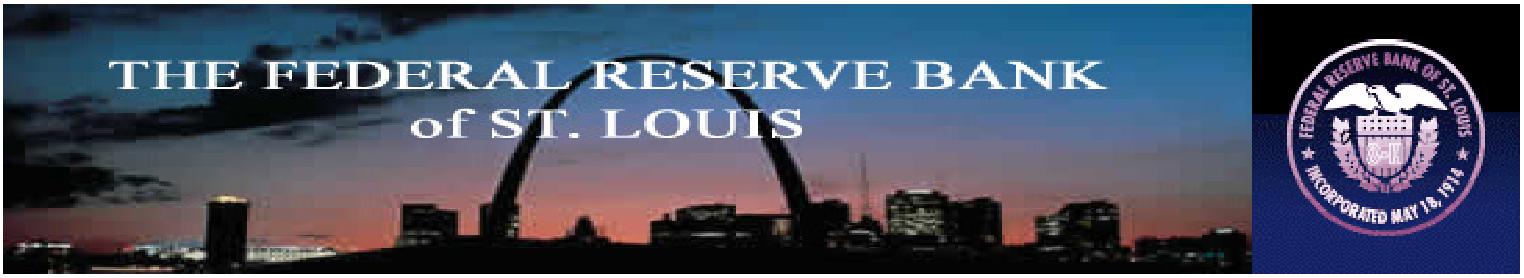


WORKING PAPER SERIES

# Near-Rational Exuberance

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Working Paper 2004-025B
http://research.stlouisfed.org/wp/2004/2004-025.pdf

October 2004
Revised September 2005

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# Near-Rational Exuberance*

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17 September 2005

**Abstract**

We study how the use of judgement or "add-factors" in macroeconomic forecasting may disturb the set of equilibrium outcomes when agents learn using recursive methods. We isolate conditions under which new phenomena, which we call *exuberance equilibria*, can exist in standard macroeconomic environments. Examples include a simple asset pricing model and the New Keynesian monetary policy framework. Inclusion of judgement in forecasts can lead to self-fulfilling fluctuations, but without the requirement that the underlying rational expectations equilibrium is locally indeterminate. We suggest ways in which policymakers might avoid unintended outcomes by adjusting policy to minimize the risk of exuberance equilibria. *JEL* codes: E520, E610. *Key words*: Learning, expectations, excess volatility, bounded rationality, monetary policy.

*An early version of this paper was presented at the ECB Conference "Monetary Policy and Imperfect Knowledge," Würzburg, Germany. We thank Martin Ellison, Roger Farmer, Petra Geraats, Eran Guse, Sharon Kozicki, Albert Marcet, Danny Quah, Bob Tetlow and the participants at the ECB conference and many other conferences and seminars for their comments.

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# 1 Introduction

## 1.1 Judgement variables in forecasting

Judgement is a fact of life in macroeconomic forecasting. It is widely understood that even the most sophisticated econometric forecasts are adjusted before presentation. This adjustment is so pervasive that it is known as the use of "add-factors"—subjective changes to the forecast which depend on the forecaster's assessment of special circumstances that are not well summarized by the variables that are included in the econometric model. A forthright discussion of how prominently judgement enters into actual macroeconomic forecasting is contained in Reifschneider, Stockton, and Wilcox (1997). As they state, "... [econometric] models are rarely, if ever, used at the Federal Reserve without at least the *potential* for intervention based on judgement. Instead, [the approach at the Federal Reserve] involves a mix of strictly algorithmic methods ("science") and judgement guided by information not available to the model ("art") (p. 2, italics in original). Recently, some authors have argued that economic theory needs to take explicit account of the effects of judgement on the behavior of macroeconomic systems.[1]

We wish to think of the news or add-factor that modifies the forecast as a qualitative, unique, commonly understood economy-wide variable: In sum, a *judgement variable*. An example of a judgemental adjustment is suggested by Reifschneider, *et al.* (1997), when they discuss the "financial headwinds" that were thought to be inhibiting U.S. economic growth in the early to mid-1990s. As they discuss, the headwinds add-factor was used to adjust forecasts over a period of many quarters. It was communicated to the public prominently in speeches by Federal Reserve Chairman Alan Greenspan. It was thus widely understood throughout the economy and was highly serially correlated. This is the type of variable we have in mind, although by no means would we wish to restrict attention to this particular example.[2] We

[1] Judgement explicitly enters the analysis in a monetary policy context in Svensson (2003, 2005) and Svensson and Tetlow (2005). Jansson and Vredin (2001) provide an empirical analysis of the impact of judgement on forecasting by the Bank of Sweden.

[2] Other examples include the Cuban Missile Crisis, wage and price controls, Hurricane

think add-factoring is occurring continuously.

Conventional wisdom among economists suggests that judgement is all to the good in macroeconomic forecasting. Models are, of course, crude approximations of reality and must be supplemented with other information not contained in the model.[3] While we have motivated our ideas in terms of macroeconomic forecasting, our framework applies more generally to economic environments where expectations and qualitative judgements about the effects of unique events play an important role.

## 1.2 Feedback from judgement

Our focus in this paper is on how the add-factor or judgemental adjustment of forecasts may create more problems than it solves. In particular, we show how such a practice can lead to the possibility of self-fulfilling fluctuations. For expositional simplicity, in the main analysis we focus on the extreme case where the judgement variable is not intrinsically related to economic fundamentals at all. Thus our results come from a situation where the forecasting judgement being added is, fundamentally speaking, not useful in forecasting the variables of interest. However, this assumption is not essential. We also demonstrate that self-fulfilling fluctuations can occur in cases where judgement is related to fundamentals.

We study systems with well-defined rational expectations equilibria. We replace rational expectations with adaptive learning using the methodology of Evans and Honkapohja (2001). We then investigate the equilibrium dynamics of the system if the econometric models of the agents are supplemented with judgement. To define an exuberance equilibrium, we first require that the perceived evolution of the economy corresponds to the actual evolution by imposing a rational expectations equilibrium with limited information, or more specifically the consistent expectations equilibrium (CEE) concept, as

Katrina, the Y2K millenium bug, the savings and loan crisis, and the September 11th, 2001 terrorist attacks in the U.S.

[3] Svensson (2003, 2005) and Svensson and Tetlow (2005), for instance, formally show how the use of judgement by policymakers can improve economic performance.

developed by Sargent (1991), Marcet and Sargent (1995) and Hommes and Sorger (1998). Under this requirement, the autocovariance generating functions of the perceived and actual evolutions correspond exactly. Secondly, we require individual rationality in individual agents' choice to include the judgement variable in their forecasting model, given that all other agents are using the judgement variable and hence causing it to influence the actual dynamics of the macroeconomy. Finally, we require learnability or expectational stability. When all three of these requirements are met, we say that an exuberance equilibrium exists. In our exuberance equilibria, all agents would be better off if the judgement variable were not being used, but as it is being used, no agent wishes to discontinue its use. We view this as a Nash equilibrium in beliefs.

## 1.3 Near-rationality

Our Nash equilibrium does not correspond exactly to a rational expectations equilibrium. This is because the judgement variable is assumed to be unavailable in the statistical part of the forecasting. We think of this as reflecting the separation of the econometric forecasting unit from the actual decision makers. Decision makers treat the econometric forecast as an input to which they are free to add the judgement variable. The judgementally adjusted forecasts are the basis for the decisions and actions of the agents, but the adjustments are not observables directly available to the econometricians.

In other words, we are assuming that the judgement variable is not one that can be extracted by the econometric forecasting unit and converted into a statistical time series that can formally be utilized in an econometric forecasting model. In a similar vein the decision makers face a dichotomy in their use of judgement: they either incorporate the variable as an add-factor or they ignore it and directly use the econometric forecast. This inability of the decision makers to transmit to the econometric forecasters in a quantitative way the judgemental aspects behind their final economic decisions is the source of the deviation from full rational expectations and the reason for our use of the term "near-rationality."[4]

### 1.4 Main findings

We isolate conditions under which exuberance equilibria exist in widely-studied dynamic frameworks in which the state of the system depends on expectations of future endogenous variables. We study two applications of a general linear model, a simple univariate asset-pricing model as well as the canonical New Keynesian model of Woodford (2003) and Clarida, Gali, and Gertler (1999). We interpret the exuberance equilibrium in the asset-pricing model as an example of "excess volatility." In the New Keynesian application, the exuberance equilibria can also exhibit considerable volatility relative to the underlying fundamental rational expectations equilibrium in which judgement does not play a role.[5]

Our results may lead one to view the possibility of exuberance equilibria as particularly worrisome, as exuberance equilibria may exist even in otherwise benign circumstances. In particular, we show that exuberance is a clear possibility even in the case where the underlying rational expectations equilibrium is unique (*a.k.a.* determinate). Thus an interesting and novel finding is the possibility of "sunspot-like" equilibria, but without requiring that the underlying rational expectations equilibrium of the model is indeterminate.[6] In a sense, we find "sunspot-like" equilibria without indeterminacy.

[4]The term "near rationality" has been used elsewhere in the literature, often to mean less-than-full maximization of utility. See, for example, Akerlof and Yellen (1985) and Caballero (1995). Ball (2000) analyzes a model where the agents use a forecasting model that does not encompass the equilibrium law of motion—a "restricted perception." Our concept is based on full optimization but subject to the restriction that some information is not quantifiable—"judgement." Our concept of near rationality is discussed further in Section 2.9.

[5]"Exuberance" (which in our equilibria leads to both positive and negative deviations from the fundamentals solution) has a long informal tradition as a potential explanation of asset price "bubbles." For its possible role in "financial fragility" see Lagunoff and Schreft (1999).

[6]Indeterminacy and sunspot equilibria are distinct concepts, as discussed in Benhabib and Farmer (1999). We consider only linear models, for which the existence of stationary sunspot equilibria requires indeterminacy—see for example Propositions 2 and 3 of Chiappori and Guesnerie (1991).

In the policy-oriented New Keynesian application, our findings suggest a new danger for policy makers: Choosing policy to induce determinacy and learnability may not be enough, because the policy maker must also avoid the prospect of exuberance equilibria.[7] We show how policy may be designed to avoid this danger. More specifically, in the cases we study, policy that is more aggressive than the requirements for determinacy and learnability is needed to avoid the possibility of exuberance equilibria.

### 1.5 Organization

We begin in the next section with a scalar case, which is simple enough to illustrate our main ideas analytically. We provide results on existence of exuberance equilibria. At the end of this section we interpret the scalar case as a simple asset pricing model, provide some simple quantitative analysis of the excess volatility associated with exuberance and discuss further the issue of near-rationality. We then turn to a multivariate linear framework. There we provide an analysis of some additional issues that arise, and discuss the concept of approximate exuberance equilibria. This section includes the New Keynesian macroeconomics application. The concluding section contains a summary of our findings and suggests some directions for additional research.

## 2 Economies with judgement

### 2.1 A scalar linear model

#### 2.1.1 Overview

Our results depend on the idea that agents participating in macroeconomic systems are learning using recursive algorithms, and that the systems under learning eventually converge. In many cases, as discussed extensively in

[7] For discussions of determinacy and learnability as *desiderata* for the evaluation of monetary policy rules, see Bullard and Mitra (2002) and Evans and Honkapohja (2003a). For a survey see Evans and Honkapohja (2003b).

Evans and Honkapohja (2001), this convergence would be to a rational expectations equilibrium. The crucial aspect for the present paper is that once agents have their macroeconometric forecast from their regression model, the forecast is then judgementally adjusted.

To fix ideas, consider an economy which may be described by

$$y_t = \beta y^e_{t+1} + u_t \tag{1}$$

where $y_t$ is the economy's state variable, $\beta$ is a scalar parameter, and $u_t$ is a stochastic noise term. For convenience we have dropped any constants in this equation. The term $y^e_{t+1}$ represents the possibly non-rational expectation of private sector agents. The novel feature of this paper is that we allow judgement, $\xi_t$, to be added to the macroeconometric forecast, $E^\star_t y_{t+1}$,

$$y^e_{t+1} = E^\star_t y_{t+1} + \xi_t. \tag{2}$$

Our goal is to understand the implications of this add-factor judgement on the nature of equilibrium in the economy, and on the convergence of the learning algorithm to equilibrium. We stress that if the judgement vector is null, the model corresponds to a version of systems analyzed extensively in Evans and Honkapohja (2001), and that the conditions for convergence to rational expectations equilibrium in that case are well-established.

### 2.1.2 The nature of judgemental adjustment

We first discuss how we model the judgemental add-factor. We view this as an attempt to allow for the impact of occasional unique events. Let $\eta_t$ represent "news" about qualitative events judged to have significant impact on the economy, where $\eta_t$ measures that part of the anticipated impact on $y_{t+1}$ that is believed not to be reflected in $E^\star_t y_{t+1}$. The forecasted future impact of this news is

$$\frac{\partial y_{t+1+j}}{\partial \eta_t} = \psi_{t,j}, \text{ for } j = 1, 2, 3, \ldots$$

Since we are here concerned with the judgemental adjustment, $\psi_{t,j}\eta_t$ measures the judgemental forecaster's view about the extent to which this news about qualitative events will fail to be reflected over time in the econometric forecast.

