(\pi_1, \pi_2, \dots, \pi_K | D_t) d\pi_1 \cdot d\pi_2 \cdot \dots \cdot d\pi_K = \frac{\sum_{k=1}^K \hat{\pi}_{k,t} (1 + g_k)^{-\gamma}}{1 + \rho}.$$

Here $\hat{\pi}_{k,t} \equiv \frac{n_{k,0} + X_{k,t}}{N_0 + t}$ and we used that $p(\pi_1, \pi_2, \dots, \pi_K | F_t)$ is also Dirichlet so $E[\pi_k | F_t] = \int_0^1 \pi_k p(\pi_k | F_t) d\pi_k = \hat{\pi}_{k,t}$ (see Gelman et al. (1995)). $p(\pi_k | F_t)$ is the marginal distribution of π_k derived from $p(\pi_1, \pi_2, \dots, \pi_K | F_t)$ and

$$\begin{aligned} B_t^{BL} &= \int_0^1 \int_0^1 \dots \int_0^1 \frac{\pi_1 (1 + g_1)^{-\gamma}}{1 + \rho} p(\pi_1, \pi_2, \dots, \pi_K | F_t) d\pi_1 \cdot d\pi_2 \cdot \dots \cdot d\pi_K + \\ &\quad + \dots + \int_0^1 \int_0^1 \dots \int_0^1 \frac{\pi_K (1 + g_K)^{-\gamma}}{1 + \rho} p(\pi_1, \pi_2, \dots, \pi_K | F_t) d\pi_1 \cdot d\pi_2 \cdot \dots \cdot d\pi_K \\ &= \frac{(1 + g_1)^{-\gamma}}{1 + \rho} \int_0^1 \pi_1 p(\pi_1 | F_t) d\pi_1 + \dots + \frac{(1 + g_K)^{-\gamma}}{1 + \rho} \int_0^1 \pi_K p(\pi_K | F_t) d\pi_K. \end{aligned}$$

As for the RL asset prices, equation (20) is easily generalized to:³⁰

$$P^{RL} \left\{ \frac{D_{t+s}}{D_t} = (1+g_1)^{X_{1,s}} (1+g_2)^{X_{2,s}} \times \dots \times (1+g_K)^{s - \sum_{k=1}^{K-1} X_{k,s}} \mid \hat{\pi}_{1,t}, \dots, \hat{\pi}_{K,t}, N_t \right\}$$

$$= \binom{s}{X_{1,s}, \dots, X_{K-1,s}} \frac{\prod_{i=0}^{X_{1,s}-1} (n_{1,t} + i) \prod_{i=0}^{X_{2,s}-1} (n_{2,t} + i) \dots \prod_{i=0}^{s - \sum_{k=1}^{K-1} X_{k,s} - 1} (N_t - \sum_{k=1}^{K-1} n_{k,t} + i)}{\prod_{i=0}^{s-1} (N_t + i)} \quad (30)$$

In fact, the compound probability of a particular path of states between periods $t+1$ and $t+s$ equals

$$\frac{\prod_{i=0}^{s-1} j_{t+i}^k \{X_{k,t+i}, N_t + i - X_{k,t+i}\}}{\prod_{i=0}^{s-1} (N_t + i)},$$

where $j_{t+i}^k \{a, b\}$ is again the selection operator that takes the value a if the growth rate equals g_k ($k = 1, \dots, K$) at time $t+i$, and otherwise equals b . $(N_t + i - X_{k,t+i})$ increases by unity each time the growth rate differs from g_k , so all paths with the same number of occurrences of the k -th state between periods $t+1$ and $t+s$ have the same probability. Equation (30) follows since there are $\binom{s}{X_{1,s}, \dots, X_{K-1,s}}$ different paths so the states characterized by growth rates $\{g_1, g_2, \dots, g_K\}$ occur $\{X_{1,s}, X_{2,s}, \dots, s - \sum_{k=1}^{K-1} X_{k,s}\}$ times between $t+1$ and $t+s$. Equation (30) simplifies to (20) when $K = 2$.

Subject to a transversality condition and $\rho > \max_k \{g_k^*\}$, the stock price under rational learning is now

$$S_t^{RL} = \left\{ \sum_{s=1}^{\infty} \beta^s \sum_{m=1}^{M(s)} G_{m,s}^*(X_{1,s}, \dots, X_{K,s}) P^{RL} (D_{m,t+s} \mid \hat{\pi}_{1,t}, \dots, \hat{\pi}_{K,t}, N_t) \right\} \cdot D_t,$$

where $M(s)$ is the number of possible end-node values for fundamentals after s steps on a K -state multinomial tree, and

$$G_{m,s}^*(X_{1,s}^m, \dots, X_{K,s}^m) \equiv \prod_{k=1}^K (1 + g_k^*)^{X_{k,s}^m} \quad m = 1, \dots, M(s)$$

corresponds to the s -step intertemporal marginal rate of substitution under the vector of states $\{X_{1,s}^m, X_{2,s}^m, \dots, s - \sum_{k=1}^{K-1} X_{k,s}^m\}$ given by the m -th combination.³¹ At this point, it is tedious but straightforward to extend Proposition 5 to show that $S_t^{BL} = S_t^{RL}$ and $B_t^{BL} = B_t^{RL}$. The BL stock price,

$$S_t^{BL} = \left\{ \sum_{s=1}^{\infty} \beta^s \sum_{m=1}^{M(s)} G_{m,s}^*(X_{1,s}, \dots, X_{K,s}) \int_0^1 \int_0^1 \dots \int_0^1 \frac{\Gamma(N_0)}{\prod_{k=1}^K \Gamma(n_{k,0})} \prod_{k=1}^K \pi_k^{n_{k,0}-1} \times \right.$$

$$\left. \times \frac{\Gamma(s)}{\prod_{k=1}^K \Gamma(X_{k,t} + X_{k,s}^m)} \prod_{k=1}^K \pi_k^{X_{k,t} + X_{k,s}^m} d\pi_1 \cdot d\pi_2 \cdot \dots \cdot d\pi_K \right\} \cdot D_t$$

³⁰In this notation $\binom{s}{X_{1,s}, \dots, X_{K-1,s}}$ is the multinomial coefficient:

$$\frac{s!}{X_{1,s}! X_{2,s}! \dots X_{K-1,s}! (s - \sum_{k=1}^{K-1} X_{k,s})!}.$$

It is easy to show for multinomial distributions that the optimal (maximum likelihood) estimator of π_k is the sample frequency, thus justifying the RL construction in (30).

³¹For instance, for $K = 3$ and $s = 2$, a trinomial lattice implies that $M(2) = 6$ and, say, $G_{2,2}^* = (1 + g_1^*)(1 + g_2^*)$ corresponds to the sequence $\{X_{1,s}^m, X_{2,s}^m, 2 - \sum_{k=1}^2 X_{k,s}^m\} = \{1, 1, 0\}$.

requires evaluating integrals of the form

$$\frac{\Gamma(N_0)}{\prod_{k=1}^K \Gamma(n_{k,0})} \frac{\Gamma(t+s)}{\prod_{k=1}^K \Gamma(X_{k,t} + X_{k,s}^m)} \int_0^1 \int_0^1 \dots \int_0^1 \prod_{k=1}^K \pi_k^{n_{k,0} + X_{k,t} + X_{k,s}^m - 1} d\pi_1 \cdot d\pi_2 \dots \cdot d\pi_K.$$

Then $\int_0^1 \int_0^1 \dots \int_0^1 \prod_{k=1}^K \pi_k^{n_{k,0} + X_{k,t} + X_{k,s}^m - 1} d\pi_1 \cdot d\pi_2 \dots \cdot d\pi_K$ ($m = 1, \dots, M(s)$) can be recognized as the kernel of a multivariate beta distribution, which can be re-written as:

$$\frac{\prod_{k=1}^K \Gamma(n_{k,0} + X_{k,t} + X_{k,s}^m)}{\Gamma(N_0 + t + s)}.$$

The quantity

$$\begin{aligned} & \frac{\Gamma(N_0)}{\prod_{k=1}^K \Gamma(n_{k,0})} \frac{\Gamma(t+s)}{\prod_{k=1}^K \Gamma(X_{k,t} + X_{k,s}^m)} \frac{\prod_{k=1}^K \Gamma(n_{0k} + X_{k,t} + X_{k,s}^m)}{\Gamma(N_0 + t + s)} \\ &= \binom{s}{X_{1,s}^m, \dots, X_{K-1,s}^m} \frac{\Gamma(N_0 + t)}{\prod_{k=1}^K \Gamma(n_{k,0} + X_{k,t})} \frac{\prod_{k=1}^K \Gamma(n_{0k} + X_{k,t} + X_{k,s}^m)}{\Gamma(N_0 + t + s)} \end{aligned} \quad (31)$$

is the probability distribution of a Dirichlet multinomial mixture with parameters $\{s, n_{1,0} + X_{1,t} + X_{1,s}, \dots, n_{K,0} + X_{K,t} + X_{K,s}\}$, which we refer to as $P^{BL}(D_{t+s} \mid \hat{\pi}_{1,t}, \dots, \hat{\pi}_{K,t}, N_t)$. The RL price is therefore given by

$$S_t^{RL} = \left\{ \sum_{s=1}^{\infty} \beta^s \sum_{m=1}^{M(s)} G_{m,s}^*(X_{1,s}, \dots, X_{K,s}) P^{BL}(D_{t+s} \mid \hat{\pi}_{1,t}, \dots, \hat{\pi}_{K,t}, N_t) \right\} \cdot D_t.$$

Clearly, $S_t^{BL} = S_t^{RL}$ if and only if $P^{BL}(D_{t+s} \mid \hat{\pi}_{1,t}, \dots, \hat{\pi}_{K,t}, N_t) = P^{RL}(D_{t+s} \mid \hat{\pi}_{1,t}, \dots, \hat{\pi}_{K,t}, N_t) \forall m$. Comparing (30) to (31) follows the lines of the proof of Proposition 5 and amounts to showing that

$$\begin{aligned} & \frac{\prod_{i=0}^{X_{1,s}-1} (n_{1,t} + i) \prod_{i=0}^{X_{2,s}-1} (n_{2,t} + i) \dots \prod_{i=0}^{s - \sum_{k=1}^{K-1} X_{k,s} - 1} (N_t - \sum_{k=1}^{K-1} n_{k,t} + i)}{\prod_{i=0}^{s-1} (N_t + i)} \\ &= \frac{\Gamma(N_0 + t)}{\prod_{k=1}^K \Gamma(n_{k,0} + X_{k,t})} \frac{\prod_{k=1}^K \Gamma(n_{0k} + X_{k,t} + X_{k,s}^m)}{\Gamma(N_0 + t + s)}. \end{aligned}$$

Once again this turns out to hold for all possible combinations of states, $m = 1, \dots, M(s)$, so

$$B_t^{RL} = B_t^{BL} = \frac{\sum_{k=1}^K \hat{\pi}_{k,t} (1 + g_k)^{-\gamma}}{1 + \rho}.$$

Finally, notice that our finding that under conjugate Dirichlet priors, equilibrium asset prices under RL and BL coincide when fundamentals follow a multinomial tree has – at least as an approximation – some additional degree of generality. From Chamberlain (1987) we know that any generic identically and independently distributed random variable $\{g_{t+k}\}$ with distribution function F_g over a support $Z_g \subset Z$ can be approximated arbitrarily well by a multinomial distribution, in the sense that for any measurable function $h : Z \rightarrow R$ such that $\int ||h|| \cdot dF < \infty$ there exists a multinomial probability measure G whose support is a finite subset of Z and $\int h \cdot dF = \int h \cdot dG$. Insofar as the process of the fundamentals growth rate is i.i.d., it follows that an appropriately selected multinomial process can represent such a process. This implies that RL and BL equilibrium prices are identical when Bayesian investors maintain a particular (conjugate) prior.

7.2. Generalization to First-Order Markov Dynamics

Many papers in the asset pricing literature have stressed the need to produce realistic equilibrium asset price and return processes when fundamentals follow smooth, i.i.d. processes (see e.g. Abel, 2002). So far we have shown that, under conjugate priors and when dividends follow a multinomial process, RL and BL asset prices coincide. Given the ability of multinomial processes to approximate other stochastic processes, this result is quite general. Still, the assumption of i.i.d. increments to dividends is quite restrictive and authors such as Mehra and Prescott (1985) have used first-order Markov processes. We therefore next show how some of our findings on RL and BL prices can be extended to this important case. After introducing details of the dividend process, we derive BL and RL equilibrium prices and then show that choices of the BL prior exist under which the two sets of prices coincide. To keep notations simple, we investigate the case with two states ($K = 2$), as in Sections 2-6.

