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Intermediaries and Payments Instruments

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We study an economy in which intermediaries have incentives to issue circulating liabilities as part of an equilibrium. We show that, with arbitrarily small transactions costs, only the liabilities of intermediaries will circulate, and not those of other private sector agents. Therefore, our model connects intermediation activity with the issuance of payments media, a connection that has not been made in earlier literature. We also describe conditions under which equilibrium outcomes may be volatile when private liabilities circulate. Finally, we use our model to suggest a resolution of the "banknote underissue puzzle" of Cagan (1963).

Key Words: Private money, banknotes, intermediation, monetary theory, endogenous volatility.
Subject Classification: JEL classification codes *E3, E4, E6.*

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1. INTRODUCTION

1.1. Overview

Throughout much of history, the term "money" has referred to a stock of commodity money, typically gold or silver, supplemented by a stock of private circulating liabilities that often constituted some kind of claim to the commodity money. Often these private circulating liabilities made up the bulk of the money supply. Today the term "money" is most commonly used to refer to a government-issued fiat money, supplemented by a stock of private liabilities (deposits) that can easily be used to make payments. Again, these private liabilities often constitute the bulk of the money supply and are typically a claim to outside money. Importantly, in either historical or modern contexts, not just anyone's liabilities have been regarded as part of the money supply. Historically the private circulating liabilities regarded as part of the money supply were notes issued by banks. In modern contexts, the private liabilities regarded as part of the money supply consist of various kinds of bank deposits.

In short, liabilities issued by certain kinds of financial intermediaries have always played a special role in making payments—a role played by the liabilities of no other agents. Moreover, the liabilities used have virtually always been those of banks: Entities that raise resources by issuing their own liabilities, and who then lend the resources they obtain.

Most modern theories of intermediation focus on the role of banks in servicing loans or in insuring depositors. In these theories, bank liabilities play no particular role in making payments.[3] These theories therefore beg the question of why the liabilities of financial intermediaries have almost always been given such a prominent role as payments instruments. That is, they do not address the question of why the activity of lending has typically been coupled with the activity of issuing a payments medium. This seems like an important omission, especially since many economists have argued that it is important to separate the activity of lending from the activity of issuing liabilities used in making payments.

In this paper we build a model in which intermediation in lending occurs only because intermediary liabilities are essential in enabling agents to make payments. In addition, in our model only intermediary liabilities are used to make payments, even though it is also feasible to use the liabilities of other

[3] Examples of the theories we have in mind include Diamond and Dybvig (1983), Williamson (1986), and Boyd and Prescott (1986).

agents for this purpose. The model also sheds light on several other issues of importance in monetary economics. For instance, it allows us to describe the nature of a general equilibrium in an economy where intermediary liabilities constitute the sole means of payment. We analyze steady states as well as dynamical equilibria. The analysis of dynamical equilibria is important since it has often been argued that the use of privately-issued, circulating liabilities in exchange is a formula for generating indeterminacy of equilibrium and excessive economic volatility.

There are many anomalies in the historical record regarding the issue of circulating liabilities by banks. For example, historically the issue of notes by banks was profitable but additional entry into banking or expansion of the stock of banknotes in circulation did not occur. Perhaps the best known instance of this phenomenon occurred under the National Banking System in the U.S., where Cagan (1963) observed that the average profits associated with note issue were high, but that at the same time, banks never issued the full quantity of notes that was possible under existing regulation. We believe that our analysis sheds light on this "banknote underissue puzzle." In particular, we show that the existence of average profits for intermediaries is consistent with limits on the quantity of notes issued. Moreover, such profits in no way imply that the welfare of agents operating intermediaries was particularly high.

Finally, historically there has always been a great deal of trade between different intermediaries. Our analysis allows us to explain why intermediaries often borrow and lend with each other, rather than expand or contract their assets and liabilities by other means.

1.2. Model Description and Main Findings

Our vehicle for addressing these issues is a model that features spatial separation and limited communication to generate a transactions role for certain kinds of privately-issued liabilities. In addition, in order to understand why the activity of lending is coupled with the issue of payments media, we need to focus on economies with active credit markets. To this end, we consider an economy consisting of an infinite sequence of three-period-lived, overlapping generations of agents. Within each generation there are agents who are endowed today but want to delay consumption, as well as agents who want to consume today by borrowing against future endowments. In the absence of spatial separation and limited communication, these agents could just borrow and lend with each other, and intermediation would be unnecessary. In order to generate a role for private circulating liabilities, we assume that the economy consists of two locations. In each location some agents who are natural borrowers and some agents who are natural lenders are born in each period. However, after one period lenders move to a different location, while borrowers remain in their original location. This complicates borrowing and lending, since borrowers and lenders who are together when young are separated when middle-aged. Under our assumption of spatial separation and limited communication, borrowers are precluded from directly repaying lenders. Moreover, the assumption of limited communication between locations implies that transactions cannot be accomplished with book credit alone.

To complicate matters further, we assume that between their second and third periods of life borrowers also change location. While this means that agents who might want to borrow and lend with each other when young are ultimately reunited, it does not facilitate borrowing and lending. This is because borrowers are endowed in their second period, when they would want to make loan repayments, and lenders want to consume in their second period, before they are reunited with the agents to whom they made loans. Indeed, if there were no other agents in this model, credit transactions would not be feasible.

However, there is another set of agents in the model. We refer to these agents as intermediaries. Intermediaries want to consume in middle age, are endowed when old, and never change location. In the absence of spatial frictions, these agents would have no economic purpose in their youth, as they are neither endowed at that time, nor do they wish to consume. But as we show, they can nevertheless play an important role in facilitating trade by issuing circulating liabilities when young. These liabilities are used to buy goods from young lenders, who then take the liabilities to their new location. There they are used to purchase goods from a new generation of young lenders. These latter agents then carry the liabilities obtained to their location of origin and redeem them. Moreover, intermediaries use the resources obtained via liability issue to make loans to young borrowers. Thus at least some borrowing and lending must be intermediated.

When intermediary liabilities are in circulation, we show that it is also feasible for young borrowers to obtain resources by issuing circulating liabilities of their own. However, in order to accomplish a particular credit transaction, the use of circulating liabilities issued by young borrowers involves more transactions than the use of circulating liabilities issued by young intermediaries. If transacting were costless, agents would not care how many transactions were required to accomplish a particular transfer of resources. However, we assume that transactions are costly. Under this assumption, agents will want to trade in a way that economizes on transactions. We will show that the way to do so is to have only intermediaries issue liabilities that circulate. Thus, even though it is feasible for other agents to issue circulating liabilities, only intermediary liabilities are used in payments.[4]

Interestingly, the presence of transactions costs, coupled with the fact that intermediaries must often borrow and lend with each other, will imply that—under certain circumstances—intermediaries can earn profits. Nonetheless, this will not induce young borrowers to issue circulating liabilities, even though they can. And, even though the issue of circulating liabilities is profitable, there will be a limit to the creation of these liabilities. This fact offers a potential resolution of Cagan's banknote underissue puzzle.

Finally, the model makes clear that, in the absence of intermediary liabilities, a number of agents are unable to transact with other agents. Thus a prohibition against issuing private circulating liabilities has very negative welfare consequences for a variety of economic actors. While such a prohibition can eliminate a number of economic phenomena that may be regarded as undesirable, doing so comes at a high economic cost.

The remainder of the paper proceeds as follows. In Section 2 we present the model in detail and describe its key features; we follow this with a description of optimal agent behavior in Section 3. We discuss general equilibrium conditions in Section 4 and steady state equilibria in Section 5. Section 6 contains our discussion of dynamical equilibria. The conclusion summarizes our findings and suggests directions for future research.

2. ENVIRONMENT

2.1. Endowments, Preferences, and Movement Patterns

We study a pure exchange economy with overlapping generations of agents who live for three periods on either of two distinct islands. Time is discrete and indexed by $t = 0, 1, 2, ...$ We index locations by $h \in \{1, 2\}$

[4] Ostroy and Starr (1974) constructed a model where the use of money economizes on transactions when agents are constrained to sequences of bilateral exchanges. Their model did not attempt to explain the use of intermediary liabilities in exchange, and we do not restrict agents to make bilateral exchange.

and $k \in \{1, 2\}$. There is movement between locations in this economy, but goods cannot be transferred across locations, nor is direct inter-location communication possible. Thus, as in other models of monetary exchange,[5] our analysis emphasizes features of spatial separation and limited communication. These features imply that all trade takes place in a set of competitive markets that operate independently of each other at any point in time. It is this lack of centralized trading that creates the potential for monetary exchange.

At each date a new young generation appears, consisting of a continuum of agents with unit mass. All agents are endowed with perfect foresight and complete knowledge of the economy in which they operate. Agents are divided among islands in a manner described below. A young agent born at date t in location h may be one of three possible types. We index these types by $j \in \{\ell, b, i\}$, where ℓ means "lender-type," b means "borrower-type," and i means "intermediary-type." These labels are meant to refer to the agents' behavior in youth and not necessarily to their behavior over their whole lifetime. An agent type has three characteristics: An endowment vector over the three periods of life, preferences defined over consumption in the three periods of life, and an itinerary describing movements over time. We let $c_{j,h,t}(t+s)$, $s = 0, 1, 2$, denote the date $t+s$ consumption of a type-j agent born in location h at date t.[6]

An agent of the first type, *lenders*, is endowed with $e > 0$ units of the perishable consumption good in youth, and has no endowment in middle age or old age. We say that they have an endowment vector $(e, 0, 0)$ over their three periods of life. Lender-type agents have preferences defined over lifetime consumption given by

$$u\left[c_{\ell,h,t}(t), c_{\ell,h,t}(t+1), c_{\ell,h,t}(t+2)\right] = c_{\ell,h,t}(t+1), \qquad (1)$$

so that lenders only wish to consume in middle age. Finally, lenders have an itinerary defined over their lifetime, which we denote as (h, k, k), with $h \neq k$. Thus a lender-type agent born in location h at time t moves to location $k \neq h$ between youth and middle-age and remains in that location during old age.

An agent of the second type, *borrowers*, has endowment vector $(0, w, 0)$,

[5] For instance Townsend (1980, 1987), Kiyotaki and Wright (1989), and Champ, Smith, and Williamson (1996).

[6] We emphasize that in this notation, the subscript t denotes the date of birth, while the date $t+s$ denotes the time at which consumption actually occurs.

Type	Endowment	Consumption	Movement
(1) Lenders	Youth	Middle age	Youth → middle age
(2) Borrowers	Middle age	Youth	Middle age → old age
(3) Intermediaries	Old age	Middle age	No movement

TABLE 1
Summary of agent types.

with $w > 0$, preferences

$$u\left[c_{b,h,t}\left(t\right), c_{b,h,t}\left(t+1\right), c_{b,h,t}\left(t+2\right)\right] = c_{b,h,t}\left(t\right), \tag{2}$$

and itinerary (h, h, k), with $h \neq k$. Thus borrowers are endowed in middle-age, wish to consume in youth, and move between middle age and old age. An agent of the third type, *intermediaries*, has endowment vector $(0, 0, a)$, with $a > 0$, preferences

$$u\left[c_{i,h,t}\left(t\right), c_{i,h,t}\left(t+1\right), c_{i,h,t}\left(t+2\right)\right] = c_{i,h,t}\left(t+1\right), \tag{3}$$

and itinerary (h, h, h). Thus intermediaries are not endowed until old age, wish to consume in middle age, and never move. The intermediary-type agents, despite their lack of income until late in life, will play an important role in facilitating trade in this economy.[7]

We summarize this situation in Table 1 and in Figure 1.

We assume throughout that the composition of agent types is the same in each generation. We let $\theta_{h,j}$ be the fraction of each young generation born in location h that is of type j. We impose $\theta_{h,j} > 0$ and require $\sum_j \theta_{1,j} = 1$ and $\sum_j \theta_{2,j} = 1$. We will also often assume that locations are symmetric, by which we mean that $\theta_{1,j} = \theta_{2,j}$ for all j.

Our objective is to show that it is "natural" in this economy for only the liabilities of certain agents—intermediaries—to circulate. In order to do so, however, it is necessary that agents care how many transactions are required to accomplish a particular transfer of resources. Indeed, our idea is that having borrowing and lending be intermediated, and having only intermediary liabilities circulate, minimizes the number of transactions required for each exchange of resources. In order to make agents care about

[7] We have assumed for convenience that each type of agent consumes in only a single period, and is endowed in only a single (but different) period. This is not essential to the analysis, but merely simplifies the exposition. See Azariadis, Bullard, and Smith (2001) and Bullard and Smith (2002) for relaxations of this assumption in related environments, but where intermediation is unnecessary.

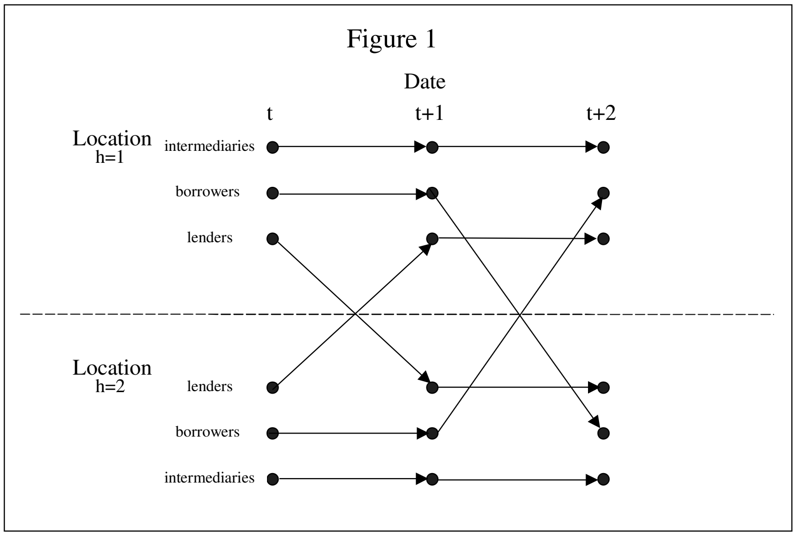

FIG. 1 Schematic movement patterns of agent types born at time t.

economizing on transactions, we assume that there is a cost associated with each transfer of either goods or liabilities. For simplicity, we assume that these costs are proportional to the value of the goods or liabilities exchanged. In particular, an exchange of z units of goods (or liabilities with equivalent real value) involves a transactions cost of αz with $\alpha > 0$. To fix ideas, we assume that transactions costs are born by agents who deliver goods.[8] Thus we have

Assumption 1. (Proportional transactions costs.) Each time goods are delivered, the agent delivering goods pays a proportional transactions cost equal to α times the value of the goods delivered.

Also, with several types of circulating liabilities trading at a given date, there may be incentives to trade one type of note for another. In order to limit the number of cases we need to consider, we introduce the following assumption:[9]

[8] Typically, in models with proportional transactions costs, resource allocations do not depend on who is assumed to bear these costs.

[9] We do not think it would add very much to the analysis to allow note-for-note trades. If there is some cost to transferring liabilities, there will be an incentive to economize on doing so, so that agents will transfer goods—and bear the associated costs in the process of note redemption—at the earliest opportunity.

Assumption 2. (No exchange of notes for notes.) Elderly borrowers must redeem any notes issued with goods.