We think of $\eta_t$ as pertaining to "unique" events and it has two components: (i) the expected effect of new qualitative events and (ii) new information about recent qualitative events that still have an impact on the economy. Since $\eta_t$ represents news we assume it to be a martingale difference sequence (which for convenience we will take to be white noise). It might often take the value zero.

The future impact $\psi_{t,j}$ of $\eta_t$ could in general have a complex time profile that reflects specific features of the unique qualitative events. For *analytical simplicity only* we make the assumption

$$\psi_{t,j} = \rho^j \text{ with } 0 < \rho < 1,$$

that is, constant geometric decay at rate $\rho$ for all $t$, $j$. Then

$$\xi_t = \sum\nolimits_{j=0}^{\infty} \psi_{t-j,j}\eta_{t-j} = \sum\nolimits_{j=0}^{\infty} \rho^j \eta_{t-j} = (1-\rho L)^{-1}\eta_t$$

and the total judgemental adjustment in $y_{t+1}^e$ satisfies

$$(1-\rho L)\,\xi_t = \eta_t \tag{3}$$

or equivalently $\xi_t = \rho\xi_{t-1} + \eta_t$. Here $L$ is the lag operator such that $Ly_t = y_{t-1}$. Thus the expected effects of the judgemental variables on $y_{t+1}$ can be summarized as $\rho\xi_{t-1}$, the expected impact of past news, plus $\eta_t$, the impact of current news.

While the AR(1) form of $\xi_t$ is convenient for our analysis, the judgemental forecasters would resist any attempt by the econometricians to reduce it to a measurable variable since they would not think it appropriate to treat past qualitative events as similar to current qualitative events, that is, they would regard it as a mistake to treat past judgments as a useful econometric time series.

We assume that $u_t$ and $\eta_t$ evolve independently, so that the judgement variable has no fundamental effect on the economy described by equation (1). This is obviously an important and extreme assumption but it is also the one that we think is the most interesting for the purpose of illustrating our main points, as it is the starkest case. Later in this section, we show that no substantive changes to our results are introduced when $\eta_t$ and $u_t$ are correlated.

### 2.1.3 Econometric forecasts

We now turn to the nature of the macroeconometric forecast. The hallmark of the recursive learning literature is the assignment of a *perceived law of motion* to the agents, so that we can view them as using recursive algorithms to update their forecasts of the future based on actual data produced by the system in which they operate.[8] A key aspect of this assignment is to keep the perceived law of motion consistent with the actual law of motion of the system, which will be generated by the interaction of equation (1) with the agents' expectations formation process. With judgement in the model, it will be apparent below that the ARMA(1,1) perceived law of motion

$$y_t = by_{t-1} + v_t - av_{t-1}, \tag{4}$$

can be consistent with the actual law of motion. Here $|b| < 1$ and $|a| < 1$ are parameters and $v_t$ is a stochastic noise term. We can write this as

$$y_t = \theta(L) v_t, \tag{5}$$

where

$$\theta(L) = \frac{1 - aL}{1 - bL}.$$

Then

$$E_t^{\star} y_{t+1} = by_t - av_t = [b\theta(L) - a] v_t \tag{6}$$

is the minimum mean square error forecast based on this perceived law of motion. We call (6) the *econometric forecast.* It is based on the econometric

[8] We can think of this as corresponding to the existence of a forecasting community using econometric-based models to guide the expectations of private sector and governmental agents. Forecasting communities like this exist in all industrialized nations. Our analysis differs from but is related to the literature in finance on strategic professional forecasting, see e.g. Ottaviani and Sørensen (2004) and the references therein.

model, the perceived law of motion, alone, and is the traditional description of the expectations formation process both under rational expectations and in the learning literature.

#### 2.1.4 Exuberance equilibrium

Since expectations in the economy are being formed via equation (2), and since these expectations affect the evolution of the economy's state through equation (1), we deduce an *actual law of motion* for this system. The forecast (2) means that the economy evolves according to

$$\begin{aligned} y_t &= \beta y_{t+1}^e + u_t \\ &= \beta \left( \frac{b-a}{1-bL} \right) v_t + \frac{\beta}{1-\rho L} \eta_t + u_t \\ &= \beta \left( \frac{b-a}{1-bL} \right) \left( \frac{1-bL}{1-aL} \right) y_t + \frac{\beta}{1-\rho L} \eta_t + u_t . \end{aligned}$$

Solving for $y_t$ implies that the actual law of motion is

$$y_t = \frac{1-aL}{\beta (a-b) + 1 - aL} \left( \frac{\beta}{1-\rho L} \eta_t + u_t \right) . \tag{7}$$

Judgement naturally influences the evolution of the state because it influences the views of economic actors concerning the future. The critical question is then whether there are conditions under which the agents would continue to use the add-factored forecast (2) when the economy is evolving according to equation (7). That is, could the agents come to perceive that the judgement variable is in fact useful in forecasting the state variable, even though by construction there is no fundamental relationship? Our main purpose in this paper is to answer this question.

In order to guide our thinking on this question, we define the concept of an *exuberance equilibrium* and seek to understand the conditions under which such an equilibrium would exist. An exuberance equilibrium is one in which the evolution of the judgement variable influences actual economic outcomes, even though there may be no fundamental impact of the judgement factor.

Our concept has three key components, all of which are discussed in detail in the subsections below. The first is that the econometric forecast should be consistent with the data generated by the model. In some sense, the econometric model should not be falsifiable. To impose this condition, we use the CEE concept.[9] The second component is that each individual agent in the economy should conclude that it is in their interests to judgementally adjust their forecast, given that all other agents are making a similar judgemental adjustment. That is, inclusion of the judgemental adjustment is actually beneficial from the point of view of each agent in the economy. The third component is that the stationary outcome is stable in the learning process being used by the agents. That is, since our systems are based on the idea that agents are using regression models for macroeconomic forecasting, we need to verify that the dynamic system created by their recursive estimation procedure is locally convergent to the proposed exuberance equilibrium.

An exuberance equilibrium can now be defined. Given the model with judgement (1), (3), (6), and (2), an *exuberance equilibrium* exists if

1. A CEE exists,

2. Individual agents rationally decide to include the (non-trivial) judgement variable in their forecasts given that all other agents are judgementally adjusting their forecasts, and

3. The CEE is learnable.

Are there conditions under which an exuberance equilibrium could exist? There are, and we argue that the conditions are in fact worrisomely plausible. In order to obtain some intuition, we turn to an analysis of each of the above conditions in the scalar model.

[9] As noted above, CEE and rational expectations equilibrium with limited information are equivalent in our linear settings.

## 2.2 Consistent expectations

The core idea of a CEE is that the econometric forecasters should see no difference between their perceived law of motion for how the economy evolves and the actual data from the economy. One way to develop conditions under which such an outcome may occur is to require that the autocovariance generating function of the perceived law of motion corresponds exactly to the autocovariance generating function of the actual law of motion.[10] We can analytically verify the existence of a solution to the equation implied by this statement for the univariate case.

The autocovariance generating function for the perceived law of motion in the scalar case is given by

$$G_{PLM}(z) = \sigma_v^2 \frac{(1-az)(1-az^{-1})}{(1-bz)(1-bz^{-1})} \tag{8}$$

where $\sigma_v^2$ is the variance of $v$, and $z$ is a complex scalar.[11] For the actual law of motion, or ALM, the autocovariance generating function is the sum of two such functions

$$G_{ALM}(z) = G_\eta(z) + G_u(z)$$

by the independence of $\eta$ and $u$. These functions are

$$G_\eta(z) = \frac{\sigma_\eta^2 \beta^2 (1-az)(1-az^{-1})}{[\beta(a-b)+1-az][\beta(a-b)+1-az^{-1}](1-\rho z)(1-\rho z^{-1})},$$

and

$$G_u(z) = \frac{\sigma_u^2 (1-az)(1-az^{-1})}{[\beta(a-b)+1-az][\beta(a-b)+1-az^{-1}]}.$$

We use these functions to demonstrate the following result in Appendix A.

**Lemma 1** *There exists a CEE with $b=\rho$ and $a \in [0,\rho]$.*

[10] See Hommes and Sorger (1998), Hommes, Sorger, and Wagener (2004) and Branch and McGough (2005).

[11] See Brockwell and Davis (1991, pp. 417-420), or Hamilton (1994, pp. 266-268).

As also shown in Appendix A, there are interesting limiting cases: when $\sigma_\eta^2 \to 0$, so that the relative variance of the judgement process is small, $a \to \rho$, while for $\sigma_u^2 \to 0$, meaning that the relative variance of the fundamental process is small, $a \to 0$. Thus the value of $a$ depends in an interesting way on the relative innovation variance $R \equiv \sigma_\eta^2/\sigma_u^2$, as well as the discount factor $\beta$ and the serial correlation $\rho$. Since a solution $a \in [0, \rho]$ always exists,[12] the conditions for a CEE can always be met in the scalar case.

We now ask whether individual rationality holds with respect to inclusion of the judgement variable in making forecasts.

## 2.3 Incentives to include judgement

When all agents in the model are making use of the judgementally adjusted forecast described in equation (2), they induce an actual law of motion for the system which is described by equation (7). An individual agent may nevertheless decide that it is possible to make more efficient forecasts by simply ignoring the judgemental adjustment. If this is possible, then it is not individually rational for all agents to use the add-factored forecast. We check this individual forecast efficiency condition by comparing the variance of the forecast error for the judgemental forecast (2) to the variance of the forecast error with judgement not included, the econometric forecast (6), under the condition that all other agents are using the judgementally adjusted forecast and thus are inducing the actual law of motion (7).

To make this calculation, we use the condition from the consistent expectations calculation that $b = \rho$. We then note that $v_t = \left(\frac{1-\rho L}{1-aL}\right) y_t$. The econometric forecast is therefore given by

$$E_t^\star y_{t+1} = \frac{\rho - a}{1 - \rho L} v_t = \frac{\rho - a}{1 - aL} y_t \tag{9}$$

[12] Appendix A also makes it clear that there is a second, negative value of $a$ that equates the two autocovariance generating functions. We found that the other conditions for exuberance equilibrium are not met at this value of $a$, and we refer to it only in passing in the remainder of the paper.

whereas the judgementally adjusted forecast is given by

$$y_{t+1}^{e} = \frac{\rho - a}{1 - aL} y_t + \frac{1}{1 - \rho L} \eta_t. \tag{10}$$

The question from an (atomistic) individual agent's point of view is then whether they should use (9) or (10) as a basis for their expectations of the future state of the economy.

Is it possible for the variance of the judgementally adjusted forecast to be lower than the variance of the econometric forecast? It is. Consider the special case when $\sigma_\eta^2 \to 0$ so that the positive root $a \to \rho$. Then it is shown in Appendix B that, apart from additive terms in $u_t$ that are identical for the two forecasts, the forecast error without judgement is

$$FE_{NJ} = \frac{\beta}{1 - \rho L} \eta_{t+1}$$

whereas the forecast error with judgement is

$$FE_J = \frac{\beta \left(1 - \beta^{-1} L\right)}{1 - \rho L} \eta_{t+1}.$$

Thus, as $\sigma_\eta^2 \to 0$ the ratio between the variances of these two forecast errors is[13]

$$\frac{Var[FE_J]}{Var[FE_{NJ}]} = 1 + \beta^{-2} - \beta^{-1} \rho.$$

This is less than one if and only if

$$\rho \beta > \frac{1}{2}. \tag{11}$$

By continuity, it follows that if $\beta > 1/2$ there are non-trivial judgement processes (with $\rho > 1/2\beta$ and $\sigma_\eta^2 > 0$ sufficiently small) for which the agents have incentives to include the process as an add factor in their forecasts. The preceding argument considered the limiting case $a \to \rho$, but as we will show below, it is not necessary for $a$ to be close to $\rho$ for our results to hold.

[13] See, for instance, Harvey (1981, p. 40). The variance of $x_t = [(1 + mL) / (1 - \ell L)] \epsilon_t$ is $\left[\left(1 + m^2 + 2\ell m\right) / \left(1 - \ell^2\right)\right] \sigma_\epsilon^2$.

We conclude that individuals will decide to use the judgementally adjusted forecast in cases where $\rho$ is relatively large, meaning that the serial correlation in the judgement variable is substantial, and when $\beta$ is simultaneously relatively high, meaning that expectations are relatively important in determining the evolution of the economy. We remark that these conditions are exactly the ones that correspond to the most likely scenario for the asset pricing example given below.