Suppose that dividends evolve on a lattice but now let the probability of a high (low) growth rate between t and $t + 1$, g_{t+1} , depend on the growth rate between $t - 1$ and t , g_t . In particular, assume that dividends follow a stationary, irreducible first-order Markov process for which $P\{g_{t+1}|F_t\} = P\{g_{t+1}|g_t\}$:

$$g_{t+1} = \begin{cases} g_h & \begin{cases} \text{with prob. } \pi_{hh} \text{ if } g_t = g_h \\ \text{with prob. } 1 - \pi_{ll} \text{ if } g_t = g_l \end{cases} \\ g_l & \begin{cases} \text{with prob. } 1 - \pi_{hh} \text{ if } g_t = g_h \\ \text{with prob. } \pi_{ll} \text{ if } g_t = g_l \end{cases} \end{cases} \quad \pi_{hh}, \pi_{ll} \in (0, 1). \quad (32)$$

We collect the transition probabilities in a matrix \mathbf{P} :

$$\mathbf{P} \equiv \begin{bmatrix} \pi_{hh} & 1 - \pi_{hh} \\ 1 - \pi_{ll} & \pi_{ll} \end{bmatrix}.$$

The information set continues to consist of the finite sample space Ω_T comprising all sequences of 0s and 1s:

$$\omega_T = \{I_{\{g_1=g_h\}}, I_{\{g_2=g_h\}}, \dots, I_{\{g_T=g_h\}}\}.$$

Sample information can be organized in a frequency count matrix \mathbf{F}_T with generic element $f_{ij,T} = \sum_{l=1}^T I_{\{g_l=g_j, g_{l-1}=g_i\}}$, $i, j = \text{'high'}, \text{'low'}$, that keeps track of the number of transitions between the states. Under BL, assume that the investor has a matrix beta prior on the unknown matrix \mathbf{P} :

$$p(\mathbf{P}|\mathbf{N}_0) = \frac{\Gamma(N_0)}{\prod_{k=1}^K \Gamma(n_{ij,0})} \prod_{i=l,h} \prod_{j=l,h} \pi_{ij}^{n_{ij,0}-1},$$

where \mathbf{N}_0 is another 2×2 matrix with generic element $n_{ij,0} > 0 \forall i, j = \text{'high'}, \text{'low'} (h, l)$ and $N_0 = \sum_{i=l,h} \sum_{j=l,h} n_{ij,0}$. Once again, the natural interpretation of this prior is that the investor has pre-sample information with $n_{ij,0}$ realizations of dividend growth rate transitions from g_i to g_j , where N_0 is the total number of such pre-sample observations. Since the matrix Beta prior is conjugate for problems of transition matrix estimation of first-order Markov chains, the resulting posterior $p(\mathbf{P}|F_t, \mathbf{N}_0)$ is also matrix Beta with parameters given by $\mathbf{N}_0 + \mathbf{F}_t$ (see e.g. Fuh and Fan (1997)):

$$p(\mathbf{P}|F_t) = \frac{\Gamma(N_0 + t)}{\prod_{k=1}^K \Gamma(n_{ij,0} + f_{ij,t})} \prod_{i=l,h} \prod_{j=l,h} \pi_{ij}^{n_{ij,0} + f_{ij,t} - 1}.$$

If the transition matrix \mathbf{P} were known, the FI equilibrium asset price could be found using methods similar to those in Mehra and Prescott (1985), so that, subject to standard transversality conditions,

$$\frac{S_t^{FI}(g_t = i)}{D_t} = \beta \sum_{k=1}^2 \pi_{ij} (1 + g_j)^{1-\gamma} \left(1 + \frac{S_t^{FI}(g_{t+1} = j)}{D_{t+1}} \right) \quad i, j = l, h, \quad (33)$$

provided $\rho > \sum_{k=1}^2 \pi_{ij} (1 + g_j^*) \forall i, j$. Equation (33) can be written as a system of two equations in two unknowns, the FI price-dividend ratios in the two possible states, Ψ_i^{FI} :

$$\begin{cases} \Psi_h^{FI} = \beta [\pi_{hh} (1 + g_h^*) (1 + \Psi_h^{FI}) + (1 - \pi_{hh}) (1 + g_l^*) (1 + \Psi_l^{FI})] \\ \Psi_l^{FI} = \beta [(1 - \pi_{ll}) (1 + g_h^*) (1 + \Psi_h^{FI}) + \pi_{ll} (1 + g_l^*) (1 + \Psi_l^{FI})] \end{cases} \quad (34)$$

Closed-form expressions exist but are not particularly insightful. Hence, we can express the FI stock price for $K = 2$ as:

$$S_t^{FI}(g_t = i) = \beta \{ \pi_{ih} (1 + g_h^*) (1 + \Psi_h^{FI}) + \pi_{il} (1 + g_l^*) (1 + \Psi_l^{FI}) \} D_t \quad i = l, h,$$

which is easy to compute once Ψ_l^{FI} and Ψ_h^{FI} are known from (34). When $\pi_{hh} = 1 - \pi_{ll} = \pi$, each of the equations produces a FI pricing kernel identical to that in (28). The equilibrium risk-free rate remains a function of the current state:

$$B^{FI}(g_t = i) = E [\beta(1 + g_{t+1})^{-\gamma} | g_t = i] = \frac{\pi_{ih}(1 + g_h)^{-\gamma} + \pi_{il}(1 + g_l)^{-\gamma}}{1 + \rho} \quad i = l, h.$$

BL asset prices are obtained as an extension of Proposition 3. Under the same assumptions, S_t^{BL} is a function of current dividend growth and is given by

$$S_t^{BL}(g_t = i) = D_t^j \cdot \frac{\Gamma(N_0 + t)}{\prod_{k=1}^K \Gamma(n_{ij,0} + f_{ij,t})} \beta \left[\int_0^1 \pi_{ih}^{n_{ih,0} + f_{ih,t}} (1 + g_h^*) (1 + \Psi_h^{FI}) d\pi_{ih} + \int_0^1 \pi_{il}^{n_{il,0} + f_{il,t}} (1 + g_l^*) (1 + \Psi_l^{FI}) d\pi_{il} \right] \quad i = l, h,$$

while

$$\begin{aligned} B^{BL}(g_t = i) &= \frac{\Gamma(N_0 + t)}{\prod_{k=1}^K \Gamma(n_{ij,0} + f_{ij,t})} \int_0^1 \int_0^1 \frac{\pi_{ih}(1 + g_h)^{-\gamma} + \pi_{il}(1 + g_l)^{-\gamma}}{1 + \rho} \prod_{k=l,h} \pi_{ik}^{n_{ik,0} + f_{ik,t} - 1} d\pi_{ih} \cdot d\pi_{il} \\ &= \frac{\Gamma(N_0 + t)}{\prod_{k=1}^K \Gamma(n_{ij,0} + f_{ij,t})} \left[\frac{(1 + g_h)^{-\gamma}}{1 + \rho} \int_0^1 \pi_{ih}^{n_{ih,0} + f_{ih,t}} d\pi_{ih} + \frac{(1 + g_l)^{-\gamma}}{1 + \rho} \int_0^1 \pi_{il}^{n_{il,0} + f_{il,t}} d\pi_{il} \right] \\ &= \frac{\hat{\pi}_{ih}(1 + g_h)^{-\gamma} + \hat{\pi}_{il}(1 + g_l)^{-\gamma}}{1 + \rho} \quad i = l, h, \end{aligned}$$

where $\hat{\pi}_{ij} \equiv \frac{n_{ij,0} + f_{ij,t}}{n_{il,0} + f_{il,t} + n_{ih,0} + f_{ih,t}}$ and we used that $p(\mathbf{P} | F_t, \mathbf{N}_0)$ is also matrix Beta so that (see e.g. Billard and Meshkani, 1995):

$$E[\pi_{ij} | F_t, \mathbf{N}_0] = \frac{\Gamma(n_{il,0} + f_{il,t} + n_{ih,0} + f_{ih,t})}{\prod_{k=1}^K \Gamma(n_{ij,0} + f_{ij,t})} \int_0^1 \pi_{ij}^{n_{ij,0} + f_{ij,t}} d\pi_{ij} = \hat{\pi}_{ij}.$$

Once again, assuming that a rational learner recursively updates the maximum likelihood estimate

$$\hat{\pi}_{ij} \equiv \frac{n_{ij,0} + f_{ij,t}}{n_{il,0} + f_{il,t} + n_{ih,0} + f_{ih,t}} \quad i = l, h$$

(where the $n_{ij,0}$ s are initial values of the frequency counters such that $n_{i,0} \equiv n_{il,0} + n_{ih,0}$ and $N_0 \equiv n_{l,0} + n_{h,0}$), under RL equation (20) generalizes to:

$$\begin{aligned} P^{RL} \left\{ \frac{D_{t+s}}{D_t} = (1 + g_h)^{(f_{hh,t+s}^\alpha - f_{hh,t}^\alpha) + (f_{lh,t+s}^\alpha - f_{lh,t}^\alpha)} (1 + g_l)^{(f_{hl,t+s}^\alpha - f_{hl,t}^\alpha) + (f_{ll,t+s}^\alpha - f_{ll,t}^\alpha)} \mid \hat{\pi}_{hh,t}, \hat{\pi}_{ll,t}, N_t, g_t = i \right\} \\ = \sum_{\alpha \in A(s, f_{hh,t+s}, f_{ll,t+s})} \frac{f_{ij_{t+1},t}^\alpha + n_{ij_{t+1},0}^\alpha}{f_{il,t}^\alpha + n_{il,0}^\alpha + f_{ih,t}^\alpha + n_{ih,0}^\alpha} \times \frac{f_{i_{t+1}j_{t+2},t+1}^\alpha + n_{i_{t+1}j_{t+2},0}^\alpha}{f_{i_{t+1}l,t+1}^\alpha + n_{i_{t+1}l,0}^\alpha + f_{i_{t+1}h,t+1}^\alpha + n_{i_{t+1}h,0}^\alpha} \times \dots \\ \dots \times \frac{f_{i_{t+s-1}j_{t+s},t+s-1}^\alpha + n_{i_{t+s-1}j_{t+s},0}^\alpha}{f_{i_{t+s-1}l,t+s-1}^\alpha + n_{i_{t+s-1}l,0}^\alpha + f_{i_{t+s-1}h,t+s-1}^\alpha + n_{i_{t+s-1}h,t+s-1}^\alpha}, \end{aligned}$$

($i = l, h$) where $A(s, f_{hh,t+s}, f_{ll,t+s})$ is the set collecting all paths of dividends such that

$$\frac{D_{t+s}}{D_t} = (1 + g_h)^{(f_{hh,t+s}^\alpha - f_{hh,t}^\alpha) + (f_{lh,t+s}^\alpha - f_{lh,t}^\alpha)} (1 + g_l)^{(f_{hl,t+s}^\alpha - f_{hl,t}^\alpha) + (f_{ll,t+s}^\alpha - f_{ll,t}^\alpha)}.$$

Once s is fixed, $f_{hh,t}$, $f_{ll,t}$, $f_{hh,t+s}$ and $f_{ll,t+s}$ are sufficient statistics for all transitions between the two states and so the set over which the probabilities need to be summed will depend only on how forward-looking the investor is (s) and on $f_{hh,t+s}$ and $f_{ll,t+s}$. In this case, the probabilities of a given dividend path $\alpha \in A(s, f_{hh,t+s}, f_{ll,t+s})$ over the horizon $[t, t+s]$, $\{I_{\{g_{t+1}^\alpha = g_{j_{t+1}}^\alpha, g_t = g_i\}}, \dots, I_{\{g_{t+s}^\alpha = g_{j_{t+s}}^\alpha, g_{t+s-1}^\alpha = g_{i_{t+s-1}}^\alpha\}}\}$, will depend on: (i) the initial state g_t ; (ii) the exact sequence of high and low growth rates.³² As a consequence of (ii), notice that (??) now no longer relies on binomial coefficients to account for the fact that there may exist multiple paths of the growth rates leading to the same final dividend level D_{t+s} . Obviously, such simplification applies whenever the RL probabilities become path-independent, in which case – because $f_{i_{t+l}j_{t+l-1},t}^\alpha = f_{j_{t+l}j_{t+l-1},t}^\alpha \equiv f_{j_{t+q},t}^\alpha \quad \forall \alpha \in A(s, f_{hh,t+s}, f_{ll,t+s}) - A(s, f_{h,t+s}, f_{l,t+s})$ reduces to $\binom{s}{f_{h,t+s} - f_{h,t}} \equiv \binom{s}{X_{h,s}}$. Interestingly, (??) can be written as:

$$\begin{aligned} P^{RL} \left\{ \frac{D_{t+s}}{D_t} = (1 + g_h)^{(f_{hh,t+s}^\alpha - f_{hh,t}^\alpha) + (f_{lh,t+s}^\alpha - f_{lh,t}^\alpha)} (1 + g_l)^{(f_{hl,t+s}^\alpha - f_{hl,t}^\alpha) + (f_{ll,t+s}^\alpha - f_{ll,t}^\alpha)} \mid \hat{\pi}_{hh,t}, \hat{\pi}_{ll,t}, N_t, g_t = i \right\} \\ = \sum_{\alpha \in A(s, f_{hh,t+s}, f_{ll,t+s})} \frac{\left[\prod_{k=0}^{f_{ih,t+q}^\alpha - f_{ih,t}^\alpha - 1} (n_{ih,0} + f_{ih,t}^\alpha + k) \right] \left[\prod_{k=0}^{s - f_{ih,t+q}^\alpha - f_{ih,t}^\alpha - 1} (N_{i,0} - n_{ih,0} + f_{il,t}^\alpha + k) \right]}{\prod_{k=0}^{s-1} (N_{i,0} + f_{il,t}^\alpha + f_{ih,t}^\alpha + k)}. \quad (35) \end{aligned}$$

³²For instance, when $s = 2$, the four expressions

$$\begin{aligned} P^{RL} \{h, l \mid \hat{\pi}_{hh,t}, \hat{\pi}_{ll,t}, N_t, g_t = h\} &= \frac{n_{hh,0} + f_{hh,t}}{N_{h,0} + f_{hl,t}^\alpha + f_{hh,t}^\alpha} \times \frac{n_{hl,0} + f_{hl,t}}{N_{h,0} + f_{hl,t}^\alpha + f_{hh,t}^\alpha + 1} \\ P^{RL} \{l, h \mid \hat{\pi}_{hh,t}, \hat{\pi}_{ll,t}, N_t, g_t = h\} &= \frac{n_{hl,0} + f_{hl,t}}{N_{h,0} + f_{hl,t}^\alpha + f_{hh,t}^\alpha} \times \frac{n_{lh,0} + f_{lh,t}}{N_{l,0} + f_{il,t}^\alpha + f_{lh,t}^\alpha} \\ P^{RL} \{h, l \mid \hat{\pi}_{hh,t}, \hat{\pi}_{ll,t}, N_t, g_t = l\} &= \frac{n_{lh,0} + f_{lh,t}}{N_{l,0} + f_{il,t}^\alpha + f_{lh,t}^\alpha} \times \frac{n_{hl,0} + f_{hl,t}}{N_{h,0} + f_{hl,t}^\alpha + f_{hh,t}^\alpha} \\ P^{RL} \{l, h \mid \hat{\pi}_{hh,t}, \hat{\pi}_{ll,t}, N_t, g_t = l\} &= \frac{n_{ll,0} + f_{ll,t}}{N_{l,0} + f_{il,t}^\alpha + f_{lh,t}^\alpha} \times \frac{n_{lh,0} + f_{lh,t}}{N_{l,0} + f_{il,t}^\alpha + f_{lh,t}^\alpha + 1} \end{aligned}$$

are all different. The sum of the first two probabilities gives $P^{RL} \left\{ \frac{D_{t+s}}{D_t} = (1 + g_h)(1 + g_l) \mid \hat{\pi}_{hh,t}, \hat{\pi}_{ll,t}, N_t, g_t = h \right\}$, while the sum of the last two gives $P^{RL} \left\{ \frac{D_{t+s}}{D_t} = (1 + g_h)(1 + g_l) \mid \hat{\pi}_{hh,t}, \hat{\pi}_{ll,t}, N_t, g_t = l \right\}$.

