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Authors	Massimo Guidolin, and Allan Timmermann
Working Paper Number	2005-007B
Revision Date	August 2007
Citable Link	https://doi.org/10.20955/wp.2005.007
Suggested Citation	Guidolin, M., Timmermann, A., 2007; Size and Value Anomalies under Regime Shifts, Federal Reserve Bank of St. Louis Working Paper 2005-007. URL https://doi.org/10.20955/wp.2005.007

Published In	Journal of Financial Econometrics
Publisher Link	https://doi.org/10.1093/jjfinec/nbm021

Size and Value Anomalies under Regime Shifts*

Massimo Guidolin
Federal Reserve Bank of St. Louis and Manchester Business School, MAGF

Allan Timmermann
University of California, San Diego

JEL code: G12, G11, C32.

Abstract

This paper finds strong evidence of time-variations in the joint distribution of returns on a stock market portfolio and portfolios tracking size- and value effects. Mean returns, volatilities and correlations between these equity portfolios are found to be driven by underlying regimes that introduce short-run market timing opportunities for investors. The magnitude of the premia on the size and value portfolios and their hedging properties are found to vary across regimes. Regimes are shown to have a large impact on the optimal asset allocation - especially under rebalancing - and on investors' utility. Regimes also have a considerable impact on hedging demands, which are positive when the investor starts from more favorable regimes and negative when starting from bad states. Recursive out-of-sample forecasting experiments show that portfolio strategies based on models that account for regimes dominate single-state benchmarks.

Keywords: optimal portfolio choice, regimes, hedging demands, size and value portfolios.

1. Introduction

Empirical evidence has linked variations in the cross-section of stock returns to firm characteristics such as market capitalization (e.g., Banz (1981), Keim (1983) Reinganum (1981), Fama and French (1992)) and book-to-market values (e.g., Fama and French (1992, 1993), Davis, Fama, and French (2000)). Cross-sectional return variations associated with these characteristics are non-trivial by conventional measures. Over the sample 1927-2005 a portfolio comprising small firms paid a return of 2.9 percent per annum in excess of the return on a portfolio composed of large firms. Similarly, firms with a high book-to-market ratio outperformed firms with a low ratio by 5.0 percent per annum. In neither case have such differences been attributed to variations in CAPM betas.

Far less is known about the extent to which the joint distribution of returns on these equity portfolios varies over time. This is clearly an important question. For a multi-period investor the economic value of investments in size and value portfolios is determined not only by their mean returns but also by their volatilities and correlations with the market portfolio and by the extent to which these vary over time. To

*We thank the editor, Rene Garcia, an associate editor and two anonymous referees for many helpful suggestions. Useful comments were provided by Fulvio Ortu, a discussant, Phelim Boyle, Christian Haefke, Hashem Pesaran, Lucio Sarno, and by seminar participants at the European Central Bank/CFS/Deutsch Bundesbank workshop, the European Financial Management Association meetings in Milan (June 2005), the Federal Reserve Bank of Atlanta, Universitat Pompeu Fabra Barcelona, University of Cambridge (CERF), University of Copenhagen, University of Waterloo (Eighth Annual Financial Econometrics Conference), and Warwick Business School. The usual disclaimer applies.

address this question, we propose in this paper a new model for the joint distribution of returns on the market portfolio and the size (SMB) and book-to-market (HML) portfolios introduced by Fama and French (1993). We find evidence of four economic regimes that capture important time-variations in mean returns, volatilities and return correlations. Two states capture periods of high volatility and "large" returns that accommodate skews and fat tails in stock returns. The other two states are associated with shifts in the distribution of size and value returns. Regimes continue to be important even if our model is extended to include the dividend yield or the 1-month T-bill rate as additional state variables.

To quantify the economic significance of regimes in returns on US equity portfolios we consider their importance from the perspective of a small investor's optimal asset allocation. Optimal allocation to size and value portfolios has received some attention in the existing literature. Brennan and Xia (2001) solve the portfolio allocation problem of a long-term Bayesian investor assuming an asset menu similar to ours. They study optimal stock holdings obtained under different priors over the size and value effects. Their calculations suggest a substantial economic value of investments in the Fama-French portfolios, on the order of 5% per annum, although the certainty equivalent value depends on the investor's coefficient of risk aversion, prior beliefs and the extent of pricing errors in the underlying asset pricing model. Pástor (2000) considers the single-period portfolio problem of a mean-variance investor. His calculations suggest that the HML portfolio should be in much greater demand than the SMB portfolio and that even investors with strong doubts about value effects should take substantial positions in the HML portfolio.[1]

Here we focus instead on the presence of predictability linked to regimes underlying the joint distribution of returns on the market, SMB and HML portfolios. The economic value of investment strategies in the anomaly portfolios is of course related to the average size and value premium but further depends on how much these vary across economic states. As pointed out by Brennan and Xia (2001), an important issue for a long-horizon investor is whether size and value effects, if genuine, can be expected to persist in the future. By allowing these effects to vary across regimes we can address this important question. Indeed we find strong evidence that optimal asset holdings vary significantly across regimes and across short and long investment horizons as investors anticipate a shift out of the current state.

We solve the asset allocation problem by extending the Monte Carlo methods in Barberis (2000) and Detemple, Garcia, and Rindisbacher (2003) to the case with regime switching in returns. This allows us to treat the states as unobservable and to characterize investors' optimal portfolio weights under imperfect information about the current state. Uncertainty about the underlying state means that investors exploit regimes less aggressively. However, most of the time investors have sufficiently precise (filtered) estimates of the states whose presence continue to affect the portfolio weights, hedging demands and certainty equivalence returns.

We study several aspects of the portfolio allocation problem, such as the importance of the rebalancing frequency, the investment horizon, and of investors' learning about unobservable states. At long horizons we find that the size and value portfolios have moderate weights in a buy-and-hold investor's optimal allocation. This finding differs from previous estimates of a more substantial role for the SMB and HML portfolios in the optimal long-run asset allocation and is a reflection of the fat-tailed return distribution captured by the presence of high-volatility states. At short horizons, we find a more significant role for these portfolios linked

[1]Lynch (2001) analyzes the effect of linear (VAR(1)) predictability from the dividend yield or the term spread on investments in size- and value-sorted portfolios as a function of the investment horizon and finds that investors with long horizons should hold less in small stocks and stocks with high book to market ratios.

to the market timing opportunities implied by the four-state model. By allowing for adjustments to portfolio weights following changes in the underlying state probabilities, rebalancing enhances the weights on the size and value portfolios in the optimal asset allocation.

We also study the hedging demand induced by regime switching and compare it to the hedging demand under predictability from the dividend yield or under learning about the drift of the asset price process. Consider the hedging demand for the market portfolio. Since shocks to the dividend yield are negatively correlated with shocks to asset prices, the market portfolio provides a hedge against shocks to future investment opportunities and the hedging demand for this portfolio is positive under predictability from the dividend yield. In contrast, when investors learn about the mean return − as assumed by Brennan and Xia (2001) − shocks to the investment opportunity set and shocks to returns are positively correlated so the hedging demand for the market portfolio will be negative. Under regime switching we see both positive and negative hedging demand depending on which state the market starts from. The hedging demand is positive when the investor starts from regimes favorable to the market portfolio – since mean-reversion to less favorable investment opportunities is anticipated – but negative when starting from "bad" states.

Consistent with findings by Barberis (2000) and Xia (2001), we find that parameter estimation uncertainty has a large effect on optimal asset holdings. Nevertheless, regime shifts continue to have a significant effect on the optimal asset allocation and expected utility even after accounting for parameter uncertainty. Furthermore, we perform a recursive out-of-sample forecasting experiment that estimates model parameters and selects portfolio weights in "real time", i.e. based only on the data available at the point in time where the forecast is computed. We find that four-state models perform better than single-state alternatives both in terms of the precision of their out-of-sample forecasts and in terms of sample estimates of mean returns and average utility.

These conclusions appear to be robust to the particular form of regime specification used in the analysis. We find that the size of the certainty equivalent return mostly hinges on the existence of regime-dependence in expected returns and less on the exact number of states. This is consistent with large expected utility losses in two-state models when expected returns are allowed to depend on the state and of small expected utility losses in four-state models with constant expected returns. The size of hedging demands depends both on the choice of the number of regimes and on time-variations in expected returns.

The outline of the paper is as follows. Section 2 presents our multivariate regime switching models for the joint distribution of returns on the market, size and book-to-market portfolios and extensions to include additional predictor variables. Section 3 presents empirical results while Section 4 sets up the asset allocation problem and Section 5 reports empirical asset allocation results. Section 6 provides utility cost calculations, considers the impact of parameter estimation uncertainty and evaluates the out-of-sample performance of a range of models. Section 7 concludes.

2. Models for Regimes in the Joint Return Process

A large literature in finance has reported evidence of predictability in stock market returns, mostly in the context of linear, constant-coefficient models, (Campbell and Shiller (1988), Fama and French (1989), Ferson and Harvey (1991), Goetzmann and Jorion (1993) and Lettau and Ludvigsson (2001).) More recently, some papers have found evidence of regimes in the distribution of returns on individual stock portfolios or pairs of these (e.g., Ang and Bekaert (2002a), Perez-Quiros and Timmermann (2000), Guidolin and Timmermann

(2006), Turner, Startz and Nelson (1989) and Whitelaw (2001)). Following this literature we model the joint distribution of a vector of n stock returns, $\mathbf{r}_t = [r_{1t}\ r_{2t}\ ...\ r_{nt}]'$ as a multivariate regime switching process driven by a common discrete state variable, S_t, that takes integer values between 1 and k :

$$\mathbf{r}_t = \boldsymbol{\mu}_{s_t} + \sum_{j=1}^{p} \mathbf{A}_{j,s_t}\mathbf{r}_{t-j} + \boldsymbol{\varepsilon}_t. \tag{1}$$

Here $\boldsymbol{\mu}_{s_t} = [\mu_{1s_t}\ ...\ \mu_{ns_t}]'$ is a vector of mean returns in state s_t, $\mathbf{A}_{j,s_t}$ is an $n \times n$ matrix of autoregressive coefficients at lag j in state s_t and $\boldsymbol{\varepsilon}_t = [\varepsilon_{1t} ... \varepsilon_{nt}]' \sim N(\mathbf{0}, \boldsymbol{\Sigma}_{s_t})$ is the vector of return innovations that are assumed to be joint normally distributed with zero mean and state-specific covariance matrix $\boldsymbol{\Sigma}_{s_t}$. Innovations to returns are thus drawn from a Gaussian mixture distribution that is known to provide a flexible approximation to a wide class of distributions (Timmermann (2000)).[2]

Each state is the realization of a first-order Markov chain governed by the $k \times k$ transition probability matrix, $\mathbf{P}$, with generic element p_{ji} defined as

$$\Pr(s_t = i | s_{t-1} = j) = p_{ji}, \quad i, j = 1, .., k. \tag{2}$$

Our estimates allow S_t to be unobserved and treat it as a latent variable.

The model (1) - (2) nests several popular models from the finance literature as special cases. In the case of a single state, $k = 1$, we obtain a linear vector autoregression (VAR) with predictable mean returns provided that there is at least one lag for which $\mathbf{A}_j \neq 0$. Absent significant autoregressive terms, the discrete-time equivalent of the Gaussian model adopted by Brennan and Xia (2001) is obtained. The model is also consistent with evidence of instability in US equity portfolio returns (Pástor (2000) and Davis et. al. (2000)).

Our model can be extended to incorporate an $l \times 1$ vector of predictor variables, $\mathbf{z}_{t-1}$, comprising variables such as the dividend yield or interest rates that have been used in recent studies on predictability of stock returns (e.g. Aït-Sahalia and Brandt (2001) and Campbell, Chan and Viceira (2003)). Define the $(l+n) \times 1$ vector of state variables $\mathbf{y}_t = (\mathbf{r}_t'\ \mathbf{z}_t')'$. Then (1) is readily extended to

$$\mathbf{y}_t = \begin{pmatrix} \boldsymbol{\mu}_{s_t} \\ \boldsymbol{\mu}_{zs_t} \end{pmatrix} + \sum_{j=1}^{p} \mathbf{A}^*_{j,s_t}\mathbf{y}_{t-j} + \begin{pmatrix} \boldsymbol{\varepsilon}_t \\ \boldsymbol{\varepsilon}_{zt} \end{pmatrix}, \tag{3}$$

where $\boldsymbol{\mu}_{zs_t} = [\mu_{z_1 s_t}\ ...\ \mu_{z_l s_t}]'$ is the intercept vector for $\mathbf{z}_t$ in state s_t, $\{\mathbf{A}^*_{j,s_t}\}_{j=1}^{p}$ are now $(n+l) \times (n+l)$ matrices of autoregressive coefficients in state s_t and $[\boldsymbol{\varepsilon}_t'\ \boldsymbol{\varepsilon}_{zt}']' \sim N(0, \boldsymbol{\Sigma}^*_{s_t})$, where $\boldsymbol{\Sigma}^*_{s_t}$ is an $(n+l) \times (n+l)$ covariance matrix. This model allows for predictability in returns through the lagged values of $\mathbf{z}_t$. It embeds a variety of single-state VAR models that have been considered in recent studies including Barberis (2000), Campbell and Viceira (1999) and Kandel and Stambaugh (1996). This model is complicated by the joint presence of linear and non-linear predictability patterns, the latter arising due to time-variations in the filtered state probabilities.

Even in the absence of autoregressive terms or predictor variables, (1) - (2) imply time-varying investment opportunities. For example, the conditional mean of asset returns is an average of the vector of mean returns, $\boldsymbol{\mu}_{s_t}$, weighted by the filtered state probabilities $[\Pr(s_t = 1|\mathcal{F}_t)\ ...\ \Pr(s_t = k|\mathcal{F}_t)]'$, conditional on information available at time t, $\mathcal{F}_t$. Since these state probabilities vary over time, the expected return will also change. Similar comments apply to higher order moments of the return distribution.

[2] Recent papers have emphasized the importance of adopting flexible models capable of capturing time-varying correlations, skewness and kurtosis in the joint distribution of asset returns, see Manganelli (2004) and Patton (2004).

Regime switching models can be estimated by maximum likelihood after putting (3) in state-space form. In particular, estimation and inferences are based on the EM algorithm which allows iterative calculation of one-step ahead forecasts of the state vector

$$\boldsymbol{\xi}_t = [I(s_t = 1|\mathcal{F}_t) \;\; I(s_t = 2|\mathcal{F}_t) \;\; ... \;\; I(s_t = k|\mathcal{F}_t)]'$$

where $I(s_t = i|\mathcal{F}_t)$ is a standard indicator variable, given the information set $\mathcal{F}_t$. Under standard regularity conditions, consistency and asymptotic normality of the ML estimator $\hat{\boldsymbol{\theta}}$ can be established (e.g. Hamilton (1989)):

$$\sqrt{T}\left(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}\right) \xrightarrow{d} N\left(\mathbf{0}, \mathcal{I}_a(\boldsymbol{\theta})^{-1}\right)$$

where $\mathcal{I}_a(\boldsymbol{\theta})$ is the asymptotic information matrix. Our empirical results apply a 'sandwich' estimator of $\mathcal{I}_a(\boldsymbol{\theta})$ of the form[3]

$$Var(\hat{\boldsymbol{\theta}}) = T^{-1}\left[\mathcal{I}_2(\hat{\boldsymbol{\theta}})\left(\mathcal{I}_1(\hat{\boldsymbol{\theta}})\right)^{-1}\mathcal{I}_2(\hat{\boldsymbol{\theta}})\right],$$

where $p(\mathbf{y}_t|\mathcal{F}_{t-1};\hat{\boldsymbol{\theta}})$ is the conditional density of the data and

$$\mathcal{I}_1(\hat{\boldsymbol{\theta}}) \equiv T^{-1}\sum_{t=1}^{T}\left[\mathbf{h}_t(\hat{\boldsymbol{\theta}})\right]\left[\mathbf{h}_t(\hat{\boldsymbol{\theta}})\right]', \quad \mathbf{h}_t(\hat{\boldsymbol{\theta}}) \equiv \frac{\partial \ln p(\mathbf{y}_t|\mathcal{F}_{t-1};\hat{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}}, \quad \mathcal{I}_2(\hat{\boldsymbol{\theta}}) \equiv -T^{-1}\sum_{t=1}^{T}\left[\frac{\partial^2 \ln p(\mathbf{y}_t|\mathcal{F}_{t-1};\hat{\boldsymbol{\theta}})}{\partial \boldsymbol{\theta}\partial \boldsymbol{\theta}'}\right].$$

Under a mean squared forecast error (MSFE) criterion, forecasting is simple in spite of the nonlinearity of the underlying process. Conditional on the parameter estimates, the conditional expectation minimizes the MSFE, i.e.

$$E[\mathbf{y}_{t+1}|\hat{\boldsymbol{\theta}},\mathcal{F}_t] = \mathbf{X}_t\hat{\boldsymbol{\Psi}}\left(\hat{\boldsymbol{\xi}}_{t+1|t} \otimes \boldsymbol{\iota}_{l+q}\right), \tag{4}$$

where $\mathbf{X}_t = [1 \; \mathbf{y}_t' ... \mathbf{y}_{t-p+1}'] \otimes \boldsymbol{\iota}_{l+n}$, $\hat{\boldsymbol{\Psi}}$ stacks the estimates of the conditional mean parameters and $\hat{\boldsymbol{\xi}}_{t+1|t}$ is the one-step ahead forecast of the latent state vector given $\mathcal{F}_t$.

3. Regimes in market, size and book-to-market returns

3.1. *The Data*

We study continuously compounded monthly returns on US stock portfolios over the sample 1927:12 - 2005:12, a total of 937 observations. The basis for our analysis is the returns on six equity portfolios formed on the intersection of two size portfolios and three book-to-market portfolios. All portfolios are value-weighted with weights that are revised at the end of June every year and held constant for the following twelve months.[4] We also use data on the value-weighted CRSP index, the dividend yield, and 1-month T-bill rates.

To simplify the asset allocation problem, we follow Fama and French (1993) and consider two portfolios tracking size and book-to-market ratio effects. The first portfolio (SMB) is long in small firms and short in big firms, controlling for the book-to-market ratio:

$$r_t^{SMB} = \frac{1}{3}(\text{Small Value} + \text{Small Neutral} + \text{Small Growth}) - \frac{1}{3}(\text{Big Value} + \text{Big Neutral} + \text{Big Growth}).$$

[3] Under the null of no misspecification, $\mathcal{I}_1(\hat{\boldsymbol{\theta}})$ and $\mathcal{I}_2(\hat{\boldsymbol{\theta}})$ should be identical. Since we do not perform misspecification tests based on the 'distance' between $\mathcal{I}_1(\hat{\boldsymbol{\theta}})$ and $\mathcal{I}_2(\hat{\boldsymbol{\theta}})$, we base our inferences on the "sandwich" form.

[4] The portfolios for July of year t to June of year $t+1$ include all NYSE, AMEX and NASDAQ stocks with market equity data available for December of year $t-1$ and June of year t, and book equity data for year $t-1$. The book-to-market ratio for June of year t is the book equity for the last fiscal year ending in $t-1$ divided by the market equity in December of year $t-1$. Further details on data construction are available from Ken French's web site at Dartmouth.

The second portfolio (HML) is long in firms with a high book-to-market ratio and short in firms with a low book-to-market ratio, controlling for size:

$$r_t^{HML} = \frac{1}{2}(\text{Small Value} + \text{Big Value}) - \frac{1}{2}(\text{Small Growth} + \text{Big Growth}).$$

Both SMB and HML are zero-investment portfolios. It is therefore appropriate to consider their simple returns as opposed to returns in excess of a T-bill rate. Conversely, we follow common practice and consider returns on the market portfolio in excess of the T-bill rate.

We first report the usual summary statistics for the two spread portfolios and the market index. The mean excess return on the market portfolio is 8% per annum. The volatility of this portfolio is 19% per annum and it also has a thick-tailed, largely symmetric distribution. The HML portfolio earns a mean return of 5% per annum and, at 13% per annum, is less volatile than the market portfolio but with strongly skewed returns. The SMB portfolio earns a mean return of 3% per annum and has lower volatility and more right-skew than the HML portfolio. Correlations between returns on the three equity portfolios vary between 0.08 and 0.33. These properties are similar to those reported by Davis et al. (2000) for a comparable sample 1929-1997.

