1. **Introduction**

The bull equity markets of the 1990's have left us with a secular (1890-1999) average 7 percent equity return in excess of risk-free bonds.[1] These figures are even more striking considering that over two periods − 1930-1942 (Great Depression and WWII) and 1974-1981 (oil shocks) − average excess equity returns have been negative (-0.41% and -1.20%). In an economy populated by risk-averse individuals, negative excess returns are as difficult to understand as high averages. This paper shows that high and variable equity premia and low interest rates can all be rationalized in a simple general equilibrium model in which agents are on a recursive, rational learning path.

Since Mehra and Prescott (1985, MP), we know that a Lucas (1978) economy with power, time-additive expected utility, complete markets, no frictions, and in which a representative agent forms rational expectations on the only source of risk (real consumption) cannot pass the test of reproducing the historical mean equity risk premium. This impasse is labeled the *equity premium puzzle*. Moreover, in MP's framework high risk aversion implies an implausibly low elasticity of intertemporal substitution that forces the real riskless rate to levels in excess of historical averages, the *risk-free rate puzzle* (cf. Weil (1989)).

A vast literature has developed after MP had first pointed out the puzzle.[2] Many papers have focused on the role of power, time-additive, expected utility preferences which constrain the elasticity of intertemporal substitution to be the inverse of the coefficient of relative risk-aversion (e.g. Constantinides (1990), Campbell and Cochrane (1999), Epstein and Zin (1989)). Efforts have been directed at removing the assumption of market completeness, showing that the additional uncertainty in individual consumption due to the absence of insurance markets for some states helps increasing the equity premium and lowers the risk-free rate (e.g. Constantinides and Duffie (1995)). Another strand of literature has assessed the importance of borrowing constraints and transaction costs (Aiyagari and Gertler (1991)).

Surprisingly, less attention has been given to the mechanism by which beliefs are formed and updated. The early literature had in fact assumed full-information rational expectations as a way to close the model and impose *some* − arbitrary − consistency on the mechanism by which beliefs are formed. This means that agents are empowered with complete knowledge of the stochastic process driving the relevant state variables (fundamentals) and escape any kind of parameter uncertainty and the need of (econometric) learning. Yet, besides that its level is high, we know two additional facts concerning excess returns on US equities. First, excess returns are subject to remarkable fluctuations. Second, high excess returns seem to be a phenomenon of the XXth century, in particular of the 1950s and 90s. Interestingly, recent empirical research has showed that both the 1930s and the early 1980s imply the presence of structural breaks in the regime followed by fundamentals (Stock and Watson (1996)). Therefore it appears that changes in the regime characterizing fundamentals tend to be followed by high equity premia, so there might be something special about the historical path followed by the US economy.

A few papers have tried to offer explanations of the puzzles that move from events unique to the US history, particularly the Great Depression. Rietz (1988) shows that biasing agents' beliefs to reflect catastrophic scenarios *not present in the historical data*, a sizeable equity premium can be generated

[1] A few recent papers have shown that the realized mean excess return is likely to be an upward biased estimate of the ex-ante, equity premium expected by investors, e.g. Pastor and Stambaugh (2000) and Fama and French (2001). These studies estimate an equity premium in the range 3-5%.

[2] Cochrane (2001), Kocherlakota (1996), and Mehra and Prescott (2003) survey the literature.

for reasonable degrees of risk aversion. However the crash state needed to deliver the result must be truly catastrophic (Mehra and Prescott (1988)). Also, matching the empirical volatility of excess returns remains difficult (Salyer (1998)). Danthine and Donaldson (1999) explore the same concept, showing that Peso problems have more dramatic effects in artificial samples in which economy crashes are not actually present. Cecchetti, Lam, and Mark (2000) study the effects of belief distortions on asset prices. Under the assumption that the rate of growth of fundamentals follows a two-state Markov switching process, they show that some degree of pessimism relative to the maximum-likelihood estimates generates plausible moments. However the origin of such pessimistic fears is unclear. This literature therefore relies on deviations between realized and subjectively perceived beliefs, often in arbitrary ways. On the opposite, we are interested in detecting situations in which rational pessimism and crash fears may arise as a consequence of the application of simple but optimal maximum likelihood methods.

Abel (2002) explores the effects of pessimism and doubt for equilibrium asset prices. He shows that pessimism reduces the risk-free rate and that a peculiar kind of pessimism (uniform) also increases the average equity premium; doubt has similar effects. This makes it possible to generate plausible risk premia for acceptable degrees of curvature of the utility function, without running into a risk-free rate puzzle. On the other hand, Abel's analysis is admittedly exploratory, in the sense that the sources of pessimism and doubt are left unspecified.[3] Our paper may be read as an attempt to endogenously generate pessimism and doubt when agents cannot form full information rational expectations, but recursively update an estimate of the distribution of future growth rates.

When agents lack full information on some parameters characterizing the environment, their subjective beliefs may rationally deviate from the empirical distribution of the state variables, without the need of postulating in an *ad-hoc* fashion that markets agree on the possibility of some disaster state. A few papers have studied the implications of recursive learning for asset pricing.[4] However the implications for the equity premium are not pursued. An exception is Brennan and Xia (2001): In a continuous-time general equilibrium setting a representative agent recursively estimates the unobservable drift of dividends. Using a risk aversion coefficient of 15 and a rate of time preference of -10%, they derive an equity premium of 6 percent and a risk-free rate of 2.5 percent; the same parameters imply volatile stock returns and realistic correlations between dividend growth and stock returns. Unfortunately, a risk aversion of 15 is high according to the literature standards and Brennan and Xia observe (p. 266) that the effects of recursive learning are of second-order magnitude, their assumption on the randomness of the drift parameter accounting for most of the effects. We show instead that first-order effects can be obtained from learning using a plausible degree of curvature of the utility function. Finally, Guidolin (2003) shows that in principle at least, rational learning and pessimism may inflate the equilibrium equity premium and lower the riskless interest rate, but he stops short of a full assessment of the quantitative implications of the model through a standard calibration exercise.

The paper pursues three objectives. First, it removes the assumption of full-information (FI) rational expectations with complete knowledge of the process of the risk factors. In particular, we focus on a restrictive learning mechanism that does not allow any expected gains from implementing trading strategies based on the impact on equilibrium prices of the future unfolding of parameter uncertainty. Such learning

[3] He writes that "(...) the next challenge is to explain why pessimism and doubt may occur." (p. 1091). We contend that departures from rationality are not necessary, while departures from complete information are.

[4] See Barsky and De Long (1993), Bullard and Duffie (2001), Veronesi (1999), and Timmermann (1993, 2001).

schemes are called rational and in asset pricing applications they imply that prices reflect all possible, future perceived distributions of the parameters' estimates (future learning).[5] Second, we prove that when a representative agent is on a learning path and the process for dividends is described by a binomial lattice calibrated on US real consumption growth, both average excess equity returns and bond yields similar to those displayed the by the post-depression (1934-1999) US data may be generated for low levels of risk aversion. Third, we show that on a recursive learning path, US investors might have rationally come to attach positive probability to crash states as a consequence of the pessimism caused by the two deep recessions of the 1930s and 1970s. This "Peso problem" situation would have arisen in an entirely rational and endogenous fashion. In a sense, we impose structure on the thought that

> "(...) the experience of the Great Depression continues to have a significant influence on the behaviour of those who experienced it directly or indirectly, even though it has not occurred in sixty-five years." (Danthine and Donaldson (1999), p. 608)

by using the Great Depression and the two oil shocks as starting points in setting the initial beliefs held by agents on a recursive learning path. In this sense, *rational* learning and *irrational* crash fears are shown to be observationally equivalent, although only learning provides a foundation for the persistence and size of the belief distortions needed to rationalize observed asset prices.

The paper has the following structure. Section 2 presents a few empirical regularities. Section 3 introduces the model. Section 4 characterizes equilibrium asset prices under full-information rational expectations. This can be considered a version of MP's results, specialized to the case of an i.i.d. binomial tree. Section 5 characterizes the rational learning scheme, and derives equilibrium expressions for asset prices. Section 6 discusses the implications for the equity premium and the risk-free rate of the two assumptions on beliefs. The FI case is shown to display non-trivial equilibrium properties that may prevent the generation of high equity premia even for high levels of risk-aversion. Section 6 goes on to show how rational learning might contribute to solve the puzzles. Section 7 conducts simulation experiments. Section 8 discusses the role of initial beliefs and performs a few additional robustness checks. Section 9 concludes.

2. **Stylized Facts**

We use the same annual data as Shiller (1990), appropriately extended to cover the 110 years of the period 1890-1999. Stock prices and dividends correspond to January levels of the Standard & Poors Composite Indices. Real stock prices and dividends are obtained by dividing the series by the consumption deflator series (non-durables and services). The risk-free rate corresponds to the return of a strategy that rolls over an investment of one dollar in 4-6 months commercial paper. The real risk-free rate is calculated by subtracting the annual inflation rate (calculated as the percentage change of the consumption deflator) from the nominal rate. The per capita consumption growth rate series concerns non-durables and services.

2.1. *Facts Concerning the Real Consumption Growth Rate*

Real consumption growth data confirm their well-known 'smoothness'. The average growth rate is 1.80% per year (identical to MP) and the standard deviation is 3.27% (lower than MP's 3.57%). The series

[5] Guidolin and Timmermann (2003b) show these learning schemes can be equivalently characterized as Bayesian learning mechanisms when some assumptions on prior beliefs are made.

exhibits a low degree of serial correlation (-0.15), which matches the figure in MP. Hence the growth of the endowment process is well approximated by an i.i.d. process. The volatility of real consumption growth substantially decreases after WWII, from 4.4% over 1890-1945 to 1.4% over 1946-1999.

Such changes in the consumption process open the possibility that fundamentals be subject to structural breaks. In particular, breaks may have been so evident to be perceived by US investors. Chu, Stinchcombe, and White (1996) develop a procedure of *real time*, recursive monitoring of structural changes in regression models. The real time nature of the algorithm allows us to locate (i.e. test for) structural breaks perceived by the agents as they were receiving new data and making decisions. Consider the following autoregressive process for g_t, the rate of growth of real consumption

$$g_t \equiv \frac{c_t - c_{t-1}}{c_{t-1}} = \mu + \sum_{j=1}^{L} \phi_j L^j g_t + \epsilon_t = \mathbf{x}_t' \boldsymbol{\theta}_t + \epsilon_t,$$

where c_t is real per-capita consumption, L is the standard lag operator, and ϵ_t a white noise process. Call ι the minimum time span over which the parameters $\boldsymbol{\theta}_t \equiv [\mu_t \ \phi_{1t} \ ... \ \phi_{Lt}]'$ are assumed to remain constant, i.e. $\boldsymbol{\theta}_{\tau+1} = \boldsymbol{\theta}_{\tau+2} = ... = \boldsymbol{\theta}_{\tau+\iota}$, where τ is the time of the last structural break detected by agents and $\mathbf{x}_t \equiv [1 \ g_{t-1} \ ... \ g_{t-L}]'$. Suppose agents aim at testing the presence of a break in the regression model at time $t > \tau + \iota$. Chu et al. suggest calculating the following 'fluctuation detector':

$$\hat{F}_t = (t - \tau)\hat{D}_\iota^{-1/2}(\hat{\theta}_t - \hat{\theta}_\iota). \tag{1}$$

Details on the structure of the test statistic and on the associated asymptotic bounds under the null of no breaks are provided in Appendix A. In practice, we can think that after at least ι observations have been received after a break in τ, the agents start the recursive calculation of $\hat{F}_t$. If at $\bar{t}$ the statistic hits the bounds, then the null of no structural breaks since τ fails to be rejected. $\bar{t}$ becomes then the time of the new structural break. After ι further observations have been received, agents start again monitoring the occurrence of breaks, etc. We apply these tests to:

$$g_t = \mu + \phi g_{t-1} + \epsilon_t,$$

a standard AR(1) model (Timmermann (2001)). We use a value $\iota = 20$ exceeding the average duration of US business cycles to prevent natural fluctuations to be interpreted as breaks. Figure 1 shows the results of tests based on the fluctuation detector (1) by plotting $|\hat{F}_t^{(\mu)}|$ and $|\hat{F}_t^{(\beta)}|$ vs. the asymptotic bound at 1 percent test size. The null of no break is unequivocally rejected for both μ and β in correspondence to the mid 1930s, the Great Depression. Indeed the mean level of $\hat{\mu}$ jumps from 3% to 2% in the mid 1930s, while the mean $\hat{\beta}$ goes from about -0.45 to -0.2; this implies a reduction in the perceived long run mean consumption growth from 2.07% to 1.67% that fits the negative real growth during 1929-1938 (-0.66%).

The middle plots of Figure 1 repeat the analysis conditioning on the occurrence of a first structural break during the 1930s: the analysis is applied to a shorter annual data set covering the 1938-1999 period. Although the evidence on a further break in μ is inconclusive, the fluctuation detector for β locates a second break in the early 1980s, in the aftermath of the oil shocks. Indeed in the period 1974-1982 the average real growth rate was 1.67%, below the 2.10% average of 1939-1973. These econometric tests suggest the presence of two structural breaks in the fundamentals' process: the first in 1938, at the conclusion of the Depression cycle, and the second in 1982, at the conclusion of the two cycles marked by the oil price shocks.

2.2. *Facts Concerning Asset Returns*

Over 1938-1999 the mean excess return on stocks has been 7.64%, above MP's 6.18%. We take this long-run average as a measure of the ex-post equity premium. On the other hand, the average level of the risk-free rate, 0.96%, is similar to the 0.80% calculated by MP. The volatility of the equity premium is 16% while the risk-free rate is stable, 3.86%. Notoriously, a high volatility of excess equity returns along with a negligible standard deviation for the risk-free rate is puzzling (Hagiwara and Herce (1997)) and has proven to be a tough stylized fact to match (Cecchetti, Lam, and Mark (1990)). During the period following the oil shocks, the equity premium climbs even higher (10.36%) despite the higher average risk-free rate (3.84%). The volatility of both interest rates and the equity premium declines to.14.42% and 1.98%, respectively.

3. **The Model**

The model is a version of the infinite horizon, representative agent, endowment economy studied by MP (1985). There are two assets: a one-period, risk-free, zero coupon bond in endogenous zero net supply, yielding an interest rate of r_t (hence $B_t = \frac{1}{1+r_t}$ where B is the bond price), and a stock index with price S_t in exogenous net unit supply. The stock pays an infinite stream of real dividends $\{D_t\}_{t=1}^{\infty}$. These dividends are perishable, they cannot be reinvested and therefore they must be consumed in the period they are received. The initial level of fundamentals D_0 is given. The real growth rate of dividends $g_t \equiv \frac{D_t}{D_{t-1}} - 1$ follows a two-state Markov chain. State 1 is characterized by a high growth rate g_h and can be identified with business cycle expansions, while state 2 is a recession state in which growth is $-1 < g_l < 0 < g_h$: During a recession fundamentals decrease. The transition matrix is

$$\Pi = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix}$$

with stationary probabilities $\left[\frac{1-q}{2-p-q}, \frac{1-p}{2-p-q}\right]$ and first-order autocorrelation $p+q-1$.