For instance,

$$\begin{aligned}
P^{RL} \{h, l | \hat{\pi}_{hh,t}, \hat{\pi}_{ll,t}, N_t, g_t=h\} &= \frac{\left[\prod_{k=0}^0 n_{hh,0} + f_{hh,t}^\alpha + k \right] \left[\prod_{k=0}^0 (N_{i,0} - n_{hh,0} + f_{hl,t}^\alpha + k) \right]}{\prod_{k=0}^1 (N_{h,0} + f_{hl,t}^\alpha + f_{hh,t}^\alpha + k)} \\
&= \frac{n_{hh,0} + f_{hh,t}^\alpha}{N_{h,0} + f_{hl,t}^\alpha + f_{hh,t}^\alpha} \times \frac{n_{hl,0} + f_{hl,t}^\alpha}{N_{h,0} + f_{hl,t}^\alpha + f_{hh,t}^\alpha + 1} \\
&= \frac{n_{hh,0} + f_{hh,t}^\alpha}{N_{h,0} + f_{hl,t}^\alpha + f_{hh,t}^\alpha} \times \frac{N_{h,0} - n_{hh,0} + f_{hl,t}^\alpha}{N_{h,0} + f_{hl,t}^\alpha + f_{hh,t}^\alpha + 1}.
\end{aligned}$$

When state transitions are independent of the current state so $P\{g_h|F_t\} = \pi \in (0, 1) \forall t$, (35) reduces to (20). For instance, using our example,

$$\begin{aligned}
P^{RL} \{h, l | \hat{\pi}_t, N_t\} &= \frac{n_{h,0} + f_{h,t}^\alpha}{N_{h,0} + f_{l,t}^\alpha + f_{h,t}^\alpha} \times \frac{N_{h,0} - n_{h,0} + f_{l,t}^\alpha}{N_{h,0} + f_{l,t}^\alpha + f_{h,t}^\alpha + 1} \\
&= \frac{n_0 + n_t}{N_0 + n_t + (t - n_t)} \times \frac{N_0 - n_0 + (t - n_t)}{N_0 + t + 1} = \frac{n_0 + n_t}{N_0 + t} \times \frac{N_0 - n_0 + (t - n_t)}{N_0 + t + 1}.
\end{aligned}$$

Subject to a transversality condition and $\rho > \{g_l^*, g_h^*\}$, the stock price under RL is then

$$\begin{aligned}
S_t^{RL}(g_t = i) &= \left\{ \sum_{s=1}^{\infty} \beta^s \sum_{\alpha \in A(s, f_{hh,t+s}, f_{ll,t+s})} G_s^*(f_{hh,t+s}^\alpha, f_{ll,t+s}^\alpha) \times \right. \\
&\quad \left. \times P^{RL} \left(I_{\{g_{t+1}^\alpha = g_{j_{t+1}}, g_t = g_i\}}, \dots, I_{\{g_{t+s}^\alpha = g_{j_{t+s}}, g_{t+s-1}^\alpha = g_{i_{t+s-1}}\}} \mid \hat{\pi}_{hh,t}, \hat{\pi}_{ll,t}, N_t, g_t = i \right) \right\} \cdot D_t
\end{aligned}$$

where

$$G_s^*(f_{hh,t+s}^\alpha, f_{ll,t+s}^\alpha) \equiv (1 + g_h^*)^{(f_{hh,t+s}^\alpha - f_{hh,t}) + (f_{lh,t+s}^\alpha - f_{lh,t})} (1 + g_l^*)^{(f_{hl,t+s}^\alpha - f_{hl,t}) + (f_{ll,t+s}^\alpha - f_{ll,t})}$$

corresponds to the s -step intertemporal marginal rate of substitution over the sample path $\{I_{\{g_{t+1}^\alpha = g_{j_{t+1}}, g_t = g_i\}}, \dots, I_{\{g_{t+s}^\alpha = g_{j_{t+s}}, g_{t+s-1}^\alpha = g_{i_{t+s-1}}\}}\}$.

Generalizing Proposition 5 to show that $S_t^{BL} = S_t^{RL}$ and $B_t^{BL} = B_t^{RL}$ requires re-writing the BL stock price as follows:

$$\begin{aligned}
S_t^{BL}(g_t=i) &= \left\{ \sum_{s=1}^{\infty} \beta^s \sum_{\alpha \in A(s, f_{hh,t+q}, f_{ll,t+q})} G_s^*(f_{hh,t+s}^\alpha, f_{ll,t+s}^\alpha) \left[I_{\{g_{t+q}^\alpha = g_h\}} \frac{\Gamma(n_{hl,0} + f_{hl,t} + n_{hh,0} + f_{hh,t})}{\prod_{k=l,h} \Gamma(n_{hk,0} + f_{hk,t})} \right. \right. \\
&\quad \times \int_0^1 \int_0^1 \pi_{hh}^{n_{hh,0} + f_{hh,t+q}} \pi_{hl}^{n_{hl,0} + f_{hl,t+q}} d\pi_{hh} d\pi_{hl} + I_{\{g_{t+q}^\alpha = g_l\}} \frac{\Gamma(n_{ll,0} + f_{ll,t} + n_{lh,0} + f_{lh,t})}{\prod_{k=l,h} \Gamma(n_{lk,0} + f_{lk,t})} \\
&\quad \left. \left. \times \int_0^1 \int_0^1 \pi_{lh}^{n_{lh,0} + f_{lh,t+q}} \pi_{ll}^{n_{ll,0} + f_{ll,t+q}} d\pi_{lh} d\pi_{ll} \right] \right\} \cdot D_t.
\end{aligned}$$

The BL price implies evaluating integrals of the form

$$\frac{\Gamma(n_{il,0} + f_{il,t} + n_{ih,0} + f_{ih,t})}{\prod_{k=l,h} \Gamma(n_{ik,0} + f_{ik,t})} \int_0^1 \int_0^1 \pi_{ih}^{n_{ih,0} + f_{ih,t+q}} \pi_{il}^{n_{il,0} + f_{il,t+q}} d\pi_{ih} d\pi_{il} \quad i = l, h, \quad \forall \alpha \in A(s, f_{hh,t+q}, f_{ll,t+q}).$$

Because $\pi_{ih} + \pi_{il} = 1$, this remains the kernel of a bivariate beta distribution:

$$\begin{aligned}
& \frac{\Gamma(n_{il,0} + f_{il,t}^\alpha + n_{ih,0} + f_{ih,t}^\alpha) \Gamma(n_{ih,0} + f_{ih,t+q}^\alpha) \Gamma(n_{il,0} + f_{il,t+q}^\alpha)}{\prod_{k=l,h} \Gamma(n_{ik,0} + f_{ik,t}^\alpha) \Gamma(n_{ih,0} + f_{ih,t+q}^\alpha + n_{il,0} + f_{il,t+q}^\alpha)} \\
&= \frac{\Gamma(n_{il,0} + f_{il,t}^\alpha + n_{ih,0} + f_{ih,t}^\alpha) \Gamma(n_{ih,0} + f_{ih,t+q}^\alpha) \Gamma(N_{i,0} - n_{ih,0} + f_{i,t+q}^\alpha - f_{ih,t+q}^\alpha)}{\Gamma(n_{ih,0} + f_{ih,t}^\alpha) \Gamma(N_{i,0} - n_{ih,0} + f_{i,t}^\alpha - f_{ih,t}^\alpha) \Gamma(n_{ih,0} + f_{ih,t+q}^\alpha + n_{il,0} + f_{il,t+q}^\alpha)}. \quad (36)
\end{aligned}$$

Equation (36) is similar to expressions appearing in the proof of Proposition 5 (see the Appendix), with t replaced by $f_{il,t} + f_{ih,t}$ (the total number of recorded transitions from state $i = l, h$ up to time t), j replaced by $f_{ih,t}^\alpha$, n_0 by $n_{ih,0}$, N_0 by $n_{il,0} + n_{ih,0}$, and i by $f_{ih,t+q}^\alpha - f_{ih,t}^\alpha$. Therefore (36) can be written as

$$\frac{\left[\prod_{q=0}^{f_{ih,t+s}^\alpha - f_{ih,t}^\alpha - 1} n_{ih,0} + f_{ih,t}^\alpha + q \right] \left[\prod_{q=0}^{s - f_{ih,t+q}^\alpha - f_{ih,t}^\alpha - 1} (N_{i,0} - n_{ih,0} + f_{il,t}^\alpha + q) \right]}{\prod_{q=0}^{s-1} (N_{i,0} + f_{il,t}^\alpha + f_{ih,t}^\alpha + q)},$$

which is the same as (35). Since, for all $s \geq 0$, the RL probabilities coincide with those under the BL predictive density for the relevant transition probabilities, it follows that $S_t^{BL} = S_t^{RL}$ and $B_t^{BL} = B_t^{RL}$.

Although the complexity of the notations increases in K , a similar logic could be applied to prove that BL and RL asset prices coincide when the BL prior is taken to be an appropriate matrix Beta for the general case with an m -th order Markov chain and $K > 2$ states.

The results so far demonstrated that equilibrium prices under BL and RL are identical for suitable choices of the priors that are conjugate with respect to the dividend process. When $\{g_t\}$ follows a binomial lattice, the prior should be Beta; when $\{g_t\}$ follows a general multinomial tree with K possible states, the prior should be Dirichlet, a vector generalization of a Beta; when $\{g_t\}$ follows a first-order Markov process with unknown transition probability matrix, the suitable prior is a matrix Beta, a further multivariate generalization of the standard Beta. One may therefore conjecture that the use of a conjugate prior might be at least sufficient (if not necessary) for BL and RL equilibrium prices to be identical. This conjecture is, however, incorrect and it is easy to construct counter-examples where this does not hold.³³

7.3. Heterogeneity

Our analysis has explored the implications of various learning schemes for equilibrium asset prices under the simplifying assumption that only the learning of the ‘average’, representative agent matters. Extending our results on the differential impact of alternative learning schemes to cover heterogeneous agents runs into three complications. First, heterogeneity may give agents an incentive to learn from the observed (aggregate) market outcomes. While this does not pose any principal difficulties, in practice it becomes more difficult to characterize the equilibrium. Second, strategic incentives may emerge if a group of agents

³³Under Gaussian IID dividend growth, the perceived distribution for the s -step cumulative dividend growth under BL or RL differ for $s \geq 2$. While for $s = 1$ the rational learner will believe that the predictive density for D_{t+1}/D_t is Gaussian, for $s \geq 2$ the predictive distribution of future dividend growth rates fails to be Gaussian and is instead a mixture of Gaussian variables, with mixing weights that depend on future realizations of the dividend growth rate.

realize that equilibrium market outcomes depend on their own beliefs and actions. Third, existing papers focus on the effect of heterogeneity in beliefs, while other forms of heterogeneity—chiefly in preference parameters—may matter. Results in Constantinides (1982) suggest that problems caused by heterogeneity in beliefs get compounded with possible differences in preferences.

Heterogeneity is undoubtedly important in practice, so it is worthwhile briefly considering how our results might be altered by such effects. Since we work with power utility, results in Rubinstein (1974) imply that an aggregation result holds whereby a representative agent exists if all individuals populating the economy have identical time preference parameters $\{\rho_i\}_{i=1}^I$, identical coefficients of relative risk aversion $\{\gamma_i\}_{i=1}^I$, and identical beliefs.³⁴ These are strong restrictions, not very dissimilar from imposing the existence of a single agent. Moreover, even if one found these assumptions acceptable, it would not resolve issues such as the no-trade theorem which implies a zero trading volume in this type of model.