3.2. *Regimes in the joint return process*

No previous work seems to have attempted to identify regimes in the joint process of returns on the market, size and value portfolios $[r_t^{MKT}\ r_t^{SMB}\ r_t^{HML}]'$. Economic theory offers little guidance on how to select the number of regimes and lags for this process. To address these issues and to make sure that there is robust evidence of regimes in the first place we conducted a thorough specification analysis.[5]

More specifically, we considered a range of values for the number of regimes ($k = 1, 2, 3, 4$, and 6). This covers very parsimonious as well as heavily parameterized models. To select among the regime specifications, we considered the Akaike (AIC) and Schwartz (SIC) information criteria. These trade off in-sample fit with a penalty for over-parameterization. Unlike formal hypothesis tests which are subject to nuisance parameter problems, these criteria do not, however, provide rigorous tests for the presence of regimes. Since the AIC tends to select overparameterized models (Fenton and Gallant (1996)), we chose the model that was selected by the SIC. In a second step we then use likelihood ratio tests to impose restrictions on mean returns and covariance matrices and see whether a more parsimonious model is supported by the data (see Section 3.3).

The preferred specification has four states but no autoregressive terms.[6] The absence of autoregressive terms is perhaps unsurprising given the lack of serial correlation in the individual return series. That four states are required to capture the dynamics of the joint returns on the market and Fama-French portfolios is consistent with our finding of three (largely common) states for the HML and SMB portfolios and two (uncorrelated) states for the market portfolio.

To assist in the economic interpretation of the four-state model, Panel B of Table 1 presents parameter estimates while Figure 1 plots the associated state probabilities. Regime 1 is a moderately persistent bear state

[5] Before undertaking the analysis of the joint distribution of returns on the three stock portfolios, we considered the presence of regimes in returns on the individual portfolios, r_t^{MKT}, r_t^{SMB} and r_t^{HML}. For each portfolio we first tested the null of a single state against the alternative of multiple states and found that the single state model was soundly rejected at the 1% significance level. Tests were performed using the statistic proposed by Davies (1977). This accounts for the fact that under the null of a single state ($k = 1$) some of the regime switching parameters are not identified. A two-state model was found to be appropriate for the market portfolio while three-state models were selected for the HML and SMB portfolios.

[6] Any finite-state model is best viewed as an approximation to a more complex and evolving data generating process with non-recurrent states (see, e.g., Pesaran, Pettenuzzo and Timmermann (2006)).

whose average duration is seven months. In this state the mean excess return on the market is significantly negative at -13% per annum. During bear markets, size and value anomalies are largely absent from the data and mean returns on the SMB and HML portfolios are not significantly different from zero. Volatility is high and return correlations closely track their unconditional counterparts listed in panel A. Figure 1 shows that this regime captures major crashes and periods with sustained declines in stock prices such as the 1929 crash, the Great Depression, the two oil shocks in the 1970s and the recent bear market of 2000-2002.

Regime 2 is a highly persistent, low-volatility bull state with an average duration of 14 months that captures long periods with growing stock prices during the 1940s, the 1950s, and the mid-1990s. Mean returns in this state are significantly positive for the market and HML portfolios (13% in excess of the riskless rate and 4% per annum, respectively) but slightly negative for the SMB portfolio. Hence the value effect is strong in this state while the size effect is absent. Returns on the HML portfolio are positively correlated with returns on the market portfolio while SMB returns are uncorrelated with both the market and HML returns.

Regime 3 is another highly persistent, low-volatility state where all equity portfolios earn positive mean returns (9%, 6%, and 4%, respectively). This state captures most of the bull markets since the mid-sixties, including the late 1990s run-up. A clear difference between regimes 2 and 3 is found in their correlation structure. In the second state the SMB portfolio provides a hedge for the performance of the market portfolio. In the third state the HML portfolio plays a similar role.

Finally, regime 4 is a highly volatile, transient state that captures stock prices during parts of the Great Depression and 1999-2000. Mean returns in this state are high (17, 10, and 12 percent per month) but not absurdly so since the average duration of this state is less than two months and volatilities in this state are also very high, i.e. 47, 52, and 49% per annum. Despite its short duration, regime 4 is clearly important for size and value effects to emerge in the data.

The steady state probabilities implied by the estimates of the transition matrix, $\hat{\mathbf{P}}$, are 22%, 27%, 50% and 1%, respectively. Furthermore, transition probabilities follow a very particular pattern in our model: The market either remains in the fourth, high return state (with a probability of about one-third) or exits to the bear/crash state (with a two-thirds probability) so that states 1 and 4 jointly identify periods with clustering of high volatility.

The states are identified using an ex-post classification scheme. This is important since it is not reasonable to expect (and we do not find) states with high ex-ante volatility and negative ex-ante mean returns for the market portfolio.[7] One factor that complicates economic interpretation of the states is that the regimes differ along several dimensions such as expected returns, volatility and magnitude of the size and value effects. It is clear, however, that state one is a recession or bear state with high volatility and mostly negative mean returns, while state four is a recovery state which together with state one captures episodes of high volatility. Markets are calmer in states 2 and 3 which also see fairly large mean returns on the market portfolio. However, whereas in state 2 the value effect is significant while the size effect is not, the size effect is somewhat larger in the third state.

Corroborating our economic interpretation, we found that 39% of the periods classified as state 1 by our model occur in an NBER recession, while the corresponding numbers are 15% or less for the other states.

[7] Note that this occurs *ex-post* in state 1 but, starting from state 1, the likelihood of moving to states with higher expected returns means that the ex-ante expected return is small but positive (one percent per annum). See also Gu (2006) for a discussion of this point.

Regressions of state probabilities on the NBER recession indicator came up with a highly significant positive coefficient for state 1 and significant but negative coefficients for states 2 and 3. Moreover, when we fitted a regime switching model to industrial production growth, again we found that state 1 in our model was associated with a zero growth, high-volatility state for industrial production. In fact, the average annual growth in industrial production in the four states is zero in state 1, 4-5% in states 2 and 3 and a staggering 40% in state 4. This clearly suggests that our states are associated with underlying economic fundamentals.

3.3. *Testing restrictions and ARCH effects*

Our very long data set on three relatively weakly correlated return series means that most parameters in Table *1* are reasonably precisely estimated. Even so, the number of parameters of the four-state model is quite large and it is worth investigating whether a more parsimonious specification can be obtained. In view of the imprecise mean return estimates often found for equity portfolios, we follow Ang and Bekaert (2002a, pp. 1147-1149) and first test a model where mean returns are restricted to be identical across regimes:

$$\mathbf{r}_t = \boldsymbol{\mu} + \boldsymbol{\varepsilon}_t \quad \boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \boldsymbol{\Sigma}_{s_t}). \tag{5}$$

We can formally test the restrictions on the mean return parameters through a likelihood-ratio test:

$$LR = 2(5422.52 - 5408.40) = 28.09.$$

The implied p-value of 0.0009 strongly rejects the state-independence of mean returns.

Next, we test whether the regime switching model can be simplified by imposing covariance restrictions. Returns in regimes 1 and 4 are highly volatile so it is natural to test the hypothesis that $\boldsymbol{\Sigma}_1 = \boldsymbol{\Sigma}_4$ which implies six parameter restrictions:

$$LR = 2(5422.52 - 5397.39) = 47.74.$$

This yields a p-value very near zero. Once again the restrictions are resoundingly rejected so we maintain the general four-state model from Table *1*.

Finally, we test whether the preferred four-state model is misspecified or needs to be extended to incorporate ARCH effects. To address this question, we estimated a bivariate Markov switching ARCH model similar to that considered by Hamilton and Lin (1996):[8]

$$\begin{aligned}
\mathbf{r}_t &= \boldsymbol{\mu}_{S_t} + \boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon}_t \sim N(0, \boldsymbol{\Sigma}_{S_t}) \\
\boldsymbol{\Sigma}_{S_t} &= \mathbf{K}_{S_t} + \boldsymbol{\Delta}_{S_t} \boldsymbol{\varepsilon}_t' \boldsymbol{\varepsilon}_t \boldsymbol{\Delta}_{S_t}'.
\end{aligned} \tag{6}$$

Here $\mathbf{K}_{S_t}$ is restricted to be symmetric and positive definite and $\boldsymbol{\Delta}_{S_t}$ captures regime-dependent effects of past shocks on current volatility. To formally test for ARCH effects, we imposed the restriction $\boldsymbol{\Delta}_{S_t} = \boldsymbol{\Delta}$, $S_t = 1, 2, 3, 4$ and obtained the likelihood ratio test

$$LR = 2\,[5447.23 - 5422.52] = 49.42.$$

The associated p-value is 0.301 so the null hypothesis of no ARCH effects fails to be rejected. We therefore maintain the simpler four-state model without ARCH effects. The absence of ARCH effects in our model can

[8]It is possible that other multivariate regime switching GARCH models may improve the fit, see e.g. Haas, Mittnik, and Paolella (2004).

be explained by the fact that, at the monthly frequency, regime switching can capture volatility clustering through time-variations in the probabilities of (persistent) states with very different levels of volatility, see Gray (1996) and Timmermann (2000).

3.4. *Predictor Variables: The Dividend Yield*

Many studies suggest that stock returns are predicted by regressors such as term and default spreads or the dividend yield, e.g. Campbell and Shiller (1988), Fama and French (1989), Ferson and Harvey (1991), Goetzmann and Jorion (1993). Most of the literature on optimal asset allocation has focused on predictability from the dividend yield, see Barberis (2000) and Kandel and Stambaugh (1996). Standard linear predictors fail to explain much of the variation in the monthly returns of size- and book-to-market sorted equity portfolios. However, the dividend yield is the predictor variable that generates the strongest variations in hedging demands. The possibility that the dividend yield might predict returns on the SMB and HML portfolios has not been considered in the context of regime switching models.

To investigate the effect on our model of adding predictor variables such as the dividend yield, again we used a battery of tests to determine the best model specification for $[r_t^{MKT}\ r_t^{SMB}\ r_t^{HML}\ dy_t]'$, where dy_t is the dividend yield in period t. Reflecting the strong persistence in the yield, the SIC suggests a VAR(1) model irrespective of the number of states, k. Even with a first order autoregressive term included, a four-state model continues to be selected.

The economic interpretation of the four regimes is aided by studying the smoothed state probabilities presented in Figure 2 and the parameter estimates reported in Panel B of Table 2. For comparison Panel A reports estimates for a single-state, VAR(1) benchmark model. The basic interpretation of the regimes remains unchanged from the simpler model reported in Table 1. The expected returns which allow for the possibility of regime switches between t and $t+1$, evaluated at the mean of the dividend yield within each state, $E[\mathbf{y}_{t+1}|s_t = i, dy_t = \overline{dy}_{s_t}]$, are as follows:

$$E[\mathbf{y}_{t+1}|s_t = 1] = [0.0002\ 0.0106\ 0.0041\ 0.0217]' \quad \text{(regime 1)}$$
$$E[\mathbf{y}_{t+1}|s_t = 2] = [0.0068\ \text{-}0.0038\ 0.0038\ 0.0210]' \quad \text{(regime 2)}$$
$$E[\mathbf{y}_{t+1}|s_t = 3] = [0.0047\ 0.0060\ \text{-}0.0004\ 0.0467]' \quad \text{(regime 3)}$$
$$E[\mathbf{y}_{t+1}|s_t = 4] = [0.0275\ 0.0346\ \text{-}0.0447\ 0.0448]' \quad \text{(regime 4)}$$

Regime 1 is a transient state with an average duration less than two months that mostly picks up bear markets such as the Great Depression, the two oil shocks in the 1970s and the more recent period 2000-2002. The main difference when compared to the bear state in the simpler model in Table 1 is that this state now has a shorter expected duration and records a relatively high, positive mean return on the SMB portfolio.

Regimes 2 and 3 continue to be persistent, low volatility states with average durations exceeding 8-10 months. Taken together, these states capture most bull markets between the 1940s and 1990s. State 2 has a low dividend yield (on average 2.1%) while state 3 has a high yield (on average 4.7%). While state 2 tracks periods with large value but small size anomalies, state 3 captures periods where only the size anomaly is present. Three of four of the coefficients of the lagged dividend yield on the SMB and HML returns are significant in these two states.

Finally, regime 4 remains an outlier state with large positive mean returns on the market and SMB portfolio although it now has negative returns on the HML portfolio. In this state the mean excess return on the market is 33% per annum while growth stocks outperform value stocks to the tune of 54% per annum and small firms outperform large firms by 42% per annum. Volatility is also high, ranging from 26% to 47% per annum for the three portfolios.

Equity return correlations continue to vary significantly across states. The correlation between the market and the SMB portfolio varies from 0.12 to 0.49, while the correlation between the market and HML portfolio varies from -0.36 to 0.69. Correlations between shocks to the dividend yield and shocks to stock returns are large and negative for the market portfolio but considerably smaller for the HML and SMB portfolios. Finally, indicating time-variations in the hedging properties of the Fama-French portfolios, Table 2 shows significant time-variations in the ability of the dividend yield to predict future stock returns. For instance, higher dividend yields forecast higher market risk premia in states 2 and 3, but negative ones in state 1 (the relationship is weak in state 4). In the case of SMB (HML), higher dividend yields forecast higher returns in states 1 and 2 (state 3 for HML), and lower returns in states 3 and 4.

Once again we considered a more parsimonious model. In particular, we estimated the following model which lets the predictive power of the dividend yield be state dependent but rules out predictability from lagged returns,

$$\begin{aligned} r_t^j &= \mu_{s_t}^j + \alpha_{s_t}^j dy_{t-1} + \varepsilon_t^j \quad j = \text{MKT, SMB, HML} \\ dy_t &= \mu_{dy,s_t} + \alpha_{dy,s_t} dy_{t-1} + \varepsilon_{dy,t}. \end{aligned} \tag{7}$$

We continue to let the covariance matrix be unrestricted, i.e. $\boldsymbol{\varepsilon}_t^* \sim N(0, \boldsymbol{\Sigma}_{s_t}^*)$, where $\boldsymbol{\varepsilon}_t^* \equiv [\varepsilon_t^{MKT}\ \varepsilon_t^{SMB}\ \varepsilon_t^{HML}\ \varepsilon_{dy,t}]'$ and assume four states. This model has 84 parameters, a reduction of 48 parameters relative to the unrestricted version of (3). Again, a test of the 48 restrictions on the state-dependent VAR matrices was strongly rejected.

4. **The Asset Allocation Problem**

So far we have documented the presence of regimes in the process underlying returns on the market portfolio and portfolios tracking size and value effects. We next explore the asset allocation implications of such regimes. Since it is clear that regime shifts generate predictability in future investment opportunities, we expect to find interesting horizon effects and hedging demands. Under the CAPM, investors should not hold the size or value portfolios. To see if this continues to be valid here, we consider the asset allocation problem of an investor with power utility over terminal wealth, W_{t+T}, coefficient of relative risk aversion, γ, and time horizon, T:

$$u(W_{t+T}) = \frac{W_{t+T}^{1-\gamma}}{1-\gamma}. \tag{8}$$

The investor is assumed to maximize expected utility by choosing at time t a portfolio allocation to the market, SMB and HML portfolios, $\boldsymbol{\omega}_t \equiv [\omega_t^{MKT}\ \omega_t^{SMB}\ \omega_t^{HML}]'$, while any residual wealth is invested in riskless, one-month T-bills. For simplicity, we assume the investor has unit initial wealth and ignores intermediate consumption. Portfolio weights are adjusted every $\varphi = \frac{T}{B}$ months at B equally spaced points t, $t+\frac{T}{B}$, $t+2\frac{T}{B}$, ..., $t+(B-1)\frac{T}{B}$. When $B=1$, $\varphi = T$, so the investor simply implements a buy-and-hold strategy.

Let $\boldsymbol{\omega}_b$ $(b = 0, 1, ..., B-1)$ be the weights on the stock portfolios at the rebalancing points. The investor's optimization problem is:[9]

$$\max_{\{\boldsymbol{\omega}_j\}_{j=0}^{B-1}} \quad E_t\left[\frac{W_B^{1-\gamma}}{1-\gamma}\right] \tag{9}$$

$$s.t.\ W_{b+1} = W_b\{(1-\omega_b^{MKT})\exp\left(\varphi r^f\right) + \omega_b^{MKT}\exp\left(R_{b+1}^{MKT} + \varphi r^f\right) + \omega_b^{SMB}\exp(R_{b+1}^{SMB}) + \omega_b^{HML}\exp(R_{b+1}^{HML})\}$$

Here $E_t[\cdot]$ denotes the conditional expectation given the information set at time t, $\mathcal{F}_t$, and R_b denotes cumulative returns over a period of φ months. The term $R_{b+1}^{MKT}+\varphi r^f$ arises since we specified our model for the vector of (continuously compounded) excess returns on the market portfolio while $(1-\omega_b^{MKT})\exp\left(\varphi r^f\right)$ arises since both SMB and HML are zero-investment portfolios that require short-selling stocks and thus depositing funds in margin accounts. If a proportion ω_b is invested in one of these portfolios, ω_b must also be invested at the riskless rate to satisfy the deposit requirement, for a total gross return of $\omega_b\exp(R_{b+1})+|\omega_b|\exp(\varphi r^f)$. Thus, as written in (9)

$$\begin{aligned} W_{b+1} &= W_b\{(1-\omega_b^{MKT}-|\omega_b^{SMB}|-|\omega_b^{HML}|)\exp\left(\varphi r^f\right) + \omega_b^{MKT}\exp\left(R_{b+1}^{MKT}+\varphi r^f\right) + \\ &\quad +\omega_b^{SMB}\exp(R_{b+1}^{SMB}) + |\omega_b^{SMB}|\exp(\varphi r^f) + \omega_b^{HML}\exp(R_{b+1}^{HML}) + |\omega_b^{HML}|\exp(\varphi r^f)\} \\ &= W_b\{(1-\omega_b^{MKT})\exp\left(\varphi r^f\right) + \omega_b^{MKT}\exp\left(R_{b+1}^{MKT}+\varphi r^f\right) + \omega_b^{SMB}\exp(R_{b+1}^{SMB}) + \omega_b^{HML}\exp(R_{b+1}^{HML})\}. \end{aligned}$$

In what follows we report the total weight on T-bills reflecting both the asset allocation decisions and margin requirements.[10]

Incorporating the predictor variables, $\mathbf{z}_b$, at the decision points, b, the derived utility of wealth is

$$J(W_b, \mathbf{y}_b, \boldsymbol{\theta}_b, \boldsymbol{\pi}_b, t_b) \equiv \max_{\{\boldsymbol{\omega}_j\}_{j=b}^{B-1}} E_{t_b}\left[\frac{W_B^{1-\gamma}}{1-\gamma}\right]. \tag{10}$$

Here $\mathbf{y}_b \equiv (\mathbf{r}_b\ \mathbf{z}_b)'$, $\boldsymbol{\theta}_b = \left(\left\{\boldsymbol{\mu}_{i,b}, \{\mathbf{A}_{j,i,b}\}_{j=1}^{p}, \boldsymbol{\Sigma}_{i,b}\right\}_{i=1}^{k}, \mathbf{P}_b\right)$ collects the parameters of the regime switching model, and $\boldsymbol{\pi}_b$ is the state probabilities at point b. Investors face a large set of state variables, most obviously the regime probabilities, $\boldsymbol{\pi}_b$, and the vector of returns and predictor variables, $\mathbf{y}_b$. The parameter vector $\boldsymbol{\theta}_b$ could also be treated as a separate state variable that gets updated at each point in time. Solving the associated problem implies using a very large set of state variables. We therefore solve a simplified version of the asset allocation program in which the model parameters are fixed at their estimated values $\boldsymbol{\theta}_b = \hat{\boldsymbol{\theta}}$ for all $b = 0, 1, ..., B-1$.[11] Treating states as unobserved is consistent with the estimation problem solved by the investor in Section 2 where the regime can only be inferred from the available data. Investors' learning process is incorporated in this setup by letting them optimally update their beliefs about the underlying state at each point in time using Bayes' rule

$$\boldsymbol{\pi}_{t+j+1}(\hat{\boldsymbol{\theta}}_{t+j}) = \frac{\boldsymbol{\pi}_{t+j}(\hat{\boldsymbol{\theta}}_{t+j})\hat{P}_{t+j} \odot \boldsymbol{\eta}(\mathbf{y}_{t+j+1}; \hat{\boldsymbol{\theta}}_{t+j})}{(\boldsymbol{\pi}_{t+j}(\hat{\boldsymbol{\theta}}_{t+j})\hat{P}_{t+j} \odot \boldsymbol{\eta}(\mathbf{y}_{t+j+1}; \hat{\boldsymbol{\theta}}_{t+j}))\boldsymbol{\iota}_k}. \tag{11}$$

[9] As is common in the empirical literature on optimal asset allocation, we assume that the risk-free rate is constant over time and also do not address market equilibrium issues so our investor is small relative to the total market. We will remove the assumption of a constant short-term rate in Section 5.4.