2.2. Liabilities, Circulating and Otherwise

The borrower-type agents born in location h at time t wish to consume at date t even though they are only endowed at date $t+1$. These agents can look to several sources for conventional, one-period consumption loans. One possible source is lender-type agents born in location h at time t. However, these agents will not be in the same location as the borrowers at date $t+1$, and therefore will not trade with the borrowers via conventional consumption loans. Another possible source is the elderly intermediary-type agents; however since they will die next period they will not trade with the young borrowers. A third possibility is the middle-aged borrower-type agents; that is, borrowers born in location h at time $t-1$. While these agents are endowed and will be alive next period, they will not be in the same location as the young borrowers at that time. We conclude that in this economy, as in Azariadis, Bullard, and Smith (2001), something other than conventional, one-period consumption loans must be used to finance borrowers' first-period consumption.

We will show that, in order to accomplish exchange, agents may wish to issue circulating liabilities—notes which are used in third-party transactions before they are redeemed—in this economy. Both borrower-type and intermediary-type agents may have incentives to issue these liabilities. In addition, the issuance of circulating liabilities may allow conventional, one-period consumption loans to take place after all. We want to distinguish between these events, and so we now turn to this task.

In a conventional one-period consumption loan, we say that the borrower issues an *IOU* to the lender. In this transaction, an agent who wishes to borrow exchanges a liability for goods today, and then delivers goods in repayment one period later.[10] Such *IOUs* are held by only one agent, and so *do not circulate*. Let $b_{j,h,t}(t+s)$ be the date $t+s$ face value of the one-period liabilities issued by type j agents born in location h at date t. By the face value we mean that these liabilities are a promise to pay $b_{j,h,t}(t+s)$ units of goods at maturity (that is, one period after being issued).

[10] Throughout we assume that contracts are costlessly enforceable, so that there is never any possibility that agents will default on their liabilities.

The use of *two-period* liabilities is associated with a long-term loan: Agents offer two-period liabilities in exchange for goods, and pay off the liability two periods later. However, a two-period liability can—and, in equilibrium, will—be transferred through a sequence of owners between the time it is issued and the time it matures. Or, in other words, two-period liabilities potentially *circulate*. We will sometimes use the following nomenclature to keep track of two-period notes in our model. When an intermediary born in location h issues a circulating liability we say that the intermediary has issued a *location h banknote*. When a borrower born in location h issues a circulating liability, we say that the borrower has issued a *location h private note*. We sometimes also refer to circulating liabilities generally as *notes.*

Let $x_{j,h,t}(t+2)$ denote the face value of the two-period liabilities issued by type j agents born in location h at date t. Here the argument $t+2$ indicates the date at which these liabilities mature.[11] And, again, by face value we mean that these liabilities are a claim to $x_{j,h,t}(t+2)$ units of the good at $t+2$. Let $R_{h,k}(t)$ denote the competitively-determined gross real rate of return between t and $t+1$ earned by an agent who is moving from location h to location k. It is easy to verify that newly-issued liabilities sell for their appropriate discounted present value. Thus, for instance, a one-period liability issued in location h at time t and repaid in location h at time $t+1$ sells for $b_{j,h,t}(t)/R_{h,h}(t)$ upon issue. Or, a two-period liability issued in location h at date t that is carried to location k at date $t+1$, and then back to location h at date $t+2$, sells for $x_{j,h,t}(t+2)/R_{h,k}(t)\,R_{k,h}(t+1)$ when issued.

As the previous discussion indicates, in addition to trading newly-issued one- and two-period liabilities, there is a market in each location where agents exchange previously-issued circulating liabilities. In particular, at date t, there are markets in which agents buy and sell liabilities that were issued at $t-1$, and that mature at $t+1$. In these markets previously-issued liabilities also sell for their discounted present value. For example, a circulating liability with one period to maturity, a face value of $x_{j,h,t-1}(t+1)$, and that matures in location h,[12] sells for $x_{j,h,t-1}(t+1)/R_{k,h}(t)$ in location k at t.

We stress that the use of some circulating (two-period) liabilities is

[11] Clearly agents can issue two-period liabilities only when young. Hence there is no need for an argument showing the date of issue.

[12] That is, the issuer will be in location h at $t+1$. Thus the liability must be carried to that location at $t+1$ in order for it to be redeemed.

essential in order for any trade to occur. As we have observed, the pattern of agents' itineraries implies that no two agents can trade solely through the use of one-period liabilities. Agents who want to borrow are spatially separated from any agents who might lend to them directly. However, potential lenders can take two-period liabilities, carry them to their new location, and then trade them to agents who will meet the issuer of the liability at a future date. These new agents can then redeem the circulating liability, which clearly functions as a payments instrument.

Our interest is in whose liabilities circulate. We begin by describing what transactions are feasible.[13]

2.3. Feasible Transactions Patterns

One possible transactions pattern in this economy is schematically described in Figure 2. Young intermediaries born at time t in location h exchange *location h banknotes* for goods with young lenders born at time t in location h. Young borrowers born at date t in location h then obtain resources for their consumption by arranging a one-period consumption loan with the young intermediaries. The borrowers and the intermediaries then proceed to middle age at time $t+1$ still in location h, whereas the lenders move to location $k \neq h$. At time $t+1$, the now-middle-aged borrowers are endowed. They use this endowment to repay the loan to the middle-aged intermediaries. The middle-aged intermediaries can then consume. Meanwhile, the lenders who moved to location k exchange the location h banknotes they hold for goods provided by the young lenders born in location k at time $t+1$. The middle-aged lenders can then consume. The young lenders in location k then move to location $h \neq k$. At time $t+2$, these now middle-aged lenders meet with old intermediaries in location h to exchange the location h banknotes the intermediaries issued at time t for goods. Elderly borrowers and lenders have no further transactions to conduct.

[13] One can view the model of Kiyotaki and Wright (1989) as an analysis of which physical objects have a high velocity of circulation. Our question is, whose liabilities have a high velocity of circulation.

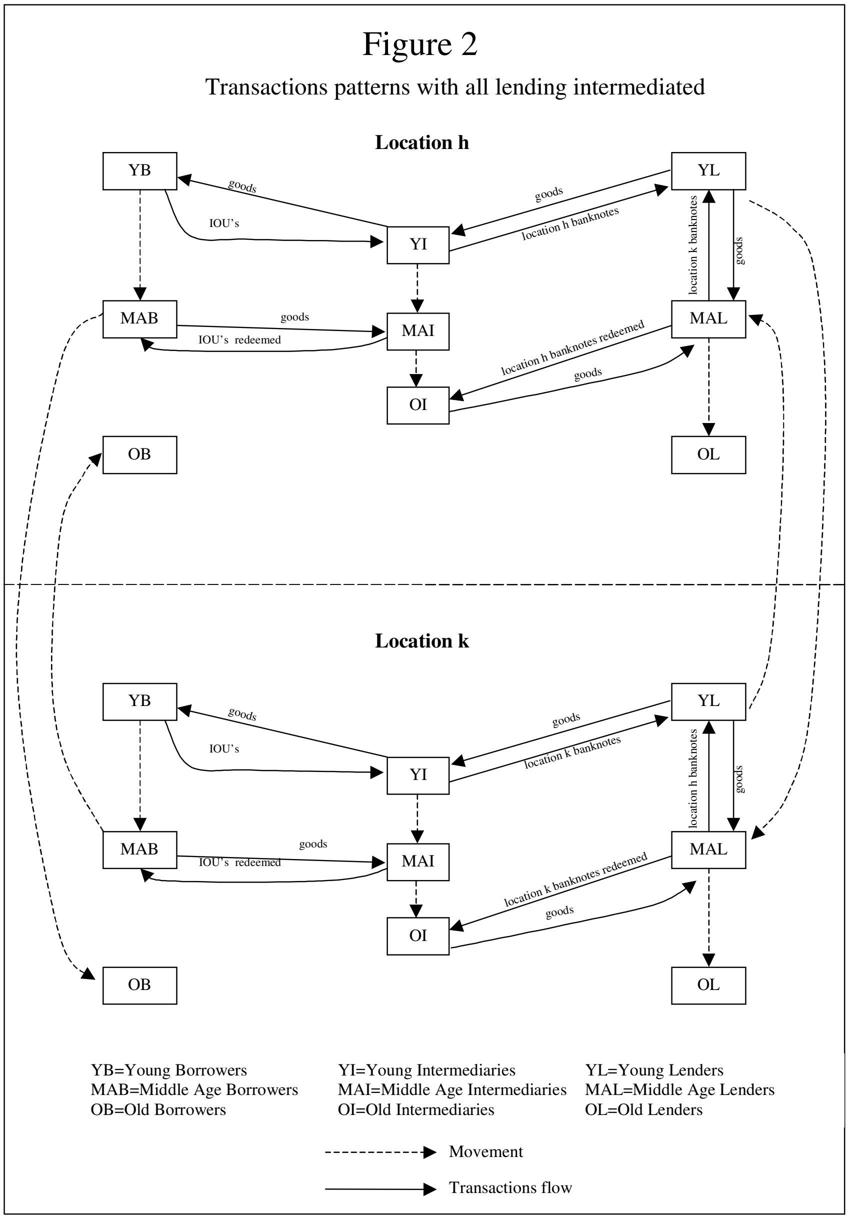

FIG 2. Schematic of transactions patterns when all borrowing and lending is intermediated. In this scenario, relatively few transactions occur. We will show how this transactions pattern arises as part of an equilibrium for this model.

Could young borrowers issue their own notes directly to young lenders, and avoid the intermediary? They could indeed, provided some intermediary liabilities circulate as well. We illustrate the second, more complicated, possible transactions pattern for this economy in Figure 3. In this scenario, young borrowers born in location h at time t issue *location h private notes* to young lenders. The young lenders carry these notes to location k at time $t+1$, and trade them for goods to the young intermediaries there. The intermediaries hold the notes for one period, and then redeem them with the old borrowers—the original issuers—arriving in location k. In order to support these events, young intermediaries must acquire real resources. A young intermediary born at time t in location h does so by issuing *location h banknotes* to young lenders, and, in this case, also to middle-aged borrowers. Both the young lenders and the middle-aged borrowers move to location k at time $t+1$. At that point, they trade the location h banknotes to young lenders, who then move to location h at time $t+2$ and redeem the notes with the old intermediaries—the original issuers—there. There is one other transaction sequence in this scenario, namely that young borrowers may issue one-period IOUs to young intermediaries, which are then repaid in the following period.

In principle, the fact that each resource transfer involves a cost would seem to make it undesirable to have intermediated transactions, if this could be avoided. In particular, having lenders transfer goods to intermediaries, who then transfer resources to borrowers, involves an extra and, to superficial appearance, avoidable transaction. However, the necessity of trading with circulating liabilities—along with our assumptions on agents' itineraries—imply that it is impossible for exchange to take place without at least some intermediary liabilities circulating. Thus the need for intermediary liabilities as payments instruments makes costly intermediation not only desirable, but essential to trade. And, as we will show, in equilibrium only intermediary liabilities will circulate. When general equilibrium consequences are taken into account, this is the transactions pattern that economizes on transactions costs.

Figure 3

Transactions pattern when some borrowing is not intermediated

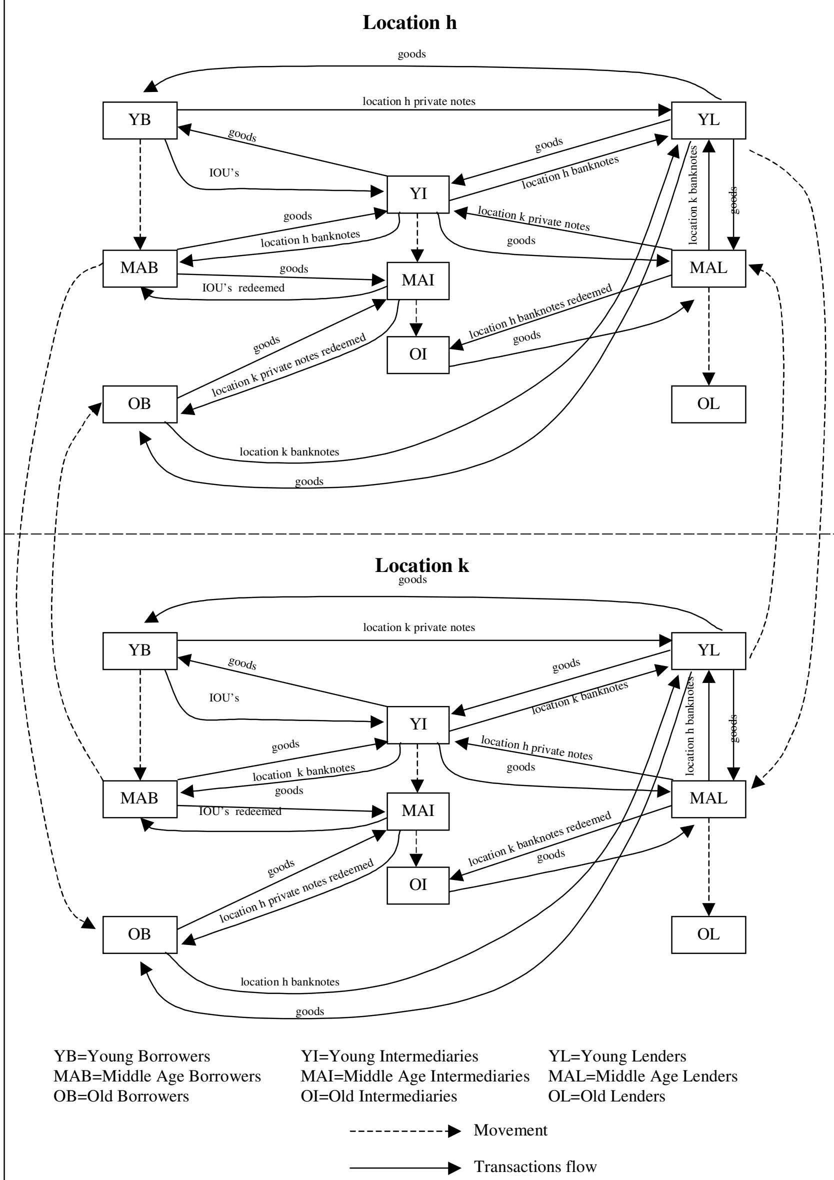

FIG 3. Schematic of transactions patterns when not all borrowing is intermediated. In this scenario, many more transactions have to occur.

3. OPTIMAL AGENT BEHAVIOR

3.1. Lenders

Young lenders born in location h supply their endowment inelastically. Since they will be separated in middle age from any agent they trade with when young, lenders hold only notes—two-period circulating liabilities. These may be newly-issued notes, or they may be location $k \neq h$ bank-notes issued in the previous period in the opposite location (by young intermediaries there). Since both types of notes must be held, in equilibrium, they bear the same rate of return between date t and date $t+1$. And, since a young lender born in location h at t moves to location $k \neq h$ at $t+1$, this gross real return is $R_{h,k}(t)$. Therefore, the middle-age consumption of a lender is

$$c_{\ell,h,t}(t+1) = (1-\alpha)\, eR_{h,k}(t)\,. \tag{4}$$

The term $(1-\alpha)$ enters this expression because the lender bears a proportional transactions cost of αe when goods are delivered in exchange for notes.

3.2. Intermediaries

Young intermediary-type agents born in location h at time t issue *location h banknotes* with a face value of $x_{i,h,t}(t+2)$.[14] These notes will be accepted in exchange only by agents with a positive endowment and at least one more period to live, namely, young lenders or middle-aged borrowers. Either type of agent would carry the notes from location h to location $k \neq h$ between t and $t+1$. The notes would therefore earn the gross real return $R_{h,k}(t)$ between those dates. Other agents—young lenders or middle-aged borrowers—will obtain the notes in location k at $t+1$, and then carry them back to location h at $t+2$ in order to redeem them. These notes must therefore earn the gross real return $R_{k,h}(t+1)$ between $t+1$ and $t+2$. Since the real value of newly-issued notes at t is the discounted present value of their face value, the issue of notes allows young intermediaries in location h at t to obtain goods in the amount $x_{i,h,t}(t+2)/R_{h,k}(t)\,R_{k,h}(t+1)$.