When $p = 1-q = 1-\pi$, the probability of switching to a given state becomes independent of the original state, the stationary probabilities of the two states reduce to $\{\pi, 1-\pi\}$, and the first-order serial correlation is nil. When confronted with smooth processes such as US consumption growth, a zero first-order autocorrelation is realistic.[6] In this case, the driving process for the endowment is a binomial tree. Also, g_h, g_l, and π may be subject to infrequent jumps, i.e. structural breaks. For simplicity, assume that structural breaks are *observable*. Events of the magnitude of the Great Depression and the world energy crises are likely to be rapidly recognized because of their deep consequences. Thus, between today and a certain future date T and conditioning on no structural breaks occurring, the continuously compounded rate of growth of dividends follows a $(T-t)$-steps binomial process by which the dividend growth rate in each period $[t, t+1]$ can be either g_h with probability π or g_l with probability $1-\pi$:

$$g_t = \begin{cases} g_h & \text{with prob. } \pi \\ g_l & \text{with prob. } 1-\pi \end{cases} \qquad \forall t \geq 1, \quad \pi \in (0,1) \tag{2}$$

and the rates of growth over time are independent. The description of the physical environment is completed by the assumption of perfect capital markets: unlimited short sales possibilities, perfect liquidity, no taxes,

[6] Abel (2002) and Barsky and De Long (1993) stress that to a first approximation dividends follow a random walk.

transaction fees, bid-ask spreads, markets are open at all points in time in which news on dividends are generated, no borrowing or lending constraints.

There exists a representative agent who has power, constant relative risk aversion preferences

$$u(c_t) = \begin{cases} \frac{C_t^{1-\gamma}-1}{1-\gamma} & \gamma \neq 1 \\ \ln C_t & \gamma = 1 \end{cases} \tag{3}$$

where C_t is real consumption. The agent maximizes the discounted value (at a rate $\rho > 0$) of the infinite stream of expected future (instantaneous) utilities deriving from consumption of real dividends.

4. **Asset Pricing Under Full Information Rational Expectations**

Assume the representative agent knows the stochastic process of dividends, i.e. its binomial structure, the parameters $\{g_h, g_l, \pi_t\}$, and that she forms rational expectations. Moreover, assume that breaks occur with such a low frequency to be safely disregarded by agents when they form expectations on future cash flows and determine current asset prices.[7] The representative agent solves:

$$\max_{\{C_{t+k}, w^s_{t+k}, w^b_{t+k}\}_{k=0}^{\infty}} E\left[\sum_{k=0}^{\infty} \beta^k u(C_{t+k}) | \digamma_t\right]$$
$$s.t.\ C_{t+k} + w^s_{t+k} S_{t+k} + w^b_{t+k} B_{t+k} = w^s_{t+k-1}(S_{t+k} + D_{t+k}) + w^b_{t+k-1}, \tag{4}$$

where $\beta = \frac{1}{1+\rho}$, and w^s_{t+k}, w^b_{t+k} represent the number of shares of stocks and bonds in the agent's portfolio as of period $t+k$. $E[\cdot|\digamma_t] \equiv E_t[\cdot]$ denotes the conditional expectation operator measurable with respect to $\digamma_t$, the information set. Standard dynamic programming methods yield the following Euler equations:

$$S_t = E\left[Q_{t+1}(S_{t+1} + D_{t+1}) \mid \digamma_t\right] \tag{5}$$
$$B_t = E\left[Q_{t+1} \mid \digamma_t\right], \tag{6}$$

where $Q_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$ is the pricing kernel (stochastic discount factor) defined as the product of the subjective discount factor and the intertemporal marginal rate of substitution in consumption. In equilibrium dividends are the only source of endowment and consumption, so $C_{t+k} = D_{t+k}\ \forall k \geq 0$.

In the full information case, a solution for asset prices can easily be obtained using the method of undetermined coefficients. It is straightforward to prove that the full information rational expectations (FI) stock price, S_t^{FI}, is given by

$$S_t^{FI} = \lim_{T\to\infty} E_t\left[\sum_{j=1}^{T}\left(\beta^j \prod_{i=1}^{j}(D_{t+i}/D_{t+i-1})^{1-\gamma}\right)\right] \cdot D_t. \tag{7}$$

The linear homogenous form of the pricing function $S_t^{FI} = \Psi_t^{FI} D_t$ is a direct implication of expected utility maximization, where Ψ_t^{FI} denotes the pricing kernel, i.e. the price-dividend ratio, see Abel (2002, p. 1079) and Brennan and Xia (2001, p. 258), a time-varying function of π_t. Assuming

$$\rho > \pi_t g_h^* + (1-\pi_t) g_l^*, \tag{8}$$

[7] Our empirical analysis has isolated only 2 breaks in a 110 years long time series, a frequency of 1.8%. Timmermann (2001) uses a monthly probability of 0.3%. Guidolin (2003) calculates by simulation equilibrium prices when future, random breaks are taken into account and concludes that closed-form solutions provide a good approximation.

where $g_l^* \equiv (1+g_l)^{1-\gamma} - 1$ and $g_h^* \equiv (1+g_h)^{1-\gamma} - 1$, Guidolin and Timmermann (2003a,b) prove that

$$S_t^{FI} = \Psi_t^{FI} D_t = \frac{1 + g_l^* + \pi_t(g_h^* - g_l^*)}{\rho - g_l^* - \pi_t(g_h^* - g_l^*)} D_t, \tag{9}$$

while the positive, time-invariant equilibrium risk-free rate, $r^{f,FI}$, is

$$r^{f,FI} = \frac{1}{B^{FI}} - 1 = \frac{1+\rho}{(1+g_l)^{-\gamma} + \pi_t\left[(1+g_h)^{-\gamma} - (1+g_l)^{-\gamma}\right]} - 1. \tag{10}$$

Since the stock price is homogeneous of degree one in dividends, it follows the same binomial tree $\{g_h, g_l, \pi\}$ as dividends. Condition (8) ensures not only $\Psi_t^{FI} > 0$ but also convergence of the series $\sum_{s=1}^{\infty} E_t[(\prod_{k=1}^{s} Q_{t+k}) D_{t+s}]$ and existence of the equilibrium. For $\rho > 0$ and given $\{g_h, g_l, \pi\}$, too low or too high levels of γ might violate this condition, meaning that there exists no finite stock price such that markets clear. When γ is too low for (8) to hold, then γ must be close to zero. A $\gamma \simeq 0$ means that the agent is nearly risk-neutral so that $\pi_t g_h^* + (1-\pi_t) g_l^* \simeq E[g_t]$ and from (10) $r^{FI} \simeq \rho$. Then condition (8) is equivalent to $r^{FI} > E[g_t]$ since a risk-neutral agent will never demand risk-free bonds when it is possible to earn a higher expected stream of cash dividends from holding the stock. No equilibrium exists as the stock price diverges to infinity in response to the excess demand while the bond price falls to zero as all agents would like to issue bonds to finance their stock holdings. A high γ can prevent satisfaction of (8) since when $g_l \leq 0$ there exists a state in which the agent's intertemporal marginal rate of substitution (IMRS) $u'(D_{t+k})/u'(D_{t+k-1})|_{g_{t+k}=g_l} = (1+g_l)^{-\gamma}$ diverges as $\gamma \to \infty$. This means that all assets paying out a positive amount of real consumption in the bad state receive an infinite valuation.

5. **Asset Prices on a Learning Path**

5.1. *Rational Learning*

Suppose instead that the representative agent is on a learning path: he knows that dividends follow a binomial lattice $\{g_h, g_l, \pi_t\}$. He also knows g_h and g_l. However, π_t is unknown and the agent estimates it using all the available information since its last change (break), time τ_b. The agent recursively gains knowledge on π_t by using the simple frequency estimator:

$$\widehat{\pi}_t^{\tau_b} = \frac{n_0^{\tau_b} + \sum_{j=\tau_b+1}^{t} I_{\{g_j = g_h\}}}{N_0^{\tau_b} + t - \tau_b + 1} = \frac{n_0^{\tau_b} + n_t^{\tau_b}}{N_0^{\tau_b} + N_t^{\tau_b}} \quad t > \tau_b, \tag{11}$$

where $I_{\{g_{t+j}=g_h\}}$ takes value 1 when at step/time j of the binomial tree dividends grow at a high rate, and zero otherwise. $n_t^{\tau_b}$ denotes the number of high growth states recorded between τ_b+1 and time t, while $N_t^{\tau_b}$ is the total number of dividend movements recorded over $[\tau_b + 1, t]$. After a break, investors are assumed to start out with beliefs synthesized by $\{n_0^{\tau_b}, N_0^{\tau_b}\}$.[8] $\widehat{\pi}_0^{\tau_b} = n_0^{\tau_b}/N_0^{\tau_b}$ reflects a starting belief agents hold on the probability of a good state, with $1/N_0^{\tau_b}$ the associated degree of precision. As stressed by Timmermann (2001, p. 305), the presence of infrequent breaks makes learning a much more plausible assumption than full information rational expectations as investors rarely will have available a large historical sample from which to derive precise estimates of the relevant parameters.

[8] If breaks are observable, agents know with certainty when to restart their learning process and discard information from the previous regime. On the other hand, if agents were uncertain as to the occurrence of breaks, explicit econometric methods to estimate the likelihood of a break at all times $\tau \leq t$ should be used. We do not pursue this extension here.

Agents are on a *rational* learning (RL) path, see Guidolin and Timmermann (2003b), i.e. they take into account that their beliefs on π will be updated for $t' > t$ and incorporate the effects of future learning in their current beliefs.[9] Consider the state vector W_t with stationary density parameterized by $\theta \in \Theta \subseteq \Re^p$, $f(W_t;\theta)$. Suppose an agent wants to calculate a T-steps ahead forecast W^f_{t+T}. If the agent does not know π but he is on a RL path, then his best forecast will be:

$$W^f_{t+T} = E\left\{...E\left[E\left(W_{t+T}|\digamma_{t+T-1}, \widehat{\theta}_{t+T-1}\right)|\digamma_{t+T-2}, \widehat{\theta}_{t+T-2}\right]...|\digamma_t, \widehat{\theta}_t\right\}. \quad (12)$$

$E\left(\cdot|\digamma_{t+k}, \widehat{\theta}_{t+k}\right)$ is a conditional expectation measurable with respect to the information structure $\digamma_{t+k}$, conditional on current knowledge on θ, some estimator $\widehat{\theta}_{t+k}$. (12) shows that the agent takes into account that her knowledge about θ will change for $t' > t$ with probability 1. The sequence $\{\widehat{\theta}_t, \widehat{\theta}_{t+1}, ..., \widehat{\theta}_{t+T-1}\}$ enters the forecasting problem and future beliefs are recognized to be state-dependent.

For instance, consider the case in which dividends follow a binomial lattice $\{g_h, g_l, \pi\}$, $(T-t) = 2$, and $(1+g_h)(1+g_l) = 1$. The agent's perception of $P_t\left\{D_{t+2} = (1+g_h)^j(1+g_l)^{2-j}D_t|\digamma_t\right\}$ depends crucially on the contents of $\digamma_t$. When π is know, i.e. under full-information rational expectations, knowledge of the history of the process is redundant as the agent has full knowledge of the lattice:

$$P^{FI}\left\{D_{t+2} = (1+g_h)^j(1+g_l)^{2-j}D_t|\digamma_t\right\} = \binom{2}{j}\pi^j(1-\pi)^{2-j} \quad j = 0, 1, 2.$$

a (transform of a) $Bi(2, \pi)$. Suppose instead that π is unknown and the agent follows a rational learning scheme based on (11). Figure 2 helps to understand the logic with which beliefs are revised.

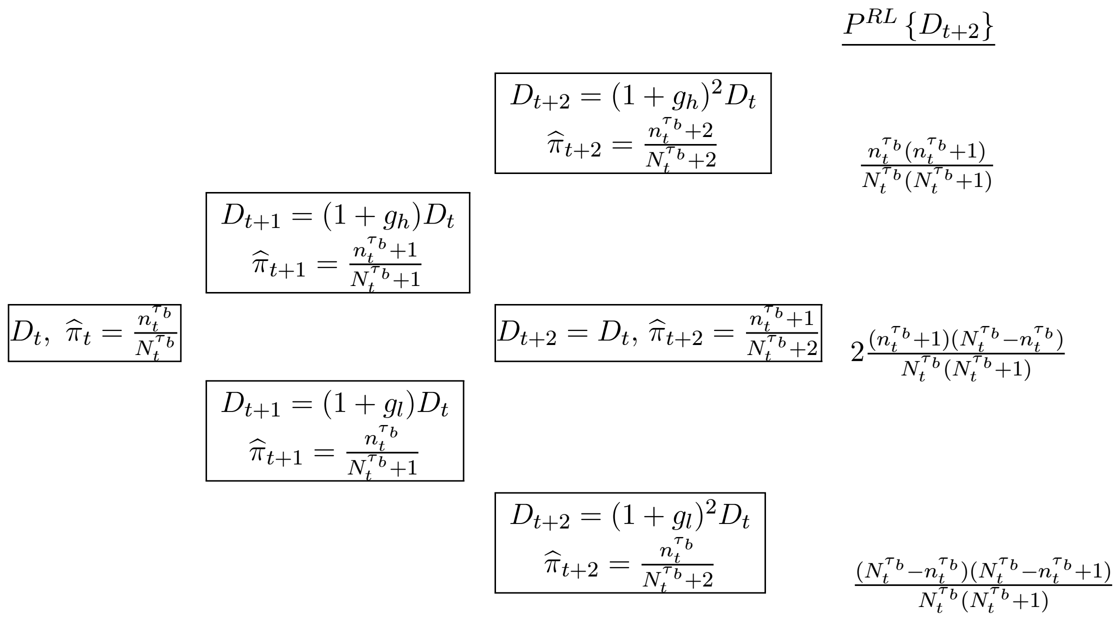


Figure 2

At time t the agent has an initial assessment $\widehat{\pi}_t^{\tau_b} = \frac{n_t^{\tau b}}{N_t^{\tau b}}$. For instance, assume that these beliefs are correct, i.e. $\widehat{\pi}_t^{\tau_b} = \pi$. In this case the calculation of $P_t\left\{D_{t+2} = (1+g_h)^2 D_t|\digamma_t\right\}$ implies recognition that to get to

[9] Most of the papers in the asset pricing literature use adaptive, least-squares learning schemes that do not allow prices to reflect the effects of future learning, i.e. belief revisions triggered by the arrival of additional information. For instance, Barsky and De Long (1993) realize that their model "(...) does not allow prices today to be influenced by investors' knowledge that they will be revising their estimate (...) in the future" (p. 299).