[10] For example, a position of -25% in SMB, and 15% in HML requires an investor to hold 40% in T-bills. Since after putting (say) 65% in the market, only 35% of the initial wealth is available, the investor will have to borrow 5% of his wealth at the T-bill rate. Therefore the *net* investment in T-bills is only 35%, i.e., $1-\omega_b^{MKT}$, consistent with (9).

[11] Barberis (2000) considers a simple example with future updating limited to two parameter estimates.

Here $\odot$ denotes the element-by-element product, $\mathbf{y}_t \equiv [r'_t \ z'_t]'$, and $\eta(y_{t+j+1})$ is a $k \times 1$ vector that gives the density of observation y_{t+j+1} in the k states at time $t+j+1$ conditional on $\hat{\theta}_{t+j}$:[12]

$$\boldsymbol{\eta}(\mathbf{y}_{t+j+1}; \hat{\boldsymbol{\theta}}_{t+j}) \equiv \begin{bmatrix} f(\mathbf{y}_{t+j+1}|s_{t+j+1} = 1, \{\mathbf{y}_{t+j-i}\}_{i=0}^{p-1}; \hat{\boldsymbol{\theta}}_{t+j}) \\ f(\mathbf{y}_{t+j+1}|s_{t+j+1} = 2, \{\mathbf{y}_{t+j-i}\}_{i=0}^{p-1}; \hat{\boldsymbol{\theta}}_{t+j}) \\ \vdots \\ f(\mathbf{y}_{t+j+1}|s_{t+j+1} = k, \{\mathbf{y}_{t+j-i}\}_{i=0}^{p-1}; \hat{\boldsymbol{\theta}}_{t+j}) \end{bmatrix}$$

$$= \begin{bmatrix} (2\pi)^{-\frac{N}{2}} |\hat{\Sigma}_1^{-1}|^{\frac{1}{2}} \exp\left[-\frac{1}{2} \left(\mathbf{r}_{t+j} - \hat{\boldsymbol{\mu}}_1 - \sum_{i=0}^{p-1} \hat{\mathbf{A}}_{1j} \mathbf{y}_{t+j-i} \right)' \hat{\Sigma}_1^{-1} \left(\mathbf{r}_{t+j} - \hat{\boldsymbol{\mu}}_1 - \sum_{i=0}^{p-1} \hat{\mathbf{A}}_{1j} \mathbf{y}_{t+j-i} \right) \right] \\ (2\pi)^{-\frac{N}{2}} |\hat{\Sigma}_2^{-1}|^{\frac{1}{2}} \exp\left[-\frac{1}{2} \left(\mathbf{r}_{t+j} - \hat{\boldsymbol{\mu}}_2 - \sum_{i=0}^{p-1} \hat{\mathbf{A}}_{2j} \mathbf{y}_{t+j-i} \right)' \hat{\Sigma}_2^{-1} \left(\mathbf{r}_{t+j} - \hat{\boldsymbol{\mu}}_2 - \sum_{i=0}^{p-1} \hat{\mathbf{A}}_{2j} \mathbf{y}_{t+j-i} \right) \right] \\ \vdots \\ (2\pi)^{-\frac{N}{2}} |\hat{\Sigma}_k^{-1}|^{\frac{1}{2}} \exp\left[-\frac{1}{2} \left(\mathbf{r}_{t+j} - \hat{\boldsymbol{\mu}}_k - \sum_{i=0}^{p-1} \hat{\mathbf{A}}_{kj} \mathbf{y}_{t+j-i} \right)' \hat{\Sigma}_k^{-1} \left(\mathbf{r}_{t+j} - \hat{\boldsymbol{\mu}}_k - \sum_{i=0}^{p-1} \hat{\mathbf{A}}_{kj} \mathbf{y}_{t+j-i} \right) \right] \end{bmatrix}. \quad (12)$$

Learning effects are important since portfolio choices depend not only on future values of asset returns and predictor variables, but also on future perceptions of the probability of being in each of the regimes. Using that W_b is known at time t_b, the scaled value function, $Q(\cdot)$, simplifies to

$$Q(\mathbf{r}_b, \mathbf{z}_b, \boldsymbol{\pi}_b, t_b) = \max_{\boldsymbol{\omega}_b} E_{t_b} \left[\left(\frac{W_{b+1}}{W_b} \right)^{1-\gamma} Q\left(\mathbf{r}_{b+1}, \mathbf{z}_{b+1}, \boldsymbol{\pi}_{b+1}, t_{b+1} \right) \right]. \quad (13)$$

Conditional on the current parameter estimates, $\hat{\boldsymbol{\theta}}_t$, the optimal portfolio weights reflect not only hedging demands due to stochastic shifts in investment opportunities but also changes in investors' beliefs concerning future state probabilities, $\boldsymbol{\pi}_{t+j}$. In the absence of predictor variables, $\mathbf{z}_t$, the investor's perception of the regime probabilities, $\boldsymbol{\pi}_t$, is the only state variable and the basic recursions simplify to

$$\begin{aligned} Q(\boldsymbol{\pi}_b, t_b) &= \max_{\boldsymbol{\omega}_b} E_{t_b} \left[\left(\frac{W_{b+1}}{W_b} \right)^{1-\gamma} Q\left(\boldsymbol{\pi}_{b+1}, t_{b+1} \right) \right], \\ \boldsymbol{\pi}_b(\hat{\boldsymbol{\theta}}_t) &= \frac{\boldsymbol{\pi}_{t_b-1}(\hat{\boldsymbol{\theta}}_t) \hat{\mathbf{P}}_t^{\varphi b} \odot \boldsymbol{\eta}(\mathbf{r}_b; \hat{\boldsymbol{\theta}}_t)}{\left(\boldsymbol{\pi}_{t_b-1}(\hat{\boldsymbol{\theta}}_t) \hat{\mathbf{P}}_t^{\varphi b} \odot \boldsymbol{\eta}(\mathbf{r}_b; \hat{\boldsymbol{\theta}}_t) \right) \boldsymbol{\iota}_k}, \end{aligned} \quad (14)$$

where $\hat{\mathbf{P}}_t^{\varphi b} \equiv \prod_{i=1}^{\varphi b} \hat{\mathbf{P}}_t$. Backward solution of (14) only requires knowledge of $\boldsymbol{\pi}_b(\hat{\boldsymbol{\theta}}_t)$, $b = 0, 1, ..., B-1$, although we allow the perceived state probabilities to be updated along each simulated path.

4.1. *Numerical Solution*

A variety of solution methods have been applied in the literature on portfolio allocation under time-varying investment opportunities. Barberis (2000) employs simulation methods and studies a pure allocation problem without interim consumption. Campbell and Viceira (1999) derive approximate analytical solutions for an infinitely lived investor when interim consumption is allowed and rebalancing is continuous. Campbell et al. (2003) extend this approach to a multivariate set-up and show that a mixture of approximations and numerical methods can be applied. Finally, some papers have derived closed-form solutions by working in continuous-time, e.g. Kim and Omberg (1996).

[12] This formula is derived in Hamilton (1994, pp. 692-693).

Ang and Bekaert (2002a) propose a Markov switching model for pairs of international stock market returns. They consider asset allocation when regimes are observable to investors, so the state variable simplifies to a set of dummy indicators. Our framework is quite different since we calculate asset allocations under optimal filtering, allowing for unobservable states. In our model investors therefore have to account for revisions in future beliefs π_{t_b+j} $(j \geq 1)$ when determining optimal asset allocations. This means that quadrature methods cannot be applied to our problem.

To solve for the portfolio weights under regime switching we use Monte-Carlo methods for integral (expected utility) approximation. For example, for a buy-and-hold investor, we approximate the integral in the expected utility functional as follows:

$$\max_{\boldsymbol{\omega}_t(T)} N^{-1} \sum_{n=1}^{N} \left\{ \frac{1}{1-\gamma} \left[(1-\omega_t^{MKT}) \exp\left(Tr^f\right) + \omega_t^{MKT} \exp\left(R_{T,n}^{MKT} + Tr^f\right) \right. \right.$$
$$\left. \left. + \omega_t^{SMB} \exp(R_{T,n}^{SMB}) + \omega_t^{HML} \exp(R_{T,n}^{HML}) \right]^{1-\gamma} \right\}.$$

Here $R_{T,n}^{j}$ (j = MKT, SMB, HML) are the cumulative returns in the n-th Monte Carlo simulation. Each simulated path of portfolio returns is generated using draws from the model (1)-(3) which allows regimes to shift randomly as governed by the transition matrix, $\hat{\mathbf{P}}$. We use $N = 30,000$ simulations and vary the investment horizon, T, between 1 and 120 months in increments of 6 months.[13] The optimal weights $\hat{\omega}_t(T)$ are determined over a three-dimensional grid, $\omega_t^i(T) = -5$, -4.99, -4.98, ..., 4.99, 5.00 for i = MKT, SMB, and HML. Fortunately, such extreme portfolio choices never appeared in our empirical results.

Since our solution does not rule out short-sales, it is possible that wealth can become negative.[14] To rule out such cases, we impose a no-bankruptcy constraint by rejecting all simulated sample paths that lead to negative wealth. Effectively our portfolio choice problem is solved by appropriately truncating the tails of the joint distribution obtained in Section 3 although such rejections account for a very small percentage of our simulation runs. As a result the general features of the joint process implied by our estimates in Section 3 and the approximate density that is compatible with finite expected utility are very similar.[15] An Appendix provides further details on the numerical techniques.

5. **Empirical Asset Allocation Results**

5.1. *Buy-and-Hold Investor*

We first consider the asset allocation strategy of a buy-and-hold investor. Consistent with choices in the literature the coefficient of relative risk aversion is set at $\gamma = 5$. The levels of the risky asset holdings clearly depend on γ although a more extensive analysis revealed robustness of our qualitative results within a broad range of values for γ.

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[13] A large number of simulations is needed to account for the occurrence of regimes with low steady-state probabilities. We varied N between 5,000 and 100,000 (in steps of 5,000) and found that random variation in the optimal portfolio weights due to sampling error in the Monte Carlo approximations becomes negligible for $N = 30,000$.

[14] This occurs when $R_b^p(\boldsymbol{\omega}_{b-1}) \leq 0$, so the marginal utility of wealth $[R_b^p(\boldsymbol{\omega}_{b-1})]^{-\gamma}$ is either not defined (if $R_b^p(\boldsymbol{\omega}_{b-1}) = 0$) or becomes negative.

[15] Using a 120-month horizon we simulated the first four moments of equity returns under two alternative settings: (i) using the original set of 30,000 random paths draws (before applying rejections), and (ii) using the 30,000 random paths after replacements due to rejections. The resulting moments are virtually indistinguishable to the fourth digit after the decimal point.

In the following, we provide intuition for the asset allocation results along two distinct dimensions. First, the presence of regimes may give an investor short-term market timing incentives since the filtered state probabilities contain information about the joint predictive density of future asset returns. Optimal portfolio weights should therefore depend on the characteristics of the underlying regimes including the conditional moments (means, variances, covariances as well as higher order moments) of asset returns within and across the four regimes. As the horizon grows, portfolio decisions increasingly reflect properties of the unconditional distribution of returns and decreasingly depend on the initial state probabilities.

Figure 3 plots the optimal portfolio weights as a function of the investment horizon. In these plots we assume that the investor knows the initial state (i.e. π_t equals one of the "unit" vectors e_1, e_2, e_3, e_4, i.e. vectors that contain a one in the j-th position and zeros elsewhere), but not the identity or sequence of any future states. Asset allocations vary significantly across regimes in the four-state model, particularly at short horizons where market timing effects are strong. Regime 1 is dominated by the negative average return on the market portfolio and by the positive mean returns on the SMB and HML portfolios. Starting from this state, the short-run allocation to the market portfolio is therefore small though it rises in T. While the weights on the SMB and HML portfolios initially rise, they decline as a function of the horizon, T, for $T \geq 6$ months.

Turning to regime 2, due to its high expected return, the market portfolio features prominently in the optimal asset allocation with a weight above 100% at short horizons. Regime 3 produces similar portfolio choices although the allocation to the market portfolio is far smaller than in regime 2, reflecting its lower mean return. An investor should also hold a long (short) position in high (low) book-to-market firms in this state. This is explained by the hedge that the HML portfolio provides with respect to the market portfolio. Finally, in the short-lived fourth regime the equity portfolios offer high mean returns and are generally held in long positions at short or medium horizons. Recalling the definition of SMB and HML, this means that short-term investors hold long positions in small value firms. The holdings in the equity portfolios are financed by some short-term borrowing in T-bills.[16]

At the 10-year horizon, almost 65% is held in the market, 15% in the HML portfolio, -25% in the SMB portfolio and 35% in T-bills. These long-run asset allocation results are broadly consistent with those reported by Pástor (2000) for a single-period exercise under a tight prior tilted towards the CAPM. Our finding that the allocation to the HML portfolio is positive in three of four states and only negative in the fourth state for very short horizons is also consistent with Pastor's results.

Our long-run allocations are also quite similar to those in Brennan and Xia based on a 50-50 mixed prior over the CAPM and the empirical distribution of asset returns which gives rise to weights on the HML, SMB and market portfolios of 14%, -3% and 35%, respectively. Hence, similar long-run allocations can be achieved either by putting a large prior on the CAPM or by adopting a model such as ours that accounts for fat tails - and thus higher risk - in the returns on the size and value portfolios.

5.2. *Uncertainty about the States*

Figure 4 reports results for the case where the investor is highly uncertain about the identity of both the initial and future states.[17] We capture this uncertainty by setting the initial state probabilities equal to their

[16] Consistent with findings reported by Ang and Bekaert (2002a), the portfolio weights tend to converge to their long-run levels at horizons of 2-3 years.

[17] Cases where none of the filtered state probabilities exceeds 0.9 occur in 19.3% of the sample.

steady state values, $\boldsymbol{\pi}_t = [0.21\ 0.25\ 0.53\ 0.01]'$.

The plots illustrate two points. First, even when the investor does not know the initial state, $\boldsymbol{\pi}_t$, we continue to find interesting shapes for the investment schedules that map the optimal portfolio weights against the investment horizon. The schedules are clearly not as steep as those in Figure 3 where the initial state was assumed to be known. Moreover, they appear to be dominated by the shapes previously observed when starting either from state 2 or 3. For instance, the optimal investment in the market and HML portfolios both decline slowly as T grows, while the weight on the SMB portfolio exhibits a positive slope. This is a reflection of the high steady-state probabilities of these states. As one would expect from the statistical definition of ergodic distribution, the long-horizon, 10-year weights are virtually identical to those reported before, i.e. 66% in the market portfolio, -26% in SMB, +13% in HML.

5.3. *Predictability from the Dividend Yield*

Both the dividend yield and the latent state variable defining regimes are able to capture interesting time-variations in the investment opportunities but do so at different frequencies. Hence it is important to investigate portfolio allocations when the dividend yield is allowed to forecast returns within a regime switching model, as in (3). To this end, Figure 5 shows the optimal asset allocation for a buy-and-hold investor when predictability from the dividend yield is incorporated in the regime switching model. For simplicity, the dividend yield and the lagged value of returns used in the VAR computations are set at their unconditional means within each state. The results are qualitatively quite similar to those shown in Figure 3. For example, the optimal allocation to the market portfolio is increasing when starting from the bear state (state 1) and decreasing from the other states. The slope of the investment demand for the SMB and HML portfolios also varies significantly across states. At short horizons the optimal allocations to the size and value portfolios remain highly sensitive to the current state probability, but quickly converge to their long-run levels as T grows. Comparing Figures 3 and 5, holdings in the SMB and HML portfolios become more extreme once the yield is included as a predictor variable.

The most notable difference with respect to the earlier results in Figure 3 is the large positive holdings in the HML portfolio and the negative holdings in the market and SMB portfolios in the bear state (regime 1) at the shortest horizons. The reason for this change is the large negative mean return on the SMB portfolio and the large positive mean return on the HML portfolio in this state. When combined with the fact that the bear state is highly transient in the extended model, this explains why the equity positions now become more extreme at the shortest horizons and why these positions quickly revert to the steady-state weights as the horizon is expanded and a regime shift is anticipated.

Finally we computed portfolio weights starting from the steady state probabilities, while the value of the dividend yield varies between plus or minus two standard deviations from its sample mean (3.8%). Figure 6 shows the results. The demand curves for the market portfolio are generally upward sloping, the only exception occurring for rather extreme values of the dividend yield in excess of 7%. The schedules progressively move up (e.g., for $T = 6$ months the optimal weights increase from 10% for almost zero dividend yield, to 50% when the dividend yield equals the historical average, to 73% for high values) as the initial dividend yield increases, which – given its remarkable persistence – forecasts future high dividend yields. Similarly, the HML schedules are upward sloping in T and not affected to a great extent by changes in the initial dividend yield. In contrast, the SMB schedules are either non-monotonic at high values of the dividend yield or simply monotonically decreasing for at or below-average values of the dividend yield.

5.4. *Stochastic Short-term Interest rate*

A number of papers have found evidence of regime switching dynamics in short-term US interest rates (e.g., Gray (1996), Ang and Bekaert (2002b), Bansal, Tauchen and Zhou (2004) and Guidolin and Timmermann (2007)). Furthermore, some studies have found that short-term rates are useful predictors of stock returns (e.g., Keim and Stambaugh, (1986)). While we previously followed the majorities of studies on predictability and portfolio choice (but see Detemple, Garcia, and Rindisbacher, 2003, for a different approach) and assumed that the short-term rate is constant and therefore riskless, we next check the robustness of our earlier conclusions, using a version of (3) in which the short-term interest rate is subject to regime switching and also forecasts equity returns:

$$\mathbf{y}_t = \begin{pmatrix} \boldsymbol{\mu}_{s_t} \\ \boldsymbol{\mu}_{rs_t} \end{pmatrix} + \mathbf{A}^*_{s_t}\mathbf{y}_{t-j} + \begin{pmatrix} \boldsymbol{\varepsilon}_t \\ \boldsymbol{\varepsilon}_{rt} \end{pmatrix}, \tag{15}$$

where now $\mathbf{y}_t \equiv [r_t^{MKT}\ r_t^{SMB}\ r_t^{HML}\ r_t^f]'$, $\boldsymbol{\varepsilon}^*_t \equiv [\varepsilon_t^{MKT}\ \varepsilon_t^{SMB}\ \varepsilon_t^{HML}\ \varepsilon_{r,t}^f]' \sim N(0, \boldsymbol{\Sigma}^*_{s_t})$, so the short T-bill rate has been added to the state vector.

To save space we do not report the parameter estimates and the smoothed state probabilities for this model.[18] However, as one would expect, these are somewhat different from the earlier values defined over a different vector of state variables. Instead we concentrate on the portfolio weights as a function of the investment horizon T for a buy-and-hold investor. These are shown in Figure 7, where we have initialized equity returns and the interest rate at their regime-specific unconditional means. The results show once again that although the presence of regimes generates market timing opportunities for short-term investors (especially when the investor has precise information about the current state), for horizons $T \geq 60$ months, the investment schedules quickly converge to their steady state values, which imply allocations of 30% to the market portfolio, no weight on the SMB portfolio, 42% to the HML portfolio, and the remainder to T-bills. Introducing regime switching in interest rates thus implies a lower demand for the market portfolio, a higher demand for the HML portfolio and a large position in T-bills. Portfolio allocations initialized in a situation of uncertainty tend to display shapes which "average out" the regime-specific schedules.

5.5. *Rebalancing and Hedging Demands*

So far we have studied the optimal asset allocation for a buy-and-hold investor. Investors may, however, have access to rebalancing opportunities. Table 3 shows the effects on optimal holdings of rebalancing every 1, 3, 6 or 12 months. If frequent rebalancing is possible, the investor's horizon matters far less than under the buy-and-hold scenario. Effectively, only the period between the current time (t) and the next rebalancing point ($t + \varphi$) induces curvature in the investment demand.[19] The investor also responds more aggressively to the current state. The reason is simple: an investor who can rebalance frequently will utilize information about the current state by taking large short positions when the return distribution indicates poor prospective returns and large long positions in states with more attractive returns. If the perceived state probabilities change next period, the investor can simply adjust the portfolio weights. Such adjustment opportunities are

[18] These are available from the authors upon request.

[19] For $\varphi \geq T$ the optimal portfolio weights are identical to the buy-and-hold values and thus omitted.

not available to the buy-and-hold investor who must consider the probability of future states during the entire holding period.