The goods obtained by issuing notes, net of transactions costs, are loaned out, as we will see, either to young borrowers or to middle-aged

[14] Young intermediaries will not issue one-period liabilities. If they did so, they would lend out the resources obtained and then repay the loan one period later. In each case a transactions cost would be incurred. And since young intermediaries would need to pay the prevailing market interest rate on loans, the issue of one-period liabilities would lead to young intermediaries incurring losses.

intermediaries. In making loans intermediaries deliver goods, thereby incurring the proportional transactions cost α. Thus young intermediaries in location h at t make one-period loans with a real value of $\frac{(1-\alpha)x_{i,h,t}(t+2)}{R_{h,k}(t)R_{k,h}(t+1)}$. Since these loans are made to agents who remain in location h between t and $t+1$, they earn the gross real rate of return $R_{h,h}(t)$. As a result, middle-aged intermediaries in location h at t collect revenue equal to $\frac{(1-\alpha)R_{h,h}(t)x_{i,h,t}(t+2)}{R_{h,k}(t)R_{k,h}(t+1)}$.

The behavior of intermediaries differs depending on the value of their middle-age consumption relative to their middle-age income. First we consider what we will call *Case 1*, a situation where middle-aged intermediaries borrow. In this case we have

$$c_{i,h,t}(t+1) > \frac{(1-\alpha)R_{h,h}(t)x_{i,h,t}(t+2)}{R_{h,k}(t)R_{k,h}(t+1)}, \tag{5}$$

meaning that the middle-age consumption of the intermediary-type agents exceeds their middle-age revenue. In *Case 2*, middle-age revenue is exactly sufficient to fund middle-age consumption, so that

$$c_{i,h,t}(t+1) = \frac{(1-\alpha)R_{h,h}(t)x_{i,h,t}(t+2)}{R_{h,k}(t)R_{k,h}(t+1)}. \tag{6}$$

Case 3 is then a situation where middle-aged intermediaries lend, because

$$c_{i,h,t}(t+1) < \frac{(1-\alpha)R_{h,h}(t)x_{i,h,t}(t+2)}{R_{h,k}(t)R_{k,h}(t+1)}. \tag{7}$$

We now turn to describing each of these cases.

In Case 1, middle-aged intermediary agents must obtain additional consumption goods by issuing one-period liabilities, or IOUs. The patterns of agents' movements implies that these liabilities must be sold to young intermediaries. Hence there is substantial scope for borrowing and lending between intermediaries in this model, as we observe historically with note-issuing banks.

Since middle-aged intermediaries borrow from agents who remain in location h, they face the gross real rate of interest $R_{h,h}(t+1)$ between $t+1$ and $t+2$. Thus the face value of the one-period liabilities issued by middle-aged intermediaries in location h at $t+1$ must satisfy

$$\frac{b_{i,h,t}(t+1)}{R_{h,h}(t+1)} = c_{i,h,t}(t+1) - \frac{(1-\alpha)R_{h,h}(t)x_{i,h,t}(t+2)}{R_{h,k}(t)R_{k,h}(t+1)}. \tag{8}$$

When old, an intermediary must redeem the two-period liabilities issued when young, and the one-period liabilities issued when middle-aged. To do so, the intermediary will deliver their old age endowment, incurring a transactions cost of αa in the process. Thus an old intermediary faces the budget constraint

$$(1-\alpha)\,a \geq x_{i,h,t}\,(t+2) + b_{i,h,t}\,(t+1)\,. \tag{9}$$

Consolidating equations (8) and (9), a young intermediary born in location h at time t will choose a value $x_{i,h,t}\,(t+2)$ to maximize $c_{i,h,t}\,(t+1)$, subject to

$$\begin{aligned}(1-\alpha)\,a \geq\ & x_{i,h,t}\,(t+2)+\\ & R_{h,h}\,(t+1)\left[c_{i,h,t}\,(t+1)-\frac{(1-\alpha)\,R_{h,h}\,(t)\,x_{i,h,t}\,(t+2)}{R_{h,k}\,(t)\,R_{k,h}\,(t+1)}\right].\end{aligned} \tag{10}$$

We therefore deduce that banknotes will be issued by young intermediaries only if

$$(1-\alpha)\,R_{h,h}\,(t)\,R_{h,h}\,(t+1) \geq R_{h,k}\,(t)\,R_{k,h}\,(t+1)\,. \tag{11}$$

Since we know that any feasible transactions pattern involves note issues by young intermediaries, equation (11) must be satisfied in equilibrium. Moreover, if inequality (8) holds, so that the note issue of young intermediaries is not expanded to the point where (8) holds with equality (that is, if Case 1 obtains), then it must be the case that (11) holds with equality. And, when (11) holds with equality, then (10) implies that

$$c_{i,h,t}\,(t+1) = \frac{(1-\alpha)\,a}{R_{h,h}\,(t+1)}. \tag{12}$$

That is, middle-aged intermediaries consume the discounted present value of their endowment, net of transactions costs. While this might seem to be an obvious outcome, we will see that a similar result does *not* obtain if middle-aged intermediaries are not borrowers.

Finally, when equations (10) and (11) hold with equality, it is easy to verify that (5) holds if and only if

$$(1-\alpha)\,a > x_{i,h,t}\,(t+2)\,. \tag{13}$$

This condition states that middle-aged intermediaries borrow if and only if their after-transactions-cost old-age endowment exceeds the face value of the notes they issue when young.

In Case 2, equation (6) holds and middle-aged intermediaries neither borrow nor lend, because their loan income is exactly sufficient to fund their consumption. In this case, the budget constraint (10) holds with equality implying that

$$(1-\alpha)\,a = x_{i,h,t}\,(t+2)\,, \tag{14}$$

that is, the net-of-transactions-cost endowment of old intermediaries is exactly enough to redeem the liabilities issued when young. Again, in order for young intermediaries to issue notes condition (11) must be satisfied. And as we show below, in order for young intermediaries *not* to want to expand their note issues above $(1-\alpha)\,a$, it is necessary that

$$(1-\alpha)^3\,R_{h,h}\,(t)\,R_{h,h}\,(t+1) \leq R_{h,k}\,(t)\,R_{k,h}\,(t+1)\,. \tag{15}$$

Thus, in order for middle-aged intermediaries to neither borrow nor lend, rates of return must satisfy

$$\begin{aligned}(1-\alpha)^3\,R_{h,h}\,(t)\,R_{h,h}\,(t+1) &\leq \\ R_{h,k}\,(t)\,R_{k,h}\,(t+1) &\leq \\ (1-\alpha)\,R_{h,h}\,(t)\,R_{h,h}\,(t+1)\,.\end{aligned} \tag{16}$$

Finally, we note that equations (6) and (14) imply

$$c_{i,h,t}\,(t+1) = \frac{(1-\alpha)^2\,R_{h,h}\,(t)\,a}{R_{h,k}\,(t)\,R_{k,h}\,(t+1)}. \tag{17}$$

Moreover, inequality (15) then implies that

$$c_{i,h,t}\,(t+1) \geq \frac{(1-\alpha)\,a}{R_{h,h}\,(t+1)}. \tag{18}$$

As we will see, this condition can hold as a strict inequality. Thus middle-aged intermediaries can consume more than the discounted present value of their endowment. This is because, when (11) holds as a strict inequality, intermediaries earn a profit on note issues up to a face value of $(1-\alpha)\,a$.

Remark 1. Under the arrangement for issuing banknotes existing under the National Banking Act in the U.S.,[15] there was an upper limit on total banknote issues. As noted by Cagan (1963) and others,[16] on average banknote issue earned in excess of normal profits. Nonetheless, the upper limit on banknote issues was never reached. This has been viewed as a major puzzle in the literature. In our model, when (11) holds as a strict

[15] This arrangement was in place from 1863-1929.

[16] See, for instance, Champ, Wallace, and Weber (1994) or Champ, Freeman, and Weber (1999).

inequality—as can happen—banknote issue earns excess profits. Even so, if (15) holds as a strict inequality, intermediaries will optimally choose not to expand the face value of their note issues beyond $(1-\alpha)\,a$. This is simply because of the transactions costs they would incur if they did so. But an outside observer, making Cagan's calculations in our economy when Case 2 obtains, would conclude that banknote issue was profitable, and yet that banks voluntarily limited the quantity of banknotes in circulation. Thus our analysis suggests a resolution of the question of why banks did not issue a larger volume of notes historically in the U.S.

We now turn to Case 3, in which inequality (7) holds and the revenue of middle-aged intermediaries exceeds their consumption. As a result, middle-aged intermediaries make loans. These loans are made, in equilibrium, to young borrowers. Since loans are made to agents who remain in location h between t and $t+1$, they earn the gross real rate of return $R_{h,h}\,(t+1)$.

Since middle-aged intermediaries deliver

$$\frac{(1-\alpha)\,R_{h,h}\,(t)\,x_{i,h,t}\,(t+2)}{R_{h,k}\,(t)\,R_{k,h}\,(t+1)} - c_{i,h,t}\,(t+1) \tag{19}$$

units of goods to borrowers, they incur a proportional transactions cost of α. Hence the actual quantity lent by a middle-aged intermediary in location h at t is

$$(1-\alpha)\left\{\frac{(1-\alpha)\,R_{h,h}\,(t)\,x_{i,h,t}\,(t+2)}{R_{h,k}\,(t)\,R_{k,h}\,(t+1)} - c_{i,h,t}\,(t+1)\right\}. \tag{20}$$

This lending then generates old age income of

$$(1-\alpha)\,R_{h,h}\,(t+1)\left\{\frac{(1-\alpha)\,R_{h,h}\,(t)\,x_{i,h,t}\,(t+2)}{R_{h,k}\,(t)\,R_{k,h}\,(t+1)} - c_{i,h,t}\,(t+1)\right\}. \tag{21}$$

This income, along with the intermediary's old age endowment, is used to redeem the notes issued when young.

To fix ideas we invoke Assumption 2 so that, when agents redeem their liabilities, they must deliver goods.[17] Then a fraction α of old intermediaries' endowments and a fraction α of their loan income is "lost" in the

[17] Alternatively, if agents redeem their own liabilities using the liabilities of other agents, they incur the same transactions costs as if they issue goods. Historically, arrangements that allowed banks to issue notes often specified that the holder of the note was entitled to redemption in a certain specified set of objects, such as specie. It is easy to give our model an interpretation where redemption of liabilities with goods is equivalent to redemption of notes with specie. Such an interpretation would follow Sargent and Wallace (1983).

form of transactions costs. Hence old intermediaries in location h at $t+2$ can redeem their note issues iff

$$(1-\alpha)^2 R_{h,h}(t+1)\left\{\frac{(1-\alpha) R_{h,h}(t)\, x_{i,h,t}(t+2)}{R_{h,k}(t)\, R_{k,h}(t+1)} - c_{i,h,t}(t+1)\right\} + (1-\alpha)\, a \geq x_{i,h,t}(t+2). \quad (22)$$

Young intermediaries thus choose $x_{i,h,t}(t+2)$ to maximize $c_{i,h,t}(t+1)$, subject to inequality (22).

An absence of arbitrage opportunities associated with banknote issues requires that

$$(1-\alpha)^3 R_{h,h}(t)\, R_{h,h}(t+1) \leq R_{h,k}(t)\, R_{k,h}(t+1) \quad (23)$$

must hold. Moreover, when inequality (7) is satisfied, then (23) must hold with equality. When that occurs, we have

$$c_{i,h,t}(t+1) = \frac{a}{(1-\alpha) R_{h,h}(t+1)}. \quad (24)$$

Thus intermediaries earn profits from note issue: Not only does their middle-age consumption exceed the discounted present value of their net-of-transactions-cost endowment, it exceeds the discounted present value of their total endowment.

Finally, condition (23) holding at equality coupled with equation (24) implies inequality (7) holds iff

$$(1-\alpha)\, a < x_{i,h,t}(t+2). \quad (25)$$

That is, the face value of banknote issues by young intermediaries must exceed their old-age, net-of-transactions-cost endowment.

Remark 2. As we observed for a Case 2 economy, this model has the potential to explain Cagan's banknote underissue puzzle. In a Case 3 economy, average profits from note issue are positive.[18] However, at the margin, additional note issues do not generate additional profits. The nonlinearity of profits in note issue is generated by the kink that arises in the budget set of intermediaries as they transit from being borrowers in middle age to being lenders. This kink is due to the presence of transactions costs.

[18] Cagan calculated average profits generated by banknote issue under the National Banking System.

3.3. Borrowers

A young borrower could, at least potentially, issue a mix of one and two-period liabilities. One-period liabilities, IOUs, are issued with a face value of $b_{b,h,t}(t)$ by young borrowers in location h at t. These liabilities must be held by agents who remain in location h between t and $t+1$; therefore they have a time t discounted present value of $b_{b,h,t}(t)/R_{h,h}(t)$. In addition, young borrowers in location h at t can issue two-period circulating liabilities, *private notes*, with a face value of $x_{b,h,t}(t+2)$. There are two possibilities regarding who might hold these liabilities, which we now take up in turn.

The first possibility is that the two-period circulating liabilities are held by agents who move from location h at t to location k, $k \neq h$, at $t+1$. If this transpires, these liabilities must remain in location k between $t+1$ and $t+2$ in order for their issuer—who will be in location k at $t+2$—to redeem them. In this situation the liabilities earn a gross real return of $R_{h,k}(t)$ between t and $t+1$, and $R_{k,k}(t+1)$ between $t+1$ and $t+2$. In this case a young borrower faces the budget constraint

$$c_{b,h,t}(t) \leq \frac{b_{b,h,t}(t)}{R_{h,h}(t)} + \frac{x_{b,h,t}(t+2)}{R_{h,k}(t)\, R_{k,h}(t+1)}. \tag{26}$$

In middle-age borrowers must complete two transactions. They must pay off any one-period liabilities issued, and they must acquire claims on consumption in location $k \neq h$ at $t+2$. These claims enable them to redeem their two-period liabilities. In order to do so, they must obtain circulating liabilities. Let $\tilde{x}_{b,h,t}(t+1)$ be the time $t+2$ value in location k of the liabilities acquired. Since these liabilities earn a gross real return of $R_{h,k}(t+1)$ between $t+1$ and $t+2$, the cost of acquiring them is $\tilde{x}_{b,h,t}(t+1)/R_{h,k}(t+1)$. The second period budget constraint of a borrower then requires that

$$(1-\alpha)\, w \geq b_{b,h,t}(t) + \frac{\tilde{x}_{b,h,t}(t+1)}{R_{h,k}(t+1)}. \tag{27}$$

Finally, since the borrower must acquire claims to pay off previously-issued circulating liabilities,

$$(1-\alpha)\, \tilde{x}_{b,h,t}(t+1) \geq x_{b,h,t}(t+2) \tag{28}$$

must hold. The presence of the term $(1-\alpha)$ on the left-hand-side of (28) reflects the fact that old borrowers exchange the claims they acquire in middle-age for goods, which are then used to redeem two-period liabilities. The delivery of these goods involves a proportional transactions cost of α.