$(1+g_h)^2 D_t$ the agent realizes that dividends have to grow at a high rate between t and $t+1$. This implies that in $t+1$ the probability belief will be revised to $\widehat{\pi}_{t+1}^{\tau_b} = \frac{n_t^{\tau_b}+1}{N_t^{\tau_b}+1}$. So the probability of two consecutive up-moves perceived at time t is not simply $\left(\frac{n_t^{\tau_b}}{N_t^{\tau_b}}\right)^2 = (\widehat{\pi}_t^{\tau_b})^2$ but instead

$$P_t^{RL}\left\{D_{t+2} = (1+g_h)^2 D_t | n_t^{\tau_b}, N_t^{\tau_b}\right\} = P_t^{RL}\left\{D_{t+1} = (1+g_h) D_t | n_t^{\tau_b}, N_t^{\tau_b}\right\} \times$$
$$P_t^{RL}\left\{D_{t+2}{=}(1{+}g_h) D_{t+1} | n_t^{\tau_b}+1, N_t^{\tau_b}+1\right\} = \frac{n_t^{\tau_b}(n_t^{\tau_b}+1)}{N_t^{\tau_b}(N_t^{\tau_b}+1)} > (\widehat{\pi}_t^{\tau_b})^2,$$

the agent 'integrates' over all possible future values of $\widehat{\pi}_{t+k}^{\tau_b}$. Despite its simplicity, this framework stresses that agents perceive their own future beliefs as random variables measurable with respect to the sequence of future information sets. Figure 2 also reports probability calculations for the other two final nodes.

In the binomial tree model the compounded probability distribution perceived under rational learning can be fully characterized. Guidolin and Timmermann (2003a,b) prove that the distribution for the number of up-states occurring between t and $t+T$, is given by

$$P^{RL}\left\{\frac{D_{t+T}}{D_t} = (1+g_h)^i (1+g_l)^{T-i} \mid \widehat{\pi}_t^{\tau_b}, N_t^{\tau_b}\right\} = \binom{T}{i} \cdot \frac{\prod_{k=0}^{i-1}(n_t^{\tau_b}+k) \prod_{k=0}^{T-i-1}(N_t^{\tau_b} - n_t^{\tau_b} + k)}{\prod_{k=0}^{T-1}(N_t^{\tau_b}+k)}, \quad (13)$$

$i = 0, ..., T$, where $\binom{T}{i} = \frac{T!}{(j-i)!\,i!}$ is the permutation operator for $T \geq i$, and $\prod_{k=0}^{-1}(\cdot) = 1$. The updated probability distribution of dividends for period $t+T$ only depends on the number of up-states occurring between periods t and $t+T-1$ and is independent of the specific path followed on the binomial lattice.

5.2. *Equilibrium Asset Prices*

Despite beliefs are recursively shaped by the learning process, the same features that simplified the solution of the model under FI are in place: Consumption and dividends must coincide in general equilibrium; from Gennotte (1986) it is known that the decision problem may be decomposed into an inference problem in which an investor derives a predictive density for the state variable, plus a consumption-portfolio program in which such a density is employed to find optimal policy functions. Solving the consumption problem and applying standard methods, the following Euler equations characterize an internal optimum:

$$\begin{aligned} B^{RL} &= \widehat{E}_t^{\tau_b}\left[Q_{t+1}\right] \\ S_t^{RL} &= \widehat{E}_t^{\tau_b}\left\{Q_{t+1}\left[D_{t+1} + \widehat{E}_{t+1}^{\tau_b}\left(Q_{t+2}\left(D_{t+2} + \widehat{E}_{t+2}(...\widehat{E}_{t+T-1}^{\tau_b}(Q_{t+T}(D_{t+T}+S_{t+T}))...\right)\right)\right]\right\} \qquad (14) \\ &= \widehat{E}_t^{\tau_b}\left(Q_{t+1}D_{t+1}\right) + \widehat{E}_{t+1}^{\tau_b}\left[Q_{t+1}\widehat{E}_{t+2}^{\tau_b}\left(Q_{t+2}D_{t+2}\right)\right] + ... + \widehat{E}_t^{\tau_b}\left\{Q_{t+1}\widehat{E}_{t+1}^{\tau_b}\left[Q_{t+2}...\widehat{E}_{t+T-1}^{\tau_b}\left(Q_{t+T}D_{t+T}\right)...\right]\right\} \end{aligned}$$

where $\widehat{E}_t^{\tau_b}[\cdot] \equiv E[\cdot|\mathcal{F}_t, \widehat{\pi}_t^{\tau_b}]$ defines the expectation operator conditional on the information available at time t and the current estimate of the unknown parameter π after the last break in $\tau_b < t$. Since the sequence of conditional expectations at the nodes $t+1$, $t+2$, ..., $t+T$ implied by (14) depends on the future states as summarized by $\widehat{\pi}_{t+1}^{\tau_b}$, $\widehat{\pi}_{t+2}^{\tau_b}$, ..., $\widehat{\pi}_{t+T-1}^{\tau_b}$, the law of iterated expectations can no longer be used as the distributions with respect to which future expectations are taken should also discount future information flows. Once this fact is recognized, Guidolin and Timmermann (2003b) show that if a transversality condition holds and $\rho > \max\{g_l^*, g_h^*\}$, the stock price under rational learning, S_t^{RL}, is

$$S_t^{RL} = \Psi_t^{RL} D_t = \left\{ \sum_{i=1}^{\infty} \beta^i \sum_{j=0}^{i} (1+g_h^*)^j (1+g_l^*)^{i-j} P^{RL}\left(D_{t+i}^j | \widehat{\pi}_t^{\tau_b}, N_t^{\tau_b} \right) \right\} \cdot D_t \tag{15}$$

where $P^{RL}\left(D_{t+i}^j = (1+g_h)^j (1+g_l)^{i-j} D_t | \widehat{\pi}_t^{\tau_b}, N_t^{\tau_b} \right)$ is given by (13). The equilibrium risk-free rate is

$$r_t^{RL}(\widehat{\pi}_t^{\tau_b}) = \frac{1+\rho}{(1+g_l)^{-\gamma} + \widehat{\pi}_t^{\tau_b}\left[(1+g_h)^{-\gamma} - (1+g_l)^{-\gamma}\right]} - 1. \tag{16}$$

These results have three implications. First, the pricing kernel is no longer a constant, depending on the cumulated knowledge on π, through $n_t^{\tau_b}$ and $N_t^{\tau_b}$. In this sense, dividend changes between time $t+k$ and $t+k+1$ acquire a self-enforcing nature: news of a certain sign will cause not only a stock price change through the linear pricing relationship $S_t^{RL} = \Psi_t^{RL} D_t$, but also through the revision of the pricing kernel, Ψ_t^{RL}. Second, while under FI the risk-free rate was a constant, on a learning path it changes as a function of the 'new' state variables $n_t^{\tau_b}$ and $N_t^{\tau_b}$. In particular, for $\gamma \leq 1$ $(1+g_h)^{-\gamma} - (1+g_l)^{-\gamma} \leq 0$ and high dividend growth raises the risk free rate by raising $\widehat{\pi}_t^{\tau_b}$; the opposite when $\gamma > 1$. Third, notice that structural breaks are reflected in equilibrium asset prices only because of their 'resetting' effects on the agents' learning clock, i.e. via $n_t^{\tau_b}$ and $N_t^{\tau_b}$. Although structural breaks are possible at all times, their infrequent nature makes the cost of ignoring their future occurrence small.

Guidolin and Timmermann (2003b) show that although the process of fundamentals is 'smooth' (i.e. i.i.d. and with low volatility) rational learning may generate stock prices with many realistic features, such as serial correlation, volatility clustering, and excess volatility. However, they fail to investigate the implications for excess stock returns and the riskless interest rate. Furthermore, it is clear that in the absence of structural breaks agents would eventually learn the process for dividends to an arbitrary accuracy, so that the pricing kernel Ψ_t^{RL} would converge to Ψ^{FI} and all learning effects would disappear. In other words, by assuming the observable occurrence of breaks, we rule out the possibility of complete information, see Timmermann (2001, p. 302).

6. **Implications for the Equity Premium**

6.1. *Full Information Rational Expectations*

In the FI case, the mapping simplifies to a relation between preferences $[\rho\ \gamma]'$ and equilibrium asset returns. It is straightforward to derive the expression for the equity premium:

$$E\left[r_t^{p,FI}\right] = (1+\rho)\left\{ \frac{1+g_l+\pi_t(g_h-g_l)}{1+g_l^*+\pi_t(g_h^*-g_l^*)} - \frac{1}{(1+g_l)^{-\gamma} + \pi_t\left[(1+g_h)^{-\gamma} - (1+g_l)^{-\gamma}\right]} \right\}. \tag{17}$$

Since dividends are i.i.d., there is no difference between the time t conditional and unconditional equity premium. Moreover, besides the parameters $\{g_h, g_l, \pi_t\}$, (17) depends on $[\rho\ \gamma]'$ only. To stress this dependency, we write $E[r_t^{p,FI}(\rho,\gamma)]$ and $r^{f,FI}(\rho,\gamma)$. It is then natural to ask what are the limits of $E\left[r_t^{p,FI}\right]$ as $[\rho\ \gamma]'$ vary, i.e. does a preference structure exist such that the stylized facts can be matched?

From Section 4, it makes no sense to ask what happens to asset returns when $\gamma \to \infty$ or $\gamma \to 0$ irrespective of ρ, i.e. to consider independent limits. Since we have assumed that $g_l < 0$, for given ρ when

$\gamma \to \infty$ we incur in a violation of (8). Therefore we restrict ourselves to a range of relative risk aversion that ensures finite stock prices, $[0, \bar{\gamma})$, where $\bar{\gamma} > 1$ is defined as the CRRA such that:

$$1 + \rho = \pi_t \left[(1 + g_h)^{1-\bar{\gamma}} \right] + (1 - \pi_t) \left[(1 + g_l)^{1-\bar{\gamma}} \right] \tag{18}$$

Therefore we will take limits as $\gamma \nearrow \bar{\gamma}$ (from the left). As for the limit of expected asset returns as $\gamma \searrow 0$ (from the right), since $g_h > 0$ it is possible to find a $\underline{\gamma} < 1$ such that:

$$1 + \rho = \pi_t \left[(1 + g_h)^{1-\underline{\gamma}} \right] + (1 - \pi_t) \left[(1 + g_l)^{1-\underline{\gamma}} \right] \tag{19}$$

and (8) fails to hold for $\gamma < \underline{\gamma}$. We write $\gamma \searrow \underline{\gamma}$, under the understanding that $\underline{\gamma}$ may be zero.

The following result characterizes the basic properties of asset returns in this artificial economy. To simplify its statement, define γ^f and γ^e as the coefficients of relative risk aversion such that:

$$-\pi_t(1 + g_h)^{-\gamma^f} \ln(1 + g_h) - (1 - \pi_t)(1 + g_l)^{-\gamma^f} \ln(1 + g_l) = 0 \tag{20}$$

$$-\pi_t(1 + g_h)^{1-\gamma^e} \ln(1 + g_h) - (1 - \pi_t)(1 + g_l)^{1-\gamma^e} \ln(1 + g_l) = 0, \tag{21}$$

if solutions to the equations exist.

Proposition 1. *Under full-information rational expectations:*

(a) Given $\rho \geq 0$, for $\gamma < \gamma^f$, $r^{f,FI}(\rho,\gamma)$ is an increasing function of γ, while for $\gamma > \gamma^f$, $r^{f,FI}(\rho,\gamma)$ decreases in γ so that $r^{f,FI}(\rho,\gamma) < 0$ is possible. If γ^f is not defined, then $r^{f,FI}(\rho,\gamma)$ is always monotone decreasing in γ.

(b) Independently of other conditions, $E[r_t^{p,FI}(\rho,\gamma)] \geq 0$, $E[r_t^{p,FI}(\rho,0)] = 0$. $\gamma^f \leq \gamma^e$ so that, given $\rho \geq 0$, there exists a $\gamma^{\max}$ ($\gamma^f \leq \gamma^{\max} \leq \bar{\gamma}$) such that for γ below $\gamma^{\max}$ $E[r_t^{p,FI}(\rho,\gamma)]$ is an increasing function of γ, while for $\gamma > \gamma^{\max}$, $E[r_t^{p,FI}(\rho,\gamma)]$ decreases in γ.

Proof: See Appendix.

The proposition offers direct implications for the possibility to produce under full information a risk-free rate and an equity premium consistent with the evidence. The naive notion that a high coefficient of relative risk aversion γ can offer a way to resolve the puzzles does not apply to our model. From (b), we can only hope that $\bar{\gamma}$ is high enough to span an interval that includes a $\gamma^{\max}$ such that a risk premium as high as in the data obtains. Section 7 examines whether in a standard calibration exercise a model with FI does stand any chance to match observed features of asset returns series.

6.2. *Rational Learning*

Under RL preferences impact on equilibrium prices in a way that depends on the state of beliefs. From (15), $S_t^{RL} = \Psi^{RL}(n_t^{\tau_b}, N_t^{\tau_b}; \rho, \gamma) D_t$. Therefore — assuming the absence of a structural break between t and $t+1$ — the excess stock return over the interval $[t, t+1]$ is:

$$r_{t+1}^{p,RL}(n_t^{\tau_b}, N_t^{\tau_b}, n_{t+1}^{\tau_b}; \rho, \gamma) = \frac{(1 + g_{t+1})}{\Psi^{RL}(n_t^{\tau_b}, N_t^{\tau_b}; \rho, \gamma)} + \frac{\Psi^{RL}(n_{t+1}^{\tau_b}, N_t^{\tau_b}+1; \rho, \gamma)}{\Psi^{RL}(n_t^{\tau_b}, N_t^{\tau_b}; \rho, \gamma)}(1+g_{t+1}) - 1 - r_t^{f,RL}(n_t^{\tau_b}, N_t^{\tau_b}; \rho, \gamma). \tag{22}$$

On a learning path, *realized* excess equity returns depend on both agent's initial beliefs $(n_t^{\tau_b}, N_t^{\tau_b})$ as well as on their change between t and $t+1$ (through the ratio $\Psi^{RL}(n_{t+1}^{\tau_b}, N_t^{\tau_b} + 1)/\Psi^{RL}(n_t^{\tau_b}, N_t^{\tau_b})$).