Another, polar opposite, special case is one where $\sigma_u^2 \to 0$ so that the positive root $a \to 0$. Then

$$FE_{NJ} = \frac{\beta}{1-\rho\beta}\eta_{t+1}$$

whereas

$$FE_J = \frac{\beta}{(1-\rho\beta)}\frac{\left(1-\beta^{-1}L\right)}{(1-\rho L)}\eta_{t+1}.$$

The difference between the variances of these two forecast errors is then

$$Var[FE_J] - Var[FE_{NJ}] = \left(\frac{\left(\beta^{-1}-\rho\right)^2}{1-\rho^2}\right)\sigma_\eta^2.$$

This can never be less than zero under maintained assumptions. We conclude that it cannot be individually rational for agents to use a judgementally adjusted forecast in the scalar case when the relative variance of the judgemental variable is very large.

By continuity we deduce from these two special cases that there are values of $R = \sigma_\eta^2/\sigma_u^2 \in (0,\infty)$ such that $a \in (0,\rho)$ and agents rationally choose to use a judgementally adjusted forecast, given that all other agents are doing so.[14] The conclusion that it can be optimal to judgementally adjust the econometric forecast is striking since this forecast already reflects the effects of judgement on the time series properties of the observable variables. By construction, the econometric forecast is the best forecasting model based on observable information.

[14] The case with $a \approx \rho$ is a near-common factor representation of the time series, but the required variances remain continuous in the parameters, as can be seen from the formulae in Appendix B.

Figure 1. Including Judgement.

Scalar case.

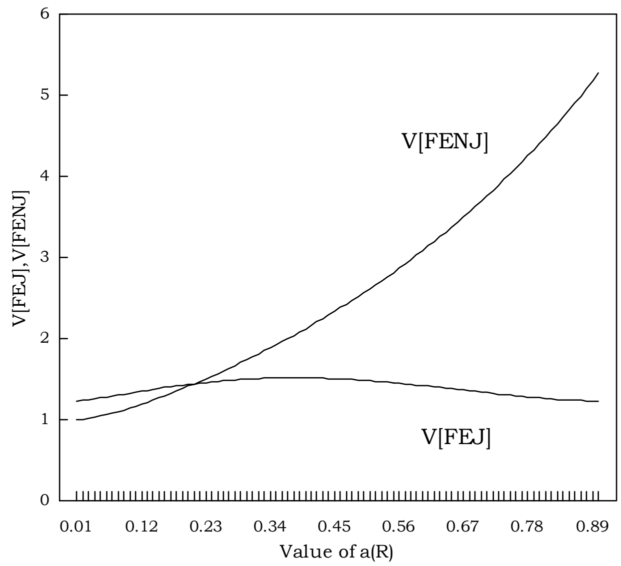


Figure drawn for ß=.9, rho=.9.

Figure 1: The variance of the forecast error, with (FEJ) and without (FENJ) judgement. The variance can be lower with judgement included, even for values of $a$ far from $\rho$.

To illustrate the point that the individual rationality constraint can be met even when $a$ is substantially less than $\rho$, we consider a numerical example. The forecast error variances in the general case involve the variance of an ARMA$(2,2)$ process. We show how to compute this variance in Appendix B, and illustrate the findings in Figure 1.

The Figure is drawn for $\beta = .9$ and $\rho = .9$, which corresponds to what might be regarded as a realistic case. The variances of the forecast errors with and without judgement are plotted on the vertical axis, while the value of $a$ is plotted on the horizontal axis. Each value of $a$ between zero and $\rho$ corresponds to a different relative variance $R = \sigma_\eta^2/\sigma_u^2$, and larger values of $R$ are associated with smaller values of $a$.[15] We have already seen from the examination of special cases that as $R \to \infty$, $a \to 0$ and we expect the forecast error variance of the econometric forecast to be smaller. This result is borne out in the Figure. In addition, we expect the variance of the judgementally adjusted forecast to be lower when $R \to 0$, in which case $a \to \rho$. This is also borne out in the Figure. But the Figure also shows intermediate cases, and indicates that $a$ does not have to be particularly close to $\rho$ for the individual rationality condition to be met. In fact, the two forecast error variances are equal at $a \approx .21$, which is far from the value of $\rho$ in this example, which is .9. We conclude that the conditions for exuberance equilibria to exist are quite likely to be met for a wide range of relative variances $R$ provided both $\beta$ and $\rho$ are relatively close to one.

This intuition can be partially verified by checking cases where $\beta$ and $\rho$ are not so large. Based on condition (11), one might conjecture that the individual rationality constraint is binding at values $\rho\beta < 1/2$. In fact, at $\rho = .7$ and $\beta = .7$, an exercise like the one behind Figure 1 shows that there are no values of $a$ that make the judgementally adjusted forecast preferable to the econometric forecast.

### 2.4 Learnability

Since we have made an assumption that the econometricians in the model are learning using recursive algorithms, we also need to impose learnability of any proposed equilibrium as a condition for plausibility. We study the stability of the system under learning following the literature on least squares learning in which the economic agents making forecasts are assumed to employ econometric models with parameters updated over time as new data becomes available.[16] The standard way to analyze systems under learning is to employ results on recursive algorithms such as recursive least squares.

[15] To draw Figure 1, we consider changes in $R$ resulting from changes in $\sigma_u^2$ with $\sigma_\eta^2$ fixed.

[16] Evans and Honkapohja (2001) gives a systematic treatment of adaptive learning and its implications in macroeconomics. Evans and Honkapohja (1999), Marimon (1997) and Sargent (1993, 1999) provide surveys of the field.

In many applications it can be shown that there is convergence to rational expectations equilibrium, provided the equilibrium satisfies a stability condition.

In the current context, the CEE formulated above takes the form of an ARMA(1,1) process. Estimation of ARMA(1,1) processes is usually done using maximum likelihood techniques, taking us beyond standard least squares estimation. Recursive maximum likelihood (RML) algorithms are available and they have formal similarities to recursive least squares estimation.[17] Because this technical analysis is relatively unfamiliar, we confine the formal details to Appendix D.2. However, the results are easily summarized. Let $a_t$ and $b_t$ denote estimates at time $t$ of the coefficients of the ARMA forecast function (6). Numerical computations using RML indicate convergence of $(a_t, b_t)$ to $(a, \rho)$, where $a > 0$ is the CEE value given in Lemma 1. Thus this CEE is indeed stable under learning. Moreover, in Section 2.5 we state a formal convergence result as part of our existence theorem.

## 2.5 Existence and properties of equilibrium

We now collect the various results above. The following theorem gives the key results about existence of an exuberance equilibrium in the univariate model and characterizes its asymptotic variance:

**Theorem 2** *Consider the univariate model with judgement and suppose that* $\beta > 1/2$. *Then*
*(i) for appropriate AR(1) judgement processes there exists an exuberance equilibrium and*
*(ii) the exuberance equilibrium has a higher asymptotic variance than the rational expectations equilibrium.*

**Proof.** (i) The preceding analysis has verified that the conditions 1 and 2 for an exuberance equilibrium defined in Section 2.1.4 are met for all $\sigma_\eta^2 > 0$

[17] They are also called Recursive Prediction Error (RPE) algorithms—see Evans and Honkapohja (1994) and Marcet and Sargent (1995) for other uses of RPE methods in learning.

sufficiently small. In Appendix D it is proved that condition 3 also holds, that is, the CEE is stable under RML learning, when $\sigma_\eta^2 > 0$ is sufficiently small.

(ii) The rational expectations equilibrium for the univariate model is $y_t = u_t$ since $0 < \beta < 1$ and $u_t$ is *iid* with mean zero. The exuberance equilibrium with $a > 0$ can be represented as the ARMA(1,1) process $y_t = \rho y_{t-1} + v_t - a v_{t-1}$ where $a$ solves equation (28) given in Appendix A. From (27) of Appendix A it can be seen that

$$\sigma_v^2 = \frac{\rho}{a(\beta(a-\rho)+1)}\sigma_u^2 > \sigma_u^2$$

since $a < \rho$ and $0 < \beta, \rho < 1$. Next, using the formula for the variance of an ARMA(1,1) process we have

$$\sigma_y^2 = \frac{1 + a^2 - 2\rho a}{1-\rho^2}\sigma_v^2$$

and since $\frac{1+a^2-2\rho a}{1-\rho^2} > 1$, the result follows. ■

The theorem states that in an exuberance equilibrium, the variance of the state variable $y_t$ is larger than it would be in a fundamental rational expectations equilibrium. This is because the REE has $y_t = u_t$, so that $\sigma_y^2 = \sigma_u^2$, but in an exuberance equilibrium $\sigma_y^2 > \sigma_u^2$.

## 2.6 An asset pricing example

A simple univariate example of the framework (1) is given by the standard present value model of asset pricing. A convenient way of obtaining the key structural equation can be based on the quadratic heterogeneous agent model of Brock and Hommes (1998). In their framework agents are myopic mean-variance maximizers who choose the quantity of riskless and risky assets in their portfolio to maximize expected value of a quadratic utility function of end of period wealth.

We modify their framework to allow for shocks to the supply of the risky asset. For convenience we assume homogeneous expectations and constant known dividends. The temporary equilibrium is given by

$$p^e_{t+1} + d - R_f p_t = s_t,$$

where $d$ is the dividend, $p_t$ is the price of the asset and $R_f > 1$ is the rate of return factor on the riskless asset. Here $s_t$ is a linear function of the random supply of the risky asset per investor, assumed *i.i.d.* for simplicity.[18] Defining $y_t = p_t - \bar{p}$, where $\bar{s} = Es_t$ and $\bar{p} = (d - \bar{s})/(R_f - 1)$, we obtain (1) with $\beta = R_f^{-1}$ and $u_t = -R_f^{-1}(s_t - \bar{s})$. We assume that $0 < \beta < 1$.

The univariate equation (1) is a benchmark model of asset pricing and there are, of course, alternative ways to derive the same equation. Because $0 < \beta < 1$ the model is said to be regular or determinate, that is, under rational expectations there is a unique nonexplosive solution, given by the "fundamentals" solution $y_t = u_t$. In particular, under rational expectations, sunspot solutions do not exist.

Theorem 2 shows that the basic asset pricing model is consistent with excess volatility. If investors incorporate judgemental factors that are strongly serially correlated, they will find that this improves their forecasts, but in an exuberance equilibrium this will also generate significant stationary asset price movements in excess of those associated with fundamental factors. The stationarity of our exuberance movements is in marked contrast to the literature on rational asset price bubbles. Because the latter are explosive, the literature on rational bubbles has been punctuated by controversy and complicated by the need to construct valid tests for non-stationary bubbles. Exuberance equilibria offer an alternative approach to modeling bubbles within a stationary time series framework.

## 2.7 Excess volatility

A natural question is whether the excess volatility associated with an exuberance equilibrium is economically meaningful, or if the exuberance conditions

[18] Using the notation of Brock and Hommes (1998) $s_t = a\sigma^2 z_{st}$, where $\sigma^2$ is the conditional variance of excess returns (assumed constant), $a$ is a parameter of the utility function and $z_{st}$ is the (random) asset supply.

**Table 1. Excess Volatility**

| | $\sigma_\xi/\sigma_u$ | | | |
|---|---|---|---|---|
| | 0.50 | 1.00 | 1.50 | 2.00 |
| $\rho = 0.70$ | 1.54 | 2.74 | – | – |
| $\rho = 0.80$ | 1.85 | 3.62 | 5.58 | – |
| $\rho = 0.90$ | 2.70 | 5.82 | 9.11 | 12.43 |
| $\rho = 0.95$ | 3.99 | 8.75 | 13.64 | 18.56 |

Table 1: Exuberance equilibria in the asset pricing model. A dash indicates that exuberance equilibrium does not exist. The entries in the table give one measure of the degree of excess volatility generated, namely, the ratio of the standard deviation of y to the standard deviation of u. The model can easily generate substantial excess volatility like that estimated by Shiller (1981).

outlined in Theorem 2 are only met for situations in which the variance $\sigma_y^2$ is just trivially larger than the fundamental variance. This is not clear from the theorem since $a$ is itself a nonlinear function of $\beta$, $\rho$, and $R = \sigma_\eta^2/\sigma_u^2$. It is also of interest to know if the excess volatility effect isolated in the theorem is large enough to be comparable to empirical estimates of the degree of excess volatility in financial data. One famous calculation due to Shiller (1981) put the ratio of the standard deviation of U.S. stock prices to the standard deviation of prices based on fundamental alone at between 5 and 13.[19]

Table 1 provides some illustrative calculations of exuberance equilibria for representative parameter values. In the Table, instead of considering the relative variance $R = \sigma_\eta^2/\sigma_u^2$, we consider the perhaps more intuitive ratio of the standard deviation of the exuberance variable to the standard deviation of the fundamental shock $\sigma_\xi/\sigma_u$.[20] Ratios of $\sigma_\xi/\sigma_u$ near unity correspond to ratios of innovation variances $\sigma_\eta^2/\sigma_u^2$ on the order of 0.1 for a high degree of serial correlation, so that the noise associated with judgement in the economy is actually quite modest. The table gives results for several possible values of $\sigma_\xi/\sigma_u$, ranging from 0.5 to 2.0. We examine the empirically realistic case

[19]Shiller (1981) actually compared the variance of equity prices to the variance of their ex post price (the present value of actual future dividends), but the latter must exceed the variance of the fundamentals price under rational expectations.