Consolidating equations (26)-(28), we obtain the lifetime budget constraint of a borrower born in location h at t when circulating liabilities are sold to young lenders or middle-aged borrowers:

$$c_{b,h,t}(t) \leq R_{h,h}(t)^{-1}\left[(1-\alpha)w - \frac{\tilde{x}_{b,h,t}(t+1)}{R_{h,k}(t+1)}\right] + \frac{(1-\alpha)\tilde{x}_{b,h,t}(t+1)}{R_{h,k}(t)R_{k,k}(t+1)}. \quad (29)$$

Borrowers choose $\tilde{x}_{b,h,t}(t+1) \geq 0$ to maximize $c_{b,h,t}(t)$, subject to (29). The no-arbitrage condition associated with this problem is

$$(1-\alpha)R_{h,h}(t)R_{h,k}(t+1) \leq R_{h,k}(t)R_{k,k}(t+1). \quad (30)$$

If (30) is a strict inequality, then borrowers optimally set $\tilde{x}_{b,h,t}(t+1) = 0$. Equation (28) then implies that young borrowers do not issue private notes if (30) is a strict inequality.

As we will see, if middle-aged intermediaries either borrow or lend, a steady state will necessarily have $R_{h,h}(t) = R_{k,k}(t+1)$. Then, at or near a steady state, (30) will hold as a strict inequality. In addition, if locations are symmetric, $R_{h,h}(t) = R_{k,k}(t+1)$ will hold in a steady state if middle-aged intermediaries neither borrow nor lend. Thus again (30) will be a strict inequality at or near a steady state. We conclude that young borrowers will not issue private notes if locations are symmetric.

The second possibility is that young borrowers issue two-period liabilities that are held by intermediaries between t and $t+1$, and therefore their liabilities remain in location h between those dates. Intermediaries then sell these liabilities to agents who move from location h to location k between t and $t+1$. Intuitively, doing this is not superior to a scenario where borrowers take a sequence of one-period loans; this is feasible since all trades involve agents with whom borrowers share the same location in the relevant periods. Moreover, a sequence of one-period loans is inferior to a single one-period loan, since it involves more transactions and thus more transactions costs.[19] Thus we can conclude that borrowers never issue private notes.

[19] It is possible to repeat the logic above and show that the no-arbitrage condition associated with two-period liabilities that are sold to intermediaries is $(1-\alpha)R_{h,h}(t)R_{h,k}(t) \leq R_{h,h}(t)R_{h,k}(t+1)$. If this condition is a strict inequality, no private notes will be issued when young. But obviously this condition will be a strict inequality at or near a steady state.

Finally, when borrowers issue no private notes, equation (29) implies that

$$c_{b,h,t}(t) = \frac{(1-\alpha)\,w}{R_{h,h}(t)}. \tag{31}$$

Young borrowers consume the discounted present value of their net-of-transactions-cost endowment.

3.4. Summary

The optimal behavior of agents indicates that young intermediaries will issue banknotes, which they use to buy goods from young lenders. These goods are then loaned in a conventional way (that is, using one-period liabilities, or IOUs) to young borrowers. Lenders take newly-issued banknotes to their new location, and use them to buy goods. Lenders also take the previously-issued banknotes obtained in this way to their location of origin and redeem them. Borrowers use the services of intermediaries rather than issue private notes themselves because issuing private notes involves an excessive number of transactions, and excessive transactions costs.

4. EQUILIBRIUM CONDITIONS

There are three general conditions that an equilibrium must satisfy. First, agents who hold circulating liabilities (young lenders) must be willing to hold all of the banknotes that are issued by young intermediaries in their current location, as well as all of the previously-issued banknotes that will mature next period in the opposite location. The demand for assets per young lender is simply their net-of-transactions-cost endowment, $(1-\alpha)\,e$. And, the fraction of the location h population that is lenders is $\theta_{h,\ell}$. Thus the aggregate demand for circulating liabilities in location h is $\theta_{h,\ell}(1-\alpha)\,e$ at each date.

At date t, each young intermediary in location h issues banknotes with a face value of $x_{i,h,t}(t+2)$. These notes have a date t discounted present value of $x_{i,h,t}(t+2)/R_{h,k}(t)\,R_{k,h}(t+1)$, since they will be carried from location h to location k between t and $t+1$, and then back again between $t+1$ and $t+2$. And young intermediaries constitute a fraction $\theta_{h,i}$ of the location h population. In addition, at $t-1$ young intermediaries in location $k \neq h$, who constitute a fraction $\theta_{k,i}$ of the population in that location, issued location k banknotes with a face value of $x_{i,k,t-1}(t+1)$. All of these notes were carried from location k to location h by young lenders, and hence at t they circulate in location h. There they are used to buy goods from young lenders. The date t discounted present value of these notes is $x_{i,k,t-1}(t+1)/R_{h,k}(t)$, since they must be carried from location h to location k. It follows that the demand for notes equals their supply if

$$\theta_{h,i}\left[\frac{x_{i,h,t}(t+2)}{R_{h,k}(t)\,R_{k,h}(t+1)}\right]+\theta_{k,i}\left[\frac{x_{i,k,t-1}(t+1)}{R_{h,k}(t)}\right]=\theta_{h,\ell}(1-\alpha)\,e \tag{32}$$

for $h=1,2$; $k=1,2$; $h\neq k$.

The nature of the remaining equilibrium conditions, which are loan market clearing and no arbitrage conditions, depend on whether middle-aged intermediaries borrow or lend. We now consider each case in turn.

4.1. Case 1: Middle-aged Intermediaries Borrow

When Case 1 obtains, the no arbitrage condition associated with banknote issue is

$$(1-\alpha)\,R_{h,h}(t)\,R_{h,h}(t+1)=R_{h,k}(t)\,R_{k,h}(t+1) \tag{33}$$

for $h=1,2$; $k=1,2$; $h\neq k$. In addition, middle-aged intermediaries—who comprise a fraction $\theta_{h,i}$ of the location h population—want to borrow

$$c_{i,h,t-1}(t)-\frac{(1-\alpha)\,R_{h,h}(t-1)\,x_{i,h,t-1}(t+1)}{R_{h,k}(t-1)\,R_{k,h}(t)} \tag{34}$$

in location h at t. Moreover, when Case 1 obtains,

$$c_{i,h,t-1}(t)=\frac{(1-\alpha)\,a}{R_{h,h}(t)}. \tag{35}$$

Young borrowers must borrow their entire young period consumption,

$$c_{b,h,t}(t)=\frac{(1-\alpha)\,w}{R_{h,h}(t)}. \tag{36}$$

These agents constitute a fraction $\theta_{h,b}$ of the location h population. Finally, the supply of credit in location h at t in a Case 1 economy is $\frac{(1-\alpha)\theta_{h,i}x_{i,h,t}(t+2)}{R_{h,k}(t)R_{k,h}(t+1)}$. This is the case since young intermediaries each obtain goods with a value of $\frac{x_{i,h,t}(t+2)}{R_{h,k}(t)R_{k,h}(t+1)}$ by issuing banknotes. The goods obtained are delivered to borrowers, so that a proportional transactions cost of α is incurred. All of these observations imply that the loan market clears in location h at t if

$$\theta_{h,b}\left[\frac{(1-\alpha)\,w}{R_{h,h}\,(t)}\right]+ \theta_{h,i}\left[\frac{(1-\alpha)\,a}{R_{h,h}\,(t)}-\frac{(1-\alpha)\,R_{h,h}\,(t-1)\,x_{i,h,t-1}\,(t+1)}{R_{h,k}\,(t-1)\,R_{k,h}\,(t)}\right]= \theta_{h,i}\left[\frac{(1-\alpha)\,x_{i,h,t}\,(t+2)}{R_{h,k}\,(t)\,R_{k,h}\,(t+1)}\right] \quad (37)$$

for $h=1,2$; $k=1,2$; $h\neq k$.

We also note that, in order for Case 1 to obtain, $x_{i,h,t}\,(t+2)<(1-\alpha)\,a$ must hold.[20]

4.2. Case 2: Middle-aged Intermediaries Neither Borrow Nor Lend

When Case 2 holds, it must be the case that young intermediaries want neither to expand nor contract their banknote issues. This requires

$$(1-\alpha)^3\,R_{h,h}\,(t)\,R_{h,h}\,(t+1)\leq R_{h,k}\,(t)\,R_{k,h}\,(t+1)\leq(1-\alpha)\,R_{h,h}\,(t)\,R_{h,h}\,(t+1) \quad (38)$$

for $h=1,2$; $k=1,2$; $h\neq k$. Loan market clearing is exactly as described before, except that middle-aged intermediaries do not borrow or lend. Thus credit supply and credit demand are equal in each location if

$$\frac{\theta_{h,b}\,(1-\alpha)\,w}{R_{h,h}\,(t)}=\frac{\theta_{h,i}\,(1-\alpha)\,x_{i,h,t}(t+2)}{R_{h,k}\,(t)\,R_{k,h}(t+1)} \quad (39)$$

for $h=1,2$; $k=1,2$; $h\neq k$. Moreover, when Case 2 obtains, the face value of banknote issues by young intermediaries equals their third-period, net-of-transactions-cost endowment: That is, $x_{i,h,t}\,(t+2)=(1-\alpha)\,a$. Then (38) becomes

$$\theta_{h,b}R_{h,k}\,(t)\,R_{k,h}\,(t+1)=\theta_{h,i}\,(1-\alpha)\,aR_{h,h}\,(t)\,. \quad (40)$$

[20] Throughout this discussion we focus only on situations where middle-aged intermediaries are either net borrowers in each location, or net lenders in each location. This must transpire if the locations are symmetric. However, if locations are sufficiently asymmetric, it could happen that middle-aged intermediaries are net borrowers in one location and net lenders in the other. This would expand the set of possible equilibrium configurations.

4.3. Case 3: Middle-aged Intermediaries Lend

When Case 3 obtains, it must again be the case that young intermediaries want neither to contract nor to indefinitely expand their banknote issue. This requires that the no-arbitrage condition

$$(1-\alpha)^3 R_{h,h}(t) R_{h,h}(t+1) = R_{h,k}(t) R_{k,h}(t+1); \tag{41}$$

for $h = 1,2$; $k = 1,2$; $h \neq k$; be satisfied. In addition, in a Case 3 economy, the demand for credit by young borrowers and the supply of credit by young intermediaries is as previously described. However, middle-aged intermediaries in location h at time t now want to lend

$$\frac{(1-\alpha) R_{h,h}(t-1) x_{i,h,t-1}(t+1)}{R_{h,k}(t-1) R_{k,h}(t)} - c_{i,h,t-1}(t) = \frac{(1-\alpha) R_{h,h}(t-1) x_{i,h,t-1}(t+1)}{R_{h,k}(t-1) R_{k,h}(t)} - \frac{a}{(1-\alpha) R_{h,k}(t)}. \tag{42}$$

Since intermediaries constitute a fraction $\theta_{h,i}$ of the location h population, it follows that the loan market clears in each location if

$$\theta_{h,b}\left[\frac{(1-\alpha) w}{R_{h,h}(t)}\right] = \theta_{h,i}\left[\frac{(1-\alpha) x_{i,h,t}(t+2)}{R_{h,k}(t) R_{k,h}(t+1)}\right] + (1-\alpha)\theta_{h,i}\left\{\frac{(1-\alpha) R_{h,h}(t-1) x_{i,h,t-1}(t+1)}{R_{h,k}(t-1) R_{k,h}(t)} - \frac{a}{(1-\alpha) R_{h,h}(t)}\right\} \tag{43}$$

for $h = 1,2$; $k = 1,2$; $h \neq k$. The second term on the right-hand side of equation (43) is multiplied by $(1-\alpha)$ since a proportional transactions cost of α is incurred when middle-aged intermediaries deliver goods to young borrowers in the process of making loans.

Finally we note that, in order for Case 3 to obtain, $x_{i,h,t}(t+2) > (1-\alpha) a$ must hold.

It remains to describe the initial conditions of our economy. In the initial period, $t = 0$, the values $x_{i,h,t-1}(1)$; $h = 1,2$ are given as initial conditions.

We now turn our attention to a more complete characterization of equilibrium. This characterization includes a description of when each case obtains.

5. STEADY STATE EQUILIBRIA

In this section we analyze steady states. We therefore omit time subscripts and arguments. In addition, we restrict our attention to economies where locations are symmetric, so that $\theta_{1,j} = \theta_{2,j}$ for $j \in \{\ell, b, i\}$; consequently, we will simply denote fractions of agent types by θ_j, $j \in \{\ell, b, i\}$. Considering only economies where locations are symmetric greatly simplifies the conditions that an equilibrium must satisfy.[21]

As in the previous section, the conditions that a steady state equilibrium must satisfy differ according to whether middle-aged intermediaries are net borrowers or net lenders. We now describe each case in turn.

5.1. Case 1: Middle-aged Intermediaries Borrow

In a Case 1 economy, the steady state version of the no-arbitrage condition (33) is simply

$$(1-\alpha)(R_{1,1})^2 = (1-\alpha)(R_{2,2})^2 = R_{1,2}R_{2,1}. \tag{44}$$

It is then immediate that $R_{1,1} = R_{2,2} = R$.[22] In addition, the steady state version of equation (37) is

$$\frac{\theta_b w + \theta_i a}{R_{h,k}} = \theta_i \left(\frac{x_{i,h}}{R_{h,k}R_{k,h}} \right) (1 + R_{h,h}). \tag{45}$$

Using (44) in (45) and rearranging terms we obtain

$$x_{i,h} = \left(\frac{\theta_b w + \theta_i a}{\theta_i} \right) \left[\frac{(1-\alpha) R_{h,h}}{1 + R_{h,h}} \right]. \tag{46}$$

Symmetry of locations therefore implies that $x_{i,1} = x_{i,2} = x$.

Using these observations in the note market clearing condition (32), along with the fact that $\theta_{h,i} = \theta_{k,i} = \theta_i$, we obtain the remaining steady state equilibrium condition in a Case 1 economy:

$$\frac{x}{R_{1,2}R_{2,1}} + \frac{x}{R_{h,k}} = \frac{\theta_\ell (1-\alpha) e}{\theta_i} \tag{47}$$

for $h = 1, 2$; $k = 1, 2$; $h \neq k$. Symmetry of locations therefore implies that $R_{1,2} = R_{2,1}$. Thus, in a symmetric Case 1 economy, appropriate rates of return and banknote issues are equal across locations.

Using $R_{1,2} = R_{2,1}$ in (47) and rearranging terms, we obtain

$$x = \left[\frac{\theta_\ell (1-\alpha) e}{\theta_i} \right] \frac{(R_{1,2})^2}{1 + R_{1,2}}. \tag{48}$$

[21] Although, as Bullard and Smith (2002) show, many interesting phenomena that can occur when locations are not symmetric *cannot* occur if locations are symmetric.

[22] Note that this result in no way depends on the symmetry of locations.

In addition, setting $R_{1,2} = R_{2,1}$ in equation (44) implies that $R_{1,2} = R_{2,1} = R\sqrt{1-\alpha}$. Using this fact in (48), and using (46) to substitute out x, we get the condition that determines the steady state value of the within location interest rate R:

$$\frac{R(1+R)}{1+R\sqrt{1-\alpha}} = \Psi \tag{49}$$

where

$$\Psi \equiv \frac{\theta_b w + \theta_i a}{\theta_\ell (1-\alpha) e}. \tag{50}$$

The quantity $(1-\alpha)\Psi/R$ has an interpretation as the ratio of the consumption of young borrowers and middle-aged lenders—all of whom demand credit in a Case 1 economy—to the supply of savings by young lenders, net of transactions costs. The unique positive solution to (49) is

$$R = \frac{1}{2}\left(\Psi\sqrt{1-\alpha}-1\right) + \frac{1}{2}\left[\left(\Psi\sqrt{1-\alpha}-1\right)^2 + 4\Psi\right]^{\frac{1}{2}}. \tag{51}$$

Of course, in order to have a Case 1 steady state, $x < (1-\alpha)a$ must hold. From (46), this requirement is equivalent to

$$R < \frac{\theta_i a}{\theta_b w}. \tag{52}$$

Equation (52) asserts that the within location steady state interest rate must be low relative to the ratio of the aggregate endowment of intermediaries to the aggregate endowment of borrowers. Intuitively, the discounted present value of banknote issue must be large enough to meet the demand for credit of young borrowers and middle-aged intermediaries. Equation (52) points out that this is possible only if interest rates are not too high. How high they can be in a Case 1 economy depends on how much young borrowers consume relative to middle-aged lenders.