A different concept is the equity premium *expected* at time t, conditional on the information on the process for dividends then available $(n_t^{\tau_b}, N_t^{\tau_b})$:

$$\hat{E}_t^{\tau_b}\left[r_{t+1}^{pRL,}(\rho,\gamma)\right] = \frac{\hat{E}_t^{\tau_b}[(1+g_{t+1})]}{\Psi_t^{RL}} + \frac{\hat{E}_t^{\tau_b}[(1+g_{t+1})\Psi_{t+1}^{RL}]}{\Psi_t^{RL}} - \frac{1+\rho}{\hat{E}_t^{\tau_b}[(1+g_{t+1})^{-\gamma}]}. \tag{23}$$

Notice that (23) represents a *subjective* notion of the equity premium as a subjective expectation operator $\hat{E}_t^{\tau_b}[\cdot]$ that depends on the agents' information set $(n_t^{\tau_b}, N_t^{\tau_b})$ is employed. Importantly,

$$E\left[r_{t+1}^{p,RL}(\rho,\gamma)\right] \neq E\left\{\hat{E}_t^{\tau_b}\left[r_{t+1}^{p,RL}(\rho,\gamma)\right]\right\},$$

i.e. the *objective* unconditional equity premium (left-hand side) on a rational learning path differs from the *objectively expected subjective* equity premium (right-hand side). This difference raises an important issue. Traditionally, the literature has discussed the circumstances under which a model generates a stationary distribution for excess returns matching sample moments. In particular, the equity premium $E[r_{t+1}^{p}(\rho,\gamma)]$ is identified with a long-run sample mean of excess equity returns. While $E[r_{t+1}^{p,RL}(\rho,\gamma)]$ can be quantified by simulating prices in (22) and (16) and taking averages over simulation trials, this quantity is in general different from the estimate of $E\left\{\hat{E}_t^{\tau_b}\left[r_{t+1}^{p,RL}(\rho,\gamma)\right]\right\}$ that can be similarly obtained by simulation. Although the average (expected) subjective equity premium may be possibly interesting in itself (see Abel (2002) and Section 8.4), it is clear that the only quantity that can be directly compared to the data is $E[r_{t+1}^{p,RL}(\rho,\gamma)]$.

While the current section reports results for the objective *conditional* expectation of the equity premium under RL, Section 7 uses simulations from a calibrated version of our model to produce results on the objective *unconditional* equity premium to be compared with the evidence from Section 2. A Lemma in Guidolin and Timmermann (2003b) shows that Ψ_t^{RL} is an nondecreasing and convex function of $\widehat{\pi}_t^{\tau_b}$ when $\gamma \leq 1$, and a decreasing and convex function of $\widehat{\pi}_t^{\tau_b}$ when $\gamma > 1$. Based on this result, it is possible to prove:

Proposition 2. *$\gamma < 1$ and pessimistic beliefs ($\widehat{\pi}_t^{\tau_b} < \pi_t$) imply*

$$\begin{aligned} r_t^{f,RL}(\rho,\gamma) &< r^{f,FI}(\rho,\gamma) \\ E_t[r_{t+1}^{p,RL}(\rho,\gamma)] &> E[r_{t+1}^{p,RL}(\rho,\gamma)]. \end{aligned}$$

Proof: See Appendix.

We are able to isolate a combination of risk-aversion and beliefs — low risk aversion and pessimism — for which the conditional risk premium is higher under RL than the (unconditional) risk premium under FI. In principle, the incorporation of learning effects points in the direction of higher equity premia for plausible preferences, provided the economy is characterized 'on average' by some degree of pessimism. Moreover, under the same assumptions, $r_t^{f,RL}(\rho,\gamma|n_t^{\tau_b}, N_t^{\tau_b}) \leq r^{f,RL}(\rho,\gamma)$ should make the occurrence of a risk-free rate puzzle unlikely.

Contrary to the FI case, under RL it is extremely difficult to characterize the behavior of equilibrium expected returns and of the equity premium as a function of preference parameters only. In fact, no analog to Proposition 1 can be proven because ρ and γ have effects on asset returns that depend on the state of beliefs. For instance, if there is pessimism and relative risk aversion is progressively *lowered* towards zero, the conditional premium *can increase*, a somewhat counter-intuitive result. The intuition is that as an economy becomes risk-neutral, the intertemporal elasticity of substitution $(1/\gamma)$ diverges and $r_t^{f,RL}(\rho,\gamma) \nearrow \rho$ as the demand for bonds increases for consumption smoothing. As for stock returns, the

RL pricing kernel increases in $\widehat{\pi}_t^{\tau_b}$ faster and faster as $\gamma \searrow 0$. However, as we approach risk-neutrality, the growing Ψ_t^{RL} reduces the contribution of dividend growth to stock returns. The first effect reflects the fact that under near-risk-neutrality the agents will revise the kernel $\Psi_t^{RL}(\widehat{\pi}_t^{\tau_b})$ more heavily as $\hat{\pi}_t \nearrow \pi$, i.e. starting from pessimistic beliefs learning gives a stronger contribution to high realized excess returns through the elimination of the undervaluation. The second effect reflects the pure decline of dividend yields as we approach risk-neutrality. Although which of the two effects dominates is a function of g_h, g_l, ρ, and $\widehat{\pi}_t^{\tau_b}$, if γ is close to zero the former effect will prevail and the conditional premium increases as $\gamma \searrow 0$. Since also the risk-free rate decreases, the equity premium increases.

7. Simulations

7.1. *Model Calibration*

We set a yearly $\rho = 0.02$, which implies an annual discount factor $\beta \simeq 0.98$, a common value in the literature. In the tradition of MP, we experiment with alternative levels of relative risk aversion. Suppose that real consumption growth changes at *quarterly* frequency. So its *annual* rate of growth follows a transformation of $Bi(4, \pi)$ process. We condition our exercise on the fact that US investors observed the two structural breaks in real consumption uncovered in Section 2.1. Therefore we perform a double calibration exercise: the first with reference to the period 1938-1981, the second with reference to the period 1982-1999.

With reference to the period 1938-1981, we calibrate the quarterly growth process by taking $g_h = +1.5\%$, $g_l = -1.25\%$, and $\pi = 0.645$. These parameters guarantee an annual mean growth of 2%, an annual volatility of 2.6%, while the annual growth rate is positive 87% of the time. On a learning path initial beliefs n_0 and N_0 concur to determine the equity premium. During the depression period 1930-1937, the average real consumption growth had been -0.5% and growth had been negative half of the time. Undoubtedly, beliefs had to be pessimistic. In particular, the $\widehat{\pi}_0^{1938}$ that makes the assumed *quarterly* process for dividends compatible with real consumption declining 50% of the time at an *annual* level is:

$$\sum_{i=0}^{4} I_{\{(1.011)^i(0.995)^{4-i}<0\}} \binom{4}{i} \left(\widehat{\pi}_0^{1938}\right)^i \left(1-\widehat{\pi}_0^{1938}\right)^{4-i} = \left(1-\widehat{\pi}_0^{1938}\right)^4 + 4\widehat{\pi}_0^{1938}\left(1-\widehat{\pi}_0^{1938}\right)^3 \cong 0.50. \tag{24}$$

$\widehat{\pi}_0^{1938} = 0.35$ approximately satisfies (24) and implies a slightly negative expected real growth of consumption (-0.3%), a pessimistic albeit not extreme belief. $\widehat{\pi}_0^{1938}$ must be assigned some weight, a measure of its strength against subsequent information. Although the experience of the Great Depression is likely to have left a big mark on collective beliefs, we limit its precision to 32, the number of quarters in the cycle 1930-1937 (NBER dating). Therefore $\hat{n}_0^{1938} = 11$ and $\hat{N}_0^{1938} = 32$ so that $\hat{n}_0^{1938}/\hat{N}_0^{1938} \simeq 0.35$. In our view, calibrating a pessimistic belief for the post-Depression era is merely a way to give content to the claim that "(...) the experience of the Great Depression (...) [had] a significant influence on the behaviour of those who experienced it directly or indirectly (...)" (Danthine and Donaldson (1999, p. 608)).

The second calibration exercise concerns the period following the structural break determined by the oil shocks of the 1970s. We calibrate the quarterly process of consumption growth by taking $g_h = +1.5\%$, $g_l = -1.25\%$, and $\pi = 0.63$. These parameters imply an annual mean of 1.9% and annual volatility of 2%. During the cycle 1974-1981, average real consumption growth has been lower than in the 1960s and recollection of the pessimistic evaluations of the growth slowdown of those years makes a parameterization of the initial beliefs reflecting some degree of pessimism plausible. It is easy to verify that $\widehat{\pi}_0^{1982} = 0.5$

produces an annual mean growth rate of 0.4%, a volatility of 2.8%, and positive rates of annual growth 69% of the time. These features closely match the process of real fundamentals during the late 1970s. We also set $\hat{N}_0^{1982} = 8$ (the number of quarters in the cycle 1980-1981) and therefore $\hat{n}_0^{1982} = 4$.

7.2. *The Equity Premium under Full Information*

In the full-information case, it is straightforward to calculate $\underline{\gamma} = 0$ and $\bar{\gamma} \simeq 58.8$: outside this interval the FI equilibrium fails to exist. Figure 3 depicts the equity premium and the risk-free rate under FI when γ changes and no breaks are imposed. Notice that under FI no simulations are required since the expectations involved by (17) can be directly evaluated. The equity premium puzzle admits no solution: an average risk-free rate below 1% per year can be attained only using a constant relative risk aversion above 57. Incidentally, the available window is also very narrow as $\bar{\gamma} \simeq 58.8$ and for $\gamma > 58$ the real risk-free rate becomes negative, which is counterfactual. However, even for risk aversion coefficients as high as 58, the ex-ante expected risk premium is at most 3.8%. Assuming that the consumption growth process is adequately described by a binomial lattice, it is impossible to find a level of risk aversion such that ex-post realized excess returns in the order of 7% are generated.

7.3. *The Equity Premium under Rational Learning*

For alternative levels of γ, we simulate $Z = 10,000$ independent, quarterly time paths for real dividends and equilibrium asset prices when the agent is on a rational learning path and breaks occur in 1938 and 1982. After each break, initial beliefs are calibrated to plausible values as of January 1938 ($\hat{n}_0^{1938} = 11$, $\hat{N}_0^{1938} = 32$) and January 1982 ($\hat{n}_0^{1982} = 4$, $\hat{N}_0^{1982} = 8$). Fundamentals are drawn according to the parameters discussed in Section 7.1. The time paths have a length equal to the post-depression period 1938-1999, 248 quarters. Since the statistical properties we match refer to annual series, after simulating 248-quarter long series for dividends and prices, we aggregate them to obtain 62-year long annual series.

One issue that arises when assessing asset pricing properties on a learning path by simulation is the existence of the equilibrium *along the entire simulated path.* Indeed the market belief $\widehat{\pi}_t^{\tau_b}$ changes as new realizations of the growth process come along. From Section 5 we know that given $\rho > 0$ and $\widehat{\pi}_t^{\tau_b}$, $\gamma \neq 1$ could be chosen either too large or too small in order for the equilibrium to exist at time t. In particular, when beliefs are strongly pessimistic ($\widehat{\pi}_t^{\tau_b} << \pi$), $\gamma >> 1$ might be excessive to support the equilibrium. It is also possible that strongly optimistic beliefs ($\widehat{\pi}_t^{\tau_b} >> \pi$) might disrupt the RL equilibrium when $\gamma << 1$. The occurrence of any violation at any point of a simulated path $t = 1, ..., T$ invalidates the ability of the path itself to represent an equilibrium outcome from an artificial RL economy. We handle the issue in a pragmatic way. First, we limit the simulations to an interval for the coefficient of relative risk aversion such that divergence of (15) is unlikely, $\gamma \in [0.3, 2]$. This is also the interval including values of the coefficient of relative risk aversion that are commonly thought of as plausible. Second, we check convergence of (15) monitoring the progressive shrinking of the contribution to S_t^{RL} of successive terms in (15).

Figure 4 plots the unconditional premium and the risk-free rate when agents are on a rational learning path and perceive two breaks vs. the FI case. In practice, we report the quantity

$$\frac{1}{Z}\sum_{j=1}^{Z}\frac{1}{T}\sum_{t=1}^{T} r_{t,j}^{p,RL}(n_{t-1,j}^{\tau_b}, N_{t-1}^{\tau_b}, n_{t,j}^{\tau_b})$$

$$r_{t,j}^{p,RL}(n_{t-1,j}^{\tau_b}, N_{t-1}^{\tau_b}, n_{t,j}^{\tau_b}) \equiv \frac{(1+g_{t,j})}{\Psi^{RL}(n_{t-1,j}^{\tau_b}, N_{t-1}^{\tau_b})} + \frac{\Psi^{RL}(n_{t,j}^{\tau_b}, N_{t-1}^{\tau_b}+1)}{\Psi^{RL}(n_{t-1,j}^{\tau_b}, N_{t-1}^{\tau_b})}(1+g_{t,j}) - 1 - r_{t,j}^{f,RL}(n_{t-1,j}^{\tau_b}, N_{t-1}^{\tau_b}),$$

where $j = 1, 2, ..., Z$ indexes simulation paths, and $n_{t,j}^{\tau_b}$ evolves randomly on each path. Other unconditional moments are defined similarly. 90% confidence bands are also plotted. As for the FI values, they are obtained by simulation as well since the occurrence of breaks slightly changes results relative to Figure 3. As shown by Proposition 2, $\gamma < 1$ combined with pessimism pushes the RL premium higher than under FI. In particular, the RL equity premium is decreasing in $\gamma < 1$, so a moderate curvature of the utility function is consistent with generating high excess returns. For instance, for $\gamma = 0.3$ the equity premium over the 62 years covered by the exercise is 5%, a remarkable result in the premium literature. A 90% confidence interval generated from the distribution of the simulated equity premia under RL is wide ($[3, 7.1]$), including premia close to the 7.6% target reported in Section 2 (and represented by a solid bar in the plots). 51% of the simulations generated equity premia in excess of 5%, and many (about 24%) were above 7%. For γs above 1.3 the equity premium becomes negative, an indication that downward revisions of Ψ_t^{RL} as $\widehat{\pi}_t^{\tau_b}$ moves (on average) towards π reduce realized excess returns. The annual rate of return on short-term bonds is always above 2 percent, above the 1 percent observed over the period 1938-1999. In fact, under both RL and FI, it takes a downward drifting endowment process in order to produce equilibrium values of the risk-free rate below 2%. However, Figure 4 also shows that — relying on a low γ — a learning-based explanation does not incur in the risk-free rate puzzle.

A point often insisted upon (Cecchetti et al. (2001)) is that proposed resolutions to the equity premium puzzle have been moderately successful at reproducing first moments but that difficulties remain when it comes to match higher-order moments, in particular variances. Under FI and regardless of the presence of breaks, equity returns inherit the stochastic properties of the endowment process. Since the growth in fundamentals is assumed to be i.i.d. and is calibrated to match the smoothness of the US economy, it implies not only a non-volatile, i.i.d. process for excess returns, but also a constant interest rate. Hence the FI model stands no chance to reproduce the excess volatility of stock returns vs. consumption growth. On the opposite, under the RL model the volatility of real stock returns and excess equity returns can be easily matched (at roughly 18%). Moreover, realistic variability in the equilibrium interest rate appears.