[20]Note that $\sigma_\xi^2 = \sigma_\eta^2/\left(1-\rho^2\right)$.

where the discount factor $\beta = 0.95$, and where the degree of serial correlation $\rho$ is relatively high, as indicated in the leftmost column. A dash in the table indicates that an exuberance equilibrium does not exist for the indicated parameter values. The entries in the table are a measure of excess volatility corresponding to Shiller's (1981) concept, namely, $\sigma_y/\sigma_u$. The results indicate that these measures are often in the range of 5 to 13 estimated by Shiller. We conclude based on this illustrative calculation that the model can generate substantial excess volatility without difficulty. We remark that if we push the discount factor $\beta$ closer to unity, the degree of excess volatility can rise to very high levels for high degrees of serial correlation, with $\sigma_y$ many hundreds of times larger than $\sigma_u$. In this sense, the model can generate arbitrarily large amounts of excess volatility.

We again emphasize that $0 < \beta < 1$ corresponds to the determinate case for this model, that is, the rational expectations equilibrium is unique. However, for $0.5 < \beta < 1$ exuberance equilibria exist even though sunspot equilibria do not exist. We think this feature of our findings is striking as it means that what would normally be regarded as benign circumstances can actually be dangerous situations, with the possibility of near-rational exuberance.

## 2.8 Correlation between judgement and fundamentals

Thus far, we have assumed that the judgement variable is not intrinsically related to the fundamentals. To widen the scope of our analysis we consider correlation between judgement and unobserved fundamentals. In this case, judgement can be viewed as imprecise knowledge of some unobserved shocks that hit the economy. In this section we will show that our results are unaffected by this extension.

The extended model is

$$y_t = \beta y_{t+1}^e + u_t + w_t, \tag{12}$$

where we have added a second unobservable shock $w_t$, which is assumed to be *iid* and independent of $u_t$ for simplicity. The judgement process is still

(3), where the "news" or innovation of the judgement process is now

$$\eta_t = f w_t + \hat{\eta}_t. \tag{13}$$

In other words, the news consists of both information on the shock $w_t$ as well as extraneous noise $\hat{\eta}_t$. (We assume that $w_t$ and $\hat{\eta}_t$ are independent.) The latter can be interpreted either as extraneous randomness or as measurement error of $w_t$.

The formal analysis can be extended a straightforward way. The ALM (which previously was (7)) can now be written as

$$y_t = \frac{1 - aL}{\beta\,(a-b) + 1 - aL}\left[(1-\rho L)^{-1}\beta(f w_t + \hat{\eta}_t) + (u_t + w_t)\right] \tag{14}$$

and the requirements for CEE, incentives to include judgement and learnability can be modified accordingly:

**Proposition 3** *Consider the univariate model with judgement correlated with fundamentals as above. If $\beta > 1/2$, then for appropriate AR(1) judgement processes there exists an exuberance equilibrium.*

Formal details are in Appendices C and D. We remark that the result goes through with a significant degree of correlation between the economic fundamental $u_t + w_t$ and the judgement innovation $\eta_t$ (see Appendix C).

## 2.9 Further discussion of near rationality

Our exuberance equilibrium is near-rational but not fully rational. There are two ways in which we have imposed assumptions that deviate from full rationality. First, the judgement process $\xi_t$ is assumed not directly available to (or usable by) econometric forecasters, who rely purely on the observables $y_t$. This seems realistic because $\xi_t$ represents the impact of "unique" qualitative events. More specifically, $\xi_t$ is the adjustment the judgemental forecasters believe is appropriate to make to the econometric forecast. This procedure thus reflects a natural division of labor in which the econometricians produce the best statistical forecast based on the observable variables of interest, and the judgemental forecasters modify these forecasts as they think appropriate to reflect additional qualitative factors. Although $\xi_t = y^e_{t+1} - E_t y^\star_{t+1}$ may possibly be obtainable by the econometricians (at least with a lag), we would expect the judgemental forecasters to resist the incorporation of $\xi_t$ into the econometric model.

Furthermore, older $\xi_{t-j}$ represent different unique events, unrelated to the current judgemental variable. Econometric models sometimes incorporate dummy variables (or other proxies) to capture the quantitative effects of qualitative events, but as the events become more distant such variables tend to get dropped and rolled into the unobserved random shocks in order to preserve degrees of freedom. The impact of recent qualitative events could be estimated by incorporating dummy variables into the econometric model, but for forecasting purposes this would be unhelpful, and would still leave the problem of forecasting the future impact of qualitative factors to the judgemental forecasters.

The second way in which our exuberance equilibrium is not fully rational is that the incentive condition is assumed dichotomous. This also seems realistic, since its inclusion is determined by the judgemental forecaster. Furthermore, econometric tests of whether "all" of $\xi_t$ should have been included would (often) have low power. Suppose we allowed for just a proportion $k \in [0, 1]$ of the judgement to be included in the forecast. It can be shown that the minimum MSE in the univariate case occurs at $k = \beta\rho$, where for an exuberance equilibrium we expect $0.5 < \beta\rho < 1$. For $\beta\rho$ near one, rationality tests using

$$y_{t+1} - y^e_{t+1} = (1 - k)\,\xi_t + \zeta_{t+1} \tag{15}$$

of the null hypothesis $H_0 : k = 1$, would have low power, and considerable data would be required to detect that not all of $\xi_t$ should be optimally included.

We illustrate this point in Table 2, which takes into account both points just discussed. Suppose that econometricians do have access ex post to the judgementally adjusted forecasts $y^e_{t+1}$, and therefore to $\xi_t = y^e_{t+1} - E^\star_t y_{t+1}$,

**Table 2. Test rejection rates**

| | $\rho$ | | | | | |
|---|---|---|---|---|---|---|
| | 0.7 | 0.8 | 0.85 | 0.9 | 0.95 | 0.99 |
| $n = 120$ | 47.6 | 5.1 | 0.1 | 0 | 0.4 | 2.0 |
| $n = 240$ | 86.8 | 15.9 | 1.4 | 0 | 0.4 | 1.7 |
| $n = 480$ | 99.7 | 48.0 | 3.5 | 0 | 0.4 | 1.0 |

Table 2: Exuberance equilibria in the asset pricing model. Percent of test rejections at 5 percent level of the null hypothesis that including judgement is fully rational, that is, Ho: k=1. Results given are based on 1000 replications.

and that they estimate (15) and test the null hypothesis $H_0 : k = 1$ that the inclusion of judgement is fully rational. For the purposes of this test we set the discount factor at $\beta = 1 - 0.05/12 = 0.9958$ in line with a real monthly risk-free rate of return of 0.05/12.[21] We also set $\sigma_\xi^2/\sigma_u^2 = 1.0$. The three sample sizes shown correspond to 10, 20 and 40 years of monthly data and the nominal significance level of the test is set at 5%. When $\rho$ is below 0.8 one would expect to eventually detect a deviation from full rationality. However it can be seen that for $\rho$ at or above 0.85, rejection of the null is unlikely even with 40 years of data. In particular, for $\rho = 0.9$ or $\rho = 0.95$ any deviation of the judgmental forecasts from full rationality would be virtually undetectable except with enormous sample sizes. Furthermore, these cases correspond to large, empirically plausible values of excess volatility: for the parameter settings of Table 2 we have excess volatility measures of $\sigma_y/\sigma_u = 8.40$ for $\rho = 0.9$ and $\sigma_y/\sigma_u = 16.39$ for $\rho = 0.95$.

From Table 2 we see that, for an exuberance equilibrium with $\rho$ values above 0.85, decision makers are likely to conclude that the functional division of labor between econometricians, who supply forecasts based on the observable variable $y_t$, and judgemental forecasters, who adjust these forecasts to take account of perceived qualitative events omitted from the econometric model, is entirely appropriate. Because an exuberance equilibrium is a CEE,

[21] In the Brock and Hommes (1998) set-up, $\beta$ is the inverse of the risk-free real rate-of-return factor. The value chosen here corresponds to an annual discount rate of 5% p.a., but the results of Table 2 are quite similar if 3% p.a. (or 7% p.a.) is used.

the econometricians are fully taking into account the predictable serial correlation properties of the variable being forecast. At the same time, the mean square forecast error is smaller for the judgemental forecasts than for the pure econometric forecast, and thus there is a clear gain to forecast performance in making use of the judgemental adjustment. Furthermore, for sufficiently serially correlated judgement processes, econometric tests of the forecast errors would not detect any deviation from full rationality of the judgementally adjusted forecasts. Exuberance equilibria thus appear to be plausible outcomes in the asset pricing model.

The uniqueness of qualitative events is also relevant to the issue at hand. Suppose, for example, that $\rho = 0.8$ and that rationality tests eventually indicate a statistically significant deviation from full rationality, with an estimated value near $k = 0.8$. It does not really seem plausible that forecasters would decide to downweight current judgemental adjustments, based on the finding that such adjustments over the last 20 years or so have been about 20% too high, since past judgemental adjustments mainly concerned different qualitative events, and since the adjustments may have been made by different judgemental forecasters. Furthermore, even if on this basis current judgement is downweighted, and even if this eventually results in the role of judgement being gradually extinguished over time, a new qualitative event will at some point suggest the need once again for judgement, with the judgement process again becoming persistent. In this sense, an economy in which exuberance equilibria exist always remains "subject to judgement." An economy subject to judgement contrasts strongly with an economy that is nonexuberant. In the latter case there is no incentive to include judgement, since unadjusted econometric forecasts have lower mean-square error, whether or not other agents include judgement.

# 3 The multivariate case

## 3.1 The linear framework

The basic features of the univariate analysis extend directly to the multivariate case. We again have

$$y_t = \beta y^e_{t+1} + u_t \tag{16}$$

where now $y_t$ is a vector of the economy's state variables, $\beta$ is a conformable matrix of parameters, and $u_t$ is a vector of stochastic noise terms. The judgement vector in the economy follows

$$(I - \rho L)\, \xi_t = \eta_t$$

where $I$ is a conformable identity matrix, $\rho$ is a conformable matrix with roots inside the unit circle, $\xi_t$ is a vector of judgement variables, and $\eta_t$ is a vector of stochastic noise terms.

One change we make to the analysis in the multivariate setting is that we now endow the agents with laws of motion that take the form of a VAR($p$) process. Recalling that in the univariate case an exact CEE is an ARMA(1,1) process, we might hypothesize that the CEE would take the form of a VARMA (vector ARMA) process in the multivariate case. However, this is formally difficult to verify. In any event VARMA procedures are not widely used and in practice the standard forecasting tool in multivariate settings is estimation of a VAR. We will show that a VAR($p$) process cannot deliver an exact CEE, but for large values of $p$ we will obtain close approximations.

The PLM is therefore specified as

$$y_t = \sum_{i=1}^{p} b_i y_{t-i} + v_t, \tag{17}$$

where $y_t$, $v_t$ are $n \times 1$ vectors, the $b_i$ are $n \times n$ matrices and $Ey'_{t-i}v_t = 0$ for $i = 1, \dots, p$. This leads to econometric forecasts $E^\star_t y_{t+1} = \sum_{i=0}^{p-1} b_{i+1} y_{t-i}$, and to the judgementally adjusted forecasts

$$y^e_{t+1} = \sum_{i=0}^{p-1} b_{i+1} y_{t-i} + \xi_t.$$

The ALM is thus

$$y_t = (I - \beta b_1)^{-1} \left\{ \sum_{i=1}^{p-1} \beta b_{i+1} y_{t-i} + \beta \xi_t + u_t \right\}. \tag{18}$$

It is easily verified that the ALM is a VARMA($p$,1) process and this is the sense in which the VAR($p$) PLM can only give an approximate CEE.

Let $b = (b_1, \ldots, b_p)$ and let $P[y_t \,|Y_{t-1}] = T(b)' Y_{t-1}$ be the linear projection of $y_t$ on $Y_{t-1}$ where $Y'_{t-1} = (y'_{t-1}, \ldots, y'_{t-p})$. Using standard results on linear projections,

$$T(b) = \left(E y_t Y'_{t-1}\right) (E Y_{t-1} Y'_{t-1})^{-1}. \tag{19}$$

An approximate CEE is defined as a value $\bar{b}$ that satisfies the equation $\bar{b} = T(\bar{b})$. We require also that all roots of $\det(I - \sum_{i=1}^{p} b_i L^i) = 0$ lie outside the unit circle so that $y_t$ is a stationary process. In an approximate CEE, for each variable the forecast errors $v_t$ of the econometric forecasters have the property that they are orthogonal to $Y'_{t-1}$. It follows that the agents are "getting right" all of the first $p$ autocovariances of the $y_t$ process. For a stationary process the autocovariances $E y_t y_{t-j} \to 0$ as $j \to \infty$ and thus stationary fixed points $\bar{b}$ deliver approximate CEE in the sense that as $p$ becomes large the econometric forecasters neglect only high order autocovariances that are vanishingly small.