Equations (51) and (52) allow us to state an exact condition under which any economy will be a Case 1 economy: An economy will be in Case 1 iff

$$\frac{1}{2}\left(\Psi\sqrt{1-\alpha}-1\right) + \frac{1}{2}\left[\left(\Psi\sqrt{1-\alpha}-1\right)^2 + 4\Psi\right]^{\frac{1}{2}} < \frac{\theta_i a}{\theta_b w}. \tag{53}$$

5.2. Case 2: Middle-aged Intermediaries Neither Borrow Nor Lend

In a Case 2 steady state, equation (38) reduces to

$$(1-\alpha)^3 \left(R_{h,h}\right)^2 \le R_{h,k}R_{k,h} \le (1-\alpha)\left(R_{h,h}\right)^2 \tag{54}$$

for $h = 1, 2$; $k = 1, 2$; $h \neq k$. In addition, loan market clearing requires

$$R_{h,h} = \left[\frac{\theta_b w}{\theta_i (1-\alpha) a}\right] R_{h,k} R_{k,h} \tag{55}$$

for $h = 1, 2$; $k = 1, 2$; $h \neq k$. Therefore symmetry of locations implies that $R_{1,1} = R_{2,2} = R$.[23]

Moreover, imposing $\theta_{h,i} = \theta_{k,i} = \theta_i$ in (32), the steady state version of the note market clearing condition takes the form

$$\frac{x_{i,h}}{R_{h,k} R_{k,h}} + \frac{x_{i,k}}{R_{h,k}} = \frac{\theta_\ell (1-\alpha) e}{\theta_i}. \tag{56}$$

Then, since $x_{i,h} = x_{i,k} = (1-\alpha) a$, equation (56) implies that $R_{1,2} = R_{2,1}$. Using this fact in equation (55), we obtain

$$R_{1,2} = R_{2,1} = \left[\left(\frac{\theta_i a}{\theta_b w}\right)(1-\alpha) R\right]^{\frac{1}{2}}. \tag{57}$$

Using this and $x_{i,h} = x_{i,k} = (1-\alpha) a$ in (56) gives the condition that determines the steady state value of R:

$$\left[\frac{\theta_i a (1-\alpha)}{\theta_b w}\right]^{\frac{1}{2}} R^{\frac{1}{2}} = \left(\frac{1}{2}\right)\left(\frac{\theta_i a}{\theta_\ell e}\right) + \left(\frac{1}{2}\right)\left[\left(\frac{\theta_i a}{\theta_\ell e}\right)^2 + 4\left(\frac{\theta_i a}{\theta_\ell e}\right)\right]^{\frac{1}{2}}. \tag{58}$$

Of course in order for the economy to actually be in Case 2, the values of R and $R_{1,2} = R_{2,1}$ given by (57) and (58) must satisfy equation (54). It is straightforward to show that they do iff

$$\frac{\theta_i a}{\theta_b w} \leq R \leq \left(\frac{\theta_i a}{\theta_b w}\right)\left(\frac{1}{1-\alpha}\right)^2. \tag{59}$$

Using equation (58) we can write (59) in terms of parameters alone: An economy has a Case 2 steady state iff

$$2\sqrt{1-\alpha}\left(\frac{\theta_i a}{\theta_b w}\right) \leq \left(\frac{\theta_i a}{\theta_\ell e}\right) + \left[\left(\frac{\theta_i a}{\theta_\ell e}\right)^2 + 4\left(\frac{\theta_i a}{\theta_\ell e}\right)\right]^{\frac{1}{2}} \leq \left(\frac{2}{\sqrt{1-\alpha}}\right)\left(\frac{\theta_i a}{\theta_b w}\right). \tag{60}$$

[23] Even if locations are not symmetric, so that $R_{1,1} \neq R_{2,2}$ can hold, it is easy to verify that equation (30) will not hold as a strict inequality (borrowers will not issue notes) so long as $(1-\alpha)\,\theta_{2,b}/\theta_{2,i} < \theta_{1,b}/\theta_{1,i} < \theta_{2,b}/(1-\alpha)\,\theta_{2,i}$.

5.3. Case 3: Middle-aged Intermediaries Lend

When middle-aged intermediaries are net lenders, the steady state version of the no-arbitrage condition is

$$(1-\alpha)^3 (R_{h,h})^2 = R_{h,k} R_{k,h} \tag{61}$$

for $h = 1, 2$; $k = 1, 2$; $h \neq k$. It is then immediate that $R_{1,1} = R_{2,2} = R$.[24] Moreover, using (59) in the steady state version of the loan market clearing condition (43) yields

$$x_{i,h} = \left[\frac{\theta_b (1-\alpha) w + \theta_i a}{\theta_i}\right] \left[\frac{(1-\alpha)^2 R}{1 + (1-\alpha) R}\right] \tag{62}$$

for $h = 1, 2$. Then, as before $x_{i,1} = x_{i,2} = x$. Finally, using $\theta_{h,i} = \theta_{k,i} = \theta_i$ in (32) gives

$$\left(\frac{\theta_{h,3} x}{R_{h,k}}\right) \left(1 + \frac{1}{R_{k,h}}\right) = \theta_\ell (1-\alpha) e \tag{63}$$

for $h = 1, 2$, $k = 1, 2$, $h \neq k$. It follows that $R_{1,2} = R_{2,1}$. Moreover, equation (61) then implies that

$$R_{1,2} = R_{2,1} = R (1-\alpha)^{\frac{3}{2}}. \tag{64}$$

Substituting (62) and (64) into (63) then gives the condition that determines the steady state value of R:

$$\frac{R [1 + (1-\alpha) R]}{1 + R (1-\alpha)^{3/2}} = \lambda, \tag{65}$$

where

$$\lambda \equiv \frac{\theta_b w (1-\alpha) + \theta_i a}{\theta_\ell e (1-\alpha)^2}. \tag{66}$$

It is easy to check that equation (65) has a unique positive solution for R:

$$R = \frac{\lambda (1-\alpha)^{3/2} - 1}{2 (1-\alpha)} + \left[\frac{1}{2 (1-\alpha)} \left\{ \left[\lambda (1-\alpha)^{3/2} - 1\right]^2 + 4 (1-\alpha) \lambda \right\}^{\frac{1}{2}}\right]. \tag{67}$$

Of course, in order to have a Case 3 steady state it is necessary that $x > (1-\alpha) a$ hold. Equation (62) implies that this condition will be satisfied iff

$$R > \left(\frac{\theta_i a}{\theta_b w}\right) \left(\frac{1}{1-\alpha}\right)^2. \tag{68}$$

[24] Again, this does not depend on the assumed symmetry of locations.

Using (67) we can write (68) in terms of parameters alone: An economy has a Case 3 steady state iff

$$\frac{\lambda (1-\alpha)^{3/2} - 1}{2} + \left(\frac{1}{2}\right)\left\{\left[\lambda (1-\alpha)^{3/2} - 1\right]^2 + 4(1-\alpha)\lambda\right\}^{\frac{1}{2}} > \left(\frac{\theta_i a}{\theta_b w}\right)\left(\frac{1}{1-\alpha}\right). \quad (69)$$

5.4. Intermediary Profits and Welfare

In a Case 2 or Case 3 steady state, the consumption of middle-aged intermediaries exceeds the discounted present value of their net-of-transactions-cost endowments. Thus, in effect, intermediaries make profits from issuing notes and lending the goods purchased with them. This observation might lead one to believe that an intermediary would prefer to live in an economy that had a Case 2 or a Case 3 steady state rather than in an economy that had a Case 1 steady state. Interestingly, this is not necessarily the case. Indeed, comparing intermediary welfare across Cases 1, 2, and 3 steady states,[25] it is possible that intermediary welfare is highest (lowest) in an economy with a Case 1 (Case 3) steady state. In effect, comparing across economies with different steady states, intermediary profits and intermediary welfare can be *inversely* related. Or, in other words, the fact that intermediaries earn profits in no way implies that they enjoy high levels of utility.

To see this we observe that, in an economy with a Case 1 steady state, the consumption of a middle-aged intermediary is $c_{i,h,t}(t+1) = \frac{(1-\alpha)a}{R_{h,h}(t)}$. Moreover, the fact that there is a Case 1 steady state implies that $R_{h,h}(t) < \theta_i a/\theta_b w$. Thus, in a Case 1 steady state, $c_{i,h,t}(t+1) > \theta_b (1-\alpha) w/\theta_i$ holds.

Now consider a different economy where parameter values imply the existence of a Case 2 steady state. Here the consumption of a middle-aged intermediary is

$$c_{i,h,t}(t+1) = \frac{(1-\alpha) R_{h,h}(t)\, x_{i,h,t}(t+2)}{R_{h,k}(t)\, R_{k,h}(t+1)} = \frac{(1-\alpha)^2 a R_{h,h}(t)}{R_{h,k}(t)\, R_{k,h}(t+1)}. \quad (70)$$

[25] That is, we think about the steady state welfare of a typical intermediary who lives in an economy with a Case 1 steady state, and compare that with the welfare of the same intermediary, living instead in economies with Case 2 and Case 3 steady states, respectively.

Moreover, in a Case 2 steady state,

$$R = \left[\frac{\theta_b w}{\theta_i (1-\alpha) a}\right] R_{1,2} R_{2,1}. \tag{71}$$

Thus, in an economy with a Case 2 steady state

$$c_{i,h,t}(t+1) = \frac{\theta_b (1-\alpha) w}{\theta_i}. \tag{72}$$

Finally, consider a third economy that has a Case 3 steady state. In such an economy, the consumption of a middle-aged intermediary is

$$c_{i,h,t}(t+1) = \frac{a}{(1-\alpha) R_{h,h}(t)}. \tag{73}$$

Moreover, in a Case 3 steady state

$$R_{h,h}(t) > \frac{\theta_i a}{\theta_b w (1-\alpha)^2} \tag{74}$$

holds. Therefore, when an economy has a Case 3 steady state,

$$c_{i,h,t}(t+1) < \frac{\theta_b w (1-\alpha)}{\theta_i} \tag{75}$$

holds.

We now note that we can fix the values a, w, and θ_j, $j \in \{\ell, b, i\}$ and can, by choosing the endowment of lenders, e, appropriately, make sure that an economy has a Case 1, 2, or 3 steady state. Thus it is quite possible that intermediary utility will be highest (lowest) in a Case 1 (Case 3) economy, even though intermediary profits are highest (lowest) in a Case 3 (Case 1) economy. We conclude that the ability of intermediaries to earn profits in economies with Case 2 or Case 3 steady states does not imply that intermediary welfare is necessarily high in such economies.

5.5. Existence and Uniqueness of Steady State Equilibria

Appendix A establishes the following result.

PROPOSITION 1. *(a) Every economy has a steady state equilibrium. (b) Every economy has only one steady state with* $R_{11} = R_{22}$.

Proof. See Appendix A. ∎

Part (b) of the proposition implies that no economy has a Case 1 and a Case 2 steady state, for example.

6. DYNAMICS

In this section we explore equilibrium dynamics. The dynamical system that governs the evolution of banknote issues and rates of return differs according to the behavior of middle-aged intermediaries. To simplify matters, we focus only on dynamics when middle-aged intermediaries always borrow, always lend, or always do neither. In particular, we do not analyze transitions between these situations. Thus, in some sense, we focus on local dynamics. As we know from the previous section, every economy has a unique steady state with $R_{1,1} = R_{2,2}$. Thus we refer to an economy that has a Case 1 steady state as a Case 1 economy, and so on. We continue to employ the assumption of symmetry of locations, so that $\theta_{1,j} = \theta_{2,j} = \theta_j$ for all j. We now analyze equilibrium dynamics in Case 1, 2, and 3 economies in turn.

6.1. A Case 1 Economy

For a Case 1 economy, the no-arbitrage condition (33) implies that the loan-market clearing condition for location h, equation (37), can be written as

$$\frac{x_{i,h,t}\left(t+2\right)}{R_{h,k}\left(t\right)R_{k,h}\left(t+1\right)} = \frac{\theta_b w + \theta_i a}{\theta_i R_{h,h}\left(t\right)} - \frac{x_{i,h,t-1}\left(t+1\right)}{\left(1-\alpha\right)R_{h,h}\left(t\right)} \tag{76}$$

for $h = 1, 2$; $k = 1, 2$; $h \neq k$. Substituting (76) into the banknote market clearing condition (32) gives

$$\frac{\theta_b w + \theta_i a}{\theta_i R_{h,h}\left(t\right)} - \frac{x_{i,h,t-1}\left(t+1\right)}{\left(1-\alpha\right)R_{h,h}\left(t\right)} + \frac{x_{i,k,t-1}\left(t+1\right)}{R_{h,k}\left(t\right)} = \frac{\theta_\ell\left(1-\alpha\right)e}{\theta_i} \tag{77}$$

for $h = 1, 2$; $k = 1, 2$; $h \neq k$. We can now solve the $h = 1$ and $h = 2$ versions of (77) for $x_{i,h,t-1}\left(t+1\right)$; $h = 1, 2$, to obtain

$$\begin{aligned} x_{i,2,t-1}\left(t+1\right) = &\left\{\left[\frac{\theta_b w + \theta_i a}{\theta_i}\right]\left[\frac{R_{1,2}\left(t\right)}{R_{1,1}\left(t\right)}\right]\left[1 + \frac{R_{2,1}\left(t\right)}{\left(1-\alpha\right)R_{2,2}\left(t\right)}\right] - \right. \\ &\left.\left(\frac{\theta_\ell}{\theta_i}\right)\left(1-\alpha\right)eR_{1,2}\left(t\right)\left[1 + \frac{R_{2,1}\left(t\right)}{\left(1-\alpha\right)R_{1,1}\left(t\right)}\right]\right\} \times \\ &\left\{\frac{R_{1,2}\left(t\right)R_{2,1}\left(t\right)}{\left(1-\alpha\right)^2 R_{1,1}\left(t\right)R_{2,2}\left(t\right)} - 1\right\}^{-1} \equiv f\left[R_{1,1}\left(t\right), R_{2,2}\left(t\right), R_{1,2}\left(t\right), R_{2,1}\left(t\right)\right] \end{aligned} \tag{78}$$

and

$$x_{i,1,t-1}(t+1) = \left(\frac{\theta_\ell}{\theta_i}\right)(1-\alpha)\, eR_{2,1}(t) - \left(\frac{\theta_b w + \theta_i a}{\theta_i}\right)\left[\frac{R_{2,1}(t)}{R_{2,2}(t)}\right] + \left[\frac{R_{2,1}(t)}{(1-\alpha)\, R_{2,2}(t)}\right] f\left[R_{1,1}(t), R_{2,2}(t), R_{1,2}(t), R_{2,2}(t)\right] \equiv g\left[R_{1,1}(t), R_{2,2}(t), R_{1,2}(t), R_{2,1}(t)\right]. \quad (79)$$

Equations (78) and (79) give banknote issues at t as a function of the complete set of time t interest rates.