7.4. *Matching Other Properties of Asset Returns*

A point often insisted upon (Cecchetti et al. (2001)) is that proposed resolutions to the equity premium puzzle have been moderately successful at reproducing levels of the risk-free rate and of the risk-premium, although difficulties remain when it comes to match higher-order moments, for instance variances. Since Table I has also reported other descriptive statistics concerning real stock returns, excess returns, and short-term yields, we engage in the same type of evaluation for asset returns simulated from the model. Given preferences $[\rho\ \gamma]'$, for each simulation trial we calculate descriptive statistics $\chi_j(\rho, \gamma|\hat{n}_0, \hat{N}_0)$ ($j = 1$, ..., Z) and report averages $\bar{\chi}(\rho, \gamma|\hat{n}_0, \hat{N}_0) \equiv Z^{-1}\sum_{j=1}^{Z}\chi_j(\rho, \gamma|\hat{n}_0, \hat{N}_0)$. Figures 5-7 plot the following (average) statistics as a function of relative risk aversion: excess returns standard deviation, the percentage of simulations for which the null hypothesis of zero serial correlation is rejected at a nominal size of 5% (using the Ljung-Box statistic of order 8), the percentage simulations for which the null of no serial correlation in squared asset returns is rejected at 5% (using the LB(8) statistic, interpreted as a test of volatility clustering), and the correlation between interest rates and excess returns.

Under FI and regardless of the presence of breaks, the model is clearly incapable of capturing some stylized facts: Equity returns simply pick up the stochastic properties of the assumed process of endowment growth. Since the growth in fundamentals is assumed to be i.i.d. and is calibrated to match the smoothness characterizing the US economy, it implies not only a non-volatile, i.i.d. process for excess returns, but also a constant interest rate. Hence the FI model stands no chance to pick up interesting stylized facts, such as excess volatility of stock returns vs. consumption growth, and the rich statistical properties of the real risk-free rate.[10] As far as RL is concerned, Figure 5 shows that for a small $\gamma < 1$ the volatility of real stock returns and excess equity returns can be easily matched. However this conclusion does not fully apply to the risk-free rate: although RL produces time variation, it is insufficient. On the other hand, it is remarkable that learning can generate sufficient variation in real stock prices at the same time matching the volatility of fundamentals and not producing excessive variation in riskless interest rates, a result that has proven elusive in previous research (see Cecchetti et al. (1990)).

Figure 6 focuses on serial correlation and volatility clustering. Table I shows that while annual real and excess returns on equities display no sign of serial correlation or ARCH, the opposite holds for the risk-free rate that has long memory. Under RL, simulated real and excess stock returns display some mild structure in the first two moments when γ is small, as evidenced by a percentage of simulations between 30 and 50 that show a significant Ljung-Box statistic at 5%. As γ increases above one, these figures rapidly increase above 90%, sign of strong and counterfactual correlation and volatility clustering. In the case of the risk-free rate, independently of γ almost 100 percent of the simulations display significant correlation both in returns and in squared returns. Finally, Figure 7 plots the average simulated correlation coefficient between excess returns and the real-risk free rate. In this case, the stylized fact to be matched is a small, negative correlation (-0.05). It is clear that FI has an advantage, since (apart from breaks) the FI risk-free rate is constant and uncorrelated with any other random variable. On the other hand, the graph shows that for small values of γ the average simulated correlation is low (-0.15), as required. We take Figure 7 as evidence favorable to a learning based explanation of US asset returns in the XXth century.

7.5. *A Path Calibration*

As a more stringent test of the model's predictive ability, we perform a path-calibration: since realized consumption growth rates are observed for every year of the period 1938-1999, we fit the binomial lattice to the data and let our representative agent learn π by using the sign of realized changes in consumption to infer whether g_t equals either $g_h > 0$ or $g_l < 0$. This calibration strategy is similar to Brennan and Xia (2001). Figure 8 shows the dynamics of $\widehat{\pi}_t^{\tau_b}$ implied by US consumption data at *annual* frequency. The effects of the Great Depression are evident, as $\widehat{\pi}_t^{\tau_b}$ starts declining in 1929. The big jump at the end of the 1930s is also caused by the assumption that agents perceived a break in 1938 and re-started their estimation from pessimistic beliefs ($\hat{n}_0^{1938} = 2$ $\hat{N}_0^{1938} = 8$, annualizing quarterly values). Despite a slowdown at the end of the 1940s, the three decades following 1938 are marked by rapid upward revisions of $\widehat{\pi}_t^{\tau_b}$. By the early 1970s, $\hat{\pi} \simeq 0.75$ implying a perceived high mean growth rate and moderate volatility. The oil shocks end this booming period and induce an early 1980s break, characterized by mild pessimism ($\hat{n}_0^{1982} = 1$ $\hat{N}_0^{1982} = 2$ consistently with previous choices).

[10] The FI statistics simply consist of straight lines that do not depend on the relative risk aversion coefficient.

The right panel shows equilibrium stock prices obtained assuming $\rho = 0.02$ and $\gamma = 0.3$. For most of the sample period, FI prices stay above the RL ones. When at the end of the 1990s the perception of π eventually catches up with values consistent with the statistical properties of US consumption data, RL and FI prices converge. The (unreported) simulated riskless rate shows that as pessimism is imposed, the RL rate remains below the FI rate, barely exceeding a plausible level of 3% on average. We extend the exercise and calculate equity premia and excess return volatility for all levels of γ in [0.3, 2].[11] Results are qualitatively similar to those in Section 7.3 and are quantitatively interesting for low risk aversion. When $\gamma = 0.3$ we find an equity premium of 3.3% and an average riskless interest rate of 3.2% for the period 1938-1999. The corresponding values under FI are 0.7% and 3.6%. As obtained before, the equity premium is monotone decreasing in γ (over the [0.3, 2] interval). Although an equity premium of 3.3% does not entirely solve MP's puzzle, the ability of the model to explain roughly half of MP's puzzle is encouraging.

7.6. *Dynamic Properties*

A further set of restrictions implied by RL can be tested: on one hand, since learning is stronger in the aftermath of structural breaks, the data should display deviations from the unconditional (full-sample) statistical properties – such as higher than average equity premia and volatility – over the periods following breaks; on the other hand, since we have calibrated initial beliefs to reflect some pessimism in the aftermath of breaks, our simulations ought to generate these stronger deviations from FI.

We study these implications in two ways. First, Figure 9 shows that there is evidence in the data of higher equity premia and volatility in the aftermath of breaks (the solid vertical lines in the plots). The top graph plots 10- and 15-year *forward* rolling window equity premia calculated by collecting partial samples at each date between 1930 and 1990 and averaging excess equity returns over these intervals.[12] The bottom graph does the same with reference to sample standard deviations. The solid horizontal lines provide unconditional benchmarks, 7.1% and 19.5% for the equity premium and volatility. Clearly, all potential breaking dates are followed by above average conditional equity premia, both on 10- and on 15-year sub-samples. For instance, using 15-year windows 1938 is followed by a 11% premium, 1982 by a 10.1% premium. Results for volatility are instead mixed: while the 1930s break is certainly followed by above-normal volatility, this does not happen for other breaks, notably for the one in the early 1980s. In this sense, our model seems to propose a plausible explanation for the high equity premium phenomenon but shows some difficulty at generating the correct dynamic volatility patterns.

Second, we use the path calibration of Section 7.5 to measure a few properties of excess returns over periods that follow the two structural breaks; we use also in this case two identical 15-year long sub-samples, i.e. 1938-1952 and 1982-1996. For ease of exposition we report a single arithmetic average over the two periods. We find indeed evidence of stronger deviations from FI in the aftermath of breaks: for $\gamma = 0.3$ the equity premium is 7.1% and the standard deviation of excess returns is 22.7%.

[11] We set the parameters as: $g_h = +3.1\%$ and $g_l = -0.7\%$; although the choice of π is irrelevant for RL prices, it matters for FI results and we pick $\pi = 0.795$. For the period 1982-1999 we set identical g_h and g_l, and $\pi = 0.78$.

[12] For instance, we consider the 1930-1939 (10-year window) and 1930-1944 (15-year window) samples, calculate mean excess equity returms, and report the results in correspondence to 1930.

8. **Discussion**

8.1. *The Role of Initial Beliefs*

We perform robustness checks on initial beliefs: Would the results be stronger if beliefs in the aftermath of the Great Depression had been even more pessimistic than assumed? To provide some answers to these questions, we set two alternative values of γ (0.2 and 2) and proceed to calculate $E[r_t^{p,RL}(\rho,\gamma)]$ and $E[r_t^{f,RL}(\rho,\gamma)]$ with $\hat{N}_0 = 32$ as in Section 7.3 but with $\hat{n}_0$ that changes between $\hat{n}_0 = 9$ ($\hat{\pi}_0 = 0.28$) and $\hat{n}_0 = 22$ ($\hat{\pi}_0 = 0.69$) in steps of two (i.e. $\hat{n}_0$=11, 13, etc.) whenever possible.[13] Figure 10 shows that when $\gamma < 1$, pessimistic initial beliefs are not required. Even when $\hat{\pi}_0 = \pi$, $\gamma = 0.2$ gives an equity premium of 2 percent, with a 90% confidence band wide enough to include premia of 3.5 percent. Moreover, should we find a reason to specify beliefs more pessimistic than $\hat{\pi}_0^{1938} = 0.35$, we would provide complete solution to the equity premium puzzle. When $\gamma = 0.2$ and $\hat{\pi}_0 = 0.28$, the equity premium is 6 percent and the 90% confidence band spans the interval $[3.2, 8]$. Finally, for $\gamma = 2$ even optimism does not help. For instance $\hat{\pi}_0 = 0.59$ produces a negative premium. Our intuition is that optimism raises the risk-free rate faster than expected stock returns, thus reducing the premium.

8.2. *Integrating out Initial Beliefs*

We study whether rational learning can contribute to our understanding of the equity premium and risk-free rate puzzles when no restrictions on initial beliefs are imposed and time τ_b beliefs are integrated out. Suppose we take an agnostic view on beliefs at the beginning of 1938, admitting pessimism and optimism in equal degrees. Assume that given $\hat{N}_0^{1938} = 32$, the initial belief $\hat{\pi}_0^{1938}$ could have been with equal probability any value in the interval [0.445, 0.845], symmetric around the unknown $\pi = 0.645$, i.e. $\hat{\pi}_0^{1938}$ is assumed to have a uniform prior density.[14] We evaluate asset returns on a rational learning path by simulating time series for the real endowment and equilibrium prices 50,000 times, when in correspondence to each simulation the initial belief is drawn afresh from a uniform prior in independent fashion. Thus while Section 7.3 has focused on $E\left[r_t^{p,RL}(\rho,\gamma|\hat{\pi}_0)\right]$ taking $\hat{\pi}_0$ as given, we now calculate:

$$E\left[r_t^{p,RL}(\rho,\gamma)\right] = \int_{0.445}^{0.845} E\left[r_t^{p,RL}(\rho,\gamma|\hat{\pi}_0)\right] d\hat{\pi}_0, \tag{25}$$

where $\hat{\pi}_0 = \hat{n}_0/\hat{N}_0$.[15] Again, we set $\rho = 0.02$ and vary γ over the interval [0.3, 2].

Figure 11 gives an encouraging picture. Even imposing no assumptions on initial beliefs, a model incorporating learning effects gives an appreciable contribution to explain the two asset pricing puzzles, provided γ is less than one. For a low γ, the equity premium exceeds 3% and the 90% confidence band is [1.7, 5.1]. The risk-free rate is 3.4%. Interestingly, these unconditional expectations are obtained without imposing absurd degrees of curvature on the utility function. This result is made possible by the combination of two factors: Firstly, the equilibrium risk-free rate is low when γ is low independently of the state of beliefs; Secondly, when $\gamma < 1$ the price-dividend ratio is increasing and convex in $\hat{\pi}_t$, implying that upward

[13] The infinite sum defining the RL pricing kernel may not converge. In fact with very low γs (such as 0.2) the equilibrium is unlikely to exist for high $\hat{n}_0 s$, i.e. optimistic beliefs.

[14] The length of this interval is arbitrary. However all other beliefs seem to be extreme and implausible. For instance, $\hat{\pi}_0 = 0.845$ implies a yearly mean growth rate of 4.4%, which is rather exceptional for a developed country.

[15] We apply a similar randomization to initial beliefs as of 1982. Given $\hat{N}_0^{1982} = 8$, the initial belief $\hat{\pi}_0$ is drawn with equal probability on the interval [0.43, 0.83], symmetric around the unknown $\pi = 0.63$.

revisions of beliefs typical of pessimistic economies will have a stronger (positive) effects on equity returns than the downward revisions that dominate in optimistic economies.

8.3. *Number and Dating of the Breaks*

Up to this point we have identified 1938 and 1982 as the dates in which breaks in the endowment process occurred. Both breaks occur during protracted and deep recession phases (according to official NBER dating): the first break between 1937 and 1938 (cycle 1933-1937), the second between 1980 and 1981 (cycle 1981-1982). However, the two dates used in our calibration were selected as the year(s) containing the end quarter of the recession periods during which the break was perceived. One might wonder about what happens to the number and nature of the breaks in the case in which the first break is associated with (say) 1932 instead of 1938. Conditional on a 1932 break, we perform a statistical analysis similar to Section 2.1 (still taking the minimum no-break period to be $\iota = 20$ years) and uncover some evidence of a break in the drift parameter μ in the early 1950s, particularly in 1954. Interestingly, the 1954 break is another 'negative' break in the sense that it can be once more characterized by a downward revision of growth expectations on the US economy: while during the New Deal and during WWII fundamentals grew at high rates (e.g. the implied unconditional growth rate is 5.4% over the interval 1933-1946), after the end of WWII the US economy experienced a structural slowdown that agents might have perceived as a break. For instance, using data for the period 1933-1954, the implied unconditional growth rate would have been 2.9% only, indication of a remarkable slowdown in 1947-1954. When we condition on a break in 1954, there is once more evidence of a third break in correspondence of the oil shocks, although some uncertainty now exists on the dating: while the drift parameter implies a break as early as 1974, the AR(1) coefficient gives weak indication for 1975 and strongly signals a break as late as 1982, after the second shock. In any event, the entire period 1974-1982 matches a famous episode of slowdown of the US economy (see Maddison (1987)).

What matters for our purposes are the asset pricing effects of a third break. Notice that the early 1950s represent a period in which the US economy cools off after the rapid growth caused by the war effort. Therefore, if perceived by the agents, the 1954 break is likely to have been accompanied by relatively pessimistic beliefs: 1948, 1949, and 1954 were all recession years with nonpositive real consumption growth; while in 1933-1946 the average annual consumption growth had been 3%, in the interval 1947-1954 it declines to only 0.4%. Similarly to Section 7.1, we set $\hat{\pi}_0^{1954} = 0.5$ and $\hat{N}_0^{1954} = 32$ (hence $\hat{n}_0^{1954} = 16$), corresponding to the sequence of short recession periods characterizing the interval 1947-1954. We then repeat the simulation experiments.[16] For $\gamma = 0.3$ the equity premium over the 68 years covered by the exercise is 4.7% while the riskless interest rate is 3.2%. The annualized volatility of excess equity returns is 18.8%. Roughly 60% of the simulations exceed 5%. Hence our results on the possibility of generating equity premia in the order of 5% and interest rates below 3% with low risk aversion do not depend on either the exact number and location of the breaks or on the details of the calibration of initial beliefs.