The rebalancing frequency can clearly have a large effect on asset holdings, most notably when the rebalancing frequency is varied from $\varphi = 3$ to $\varphi = 1$ in state four (e.g., the weight on the market portfolio increases from 12 to 229 percent while the weight on the HML portfolio declines from 56 to -13 percent). The fourth state only has a 'stayer' probability of one-third and exits to the 'bear' state with a two-thirds probability. Under monthly rebalancing, an investor will increase holdings in the market portfolio (compared to scenarios with higher values of φ) largely by lowering investments in the HML portfolio whose returns have a lower mean and are strongly correlated with market returns in the fourth state. Conversely, moving to $\varphi = 3$ and starting from the fourth state, a switch to the bear state will almost certainly occur prior to the next rebalancing point. Since the bear state has low mean returns on the market and SMB portfolios but high mean returns on the HML portfolio, the weights on the former assets are reduced while the weight on the HML portfolio increases substantially compared to the case with $\varphi = 1$.

We continue to observe large variations across states in the portfolio weights under rebalancing. For instance, when $\varphi = 1$, the long-run allocation to the market goes from a negative value in state 1 (-31%) to values in excess of 100% in the bull states (from 115% to 228%). Starting from the first (bear) state, as rebalancing happens more frequently the allocation to the market portfolio declines and becomes negative. Conversely, in states two and three the demand for the market portfolio rises as φ is lowered while the non-monotonicities found for state four are explained by the high probability of going from state four to the low return state (state 1). State two (four) is associated with very large negative (positive) holdings in the SMB portfolio. The SMB weight increases with the rebalancing frequency in regimes one and four while the opposite happens in regimes two and three. Less variation across states is generally observed in the holdings of the HML portfolio.

These results allow us to measure the optimal hedging demand defined as the difference $\hat{\omega}_t^i(T) - \hat{\omega}_t^i(1)$ ($i =$ MKT, SMB, HML) for $T \geq 2$ and $\varphi = 1$ month, i.e. when rebalancing occurs at the same frequency as the data is observed (see Ingersoll (1987, p. 245)). Intertemporal hedging demands arise from an investor's desire to protect portfolio performance from adverse shocks, when there is time variation in investment opportunities and when the asset menu includes assets whose returns are correlated with changes in investment opportunities. Results are reported in separate rows in Table 3. Hedging demands for the market and SMB portfolios are substantially larger than hedging demands for the HML portfolio. The sign of the hedging demand for the market portfolio has an intuitive interpretation. Starting from the bear state, future changes in regimes will improve investment opportunities and raise future expected returns on the market portfolio so the hedging demand is negative and quite large (-32 percent); similarly, hedging demands are slightly negative in regime three. Conversely, shifts away from the high mean return states (two and four) imply a worsening of the investment opportunities, so hedging demands for the market portfolio are positive when starting from these states because the risk-return trade-off in the bear regime is better for the market portfolio than for the SMB and HML portfolios. Because the risk premium on the market portfolio tends to positively correlate with future regime shifts in a switching model, the result under steady-state probabilities is that hedging demand is negative and of magnitude (between -14% and -16%) comparable to typical results in the linear predictability literature: when regime shifts are taken into account the market portfolio gets riskier as the horizon grows.

To compare hedging demands under multiple regimes with those derived under a VAR benchmark, Table

3 also reports buy-and-hold allocations and hedging demands under linear predictability. For simplicity, calculations are performed when all the variables in $\mathbf{y}$ (i.e. portfolio returns and the dividend yield) are set at their unconditional means. For the market portfolio, the hedging demand is positive but moderate (15%). This is consistent with findings in Barberis (2000) and Campbell and Viceira (1999). The reason for the positive hedging demand is the negative covariance between shocks to the dividend yield and stock market returns which leads investors with a long horizon to hold more in stocks.

In the case of the SMB and HML portfolios, it is interesting to note the contrast between the rather sizeable (46% and -40%, respectively) hedging demand under the VAR(1) model and the more modest ones under regime switching. Though small, systematic patterns remain in these hedging demands which are positive in state 1 and negative in state 4. The signs of these hedging demands are sensible: Both the SMB and HML portfolios have small but positive expected returns in the bear state (regime 1) and thus provide some hedging against negative shocks to market-wide returns. Starting from steady-state probabilities, the hedging demand is small but positive (11%) for the SMB portfolio and essentially zero (-3%) for the HML portfolio.

To further explore what induces the hedging demands in the regime switching model, we considered three restricted models, namely (i) a four-state model whose expected returns are constrained to be identical across all states; (ii) a four-state model whose covariance matrices are constrained to be identical across all states; and (iii) a two-state model. The hedging demands in the first model, at -0.5%, 3.5% and -0.5% for the market, SMB and HML portfolios are quite low, showing that the greater variation across regimes captured by the four-state model is important for hedging demands. Hedging demands were also very small (always below 2%) in the four-state model whose mean returns are restricted, while they remained large in the four-state model with constrained covariance matrices, although this model is clearly misspecified. Large variations in expected returns across states are thus key to the hedging demands for the three stock portfolios.[20]

The difference between the hedging demands generated in the presence of regimes and under a VAR(1) model can be interpreted in the context of the differences between single- and multi-state models. A linear VAR model employing the highly persistent dividend yield as a predictor explains the dynamics in returns by exploiting the correlation between equity returns and the slowly changing yields. As such, the time-variation in expected returns is predicted to be highly persistent and implies sizeable hedging demands. Conversely, a regime switching model exploits regimes to capture time variations in expected returns, and so the resulting patterns of predictability in expected stock returns may be less persistent because they are controlled by the evolution in the latent state variable that – although persistent – can quickly switch values. This explains the weaker hedging demand figures that we find.

5.6. *Summary of Findings*

To compare asset allocations under a broader set of models and to isolate the effect of regime switching, Figure 8 shows optimal portfolio weights as a function of the investment horizon under three alternative specifications, namely regime switching without the dividend yield (MS), regime switching with the dividend

[20] We also computed hedging demands under the regime switching model with stochastic interest rates. Starting from steady-state probabilities, at the ten-year horizons they are 9.8% for the market portfolio, 3.5% for the SMB portfolio, and 5.3% for the HML portfolio.

yield included (MS-VAR(1)) and a VAR(1) model:

$$\mathbf{y}_t = \begin{pmatrix} \boldsymbol{\mu} \\ \boldsymbol{\mu}_{dy} \end{pmatrix} + \mathbf{A}^* \mathbf{y}_{t-1} + \begin{pmatrix} \varepsilon_t \\ \varepsilon_{dyt} \end{pmatrix}, \quad (\varepsilon_t' \ \varepsilon_{dyt})' \sim N(0, \boldsymbol{\Sigma}^*). \tag{16}$$

Estimates of this model can be found in panel A of Table 3.

Under the VAR(1) model the allocation to the market portfolio rises as a function of the investment horizon. For this model we also find that the allocations to the HML and SMB portfolios grow as a function of the investment horizon.

The portfolio weights under the single state model (16) are quite different from those obtained under the four-state model irrespective of whether this includes the dividend yield. Most notably, the four regimes introduce short-run market timing effects while the single-state model is driven by slower, long-run movements in the dividend yield. Asset demand curves are therefore steeper at horizons shorter than six months under the four-state model.

The large positive demands for the SMB portfolio and T-bills and the zero or negative demand for the market and HML portfolios at short horizons under the MS-VAR(1) model are explained by the large negative mean returns of the market and HML portfolios in the short-lived states 1 and 4 which – due to the high marginal utility in this state – dominates results for this model. Increasing the investment horizon from one to six months leads to an increased demand for the market and HML portfolios and a lower demand for the SMB portfolio under the four state MS-VAR(1) model.

6. **Economic Importance of Regimes**

So far we have shown that regimes can have a large effect on the optimal asset allocation. This continues to hold even when investors do not know the identity of the current state. However, it does not necessarily follow that ignoring regimes leads to a loss in expected utility that is sufficiently large to encourage investors to use the more complicated model that we propose. To assess whether the differences between single- and multi-state portfolio weights is dominated by the larger parameter estimation errors associated with the four-state model, this section investigates the effect of parameter estimation errors on the optimal portfolio weights (Section 6.1). We then undertake utility cost calculations to quantify the economic significance of regimes, first, by computing the reduction in expected utility resulting from ignoring regimes (Section 6.2); second, by evaluating the out-of-sample performance of a variety of model specifications including regime-switching, single-state, and VAR models (Section 6.3).

6.1. *Parameter Estimation Error*

Large standard errors surrounding parameter estimates tend to result in imprecisely determined portfolio weights. Aït-Sahalia and Brandt (2001) refer to this as the "Achille's heel" of models of conditional asset allocation. Although the portfolio weights reported so far are determined by solving a complicated dynamic programming problem, these weights condition on the parameter estimates, $\widehat{\boldsymbol{\theta}}$, and are therefore themselves random variables. We quantify the effect of estimation uncertainty by forming confidence intervals for the optimal portfolio weights as follows. From asymptotic analysis (e.g., Hamilton, 1994)

$$\sqrt{T}\left(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0\right) \overset{A}{\sim} N(\mathbf{0}, V_\theta), \tag{17}$$

where $\boldsymbol{\theta}_0$ denotes the true but unknown vector of parameters. We utilize this result in the following algorithm:

1. For a particular trial, j, draw a vector $\widehat{\widehat{\boldsymbol{\theta}}}^j$ from the distribution $N(\widehat{\boldsymbol{\theta}}, T^{-1}\hat{V}_\theta)$, where $\hat{V}_\theta$ is the estimated covariance matrix of $\widehat{\boldsymbol{\theta}}$.

2. Conditional on $\widehat{\widehat{\boldsymbol{\theta}}}^j$, solve (9) to obtain a new vector of portfolio weights $\widehat{\widehat{\boldsymbol{\omega}}}^j$.

3. Repeat steps 1-2 a large number of times, $j = 1, 2, ..., J$.

4. Form $(1 - \alpha)$ percent confidence intervals for the optimal asset allocation $\hat{\boldsymbol{\omega}}_t$ from the simulated distribution for $\widehat{\widehat{\boldsymbol{\omega}}}^j$, $j = 1, 2, ..., J$. For example, the fifth quantile, $\widehat{\widehat{\boldsymbol{\omega}}}_{0.05}$, and the 95th quantile, $\widehat{\widehat{\boldsymbol{\omega}}}_{0.95}$, form the lower and upper bounds of a 90% confidence interval for the optimal weights.

Table 4 presents 90% confidence intervals for the portfolio weights based on $J = 1,000$ simulations. We consider scenarios starting from each of the four states and the steady-state probabilities and study investment horizons of 1, 6, 60 and 120 months. For comparison, we also report confidence bands under the assumption of an IID process for returns.[21]

At the short horizon there is less uncertainty about the market weight under regime shifts than under the IID model, while the uncertainty about the weights on the SMB and HML portfolios is comparable under the two models. The degree of uncertainty about $\boldsymbol{\omega}_t$ varies significantly across states, however, with the fourth regime associated with the greatest uncertainty. This reflects the short duration of this state and the fact that a small change in the transition probabilities changes the likelihood of a transition to the low-return bear state (state 1).

Wide confidence intervals at short horizons are unsurprising: Aït-Sahalia and Brandt (2001) also report large standard errors for portfolio weights, especially when investment in cash is allowed as in our paper. Furthermore, as pointed out by Campbell, Chan and Viceira (2003) the parameters governing the dynamics of asset returns can have large effects on the optimal asset holdings so that any uncertainty about their values tends to have a large effect on portfolio weights.

At the longest horizons the confidence bands for the portfolio weights derived under the IID model continue to be very wide, while they narrow distinctly under the four-state model. Under the latter model the typical width of the 90% confidence intervals at the longest horizon is 0.30 for the market, 0.40 for the SMB portfolio and 0.50 for the HML portfolio compared to widths of 1.64, 0.92 and 1.12, respectively, under the IID model.

Despite this uncertainty, ignoring regimes would clearly lead to a suboptimal portfolio allocation: most of the four-state intervals for the weights on the market and SMB portfolios do not overlap with the confidence intervals obtained from the IID model. Ignoring regimes would lead an investor to invest too little in the market portfolio and too much in the SMB portfolio. These conclusions remain valid when the intervals are calculated from the steady-state probabilities and are thus not sensitive to the initial state and the fact that regimes are best thought of as unobservable.

6.2. *Utility Cost Calculations*

Disregarding regimes or predictability from the dividend yield is equivalent to constraining investors to choose optimal portfolio weights, $\hat{\boldsymbol{\omega}}_t^{IID}$, under the assumption that asset returns are drawn from a single-state model.

[21] More precisely, we apply a simulation methodology adjusted to the single-state case, employing $\widehat{\boldsymbol{\theta}}$ from panel A in Table 1.

To quantify the costs of this constraint, we compute the increase in initial wealth η_t^{IID} − or compensatory variation − an investor requires to derive the same level of expected utility from the IID and unconstrained asset allocation problems:

$$(1+\eta_t^{IID})^{1-\gamma}\left\{\sum_{b=0}^{B} E_t\left[(W_b)^{1-\gamma}\right]\right\} = Q(\mathbf{y}_b, S_b),$$

where $Q(\mathbf{y}_b, S_b)$ is the scaled value function defined in equation (13). Solving for η_t^{IID},

$$\tilde{\eta}_t^{IID} = \left\{\frac{Q(\mathbf{y}_b, S_b)}{\sum_{b=0}^{B} E_t\left[(W_b)^{1-\gamma}\right]}\right\}^{\frac{1}{1-\gamma}} - 1. \tag{18}$$

To avoid overstating the economic importance of regimes, once again we assume that the investor does not know the identity of the states whose probabilities are set at their steady state values. The compensatory variation − plotted in Figure 9 as an annualized percentage rate − ranges from about eleven percent at the one-month horizon to about two percent at the ten-year horizon. Figure 9 also shows 90% confidence intervals obtained by simulation. Although the confidence bands are quite wide there is no question that regimes in the return process for the market, size and value portfolios are economically important. The lower band goes from 10 percent at the 1-month to a minimum of about 25 basis points at the ten-year horizon. The upper band suggests much higher values.

Our estimates of the utility costs of ignoring regimes are higher than those reported by Ang and Bekaert (2002a) for a study of international equity portfolios. This is easy to explain due to our finding of larger and more significant mean return effects and the coincidence of the low mean return state with the high volatility state (state 1). Reducing the allocation to equity portfolios during this state will be highly beneficial to the investor, particularly if a risk-free asset is present as we assume here. Furthermore, although relatively high, our estimate of the annualized utility loss is well within the range of values reported in the literature. For instance, Brennan and Xia (2001) report a certainty equivalence value of investing in the HML and SMB portfolio that exceeds 8% per annum even in the presence of parameter estimation uncertainty. Our estimates suggest that the utility costs arising from ignoring time-variations in the joint distribution of portfolio returns due to regime switching is roughly of a similar magnitude.

Figure 10 provides a comparison of the utility costs of ignoring predictability across different models, including a VAR(1) model and a model that includes regimes and predictability from the dividend yield. Linear predictability from the dividend yield gives rise to a 10-year compensatory variation of 140 basis points per annum. The corresponding figure exceeds 200 basis points for an investor who accounts for predictability induced by regime switching but disregards predictability from the dividend yield. Regime shifts thus appear to have a slightly larger effect on utility costs than predictability from the dividend yield.

Interestingly, including both types of predictability appears to have a compounding effect, indicating that regime switching mostly identifies short- or medium-term predictability while variations in the dividend yield identify longer-term predictability. Taken together, a 10-year investor would require a compensatory return of 6% per annum to ignore both the evidence of regimes and predictability from the dividend yield. Though relatively high, this estimate of the annualized utility loss is within the range of values recently reported in the literature and well below those reported by Campbell and Viceira (1999) and Lynch (2001).

We also performed utility cost calculations under monthly rebalancing. Consistent with the existence of substantial market timing opportunities, at short horizons the utility loss from ignoring regimes was found to be very large (e.g., 25, 10, 6, and 112 percent per year in regimes 1 through 4 for $T = 6$ months). For a 10-year long-horizon investor these losses remain quite considerable, 7, 8, 6, and 11 percent if starting from each of the four states and 6 percent when starting from the steady-state probabilities.

6.3. *Out-of-sample performance*

Although our models suggest sizeable utility losses from ignoring regime shifts, they may be difficult to use in 'real time' due to parameter estimation errors which could translate into implausible time-variations in the portfolio weights. This concern is related to the prediction model's out-of-sample asset allocation performance, see Brennan, Schwartz, and Lagnado (1997), Campbell, Chan, and Viceira (2003) and Detemple, Garcia, and Rindisbacher (2003).

To address this point, we perform a "real time" asset allocation experiment for the period 1980:01-2005:12, a total of 312 months. To make the experiment computationally feasible, we focus on the buy-and-hold portfolio problem at horizons $T = 1$, 12, and 120 months. We compare the performance of the four-state regime switching model, the VAR(1) model, a four-state regime switching model that includes predictability from the dividend yield, a two-state regime-switching model and a simple IID model. The investor is precluded from having any benefit of hindsight. For instance, to predict the return distribution for 1980:01, the parameter estimates are based only on information up to 1979:12. These estimates are then updated recursively as the point of the forecast progresses through time.

To measure investment performance we consider realized portfolio returns as well as realized utility under the different models, each of which is associated with a particular portfolio weight $\hat{\boldsymbol{\omega}}_t^T$ and hence a different realized utility:

$$V(\hat{\boldsymbol{\omega}}_t^T) \equiv \frac{\left[W_T(\hat{\boldsymbol{\omega}}_t^T)\right]^{1-\gamma}}{1-\gamma} = \frac{1}{1-\gamma}\left[(1-\hat{\omega}_t^{MKT})\exp\left(Tr^f\right) + \hat{\omega}_t^{MKT}\exp\left(R_T^{MKT} + Tr^f\right) + \right. \\ \left. +\hat{\omega}_t^{SMB}\exp(R_T^{SMB}) + \hat{\omega}_t^{HML}\exp(R_T^{HML})\right]^{1-\gamma}. \quad (19)$$

Here R_T^{MKT}, R_T^{SMB}, and R_T^{HML} are realized (cumulated) returns between $t+1$ and $t+T$. The period-t weights, $\hat{\boldsymbol{\omega}}_t^T$, are computed by maximizing the objective $E_t[W_T^{1-\gamma}/1-\gamma]$ so that for each investment horizon, T, and each portfolio selection model we obtain time series $\{W_T(\hat{\boldsymbol{\omega}}_\tau^T), V(\hat{\boldsymbol{\omega}}_\tau^T)\}$, $\tau =$1980:01,...,2005:12-T of realized wealth levels and utilities. Figure 11 shows the sequence of portfolio weights for an investment horizon of $T = 12$ months. The weights seem quite sensible with some short-term variability due to parameter estimation error and in some cases also long, persistent swings reflecting changes in the investment opportunity set.

Table 5 reports summary statistics for the distribution of net returns $\{W_T(\hat{\boldsymbol{\omega}}_t^T)-1\}$ (Panel A) and 'realized utility' $\{V^T(\hat{\boldsymbol{\omega}}_t^T)\}$ (Panel B) with smaller absolute values indicating higher utility. Following Guidolin and Timmermann (2007), we use a block bootstrap (with 50,000 simulation trials) for the empirical distribution of the objects of interest to account for the fact that realized utility levels are likely to be serially dependent since time-variations in the conditional distribution of asset returns may translate into dependencies in the portfolio weights and hence in realized utilities.

First consider the results for the return distribution. The four-state models generate high mean returns ranging from 15% to 31% per year. However, the returns produced by the regime switching models also tend to be volatile, especially when the dividend yield is part of the model. At short horizons, single-state models are clearly dominated and there is even evidence that their 10% bootstrapped confidence bands fail to overlap with those of the best regime switching model. Consistent with the need to work with a four-state model, mean returns decline when we move to a two-state model.

The realized power utility results reported in Panel B offer a better way to compare the different models since the portfolio weights have been chosen ex ante to maximize expected utility. Once again the results support the simple four-state regime switching specification which dominates the other models in terms of average out-of-sample utility. At 11.0, 10.3 and 14.5 percent per annum for investment horizons of one, 12, and 120 months, this model also generates the highest certainty equivalent returns. Based on certainty equivalence returns, the single-state models continue to perform relatively poorly.

7. Conclusion

This paper documented the presence of four regimes in the joint distribution of equity returns on market, size and value portfolios. A single-state model appears to be misspecified as means, correlations and volatilities of returns on these portfolios vary significantly across states. This finding is perhaps not so surprising given the very different episodes and market conditions – such as the Great Depression, World War II and the oil shocks of the 1970s – that occurred during the sample (1927-2005). It is difficult to imagine that the same single-state model is able to capture episodes of such diversity.