Recent papers provide pointers for how investor heterogeneity may affect equilibrium asset prices. Kurz et al. (2005) present a model in which the dynamics of diversity of beliefs is the primary propagation mechanism in asset markets. In their model the distribution of the conditional probabilities of future market states is an endogenous element of the state vector. Interestingly, this requires that each investor must forecast the beliefs of the other investors, which is a typical Keynesian “Beauty Contest” effect. Calibrations suggest that such a model is successful not only at matching moments of asset returns (including the equity premium), but also can match the predictability features of US stock returns and generate stochastic volatility. A key difference between this paper and ours is that Kurz et al. (2005) use the rational belief principle in Kurz (1994) to model how agents use subjective models.

Dumas et al. (2005) study a filtering problem with two classes of agents that receive a public signal that is informative about future dividend growth. One class of agents uses the correct model specification while the other uses a misspecified model and overreacts to information, alternating between being excessively pessimistic and optimistic. When the rational traders fail to dominate the economy, asset prices can be strongly affected by the overconfident agents and prices become excessively volatile. Moreover, irrational traders may survive for a long time before being driven out of the market by the rational investors—see also Buraschi and Jiltsov (2005).

8. Conclusion

We have characterized equilibrium asset prices under three learning schemes in the context of an infinite-horizon equilibrium model where dividends evolve on a binomial lattice and agents have power utility. Since asset prices are a function of agents’ beliefs, the probability distribution of asset prices reflect agents’ learning.

The binomial lattice model analyzed in this paper is the standard tool used to derive the Black-Scholes

³⁴These assumptions can be relaxed under log-utility ($\gamma = 1$), since markets are complete (learning does not affect equilibrium outcomes in this case): aggregation would follow simply from imposing that all individuals have identical resources and time preference parameters, see Rubinstein (1974, p. 232). Otherwise markets are complete under FI (see Cox et al. (1979)), although they are possibly perceived as incomplete on a rational learning path (i.e. for finite T) because the risk posed by future variations in the price-dividend ratio is not tradable.

option pricing formula as the continuous time limit of a discrete time model. When learning is introduced into the model, option prices will change because the level and volatility of the underlying asset price change. Empirical researchers have found systematic biases when attempting to fit the Black Scholes model to a cross-section of option prices. Guidolin and Timmermann (2003) derive option prices based on the recursive learning model discussed here and find that the model is able to generate implied volatility smiles. They find that option pricing models that incorporate recursive learning effects produce equally good or better forecasts than several benchmarks provided by the empirical option pricing literature.

Although learning will disappear asymptotically in the current setting (see the discussion in Lewellen and Shanken (2002)), it is easy to modify the setup to prevent this from happening. For example, the true π may be subject to occasional structural breaks which would reset the learning clock so learning effects do not die out but recur after a break (c.f. Timmermann (2001) and Beck and Wieland (2002)). Alternatively agents may believe that the true value of π is subject to slow changes and use a rolling window rather than an expanding window to estimate π as a means of robustifying their beliefs with respect to nonstationarities in the fundamentals process. These extensions are trivial conceptually, but complicate the derivation of asset prices. For example, the use of a rolling estimation window for π introduces path dependence in agents' beliefs and makes an analytical treatment difficult.

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Appendix A

Derivation of Equation (22). Let X_{t+k} be a counter that measures the number of realizations of the high-growth state up to period $t+k$:

$$X_{t+k} = X_{t+k-1} + \iota_{t+k}, \quad k \geq 1.$$

Here ι_{t+k} is an indicator function taking the value one if the high growth state occurs in period $t+k$, and zero otherwise. The compound probability of a particular path of states between periods $t+1$ and $t+s$ equals

$$\frac{\prod_{k=0}^{s-1} \mathcal{I}_{t+k}\{X_{t+k}, N_t + k - X_{t+k}\}}{\prod_{k=0}^{j-1} (N_t + k)},$$

where $\mathcal{I}_{t+k}\{a, b\}$ is a selection operator that takes the value a if the high growth state occurred at time $t+k$, otherwise is b . $(N_t + k - X_{t+k})$ increases by one each time the low growth state appears, so all paths with the same number of low growth states between periods $t+1$ and $t+k$ have the same probability. Equation (22) follows since there are $\binom{s}{i}$ different paths with i up-states and $(s-i)$ down-states occurring between $t+1$ and $t+s$. \square

Proof of Proposition 1. Iterating on the Euler equation (8), from the law of iterated expectations

$$S_t^{FI} = E_t [Q_{t+1}(S_{t+1} + D_{t+1})] = \sum_{s=1}^T E_t \left[\left(\prod_{k=1}^s Q_{t+k} \right) D_{t+s} \right] + E_t \left[\left(\prod_{k=1}^T Q_{t+k} \right) S_{t+T} \right].$$

Taking the limit as $T \rightarrow \infty$, imposing the transversality condition $\lim_{T \rightarrow \infty} E_t[(\prod_{k=1}^T Q_{t+k}) S_{t+T}] = 0$, and dividing and multiplying by D_t , we obtain

$$S_t^{FI} = \sum_{s=1}^{\infty} E_t \left[\left(\prod_{k=1}^s Q_{t+k} \right) D_{t+s} \right] = D_t \sum_{s=1}^{\infty} E_t \left[\prod_{k=1}^s Q_{t+k} \frac{D_{t+k}}{D_{t+k-1}} \right] = \Psi^{FI}(g_h, g_l, \pi, \gamma, \rho) D_t.$$

Under power utility the pricing kernel is $Q_{t+k} = \beta(1 + g_{t+k})^{-\gamma}$. Using that dividend growth follows an i.i.d. two-point distribution, we have

$$\Psi^{FI}(g_h, g_l, \pi, \gamma, \rho) = \sum_{s=1}^{\infty} (1 + \rho)^{-s} \prod_{k=1}^s \left[\pi (1 + g_h)^{1-\gamma} + (1 - \pi) (1 + g_l)^{1-\gamma} \right] = \frac{1 + g_l^* + \pi (g_h^* - g_l^*)}{\rho - g_l^* - \pi (g_h^* - g_l^*)}.$$

Here we used that $\rho > g_l^* + \pi (g_h^* - g_l^*)$ guarantees that the sum converges and is positive. Since $g_l > -1$, it follows that $E_t[(1 + g_{t+k})^{1-\gamma}] = 1 + g_l^* + \pi (g_h^* - g_l^*) > 0$. Finally, we check if the transversality condition $\lim_{T \rightarrow \infty} E_t[\prod_{k=1}^T Q_{t+k} S_{t+T}] = 0$ imposes additional restrictions on the parameter space. From the definition of Q_{t+k} and the expression for S_{t+T} , we have

$$\begin{aligned} E_t \left\{ \left[\prod_{k=1}^T Q_{t+k} \right] S_{t+T} \right\} &= E_t \left\{ \left[\beta^T \prod_{k=1}^T (1 + g_{t+k})^{-\gamma} \right] \Psi^{FI} \prod_{k=1}^T (1 + g_{t+k})^{-\gamma} D_t \right\} \\ &= \Psi^{FI} D_t \left\{ \frac{1 + g_l^* + \pi (g_h^* - g_l^*)}{1 + \rho} \right\}^T. \end{aligned}$$

In the limit as $T \rightarrow \infty$, this is zero if and only if $\rho > g_l^* + \pi(g_h^* - g_l^*)$, as assumed.

The equilibrium risk-free rate follows from the Euler equation (9):

$$B_t^{FI} = E_t [\beta(1 + g_{t+1})^{-\gamma}] = \frac{(1 + g_l)^{-\gamma} + \pi[(1 + g_h)^{-\gamma} - (1 + g_l)^{-\gamma}]}{1 + \rho}.$$

Since $1 + g_{t+1}$ is positive, $\rho > -1$ is necessary and sufficient to obtain a positive bond price. Under the restriction $1 + \rho > (1 + g_l)^{-\gamma} + \pi[(1 + g_h)^{-\gamma} - (1 + g_l)^{-\gamma}]$, $B^{FI} < 1$ and the risk-free rate is always positive. But this is satisfied as $1 + \rho > 1 + g_l^* + \pi(g_h^* - g_l^*) = E_t[(1 + g_{t+k})^{1-\gamma}]$. \square

Proof of Lemma 1. The Bayes risk of δ is

$$\begin{aligned} \Lambda(\pi, \delta) &= \int_{\Pi} \int_{D_t} L(\pi, \delta(D_t)) p(\pi|D_t) m(D_t) dD_t d\pi \\ &= \int_{D_t} \left[\int_{\Pi} L(\pi, \delta(D_t)) p(\pi|D_t) d\pi \right] m(D_t) dD_t = \int_{D_t} \Lambda(\pi, a_j) m(D_t) dD_t. \end{aligned}$$

Since \hat{a}_j minimizes $\Lambda(\pi, \delta(D_t^j))$, then $\delta^P = \hat{a} \equiv [\hat{a}_0 \ \hat{a}_1 \ \dots \ \hat{a}_t]'$ also minimizes $\Lambda(\pi, \delta)$. \square

Proof of Proposition 2. Using the Euler equation (8), taking the limit as $T \rightarrow \infty$ under the transversality condition, we obtain

$$S_t^{RL} = D_t \cdot \lim_{T \rightarrow \infty} \left\{ \sum_{s=1}^T \beta^s \sum_{i=0}^s (1 + g_h^*)^i (1 + g_l^*)^{s-i} P^{RL}(D_{t+s}^i | \hat{\pi}_t, N_t) \right\} = D_t \cdot \Psi_t^{RL}(\hat{\pi}_t, N_t; g_h, g_l, \gamma, \rho).$$

The bond price follows directly from (9) and using $\rho > -1$, $0 \leq \hat{\pi}_t \leq 1$:

$$f_t^{RL} = -\ln B_t^{RL}(\hat{\pi}_t) = \ln \left(\frac{1 + \rho}{(1 + g_l)^{-\gamma} + \hat{\pi}_t [(1 + g_h)^{-\gamma} - (1 + g_l)^{-\gamma}]} \right).$$

The final part of the result is the convergence of the infinite sum (21) or, equivalently, the existence of the RL equilibrium. $\rho > g_l^*$ is necessary and sufficient for the equilibrium to exist when $\gamma > 1$. Indeed, when $\gamma > 1$, $g_l^* > g_h^*$ so the highest price arises when $P^{RL}(D_{t+s}^0 | n_t, N_t) = 1$. $\Psi_t^{RL} = \sum_{s=1}^{\infty} \left(\frac{1+g_l^*}{1+\rho} \right)^s$ converges to $\frac{1+g_l^*}{\rho-g_l^*}$ if and only if $\rho > g_l^*$. When pricing the stock, agents must also integrate over the sequence of degenerate distributions $P^{RL}(D_{t+s}^0 | n_t, N_t) = 1$, so $\rho > g_l^*$ is necessary and sufficient. When $\gamma < 1$, $\rho > g_h^*$ is necessary and sufficient. In this case $g_h^* > g_l^*$, so the highest price arises when $P^{RL}(D_{t+s}^s | n_t, N_t) = 1$. Ψ_t^{RL} converges to $\frac{1+g_h^*}{\rho-g_h^*}$ if and only if $\rho > g_h^*$. Since this inequality must hold for all future periods, $\rho > g_h^*$ is necessary and sufficient for existence of the equilibrium when $\gamma < 1$. Since

$$\max\{g_l^*, g_h^*\} = \begin{cases} g_h^* & \text{if } \gamma < 1 \\ g_l^* & \text{if } \gamma > 1 \end{cases},$$

$\rho > \max\{g_l^*, g_h^*\}$ is necessary and sufficient for the equilibrium to exist.³⁵ \square

³⁵When $\gamma = 1$, $g_h^* = g_l^* = 0$, and $\rho > 0$ is sufficient for the equilibrium to exist since the pricing kernel simplifies to

$$\Psi^{RL} = \sum_{s=1}^{\infty} \beta^s \sum_{i=1}^s P^{RL}(D_{t+s}^i | n_t, N_t) = \sum_{s=1}^{\infty} \beta^s = \frac{1}{\rho} = \Psi^{FI}.$$