8.4. *Doubt, Pessimism and Rational Crash Fears*

Abel (2002) shows that pessimism and doubt on the distribution of future consumption growth rates may provide a solution to the puzzles. It is therefore interesting to link our findings to Abel's and show that rational learning endogenously generates pessimism and doubt. Figure 12 shows the evolution over the

[16] We set $g_h = +1.5\%$, $g_l = -1.25\%$, $\pi = 0.635$ (1932-1954), $\pi = 0.665$ (1955-1981), and $\pi = 0.63$ (1982-1999).

interval 1938-1999 of the subjectively perceived distribution of the 5- and 50-years ahead real consumption growth rate when the representative agent is under RL, $\{P^{RL}(C^j_{t+i}/C_t|\widehat{\pi}^{\tau_b}_t, N^{\tau_b}_t)\}^i_{j=0}$, $i = 20$, 200. These distributions are obtained as averages of distributions calculated according to (13) along the 10,000 sample paths of Section 7.3. The same calibration is used. Right plots show a few selected subjective distributions compared to the FI (approximately) normal benchmark. The support of the distributions has been re-scaled to display annualized growth rates.

Abel (2002) defines pessimism as the case in which the RL predictive distribution is first-order stochastically dominated by the FI one. Pessimism reduces the equilibrium risk-free rate. Figure 12 shows that under rational learning pessimism clearly dominates. Although not reported, the implied cumulative distribution functions display the desired pattern of stochastic dominance. Of course, the effect is stronger in the 1940s and again in the 1980s, but it seems that a rational agent might have underestimated the overall location of the distribution of future growth rates for long periods. The effects on the risk-free rate are qualitatively similar, as shown by Proposition 4, provided $\gamma < 1$. Abel (2002) strengthens his definition to uniform pessimism, when the subjective distribution lies entirely to the left of the objective one, with no contact points. Uniform pessimism is sufficient to inflate the equity premium. Figure 12 stresses that uniform pessimism obtains at many dates in our calibration. The effect on the equity premium is similar and obtains through the convexity of the RL pricing kernel for $\gamma < 1$: since upward revision of beliefs increase prices more than downward revisions, in a pessimistic economy the former are more likely than the latter and this impresses a substantial upward drift to equilibrium stock prices.

Abel defines doubt as the case in which the RL predictive distribution is a mean-preserving spread of the FI one. He shows that since the pricing kernel is convex, doubt will decrease the risk-free rate and increase the equity premium. Figure 12 provides evidence that, independently of their location, $\{P^{RL}(C^j_{t+i}/C_t|\widehat{\pi}^{\tau_b}_t, N^{\tau_b}_t)\}^i_{j=0}$ describes a leptokurtic distribution with much thicker tails than the FI benchmark. Once more, in our framework doubt is reflected in a higher equity premium because for $\gamma < 1$ the pricing kernel Ψ^{RL}_t is a convex function of $\widehat{\pi}_t$. Therefore when rational learning is supplemented with historical evidence on the US economy, pessimism and doubt do emerge in endogenous fashion, increasing the risk premium on equities for moderate degrees of curvature of the utility function.

9. **Conclusion**

This paper shows that there exists an alternative way in which extreme events such as the Great Depression or the oil shocks can generate high equity premia. While previous literature has focused on the induced, permanent biases in the stationary beliefs of investors in an *ad hoc* fashion, we show that if agents are on a recursive learning path, tail events may produce long-lasting effects on equilibrium prices. For our calibration of beliefs in the aftermath of the depression and the oil crises, we obtain that equity premia in the order of 4 to 5 percent are compatible with complete markets, the absence of friction, and power utility with a reasonable degree of curvature. These figures come close to the original size of the equity premium pointed out by Mehra and Prescott and explain more than 60 percent of the average excess returns on stocks for the post-depression period 1938-1999. The resulting confidence bands for the equity premium expected as of 1938 are wide, including premia in the order of 8 percent. The equilibrium risk-free rate is in the order of a realistic 2 percent. The model also matches the observed variance of the risk premium and of real stock returns over the period 1938-1999, thus showing that the high volatility of real stock returns in excess of real consumption growth is no puzzle.

Section 8.2 has made our case stronger by showing that the results are only slightly weakened when no restrictions are imposed on initial beliefs and the artificial economy is simulated starting from beliefs drawn from an ignorance prior. In this case we generate equity premia in the order of 3 percent, with confidence bands wide enough to include 7 percent premia. These findings require the use of low risk aversion levels, so that we avoid falling in the criticized set of explanations that rely on high risk aversion.

References

[1] Abel, A., 2002, “An Exploration of the Effects of Pessimism and Doubt on Asset Returns”, *Journal of Economic Dynamics and Control*, 26, 1075-1092.

[2] Aiyagari, R., and M. Gertler, 1991, “Asset Returns with Transactions Costs and Uninsured Individual Risk”, *Journal of Monetary Economics*, 27, 311-331.

[3] Barsky, R., and B., De Long, 1993, “Why Does the Stock Market Fluctuate?”, *Quarterly Journal of Economics*, 108, 291-311.

[4] Brennan, M., and Y., Xia, 2001, “Stock Price Volatility and Equity Premium”, *Journal of Monetary Economics*, 47, 249-283.

[5] Bullard, J., and J., Duffy, 2001, “Learning and Excess Volatility”, *Macroeconomic Dynamics*, 5, 272-302.

[6] Campbell, J., and J., Cochrane, 1999, “By Force of Habit: A Consumption-Based Explanation of Aggregate Stock Market Behavior”, *Journal of Political Economy*, 107, 205-251.

[7] Cecchetti, S., P. Lam, and N., Mark, 1990, “Mean Reversion in Equilibrium Asset Prices”, *American Economic Review*, 80, 211-242.

[8] Cecchetti, S., P. Lam, and N., Mark, 2000, “Asset Pricing with Distorted Beliefs: Are Equity Returns Too Good To Be True?”, *American Economic Review*, 90, 787-805.

[9] Chu, C., M., Stinchcombe, and H., White, 1996, “Monitoring Structural Change”, *Econometrica*, 64, 1045-1065.

[10] Cochrane, J., 2001, *Asset Pricing*, Princeton University Press.

[11] Constantinides, G., 1990, “Habit Formation: A Resolution of the Equity Premium Puzzle”, *Journal of Political Economy*, 98, 519-543.

[12] Constantinides, G., and D., Duffie, 1996, “Asset Pricing with Heterogenous Consumers”, *Journal of Political Economy*, 104, 219-240.

[13] Danthine, J.-P., and J. Donaldson, 1999, “Non-Falsified Expectations and General Equilibrium Asset Pricing: The Power of the Peso”, *Economic Journal*, 109, 607-635.

[14] Epstein, L., and S. Zin, 1989, “Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns II: An Empirical Analysis”, *Journal of Political Economy*, 99, 263-286.

[15] Fama, E., and K., French, 2002, “The Equity Premium”, *Journal of Finance*, 57, 637-659.

[16] Gennotte, G., 1986, “Optimal Portfolio Choice Under Incomplete Information”, *Journal of Finance*, 41, 733-746.

[17] Grossman, S., and R., Shiller, 1981, “The Determinants of the Variability of Stock Market Prices”, *American Economic Review*, 71, 222-227.

[18] Guidolin, M., 2003, “High Equity Premia and Crash Fears. Rational Foundations”, mimeo, University of Virginia.

[19] Guidolin, M., and A., Timmermann, 2003a, “Option Prices Under Bayesian Learning: Implied Volatility Dynamics and Predictive Densities”, *Journal of Economic Dynamics and Control*, 27, 717-769.

[20] Guidolin, M., and A., Timmermann, 2003b, “Properties of Asset Prices Under Alternative Learning Schemes”, mimeo, University of Virginia and UCSD.

[21] Hagiwara, M., and M. Herce, 1997, “Risk Aversion and Stock Price Sensitivity to Dividends”, *American Economic Review*, 87, 738-745.

[22] Hansen, L. P., and K. Singleton, 1983, “Stochastic Consumption, Risk Aversion, and the Temporal Behavior of Asset Returns”, *Journal of Political Economy*, 91, 249-265.

[23] Kocherlakota, N., 1996, “The Equity Premium: It’s Still a Puzzle”, *Journal of Economic Literature*, 34, 42-71.

[24] Lucas, D., 1994, “Asset Pricing with Undiversifiable Risk and Short Sales Constraints: Deepening the Equity Premium Puzzle”, *Journal of Monetary Economics*, 34, 325-342.

[25] Lucas, R., 1978, “Asset prices in an Exchange Economy”, *Econometrica,* 46, 1429-1445.

[26] Maddison, A., 1987, “Growth and Slowdown in Advanced Capitalist Economies: Techniques of Quantitative Assessment”, *Journal of Economic Literature,* 25, 649-698.

[27] Mehra, R., and E., Prescott, 1985, ”The Equity Premium: A Puzzle”, *Journal of Monetary Economics*, 10, 335-359.

[28] Mehra, R., and E., Prescott, 1988, ”The Equity Risk Premium: A Solution?”, *Journal of Monetary Economics*, 22, 133-136.

[29] Mehra, R., and E., Prescott, 2003, “The Equity Premium in Retrospect”, NBER W.P. 9525.

[30] Pastor, L., and R., Stambaugh, 2001, “The Equity Premium and Structural Breaks”, *Journal of Finance*, 56, 1207-1239.

[31] Rietz, T., 1988, “The Equity Risk Premium: A Solution”, *Journal of Monetary Economics*, 22, 117-132.

[32] Salyer, K., 1998, “Crash States and the Equity Premium: Solving One Puzzle Raises Another”, *Journal of Economic Dynamics and Control*, 22, 955-965.

[33] Stock, J., and M., Watson, 1996, “Evidence of Structural Instability in Macroeconomic Time Series Relations”, *Journal of Business and Economic Statistics*, 14, 11-30.

[34] Shiller, R., 1990, *Market Volatility*, MIT Press.

[35] Timmermann, A., 1993, "How Learning in Financial Markets Generates Volatility and Predictability in Stock Prices", *Quarterly Journal of Economics*, 108, 1135-1145.

[36] Timmermann, A., 2001, "Structural Breaks, Incomplete Information and Stock Prices", *Journal of Business and Economic Statistics*, 19, 299-314.

[37] Veronesi, P., 1999, "Stock Market Overreaction to Bad News in Good Times: A Rational Expectations Equilibrium Model", *Review of Financial Studies*, 12, 975-1007.

[38] Weil, P., 1989, "The Equity Premium Puzzle and the Risk-Free Rate Puzzle", *Journal of Monetary Economics*, 24, 401-421.

Appendix A

Consider an autoregressive process for g_t, the rate of growth of real, per-capita consumption

$$g_t \equiv \frac{c_t - c_{t-1}}{c_{t-1}} = \mu + \sum_{j=1}^{L} \beta_j L^j g_t + \epsilon_t = \mathbf{x}_t' \boldsymbol{\theta}_t + \epsilon_t,$$

where c_t is real per-capita consumption, L is the standard lag operator, and ϵ_t a white noise process. Call ι the minimum time span over which the parameters $\boldsymbol{\theta}_t \equiv [\mu_t \ \beta_{1t} \ ... \ \beta_{Lt}]'$ are assumed to remain constant, i.e. $\boldsymbol{\theta}_{\tau+1} = \boldsymbol{\theta}_{\tau+2} = ... = \boldsymbol{\theta}_{\tau+\iota}$, where τ is the time of the last structural break detected by agents and $\boldsymbol{\theta}_t$ is a $L+1$ column vector while $\mathbf{x}_t \equiv [1 \ g_{t-1} \ ... \ g_{t-L}]'$. Suppose agents aim at testing the presence of a break in the regression model at time $t > \tau + \iota$ using a sequential procedure with power one.[17] Chu et al. suggest calculating the following 'fluctuation detector':

$$\hat{F}_t = (t-\tau)\hat{D}_\iota^{-1/2}\left(\hat{\theta}_t - \hat{\theta}_\iota\right),$$

where $\hat{D}_\iota = \hat{M}_\iota^{-1}\hat{V}\hat{M}_\iota^{-1}$, $\hat{M}_\iota$ is a consistent estimator of $(\iota - \tau - 1)^{-1}\sum_{j=\tau+1}^{\iota} x_j x_j'$ and $\hat{V}$ is a consistent estimator of the moment matrix $\lim_{\iota\to\infty}(\iota - \tau - 1)^{-1}E[S_\iota S_\iota']$, with $S_\iota = \sum_{j=\tau+1}^{\iota} x_j' \epsilon_j$. Under the null of no break and a few regularity conditions, $\forall t \geq \iota$, Chu et al. (1996) provide asymptotic bounds for the statistic $|\hat{F}_t^{(k)}|$, $k = 1, 2, ..., L+1$:[18]

$$\lim_{\iota\to\infty} P\left\{|\hat{F}_t^{(k)}| \geq \frac{t-\iota}{\sqrt{\iota}}\left[\left(\frac{\iota}{t-\iota}\right)\left(a^2 + \ln\iota - \ln(t-\iota)\right)\right]^{1/2}\right\} = 1 - \left[1 - 2\left[1 - \Phi(a) + a\phi(a)\right]\right]^{L+1}, \quad (26)$$

where $\Phi(\cdot)$ and $\phi(\cdot)$ are the cdf and the pdf of a standard normal random variable, respectively. a is a constant characterizing the monitoring boundary that is to be selected conveniently to guarantee a desired type I error of incorrect detection of a break when there is none. For instance, when there is only one regressor, $\alpha = 5\%$, then $a \simeq 2.79$.In practice, we can think that after at least ι observations have been received in the aftermath of a break at τ, the agents start the recursive calculation of $\hat{F}_t$ (replacing unknown moments with their sample counterparts). Given a certain α, if at $\bar{t}$ $|\hat{F}_{\bar{t}}|$ hits the monitoring boundary in (26), then the null of no structural breaks since τ fails to be rejected. $\bar{t}$ becomes then the time of the new structural break.

[17] A constant level α sequential test of power one is a stopping rule such that $P(\tau < \infty | H_0) \leq \alpha$ and $P(\tau < \infty | H_1) = 1$, where H_0 corresponds to the null of no structural break.

[18] Chu et al. (1996) show that for small ιs the finite sample size of the test is substantially lower than the nominal one. Hence a large enough ι is required.