We quantified the economic value of investing in the three equity portfolios under regime switching by considering the optimal asset allocations of an investor with power utility. Economically large variations were found in the optimal portfolio weights as a function of the economic state and the investment horizon. Rebalancing opportunities make the investor respond more aggressively to the current state probabilities since portfolio weights can be adjusted rapidly should the state probabilities change. This option is not open to a buy-and-hold investor. The loss in expected utility from ignoring regimes turns out to be substantial across a range of regime switching models. Overall, our estimates suggest that it is important to account for regimes when analyzing investments in returns on the market, size and value portfolios. Furthermore, regimes and the dividend yield appear to identify quite different predictable components in stock returns. Finally, our out-of-sample recursive analysis suggests that models that account for the presence of regimes lead to higher average realized utility even after accounting for parameter estimation error.

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Appendix - Solution of the Dynamic Asset Allocation Problem by Monte Carlo Methods

A.1. No Predictor Variables

Suppose the optimization problem has been solved backwards at the rebalancing points t_{B-1}, ..., t_{b+1} so that $Q(\boldsymbol{\pi}^i_{b+1}, t_{b+1})$ is known for all values $i = 1, 2, ..., G^{k-1}$ on the discretization grid. At each point, $\boldsymbol{\pi}_b = \boldsymbol{\pi}^i_b$, it is possible to find $Q(\boldsymbol{\pi}^i_b, t_b)$ at time t_b. Monte Carlo approximation of the expectation

$$E_{t_b}[\{(1-\omega_b^{MKT})\exp\left(\varphi r^f\right) + \omega_b^{MKT}\exp\left(R^{MKT}_{b+1,n} + \varphi r^f\right) + \omega_b^{SMB}\exp(R^{SMB}_{b+1,n}) + \\ + \omega_b^{HML}\exp(R^{HML}_{b+1,n})\}^{1-\gamma} Q(\boldsymbol{\pi}^i_{b+1}, t_{b+1})]$$

requires drawing N random samples of asset returns $\{\mathbf{R}_{b+1,n}(\boldsymbol{\pi}^i_b)\}_{n=1}^N$ from the $(b+1)\varphi$-step joint density conditional on the period-t parameter estimates, $\widehat{\boldsymbol{\theta}}_t = \left(\{\hat{\boldsymbol{\mu}}_s, \hat{\boldsymbol{\Sigma}}_s\}_{s=1}^k, \hat{\mathbf{P}}\right)$ assuming that, at each point, $\boldsymbol{\pi}^i_b$ is updated to $\boldsymbol{\pi}_{b+1}(\boldsymbol{\pi}^i_b)$. The algorithm consists of the following steps:

1. For each possible value of the current regime, S_b, simulate N $\varphi-$period returns $\{\mathbf{R}_{b+1,n}(S_b)\}_{n=1}^N$ in calendar time from the regime switching model

$$\mathbf{r}_{t_b+j,n}(S_b) = \boldsymbol{\mu}_{s_{t_b+j}} + \boldsymbol{\varepsilon}_{t_b+j,n},$$

   where $\mathbf{R}_{b+1,n}(S_b) \equiv \sum_{j=1}^{\varphi} \mathbf{r}_{t_b+j,n}(S_b)$ and $\boldsymbol{\varepsilon}_{t_b+j,n} \sim N(0, \boldsymbol{\Sigma}_{s_{t_b+j}})$.[22] At all rebalancing points this simulation allows for regime switching as governed by the transition matrix $\hat{\mathbf{P}}_t$. For example, starting in state 1, the probability of switching to state 2 between t_b and t_b+1 is $\hat{p}_{12} \equiv \mathbf{e}_1'\hat{\mathbf{P}}_t\mathbf{e}_2$, while the probability of remaining in state 1 is $\hat{p}_{11} \equiv \mathbf{e}_1'\hat{\mathbf{P}}_t\mathbf{e}_1$. At each point in time $\hat{\mathbf{P}}_t$ governs possible state transitions.

2. Combine the simulated $\varphi-$period returns $\{\mathbf{R}_{b+1,n}\}_{n=1}^N$ into a random sample of size N, using the probability weights contained in the row vector $\boldsymbol{\pi}^j_b$

$$\mathbf{R}_{b+1,n}(\boldsymbol{\pi}^i_b) = \sum_{j=1}^{k}(\boldsymbol{\pi}^i_b\mathbf{e}_j)\mathbf{R}_{b+1,n}(S_b = j).$$

[22] The notation $\mathbf{R}_{b+1,n}(S_b)$ does not imply that future asset returns are directly a function of the current state S_b. In fact, the parameters $\boldsymbol{\mu}_{s_{t_b+j}}$ and $\boldsymbol{\Sigma}_{s_{t_b+j}}$ are a function of future states, S_{t_b+j}, $j = 1, 2, ..., \varphi$. However, the expression $\mathbf{R}_{b+1,n}(S_b)$ indicates that the transition probabilities to future states are a function of the current state.

3. Update the future regime probabilities perceived by the investor using the formula

$$\boldsymbol{\pi}_{b+1,n}(\boldsymbol{\pi}_b^i) = \frac{(\boldsymbol{\pi}_b^i)'\hat{\mathbf{P}}^{\varphi} \odot \boldsymbol{\eta}(\mathbf{R}_{b+1,n}(\boldsymbol{\pi}_b^i); \hat{\boldsymbol{\theta}}_t)}{\left((\boldsymbol{\pi}_b^i)'\hat{\mathbf{P}}^{\varphi} \odot \boldsymbol{\eta}(\mathbf{R}_{b+1,n}(\boldsymbol{\pi}_b^i); \hat{\boldsymbol{\theta}}_t)\right) \boldsymbol{\iota}_k}.$$

This gives an $N \times k$ matrix $\{\boldsymbol{\pi}_{b+1,n}(\boldsymbol{\pi}_b^i)\}_{n=1}^N$, whose rows correspond to simulated vectors of perceived regime probabilities at time t_{b+1}.

4. For all $n = 1, 2, ..., N$, calculate the value $\tilde{\boldsymbol{\pi}}_{b+1,n}^i$ on the discretization grid ($i = 1,\ 2, ...,\ G^{k-1}$) closest to $\boldsymbol{\pi}_{b+1,n}(\boldsymbol{\pi}_b^i)$ using the distance measure $\sum_{j=1}^{k-1} |\boldsymbol{\pi}_{b+1}^i \mathbf{e}_j - \boldsymbol{\pi}_{b+1,n}\mathbf{e}_j|$, i.e.

$$\tilde{\boldsymbol{\pi}}_{b+1,n}^i(\boldsymbol{\pi}_b^i) \equiv \underset{\mathbf{x}\in\times_{j=1}^{k-1}[0,1]}{\arg\min} \sum_{j=1}^{k-1} |\mathbf{x}\mathbf{e}_j - \boldsymbol{\pi}_{b+1,n}\mathbf{e}_j|.$$

Knowledge of the vector $\{\tilde{\boldsymbol{\pi}}_{b+1,n}^i(\boldsymbol{\pi}_b^i)\}_{n=1}^N$ allows us to build $\{Q(\boldsymbol{\pi}_{b+1}^{(i,n)}, t_{b+1})\}_{n=1}^N$, where $\boldsymbol{\pi}_{b+1}^{(i,n)} \equiv \tilde{\boldsymbol{\pi}}_{b+1,n}^i(\boldsymbol{\pi}_b^i)$ is a function of the assumed, initial vector of regime probabilities $\boldsymbol{\pi}_b^i$.[23]

5. Solve the program

$$\max_{\boldsymbol{\omega}_b(\boldsymbol{\pi}_b^i)} N^{-1} \sum_{n=1}^{N} \Big\{ \Big[(1-\omega_b^{MKT}) \exp\left(\varphi r^f\right) + \omega_b^{MKT} \exp\left(R_{b+1,n}^{MKT} + \varphi r^f\right) + \omega_b^{SMB} \exp(R_{b+1,n}^{SMB}) + \\ + \omega_b^{HML} \exp(R_{b+1,n}^{HML}) \Big]^{1-\gamma} Q(\boldsymbol{\pi}_{b+1}^{(i,n)}, t_{b+1}) \Big\}.$$

For large values of N this provides an arbitrarily precise Monte-Carlo approximation to $E[\{(1-\omega_b^{MKT}) \exp\left(\varphi r^f\right) + \omega_b^{MKT} \exp\left(R_{b+1,n}^{MKT} + \varphi r^f\right) + \omega_b^{SMB} \exp(R_{b+1,n}^{SMB}) + \omega_b^{HML} \exp(R_{b+1,n}^{HML})\}^{1-\gamma} Q(\boldsymbol{\pi}_{b+1}^i, t_{b+1})]$. The value function evaluated at the optimal portfolio weights $\hat{\boldsymbol{\omega}}_b(\boldsymbol{\pi}_b^i)$ gives $Q(\boldsymbol{\pi}_b^i, t_b)$ for the i-th point on the initial grid. We also check if $(1-\omega_b^{MKT}) \exp\left(\varphi r^f\right) + \omega_b^{MKT} \exp\left(R_{b+1,n}^{MKT} + \varphi r^f\right) + \omega_b^{SMB} \exp(R_{b+1,n}^{SMB})$ $+\omega_b^{HML} \exp(R_{b+1,n}^{HML})$ is negative and reject the corresponding sample paths.

The algorithm is applied to all possible values $\boldsymbol{\pi}_b^i$ on the discretization grid until all values of $Q(\boldsymbol{\pi}_b^i, t_b)$ are obtained for $i = 1,\ 2, ...,\ G^{k-1}$. It is then iterated backwards until $t_{b+1} = t + \varphi$. At that stage the algorithm is applied one last time, taking $Q(\boldsymbol{\pi}_{t+1}^i, t+\varphi)$ as given and using the actual row vector of smoothed regime probabilities, $\boldsymbol{\pi}_t$. The resulting vector $\hat{\boldsymbol{\omega}}_t$ gives the desired optimal portfolio allocation at time t, while $Q(\boldsymbol{\pi}_t, t)$ is the optimal value function.

Under the buy-and-hold strategy, step 1 is replaced with a simulation routine that for each possible current regime, S_t, simulates N asset returns of length T, $\{\mathbf{R}_{T,n}(S_t)\}_{n=1}^N$ from the model

$$\mathbf{r}_{t+j,n}(S_t) = \boldsymbol{\mu}_{s_{t+j}} + \boldsymbol{\varepsilon}_{t+j,n}, \quad \boldsymbol{\varepsilon}_{t+j,n} \sim N(\mathbf{0}, \boldsymbol{\Sigma}_{s_{t+j}}),$$

where $\mathbf{R}_{T,s}(S_t) \equiv \sum_{j=1}^T \mathbf{r}_{t+j,n}(S_t)$. State transitions can again occur in accordance with the probability matrix $\hat{\mathbf{P}}$. Hence we draw a matrix of monthly returns $\left\{\{\mathbf{r}_{t+j,n}(S_t)\}_{n=1}^N\right\}_{i=1}^T$ and sum these into N long-term

[23] This step may be avoided when $Q(\boldsymbol{\pi}_{b+1}^i, t_{b+1})$ is constant for all values on the discretization grid. This happens when $t_{b+1} = T$ and implies that the portfolio weights determined at step $b+1$ $\{\hat{\boldsymbol{\omega}}_{b+1}(\boldsymbol{\pi}_{b+1}^i)\}$ are invariant to changes in $\boldsymbol{\pi}_{b+1}^i$.

asset returns $\{\mathbf{R}_{T,n}(\boldsymbol{\pi}_t)\}_{n=1}^N$ using $\mathbf{R}_{T,n}(\boldsymbol{\pi}_t) = \sum_{j=1}^k (\boldsymbol{\pi}_t^i \mathbf{e}_j)\mathbf{R}_{T,n}(S_t = j)$. Steps 3-5 are irrelevant under buy-and-hold since the objective simplifies to:

$$\max_{\boldsymbol{\omega}_t(T)} N^{-1} \sum_{n=1}^N \left\{ \frac{1}{1-\gamma} \left[(1-\omega_t^{MKT}) \exp\left(Tr^f\right) + \omega_t^{MKT} \exp\left(R_{T,n}^{MKT} + Tr^f\right) + \right.\right.$$
$$\left.\left. + \omega_t^{SMB} \exp(R_{T,n}^{SMB}) + \omega_t^{HML} \exp(R_{T,n}^{HML}) \right]^{1-\gamma} \right\}.$$

(after rejecting simulation paths that lead to zero or negative wealth). This makes computations much faster under the buy-and-hold scheme.

A.2. Predictor Variables

Only minor modifications are required to extend our approach to allow for linear predictability $-$ $\mathbf{y}_t = \boldsymbol{\mu}_{s_t}^* + \mathbf{A}_{s_t}^* \mathbf{y}_{t-1} + \boldsymbol{\varepsilon}_t^*$, $\mathbf{y}_t \equiv [\mathbf{r}_t \ \mathbf{z}_t]'$. Suppose the optimization problem has been solved backwards at the rebalancing points t_{B-1}, ..., t_{b+1} so that $Q(\boldsymbol{\pi}_{b+1}^i, \mathbf{y}_{b+1}^i, t_{b+1})$ is known for all values $i = 1, 2, ..., G_z^m \times G_\pi^{k-1}$ on the discretization grid. Notice that in this case Q is also a function of the state variables (lagged portfolio returns and predictor variables) entering the model so the total number of grid points must be adjusted by multiplying the original number of grid points by the additional points used to span the values of the prediction variables, G_z^m. At each point $[\boldsymbol{\pi}_b^i \ \mathbf{y}_b^i]$, it is then possible to evaluate $Q(\boldsymbol{\pi}_b^i, \mathbf{y}_{b+1}^i, t_b)$ at time t_b. Monte Carlo approximation of the expectation

$$E_{t_b}[\{(1-\omega_b^{MKT}) \exp\left(\varphi r^f\right) + \omega_b^{MKT} \exp\left(R_{b+1,n}^{MKT} + \varphi r^f\right) + \omega_b^{SMB} \exp(R_{b+1,n}^{SMB}) +$$
$$+ \omega_b^{HML} \exp(R_{b+1,n}^{HML})\}^{1-\gamma} Q(\pi_{b+1}^i, \mathbf{y}_{b+1}^i, t_{b+1})]$$

now requires drawing N random samples of the state variables $\{\mathbf{y}_{b+1,n}(\boldsymbol{\pi}_b^i)\}_{n=1}^N$ from the $(b+1)\varphi$-step-ahead joint density conditional on period-t parameter estimates, $\widehat{\boldsymbol{\theta}}_t = \left(\{\hat{\boldsymbol{\mu}}_s, \hat{\mathbf{A}}_s, \hat{\boldsymbol{\Sigma}}_s\}_{s=1}^k, \hat{\mathbf{P}}\right)$ assuming that, at each point, $\boldsymbol{\pi}_b^i$ is optimally updated to $\boldsymbol{\pi}_{b+1}(\boldsymbol{\pi}_b^i, \mathbf{y}_b^i)$. The algorithm consists of steps very similar to those described earlier. The main differences are that in step 1, N $\varphi-$period returns $\{\mathbf{R}_{b+1,n}(S_b, \mathbf{y}_b^i)\}_{n=1}^N$ need to be simulated from the regime switching model

$$\mathbf{y}_{t_b+j,n}(S_b) = \boldsymbol{\mu}_{s_{t_b+j}}^* + \mathbf{A}_{s_{t_b+j}}^* \mathbf{y}_{t_b+j-1,n} + \boldsymbol{\varepsilon}_{t_b+j,n}^*, \quad \boldsymbol{\varepsilon}_{t_b+j,n}^* \sim N(0, \hat{\boldsymbol{\Sigma}}_{s_{t_b+j}}^*).$$

Updates in the perceived state probabilities now make use of the formula

$$\boldsymbol{\pi}_{b+1,n}(\boldsymbol{\pi}_b^i, \mathbf{y}_b^i) = \frac{(\boldsymbol{\pi}_b^i)'\hat{\mathbf{P}}^\varphi \odot \boldsymbol{\eta}(\mathbf{y}_{b+1,n}(\boldsymbol{\pi}_b^i, \mathbf{y}_b^i); \hat{\boldsymbol{\theta}}_t)}{\left((\boldsymbol{\pi}_b^i)'\hat{\mathbf{P}}^\varphi \odot \boldsymbol{\eta}(\mathbf{y}_{b+1,n}(\boldsymbol{\pi}_b^i, \mathbf{y}_b^i); \hat{\boldsymbol{\theta}}_t)\right) \boldsymbol{\iota}_k}.$$

This incorporates the realized values of the prediction variables $\mathbf{z}_{b+1,n}$. Step 4 proceeds similarly with the only difference that the measure of 'closeness' to a grid point now refers to the entire vector $[\boldsymbol{\pi}_{b+1,n}'(\boldsymbol{\pi}_b^i, \mathbf{y}_b^i)$ $\mathbf{y}_{b+1,n}(\boldsymbol{\pi}_b^i, \mathbf{y}_b^i)]'$.

Table 1

Parameter Estimates of Regime Switching Model for Market, SMB and HML Returns

This table reports parameter estimates for the multivariate regime switching model:

$$\mathbf{r}_t = \boldsymbol{\mu}_{s_t} + \boldsymbol{\varepsilon}_t$$

where $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state S_t and $\boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \Sigma_{s_t})$ is the vector of unpredictable return innovations. The unobserved state variable S_t is governed by a first-order Markov chain that can assume *k=4* values. The three series are net returns on Fama and French's (1993) SMB and HML portfolios and excess returns on the value-weighted market portfolio. The sample period is 1927:12 – 2005:12. Panel A (*k* = 1) represents the single-state benchmark, while panel B refers to the four-state model. Values reported on the diagonals of the correlation matrices are annualized volatilities. All other estimates are monthly. Standard errors are shown in parentheses for mean coefficients and transition probabilities.

	Panel A – Single State Model			
	Market Portfolio	SMB Portfolio	HML Portfolio	
1. Mean excess return	0.0062 (0.0018)	0.0024 (0.0011)	0.0042 (0.0012)	
2. Correlations/Volatilities				
Market Portfolio	0.1899***			
SMB Portfolio	0.2028*	0.1167***		
HML Portfolio	0.3300**	0.0851	0.1244***	
	Panel B – Four State Model			
	Market Portfolio	SMB Portfolio	HML Portfolio	
1. Mean excess return				
Regime 1	-0.0108 (0.0063)	-0.0009 (0.0033)	0.0018 (0.0039)	
Regime 2	0.0105 (0.0023)	-0.0025 (0.0012)	0.0024 (0.0012)	
Regime 3	0.0075 (0.0022)	0.0048 (0.0016)	0.0034 (0.0014)	
Regime 4	0.1735 (0.0505)	0.0982 (0.0495)	0.1204 (0.0521)	
2. Correlations/Volatilities				
Regime 1:				
Market Portfolio	0.2793***			
SMB Portfolio	0.3013**	0.1420***		
HML Portfolio	0.1196*	0.0933	0.1770***	
Regime 2:				
Market Portfolio	0.1104***			
SMB Portfolio	-0.0986	0.0534***		
HML Portfolio	0.3452***	0.1607**	0.0579***	
Regime 3:				
Market Portfolio	0.1339***			
SMB Portfolio	0.3702***	0.0930***		
HML Portfolio	-0.3249***	-0.1739**	0.0820***	
Regime 4:				
Market Portfolio	0.4655***			
SMB Portfolio	0.1005	0.5170***		
HML Portfolio	0.7741***	-0.0775	0.4847***	
3. Transition probabilities	Regime 1	Regime 2	Regime 3	Regime 4
Regime 1	0.8468 (0.0168)	0.0289 (0.0111)	0.0612 (0.0133)	0.0531
Regime 2	0.0370 (0.0119)	0.9384 (0.0423)	0.0246 (0.0114)	0.0000
Regime 3	0.0362 (0.0124)	0.0234 (0.0084)	0.9404 (0.0047)	0.0000
Regime 4	0.6452 (0.1659)	0.0002 (0.0239)	0.0031 (0.0269)	0.3515

* significance at 10% level, ** significance at 5%, *** significance at 1%.

Table 2

Estimates of Regime Switching Model for Stock Returns and the Dividend Yield

This table shows parameter estimates for the regime switching VAR(1) model

$$y_t = \mu_{s_t} + A_{s_t} y_{t-1} + \varepsilon_t$$

where $\mathbf{y}_t$ is a 4×1 vector collecting the market, SMB and HML portfolio returns in the first three positions and the dividend yield in the fourth. $\boldsymbol{\mu}_{s_t}$ is the intercept vector in state s_t, A_{js_t} is the matrix of autoregressive coefficients in state s_t and $\varepsilon_t \sim N(\mathbf{0}, \Sigma_{s_t})$. The unobservable state S_t is governed by a first-order Markov chain that can assume one of four distinct values. The sample period is 1927:12 – 2005:12. Panel A refers to the single-state case while panel B covers the four-state model. Values reported on the diagonals of the correlation matrices are annualized volatilities. All other estimates are monthly. Standard errors are shown in parentheses.