Proof of Proposition 3. The Euler equations under BL are

$$\begin{aligned} S_t^{BL} &= E_{\pi|D_t} \{E_{t,\pi} [Q_{t+1}(S_{t+1} + D_{t+1}) | D_t, \pi]\} \\ B_t^{BL} &= E_{\pi|D_t} \{E_{t,\pi} [Q_{t+1} | D_t, \pi]\}, \end{aligned}$$

where $E_{t,\pi}[\cdot]$ is the expectation conditional on D_t and π . Throughout the proof we suppress the dependence of D_t on j , the number of ‘up’ moves. Iterating on these Euler equations, we have:

$$\begin{aligned} S_t^{BL} &= E_{\pi|D_t} \{E_{t,\pi} [Q_{t+1}(S_{t+1}^{BL} + D_{t+1})]\} \\ &= E_{\pi|D_t} \{E_{t,\pi} [Q_{t+1}D_{t+1}] + E_{t,\pi} [Q_{t+1}E_{t+1,\pi} [Q_{t+2}(S_{t+2}^{BL} + D_{t+2})]]\} \\ &= E_{\pi|D_t} \left\{ \sum_{s=1}^T E_{t,\pi} \left[\left(\prod_{k=1}^s Q_{t+k} \right) D_{t+s} \right] \right\} + E_{\pi|D_t} \left\{ E_{t,\pi} \left[\left(\prod_{k=1}^T Q_{t+k} \right) S_{t+T}^{BL} \right] \right\}, \end{aligned}$$

where we used that

$$E_{t,\pi} [p(\pi|D_{t+k})] = p(\pi|D_t) \quad \forall k \geq 0,$$

is a martingale process (c.f. Bray and Kreps (1987, p. 609)). In the limit as $T \rightarrow \infty$, from the transversality condition $\lim_{T \rightarrow \infty} E_{t,\pi}[(\prod_{k=1}^T Q_{t+k})S_{t+T}] = 0$, we obtain

$$S_t^{BL} = D_t E_{\pi|D_t} \left\{ \sum_{s=1}^{\infty} E_{t,\pi} \left[\prod_{k=1}^s Q_{t+k} \frac{D_{t+k}}{D_{t+k-1}} \right] \right\} = D_t E_{\pi|D_t} [\Psi_{t,\pi}(g_h, g_l, \gamma, \rho)],$$

so the stock price under Bayesian learning is

$$\begin{aligned} S_t^{BL} &= D_t E_{\pi|D_t} \left[\frac{1 + g_l^* + \pi(g_h^* - g_l^*)}{\rho - g_l^* - \pi(g_h^* - g_l^*)} \right] = D_t \int_0^1 \frac{1 + g_l^* + \pi(g_h^* - g_l^*)}{\rho - g_l^* - \pi(g_h^* - g_l^*)} p(\pi|D_t) d\pi \\ &= D_t \frac{\Gamma(N_0 + t)}{\Gamma(j + n_0) \Gamma(t + N_0 - n_0 - j)} \int_0^1 \frac{1 + g_l^* + \pi(g_h^* - g_l^*)}{\rho - g_l^* - \pi(g_h^* - g_l^*)} \pi^{j+n_0-1} (1 - \pi)^{t+N_0-n_0-j-1} d\pi. \end{aligned}$$

The equilibrium risk-free rate follows from the Euler equation for B_t^{BL} :

$$B_t^{BL} = \int_0^1 \frac{(1 + g_l)^{-\gamma} + \pi[(1 + g_h)^{-\gamma} - (1 + g_l)^{-\gamma}]}{1 + \rho} p(\pi|D_t) d\pi = \frac{(1 + g_l)^{-\gamma} + \hat{\pi}_t [(1 + g_h)^{-\gamma} - (1 + g_l)^{-\gamma}]}{1 + \rho}.$$

where $\hat{\pi}_t \equiv \frac{n_0+j}{N_0+t}$ and we used that $\pi|D_t^j \sim \text{beta}(c, d)$, so

$$\int_0^1 \pi p(\pi|D_t) d\pi = E[\pi|D_t^j] = \frac{c}{c+d}.$$

Once again, $B_t^{BL}(j) < 1$ and the risk-free rate is always positive. \square

Proof of Corollary 1. Under the RL equilibrium, the gross return on the stock index is

$$\frac{S_{t+1} + D_{t+1}}{S_t} = \frac{(1 + \Psi_{t+1}^{RL})D_{t+1}}{\Psi_t^{RL} D_t}.$$

When $\gamma = 1$, $\Psi_t^{RL} = \Psi^{FI} = \frac{1}{\rho}$. When $\gamma < 1$, using a result from Proposition 7 it follows that under pessimism ($\hat{\pi}_t < \pi$) $\Psi_t^{RL} < \Psi^{FI}$, while optimism ($\hat{\pi}_t \geq \pi$) leads to $\Psi_t^{RL} \geq \Psi^{FI}$. When $\gamma > 1$, pessimism implies $\Psi_t^{RL} > \Psi^{FI}$, while optimism means $\Psi_t^{RL} \leq \Psi^{FI}$.

It is straightforward to prove that, under FI, $g_h > g_l > -1$ is sufficient for the absence of arbitrage opportunities. The strategy of our proof therefore consists of establishing, when possible, FI bounds for RL gross stock returns. When these bounds cannot be proved, the conditions in Corollary 1 are required to prevent the existence of arbitrage opportunities. When $\gamma = 1$, FI and RL stock prices coincide so that no further conditions are needed. For $\gamma \neq 1$, we study four different cases, using the definitions $\hat{\pi}_{t+1}^u \equiv \frac{1}{N_{t+1}}(N_t \hat{\pi}_t + 1)$ and $\hat{\pi}_{t+1}^d \equiv \frac{N_t}{N_{t+1}} \hat{\pi}_t$.

(a) $\gamma < 1$ and $\hat{\pi}_t < \pi$. $\Psi_t^{RL} < \Psi^{FI}$ and Ψ_{t+1}^{RL} is increasing in $\hat{\pi}_{t+1}$ (see Proposition 7), so

$$\begin{aligned} \frac{1 + \Psi^{RL}(\hat{\pi}_{t+1}^u, N_{t+1})}{\Psi^{RL}(\hat{\pi}_t, N_t)}(1 + g_h) &> \frac{1 + \Psi^{RL}(\hat{\pi}_t, N_t)}{\Psi^{RL}(\hat{\pi}_t, N_t)}(1 + g_h) \\ &= \left[1 + \frac{1}{\Psi^{RL}(\hat{\pi}_t, N_t)} \right] (1 + g_h) > \left[1 + \frac{1}{\Psi^{FI}} \right] (1 + g_h). \end{aligned}$$

The absence of arbitrage opportunities under FI is sufficient for the right hand side of this equation to hold under the stated assumptions, i.e. using $\hat{\pi}_t < \pi$ and $(1 + g_h)^{-\gamma} < (1 + g_l)^{-\gamma}$:

$$\left[1 + \frac{1}{\Psi^{FI}} \right] (1 + g_h) > \frac{1 + \rho}{(1 + g_l)^{-\gamma} + \pi [(1 + g_h)^{-\gamma} - (1 + g_l)^{-\gamma}]} > \frac{1 + \rho}{(1 + g_l)^{-\gamma} + \hat{\pi}_t [(1 + g_h)^{-\gamma} - (1 + g_l)^{-\gamma}]},$$

However this does not hold for the left hand side of (23) so we must impose the restriction

$$\frac{1 + \Psi^{RL}(\hat{\pi}_{t+1}^d, N_{t+1})}{\Psi^{RL}(\hat{\pi}_t, N_t)}(1 + g_l) < \frac{1 + \rho}{(1 + g_l)^{-\gamma} + \hat{\pi}_t [(1 + g_h)^{-\gamma} - (1 + g_l)^{-\gamma}]}.$$

(b) $\gamma < 1$ and $\hat{\pi}_t \geq \pi$. $\Psi_t^{RL} \geq \Psi^{FI}$ and Ψ_{t+1}^{RL} is increasing in $\hat{\pi}_{t+1}$, so

$$\begin{aligned} \frac{1 + \Psi^{RL}(\hat{\pi}_{t+1}^d, N_{t+1})}{\Psi^{RL}(\hat{\pi}_t, N_t)}(1 + g_l) &< \frac{1 + \Psi^{RL}(\hat{\pi}_t, N_t)}{\Psi^{RL}(\hat{\pi}_t, N_t)}(1 + g_l) \\ &= \left[1 + \frac{1}{\Psi^{RL}(\hat{\pi}_t, N_t)} \right] (1 + g_l) \leq \left[1 + \frac{1}{\Psi^{FI}} \right] (1 + g_l). \end{aligned}$$

By arguments similar to those in (a), absence of arbitrage opportunities under FI implies their absence under rational learning as well. However this does not hold for the right hand side of (23) and restrictions on the parameters must be imposed.

(c) $\gamma > 1$ and $\hat{\pi}_t < \pi$. Since $\Psi_t^{RL} > \Psi^{FI}$ and Ψ_{t+1}^{RL} is decreasing in $\hat{\pi}_{t+1}$, a series of inequalities similar to those in (b) imply that absence of arbitrage opportunities under FI is sufficient for the left hand side of the rational learning no-arbitrage conditions to hold. This does not hold for the right hand side.

(d) $\gamma > 1$ and $\hat{\pi}_t \geq \pi$. Since $\Psi_t^{RL} \leq \Psi^{FI}$ and Ψ_{t+1}^{RL} is decreasing in $\hat{\pi}_{t+1}$, a series of inequalities identical to those in (a) imply that the absence of arbitrage opportunities under FI guarantees that the right hand side of (23) holds. This does not apply to the left hand side.

When $\gamma < 1$, (a) - (b) imply that on a rational learning path where optimism and pessimism can alternate, the condition in the corollary should be imposed and checked at all nodes of the binomial lattice. (c) and (d) have the same implication when $\gamma > 1$. \square

Proof of Proposition 4. Under the assumed transversality condition, the BL stock price is³⁶

$$\begin{aligned}
S_t^{BL} &= D_t E_{\pi|D_t} \left\{ \sum_{s=1}^{\infty} E_{t,\pi} \left[\prod_{k=1}^s Q_{t+k} \frac{D_{t+k}}{D_{t+k-1}} \right] \right\} \\
&= D_t \sum_{s=1}^{\infty} \int_0^1 \left[\sum_{i=0}^s \beta^s (1 + g_l^*)^{s-i} (1 + g_h^*)^i \binom{s}{i} (1 - \pi)^{s-i} \pi^i \right] p(\pi|D_t) d\pi \\
&= D_t \sum_{s=1}^{\infty} \sum_{i=0}^s \beta^s (1 + g_l^*)^{s-i} (1 + g_h^*)^i \int_0^1 \binom{s}{i} (1 - \pi)^{s-i} \pi^i \\
&\quad \times \frac{\Gamma(N_0 + t)}{\Gamma(j + n_0) \Gamma(t + N_0 - n_0 - j)} \pi^{j+n_0-1} (1 - \pi)^{t+N_0-n_0-j-1} d\pi.
\end{aligned}$$

The integral is

$$\begin{aligned}
&\int_0^1 \binom{s}{i} \frac{\Gamma(N_0 + t)}{\Gamma(j + n_0) \Gamma(t + N_0 - n_0 - j)} \pi^{j+i+n_0-1} (1 - \pi)^{t+N_0+s-n_0-j-i-1} d\pi \\
&= \binom{s}{i} \frac{\Gamma(N_0 + t)}{\Gamma(j + n_0) \Gamma(t + N_0 - n_0 - j)} \int_0^1 \pi^{j+i+n_0-1} (1 - \pi)^{t+N_0+s-n_0-j-i-1} d\pi \\
&= \binom{s}{i} \frac{\Gamma(N_0 + t)}{\Gamma(n_0 + j) \Gamma(N_0 + t - n_0 - j)} \frac{\Gamma(j + i + n_0) \Gamma(t + N_0 + s - n_0 - j - i)}{\Gamma(N_0 + t + s)},
\end{aligned}$$

since $\int_0^1 \pi^{j+i+n_0-1} (1 - \pi)^{t+N_0+s-n_0-j-i-1} d\pi$ is a beta function $\beta f(c, d)$ and $\beta f(c, d) = \Gamma(c) \Gamma(d) / \Gamma(c + d)$. The last line is the probability function of a beta-binomial distribution with parameters $(s, N_0 + t, N_0 + t - n_0 - j)$. Defining $P^{BL}(D_{t+s})$ as this discrete probability distribution,

$$\begin{aligned}
P^{BL}(D_{t+s}) &= (1 + g_l)^{t+s-j+i} (1 + g_h)^{j+i} D_0 \\
&= \binom{s}{i} \frac{\Gamma(N_0 + t)}{\Gamma(n_0 + j) \Gamma(N_0 + t - n_0 - j)} \frac{\Gamma(j + i + n_0) \Gamma(t + N_0 + s - n_0 - j - i)}{\Gamma(N_0 + t + s)},
\end{aligned}$$

the equilibrium stock price under BL can be written as:

$$S_t^{BL}(j) = D_t^j \left[\sum_{s=1}^{\infty} \sum_{i=0}^s \beta^s (1 + g_l^*)^{s-i} (1 + g_h^*)^i P^{BL}(D_{t+s}^i) \right]. \quad \square$$

Proof of Proposition 5. Comparing P^{RL} to the expression for P^{BL} , the only difference between the asset price under rational and Bayesian learning arises from the terms

$$\binom{s}{i} \times \frac{\prod_{k=0}^{i-1} (n_0 + \sum_{j=1}^t I_{\{g_j=g_h\}} + k) \prod_{k=0}^{s-i-1} (N_0 - n_0 + t - \sum_{j=1}^t I_{\{g_j=g_h\}} + k)}{\prod_{k=0}^{s-1} (N_0 + t + k)}$$

and the ratios of gamma products

$$\frac{\Gamma(N_0 + t)}{\Gamma(n_0 + j) \Gamma(N_0 + t - n_0 - j)} \frac{\Gamma(j + i + n_0) \Gamma(t + N_0 + s - n_0 - j - i)}{\Gamma(N_0 + t + s)}.$$

³⁶Throughout, we suppress the dependence of D_t on j (the number of ‘up’ moves).