Appendix B

Proof of Proposition 1. (a) Recall that:

$$r_t^{f,FI} = \frac{(1+\rho)}{(1+g_l)^{-\gamma} + \pi\left[(1+g_h)^{-\gamma} - (1+g_l)^{-\gamma}\right]} - 1$$

Therefore $r_t^{f,FI}$ depends on γ only through the denominator. It increases in γ as long as the denominator decreases, and viceversa. We study the first derivative of the denominator as a function of $\gamma \geq 0$, $d(\gamma)$:

$$d'(\gamma) = -\pi(1+g_h)^{-\gamma}\ln(1+g_h) - (1-\pi)(1+g_l)^{-\gamma}\ln(1+g_l)$$

Since $g_l < 0 < g_h$, $-(1-\pi)(1+g_l)^{-\gamma}\ln(1+g_l)$ is positive and increases in γ, while the first term of the expression for $d'(\gamma)$ is negative and decreasing in γ. In particular, $\underset{\gamma\to+\infty}{\ell im} -\pi(1+g_h)^{-\gamma}\ln(1+g_h) = 0^-$ while $\underset{\gamma\to+\infty}{\ell im}(1-\pi)(1+g_l)^{-\gamma}\ln(1+g_l) = +\infty$ and the second term dominates by diverging. Two cases are then possible: Either $d'(0) > 0$ (no crossing) so that γ^f cannot be defined and $r_t^{f,FI}$ decreases $\forall\gamma \geq 0$, or $d'(0) < 0$ and the two functions defining $d'(\gamma)$ cross at $\gamma = \gamma^f$ such that $d'(\gamma^f) = 0$, $r^{f,FI}$ increases over $[0, \gamma^f)$ and decreases for higher γs.

(b) We show that $\gamma^f \leq \gamma^p$. Define

$$k^f(\gamma^f) = -\pi(1+g_h)^{-\gamma^f}\ln(1+g_h) - (1-\pi)(1+g_l)^{-\gamma^f}\ln(1+g_l) = 0$$
$$k^e(\gamma^e) = -\pi(1+g_h)^{1-\gamma^p}\ln(1+g_h) - (1-\pi)(1+g_l)^{1-\gamma^p}\ln(1+g_l) = 0$$

The cases $\gamma^f = 0 \leq \gamma^e = 0$ and $\gamma^e = +\infty \leq \gamma^e = +\infty$ are trivial and obviously do not contradict the statement of the proposition. Suppose instead that both quantities are negative at $\gamma = 0$, so that at least initially they increase in γ. Assume there exists γ^f such that $k^f(\gamma^f) = 0$. We show that $k^e(\gamma^f) < 0$, implying that γ must be increased even more in order to reach $k^e(\gamma^e) = 0$.

$$\begin{aligned}
k^e(\gamma^f) &= -\pi(1+g_h)(1+g_h)^{-\gamma^f}\ln(1+g_h) - (1-\pi)(1+g_l)(1+g_l)^{-\gamma^p}\ln(1+g_l) \\
&= -\pi(1+g_h)^{-\gamma^f}\ln(1+g_h) - (1-\pi)\frac{(1+g_l)}{(1+g_h)}(1+g_l)^{-\gamma^p}\ln(1+g_l) \\
&< -\pi(1+g_h)^{-\gamma^f}\ln(1+g_h) - (1-\pi)(1+g_l)^{-\gamma^p}\ln(1+g_l) = k^f(\gamma^f) = 0
\end{aligned}$$

This follows from $\frac{(1+g_l)}{(1+g_h)} < 1$ and $\ln(1+g_l) < 0$. Thus $k^e(\gamma^f) < 0$. Since $k^e(\gamma)$ is increasing in γ it then takes a $\gamma^e > \gamma^f$ in order for $k^e(\gamma^e) = 0$. An implication is that $k^f(0) < 0$ is sufficient for $k^e(0) < 0$.
If $\bar{\gamma}$ can be found, since $g_l < 0 < g_h$ and $k^f(0) < 0$, we know that $\gamma^e > \gamma^f$. Therefore there must exists a $\gamma^{\max} > \gamma^f$ such that $E[r_t^{p,FI}(\rho,\gamma)]$ is maximized by $\gamma = \gamma^{\max}$. In the interval (γ^f, γ^e) the expected equity premium keeps growing and it is easy to see that the CRRA maximizing $E[r_t^{p,FI}(\rho,\gamma)]$ must lie on the right of γ^f. The same holds when $k^f(0) > 0$ (so that $\gamma^f = 0$). However, when $k^f(0)$, $k^e(0) > 0$ then $\gamma^{\max} = \bar{\gamma}$ as we know that $E[r_t^{p,FI}(\rho,\gamma)] \geq 0$, ruling out $\gamma^{\max} = 0$ as a maximizer: if $r^{f,FI}(\rho,\gamma)$ grows in γ faster than $E[r_t^{e,FI}(\rho,\gamma)]$ the equity premium would end up being negative for a large enough γ. $\square$

Proof of Proposition 2. All the expectations are taken with respect to true probability measure π_t. Therefore, in the case of (a) start with:

$$E_t\left[1 + r_{t+1}^{e,RL}(\rho,\gamma)\right] = \frac{E_t[(1+g_{t+1})]}{\Psi_t^{RL}} + \frac{E_t[(1+g_{t+1})\Psi_{t+1}^{RL}]}{\Psi_t^{RL}}$$

Since for $\gamma \leq 1$ Ψ_t^{RL} is nondecreasing in $\hat{\pi}_t^{\tau_b}$ and pessimism is defined as $\hat{\pi}_t^{\tau_b} \leq \pi$, it follows that $\Psi_t^{RL} \leq \Psi^{FI}$. Hence:

$$\begin{aligned} E_t\left[1 + r_{t+1}^{e,RL}(\rho,\gamma)\right] &= \frac{E_t[(1+g_{t+1})]}{\Psi_t^{RL}} + \frac{E_t[(1+g_{t+1})\Psi_{t+1}^{RL}]}{\Psi_t^{RL}} \\ &\geq \frac{E_t[(1+g_{t+1})]}{\Psi^{FI}} + \frac{E_t[(1+g_{t+1})\Psi_{t+1}^{RL}]}{\Psi_t^{RL}} \geq \frac{E_t[(1+g_{t+1})]}{\Psi^{FI}} + \frac{E_t[\Psi_{t+1}^{RL}]}{\Psi_t^{RL}} E_t[(1+g_{t+1})] \end{aligned}$$

where the last line derives from an application of the covariance inequality (as Ψ_{t+1}^{RL} can be seen as a nondecreasing function of g_{t+1}). As for the term $\frac{E_t[\Psi_{t+1}^{RL}]}{\Psi_t^{RL}}$, notice that:

$$E_t[\Psi_{t+1}^{RL}] = \pi_t \Psi_{t+1}^{RL}(\hat{\pi}_{t+1}^u) + (1-\pi_t)\Psi_{t+1}^{RL}(\hat{\pi}_{t+1}^d) \geq \hat{\pi}_t^{\tau_b}\Psi_{t+1}^{RL}(\hat{\pi}_{t+1}^u) + (1-\hat{\pi}_t^{\tau_b})\Psi_{t+1}^{RL}(\hat{\pi}_{t+1}^d)$$

where $\hat{\pi}_{t+1}^u = \frac{N_t}{N_t+1}\hat{\pi}_t^{\tau_b} + \frac{1}{N_t+1} > \hat{\pi}_t^{\tau_b} > \frac{N_t}{N_t+1}\hat{\pi}_t^{\tau_b}$ and the last line uses the fact that $\hat{\pi}_t^{\tau_b} \leq \pi_t$ whereas $\Psi_{t+1}^{RL}(\hat{\pi}_{t+1}^u) > \Psi_{t+1}^{RL}(\hat{\pi}_{t+1}^d)$ from the fact that $\Psi_t^{RL}(\hat{\pi}_t^{\tau_b})$ is nondecreasing in $\hat{\pi}_t^{\tau_b}$. From the convexity of Ψ_{t+1}^{RL} it follows that:

$$\hat{\pi}_t^{\tau_b}\Psi_{t+1}^{RL}(\hat{\pi}_{t+1}^u) + (1-\hat{\pi}_t^{\tau_b})\Psi_{t+1}^{RL}(\hat{\pi}_{t+1}^d) \geq \Psi_{t+1}^{RL}\left(\hat{\pi}_t^{\tau_b}\hat{\pi}_{t+1}^u + (1-\hat{\pi}_t^{\tau_b})\hat{\pi}_{t+1}^d\right) = \Psi_{t+1}^{RL}\left(\hat{\pi}_{t+1}^*\right)$$

where

$$\hat{\pi}_{t+1}^* = \hat{\pi}_t^{\tau_b}\hat{\pi}_{t+1}^u + (1-\hat{\pi}_t^{\tau_b})\hat{\pi}_{t+1}^d = \hat{\pi}_{t+1}^d + \hat{\pi}_t^{\tau_b}(\hat{\pi}_{t+1}^u - \hat{\pi}_{t+1}^d) = \frac{N_t}{N_t+1}\hat{\pi}_t^{\tau_b} + \frac{1}{N_t+1}\hat{\pi}_t^{\tau_b} = \hat{\pi}_t^{\tau_b}$$

Therefore $E_t[\Psi_{t+1}^{RL}] \geq \Psi_{t+1}^{RL}\left(\hat{\pi}_t^{\tau_b}\right)$[19] so that $\frac{E_t[\Psi_{t+1}^{RL}]}{\Psi_t^{RL}} \geq 1$. Using this fact and $\frac{\Psi^{FI}}{\Psi^{FI}} = 1$, we conclude that:

$$\begin{aligned} E_t\left[1 + r_{t+1}^{e,RL}(\rho,\gamma)\right] &\geq \frac{E_t[(1+g_{t+1})]}{\Psi_t^{FI}} + \frac{E_t[\Psi_{t+1}^{RL}]}{\Psi_t^{RL}} E_t[(1+g_{t+1})] \\ &\geq \left[1 + \frac{1}{\Psi^{FI}}\right] E_t[(1+g_{t+1})] = E\left[1 + r_{t+1}^{e,FI}(\rho,\gamma)\right], \end{aligned}$$

the FI expected gross return on stocks. Trivially this implies $E_t\left[r_{t+1}^{e,RL}(\rho,\gamma)\right] \geq E\left[r_{t+1}^{e,FI}(\rho,\gamma)\right]$. On the other hand,

$$r_t^{f,RL}(\rho,\gamma) = \frac{1+\rho}{\hat{\pi}_t^{\tau_b}(1+g_h)^{-\gamma} + (1-\hat{\pi}_t^{\tau_b})(1+g_l)^{-\gamma}} \leq \frac{1+\rho}{\pi_t(1+g_h)^{-\gamma} + (1-\pi_t)(1+g_l)^{-\gamma}} = r_t^{f,FI}(\rho,\gamma),$$

the FI equilibrium risk-free rate, since $(1+g_h)^{-\gamma} < (1+g_l)^{-\gamma}$ and $\hat{\pi}_t^{\tau_b} \leq \pi_t$. Finally, since the equity premium is the difference between expected stock returns and the equilibrium risk-free rate, and we have shown that $E_t\left[r_{t+1}^{e,RL}(\rho,\gamma)\right] \geq E\left[r_{t+1}^{e,FI}(\rho,\gamma)\right]$ and $r_t^{f,RL}(\rho,\gamma) \leq r_t^{f,FI}(\rho,\gamma)$, then $E_t\left[r_{t+1}^{p,RL}(\rho,\gamma)\right] \geq E\left[r_{t+1}^{p,FI}(\rho,\gamma)\right]$ follows. □

[19] The fact that $\Psi_{t+1}^{RL}\left(\hat{\pi}_t^{\tau_b}\right)$ employs a precision $N_{t+1} = N_t + 1$ just makes the effect stronger, as a pessimistic belief $\hat{\pi}_t^{\tau_b} < \pi_t$ reduces more the RL pricing kernel for $N_t + 1$ than for N_t. The real simplification here is that we ignore integer problems.

Figure 1

Structural Breaks in the Process of Real Consumption Growth

The plots report the outcomes of real-time tests of structural breaks in the process of real consumption growth. The structural break test applied is the 'fluctuation detector' by Chu, Stinchcombe, and White (1996). The estimates on which the fluctuation detector is based are produced by an AR(1) model:

$$g_t = \mu + \beta\, g_{t-1} + \varepsilon_t$$

with $E[\varepsilon_t]=0$ and constant variance. The minimum time span ι used to initialize the estimates over which the estimates are not subject to breaks is 20 years. The left plots refer to the drift μ. The upper plots refers to the entire sample period 1890-1999, the middle ones to the more recent period 1938-1999, the bottom plot to the recursive estimates of the coefficient over the two sample periods. The right plots refer instead to the AR(1) coefficient β.

Figure 3

Equity Premium and Real Risk-Free Rate Under Full-Information (No Breaks)

The plots report the equity premium and the real risk-free rate under full information rational expectations as a function of the coefficient of relative risk aversion γ. The top graph plots the equity premium and the real risk-free rate together, while the two smaller graphs at the bottom plots these two quantities separately and vs. a benchmark represented by the sample means over 1938-1999 (annual frequency), the horizontal lines. The model is calibrated by assuming g_h = 1.5%, g_l = -1.25%, π = 0.645, and ρ = 0.50% on a quarterly basis. This ρ implies that the annualized discount factor is approximately equal to β = 0.98.

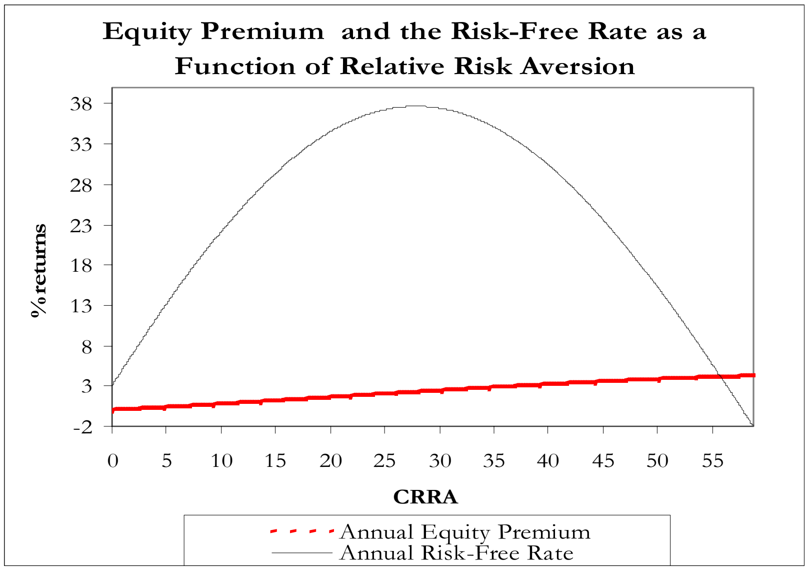


Figure 4

Asset Returns Under Rational Learning vs. Full Information (Two Breaks)

The plots report the equity premium and the real risk-free rate under rational learning and full information rational expectations as a function of the coefficient of relative risk aversion γ. The model is simulated (on a quarterly basis) by assuming g_h = 1.5%, g_l = -1.25%, π = 0.645 for 1938-1981 and π = 0.63 for 1982-1999. ρ = 0.50% to imply an annualized discount factor approximately equal to β = 0.98. Under rational learning we also calibrate initial beliefs as of January 1938 (n = 11, N = 32, $\hat{\pi} = 0.34$) and as of January 1982 (n = 4, N = 8, $\hat{\pi} = 0.5$). FI values are calculated under the assumption of breaks occurring and being observable. Solid bars represent empirical values.