To compute $T(b)$ one can write the system in first order form

$$z_t = B z_{t-1} + D \begin{pmatrix} u_t \\ \eta_t \end{pmatrix}$$

with $z_t = (Y'_t, \xi_t)'$. The relevant values for $\left(E y_t Y'_{t-1}\right)$ and $(E Y_{t-1} Y'_{t-1})$ can be obtained from the equation

$$\text{vec}(\text{Var}(z_t)) = [I - B \otimes B]^{-1} \text{vec}(D \left[ \text{Var} \begin{pmatrix} u_t \\ \eta_t \end{pmatrix} \right] D').$$

Here $\text{Var}\,(z_t)$ is the covariance matrix of $z_t$, $\text{vec}(K)$ is the vectorization of a matrix $K$ and $\otimes$ is the Kronecker product. The equilibrium $\bar{b}$ can then be calculated by the *E-stability algorithm*

$$b_s = b_{s-1} + \gamma(T(b_{s-1}) - b_{s-1}), \tag{20}$$

where $\gamma$ is chosen to be a small positive constant.

This procedure will automatically give us learnable equilibria in the following sense. The econometricians are estimating a VAR($p$) PLM for $y_t$ and are assumed to update their parameter estimates over time using recursive least squares (RLS). As previously explained, the decision makers add their judgemental adjustment to the econometricians' forecast and, together with the variable $u_t$, the current value of $y_t$ is determined. The vector $T(b)$ denotes the true coefficients projection for a given forecast coefficients $b$. Under RLS learning it can be shown that the estimates $b_t$ at time $t$ on average move in the direction $T(b_t)$. Equation (20) describes this adjustment in notional time $s$. Using the techniques of Evans and Honkapohja (2001), it can be shown that RLS learning converges locally to $\bar{b}$ if it is a locally asymptotically stable fixed point of (20), for sufficiently small $\gamma > 0$. Formal details of the RLS algorithm and learning are outlined in Appendix D.1.

Finally, we also require the condition concerning the incentives to include judgement. In other words, under what conditions is the covariance matrix of $y^e_{t+1} - y_{t+1}$ in some sense smaller than the covariance matrix of $E^\star_t y_{t+1} - y_{t+1}$?[22] Denote the covariance matrix without judgement as $\mathcal{M}(0)$ and with judgement as $\mathcal{M}(1)$. We will usually interpret the incentive to include judgement condition to mean that the component by component comparison of the matrices along the diagonal are all smaller for $\mathcal{M}(1)$. That is, we require that

$$\mathcal{M}(0)_i - \mathcal{M}(1)_i > 0$$

for all diagonal components $i$. By setting up the model in first-order state space form, and including in the state the forecast errors with and without judgement, it is straightforward to compute $\mathcal{M}(0) - \mathcal{M}(1)$ and test numerically for the existence of exuberance equilibria.

When an approximate CEE is stable under learning and satisfies the incentives to include judgement, then we refer to it as an *approximate exuberance equilibrium*.

[22] We implicitly are assuming that $y^e_{t+1}$ and $E^\star_t y_{t+1}$ have the same mean as $y_{t+1}$, so that variance of the forecast error is the same as the mean squared error. This will always hold in our analysis.

A simple illustration of an approximate exuberance equilibrium can be given by returning to the univariate case. Using the procedure just described we compute an AR(3) approximate CEE for $\beta = 0.9$, $\rho = 0.7$ and $R = 1$. The exact ARMA(1,1) CEE given in Lemma 1 is $b = 0.7$ and $a = 0.180814$. Computing the approximate AR(3) CEE using (20) we obtain

$$b_1 = 0.517259,\ b_2 = 0.0929807,\ b_3 = 0.0157536.$$

These values are fairly close approximations to the first three terms of the series expansion of $(1 - bz)/(1 - az)$ and thus provide an approximate CEE. Since for this solution the incentive and learnability conditions are also met, this describes an approximate exuberance equilibrium.

Returning to the incentives issue, in the multivariate case we sometimes refer to alternative versions of the incentives condition as a method of categorizing our results. If the individual rationality condition is met in the sense that the difference between the two covariance matrices is a positive definite matrix, in conjunction with the other two requirements, we say that a *strong exuberance equilibrium* exists. If some diagonal components of the difference between the two covariance matrices are positive, while others are negative, when all other conditions are met, this means that the agents may or may not come to the conclusion that including the judgemental adjustment is valuable. We will refer to this case as *indefinite.*

Another possibility is that the diagonal components of the difference between the two covariance matrices are all negative when all other conditions are met. In this case the agents would most likely conclude that the inclusion of judgement was not valuable. We call this case one of *non-exuberance.* Finally, to be complete, the difference could be a negative definite matrix in which case we say that there is *strong non-exuberance.*

## 3.2 Exuberance and monetary policy

As an example we now study exuberance equilibria in a New Keynesian macroeconomic model suggested by Woodford (2003) and Clarida, Gali, and

Gertler (1999). We use a simple, three-equation version given by

$$x_t = x_{t+1}^e - \sigma^{-1}\left[r_t - \pi_{t+1}^e\right] + \tilde{u}_{x,t}, \tag{21}$$

$$\pi_t = \kappa x_t + \delta \pi_{t+1}^e + \tilde{u}_{\pi,t}, \tag{22}$$

$$r_t = \varphi_\pi \pi_t + \varphi_x x_t. \tag{23}$$

In these equations, $x_t$ is the output gap, $\pi_t$ is the deviation of inflation from target, and $r_t$ is the deviation of the nominal interest rate from the value that is consistent with inflation at target and output at potential. All variables are expressed in percentage point terms and the steady state is normalized to zero. The terms $\tilde{u}_{x,t}$ and $\tilde{u}_{\pi,t}$ represent stochastic disturbances to the economy. The parameter $\sigma^{-1}$ is related to the elasticity of intertemporal substitution in consumption of a representative household. The parameter $\kappa$ is related to the degree of price stickiness in the economy, and $\delta$ is the discount factor of a representative household.[23] The third equation describes the Taylor-type policy rule in use by the policy authority, in which the parameters $\varphi_\pi$ and $\varphi_x$ are assumed to be positive. In the formulation (21)-(23), only private sector expectations affect the economy.

Substituting (23) into (21) and writing the system in matrix form gives (16) where $y_t = [x_t, \pi_t]'$, $y_{t+1}^e = \left[x_{t+1}^e, \pi_{t+1}^e\right]'$, $u_t = C\tilde{u}_t$, $\tilde{u}_t = [\tilde{u}_{x,t}, \tilde{u}_{\pi,t}]'$ with covariance matrix

$$\Sigma_u = \begin{bmatrix} \sigma_{u,11}^2 & \sigma_{u,12}^2 \\ \sigma_{u,21}^2 & \sigma_{u,22}^2 \end{bmatrix},$$

$$\beta = \frac{1}{\sigma + \varphi_x + \kappa\varphi_\pi} \begin{bmatrix} \sigma & 1 - \delta\varphi_\pi \\ \kappa\sigma & \kappa + \delta\left(\sigma + \varphi_x\right) \end{bmatrix},$$

and

$$C = \frac{1}{\sigma + \varphi_x + \kappa\varphi_\pi} \begin{bmatrix} \sigma & -\varphi_\pi \\ \kappa\sigma & \sigma + \varphi_x \end{bmatrix}.$$

[23]This formulation of the model is based upon individual Euler equations under (identical) private sector expectations. Other models of bounded rationality are possible, see, for instance, Preston (2005) for a formulation in which long-horizon expectations directly affect individual behavior.

## 3.3 Results

### 3.3.1 A Taylor-type monetary policy rule

We now illustrate the possibility of approximate exuberance equilibria in the New Keynesian model. We use Woodford's (2003) calibration $\sigma = 0.157$, $\kappa = 0.024$, and $\delta = 0.99$. For the exuberance variable we assume the matrix describing the degree of serial correlation is $\rho = \text{diag}(0.99, 0.95)$ and $\Sigma_\eta = \text{diag}(0.0035, 0.0035)$.[24] The variances of the fundamental shocks are assumed to be $\Sigma_{\tilde{u}} = \text{diag}(1.1, 0.03)$. No real attempt has been made to calibrate the shocks except to choose values that, in the exuberance equilibrium, roughly match U.S. inflation and output-gap variances measured in percent.

The policy parameters $\varphi_\pi$ and $\varphi_x$ can be varied and we are interested in values of $\varphi_\pi$ and $\varphi_x$ that might be consistent with exuberance equilibrium. Consider $\varphi_\pi = 1.05$ and $\varphi_x = 0.05$. These values satisfy the Taylor principle and deliver a determinate rational expectations equilibrium in the usual set-up; see Bullard and Mitra (2002). Suppose that econometricians estimate a VAR(3). In the approximate CEE the coefficients of the vector autoregression are approximately

$$\begin{aligned} b_1 &= \begin{pmatrix} 0.0975 & -0.3319 \\ 0.0759 & 0.8775 \end{pmatrix}, \\ b_2 &= \begin{pmatrix} 0.0976 & 0.0108 \\ 0.0012 & 0.0902 \end{pmatrix}, \end{aligned}$$

and

$$b_3 = \begin{pmatrix} 0.0586 & 0.0731 \\ -0.0037 & 0.0071 \end{pmatrix}.$$

This corresponds to a stationary process. The output variance is approximately 2.54 and the inflation variance is approximately 6.14. The matrix describing the key condition for individual rationality, $\mathcal{M}(0) - \mathcal{M}(1)$, is positive definite, hence the CEE is strongly exuberant. As in the scalar model the exuberance equilibrium exhibits excess volatility. In fact, the ratio of the output-gap standard deviation in the exuberance equilibrium to its standard

[24] Somewhat lower values of the $\rho$ parameters delivered qualitatively similar results.

deviation in the fundamental rational expectations equilibrium is about 1.5 and for the standard deviation of inflation the corresponding ratio is almost 16!

In this example we can also show that a change in the Taylor-rule coefficients can diminish the likelihood of exuberance equilibria. When $\varphi_\pi$ is increased to 1.1 the equilibrium is no longer strongly exuberant but it does remain exuberant. However, if $\varphi_\pi$ is increased to 1.5 and $\varphi_x$ is increased to 0.1, the possibility of an exuberance equilibrium is eliminated. In this sense, a more aggressive policy tends to reduce the likelihood of an exuberance equilibrium.

We next analyze the idea that more aggressive policy is less likely to be associated with the existence of exuberance equilibrium more systematically. For this, we calculate the conditions for exuberance equilibrium using the calibration given above but allowing the Taylor rule coefficients to vary. The results are given in Figure 2, where $\varphi_\pi \in (0, 1.25)$ and $\varphi_x \in (0, 0.25)$ at selected grid points. The open squares indicate the points where determinacy and learnability of the rational expectations equilibrium hold for this model.[25] The Figure displays the points at which exuberance equilibria exist. These points tend to be for values of $\varphi_x$ less than about 0.08, and for values of $\varphi_\pi$ up to 1.25. Again, these exuberance equilibria exist in the region associated with determinacy, and therefore can arise in parameter regions where sunspot equilibria are ruled out.

While Figure 2 illustrates where exuberance equilibria exist in this economy, it is not comforting regarding the possibility that policymakers may be able to choose policy parameters so as to rule out exuberance equilibria. According to the Figure, either an exuberance equilibrium exists or the indefinite case arises (plain open boxes in the Figure). However, if we expand the space of points considered, it becomes apparent that more aggressive policy can produce situations characterized by non-exuberance. This is shown in Figure 3, where the region of the policy parameter space has been expanded

[25] The blank area to the left in this figure is associated with indeterminacy of rational expectations equilibrium.