While these terms may appear to be very different, notice that

$$\begin{aligned}
& \frac{\Gamma(N_0 + t)}{\Gamma(n_0 + j)\Gamma(N_0 + t - n_0 - j)} \frac{\Gamma(j + i + n_0)\Gamma(t + N_0 + s - n_0 - j - i)}{\Gamma(N_0 + t + s)} \\
&= [(t + N_0 + s - n_0 - j - i - 1) \times (t + N_0 + s - n_0 - j - i - 2) \times \dots \times (N_0 + t - n_0 - j)] \\
&\quad \times \frac{[(n_0 + j + i - 1) \times (n_0 + j + i - 2) \times \dots \times (n_0 + j)]}{(N_0 + t + s - 1) \times (N_0 + t + s - 2) \times \dots \times (N_0 + t)} \\
&= \frac{\left[\prod_{k=0}^{i-1} (n_0 + \sum_{k=1}^t I_{\{g_k = g_h\}} + k) \right] \left[\prod_{k=0}^{s-i-1} (t - n_0 - \sum_{k=1}^t I_{\{g_k = g_h\}} + k) \right]}{\prod_{k=0}^{s-1} (N_0 + t + k)},
\end{aligned}$$

where we used $\sum_{k=1}^t I_{\{g_k = g_h\}} = j$. Hence $P^{RL}\{D_{t+s}\} = P^{BL}(D_{t+s})$ and the BL stock price is

$$S_t^{BL} = D_t^j \left[\sum_{s=1}^{\infty} \sum_{i=0}^s \beta^s (1 + g_l^*)^{s-i} (1 + g_h^*)^i P^{RL}(D_{t+s}) \right].$$

The second part of the proof follows from noting that the above factorization does not work if a non-beta prior is used. \square

Proof of Proposition 6. We need to establish conditions under which the compound probability of i up-states and $s - i$ down-states under the rational learning rule exceeds the probability under the adaptive learning rule. This holds when

$$\binom{s}{i} \cdot \frac{\prod_{k=0}^{i-1} (n_t + k) \prod_{k=0}^{s-i-1} (N_t - n_t + k)}{\prod_{k=0}^{s-1} (N_t + k)} > \binom{s}{i} \cdot \frac{\prod_{k=0}^{i-1} n_t \prod_{k=0}^{s-i-1} (N_t - n_t)}{\prod_{k=0}^{s-1} N_t}, \quad i = 0, \dots, s.$$

Dividing through by the expression on the right hand side and taking logarithms, we get

$$\sum_{k=0}^{i-1} \ln \left(1 + \frac{k}{n_t} \right) + \sum_{k=0}^{s-i-1} \ln \left(1 + \frac{k}{N_t - n_t} \right) - \sum_{k=0}^{s-1} \ln \left(1 + \frac{k}{N_t} \right) > 0.$$

It is easily verified that this expression is positive when i is either very large (“close to s ”) or very small (“close to zero”). Also, the expression is monotonically decreasing as a function of i for $i < (s - 1)n_t/N_t$, and increases for $i > (s - 1)n_t/N_t$. Both sides of the first inequality sum up to one and the left hand side has larger probability mass in the tails, so it follows that the probability distribution over the proportion of up-states under RL is a mean-preserving spread relative to the probability distribution under AL. \square

Proof of Proposition 7. Consider the RL pricing kernel,

$$\Psi^{RL}(\hat{\pi}_t, N_t) \equiv \sum_{s=1}^{\infty} \beta^s \sum_{i=0}^s (1 + g_h^*)^i (1 + g_l^*)^{s-i} P^{RL}(i|\hat{\pi}_t, N_t) = \sum_{s=1}^{\infty} \beta^s (1 + g_l^*)^s \sum_{i=0}^s \alpha^i P^{RL}(i|\hat{\pi}_t, N_t),$$

where $\alpha \equiv \frac{1+g_h^*}{1+g_l^*}$ and $P^{RL}(i|\hat{\pi}_t, N_t) \equiv P^{RL}(D_{t+s} = (1 + g_h)^i (1 + g_l)^{s-i} | \hat{\pi}_t, N_t)$. Clearly, $\alpha \geq 1$ if and only if $\gamma \leq 1$. Fix $s \geq 1$, so

$$\sum_{i=0}^s \alpha^i P^{RL}(i|\hat{\pi}_t, N_t) = E[\alpha^i | \hat{\pi}_t, N_t],$$

where i is the number of up moves of the dividend out of a total of s steps. i is a realization of a random variable with discrete distribution $\{P^{RL}(i|\hat{\pi}_t, N_t)\}_{i=0}^T$.

When $\gamma < 1$, $\alpha > 1$ so α^i is strictly increasing in i . For $\hat{\pi}'_t > \hat{\pi}_t$, $E[\alpha^i|\hat{\pi}'_t, N_t] > E[\alpha^i|\hat{\pi}_t, N_t]$, and

$$\Psi^{RL}(\hat{\pi}'_t, N_t) = \sum_{s=1}^{\infty} \beta^s (1 + g_l^*)^s E[\alpha^i|\hat{\pi}'_t, N_t] > \sum_{s=1}^{\infty} \beta^s (1 + g_l^*)^s E[\alpha^i|\hat{\pi}_t, N_t] = \Psi^{RL}(\hat{\pi}_t, N_t).$$

The RL pricing kernel is therefore monotonically increasing in $\hat{\pi}_t$. Conversely, when $\gamma > 1$, $\alpha < 1$ and α^i is strictly decreasing in i . It then follows that $\Psi^{RL}(\hat{\pi}'_t, N_t) < \Psi^{RL}(\hat{\pi}_t, N_t)$, as claimed.

To establish convexity of $\Psi^{RL}(\hat{\pi}_t, N_t)$, we need to show that

$$\hat{\pi}_t \Psi^{RL}(\hat{\pi}^u) + (1 - \hat{\pi}_t) \Psi^{RL}(\hat{\pi}^d) > \Psi^{RL}(\hat{\pi}_t \hat{\pi}^u + (1 - \hat{\pi}_t) \hat{\pi}^d) = \Psi^{RL}(\hat{\pi}^*) = \Psi^{RL}(\hat{\pi}_t),$$

where $\hat{\pi}^d = \hat{\pi}_t - \frac{1}{N_t}$ and $\hat{\pi}^u = \hat{\pi}_t + \frac{1}{N_t}$. The last equality follows from

$$\hat{\pi}^* = \hat{\pi}_t \hat{\pi}^u + (1 - \hat{\pi}_t) \hat{\pi}^d = \frac{n_t}{N_t} \frac{n_t + 1}{N_t} + \frac{N_t - n_t}{N_t} \frac{n_t - 1}{N_t} = \frac{n_t}{N_t} \left[\frac{n_t + 1}{N_t} + \frac{N_t - n_t}{N_t} - \frac{1}{N_t} \right] = \hat{\pi}_t.$$

To verify the inequality, notice that

$$\begin{aligned} \hat{\pi}_t \Psi^{RL}(\hat{\pi}^u) &= \sum_{s=1}^{\infty} \beta^s \sum_{i=0}^s (1 + g_h^*)^i (1 + g_l^*)^{s-i} \left[\hat{\pi}_t \frac{n_t + i}{n_t} \frac{N_t - n_t - 1}{N_t - n_t + s - i - 1} \right] P^{RL}(i|\hat{\pi}_t), \\ (1 - \hat{\pi}_t) \Psi^{RL}(\hat{\pi}^d) &= \sum_{s=1}^{\infty} \beta^s \sum_{i=0}^s (1 + g_h^*)^i (1 + g_l^*)^{s-i} \left[(1 - \hat{\pi}_t) \frac{n_t - 1}{n_t + i - 1} \frac{N_t - n_t + s - i}{N_t - n_t} \right] P^{RL}(i|\hat{\pi}_t). \end{aligned}$$

Adding these together, we have

$$\begin{aligned} \hat{\pi}_t \Psi^{RL}(\hat{\pi}^u) + (1 - \hat{\pi}_t) \Psi^{RL}(\hat{\pi}^d) &= \sum_{s=1}^{\infty} \beta^s \sum_{i=0}^s (1 + g_h^*)^i (1 + g_l^*)^{s-i} P^{RL}(i|\hat{\pi}_t) \\ &\quad \times \left[\hat{\pi}_t \frac{n_t + i}{n_t} \frac{N_t - n_t - 1}{N_t - n_t + s - i - 1} + (1 - \hat{\pi}_t) \frac{n_t - 1}{n_t + i - 1} \frac{N_t - n_t + s - i}{N_t - n_t} \right]. \end{aligned}$$

The probability distribution

$$\check{P}^{RL}(i|\hat{\pi}_t) \equiv P^{RL}(i|\hat{\pi}_t) \left[\frac{n_t + i}{n_t} \frac{N_t - n_t - 1}{N_t - n_t + s - i - 1} + \frac{n_t - 1}{n_t + i - 1} \frac{N_t - n_t + s - i}{N_t - n_t} \right]$$

can be shown to represent a mean-preserving spread of $P_t^{RL}(j|\hat{\pi}_t)$:

$$\begin{aligned} \sum_{i=0}^s \left[\frac{[\hat{\pi}_t \hat{\pi}_t^u + (1 - \hat{\pi}_t) \hat{\pi}_t^d] N_t + i}{N_t + s} \right] \check{P}_t^{RL}(i|\hat{\pi}_t) \\ = \hat{\pi}_t \sum_{i=0}^s \left[\frac{\hat{\pi}_t^u N_t + i}{N_t + s} \right] \binom{s}{i} \frac{\prod_{k=0}^i (n_t + k + 1) \prod_{k=0}^{s-i-1} (N_t - n_t + k - 1)}{\prod_{k=0}^s (N_t + k)} \\ + (1 - \hat{\pi}_t) \sum_{i=0}^s \left[\frac{\hat{\pi}_t^d N_t + i}{N_t + s} \right] \binom{s}{i} \frac{\prod_{k=0}^i (n_t + k - 1) \prod_{k=0}^{s-i-1} (N_t - n_t + k + 1)}{\prod_{k=0}^s (N_t + k)} = \hat{\pi}_t \hat{\pi}_t^u + (1 - \hat{\pi}_t) \hat{\pi}_t^d = \hat{\pi}_t. \end{aligned}$$

This has the same mean as $P^{RL}(j|\hat{\pi}_t)$. However, the term

$$\left[\hat{\pi}_t \frac{n_t + i}{n_t} \frac{N_t - n_t - 1}{N_t - n_t + s - i - 1} + (1 - \hat{\pi}_t) \frac{n_t - 1}{n_t + i - 1} \frac{N_t - n_t + s - i}{N_t - n_t} \right]$$

is greater than one when i is either close to zero or s and is otherwise less than one, showing that probability mass is shifted from the center of the distribution to the tails. Further calculations show that as long as i is small, $\check{P}_t^{RL}(i|\hat{\pi}_t) > P_t^{RL}(i|\hat{\pi}_t)$. Likewise, it is possible to show that when i is large, $\check{P}_t^{RL}(i|\hat{\pi}_t) > P_t^{RL}(i|\hat{\pi}_t)$.

Since $\hat{\pi}_t \Psi^{RL}(\hat{\pi}_t^u) + (1 - \hat{\pi}_t) \Psi^{RL}(\hat{\pi}_t^d)$ is an infinite sum involving terms such as $\sum_{i=0}^s (1 + g_h^*)^i (1 + g_l^*)^{s-i} \check{P}_t^{RL}(i|\hat{\pi}_t)$, the function $h(i) = (1 + g_h^*)^i (1 + g_l^*)^{s-i}$ ($i \leq s$) is increasing and convex in i , and $\check{P}_t^{RL}(i|\hat{\pi}_t)$ is a mean preserving spread of $P_t^{RL}(i|\hat{\pi}_t)$. Moving probability mass to the good states $(1 + g_h^*)$ thus more than compensates shifting mass to the bad states $(1 + g_l^*)$ and the result follows from

$$\begin{aligned} \hat{\pi}_t \Psi^{RL}(\hat{\pi}_t^u) + (1 - \hat{\pi}_t) \Psi^{RL}(\hat{\pi}_t^d) &= \sum_{s=1}^{\infty} \beta^s \sum_{i=0}^s (1 + g_h^*)^i (1 + g_l^*)^{s-i} \check{P}_t^{RL}(i|\hat{\pi}_t) > \\ &> \sum_{s=1}^{\infty} \beta^s \sum_{i=0}^s (1 + g_h^*)^i (1 + g_l^*)^{s-i} P_t^{RL}(i|\hat{\pi}_t) = \Psi^{RL}(\hat{\pi}_t). \end{aligned}$$

When $\gamma = 1$, the proof is trivial since $\Psi^{RL}(\hat{\pi}_t) = \frac{1}{\rho}$, so Ψ_t^{RL} is a constant that does not depend on $\hat{\pi}_t$ and therefore is convex.

When $\gamma > 1$, $(1 + g_h^*) < (1 + g_l^*)$ so $\hat{\pi}_t' > \hat{\pi}_t$ shifts probability mass from the good to the bad state. Therefore Ψ_t^{RL} decreases in $\hat{\pi}_t$. Since $\hat{\pi}_t \Psi^{RL}(\hat{\pi}_t^u) + (1 - \hat{\pi}_t) \Psi^{RL}(\hat{\pi}_t^d)$ is an infinite sum involving terms such as $\sum_{i=0}^s (1 + g_h^*)^i (1 + g_l^*)^{s-i} \check{P}_t^{RL}(i|\hat{\pi}_t)$, the function $h(i) = (1 + g_h^*)^i (1 + g_l^*)^{s-i}$ ($i \leq s$) is decreasing and convex in i , and $\check{P}_t^{RL}(i|\hat{\pi}_t)$ is a mean preserving spread of $P_t^{RL}(i|\hat{\pi}_t)$. It follows that moving probability mass to the ‘good’ states more than compensates shifting mass to the ‘bad’ states so that $\hat{\pi}_t \Psi^{RL}(\hat{\pi}_t^u) + (1 - \hat{\pi}_t) \Psi^{RL}(\hat{\pi}_t^d) > \Psi^{RL}(\hat{\pi}_t)$. \square

Proof of Proposition 8. Returns can be expressed as

$$r_{t+s}^{FI} \equiv \ln \left(\frac{S_{t+s}^{FI} + D_{t+s}}{S_{t+s-1}^{FI}} \right) = \ln \left[\frac{(1 + \Psi^{FI}) D_{t+s}}{\Psi^{FI} D_{t+s-1}} \right] = k + \ln(1 + g_{t+s}),$$

where $k \equiv \ln \left(\frac{1 + \Psi^{FI}}{\Psi^{FI}} \right)$ is a nonlinear function of the FI price-dividend ratio. Let $Cov(\cdot)$ denote the covariance under the true (but unknown) probability measure π . Now

$$Cov(r_{t+s}^{FI}, r_t^{FI}) = Cov[\ln(1 + g_{t+s}), \ln(1 + g_t)] = 0 \quad \forall j \geq 1,$$

since the dividend growth rate is assumed to be independently distributed over time, and, for independently distributed variables, X, Y , $E(g(X)h(Y)) = E(g(X))E(h(Y))$ for the functions $g(\cdot)$ and $h(\cdot)$ that we are using. Analogously,

$$Cov[(r_{t+s}^{FI})^2, (r_t^{FI})^2] = Cov\left([\ln(1 + g_{t+j})]^2, [\ln(1 + g_t)]^2\right) = 0 \quad \forall j \geq 1.$$

Hence under FI, there is neither serial correlation nor heteroskedasticity in stock returns.