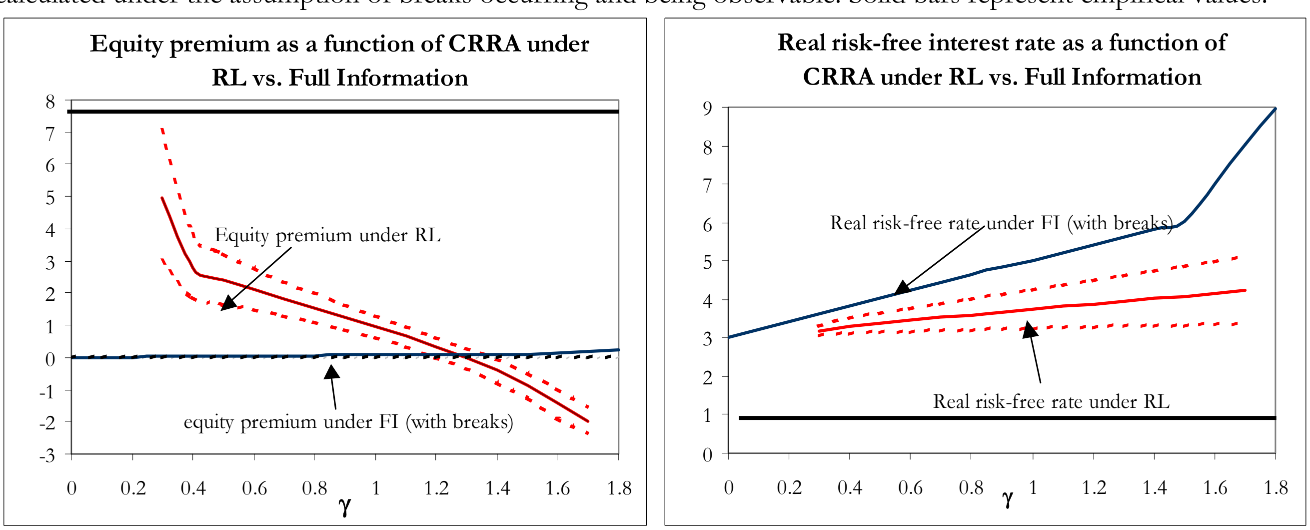
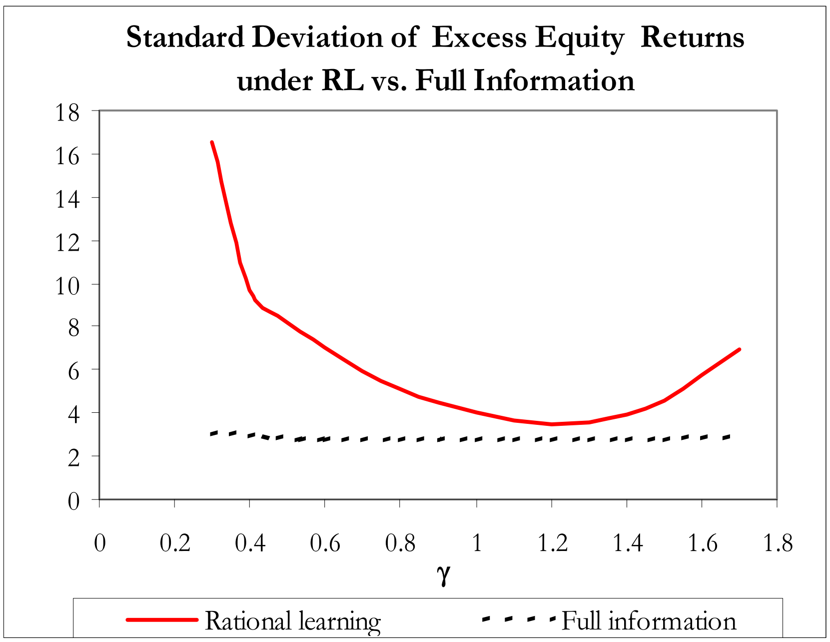


Figure 5

Volatility of Excess Equity Returns and Real Risk-Free Rate Under Rational Learning vs. Full Information (Under Two Breaks)

The plots report the standard deviation of excess equity returns and the real risk-free rate under rational learning and full information rational expectations as a function of the coefficient of relative risk aversion γ. The model is simulated (on a quarterly basis) by assuming g_h = 1.5%, g_l = -1.25%, π = 0.645 for 1938-1981, and g_h = 1.5%, g_l = -1.3%, π = 0.63 for 1982-1999. ρ = 0.50% on a quarterly basis. This ρ implies that the annualized discount factor is approximately equal to β = 0.98. Under rational learning we also calibrate initial beliefs as of January 1938 (n = 11, N = 32, $\hat{\pi} = 0.34$) and as of January 1982 (n = 4, N = 8, $\hat{\pi} = 0.5$). FI values are calculated by simulations under the assumption of breaks occurring and being observed.



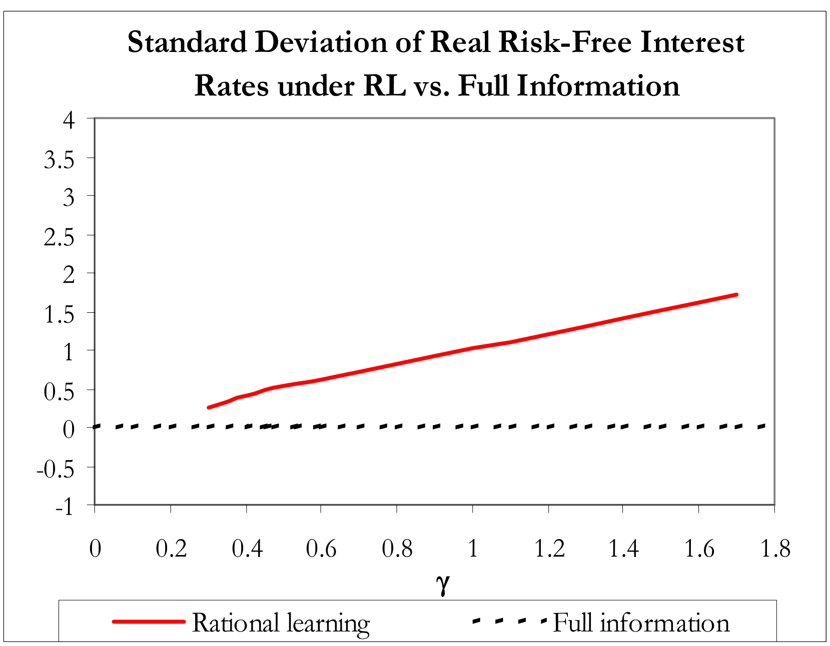


Figure 6

Percentage of Simulations (Ex-Ante Probability) of Rejecting the Null that Excess Equity Returns and Real Risk-Free Rate Are I.I.D. (Under Two Breaks)

The plots report the percentage of simulations under RL and FI for which the null hypothesis of zero serial correlations in the levels and squares of asset returns is rejected. The hypothesis is tested by assessing whether the portmanteau Lijung-Box statistics are significant at 5%. In the case of squared asset returns, rejection of zero serial correlation is taken as indication of volatility clustering (ARCH). These percentages are plotted as a function of the coefficient of relative risk aversion γ. The model is simulated (on a quarterly basis) by assuming g_h = 1.5%, g_l = -1.25%, π = 0.645 for 1938-1981, and π = 0.63 for 1982-1999. ρ = 0.50% on a quarterly basis. Under rational learning we also calibrate initial beliefs as of January 1938 (n = 11, N = 32, $\hat{\pi} = 0.34$) and as of January 1982 (n = 4, N = 8, $\hat{\pi} = 0.5$). FI values are calculated by simulations under the assumption of breaks occurring.

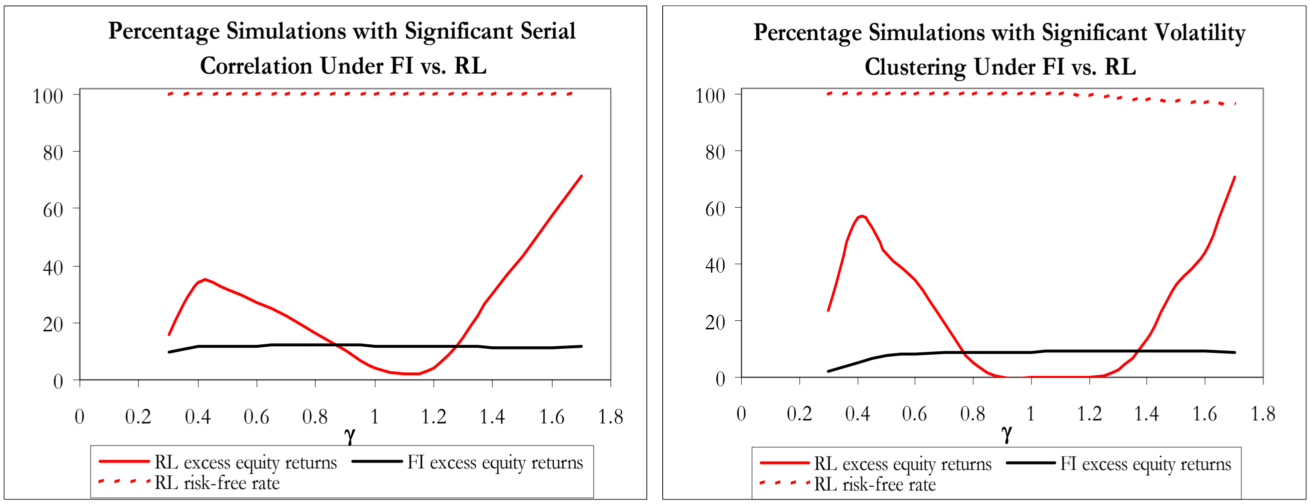


Figure 7

Correlation Between Excess Equity Returns and Real Risk-Free Rate (Under Two Breaks)

The graph reports the average simultaneous correlation between excess equity returns and the short-term, real interest rate under RL and FI as a function of the coefficient of relative risk aversion γ. The model is simulated (on a quarterly basis) by assuming g_h = 1.5%, g_l = -1.25%, π = 0.645 for 1938-1981, π = 0.63 for 1982-1999. ρ = 0.50% on a quarterly basis. Under rational learning we also calibrate initial beliefs as of January 1938 (n = 11, N = 32, $\hat{\pi} = 0.34$) and as of January 1982 (n = 4, N = 8, $\hat{\pi} = 0.5$).

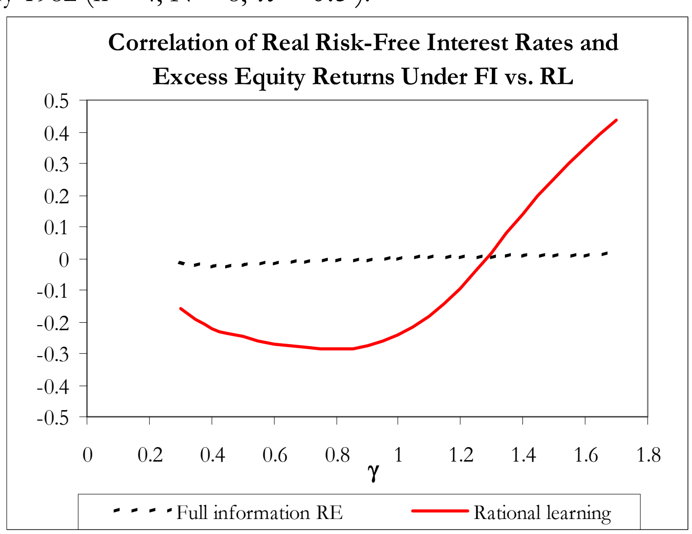


Figure 8

Equilibrium Asset Prices on a Path Calibration

The plots report the dynamics of the beliefs about π and of the equilibrium stock price when a representative agent is on a rational learning path and perceives two breaks, in 1938 and 1981. Under full information the model is calibrated by assuming g_h = 3.1%, g_l = -0.7%, π = 0.795 for 1938-1981, and g_h = 2.8%, g_l = -0.3%, π = 0.78 for 1982-1999. The subjective rate of time preference ρ is set to 2%, while the coefficient of relative risk aversion γ to 0.3. I Under rational learning we calibrate initial beliefs as of January 1938 (n = 2, N = 8, $\hat{\pi} = 0.25$) and as of January 1982 (n = 1, N = 2, $\hat{\pi} = 0.5$). On a learning path, the random dividend process is calibrated on the actual historical path of the US real consumption growth rate, setting $I_{\{g_t=g_h\}}$ = 1 when realized consumption growth is positive.

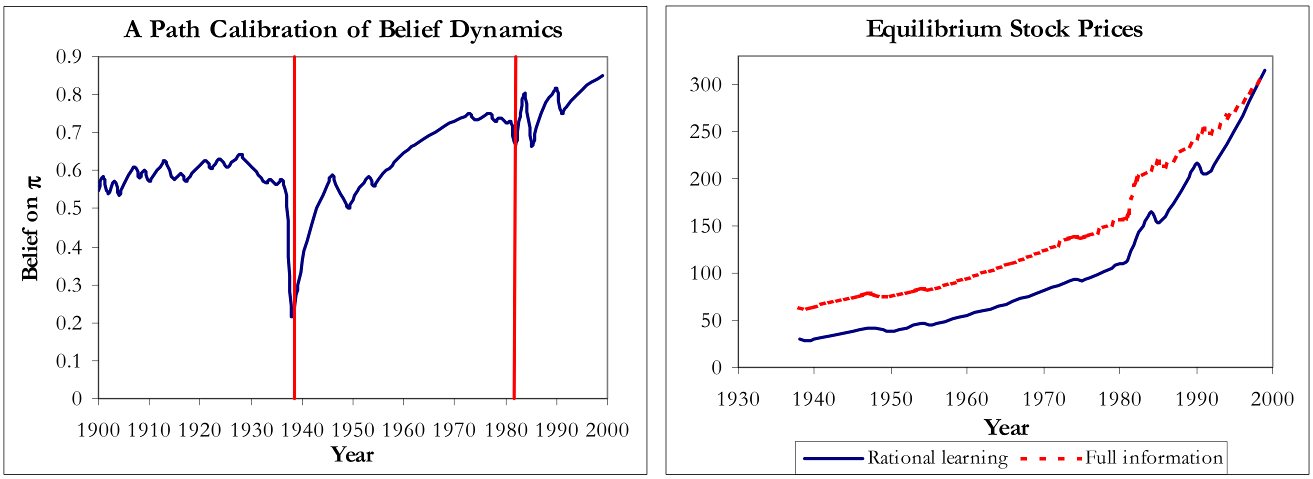


Figure 9

Rolling Window 10- and 15-Year Period Equity Premium and Volatility

The plots report the mean and standard deviation of excess returns over sub-samples of 10 and 15 years after each date (included in the sub-sample) in the interval 1930-1999. The solid, bold line represents the equity premium (mean excess returns) and the volatility of excess returns over the entire sample period 1938-1999, 7.6% and 16%, respectively.