Now substitute equations (78) and (79) into the $h = 1$ and $h = 2$ versions of equation (32) to obtain

$$\frac{g\left[R_{1,1}(t+1), R_{2,2}(t+1), R_{1,2}(t+1), R_{2,1}(t+1)\right]}{R_{1,2}(t)\, R_{2,1}(t+1)} + \frac{f\left[R_{1,1}(t), R_{2,2}(t), R_{1,2}(t), R_{2,1}(t)\right]}{R_{1,2}(t)} = \left(\frac{\theta_\ell}{\theta_i}\right)(1-\alpha)\, e. \quad (80)$$

and

$$\frac{f\left[R_{1,1}(t+1), R_{2,2}(t+1), R_{1,2}(t+1), R_{2,1}(t+1)\right]}{R_{2,1}(t)\, R_{1,2}(t+1)} + \frac{g\left[R_{1,1}(t), R_{2,2}(t), R_{1,2}(t), R_{2,1}(t)\right]}{R_{2,1}(t)} = \left(\frac{\theta_\ell}{\theta_i}\right)(1-\alpha)\, e. \quad (81)$$

Equations (80), (81), and the two no-arbitrage conditions (33) constitute a complete set of equilibrium conditions determining the evolution of interest rates. Once these are determined equations (78) and (79) describe the evolution of banknote issues. Equations (33), (78), and (79) comprise a system of four first-order difference equations.

These equations can be rewritten in explicit first-order form. Define $\pi_1 = \frac{\theta_b w + \theta_i a}{\theta_i}$, $\pi_2 = \frac{\theta_\ell e(1-\alpha)}{\theta_i}$, and $\pi_3 = \sqrt{1-\alpha}$, as well as $d_1 = \pi_2 + \pi_1 (\pi_3)^2$, $d_2 = (\pi_3)^4 (\pi_1 + \pi_2)$, $d_3 = \pi_2 - \pi_2 (\pi_3)^2$, and $d_4 = \pi_1 \left((\pi_3)^2 - 1\right)$. Then the system is given by

$$R_{1,1}(t+1) = \frac{\pi_1 \left[R_{1,2}(t)\, R_{2,1}(t) - (\pi_3)^4 R_{1,1}(t)\, R_{2,2}(t)\right]}{A_1 + R_{1,2}(t)\, R_{2,1}(t)\left[d_1 - \pi_2 R_{2,1}(t)\right] - R_{1,1}(t)\, R_{2,2}(t)\, d_2} \quad (82)$$

where

$$A_1 = R_{1,1}(t) R_{2,1}(t) \left\{ R_{1,2}(t) d_3 + (\pi_3)^2 [d_4 + \pi_2 R_{2,2}(t)] \right\}, \tag{83}$$

$$R_{2,2}(t+1) = \frac{\pi_1 \left[R_{1,2}(t) R_{2,1}(t) - (\pi_3)^4 R_{1,1}(t) R_{2,2}(t) \right]}{A_2 - \pi_2 R_{1,2}(t)^2 R_{2,1}(t) - d_2 R_{1,1}(t) R_{2,2}(t)} \tag{84}$$

where

$$\begin{aligned} A_2 = {} & R_{1,2}(t) \times \\ & \left\{ (\pi_3)^2 R_{2,2}(t) [d_4 + \pi_2 R_{1,1}(t)] + R_{2,1}(t) [d_1 + d_3 R_{2,2}(t)] \right\}, \end{aligned} \tag{85}$$

$$R_{1,2}(t+1) = \frac{\pi_1 (\pi_3)^2 R_{2,2}(t) \left[R_{1,2}(t) R_{2,1}(t) - (\pi_3)^4 R_{1,1}(t) R_{2,2}(t) \right]}{A_3 - \pi_2 R_{1,2}(t)^2 R_{2,1}(t)^2 - d_2 R_{1,1}(t) R_{2,2}(t) R_{2,1}(t)} \tag{86}$$

where

$$\begin{aligned} A_3 = {} & R_{1,2}(t) R_{2,1}(t) \times \\ & \left\{ (\pi_3)^2 [d_4 + \pi_2 R_{1,1}(t)] R_{2,2}(t) + R_{2,1}(t) [d_1 + d_3 R_{2,2}(t)] \right\}, \end{aligned} \tag{87}$$

and

$$R_{2,1}(t+1) = \frac{\pi_1 (\pi_3)^2 R_{1,1}(t) \left[R_{1,2}(t) R_{2,1}(t) - (\pi_3)^4 R_{1,1}(t) R_{2,2}(t) \right]}{R_{1,2}(t)^2 R_{2,1}(t) [d_1 - \pi_2 R_{2,1}(t)] + A_4} \tag{88}$$

where

$$\begin{aligned} A_4 = {} & R_{1,1}(t) R_{1,2}(t) \times \\ & \left\{ R_{2,1}(t) \left[d_3 R_{1,2}(t) + (\pi_3)^2 (d_4 + \pi_2 R_{2,2}(t)) \right] - d_2 R_{2,2}(t) \right\}. \end{aligned} \tag{89}$$

Letting R represent the steady state value of $R_{1,1} = R_{2,2}$ in Case 1, the Jacobian matrix associated with this system is given by

$$J = A_5 \begin{bmatrix} J_{11} & \pi_3 J_{11} & J_{13} & -J_{11} \\ \pi_3 J_{11} & J_{11} & -J_{11} & J_{13} \\ (\pi_3)^2 J_{11} & J_{32} & -\pi_3 J_{11} & J_{34} \\ J_{32} & (\pi_3)^2 J_{11} & J_{34} & -\pi_3 J_{11} \end{bmatrix} \tag{90}$$

where

$$A_5 = \frac{1}{(\pi_3 - 1) R (\pi_2 - \pi_1 \pi_3 + \pi_2 R)^2}, \tag{91}$$

$$J_{11} = \pi_1 \left(\pi_2 R - \pi_1 \pi_3\right), \tag{92}$$

$$J_{13} = \pi_1 \left(\pi_1 - \pi_2 R\right), \tag{93}$$

$$J_{32} = -\pi_1 \pi_3 \left[\pi_1 \left(\pi_3\right)^2 - \pi_2 \left(\pi_3 R + \pi_3 - 1\right)\right], \tag{94}$$

and

$$J_{34} = \pi_1 \left\{\pi_1 \left(\pi_3\right)^2 + \pi_2 \left[1 + R - \pi_3 \left(1 + 2R\right)\right]\right\}. \tag{95}$$

The associated eigenvalues, denoted by μ_n, $n = 1, 2, 3, 4$ are given by

$$\mu_1 = \frac{\pi_1 \left(\pi_2 + \pi_1 \pi_3\right)}{2R \left(\pi_2 - \pi_1 \pi_3 + \pi_2 R\right)^2} - \frac{\sqrt{\left(\pi_1\right)^2 \left[\left(\pi_1\right)^2 \left(\pi_3\right)^2 + 2\pi_1 \pi_2 \pi_3 \left(1 - 2R\right) + \left(\pi_2 + 2\pi_2 R\right)^2\right]}}{2R \left(\pi_2 - \pi_1 \pi_3 + \pi_2 R\right)^2}, \tag{96}$$

$$\mu_2 = \frac{\pi_1 \left(\pi_2 + \pi_1 \pi_3\right)}{2R \left(\pi_2 - \pi_1 \pi_3 + \pi_2 R\right)^2} + \frac{\sqrt{\left(\pi_1\right)^2 \left[\left(\pi_1\right)^2 \left(\pi_3\right)^2 + 2\pi_1 \pi_2 \pi_3 \left(1 - 2R\right) + \left(\pi_2 + 2\pi_2 R\right)^2\right]}}{2R \left(\pi_2 - \pi_1 \pi_3 + \pi_2 R\right)^2}, \tag{97}$$

$$\mu_3 = \frac{-\pi_1 \left(\pi_2 - \pi_1 \pi_3 + 2\pi_2 R\right) - \pi_1 \left(\pi_2 - \pi_1 \pi_3\right)}{2R \left(\pi_2 - \pi_1 \pi_3 + \pi_2 R\right)^2}, \tag{98}$$

and

$$\mu_4 = \frac{-2\pi_1 \pi_2 R}{2R \left(\pi_2 - \pi_1 \pi_3 + \pi_2 R\right)^2}. \tag{99}$$

From these expressions it is easy to verify that $-1 = \mu_3 < \mu_4 < 0$, and that $\mu_2 < 0 < \mu_1$. However, in order to say more concerning local dynamics it is necessary to proceed numerically. We now describe how we do so.

To evaluate the eigenvalues just stated numerically, we use a strategy that samples from all possible economies in the class that we are studying. This strategy works as follows. An economy is a list of three endowments, three masses of agent types, and a transactions cost, which we denote by $\{a, w, e, \theta_b, \theta_\ell, \theta_i, \alpha\}$. We wish to assign numerical values to these parameters, evaluate the eigenvalues, and repeat the process a large number of times in order to see if we can make general statements about the nature of the eigenvalue configuration. In assigning numerical values, we wish to be quite general. As is usual in overlapping generations contexts, it is relative endowments that matter, not absolute endowments. Therefore, we chose values for a, e, and w randomly from a uniform distribution defined on the unit interval. This means that each given pair of relative endowments could be near zero or very large, effectively spanning the space of possible endowment configurations. The transactions cost parameter, α, can be viewed as a proportional cost, and so we chose it from another uniform distribution defined on the unit interval. The masses of agents must sum to one. We chose random numbers r_1, r_2, and r_3 from a uniform distribution on $(0,1)$, and then set $\theta_b = r_1/(r_1+r_2+r_3)$, $\theta_i = r_2/(r_1+r_2+r_3)$, and $\theta_\ell = r_3/(r_1+r_2+r_3)$. Again, this allows us to sample from all possible masses. For each draw of the parameter vector $\{a, w, e, \theta_b, \theta_\ell, \theta_i, \alpha\}$, we checked that the condition for a Case 1 economy held before proceeding. We sampled 1,000 economies in this manner.

We were able to identify a number of regularities concerning the eigenvalues using this procedure. As we know, the eigenvalues are always real, and one eigenvalue is equal to -1.[26] The final eigenvalue is negative and can be in the interval $(-1,0)$, or in the interval $(-\infty,-1)$. A sufficient condition for this last eigenvalue to lie in $(-\infty,-1)$ is that $R\sqrt{1-\alpha} > 1$. The term $R\sqrt{1-\alpha} = R_{1,2} = R_{2,1}$, so that this condition is associated with efficiency.

The presence of an eigenvalue with modulus one means that the linear approximation is insufficient to fully describe the local dynamics of these systems. We need to proceed to a second order Taylor's series approximation of the system in order to determine if the motion on the center manifold is damped or explosive. The details of this calculation are given in Appendix F of Azariadis, Bullard, and Smith (2001) and so will not be repeated here.[27] We carried out this calculation numerically for 1,000 economies, using the numerical strategy described above. We found that the motion on the center manifold could be either stable or unstable, and that which situation prevailed was not related in any obvious way to the characteristics of the economy (such as the value of the cross-location steady state interest rate).

We now have enough information to fully describe the qualitative local dynamics near a steady state for Case 1 economies. When $R\sqrt{1-\alpha} > 1$, that is, the cross-location steady state interest rate is greater than one, we have one stable eigenvalue, a unit eigenvalue, and two initial conditions. We have found that the center manifold may be stable or unstable

[26] This finding is in accordance with the analysis of similar economies in Azariadis, Bullard, and Smith (2001) and in Bullard and Smith (2002).

[27] The calculations are based on the approach described in Wiggins (1990).

in this situation. Therefore, the steady state may be unapproachable, or equilibrium may be determinate. When equilibrium is determinate, paths approach the steady state via oscillatory motion owing to the presence of negative eigenvalues, especially the eigenvalue which is equal to -1. This motion is damped, but only by second-order effects, so that the oscillations are quite persistent.

When $R\sqrt{1-\alpha} < 1$, the system may again have one stable eigenvalue, a unit eigenvalue, and two initial conditions. In this case the dynamics are as described in the previous paragraph. Otherwise, the system has two stable eigenvalues, a unit eigenvalue, and two initial conditions. We have found that the motion on the center manifold may be stable or unstable in this situation. We conclude that the steady state can always be approached, but that equilibrium may be indeterminate if the center manifold is stable. Equilibrium motion is again oscillatory.

Part of Friedman's (1960) argument concerning privately-issued close currency substitutes was that allowing such issuance would create excessive volatility in the economy. Our Case 1 economies are characterized by volatility both in the sense that equilibrium motion is oscillatory, and in the sense that indeterminacies can arise.

6.2. A Case 2 Economy

The local dynamics of a Case 2 economy are, as we will show, qualitatively different from those of a Case 1 or a Case 3 economy. In addition, the dynamical system is simpler to analyze in Case 2.

In a Case 2 economy, the banknote market clearing condition (32) and the loan market clearing condition (39) imply that

$$\frac{\theta_b w}{\theta_i R_{1,1}(t)} + \frac{\theta_b w R_{2,1}(t-1)}{\theta_i R_{2,2}(t-1)} = \frac{\theta_\ell}{\theta_i}\left(1-\alpha\right) e \tag{100}$$

and

$$\frac{\theta_b w}{\theta_i R_{2,2}(t)} + \frac{\theta_b w R_{1,2}(t-1)}{\theta_i R_{1,1}(t-1)} = \frac{\theta_\ell}{\theta_i}\left(1-\alpha\right) e. \tag{101}$$

In addition, equation (40) implies

$$R_{1,1}(t) = \left[\frac{\theta_b w}{\theta_i \left(1-\alpha\right) a}\right] R_{1,2}(t) R_{2,1}(t+1) \tag{102}$$

and

$$R_{2,2}(t) = \left[\frac{\theta_b w}{\theta_i \left(1-\alpha\right) a}\right] R_{2,1}(t) R_{1,2}(t+1). \tag{103}$$

The dynamical system consisting of equations (100)-(103) constitutes a set of four first-order difference equations.

It turns out to be convenient to rewrite this system in an explicit first-order form. This form is given by

$$\frac{w}{R_{1,1}(t+1)} = \left(\frac{\theta_\ell}{\theta_b}\right)(1-\alpha)\,e - w\left[\frac{R_{2,1}(t)}{R_{2,2}(t)}\right], \tag{104}$$

$$\frac{w}{R_{2,2}(t+1)} = \left(\frac{\theta_\ell}{\theta_b}\right)(1-\alpha)\,e - w\left[\frac{R_{1,2}(t)}{R_{1,1}(t)}\right], \tag{105}$$

$$R_{2,1}(t+1) = \left(\frac{\theta_i}{\theta_b}\right)\left[\frac{(1-\alpha)\,a}{w}\right]\frac{R_{1,1}(t)}{R_{1,2}(t)}, \tag{106}$$

and

$$R_{1,2}(t+1) = \left(\frac{\theta_i}{\theta_b}\right)\left[\frac{(1-\alpha)\,a}{w}\right]\frac{R_{2,2}(t)}{R_{2,1}(t)}. \tag{107}$$

We now consider local dynamics of this economy, in the neighborhood of a Case 2 steady state. To do so, we linearize the system given by equations (104)-(107), and evaluate the linearized system at the steady state. The local dynamics are then governed by the eigenvalues of the associated Jacobian matrix J of the system (104)-(107). This matrix is displayed in Appendix B. The appendix also contains the proof of the following proposition.