Figure 2

Existence of Exuberance Equilibria

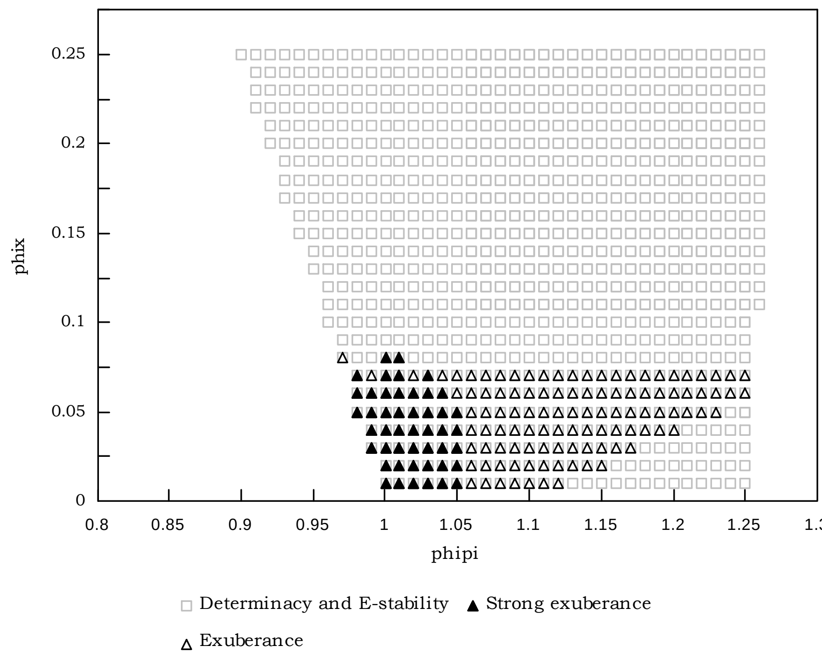


Figure 2: Exuberance equilibria in the New Keynesian model. Open boxes indicate points where the REE is determinate. Triangles indicate points where exuberance equilibria exist.

so that $\varphi_\pi \in (0, 2.5)$ and $\varphi_x \in (0, 0.45)$ at selected grid points. In this Figure, the region associated with exuberance from Figure 2 appears near the point $(1, 0)$. However, there is now a region of the policy parameter space that is associated with non-exuberance. This part of the space involves more aggressive reactions to both inflation deviations and the output gap. In this sense, a more aggressive policy can mitigate the possibility of exuberance equilibrium in this economy.

Figure 3

Existence of Exuberance Equilibria

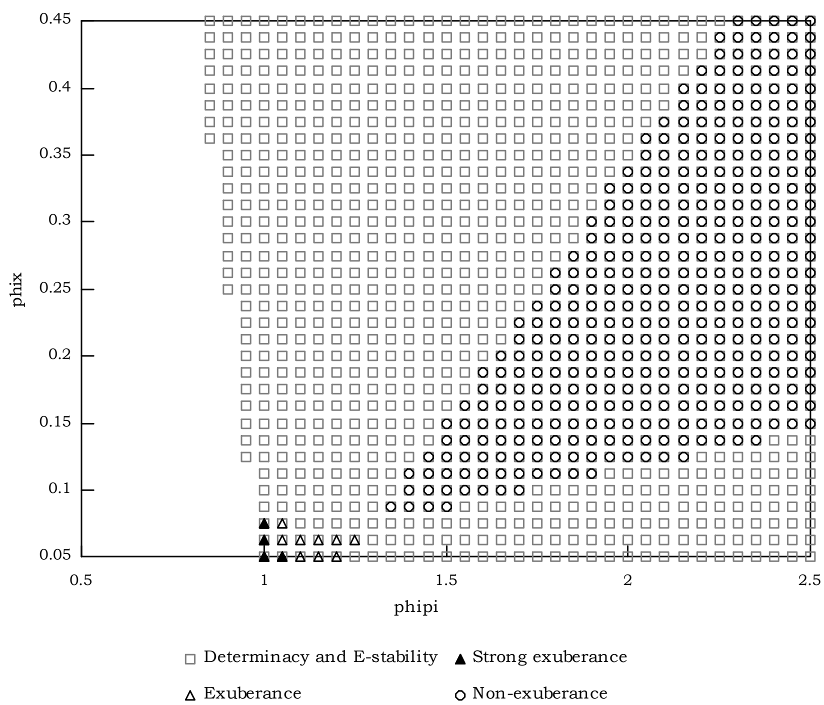


Figure 3: A sufficiently aggressive Taylor-type policy is associated with non-exuberance, denoted by open circles.

### 3.3.2 A forward-looking monetary policy rule

It is also of interest to investigate an alternative Taylor-type interest rate rule,

$$r_t = \varphi_\pi \pi^e_{t+1} + \varphi_x x^e_{t+1}, \tag{24}$$

in which policymakers react to forecasts of future values of the inflation deviation and the output gap. Interest-rate rules depending on expectations of future inflation and the output gap have been discussed extensively in the monetary policy literature and are subject to various interpretations. Here we are assuming that the monetary authorities form forecasts in the same way as the private sector, that is, by constructing an econometric forecast to which they consider adding the same judgement variable. We might hope that by reacting aggressively enough to expectations such a rule would diminish the likelihood of exuberance equilibria. With the policy rule (24) the reduced form system is the same except that the coefficient matrices become

$$\beta = \begin{bmatrix} 1-\sigma^{-1}\varphi_x & \sigma^{-1}(1-\varphi_\pi) \\ \kappa(1-\sigma^{-1}\varphi_x) & \delta+\kappa\sigma^{-1}(1-\varphi_\pi) \end{bmatrix}$$

and

$$C = \begin{bmatrix} 1 & 0 \\ \kappa & 1 \end{bmatrix}.$$

Using the same calibration, we calculate whether the conditions for exuberance equilibria hold for $\varphi_\pi \in (0, 3.5)$ and $\varphi_x \in (0, 0.35)$ at selected grid points. The results are plotted in Figure 4.[26] The open squares again indicate the points where determinacy and learnability of the rational expectations equilibrium hold for this model.[27] As with the standard Taylor-type rule, the Figure indicates that exuberance equilibria exist near the point $(1, 0)$. Again, more aggressive policy delivers non-exuberance. Comparing this performance to that of the contemporaneous rule, we see that non-exuberance begins to arise for smaller values of $\varphi_\pi$ and $\varphi_x$ – see Figures 3 and 4. In particular, with the forward-looking rule even very small values of $\varphi_x$ are sufficient to yield non-exuberance if $\varphi_\pi$ is greater than (approximately) 1.8. In this sense the performance of the forward-looking rule appears superior, which provides one potential justification for their use by central banks.[28] We conclude that by following an explicit policy of reacting against the de-

[26]There is a subtlety in this example due to the fact that the central bank has non-negligible macroeconomic effects. We assume that in comparing the performance of forecasts with and without judgement they compare forecasts to actual, realized, data.

[27]For the forward-looking rule, indeterminacy of the fundamental rational expectations equilibrium occurs not only in the blank area to the left in the figure, but also in the blank area toward the top of the figure.

[28]This example in Figure 4 also produces strong non-exuberance for large enough values of $\varphi_\pi$, approximately 3.25 or greater in this figure, depending on the value of $\varphi_x$.

Figure 4

Exuberance with Forward-Looking Rules

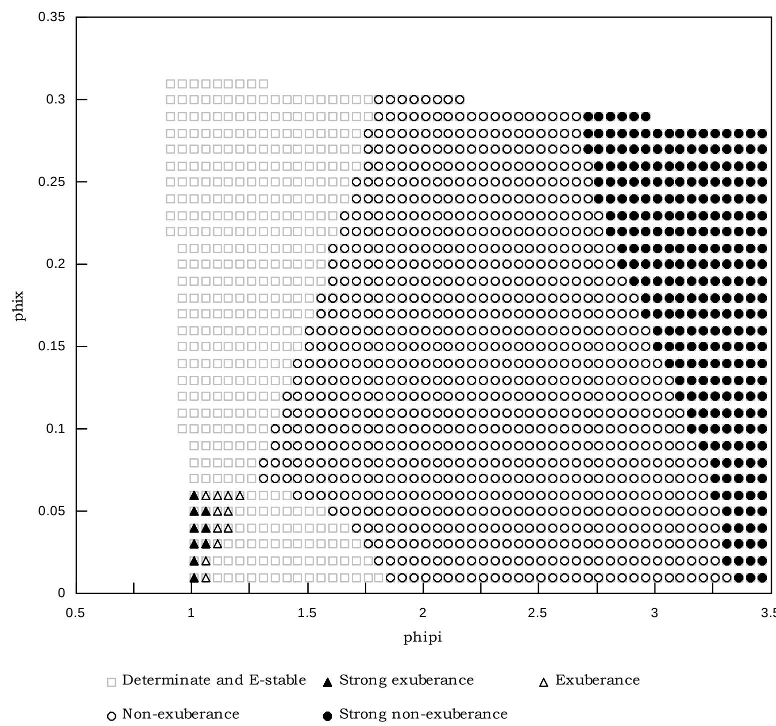


Figure 4: Sufficiently aggressive policy is again associated with non-exuberance when the policy rule is forward-looking.

viations of expectations from the values justified by the fundamental shocks, monetary authorities enhance the stability of the economy.

### 3.3.3 Optimal monetary policy rules

Finally, we discuss optimal discretionary policy as in Evans and Honkapohja (2003). They assign a standard quadratic objective to the policymaker with weight $\alpha$ on output gap variance. They write the resulting optimal policy as a Taylor-type rule in the expected output gap and the expected inflation deviation, along with reactions to fundamental shocks in the economy. Their policy rule delivers determinacy, and the unique stationary rational expectations equilibrium is stable under least squares learning for all values of structural parameters and the policy weight. We can denote this optimal policy rule as

$$r_t = \varphi_\pi^\star \pi_{t+1}^e + \varphi_x^\star x_{t+1}^e + \varphi_{u,x}^\star \tilde{u}_{x,t} + \varphi_{u,\pi}^\star \tilde{u}_{\pi,t}. \tag{25}$$

where the optimal values $\varphi_x^\star = \varphi_{u,x}^\star = \sigma$, and the matrices $\beta$ and $C$ become

$$\beta = \begin{bmatrix} 0 & \sigma^{-1}\left(1-\varphi_\pi^\star\right) \\ 0 & \delta + \kappa\sigma^{-1}\left(1-\varphi_\pi^\star\right) \end{bmatrix}$$

and

$$C = \begin{bmatrix} 0 & -\sigma^{-1}\varphi_u^\star \\ 0 & 1-\sigma^{-1}\kappa\varphi_u^\star \end{bmatrix}.$$

The relationship between $\varphi_{u,\pi}^\star$ and $\varphi_\pi^\star$ is given by $\varphi_{u,\pi}^\star = \delta^{-1}\left(\varphi_\pi^\star - 1\right)$. This leaves only the optimal choice of $\varphi_\pi^\star$, which depends on $\alpha$. A small policy weight on output gap variability $\alpha \to 0$ (an inflation hawk), is associated with an optimal value $\varphi_\pi^\star = 1 + \sigma\delta\kappa^{-1} \approx 7.47$. A large weight on output gap variability, $\alpha \to \infty$, (an inflation dove), is associated with an optimal value $\varphi_\pi^\star \to 1$. Thus we can calculate whether exuberance equilibria exist for all possible values of the policymaker weight $\alpha$ by choosing values for $\varphi_\pi^\star \in (1, 7.47)$.

The results of this calculation[29] indicate that for values of $\varphi_\pi^\star \in (1, \bar{\varphi}_\pi)$ the equilibrium is in the indefinite region. For values $\varphi_\pi^\star \in (\bar{\varphi}_\pi, 7.47)$, the equilibrium is non-exuberant. The cutoff value is $\bar{\varphi}_\pi \approx 1.557$. Thus standard optimal policy calculations alone are not enough to ensure non-exuberance. To move into the non-exuberance region, policymakers must have a sufficiently small weight on output gap variability. The policy weight value associated with $\varphi_\pi^\star = 1.557$ is quite low, approximately $\alpha \approx 0.00612$. More weight than this on output gap variance implies a value for $\varphi_\pi^\star$ that is too low, in the sense that it places the equilibrium in the indefinite region.[30]

[29] The exact optimal policy rule would create perfect multicollinearity in this system. To avoid this complication, we set $\varphi_x = 1.01\sigma$, slightly higher than the optimal value.

[30] If we assume that the policymaker has the same preferences as the representative

# 4 Conclusions and possible extensions

We have studied how a new phenomenon, *exuberance equilibria*, may arise in standard macroeconomic environments. We assume that agents are learning in the sense that they are employing and updating econometric models used to forecast the future values of variables they care about.[31] Unhindered, this learning process would converge to a rational expectations equilibrium in the economies we study. We investigate the idea that decision-makers may be tempted to include judgemental adjustments to their forecasts if all others in the economy are similarly judgementally adjusting their forecasts. The judgemental adjustment, or add factor, is a pervasive and widely-acknowledged feature of actual macroeconometric forecasting in industrialized economies. We obtain conditions under which such add-factoring can become self-fulfilling, altering the actual dynamics of the economy significantly, but in a way that remains consistent with the econometric model of the agents.

In order to develop our central points we have made some strong simplifying assumptions. We have assumed that the exuberance or judgement variables take a simple autoregressive form, but this assumption is mainly made for convenience. While we do believe that judgemental adjustments exhibit strong positive serial correlation, a more complicated stationary stochastic process could instead be used and in principle even time varying distributions could be incorporated into our framework. The assumption of identical judgements of different (representative) agents is correspondingly restrictive. Allowing for differences in judgements by individual agents would probably make the conditions for exuberance equilibrium more difficult to achieve. On the other hand, this could create new phenomena, such as momentum effects

household, we obtain a value of $\alpha \approx .00313$ at the calibrated values of Woodford (2003). (This is calculated as $\kappa/\theta = 0.024/7.67$, where $\theta$ is the parameter controlling the price elasticity of demand.) The value of $\varphi_\pi^\star$ for any specified $\alpha$ is $1 + \kappa\delta\sigma\left(\alpha + \kappa^2\right)$. This would suggest an optimal value of $\varphi_\pi^\star \approx 2.0$, large enough to imply non-exuberance.