Under RL, stock returns can be expressed as:

$$r_{t+s}^{RL} \equiv \ln \left(\frac{S_{t+s}^{RL} + D_{t+s}}{S_{t+s-1}^{RL}} \right) = \ln(1 + \Psi_{t+s}^{RL}) - \ln \Psi_{t+s-1}^{RL} + \ln(1 + g_{t+s}),$$

where $\Psi_{t+s}^{RL} = \Psi^{RL}(\hat{\pi}_{t+s}, N_{t+s})$. Furthermore

$$\begin{aligned} Cov(r_{t+s}^{RL}, r_t^{RL}) &= Cov[\ln(1 + \Psi_{t+s}^{RL}), \ln(1 + \Psi_t^{RL})] - Cov[\ln(1 + \Psi_{t+s}^{RL}), \ln \Psi_{t-1}^{RL}] \\ &\quad + Cov[\ln(1 + \Psi_{t+s}^{RL}), \ln(1 + g_t)] - Cov[\ln \Psi_{t+s-1}^{RL}, \ln(1 + \Psi_t^{RL})] \\ &\quad + Cov[\ln \Psi_{t+s-1}^{RL}, \ln \Psi_{t-1}^{RL}] - Cov[\ln \Psi_{t+s-1}^{RL}, \ln(1 + g_t)] \end{aligned}$$

since $Cov[\ln(1 + g_{t+s}), \ln(1 + g_t)] = Cov[\ln(1 + \Psi_t^{RL}), \ln(1 + g_{t+s})] = Cov[\ln \Psi_{t-1}^{RL}, \ln(1 + g_{t+s})] = 0$. For all j , $\hat{\pi}_{t+s}$ is positively correlated with $\hat{\pi}_t$:

$$\hat{\pi}_{t+s} = \frac{n_{t+s}}{N_{t+s}} = \frac{n_t + \sum_{i=1}^s I_{\{g_{t+i}=g_h\}}}{N_{t+s}} = \frac{n_t}{N_t \frac{N_{t+s}}{N_t}} + \frac{\sum_{i=1}^s I_{\{g_{t+i}=g_h\}}}{N_{t+s}} = \frac{1}{N_{t+s}} \left[N_t \hat{\pi}_t + \sum_{i=1}^s I_{\{g_{t+i}=g_h\}} \right].$$

This implies that

$$\begin{aligned} Cov(\hat{\pi}_{t+s}, \hat{\pi}_t) &= E \left[\frac{N_t}{N_{t+s}} \hat{\pi}_t^2 + \frac{\hat{\pi}_t}{N_{t+s}} \sum_{i=1}^s I_{\{g_{t+i}=g_h\}} \right] - \pi^2 \\ &= \frac{N_t}{N_{t+s}} E[\hat{\pi}_t^2] + \frac{s}{N_{t+s}} \pi^2 - \pi^2 > \frac{N_t}{N_{t+s}} \pi^2 + \frac{s}{N_{t+s}} \pi^2 - \pi^2 = 0. \end{aligned}$$

Unbiasedness of $\hat{\pi}_t$ ensures that $E[\hat{\pi}_t] = \pi$, $\hat{\pi}_t = \frac{\sum_{i=1}^t I_{\{g_i=g_h\}}}{N_t}$ and $\sum_{i=1}^s I_{\{g_{t+i}=g_h\}}$ are independent by construction. By Jensen's inequality, $E[\hat{\pi}_t^2] > \{E[\hat{\pi}_t]\}^2 = \pi^2$. Since Ψ_t^{RL} is a monotonic function of $\hat{\pi}_t$, and the transformations $\ln(1 + \Psi_{t+s}^{RL})$ and $\ln(\Psi_{t+s}^{RL})$ are monotonically increasing, it follows that $Cov[\ln(1 + \Psi_{t+s}^{RL}), \ln(1 + \Psi_t^{RL})]$ and $Cov[\ln(1 + \Psi_{t+j}^{RL}), \ln \Psi_{t-1}^{RL}]$ are nonnegative. Furthermore, $Cov[\ln(1 + \Psi_{t+s}^{RL}), \ln(1 + g_t)]$ is positive when $\gamma < 1$, and negative when $\gamma > 1$. In general $Cov(r_{t+s}^{RL}, r_t^{RL})$ will therefore be nonzero.

This result also holds for transformations of r_{t+s}^{RL} and r_t^{RL} , $g(r_{t+s}^{RL})$, that lend themselves to a Taylor series expansion. Consider a first-order Taylor expansion of $g(r_{t+s}^{RL})$ around π :

$$\begin{aligned} g(r_{t+s}^{RL}) &\approx g(r_{t+s}^{RL}(\pi)) + \frac{\partial g(r_{t+s}^{RL}(\pi))}{\partial r_{t+s}^{RL}} \frac{\partial \Psi_{t+s}^{RL}(\hat{\pi}_{t+s})}{\partial \hat{\pi}_{t+s}} \frac{1}{1 + \Psi_{t+s}^{RL}(\hat{\pi}_{t+s})} (\hat{\pi}_{t+s} - \pi) \\ &\quad - \frac{\partial g(r_{t+s-1}^{RL}(\pi))}{\partial r_{t+s-1}^{RL}} \frac{\partial \Psi_{t+s-1}^{RL}(\hat{\pi}_{t+s-1})}{\partial \hat{\pi}_{t+s-1}} \frac{1}{\Psi_{t+s-1}^{RL}(\hat{\pi}_{t+s-1})} (\hat{\pi}_{t+s-1} - \pi). \end{aligned}$$

$g(r_t^{RL})$ can be similarly expressed by means of $\hat{\pi}_t$ and $\hat{\pi}_{t-1}$. Using this, we get

$$\begin{aligned} Cov[g(r_{t+s}^{RL}), g(r_t^{RL})] &\approx G_{j,0} \frac{1}{1 + \Psi_{t+s}^{RL}(\hat{\pi}_{t+s})} \frac{1}{1 + \Psi_t^{RL}(\hat{\pi}_t)} Cov(\hat{\pi}_{t+s}, \hat{\pi}_t) \\ &\quad - G_{j,-1} \frac{1}{1 + \Psi_{t+s}^{RL}(\hat{\pi}_{t+s})} \frac{1}{\Psi_{t-1}^{RL}(\hat{\pi}_{t-1})} Cov(\hat{\pi}_{t+s}, \hat{\pi}_{t-1}) \\ &\quad - G_{j-1,0} \frac{1}{\Psi_{t+s-1}^{RL}(\hat{\pi}_{t+s-1})} \frac{1}{1 + \Psi_t^{RL}(\hat{\pi}_t)} Cov(\hat{\pi}_{t+s-1}, \hat{\pi}_t) \\ &\quad + G_{j-1,-1} \frac{1}{\Psi_{t+s-1}^{RL}(\hat{\pi}_{t+s-1})} \frac{1}{\Psi_{t-1}^{RL}(\hat{\pi}_{t-1})} Cov(\hat{\pi}_{t+s-1}, \hat{\pi}_{t-1}) \end{aligned}$$

where $G_{j,i} \equiv \frac{\partial g(r_{t+j}^{RL})}{\partial r_{t+j}^{RL}} \frac{\partial \Psi_{t+j}^{RL}}{\partial \hat{\pi}_{t+j}} \frac{\partial g(r_{t+i}^{RL})}{\partial r_{t+i}^{RL}} \frac{\partial \Psi_{t+i}^{RL}}{\partial \hat{\pi}_{t+i}}$. Since $Cov(\hat{\pi}_{t+j}, \hat{\pi}_{t+i}) \geq 0$, in general $Cov(g(r_{t+j}^{RL}), g(r_t^{RL})) \neq 0$.

Next consider $Cov [(r_{t+s}^{RL})^2, (r_t^{RL})^2]$ when $Cov(r_{t+s}^{RL}, r_t^{RL}) > 0$. Applying the Lyapunov inequality,

$$\{E [|r_{t+s}^{RL} \cdot r_t^{RL}|^2]\}^{\frac{1}{2}} \geq E [|r_{t+s}^{RL} \cdot r_t^{RL}|]$$

so

$$E [(r_{t+s}^{RL})^2 (r_t^{RL})^2] \geq \{E [|r_{t+s}^{RL} \cdot r_t^{RL}|]\}^2 \geq \{E [r_{t+s}^{RL} \cdot r_t^{RL}]\}^2.$$

The last inequality follows from the Cauchy-Schwarz inequality. Therefore,

$$\begin{aligned} Cov [(r_{t+s}^{RL})^2, (r_t^{RL})^2] &= E [(r_{t+s}^{RL})^2 (r_t^{RL})^2] - E[(r_{t+s}^{RL})^2]E[(r_t^{RL})^2] \\ &\geq (E [r_{t+s}^{RL} \cdot r_t^{RL}])^2 - E[(r_{t+s}^{RL})^2]E[(r_t^{RL})^2] \\ &= (E [r_{t+s}^{RL} \cdot r_t^{RL}])^2 + (E[r_{t+s}^{RL}]E[r_t^{RL}])^2 - (E[r_{t+s}^{RL}]E[r_t^{RL}])^2 - E[(r_{t+s}^{RL})^2]E[(r_t^{RL})^2] \end{aligned}$$

Since $g(x) = x^2$ is convex, Jensen's inequality implies that:

$$E[(r_{t+s}^{RL})^2]E[(r_t^{RL})^2] \geq \{E[r_{t+s}^{RL}]\}^2 \{E[r_t^{RL}]\}^2 = (E[r_{t+s}^{RL}]E[r_t^{RL}])^2,$$

so that

$$\begin{aligned} Cov [(r_{t+s}^{RL})^2, (r_t^{RL})^2] &\geq (E [r_{t+s}^{RL} \cdot r_t^{RL}])^2 + (E[r_{t+s}^{RL}]E[r_t^{RL}])^2 - 2 (E[r_{t+s}^{RL}]E[r_t^{RL}])^2 \\ &= \{E [r_{t+s}^{RL} \cdot r_t^{RL}] - E[r_{t+s}^{RL}]E[r_t^{RL}]\}^2 \\ &\quad + 2E [r_{t+s}^{RL} \cdot r_t^{RL}] E[r_{t+s}^{RL}]E[r_t^{RL}] - 2 (E[r_{t+s}^{RL}]E[r_t^{RL}])^2 \\ &= (Cov[r_{t+s}^{RL}, r_t^{RL}])^2 + 2E[r_{t+s}^{RL}]E[r_t^{RL}]Cov[r_{t+s}^{RL}, r_t^{RL}] \\ &= Cov[r_{t+s}^{RL}, r_t^{RL}] \{Cov[r_{t+s}^{RL}, r_t^{RL}] + 2E[r_{t+s}^{RL}]E[r_t^{RL}]\}. \end{aligned}$$

Clearly, when $E[r_{t+s}^{RL}]$ and $E[r_t^{RL}]$ have the same sign and $Cov [r_{t+s}^{RL}, r_t^{RL}] > 0$ it follows that $Cov[(r_{t+s}^{RL})^2, (r_t^{RL})^2] > 0$. \square

Proof of Proposition 9. Proposition 7 showed that, for $\gamma < 1$, $\Psi_t^{RL}(\hat{\pi}_t)$ is an increasing and convex function of $\hat{\pi}_t$. Therefore,

$$\begin{aligned} E[S_t^{RL}] &= E[\Psi_t^{RL}(\hat{\pi}_t) \cdot D_t] \\ &= E[\Psi_t^{RL}(\hat{\pi}_t)]E[D_t] + Cov [\Psi_t^{RL}(\hat{\pi}_t), D_t] \\ &> E[\Psi_t^{RL}(\hat{\pi}_t)]E[D_t] > \Psi_t^{RL}(E[\hat{\pi}_t])E[D_t] = \Psi^{FI}E[D_t] = E[S_t^{FI}]. \end{aligned}$$

The first inequality follows from the fact that when $\Psi_t^{RL}(\hat{\pi}_t)$ is strictly increasing in $\hat{\pi}_t$ and $\hat{\pi}_t$ is positively correlated with D_t , $Cov [\Psi_t^{RL}(\hat{\pi}_t), D_t] > 0$; the second inequality follows from an application of Jensen's inequality to the strictly convex function $\Psi_t^{RL}(\hat{\pi}_t)$.