PROPOSITION 2. *The eigenvalues of* J *are* 0, 0, $-1-\left[\frac{\theta_i(1-\alpha)a}{\theta_b w}\right]^{\frac{1}{2}}R^{\frac{1}{2}}$, *and* $-1+\left[\frac{\theta_i(1-\alpha)a}{\theta_b w}\right]^{\frac{1}{2}}R^{\frac{1}{2}}$, *where* $R = R_{1,1} = R_{2,2}$ *is given by equation (58).*

Proof. See Appendix B. ∎

It is immediate from Proposition 2 that J has two eigenvalues inside and two eigenvalues outside the unit circle. Since there are two given initial conditions, the steady state is determinate. There is no scope for fluctuations along dynamical equilibrium paths approaching the steady state.

The equilibrium dynamics in Case 2 are qualitatively different from those of a Case 1 or a Case 3 economy. As we will show below, a Case 3 economy behaves much like a Case 1 economy with respect to equilibrium dynamics. In either Case 1 or Case 3, when a determinate equilibrium exists, it is necessarily characterized by fluctuating motion along the equilibrium path. Such motion cannot occur in a Case 2 economy.

6.3. A Case 3 Economy

We now turn our attention to a Case 3 economy, in which middle-aged intermediary-type agents lend. The analysis in this case is similar to that for Case 1. We begin with a definition of the function $F[R_{1,1}(t), R_{2,2}(t), R_{1,2}(t), R_{2,1}(t)]$ by

$$F[R_{1,1}(t), R_{2,2}(t), R_{1,2}(t), R_{2,1}(t)] \equiv \left\{ \left(\frac{\theta_\ell}{\theta_i}\right) e \left[(1-\alpha) R_{1,2}(t) + \frac{R_{1,2}(t) R_{2,1}(t)}{(1-\alpha) R_{1,1}(t)} \right] - \left[\frac{\theta_b w + \theta_i \frac{a}{(1-\alpha)}}{\theta_i} \right] \left[\frac{R_{1,2}(t)}{R_{1,1}(t)} + \frac{R_{1,2}(t) R_{2,1}(t)}{(1-\alpha)^2 R_{1,1}(t) R_{2,2}(t)} \right] \right\} \times \left\{ 1 - \frac{R_{1,2}(t) R_{2,1}(t)}{(1-\alpha)^4 R_{1,1}(t) R_{2,2}(t)} \right\}^{-1}. \quad (108)$$

Define the function $G[R_{1,1}(t), R_{2,2}(t), R_{1,2}(t), R_{2,1}(t)]$ by

$$G[R_{1,1}(t), R_{2,2}(t), R_{1,2}(t), R_{2,1}(t)] \equiv \left(\frac{\theta_\ell}{\theta_i}\right) (1-\alpha) e R_{2,1}(t) + \left[\frac{R_{2,1}(t)}{(1-\alpha)^2 R_{2,2}(t)} \right] F[R_{1,1}(t), R_{2,2}(t), R_{1,2}(t), R_{2,1}(t)] - \left[\frac{\theta_b w + \theta_i \left(\frac{a}{1-\alpha}\right)}{\theta_i} \right] \frac{R_{2,1}(t)}{R_{2,2}(t)}. \quad (109)$$

Then using equations (32), (41), and (43) and repeating the logic pursued in Case 1, it is easy to show that a Case 3 economy has

$$x_{i,2,t-1}(t+1) = F[R_{1,1}(t), R_{2,2}(t), R_{1,2}(t), R_{2,1}(t)] \quad (110)$$

and

$$x_{i,1,t-1}(t+1) = G[R_{1,1}(t), R_{2,2}(t), R_{1,2}(t), R_{2,1}(t)] \quad (111)$$

$\forall t$. Moreover, substituting (110) and (111) into the $h = 1$ and $h = 2$ versions of (32) one obtains

$$\frac{G[R_{1,1}(t+1), R_{2,2}(t+1), R_{1,2}(t+1), R_{2,1}(t+1)]}{R_{1,2}(t) R_{2,1}(t+1)} + \frac{F[R_{1,1}(t), R_{2,2}(t), R_{1,2}(t), R_{2,1}(t)]}{R_{1,2}(t)} = \left(\frac{\theta_\ell}{\theta_i}\right) (1-\alpha) e \quad (112)$$

and

$$\frac{F\left[R_{1,1}(t+1), R_{2,2}(t+1), R_{1,2}(t+1), R_{2,1}(t+1)\right]}{R_{2,1}(t)\, R_{1,2}(t+1)}+ \frac{G\left[R_{1,1}(t), R_{2,2}(t), R_{1,2}(t), R_{2,1}(t)\right]}{R_{2,1}(t)} = \left(\frac{\theta_\ell}{\theta_i}\right)(1-\alpha)\, e. \quad (113)$$

Equations (41) for $h = 1, 2$, equation (112), and equation (113) describe the equilibrium law of motion for $R_{1,1}(t)$, $R_{2,2}(t)$, $R_{1,2}(t)$, and $R_{2,1}(t)$. Once again, this is a system of four first-order difference equations.

These equations can be written in explicit first-order form. First, define $\xi_1 = \frac{\theta_b w + \theta_i a(1-\alpha)^{-1}}{\theta_i}$, $\xi_2 = \frac{\theta_\ell e(1-\alpha)}{\theta_i}$, and $\xi_3 = (1-\alpha)^{\frac{1}{2}}$. Then

$$R_{1,1}(t+1) = \frac{\xi_1\left[R_{1,2}(t)\, R_{2,1}(t) - (\xi_3)^8 R_{1,1}(t)\, R_{2,2}(t)\right]}{B_0 - R_{1,1}(\xi_3)^2\left[R_{2,2}(t)(\xi_3)^6\left(\xi_1(\xi_3)^2 + \xi_2\right) - B_1\right]} \quad (114)$$

where

$$B_0 = R_{1,2}(t)\, R_{2,1}(t)\left[\xi_1(\xi_3)^4 - \xi_2 R_{2,1}(t) + \xi_2\right] \quad (115)$$

and

$$B_1 = R_{2,1}(t) \times \left[\left(R_{2,2}(t)\,\xi_2 + \xi_1\left((\xi_3)^2 - 1\right)\right)(\xi_3)^4 + R_{1,2}(t)\left(\xi_2 - \xi_2(\xi_3)^2\right)\right], \quad (116)$$

$$R_{2,2}(t+1) = \frac{\xi_1\left[R_{1,2}(t)\, R_{2,1}(t) - (\xi_3)^8 R_{1,1}(t)\, R_{2,2}(t)\right]}{R_{1,2}(t)\, B_2 - \xi_2 R_{1,2}(t)^2 R_{2,1}(t) - (\xi_3)^8\left(\xi_1(\xi_3)^2 + \xi_2\right) R_{1,1}(t)\, R_{2,2}(t)} \quad (117)$$

where

$$B_2 = R_{2,2}(t)\left[R_{1,1}(t)\,\xi_2 + \xi_1\left((\xi_3)^2 - 1\right)\right](\xi_3)^6 + R_{2,1}(t)\left[\xi_1(\xi_3)^4 - R_{2,2}(t)\,\xi_2\left((\xi_3)^2 - 1\right)(\xi_3)^2 + \xi_2\right], \quad (118)$$

$$R_{1,2}(t+1) = \frac{\xi_1(\xi_3)^6 R_{2,2}(t)\left[(\xi_3)^8 R_{1,1}(t)\, R_{2,2}(t) - R_{1,2}(t)\, R_{2,1}(t)\right]}{R_{2,1}(t)\left\{B_4 + \left(R_{1,2}(t)\right)^2 R_{2,1}(t)\,\xi_2 + R_{1,2}(t)\, B_3\right\}} \quad (119)$$

where

$$B_3 = R_{2,2}(t)\left[-\xi_1(\xi_3)^2+\xi_1-R_{1,1}(t)\,\xi_2\right](\xi_3)^6 + \\ R_{2,1}\left[-\xi_1(\xi_3)^4+R_{2,2}(t)\,\xi_2\left((\xi_3)^2-1\right)(\xi_3)^2-\xi_2\right] \quad (120)$$

and

$$B_4 = R_{1,1}(t)\,R_{2,2}(t)\left(\xi_1(\xi_3)^2+\xi_2\right)(\xi_3)^8, \quad (121)$$

and finally

$$R_{2,1}(t+1) = \\ \frac{\xi_1(\xi_3)^6 R_{1,1}(t)\left[R_{1,2}(t)\,R_{2,1}(t)-(\xi_3)^8 R_{1,1}(t)\,R_{2,2}(t)\right]}{R_{1,2}(t)\left\{R_{1,2}(t)\,R_{2,1}(t)\left[\xi_1(\xi_3)^4-R_{2,1}(t)\,\xi_2+\xi_2\right]-R_{1,1}(t)(\xi_3)^2 B_5\right\}} \quad (122)$$

where

$$B_5 = R_{2,2}(t)(\xi_3)^6\left(\xi_1(\xi_3)^2+\xi_2\right) - \\ R_{2,1}(t)\left[\left(R_{2,2}(t)\,\xi_2+\xi_1\left((\xi_3)^2-1\right)\right)(\xi_3)^4+R_{1,2}(t)\left(\xi_2-\xi_2(\xi_3)^2\right)\right]. \quad (123)$$

Letting R represent the steady state value of $R_{1,1} = R_{2,2}$ in Case 3, the Jacobian matrix associated with this system is given by

$$J = B_6\begin{bmatrix} J_{11} & \xi_3 J_{11} & J_{13} & -J_{11}/(\xi_3)^2 \\ \xi_3 J_{11} & J_{11} & -J_{11}/(\xi_3)^2 & J_{13} \\ (\xi_3)^4 J_{11} & J_{32} & -\xi_3 J_{11} & J_{34} \\ J_{32} & (\xi_3)^4 J_{11} & J_{34} & -\xi_3 J_{11} \end{bmatrix} \quad (124)$$

where

$$B_6 = \frac{1}{R(\xi_3-1)\left(\xi_2+R\xi_2(\xi_3)^2-\xi_1(\xi_3)^3\right)^2}, \quad (125)$$

$$J_{11} = \xi_1(\xi_3)^2(\xi_2 R-\xi_1\xi_3), \quad (126)$$

$$J_{13} = \xi_1(\xi_1-\xi_2 R), \quad (127)$$

$$J_{32} = \xi_1(\xi_3)^3\left[-(\xi_3)^4\xi_1+\xi_2\left(R(\xi_3)^3+\xi_3-1\right)\right], \quad (128)$$

and

$$J_{34} = \xi_1(\xi_3)^3\left[\xi_1\xi_3-\xi_2 R\right]. \quad (129)$$

The associated eigenvalues are given by

$$\mu_1 = -\frac{\xi_1 \xi_2 (\xi_2)^2}{\Delta^2}, \tag{130}$$

$$\mu_2 = \frac{\xi_1}{\Delta R}, \tag{131}$$

$$\mu_3, \mu_4 = \frac{\xi_1 \xi_2 + (\xi_1)^2 (\xi_3)^3}{2R\Delta^2} \pm \frac{\xi_1 \sqrt{(\xi_1)^2 (\xi_3)^6 + 2\xi_1 \xi_2 (\xi_3)^3 \left(1 - 2R(\xi_3)^2\right) + (\xi_2)^2 \left(1 + 2R(\xi_3)^2\right)^2}}{2R\Delta^2}, \tag{132}$$

where

$$\Delta = \xi_1 (\xi_3)^3 - \xi_2 \left(1 + R(\xi_3)^2\right). \tag{133}$$

We again turn to numerical methods to study the properties of these eigenvalues over a set of parameter possibilities that spans the space of economies we are considering.[28] Our numerical sampling methodology is the same as described above for Case 1. We again found a simple characterization of the qualitative eigenvalue configurations.

The qualitative situation with respect to local dynamics in Case 3 is in fact identical to the situation for Case 1. The eigenvalues are always real. One eigenvalue is positive and greater than one. A second eigenvalue is equal to -1. A third eigenvalue lies in the interval $(-1, 0)$. The final eigenvalue is negative and can be in the interval $(-1, 0)$, or in the interval $(-\infty, -1)$. A sufficient condition for this last eigenvalue to lie in $(-\infty, -1)$ is that $R(1-\alpha)^{3/2} > 1$. The term $R(1-\alpha)^{3/2} = R_{1,2} = R_{2,1}$, the cross-location steady state interest rate.

Again, the presence of an eigenvalue with unit modulus means that the first order approximation does not yield enough information to fully characterize the local dynamics. We again calculated an approximation to the motion on the center manifold using the techniques described in Appendix F of Azariadis, Bullard, and Smith (2001). As in Case 1, we found that the motion may be stable or unstable, and that which situation prevailed was not related in any obvious way to the characteristics of the economy.

[28] It is straightforward to prove results concerning the properties of the eigenvalues exactly analogous to those stated for Case 1.

Center manifold motion	$R_{12} = R_{21} > 1$, OR $R_{12} = R_{21} < 1$, with 1 stable eigenvalue	$R_{12} = R_{21} < 1$, with 2 stable eigenvalues
Stable	Determinate	Indeterminate
Unstable	Steady state unapproachable	Determinate

TABLE 2
Summary of possible equilibrium configurations. This summary is valid for both Case 1 and Case 3 economies. An entry of "determinate" means equilibrium is determinate, and so on. All four possibilities can occur.

We now have enough information to characterize the local dynamics near the steady state of a Case 3 economy. When $R(1-\alpha)^{3/2} > 1$, that is, the steady state cross-location interest rate is larger than one, the system has one stable eigenvalue, a unit eigenvalue, and two initial conditions. When the center manifold is stable, equilibrium is determinate and the equilibrium motion is oscillatory with second order damping effects. When the center manifold is unstable, the steady state cannot be approached. Both situations can occur.

When $R(1-\alpha)^{3/2} < 1$ and there is one stable eigenvalue, the dynamics are as described in the previous paragraph. Otherwise, we have two stable eigenvalues, a unit eigenvalue, and two initial conditions. When the center manifold is stable, as it can be, then indeterminacy arises. When the center manifold is unstable, equilibrium is determinate. Equilibrium motion is always oscillatory.

As in Case 1, Friedman's (1960) argument concerning the level of volatility in economies which permit close currency substitutes to be issued by the private sector seems to be borne out. Table 2 summarizes the possible equilibrium configurations for Cases 1 and 3.

7. CONCLUSION

The liabilities of certain intermediaries—banks in particular—play a central role in making payments. This is true of the liabilities of no other private agents. This paper has posed the question of why this should be the case. And, more specifically, we have posed the question of why the activity of lending or intermediated lending should be coupled—as it has always been—with the issue of payments instruments.

We have produced a model in which credit markets cannot function unless agents issue liabilities that circulate. In addition, the model described above assumes a pattern of meeting and communication among agents that implies that credit markets cannot function unless there is some intermediated lending, and unless intermediaries issue circulating liabilities. In effect, intermediated lending occurs because the circulating liabilities issued by intermediaries are essential in making payments. Thus, in contrast to the existing literature on banking, which emphasizes how banks can help overcome informational asymmetries in lending or can help insure depositors, we build a model of banks based on their role in issuing media of exchange.

In our model, as long as there are some circulating liabilities issued by intermediaries, it is feasible for some other agents to issue circulating liabilities as well. However, in equilibrium they will not do so, even if intermediaries profit by issuing such liabilities. This is because the use of intermediary liabilities economizes on transactions, which is desirable when transacting is costly. And, it bears emphasis, transactions costs of any magnitude will do—we do not need them to be large.

Our analysis also explains how intermediaries can profit by issuing notes, and how at the same time intermediary note issues will be limited. The model thus goes some way towards resolving the bank note underissue puzzle of Cagan (1963). It also explains why intermediaries borrow and lend so much with each other, when they could raise more funds, if they so desired, by issuing more notes.