	Panel A – VAR(1) (single state) Model			
	Market	SMB	HML	Dividend Yield
1. Intercept term	-0.0043 (0.0045)	-0.0029 (0.0027)	-0.0022 (0.0029)	0.0008(0.0003)
2. VAR(1) Matrix				
Market Portfolio	0.1067 (0.0351)	-0.0267 (0.0558)	0.1075(0.0506)	0.2463(0.1075)
SMB Portfolio	0.1649 (0.0210)	-0.0272 (0.0333)	0.0709 (0.0302)	0.1049 (0.0641)
HML Portfolio	0.0299 (0.0227)	-0.0938 (0.0361)	0.1831 (0.0327)	0.1484 (0.0695)
Dividend Yield	-0.0059 (0.0020)	0.0036 (0.0032)	-0.0110(0.0029)	0.9812 (0.0062)
3. Correlations/Volatilities				
Market Portfolio	0.1877***			
SMB Portfolio	0.3076***	0.1120**		
HML Portfolio	0.1853**	0.0573	0.1214**	
Dividend Yield	-0.8494***	-0.1984***	-0.3190**	0.0108***
	Panel B – Four State Model			
	Market	SMB	HML	Dividend Yield
1. Intercept term				
Regime 1	-0.0342 (0.0026)	0.0002 (0.0019)	0.0403 (0.0035)	0.0005 (0.0001)
Regime 2	-0.0003 (0.0005)	-0.0139 (0.0014)	0.0029 (0.0015)	0.0006 (1.8e-05)
Regime 3	-0.0061 (0.0015)	0.0101 (0.0015)	-0.0041 (0.0012)	0.0014 (0.0001)
Regime 4	0.0497 (0.0085)	0.0721 (0.0094)	-0.0657 (0.0084)	0.0014 (0.0004)
2. VAR(1) Matrix				
Regime 1				
Market Portfolio	0.0912 (0.0367)	0.0817 (0.0394)	-0.0079 (0.0105)	-0.1502 (0.0560)
SMB Portfolio	0.0548 (0.0258)	-0.1871 (0.0611)	0.1009 (0.0400)	0.4454 (0.0874)
HML Portfolio	-0.0816 (0.0533)	0.0010 (0.0174)	-0.0121 (0.0352)	-1.1928 (0.836)
Dividend Yield	-0.0049 (0.0014)	-0.0022 (0.0024)	-0.0028 (0.0013)	1.0023 (0.0024)
Regime 2				
Market Portfolio	-0.0056 (0.0123)	-0.0293 (0.0251)	-0.0955 (0.0275)	0.3899 (0.0193)
SMB Portfolio	0.2013 (0.0270)	0.1574 (0.0450)	-0.0223 (0.0504)	0.3795 (0.0847)
HML Portfolio	0.0562 (0.0288)	-0.0043 (0.4637)	0.2595 (0.0420)	-0.0158 (0.0231)
Dividend Yield	0.0005 (0.0006)	0.0010 (0.0006)	0.0031 (0.0013)	0.9760 (0.0007)
Regime 3				
Market Portfolio	0.0596 (0.0205)	-0.2320 (0.0334)	0.0536 (0.0311)	0.3770 (0.0190)
SMB Portfolio	0.1040 (0.0305)	0.1354 (0.0499)	0.0395 (0.0505)	-0.1614 (0.0342)
HML Portfolio	0.0462 (0.0234)	-0.1185 (0.0540)	0.22263 (0.0467)	0.1059 (0.0268)
Dividend Yield	-0.0035 (0.0014)	0.0077 (0.0015)	-0.0026 (0.0018)	0.9885 (0.0008)

* significance at 10% level, ** significance at 5%, *** significance at 1%.

Table 2 (continued)

	Panel B (continued)			
	Market	SMB	HML	Dividend Yield
2. VAR(1) Matrix (cont'd)				
Regime 4				
Market Portfolio	-0.1773 (0.0435)	-0.0191 (0.0652)	0.8122 (0.0550)	-0.0522 (0.1184)
SMB Portfolio	0.0343 (0.0612)	-0.0511 (0.1136)	0.4729 (0.0943)	-0.7399 (0.1003)
HML Portfolio	0.1694 (0.0606)	-0.1910 (0.1073)	0.2225 (0.0880)	0.1625 (0.1057)
Dividend Yield	-0.0089 (0.0037)	0.0061 (0.0068)	-0.0625 (0.0053)	0.9552 (0.0049)
3. Correlations/Volatilities				
Regime 1				
Market Portfolio	0.2440***			
SMB Portfolio	0.4940***	0.1236***		
HML Portfolio	-0.0327	0.1928*	0.1519***	
Dividend Yield	-0.7871***	-0.4058***	-0.1002	0.0126***
Regime 2				
Market Portfolio	0.1301***			
SMB Portfolio	0.2773**	0.0868***		
HML Portfolio	-0.3556***	-0.2648***	0.0786***	
Dividend Yield	-0.9431***	-0.2746**	0.3011***	0.0041***
Regime 3				
Market Portfolio	0.1211***			
SMB Portfolio	0.2522**	0.0673***		
HML Portfolio	0.3016***	0.2018**	0.0762***	
Dividend Yield	-0.9403***	-0.2583***	-0.3414***	0.0060***
Regime 4				
Market Portfolio	0.4562***			
SMB Portfolio	0.1169**	0.2558***		
HML Portfolio	0.6929***	-0.1589**	0.3126***	
Dividend Yield	-0.9260***	-0.1184***	-0.5820***	0.0358***
3. Transition probabilities	Regime 1	Regime 2	Regime 3	Regime 4
Regime 1	0.6503 (0.1640)	0.0471(0.1049)	0.2489 (0.0904)	0.0537
Regime 2	0.0176 (0.0450)	0.9762 (0.0703)	0.0001 (0.0434)	0.0060
Regime 3	0.1268(0.0649)	0.0046 (0.0124)	0.8686 (0.0658)	3.40 e-08
Regime 4	0.2282(0.1749)	8.16 e-05 (0.0394)	0.0412 (0.0859)	0.7306

* significance at 10% level, ** significance at 5%, *** significance at 1%.

Table 3

Optimal Portfolio Weights under Rebalancing

This table reports optimal weights on the market (Panel A), size (Panel B) and value (Panel C) portfolios as a function of the rebalancing frequency φ for an investor with a coefficient of relative risk aversion of 5. Returns are assumed to be generated by a four-state regime switching model. Allocations marked as 'NA' have φ ≤ T and imply portfolio weights identical to the buy-and-hold case. For comparison, portfolio weights under a Gaussian VAR(1) model (where the dividend yield and portfolio returns are set at their unconditional sample mean) are also shown.

Panel A: Market Portfolio

Rebalancing Frequency φ	Investment Horizon T (months)					
	T=1	T=6	T=12	T=24	T=60	T=120
	Gaussian VAR(1) (Linear Predictability)					
φ = T (buy-and-hold)	0.54	0.59	0.61	0.68	0.78	1.15
Hedging demand	NA	0.15	0.15	0.15	0.15	0.16
	Regime 1					
φ = T (buy-and-hold)	0.01	0.29	0.40	0.51	0.61	0.63
φ = 12 months	NA	NA	NA	0.18	0.18	0.18
φ = 6 months	NA	NA	0.06	0.06	0.06	0.06
φ = 3 months	NA	-0.04	-0.02	-0.02	-0.02	-0.02
φ = 1 month	NA	-0.31	-0.31	-0.31	-0.31	-0.31
Hedging demand	NA	-0.32	-0.32	-0.32	-0.32	-0.32
	Regime 2					
φ = T (buy-and-hold)	2.15	1.39	1.16	0.95	0.75	0.64
φ = 12 months	NA	NA	NA	0.80	0.78	0.78
φ = 6 months	NA	NA	1.10	1.06	1.06	1.06
φ = 3 months	NA	1.46	1.40	1.32	1.30	1.30
φ = 1 month	NA	2.21	2.21	2.21	2.20	2.20
Hedging demand	NA	0.06	0.06	0.06	0.05	0.05
	Regime 3					
φ = T (buy-and-hold)	1.18	0.87	0.78	0.73	0.69	0.68
φ = 12 months	NA	NA	NA	0.54	0.54	0.54
φ = 6 months	NA	NA	0.62	0.62	0.62	0.62
φ = 3 months	NA	0.76	0.76	0.76	0.76	0.76
φ = 1 month	NA	1.15	1.15	1.15	1.15	1.15
Hedging demand	NA	-0.03	-0.03	-0.03	-0.03	-0.03
	Regime 4					
φ = T (buy-and-hold)	2.12	0.32	0.37	0.47	0.57	0.62
φ = 12 months	NA	NA	NA	0.22	0.22	0.22
φ = 6 months	NA	NA	0.14	0.14	0.14	0.14
φ = 3 months	NA	0.12	0.12	0.12	0.12	0.12
φ = 1 month	NA	2.29	2.29	2.29	2.29	2.28
Hedging demand	NA	0.17	0.17	0.17	0.17	0.16
	Steady-state probabilities					
φ = T (buy-and-hold)	0.92	0.84	0.79	0.74	0.70	0.68
φ = 12 months	NA	NA	NA	0.22	0.22	0.22
φ = 6 months	NA	NA	0.14	0.14	0.14	0.14
φ = 3 months	NA	0.12	0.12	0.12	0.12	0.12
φ = 1 month	NA	0.78	0.78	0.78	0.77	0.76
Hedging demand	NA	-0.14	-0.14	-0.14	-0.15	-0.16

Panel B – SMB (size) Portfolio

Rebalancing Frequency φ	Investment Horizon T (months)					
	T=1	T=6	T=12	T=24	T=60	T=120
	Gaussian VAR(1) (Linear Predictability)					
φ = T (buy-and-hold)	-0.59	-0.63	-0.58	-0.50	-0.37	-0.29
Hedging demand	NA	0.46	0.46	0.46	0.46	0.46
	Regime 1					
φ = T (buy-and-hold)	-0.10	0.03	-0.03	-0.12	-0.19	-0.22
φ = 12 months	NA	NA	NA	0.02	0.02	0.02
φ = 6 months	NA	NA	0.12	0.12	0.12	0.12
φ = 3 months	NA	0.18	0.18	0.18	0.18	0.18
φ = 1 month	NA	0.14	0.14	0.14	0.14	0.14
Hedging demand	NA	0.24	0.24	0.24	0.24	0.24
	Regime 2					
φ = T (buy-and-hold)	-2.27	-1.01	-0.72	-0.54	-0.35	-0.33
φ = 12 months	NA	NA	NA	-0.32	-0.32	-0.32
φ = 6 months	NA	NA	-0.77	-0.76	-0.76	-0.76
φ = 3 months	NA	-1.44	-1.40	-1.32	-1.30	-1.30
φ = 1 month	NA	-2.15	-2.15	-2.15	-2.15	-2.14
Hedging demand	NA	0.12	0.12	0.12	0.12	0.11
	Regime 3					
φ = T (buy-and-hold)	-0.90	-0.58	-0.49	-0.37	-0.32	-0.30
φ = 12 months	NA	NA	NA	-0.22	-0.22	-0.22
φ = 6 months	NA	NA	-0.40	-0.40	-0.40	-0.40
φ = 3 months	NA	-0.64	-0.64	-0.64	-0.64	-0.64
φ = 1 month	NA	-0.93	-0.93	-0.94	-0.94	-0.94
Hedging demand	NA	-0.03	-0.03	-0.04	-0.04	-0.04
	Regime 4					
φ = T (buy-and-hold)	0.75	0.33	0.19	0.04	-0.08	-0.26
φ = 12 months	NA	NA	NA	0.20	0.20	0.20
φ = 6 months	NA	NA	0.34	0.34	0.34	0.34
φ = 3 months	NA	0.52	0.52	0.52	0.52	0.52
φ = 1 month	NA	0.70	0.70	0.70	0.70	0.70
Hedging demand	NA	-0.05	-0.05	-0.05	-0.05	-0.05
	Steady-state probabilities					
φ = T (buy-and-hold)	-0.59	-0.48	-0.43	-0.32	-0.27	-0.26
φ = 12 months	NA	NA	NA	-0.21	-0.21	-0.21
φ = 6 months	NA	NA	-0.28	-0.28	-0.28	-0.28
φ = 3 months	NA	-0.40	-0.35	-0.33	-0.33	-0.33
φ = 1 month	NA	-0.48	-0.48	-0.48	-0.48	-0.48
Hedging demand	NA	0.11	0.11	0.11	0.11	0.11

Panel C – HML (Book-to-market) Portfolio

Rebalancing Frequency φ	Investment Horizon T (months)					
	T=1	T=6	T=12	T=24	T=60	T=120
	Gaussian VAR(1) (Linear Predictability)					
φ = T (buy-and-hold)	-0.00	0.01	0.04	0.13	0.25	0.37
Hedging demand	NA	-0.40	-0.40	-0.40	-0.40	-0.40
	Regime 1					
φ = T (buy-and-hold)	-0.03	0.17	0.15	0.12	0.07	0.07
φ = 12 months	NA	NA	NA	0.20	0.20	0.20
φ = 6 months	NA	NA	0.26	0.26	0.26	0.26
φ = 3 months	NA	0.26	0.26	0.26	0.26	0.26
φ = 1 month	NA	0.03	0.03	0.03	0.03	0.03
Hedging demand	NA	0.06	0.06	0.06	0.06	0.06
	Regime 2					
φ = T (buy-and-hold)	-0.42	-0.17	-0.02	0.07	0.10	0.12
φ = 12 months	NA	NA	NA	0.00	0.00	0.00
φ = 6 months	NA	NA	-0.01	-0.01	-0.01	-0.01
φ = 3 months	NA	-0.10	-0.08	-0.10	-0.10	-0.10
φ = 1 month	NA	-0.47	-0.47	-0.47	-0.47	-0.47
Hedging demand	NA	-0.05	-0.05	-0.05	-0.05	-0.05
	Regime 3					
φ = T (buy-and-hold)	0.27	0.14	0.13	0.14	0.17	0.18
φ = 12 months	NA	NA	NA	0.06	0.06	0.06
φ = 6 months	NA	NA	0.05	0.05	0.05	0.05
φ = 3 months	NA	0.00	0.00	0.00	0.00	0.00
φ = 1 month	NA	0.26	0.26	0.26	0.26	0.26
Hedging demand	NA	-0.01	-0.01	-0.01	-0.01	-0.01
	Regime 4					
φ = T (buy-and-hold)	-0.09	0.44	0.34	0.27	0.20	0.15
φ = 12 months	NA	NA	NA	0.36	0.36	0.36
φ = 6 months	NA	NA	0.45	0.45	0.45	0.45
φ = 3 months	NA	0.56	0.56	0.56	0.56	0.56
φ = 1 month	NA	-0.13	-0.13	-0.13	-0.13	-0.13
Hedging demand	NA	-0.04	-0.04	-0.04	-0.04	-0.04
	Steady-state probabilities					
φ = T (buy-and-hold)	0.30	0.25	0.21	0.15	0.12	0.13
φ = 12 months	NA	NA	NA	NA	0.14	0.14
φ = 6 months	NA	NA	0.18	0.18	0.18	0.18
φ = 3 months	NA	0.21	0.21	0.21	0.21	0.21
φ = 1 month	NA	0.27	0.27	0.27	0.27	0.27
Hedging demand	NA	-0.03	-0.03	-0.03	-0.03	-0.03

Table 4

Effects of Parameter Estimation Uncertainty

This table reports 90% confidence intervals for a buy-and-hold investor's optimal portfolio weights at different investment horizons, T, assuming a constant relative risk aversion coefficient of 5. Intervals are calculated by simulation. Under regime switching, portfolio returns are assumed to be generated by the model

$$r_t = \mu_{s_t} + \varepsilon_t$$

where $\boldsymbol{\mu}_{s_t}$ are the intercepts in state s_t and $\boldsymbol{\varepsilon}_t \sim N(\mathbf{0}, \Sigma_{s_t})$ is the vector of return innovations.

		Investment Horizon T				Investment Horizon T			
		T=1	T=6	T=60	T=120	T=1	T=6	T=60	T=120
		A: Allocation to the Market Portfolio				B: Allocation to the SMB (Size) Portfolio			
I.I.D.	Upper 90% band	0.68	0.68	0.68	0.68	1.18	1.18	1.18	1.18
	Mean	0.01	0.01	0.01	0.01	0.61	0.61	0.61	0.61
	Lower 90% band	-0.96	-0.96	-0.96	-0.96	0.26	0.26	0.26	0.26
Regime 1	Upper 90% band	0.26	0.48	0.76	0.78	0.44	0.34	0.04	0.01
	Mean	0.01	0.29	0.61	0.63	-0.10	0.03	-0.19	-0.22
	Lower 90% band	-0.25	0.11	0.48	0.50	-0.62	-0.26	-0.43	-0.44
Regime 2	Upper 90% band	2.19	1.78	0.93	0.83	-1.62	-0.25	-0.05	-0.11
	Mean	2.15	1.39	0.75	0.64	-2.27	-1.01	-0.35	-0.33
	Lower 90% band	2.03	1.09	0.56	0.49	-2.63	-1.82	-0.70	-0.57
Regime 3	Upper 90% band	1.58	1.04	0.84	0.83	-0.41	-0.20	-0.06	-0.07
	Mean	1.18	0.87	0.69	0.68	-0.90	-0.58	-0.32	-0.30
	Lower 90% band	0.78	0.70	0.53	0.53	-1.44	-0.96	-0.57	-0.52
Regime 4	Upper 90% band	2.61	0.54	0.73	0.77	1.87	0.63	0.15	-0.04
	Mean	2.12	0.32	0.57	0.62	0.75	0.33	-0.08	-0.26
	Lower 90% band	1.06	0.10	0.42	0.49	-0.07	0.01	-0.29	-0.47
Steady-state probabilities	Upper 90% band	1.12	1.03	0.87	0.84	-0.25	-0.12	0.01	-0.03
	Mean	0.92	0.84	0.70	0.68	-0.59	-0.48	-0.27	-0.26
	Lower 90% band	0.72	0.66	0.53	0.50	-0.97	-0.83	-0.55	-0.50
		C: Allocation to the HML (Book-to-Market) Portfolio				D: Allocation to T-bills			
I.I.D.	Upper 90% band	0.94	0.94	0.94	0.94	1.73	1.73	1.73	1.73
	Mean	0.45	0.45	0.45	0.45	0.99	0.99	0.99	0.99
	Lower 90% band	-0.18	-0.18	-0.18	-0.18	0.39	0.39	0.39	0.39
Regime 1	Upper 90% band	0.35	0.48	0.35	0.33	1.49	1.10	0.73	0.69
	Mean	-0.03	0.17	0.07	0.07	0.99	0.71	0.39	0.37
	Lower 90% band	-0.37	-0.09	-0.19	-0.16	0.46	0.33	0.04	0.03
Regime 2	Upper 90% band	0.30	0.20	0.35	0.36	-0.38	0.36	0.65	0.69
	Mean	-0.42	-0.17	0.10	0.12	-1.15	-0.39	0.25	0.36
	Lower 90% band	-1.31	-0.54	-0.17	-0.11	-1.99	-1.16	-0.08	0.03
Regime 3	Upper 90% band	0.93	0.45	0.44	0.45	0.73	0.69	0.70	0.66
	Mean	0.27	0.14	0.17	0.18	-0.18	0.13	0.31	0.32
	Lower 90% band	-0.30	-0.15	-0.06	-0.04	-1.19	-0.48	-0.07	0.00
Regime 4	Upper 90% band	1.06	0.76	0.47	0.39	-0.08	1.13	0.77	0.71
	Mean	-0.09	0.44	0.20	0.15	-1.12	0.68	0.43	0.38
	Lower 90% band	-0.93	0.14	-0.03	-0.09	-2.29	0.26	0.09	0.06
Steady-state probabilities	Upper 90% band	0.68	0.56	0.41	0.41	0.49	0.68	0.69	0.65
	Mean	0.30	0.25	0.12	0.13	0.08	0.16	0.30	0.32
	Lower 90% band	-0.06	-0.04	-0.13	-0.10	-0.37	-0.35	-0.09	-0.03

Table 5

Real-time Out-of-Sample Performance of Predictability Models

This table reports out-of-sample performance measures for three investment horizons, T = 1, 12, and 120 months. The performance measures are computed under alternative models for the joint process of portfolio returns and the predictor variable (the dividend yield). The realized power utility results assume a coefficient of relative risk aversion, γ = 5 for a buy-and-hold investor.