[31] In line with this literature, the econometric forecasts are based on reduced form models. It would also be of interest to examine the questions we have studied in the context of econometric forecasts based on structural models.

arising when a large fraction of agents begin to agree in their judgements.

The incorporation of judgment into decisions, in the form of adjustments to econometric forecasts, can have a self-fulfilling feature in the sense that decisions makers would believe *ex post* that their judgement had improved their forecasts. This result is similar in spirit to the self-fulfilling nature of sunspot equilibria, but with the novel feature that it can arise in determinate models in which there is a unique rational expectations equilibrium that depends only on fundamentals. This widens the set of models in which self-fulfilling fluctuations might plausibly emerge. In particular, we have shown that exuberance equilibria can arise in the standard asset-pricing model, generating substantial excess volatility. Exuberance equilibria can also arise in New Keynesian models, with monetary policymakers following standard interest-rate rules, but can be eradicated if policymakers take an appropriately aggressive stance.

# Appendices

## A Conditions for CEE in the scalar case

The sum of the two functions $G_\eta(z)$ and $G_u(z)$ is

$$G_{ALM}(z) = \frac{(1-az)\left(1-az^{-1}\right)}{(1-\rho z)\left(1-\rho z^{-1}\right)}\times \\ \left\{\frac{\beta^2\sigma_\eta^2 + (1-\rho z)\left(1-\rho z^{-1}\right)\sigma_u^2}{\left[\beta(a-b)+1-az\right]\left[\beta(a-b)+1-az^{-1}\right]}\right\}.$$

It can be seen from the form of $G_{ALM}(z)$ that, for arbitrary $a$ and $b$, the ALM is an ARMA(2,2) process. As we will now show, there are choices of $a$ and $b$ that yield $G_{PLM}(z) = G_{ALM}(z)$. These choices of $a$ and $b$ also have the property that the corresponding ALM takes an ARMA(1,1) form that matches the PLM. This is possible if $a$ and $b$ are chosen so that there is a common factor in the numerator and denominator of the expression on the right-hand side of $G_{ALM}(z)$.

We now set $G_{PLM}(z) = G_{ALM}(z)$, under the condition that $b = \rho$ so that the poles of the autocovariance generating functions agree. This yields

$$\sigma_v^2\left[\beta(a-\rho)+1-az\right]\left[\beta(a-\rho)+1-az^{-1}\right] = \\ \beta^2\sigma_\eta^2 + (1-\rho z)\left(1-\rho z^{-1}\right)\sigma_u^2.$$

This equation can be written as

$$\sigma_v^2\left\{\left[1+\beta(a-\rho)\right]^2 + a^2\right\} - \sigma_v^2 a\left[\beta(a-\rho)+1\right]\left(z+z^{-1}\right) = \\ \beta^2\sigma_\eta^2 + \sigma_u^2\left(1+\rho^2\right) - \sigma_u^2\rho\left(z+z^{-1}\right).$$

For the autocovariances of the perceived and actual laws of motion to be equal, the coefficients on the powers of $z$ in this equation must be equal. Equating these we obtain the two equations

$$\sigma_v^2\left\{\left[1+\beta(a-\rho)\right]^2 + a^2\right\} = \beta^2\sigma_\eta^2 + \sigma_u^2\left(1+\rho^2\right) \qquad (26)$$

and

$$\sigma_v^2 a \left[\beta \left(a-\rho\right)+1\right]=\sigma_u^2 \rho. \tag{27}$$

We wish to solve for a value of $a$ such that $|a|<1$. Solving equation (27) for $\sigma_v^2$ and substituting the result into equation (26), and in addition defining $s \equiv \beta^2 \sigma_\eta^2 + \sigma_u^2 \left(1+\rho^2\right)$, we obtain the quadratic equation

$$F\left(a\right) \equiv c_2 a^2 + c_1 a + c_0 = 0 \tag{28}$$

with

$$\begin{aligned} c_2 &\equiv s\beta - \rho\left(1+\beta^2\right)\sigma_u^2, \\ c_1 &\equiv s\left(1-\rho\beta\right) - 2\rho\beta\left(1-\rho\beta\right)\sigma_u^2, \\ c_0 &\equiv -\rho\left(1-\rho\beta\right)^2\sigma_u^2. \end{aligned}$$

We deduce that $F\left(0\right)<0$, and that

$$F\left(1\right)=\sigma_\eta^2\beta^2\left[1+\left(1-\rho\right)\beta\right]+\sigma_u^2\left(1-\rho\beta\right)\left(1+\beta\right)\left[\left(\rho-1\right)^2\right]>0.$$

These inequalities imply that there exists a positive root $a\in[0,1]$ to (28). Moreover, it is easy to compute that $F(\rho)>0$, so that the root must be less than $\rho$. We also note that for $\sigma_\eta^2\to 0$, $a=\rho$ solves equation (28), while for $\sigma_u^2\to 0$, $a=0$ is a solution. There can be a second, negative root. However, our numerical results indicate that the CEE corresponding to the negative root is not learnable.

# B Judgement in the scalar case

The induced actual law of motion, as depicted in equation (7), is

$$y_t = \frac{1-aL}{\beta\left(a-\rho\right)+1-aL}\left(\frac{\beta}{1-\rho L}\eta_t + u_t\right). \tag{29}$$

By substituting equation (29) into both (9) and (10), we can write the two types of forecasts in terms of the shocks $u_t$ and $\eta_t$. These expressions become

$$E_t^\star y_{t+1} = \frac{\rho-a}{\beta\left(a-\rho\right)+1-aL}\left(\frac{\beta}{1-\rho L}\eta_t + u_t\right)$$

in the case of no judgement, and

$$y^e_{t+1} = \frac{\rho - a}{\beta(a-\rho) + 1 - aL}\left(\frac{\beta}{1-\rho L}\eta_t + u_t\right) + \frac{1}{1-\rho L}\eta_t$$

in the case of the judgementally adjusted forecast. The actual state of the economy at time $t+1$ is, from equation (29),

$$y_{t+1} = \frac{1 - aL}{\beta(a-\rho) + 1 - aL}\left(\frac{\beta}{1-\rho L}\eta_{t+1} + u_{t+1}\right). \tag{30}$$

We can therefore compute forecast errors in each of the two cases. When computing these forecast errors, we save on clutter by ignoring the terms involving $u$, as these will be the same whether or not the agent judgementally adjusts the forecast. The forecast error in the case of no judgement can be written as

$$FE_{NJ} \equiv [y_{t+1} - E^{\star}_t y_{t+1}]\,|_{u=0} = \frac{\beta}{1+\beta(a-\rho)} \frac{1}{\left[1 - \left(\frac{a}{1+\beta(a-\rho)}\right)L\right]}\eta_{t+1} \tag{31}$$

whereas in the case of a judgementally adjusted forecast it is

$$FE_J \equiv \left[y_{t+1} - y^e_{t+1}\right]|_{u=0} =$$
$$\frac{\beta}{1+\beta(a-\rho)} \times \frac{1 - \left(a + \beta^{-1}\right)L + a\beta^{-1}L^2}{1 - \left(\frac{a+\rho[1+\beta(a-\rho)]}{1+\beta(a-\rho)}\right)L + \left(\frac{a\rho}{1+\beta(a-\rho)}\right)L^2}\eta_{t+1}. \tag{32}$$

These equations simplify to those given in the text for the case $a \to \rho$.

Apart from the lead coefficient $\beta/(1+\beta(a-\rho))$, each forecast error process is in the generic class

$$x_t = \frac{1 + m_1 L + m_2 L^2}{1 - \ell_1 L - \ell_2 L^2}\epsilon_t,$$

and the variance of $x_t$ is given by

$$Var(x_t) = \frac{x_{num}}{x_{den}}\sigma^2_{\epsilon}. \tag{33}$$

where

$$x_{num} = \frac{(1+\ell_2)\,\ell_1\,(m_1+m_2\ell_1+m_2m_1)}{1-\ell_2} + (m_1+m_2\ell_1)\,(\ell_1+m_1) + \left(1+2m_2\ell_2+m_2^2\right)$$

and

$$x_{den} = 1 - \frac{\ell_1^2}{1-\ell_2} - \frac{\ell_2\ell_1^2}{1-\ell_2} - \ell_2^2.$$

Considering the forecast error in the case without judgement included, equation (31), we set $m_1 = m_2 = \ell_2 = 0$ and $\ell_1 = a/\left[1+\beta\,(a-\rho)\right]$ in equation (33). For the case with judgement, we set

$$\begin{aligned} m_1 &= -\,(1+a\beta)\,\beta^{-1}, \\ m_2 &= a\beta^{-1}, \\ \ell_1 &= \frac{a+\rho\,[1+\beta\,(a-\rho)]}{1+\beta\,(a-\rho)}, \\ \ell_2 &= \frac{-a\rho}{1+\beta\,(a-\rho)}, \end{aligned}$$

and $a$ is determined by $\beta$, $\rho$ and $R = \sigma_\eta^2/\sigma_u^2$ as described in Appendix A.

# C The correlated case

The formal analysis in Appendices A and B is modified as follows. First, the autocovariance generating function for the ALM (14) is

$$\begin{aligned} G_{ALM}(z) &= \frac{(1-az)\,(1-az^{-1})}{(1-\rho z)\,(1-\rho z^{-1})} \times \\ &\left\{ \frac{\beta^2\sigma_{\hat{\eta}}^2 + (1-\rho z)\,(1-\rho z^{-1})\,\sigma_u^2}{[\beta\,(a-b)+1-az]\,[\beta\,(a-b)+1-az^{-1}]} \right. \\ &\left. + \frac{\sigma_w^2(f\beta+1-\rho z)(f\beta+1-\rho z^{-1})}{(\beta(1-a)+1-az)(\beta(1-a)+1-az^{-1})} \right\}. \end{aligned}$$

At a CEE $a$ solves

$$\hat{F}_f(a) \equiv \hat{c}_2 a^2 + \hat{c}_1 a + \hat{c}_0 = 0$$

with

$$\begin{aligned}
\hat{c}_2 &= -\hat{t}(1+\beta^2)+\hat{s}\beta, \\
\hat{c}_1 &= -2\hat{t}\beta(1-\rho\beta)+\hat{s}(1-\rho\beta), \\
\hat{c}_0 &= -\hat{t}(1-\rho\beta)^2
\end{aligned}$$

where $\hat{s}=\beta^2\sigma_{\hat{\eta}}^2+(1+\rho^2)\sigma_u^2+\sigma_w^2((1+f\beta)^2+\rho^2)$ and $\hat{t}=\sigma_u^2\rho+\sigma_w^2\rho(1+f\beta)$. When $f=0$ we have the previous case. Moreover, $\hat{F}_f(1)$ is increasing in $f$ and thus there there exists a CEE with $b=\rho$ and $a\in[0,1]$. It can also be shown that $a<\rho$ and that $a\to\rho$ when $\sigma_{\hat{\eta}}^2\to 0$, $\sigma_w^2\to 0$.

Next, consider the incentives to include judgement. It can be computed that

$$\begin{aligned}
y_{t+1}-y_{t+1}^e &= \frac{1}{1+\beta(a-\rho)-aL}\times \\
&\quad \left\{\left[\frac{\beta f(1-aL)-(\rho-a)\beta fL}{1-\rho L}+(1-aL)\right]w_{t+1}+\right. \\
&\quad \left.\frac{\beta(1-aL)-(\rho-a)\beta L}{1-\rho L}\hat{\eta}_{t+1}\right\} \\
&\quad -\frac{kfL}{1-\rho L}w_{t+1}-\frac{kL}{1-\rho L}\hat{\eta}_{t+1} \\
&\quad +\text{ term in } u_{t+1},
\end{aligned}$$

where $k=1$ if judgement is included, and zero otherwise. When $\sigma_{\hat{\eta}}^2\to 0$, $\sigma_w^2\to 0$, the relevant terms in the forecast error for assessing judgement are:

$$\frac{1+\beta f-(\rho+kf)L}{1-\rho L}w_{t+1}+\frac{\beta-kL}{1-\rho L}\hat{\eta}_{t+1}.$$

For the second term the comparison is as before. For the term involving $w_{t+1}$ we get for the relevant variances

$$Var\,|_{k=0}-Var\,|_{k=1}\asymp\frac{f}{1+\beta f}\left(2\rho-\frac{2\rho+f}{1+\beta f}\right),$$

where $\asymp$ means "is positively proportional to." It is seen that the term in the brackets is positive for all $f$ when $\beta\rho>1/2$. This implies that adding a small correlation between judgement and unobserved fundamentals does not alter the incentive condition for inclusion of judgement. In other words, if $\beta\rho > 1/2$ an individual agent will make the judgemental adjustment to the forecast for sufficiently small values of $\sigma^2_{\hat{\eta}}$ and $\sigma^2_w$.