Variances can be ranked as follows:

$$\begin{aligned} Var \left[\ln \left(\frac{(1 + \Psi^{FI})D_t}{\Psi^{FI}D_{t-1}} \right) \right] &= Var [\ln(1 + g_t)] < Var \left[\ln \left(\frac{(1 + \Psi_t^{RL}(\hat{\pi}_t))D_t}{\Psi_{t-1}^{RL}(\hat{\pi}_{t-1})D_{t-1}} \right) \right] \\ &= Var [\ln(1 + g_t)] + Var \left[\ln \left(\frac{1 + \Psi_t^{RL}(\hat{\pi}_t)}{\Psi_{t-1}^{RL}(\hat{\pi}_{t-1})} \right) \right] + \\ &\quad + 2Cov \left[\ln(1 + g_t), \ln \left(\frac{1 + \Psi_t^{RL}(\hat{\pi}_t)}{\Psi_{t-1}^{RL}(\hat{\pi}_{t-1})} \right) \right]. \end{aligned}$$

This holds since $Cov \left[\ln(1 + g_t), \ln \left(\frac{1 + \Psi_t^{RL}(\hat{\pi}_t)}{\Psi_{t-1}^{RL}(\hat{\pi}_{t-1})} \right) \right] > 0$ because $\ln(1 + g_t)$ and $\ln \left(\frac{1 + \Psi_t^{RL}(\hat{\pi}_t)}{\Psi_{t-1}^{RL}(\hat{\pi}_{t-1})} \right)$ are both monotonically increasing functions of g_t . \square

Figure 1. Perceived probability distribution of the proportion of up-states as a function of the forecast horizon $\{i\}$ under adaptive and rational learning. The belief is initialized at $\hat{\pi}_t = \frac{6}{10}$.

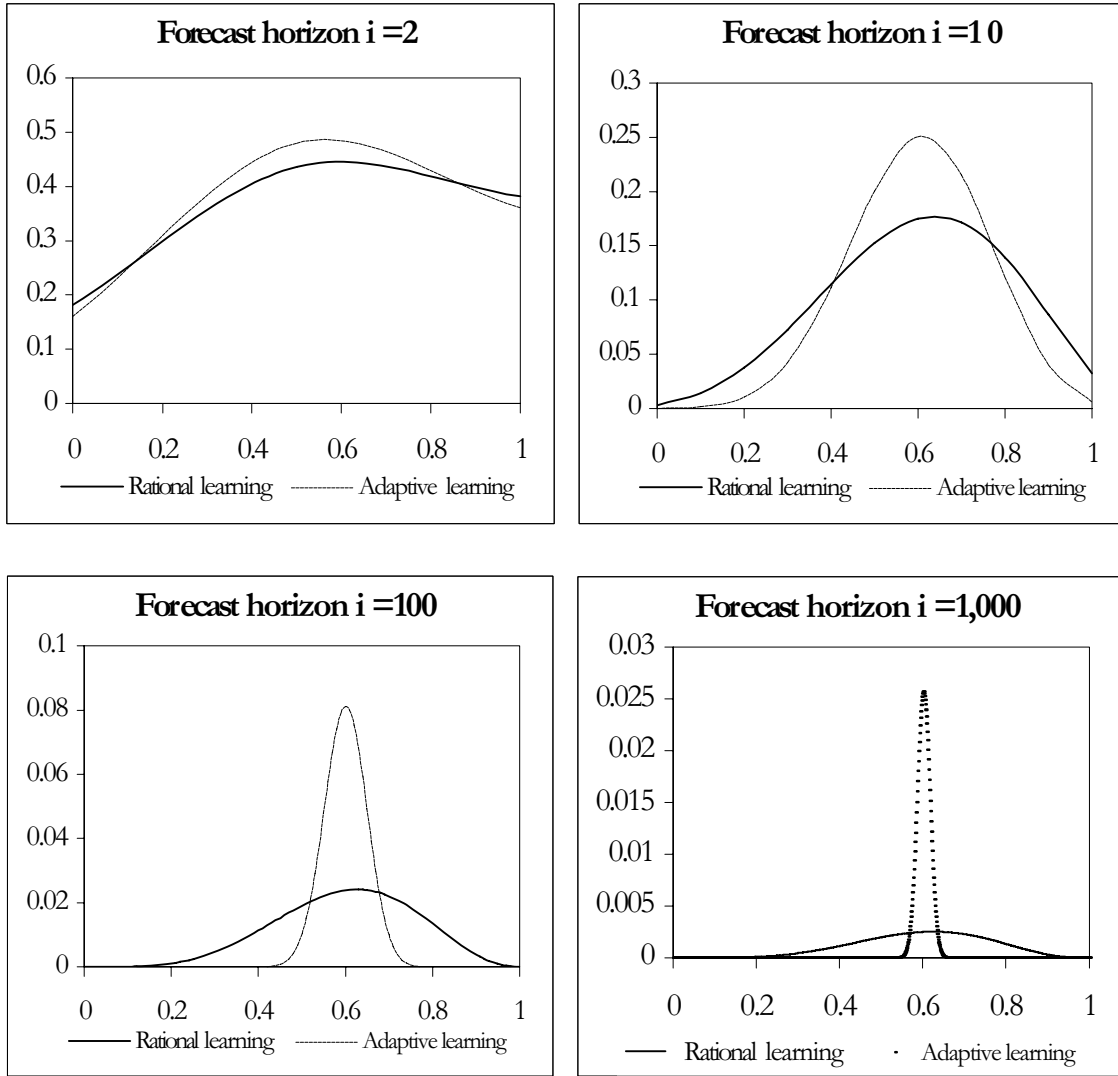


Figure 2. Rational learning pricing kernel $\Psi^{RL}(\hat{\pi}_t, 100)$ as a function of $\hat{\pi}_t$ for $\gamma = \frac{1}{2}$ and $\gamma = 1\frac{1}{2}$. For comparison the full information rational expectation (FI) pricing kernels Ψ^{FI} are also plotted. The figures assume $g_h = +6\%$, $g_l = -4\%$, $\pi = 0.6$, and $\rho = 6\%$.

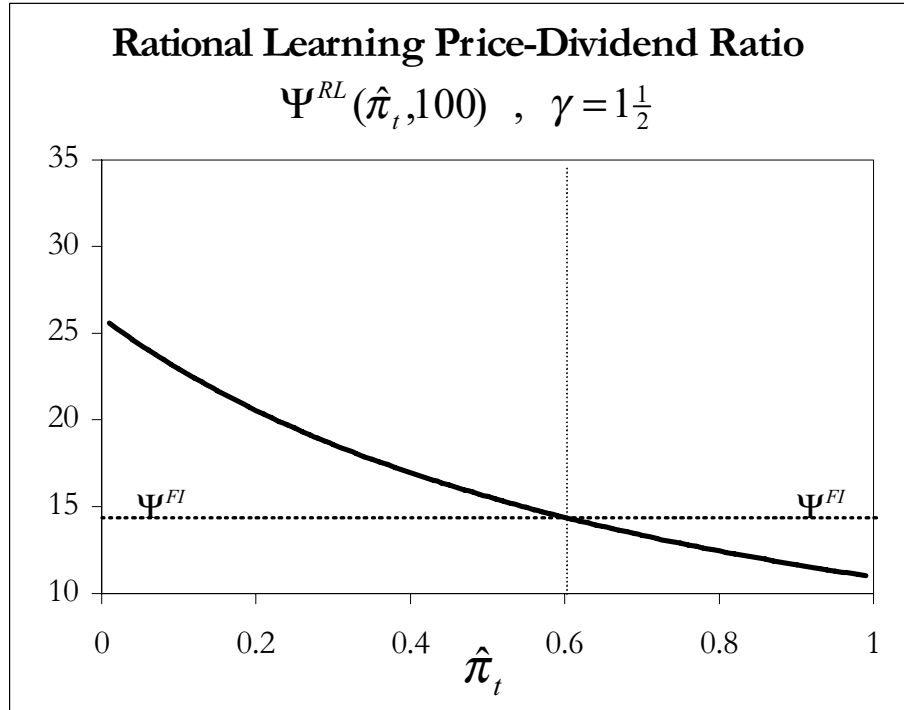
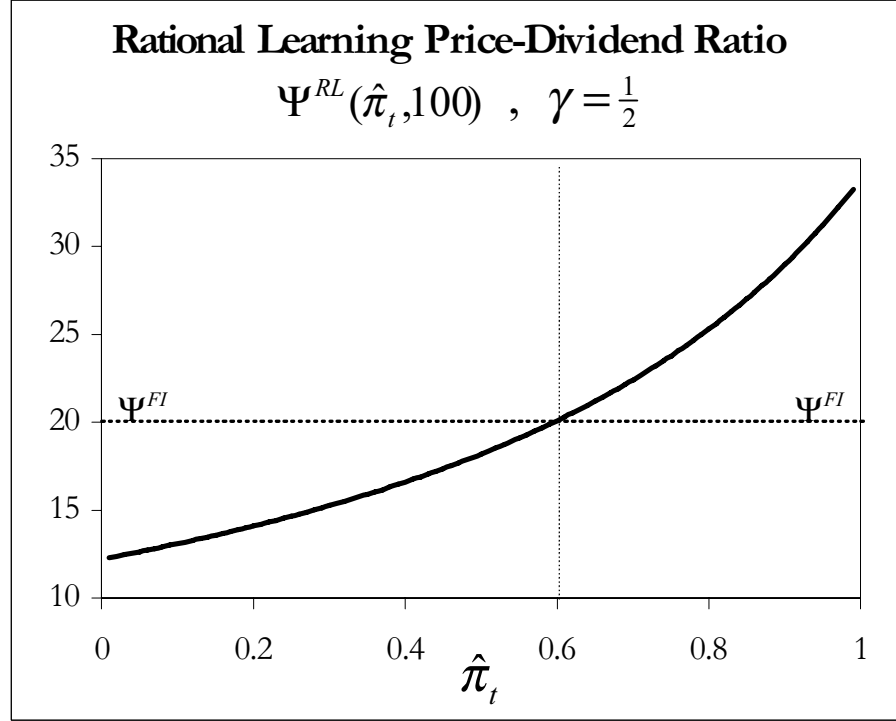


Table 1

Properties of Asset Returns Under Alternative Learning Schemes

The table reports annualized means and standard deviations for stock returns, short-term real interest rates and excess stock returns using actual data over the quarterly sample 1950:I - 2003:IV as well as simulated data. We consider a range of learning schemes and different levels of the coefficient of relative risk aversion. Columns 8-9 report Ljung-Box (LB) tests of serial correlation in returns or their squares, using 8 lags. Asymptotically, the LB(8) test statistic has a chi-square distribution with 8 degrees of freedom and a 95th percentile of 15.5. Columns 10-11 report variance ratio statistics at horizons of $q = 2$ and 5. Columns 12-15 show results for return predictability regressions using the lagged dividend yield as a predictor variable.

	Real stock returns		Short-term real interest rate		Excess stock returns		P/D ratio	Ljung-Box tests		Variance ratios		Predictability regressions			
	Mean		Stdv.		Mean		Mean	levles		q = 2		Coeff.		R ²	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)
Data	7.16	15.14	1.53	1.34	5.63	14.89	35.6	15.84	18.74	1.04	0.92	0.17	0.08	0.26	0.19
Panel A - $\gamma = 0.5$															
Full information	4.04	3.23	4.02	0.00	0.02	3.23	113.2	12.65	11.76	0.97	0.99	-0.79	0.03	-2.13	0.04
Adaptive learning	5.93	11.25	3.76	0.42	2.18	11.39	136.0	19.65	28.26	0.98	0.78	12.44	0.10	31.13	0.26
Rational learning	6.04	12.62	3.76	0.42	2.29	12.76	141.0	24.09	34.96	0.88	0.65	7.83	0.07	20.12	0.23
BL with non-conjugate priors	4.75	7.77	3.76	0.42	0.99	7.85	117.5	18.67	29.49	0.84	0.60	-17.10	0.36	-3.02	0.37
Panel B - $\gamma = 0.9$															
Full information	4.53	3.24	4.48	0.00	0.05	3.24	101.9	12.65	11.75	0.98	0.98	-0.80	0.03	-2.14	0.04
Adaptive learning	5.06	4.43	4.00	0.76	1.06	4.61	110.3	14.05	15.75	0.99	0.83	13.25	0.07	36.07	0.21
Rational learning	4.95	4.49	4.00	0.76	0.95	4.67	120.5	13.86	16.03	0.95	0.79	10.93	0.06	31.58	0.20
BL with non-conjugate priors	4.66	4.23	4.00	0.76	0.09	4.25	107.5	16.10	14.55	0.95	0.75	-4.16	0.06	1.20	0.09
Panel C - $\gamma = 1.5$															
Full information	5.32	3.25	5.22	0.00	0.10	3.25	88.45	12.65	11.74	0.97	0.98	-0.80	0.03	-2.14	0.04
Adaptive learning	6.57	6.18	5.70	1.04	0.87	6.04	96.58	27.30	45.79	0.70	0.38	-1.87	0.03	-3.64	0.12
Rational learning	5.93	5.24	5.70	1.04	0.23	4.88	98.32	31.97	52.12	0.74	0.40	-0.24	0.06	0.52	0.16
BL with non-conjugate priors	5.75	5.20	5.70	1.04	0.05	4.56	171.35	16.22	14.71	1.02	0.92	-19.45	0.40	-12.51	0.40