	Gaussian IID			VAR(1)			Four-state			Four-state VAR(1) w/DY			Two-state	
	T=1	T=12	T=120	T=1	T=12	T=120	T=1	T=12	T=120	T=1	T=12	T=120	T=1	T=12
							Panel A -- Portfolio Returns							
Mean (annualized)	0.056	0.048	0.110	0.102	0.107	0.282	0.181	0.178	0.290	0.310	0.147	0.261	0.120	0.083
10% conf. interval -- lower bound	0.019	-0.001	0.072	0.064	0.068	0.244	0.114	0.126	0.259	0.170	0.088	0.126	0.089	0.071
10% conf. interval -- upper bound	0.141	0.094	0.157	0.148	0.143	0.321	0.247	0.233	0.321	0.435	0.209	0.228	0.138	0.094
St. Dev. (annualized)	0.102	0.100	0.150	0.117	0.112	0.363	0.179	0.179	0.338	0.363	0.184	0.350	0.104	0.058
Sharpe ratio (per month)	0.053	0.033	0.167	0.160	0.182	0.219	0.225	0.229	0.243	0.207	0.181	0.208	0.224	0.221
							Panel B -- Realized Utility							
Mean	-0.253	-0.294	-0.034	-0.249	-0.196	-0.002	-0.246	-0.174	-0.001	-0.265	-0.203	-0.009	-0.265	-0.195
10% conf. interval -- lower bound	-0.256	-0.338	-0.056	-0.254	-0.244	-0.003	-0.248	-0.212	-0.002	-0.285	-0.265	-0.014	-0.270	-0.205
10% conf. interval -- upper bound	-0.258	-0.256	-0.019	-0.235	-0.161	-0.001	-0.242	-0.131	-0.001	-0.237	-0.147	-0.003	-0.263	-0.185
St. Dev.	0.030	0.134	0.013	0.044	0.213	0.002	0.050	0.136	0.001	0.270	0.244	0.010	0.026	0.023
Certainty Equivalent (annual)	-3.09	-3.73	-4.25	6.50	6.33	13.5	11.0	10.3	14.5	1.17	5.89	8.77	9.44	7.19

Figure 1

Smoothed State Probabilities: Four-State Model for Returns on SMB, HML and Market Portfolios

The graphs plot the smoothed state probabilities for the multivariate four-state Markov Switching model comprising monthly return series on Fama and French's (1993) SMB and HML portfolios and excess returns on the value-weighted market portfolio. The sample period is 1927:12 – 2005:12. Parameter estimates underlying these plots are reported in Table 1.

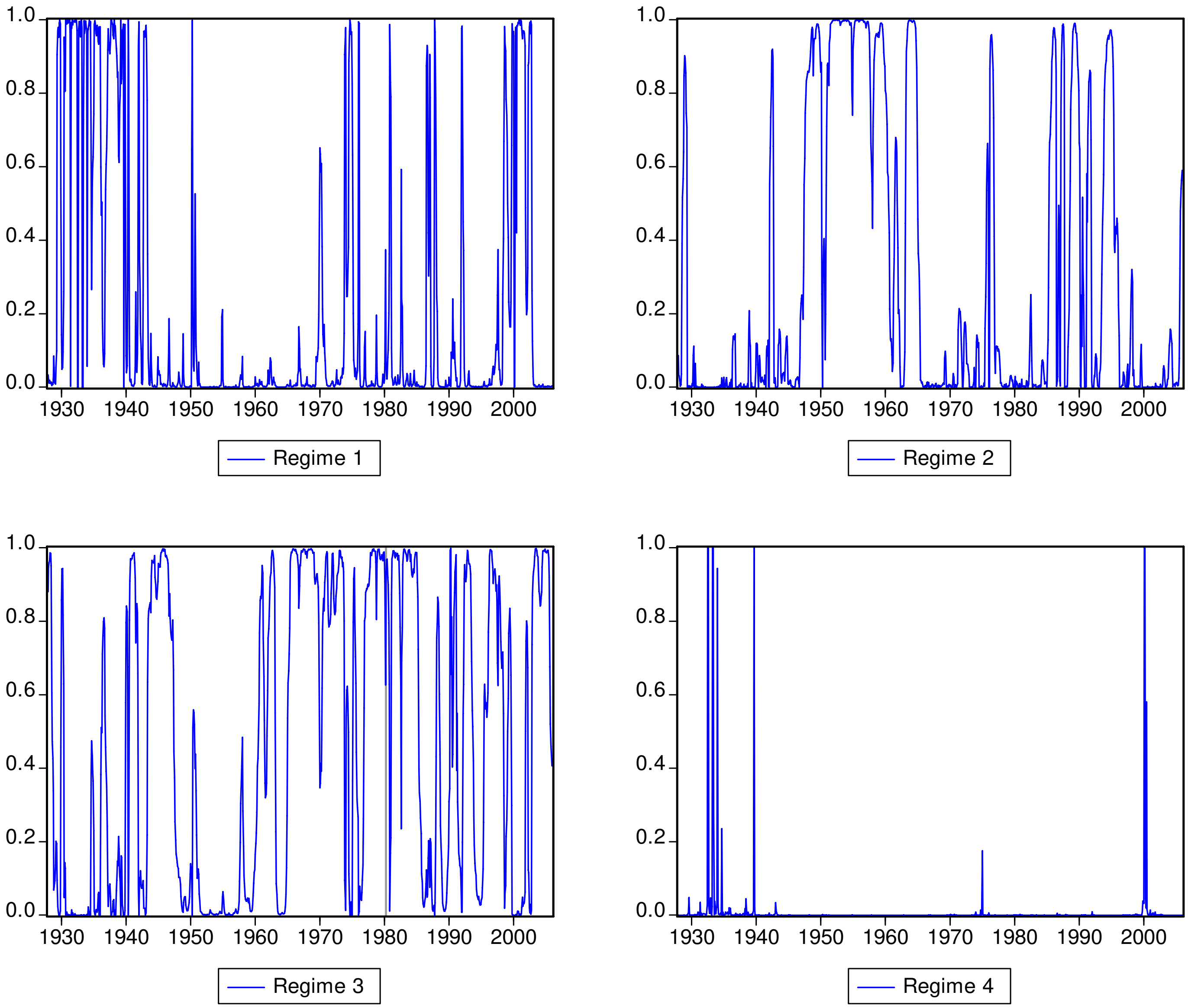

Figure 2

Smoothed State Probabilities: Four-State Model for Stock Returns and the Dividend Yield

The graphs plot the smoothed state probabilities for the multivariate four-state regime switching VAR(1) model comprising monthly returns on the SMB and HML portfolios, the value-weighted market portfolio and the dividend yield. The sample period is 1927:12 – 2005:12. Parameter estimates underlying these plots are reported in Table 2.

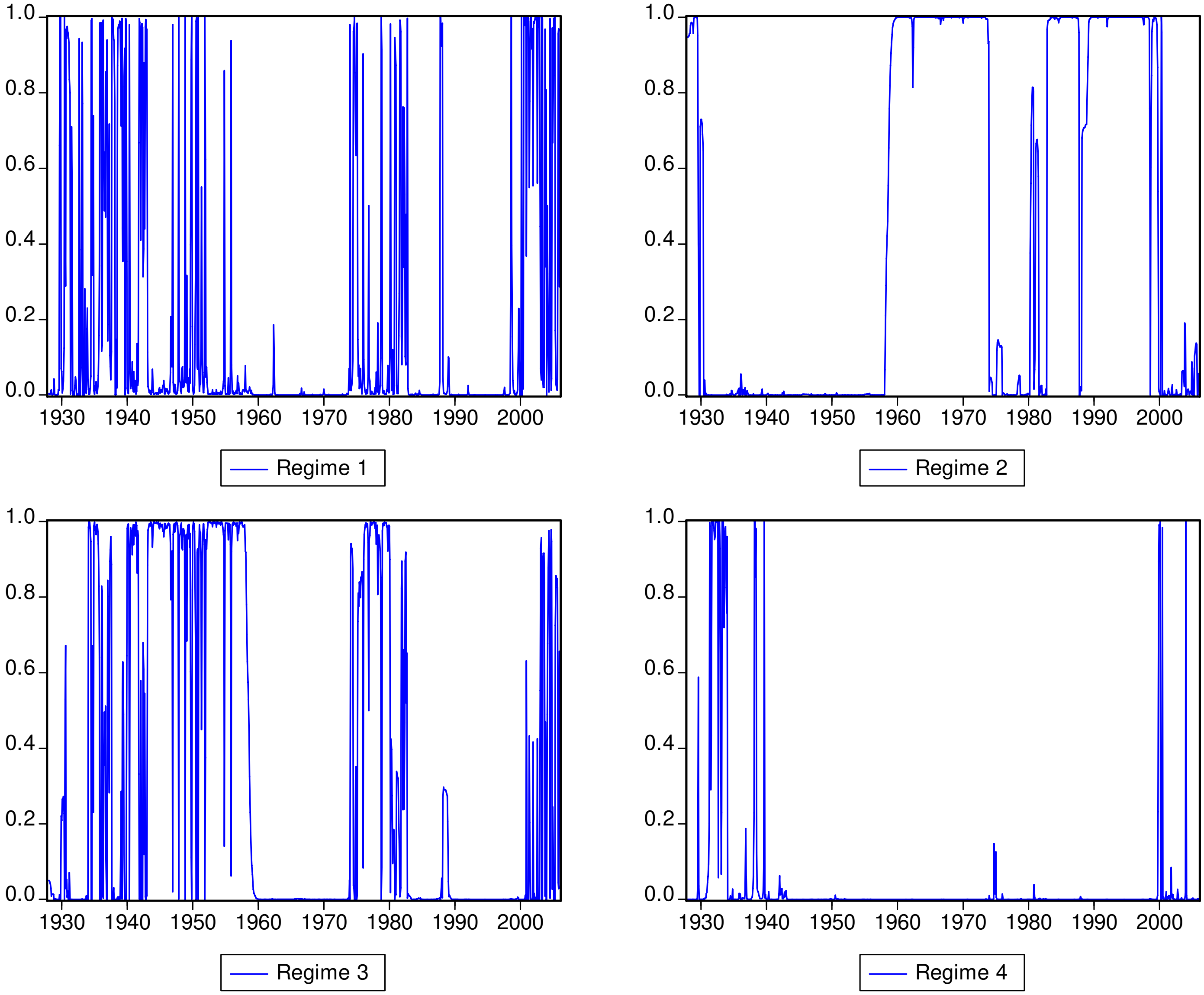

Figure 3

Optimal Asset Allocation as a Function of the Investment Horizon

The graphs show the optimal allocation to equity portfolios (market, SMB and HML) and risk-free T-bills under a four-state regime switching model as a function of the investment horizon for an investor with constant coefficient of relative risk aversion $\gamma = 5$. Each schedule corresponds to a different value of the initial state probabilities, while future states remain unknown and unobservable to investors.

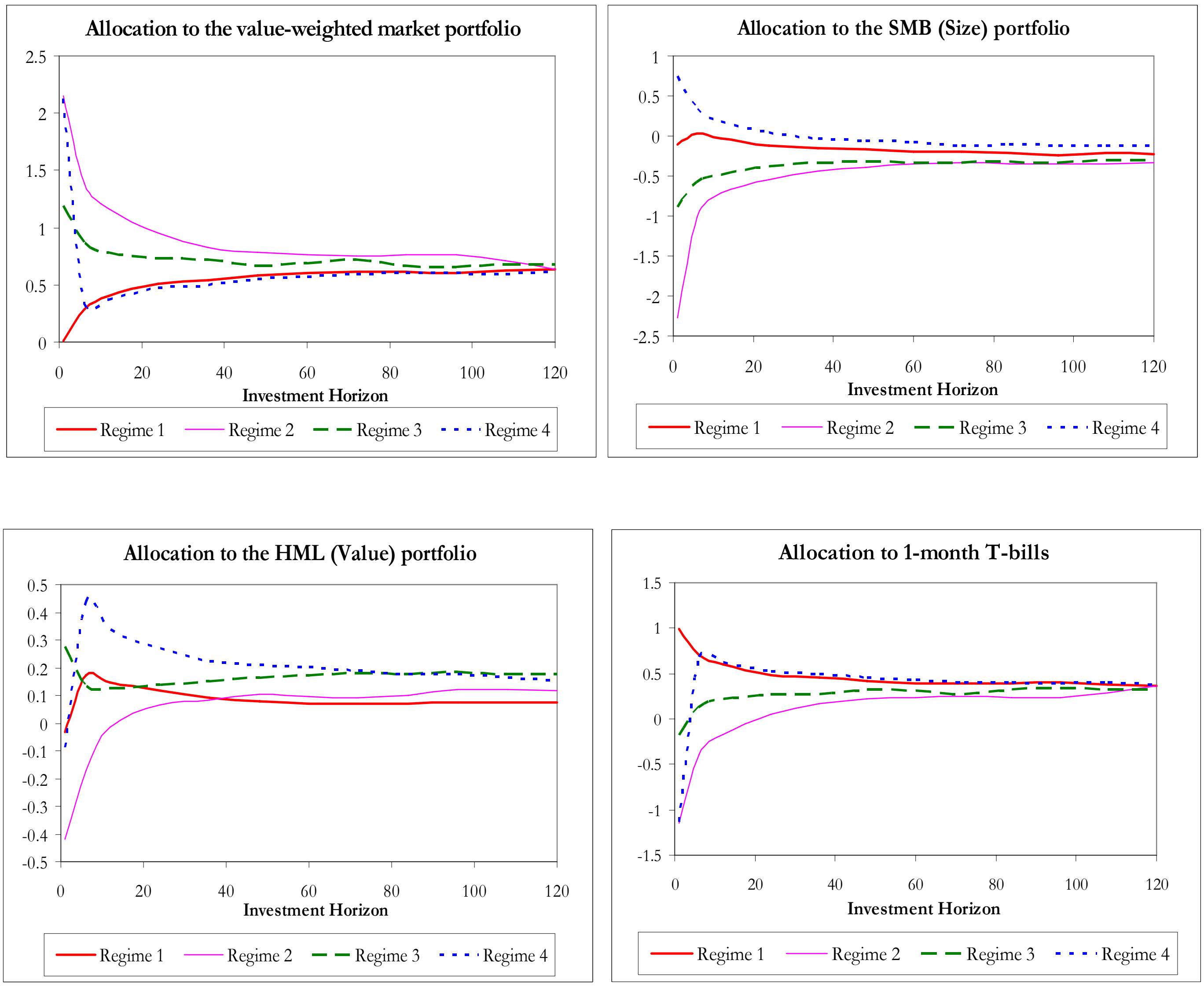

Figure 4

Optimal Asset Allocation under Uncertainty about the Initial State

The graphs show the optimal allocation to equity portfolios (market, SMB and HML) and T-bills under a four-state regime switching model in which the initial state probabilities are set at their steady-state values of [0.21 0.25 0.53 0.01]'. The graphs plot optimal portfolio shares as a function of the investment horizon for an investor with constant coefficient of relative risk aversion $\gamma = 5$.

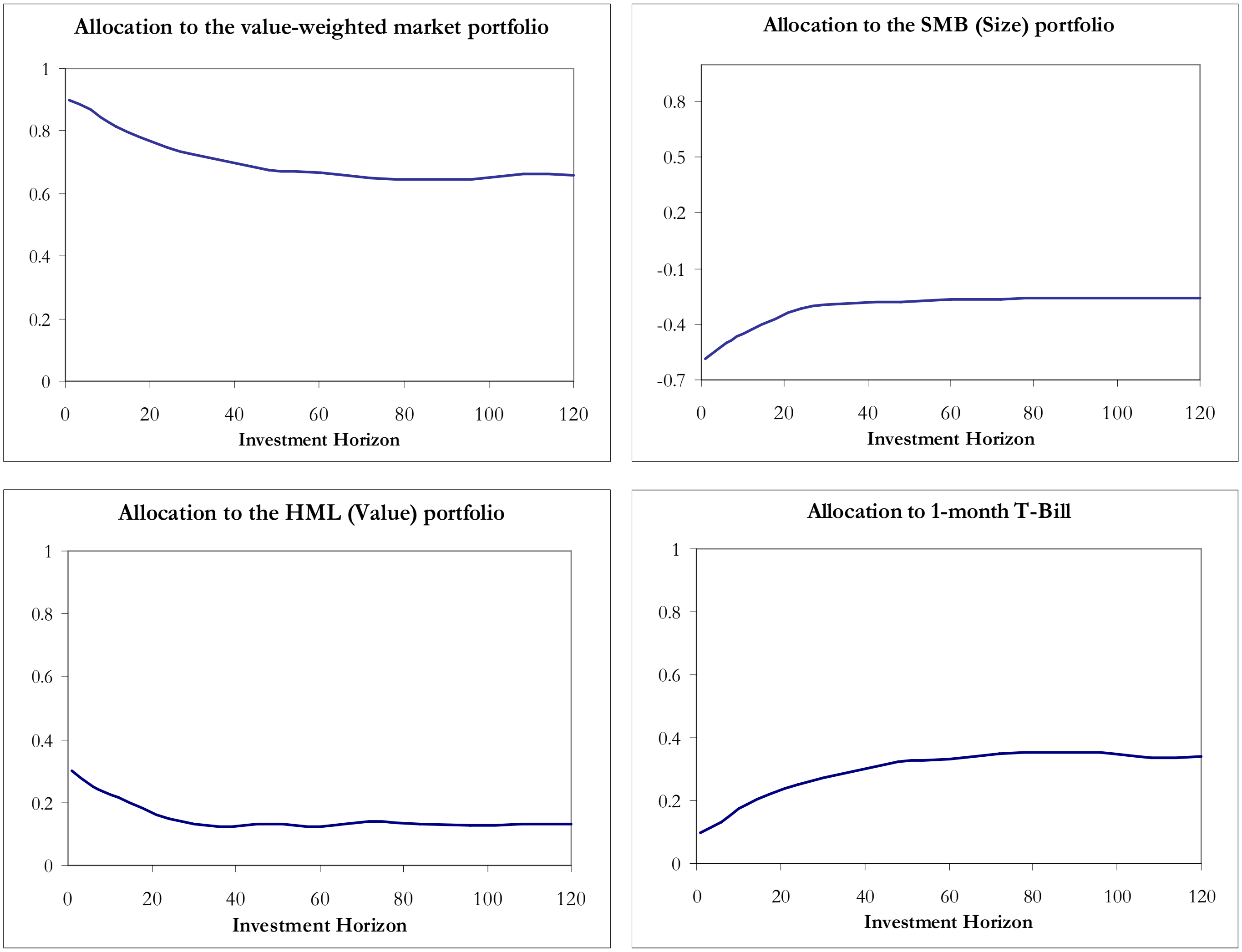

Figure 5

Optimal Allocations under Predictability from the Dividend Yield: Effects of the Regimes

The graphs show the optimal allocation to equity portfolios (market, SMB and HML) and T-bills under a four-state regime switching model in which the dividend yield predicts portfolio returns as a function of the investment horizon for an investor with constant coefficient of relative risk aversion $\gamma = 5$. Lagged values of returns and the dividend yield are set at their regime-specific unconditional means. Each schedule corresponds to a different value of the initial state, while future states remain unknown and unobservable to investors.