To examine the amount of correlation between the fundamental $u_t + w_t$ and the judgement innovation $\eta_t$, we also considered the limit $\sigma^2_{\hat{\eta}} \to 0$ and computed the correlation for different values of $\sigma^2_u$, $\sigma^2_w$ and $f$ under the constraints that learning convergence and inclusion of judgement is a CEE. For example, if $\beta = \rho = 0.95$, $f = 1$ and $\sigma^2_{\hat{\eta}}$ very small, the correlation can be pushed beyond 0.9 before the conditions start to fail.

Finally, we show in Appendix D.2 that the learnability requirement is also met in the extended model. Indeed, the proof of convergence in Appendix D.2 is worked out for the extended model, with $\sigma^2_w = 0$ treated as a special case.

# D Recursive learning

## D.1 Recursive least squares

For simplicity, we develop the details in the univariate setting. Econometricians estimate the PLM

$$y_t = \sum_{i=1}^{p} b_i y_{t-i} + v_t$$

using recursive least squares. Let $b_t = (b_{1,t}, ..., b_{p,t})$ denote the parameter estimates at time $t$ and let $Y'_{t-1} = (y_{t-1}, ..., y_{t-p})$ be the vector of state variables. The RLS algorithm is

$$\begin{aligned} b'_t &= b'_{t-1} + t^{-1} R_t^{-1} Y_{t-1}(y_t - b_{t-1} Y_{t-1}) \\ R_t &= R_{t-1} + t^{-1}(Y_{t-1} Y'_{t-1} - R_{t-1}), \end{aligned}$$

where $y_t$ is given by the univariate version of the ALM (18) with $b_i$ replaced by $b_{i,t-1}$. Here $R_t$ is an estimate of the matrix of second moments of $Y_{t-1}$ and the first equation is just the recursive form of the usual least squares formula.

Note that assumptions about timing are as follows. At the end of period $t-1$ econometricians update their parameter estimates to $b_{t-1}$ using data up to $t-1$. At time $t$ econometricians use these parameter estimates and observed $Y_t$ to make their forecast $E_t^{\star} y_{t+1}$.[32] At the end of time $t$ econometricians update the parameters to $b_t$. For further discussion of RLS learning see Chapters 2 and 8 of Evans and Honkapohja (2001).

The question of interest is whether $\lim_{t\to\infty} b_t \to \bar{b}$, where $\bar{b} = (\bar{b}_1, ..., \bar{b}_p)$ denotes the approximate CEE . In this case $\bar{b}$ is said to be locally learnable. It can be shown that the asymptotic dynamics of $(b_t', R_t)$ are governed by an associated differential differential equation and that, in particular, the asymptotic dynamics of $b_t$ are governed by

$$\frac{db}{d\tau} = \left[Ey_t(b)Y_{t-1}(b)'\right]\left[EY_t(b)Y_{t-1}(b)'\right]^{-1} - b = T(b) - b.$$

Here $\tau$ denotes notional or virtual time, $y_t(b)$ is the stationary stochastic process given by (18) for fixed $b$ and $Y_{t-1}(b)' = (y_{t-1}(b), ..., y_{t-p}(b))$. Numerically, convergence can be verified using the E-stability algorithm (20), which can also be used to compute the approximate CEE.

The above procedure can easily be generalized to the multivariate case in which the PLM is a VAR($p$) process.

## D.2 Recursive maximum likelihood

We now consider recursive estimation when the PLM is an ARMA(1,1) process, that is,

$$y_t = by_{t-1} + v_t + cv_{t-1},$$

where $y_t$ is observed but the white noise process $v_t$ is not observed. Let $b_t$ and $c_t$ denote the estimates of $b$ and $c$ using data through time $t-1$. The econometricians are assumed to use a recursive maximum likelihood (RML) algorithm, which we now describe.[33]

[32] Note that $y_t$ and $E_t^{*} y_{t+1}$ are simultaneously determined. Alternative information assumptions could be made but would not affect our main results.

[33] For further details on the algorithm see Section 2.2.3 of Ljung and Soderstrom (1983). The algorithm is often called a recursive prediction error algorithm.

Let $\phi_t' = (b_t, c_t)$. To implement the algorithm an estimate $\varepsilon_t$ of $v_t$ is required. Let $\varepsilon_t = y_t - x_{t-1}'\phi_{t-1}$, where $x_{t-1}' = (y_{t-1}, \varepsilon_{t-1})$. $y_t$ is given by $y_t = \beta\left[E_t^\star y_{t+1} + \xi_t\right] + u_t + w_t$, where $E_t^\star y_{t+1} = b_{t-1}y_t + c_{t-1}\varepsilon_t$. Thus the analysis below holds also for the extended model (12)-(13) (our basic model sets $f = \sigma_w^2 = 0$). The RML algorithm is as follows

$$\begin{aligned}
\psi_t &= -c_{t-1}\psi_{t-1} + x_t \\
\phi_t &= \phi_{t-1} + t^{-1}R_{t-1}^{-1}\psi_{t-1}\varepsilon_t \\
R_t &= R_{t-1} + t^{-1}(\psi_{t-1}\psi_{t-1}' - R_{t-1}).
\end{aligned}$$

Again the question of interest is whether $\phi_t$ converges to an exact CEE. Convergence can be studied using the associated ordinary differential equation

$$\frac{d\phi}{d\tau} = R^{-1}E\psi_t(\phi)\varepsilon_t(\phi) \tag{34}$$

$$\frac{dR}{d\tau} = E\psi_t(\phi)\psi_t(\phi)' - R. \tag{35}$$

Here $y_t(\phi)$, $\psi_t(\phi)$ and $\varepsilon_t(\phi)$ denote the stationary processes for $y_t$, $\psi_t$ and $\varepsilon_t$ with $\phi_t$ set at a constant value $\phi$. Using the stochastic approximation tools discussed in Marcet and Sargent (1989), Evans and Honkapohja (1998) and Chapter 6 of Evans and Honkapohja (2001), it can be shown that the RML algorithm locally converges provided the associated ordinary differential equation is locally asymptotically stable (analogous instability results are also available). Numerically, convergence of (34)-(35) can be verified using a discrete time version of the differential equation. A first-order state space form is convenient for computing the expectations $E\psi_t(\phi)\varepsilon_t(\phi)$ and $E\psi_t(\phi)\psi_t(\phi)'$ and this procedure was used for the numerical illustrations given in the main text.

We now prove convergence analytically for all $0 < \beta, \rho < 1$ with $\sigma_{\hat{\eta}}^2$ and $\sigma_w^2$ sufficiently small. This completes the proof of part (i) in Theorem 1. We rewrite the system (34)-(35) in the form

$$\begin{aligned}\frac{d\phi}{d\tau} &= (R)^{-1}g(\phi)\\ \frac{dR}{d\tau} &= M_\psi(\phi) - R\end{aligned}$$

where we have introduced the simplifying notation $g(\phi) = E\psi_t(\phi)\varepsilon_t(\phi)$ and $M_\psi(\phi) = E\psi_t(\phi)\psi_t(\phi)'$. An equilibrium $\bar{\phi}, \bar{R}$ of the system is defined by $g(\bar{\phi}) = 0$ and $\bar{R} = M_\psi(\bar{\phi})$. As mentioned in Appendix A, there can be two equilibrium values $\bar{\phi}' = (\rho, -a)$ determined by the solutions to the quadratic (28), but we here focus on the solution with $0 < a < 1$. Recall that for this solution $a \to \rho$ as $\sigma_\eta^2 \to 0$.

Linearizing the system at the equilibrium point, it can be seen that the linearized system has a block diagonal structure, in which one block has the eigenvalues equal to $-1$ (with multiplicity four) and the eigenvalues of the other block are equal to those of the "small" differential equation

$$\frac{d\phi}{d\tau} = (\bar{R})^{-1}J(\bar{\phi})(\phi - \bar{\phi}), \tag{36}$$

where $J(\phi)$ is the Jacobian matrix of $g(\phi)$. The system (34)-(35) is therefore locally asymptotically stable if the coefficient matrix $(\bar{R})^{-1}J(\bar{\phi})$ of the two-dimensional linear system (36) has a negative trace and a positive determinant. Since $(\bar{R})^{-1} = (\det(\bar{R}))^{-1}adj(\bar{R})$ we have

$$\begin{aligned}Tr[(\bar{R})^{-1}J(\bar{\phi})] &= (\det(\bar{R}))^{-1}Tr[adj(\bar{R})J(\bar{\phi})] \text{ and}\\ \det[(\bar{R})^{-1}J(\bar{\phi})] &= \det[(\bar{R})^{-1}]\det[J(\bar{\phi})].\end{aligned}$$

Now $\det(\bar{R}) > 0$ as $\bar{R}$ is a matrix of second moments and thus positive definite for $\sigma_\eta^2 > 0$. It thus remains to prove that $Tr[adj(\bar{R})J(\bar{\phi})] < 0$ and $\det[J(\bar{\phi})] > 0$ when $\sigma_\eta^2 > 0$ is sufficiently small.

We consider the values of $Tr[adj(\bar{R})J(\bar{\phi})]$ and $\det[J(\bar{\phi})]$ when $\sigma_\eta^2 \to 0$. Using the definition of $\varepsilon_t$, the explicit form of $g(\phi)$ is

$$\begin{aligned}g(\phi) &= E\psi_{t-1}(\phi)x_{t-1}'\left[(1-\beta b-\beta c)^{-1}\beta\begin{pmatrix}-bc\\ -c^2\end{pmatrix} - \begin{pmatrix}b\\ c\end{pmatrix}\right]\\ &\quad +(1-\beta b-\beta c)^{-1}\beta\rho E\psi_{t-1}(\phi)\xi_{t-1},\end{aligned}$$

where the moment matrices $E\psi_{t-1}(\phi)x'_{t-1}$ and $E\psi_{t-1}(\phi)\xi_{t-1}$ can be computed from the state space form

$$AX_t = CX_{t-1} + H\begin{bmatrix} u_t \\ \hat{\eta}_t \\ w_t \end{bmatrix}, \text{ with } X_t = \begin{pmatrix} y_t \\ \varepsilon_t \\ \xi_t \\ \psi_t \\ \psi_{t-1} \end{pmatrix},$$

$$A = \begin{pmatrix} 1 & -(1-\beta b)^{-1}\beta c & -(1-\beta b)^{-1}\beta & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ -b & -c & 0 & 0 & 0 \\ 0 & 0 & \rho & 0 & 0 \\ 0 & 0 & 0 & -c & 0 \\ 0 & 0 & 0 & 0 & -c \end{pmatrix}, H = \begin{pmatrix} (1-\beta b)^{-1} & 0 & (1-\beta b)^{-1} \\ 0 & 0 & 0 \\ 0 & 1 & f \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For the limit $\sigma^2_{\hat{\eta}} \to 0$ and $\sigma^2_w \to 0$ we first set $\sigma^2_w = \lambda\sigma^2_{\hat{\eta}}$, where $\lambda > 0$ is arbitrary. It can be computed using Mathematica (routine available on request) that $Tr[adj(\bar{R})J(\bar{\phi})]$ and $\det[J(\bar{\phi})]$ have the following properties as functions of (using temporary notation) $\omega \equiv \sigma^2_{\hat{\eta}}$:

$$\begin{aligned} \lim_{\omega\to 0} Tr[adj(\bar{R})J(\bar{\phi})] &= \lim_{\omega\to 0} \frac{d}{d\omega} Tr[adj(\bar{R})J(\bar{\phi})] = 0, \\ \lim_{\omega\to 0} \det[J(\bar{\phi})] &= \lim_{\omega\to 0} \frac{d}{d\omega} \det[J(\bar{\phi})] = 0, \end{aligned}$$

$$\lim_{\omega\to 0} \frac{d^2}{d\omega^2} Tr[adj(\bar{R})J(\bar{\phi})] = -\frac{4\beta^2\rho^2[\beta + f\lambda(1+f\beta-\rho^2)]^2}{(1-\beta\rho)(\rho^2-1)^6} < 0 \text{ and}$$

$$\lim_{\omega\to 0} \frac{d^2}{d\omega^2} \det[J(\bar{\phi})] = \frac{2\beta^2\rho^2[\beta + f\lambda(1+f\beta-\rho^2)]^2}{(\rho^2-1)^6} > 0.$$

Expressing $Tr[adj(\bar{R})J(\bar{\phi})]$ and $\det[J(\bar{\phi})]$ in terms of Taylor series these results show that

$$Tr[adj(\bar{R})J(\bar{\phi})] < 0 \text{ and } \det[J(\bar{\phi})] > 0$$

for $\sigma^2_\eta > 0$ sufficiently small. *Q.E.D.*

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