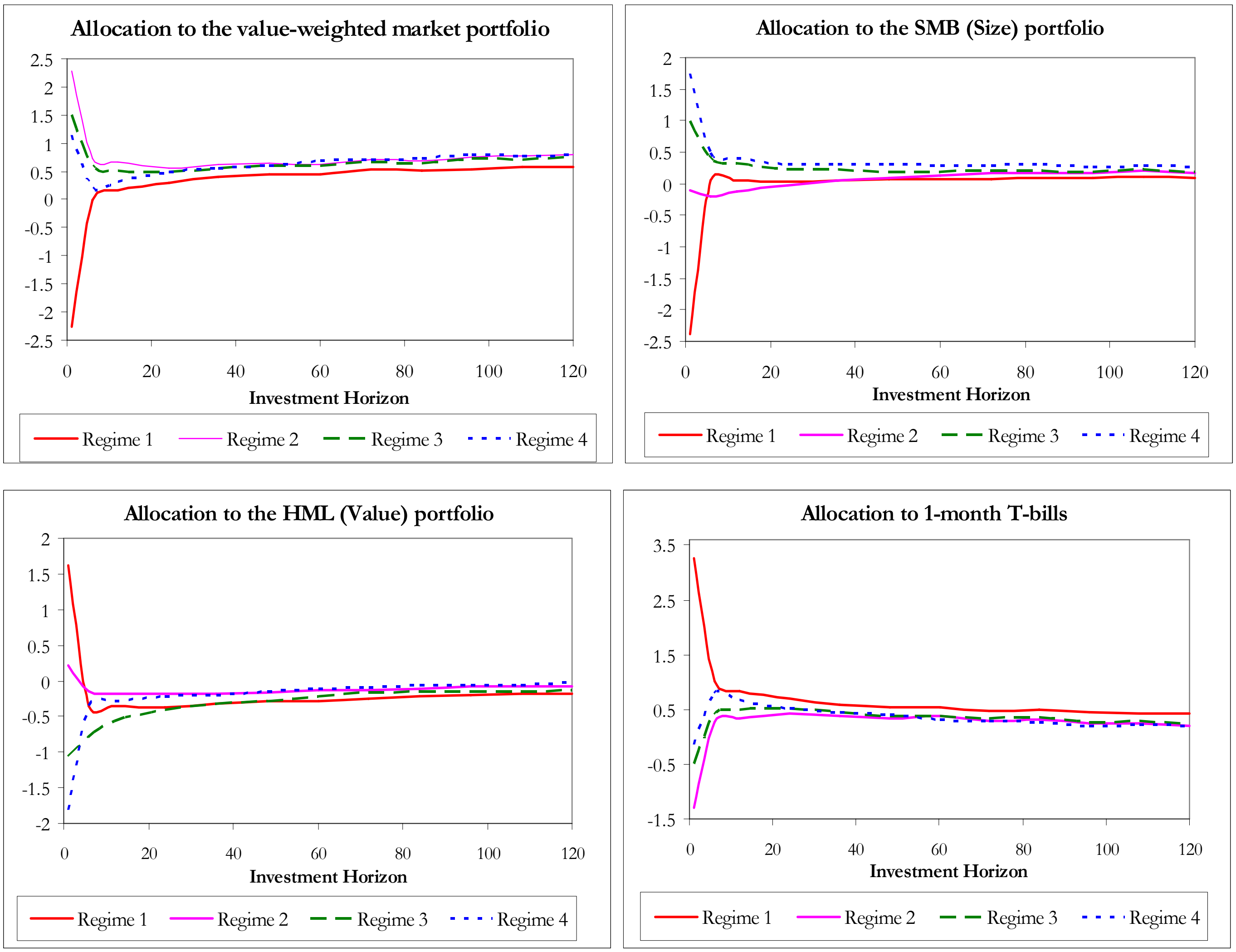

Figure 6

Optimal Allocations under Predictability from the Dividend Yield: Effects of the Initial Value of the Dividend Yield

The graphs show the optimal allocation to equity portfolios (market, SMB and HML) and T-bills as a function of the investment horizon for an investor with constant coefficient of relative risk aversion $\gamma = 5$. The calculations assume a four-state regime switching model in which the dividend yield predicts portfolio returns. Lagged values of returns are set at their regime-specific unconditional means while initial beliefs match the ergodic state probabilities. Each schedule corresponds to a different initial value of the dividend yield, i.e. 0.50% (Very Low, two standard deviations below the sample mean), 2.16% (Low, one standard deviation below the sample mean), 3.83% (Average), 5.49% (High, one standard deviation above the sample mean), and 7.15% (Very High, two standard deviations below the sample mean).

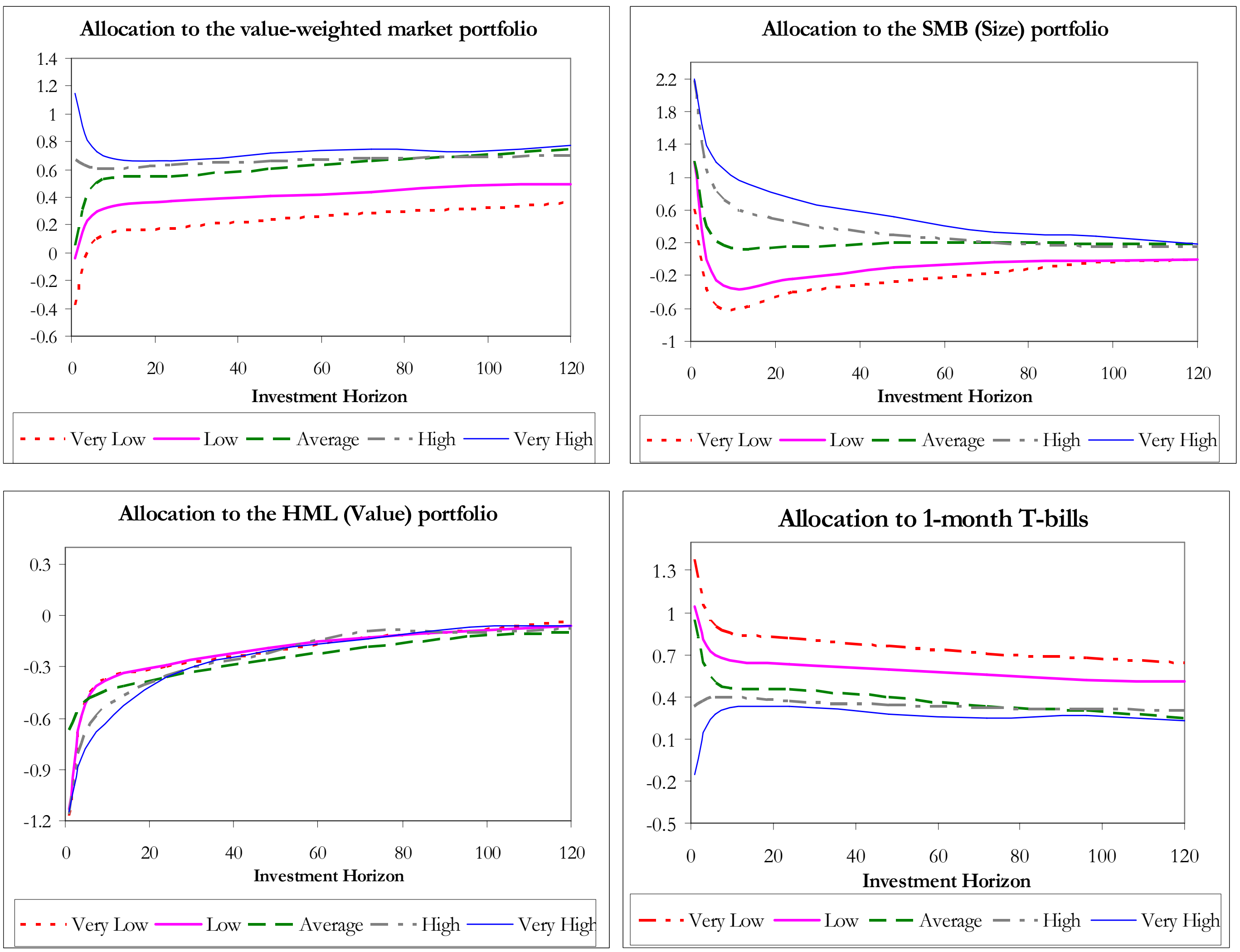

Figure 7

Optimal Asset Allocation under Predictability from the 1-month T-Bill Rate

The graphs show the optimal allocation to equity portfolios (market, SMB and HML) and T-bills as a function of the investment horizon for an investor with constant coefficient of relative risk aversion $\gamma = 5$. The calculations assume a four-state regime switching model in which the T-bill rate predicts portfolio returns. Lagged values of returns and the short-term rate are set at their regime-specific unconditional means. Each of the schedules labeled Regime 1 through Regime 4 corresponds to a different value of the initial state, while future states remain unknown and unobservable.

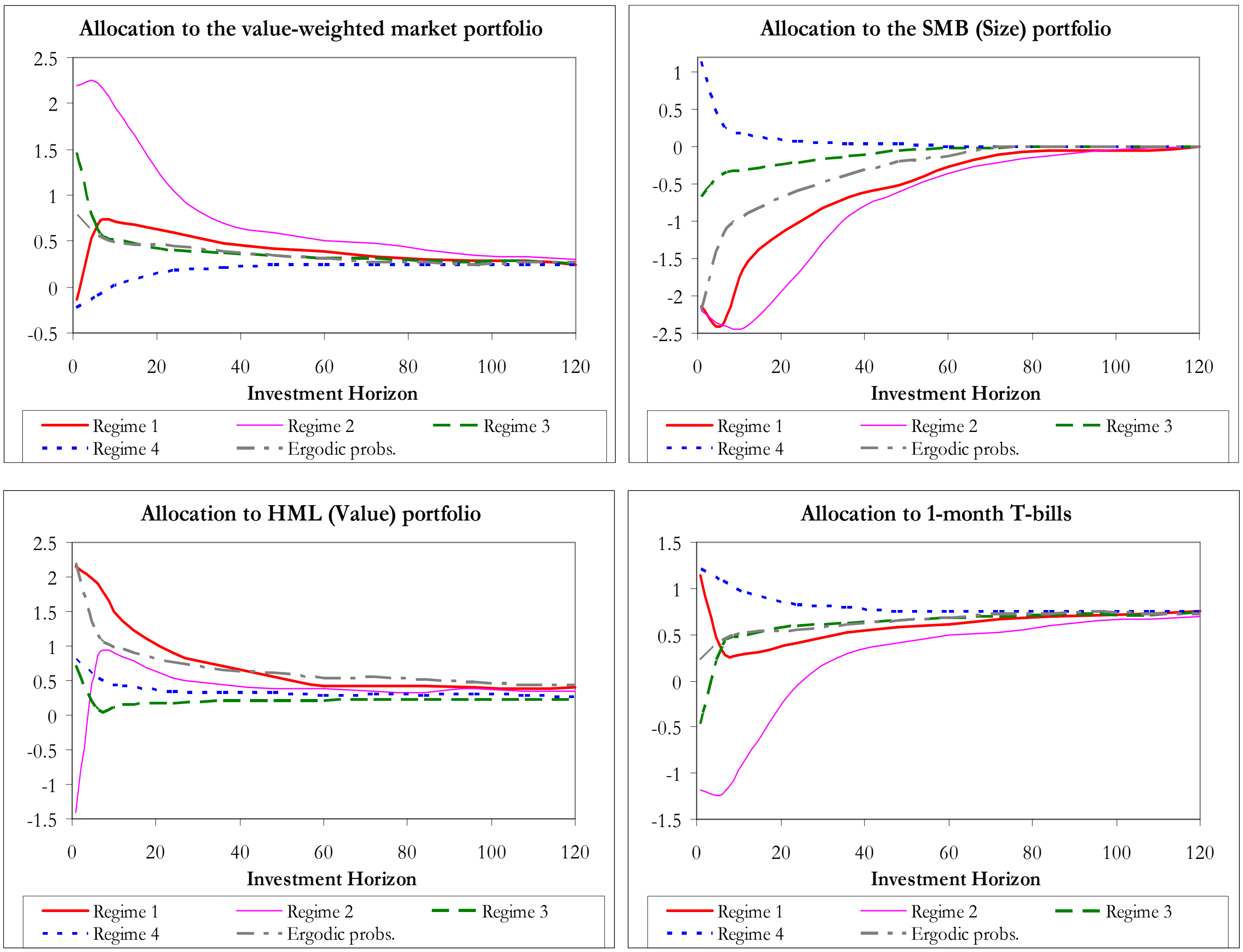

Figure 8

Comparison of Optimal Asset Allocation Across Models

The graphs show the optimal allocation to equity portfolios (market, SMB and HML) and T-bills as a function of the investment horizon for an investor with constant coefficient of relative risk aversion $\gamma = 5$. The VAR(1) model assumes predictability from the dividend yield. The MS model assumes the presence of four states while the MS-VAR(1) model allows for four regimes and predictability from the dividend yield. In VAR-type models, lagged values of returns and the dividend yield are set at their regime-specific unconditional means.

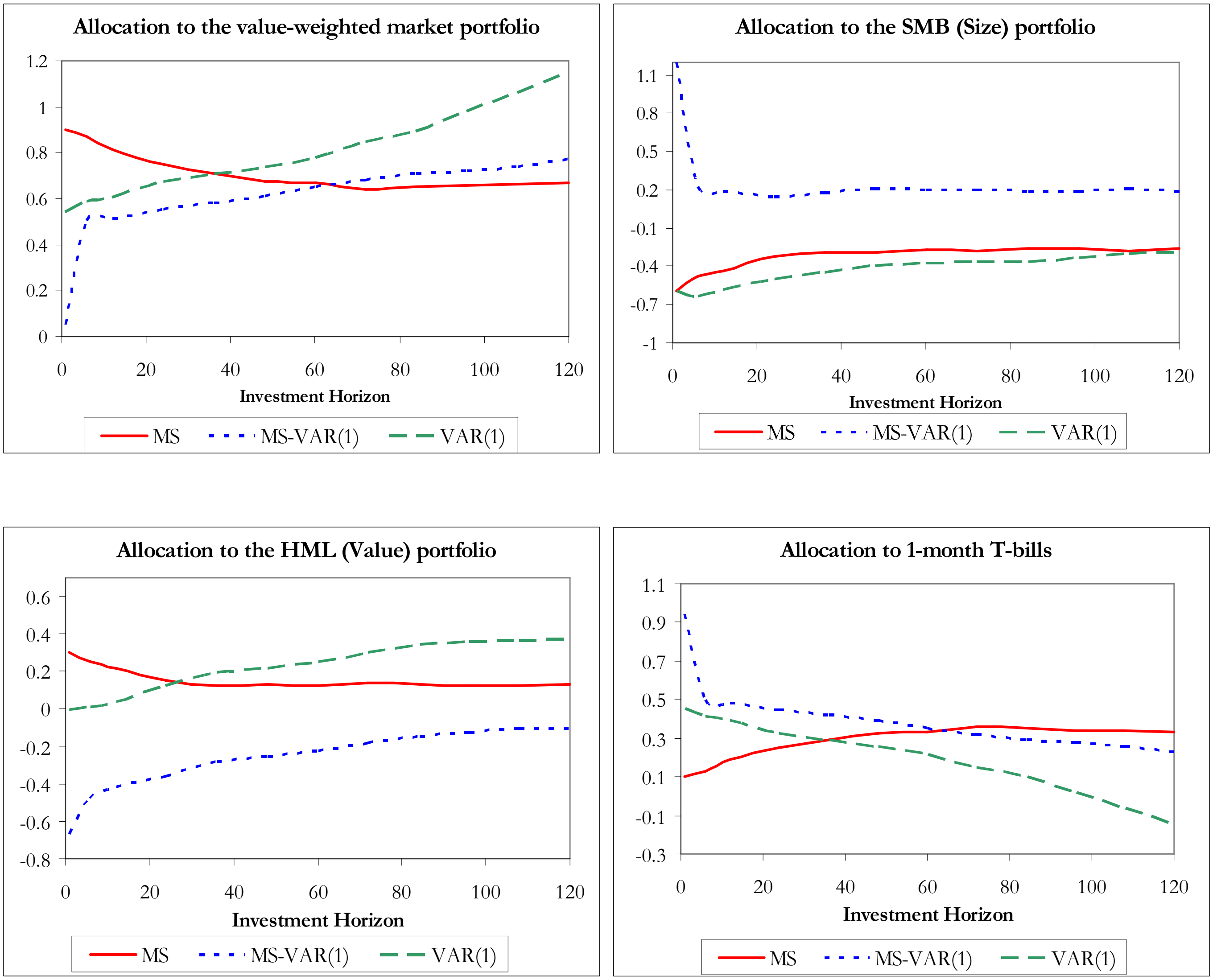

Figure 9

Utility Costs of Ignoring Regimes

This graph shows the compensation required for a buy-and-hold investor with power utility (γ = 5) to be willing to ignore regimes in asset returns starting from steady-state values of [0.21 0.25 0.53 0.01]'.

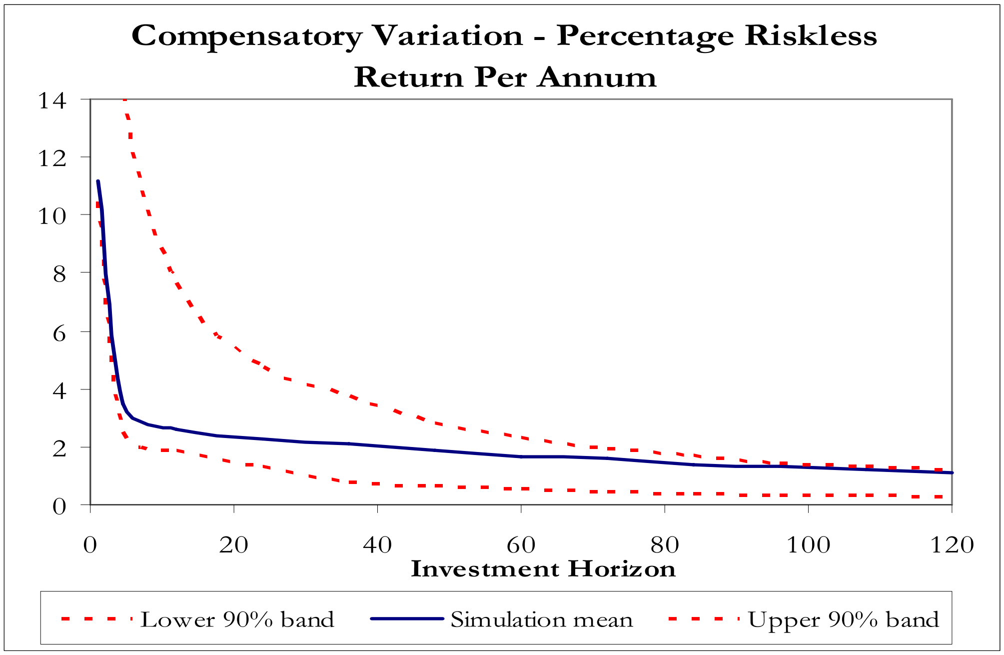

Figure 10

Comparison of Utility Costs across Models

This graph compares the utility costs from ignoring predictability arising from the dividend yield and the presence of regimes. The VAR(1) model assumes predictability from the dividend yield. The MS model assumes the presence of four states while the MS-VAR(1) model allows for four regimes and predictability from the dividend yield.

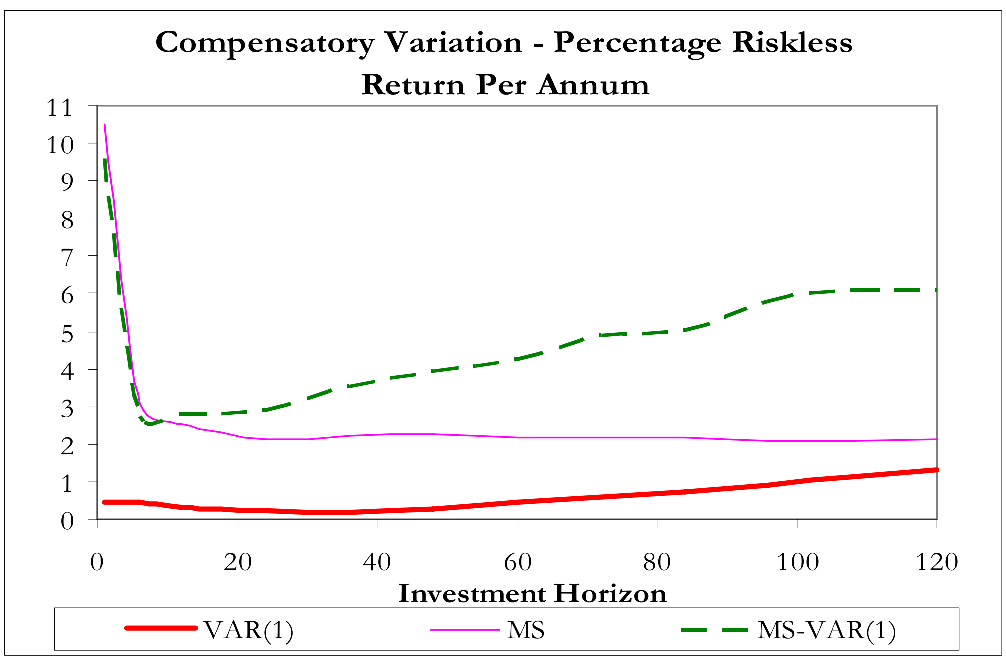

Figure 11

Recursive Portfolio Weights under Alternative Models

The graphs show the evolution in the allocation to stock portfolios (market, SMB and HML) for an investor with constant coefficient of relative risk aversion $\gamma = 5$ and a 12-month horizon. Models and weights are updated recursively over the period 1980:01 – 2005:12. The three models are a single-state Gaussian VAR(1), a four-state regime switching model, and a four-state VAR(1) regime switching model in which the dividend yield serves as a predictor variable.