It has often been argued that allowing agents who lend to issue payments media is a formula for generating indeterminacies of equilibrium and excessive economic volatility. Our analysis of equilibrium dynamics allows us to evaluate this argument. In our Case 2 economy—middle-aged intermediaries neither borrow nor lend—volatility is not a concern either in the sense of indeterminacy or in the sense of equilibrium dynamics. For the less special Case 1 and Case 3 economies, however, equilibrium dynamics, should they be determinate, are never monotonic.

An interesting question is how the introduction of outside money, or the sustained expansion of the outside money stock, affects the use of intermediary liabilities in exchange. We have not included an investigation of this topic in this paper, but in previous analyses of frameworks related to this one (namely, Azariadis, Bullard, and Smith (2001) and Bullard and Smith

(2002)) we have found that the introduction of outside money cannot supplant, in the absence of other interventions, the use of private circulating liabilities. Instead, these liabilities are essential in allowing credit markets to function. A government can, of course, engage in restrictions on the issue of private circulating liabilities or can explicitly prohibit their use. However, this model illustrates how doing so could inhibit the ability of many agents to transact at all, and how doing so could have very adverse welfare consequences.

Naturally our analysis abstracts from a number of issues. One is private information. In our model all agents know the issuers of private circulating liabilities, and there are no problems with fraud or with enforcement of contracts when these liabilities are presented for redemption. A whole array of economists, from Adam Smith (1776) to Milton Friedman (1960), have emphasized the potential for fraudulent note issue to generate a role for government regulation of note issue. It would be interesting to extend the model to consider this possibility.

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APPENDIX A: PROOF OF PROPOSITION 5.1

Under the symmetry assumption we have $\theta_{h,j} = \theta_{k,j}$ for $j \in \{\ell, b, i\}$, $h = 1,2$; $k = 1,2$. Thus we will simply denote the fractions of agent types by θ_ℓ, θ_b, and θ_i. Also, in order to economize on notation, we define the following variables:

$$z \equiv \frac{\theta_i a}{\theta_\ell e} \tag{134}$$

and

$$y \equiv \frac{\theta_i a}{\theta_b w}. \tag{135}$$

Then

$$\Psi \equiv \frac{\theta_b w + \theta_i a}{\theta_\ell \left(1-\alpha\right) e} \equiv \frac{z(1+y)}{\left(1-\alpha\right) y} \equiv Q(z,y) \tag{136}$$

and

$$\lambda \equiv \frac{\theta_b \left(1-\alpha\right) w + \theta_i a}{\theta_\ell e \left(1-\alpha\right)^2} \equiv \left(\frac{z}{1-\alpha}\right) \left[\frac{y+\left(1-\alpha\right)}{\left(1-\alpha\right) y}\right] \equiv H(z,y). \tag{137}$$

In addition, define

$$\mu_1 \equiv \left(\frac{1-\alpha}{2}\right) \times \left\{ \left[\left(1-\alpha\right)^{1.5} H(z,y) - 1\right] + \left[\left[\left(1-\alpha\right)^{1.5} H(z,y) - 1\right]^2 + 4\left(1-\alpha\right) H(z,y)\right]^{\frac{1}{2}} \right\} \equiv h(z,y), \tag{138}$$

$$\mu_2 \equiv \left(\frac{1}{2}\right) \left[Q(z,y)\sqrt{1-\alpha} - 1\right] + \left(\frac{1}{2}\right) \left\{ \left[Q(z,y)\sqrt{1-\alpha} - 1\right]^2 + 4Q(z,y) \right\}^{\frac{1}{2}} \equiv v(z,y), \tag{139}$$

$$\mu_3 \equiv \left(\frac{\sqrt{1-\alpha}}{2}\right) \left[z + \left(z^2+4z\right)^{\frac{1}{2}}\right] \tag{140}$$

$$\mu_4 \equiv \left(\frac{1}{2\sqrt{1-\alpha}}\right) \left[z + \left(z^2+4z\right)^{\frac{1}{2}}\right]. \tag{141}$$

We also note that $\mu_3 \equiv \left(1-\alpha\right) \mu_4$.

As noted in the text, there is a Case 1 steady state iff

$$y > \mu_2. \tag{142}$$

There is a Case 2 steady state iff

$$\mu_3 < y < \mu_4. \tag{143}$$

And, there is a Case 3 steady state iff

$$y < \mu_1. \tag{144}$$

We begin with existence. Suppose that there is *no* steady state. Then either

$$\mu_2 > y > \mu_1 \tag{145}$$

and

$$y < \mu_3 \tag{146}$$

hold, or (145) and

$$y > \mu_4 \tag{147}$$

hold. We now rule out each of these situations.

Suppose first that (145) and (146) are satisfied. These equations imply the existence of a value $\gamma < 1$ such that

$$\mu_1 = h(z, y) = \gamma y < y, \tag{148}$$

and a value $\beta > 1$ such that

$$\mu_3 = \left(\frac{\sqrt{1-\alpha}}{2}\right)\left[z + (z^2 + 4z)^{\frac{1}{2}}\right] = \beta y > y. \tag{149}$$

Equation (149) can be rearranged to yield

$$z = \frac{\beta^2 y^2}{(1-\alpha) + \left(\beta\sqrt{1-\alpha}\right) y}. \tag{150}$$

Equation (148) and the definition of $H(z, y)$ can be used to obtain the relation

$$z = \frac{\gamma\,(1-\alpha)\,y^2\left[\left(\frac{\gamma y}{1-\alpha}\right) + 1\right]}{[y + (1-\alpha)]\left[(1-\alpha) + \left(\gamma\sqrt{1-\alpha}\right) y\right]}. \tag{151}$$

Equations (151) and (150) together then imply that

$$\begin{aligned}&\left(\beta\gamma\sqrt{1-\alpha}\right)(\beta-\gamma)\,y^2 + \\ &\quad\left[\beta^2\gamma\,(1-\alpha)^{1.5} - \beta\gamma\,(1-\alpha)^{1.5} + (1-\alpha)\left(\beta^2-\gamma^2\right)\right] y + \\ &\qquad (1-\alpha)\left(\beta^2-\gamma^2\right) = 0.\end{aligned} \tag{152}$$

But $\beta > 1 > \gamma$ imply that the left-hand side of (152) is positive $\forall y \geq 0$. This contradicts the assumption that (145) and (146) hold simultaneously. Thus (145) and (146) cannot both hold.

Then suppose that (145) and (147) are both satisfied. Equation (145) implies the existence of a value $\beta > 1$ such that

$$Q(z,y)\sqrt{1-\alpha} - 1 + \left\{\left[Q(z,y)\sqrt{1-\alpha} - 1\right]^2 + 4Q(z,y)\right\}^{\frac{1}{2}} = 2\beta y. \tag{153}$$

Equation (147) implies the existence of a value $\delta < 1$ such that

$$\mu_4 = \left(\frac{1}{2\sqrt{1-\alpha}}\right)\left[z + \left(z^2 + 4z\right)^{\frac{1}{2}}\right] = \delta y < y. \tag{154}$$

Equation (154) can be rearranged to yield

$$z = \frac{\delta^2 (1-\alpha) y^2}{1 + \left(\delta\sqrt{1-\alpha}\right) y}. \tag{155}$$

Equation (153) and the definition of $Q(z, y)$ imply

$$z = \frac{\beta(1-\alpha)y^2(\beta y + 1)}{(1+y)\left[1 + \left(\beta\sqrt{1-\alpha}\right) y\right]}. \tag{156}$$

Together equations (156) and (155) imply

$$\left(\beta\delta\sqrt{1-\alpha}\right)(\beta - \delta) y^2 + \left[\left(\beta\delta\sqrt{1-\alpha}\right)(1-\delta) + \left(\beta^2 - \delta^2\right)\right] y + \beta - \delta^2 = 0. \tag{157}$$

But $\beta > 1 > \delta$ implies that the left-hand side of (157) is positive $\forall y \geq 0$. This contradicts the assumption that (145) and (147) are satisfied simultaneously. Thus (145) and (147) cannot both hold. This establishes the existence of a steady state.

We now turn to uniqueness. We first observe that $\mu_2 > \mu_1$holds. (This follows immediately from $\Psi > (1-\alpha)\lambda$.) Since the existence of a Case 1 steady state requires that $y > \mu_2$ and the existence of a Case 3 steady state requires $y < \mu_1$, it is then immediate that no economy can have both a Case 1 and a Case 3 steady state. Therefore, if an economy has two steady states with $R_{1,1} = R_{2,2}$, it has either a Case 2 and a Case 3 steady state, or it has a Case 2 and a Case 1 steady state. We now rule out each possibility.

Suppose first that an economy has a Case 2 and a Case 3 steady state. The existence of a Case 3 steady state implies $\mu_1 > y$. Then there exists a value $\Phi > 1$ such that

$$\mu_1 = \Phi y > y. \tag{158}$$

Similarly, the existence of a Case 2 steady state implies that

$$\mu_4 > y = \frac{\mu_1}{\Phi} > \mu_3. \tag{159}$$

Therefore, there exists a value δ such that

$$\mu_4 = \delta\mu_1 \tag{160}$$

and, by definition, $\mu_3 = (1-\alpha)\,\mu_4 = \delta\,(1-\alpha)\,\mu_1$. Then $y \in (\mu_3, \mu_4)$ implies that

$$\Phi\delta > 1 > \delta\,(1-\alpha)\,\Phi. \tag{161}$$

Moreover, since $\mu_3 < \mu_1$must hold, so must $\delta\,(1-\alpha) < 1$.

Now equations (138) and (158) imply that there exist values z and y such that

$$h(z,y) = \Phi y. \tag{162}$$

Similarly, equations (138), (141), and (160) imply that the same values (z, y) must satisfy

$$\delta h(z,y) = \left(\frac{1}{2\sqrt{1-\alpha}}\right)\left[z + \left(z^2+4z\right)^{\frac{1}{2}}\right]. \tag{163}$$

Or, substituting (162) into (163), z and y must satisfy (162) and

$$z + \left(z^2+4z\right)^{\frac{1}{2}} = (2\sqrt{1-\alpha})\Phi\delta y. \tag{164}$$

If we now use the definition of $h(z,y)$ and $H(z,y)$ in equation (162), we obtain the equivalent condition

$$z = \frac{\Phi^2 y^3 + \Phi\,(1-\alpha)\,y^2}{[y+(1-\alpha)]\left[\left(\Phi\sqrt{1-\alpha}\right)y + (1-\alpha)\right]} \tag{165}$$

In addition, (164) can be rearranged to yield

$$z = \frac{(1-\alpha)\,(\Phi\delta y)^2}{1+\left(\Phi\delta\sqrt{1-\alpha}\right)y}. \tag{166}$$

Together equations (165) and (166) imply that there must exist a value $y > 0$ such that

$$\begin{aligned}&\left(\Phi^2\delta\sqrt{1-\alpha}\right)[1-\delta\,(1-\alpha)]\,y^2 + \\ &\quad\left[\Phi\delta\,(1-\alpha)^{1.5} + \Phi - \Phi\delta^2\,(1-\alpha)^2 - \delta^2\Phi^2\,(1-\alpha)^{2.5}\right]y + \\ &\qquad(1-\alpha)\left[1-\Phi\delta^2\,(1-\alpha)^2\right] = 0.\end{aligned} \tag{167}$$

But $1 > \delta\alpha$, $1 > \Phi\delta^2\,(1-\alpha)^2$, and $1 > [\delta\,(1-\alpha)]^2$ all hold. Therefore, the left-hand side of (167) is strictly positive $\forall y \geq 0$. This contradicts the assumption that an economy simultaneously has a Case 2 and a Case 3 steady state.

Then suppose that some economy has a Case 1 and a Case 2 steady state. The existence of a Case 1 steady state implies the existence of a value $\Phi < 1$ with

$$\mu_2 = \Phi y < y. \tag{168}$$

The existence of a Case 2 steady state implies that $\mu_4 > y > \mu_3$. Therefore, there exists a value $\delta \geq 1$ such that

$$\mu_4 = \delta \mu_2. \tag{169}$$

Moreover, by definition, $\mu_3 = (1-\alpha)\,\mu_4 = \delta\,(1-\alpha)\,\mu_2$. Thus

$$\mu_3 = \delta\,(1-\alpha)\,\mu_2 < y = \frac{\mu_2}{\Phi} \leq \mu_4 = \delta\mu_2 \tag{170}$$

must hold. Equation (170) implies that $\Phi\delta\,(1-\alpha) < 1 \leq \Phi\delta$ necessarily obtains.

Now observe that equations (139) and (168) imply that values z and y exist such that

$$v(z,y) = \Phi y. \tag{171}$$

In addition, equations (141) and (169) imply that the same (z,y) pair must satisfy

$$z + (z^2+4z)^{\frac{1}{2}} = \left(2\delta\sqrt{1-\alpha}\right)\Phi y = \left(2\sqrt{1-\alpha}\right)\delta\nu(z,y). \tag{172}$$

Using the definition of $\nu(z,y)$ and $Q(z,y)$, equation (171) has the equivalent representation

$$z = \frac{\Phi\,(1-\alpha)\,y^2(\Phi y+1)}{(1+y)\left[1+\left(\Phi\sqrt{1-\alpha}\right)y\right]}. \tag{173}$$

Similarly, equation (172) has the equivalent representation

$$z = \frac{(\Phi\delta)^2\,(1-\alpha)y^2}{1+\left(\Phi\delta\sqrt{1-\alpha}\right)y}. \tag{174}$$

Together equations (173) and (174) imply the existence of a value $y > 0$ such that

$$\left(\Phi^2\delta\sqrt{1-\alpha}\right)(\delta-1)\,y^2 + \left[\Phi^2\delta^2\sqrt{1-\alpha} + \Phi\delta^2 - \Phi\delta\sqrt{1-\alpha} - \Phi\right]y + \Phi\delta^2 - 1 = 0. \tag{175}$$

But $\delta\,(\Phi\delta) > 1$ and $\delta \geq 1$ both hold. This observation implies that the left-hand side of (175) is strictly positive $\forall y \geq 0$. This contradicts the assumption that any economy has a Case 1 and a Case 2 steady state. This establishes the proposition.■

APPENDIX B: PROOF OF PROPOSITION 6.1

Since $R_{11} = R_{22} = R$ at a steady state, with R given by (58), and since $R_{1,2} = R_{2,1}$ at a steady state, it is easy to show that the Jacobian matrix J has the form

$$J = \begin{bmatrix} 0 & -R_{1,2} & 0 & R \\ -R_{1,2} & 0 & R & 0 \\ 0 & \frac{\theta_i(1-\alpha)a}{\theta_b w R_{1,2}} & 0 & -1 \\ \frac{\theta_i(1-\alpha)a}{\theta_b w R_{1,2}} & 0 & -1 & 0 \end{bmatrix}. \tag{176}$$

The associated characteristic equation, with ν denoting eigenvalues, is

$$\left(\nu^2 - 1\right)\left[\nu^2 - (R_{1,2})^2\right] - 2R\left[\frac{\theta_i(1-\alpha)a}{\theta_b w R_{1,2}}\right]\nu^2 - 2R\left[\frac{\theta_i(1-\alpha)a}{\theta_b w}\right] + R^2\left[\frac{\theta_i(1-\alpha)a}{\theta_b w R_{1,2}}\right] = 0. \tag{177}$$

Substituting $(R_{1,2})^2 = \theta_i(1-\alpha)aR/\theta_b w$ into (177) yields the equivalent condition

$$\nu^2\left[\nu^2 - \left\{1 + \left[\frac{\theta_i(1-\alpha)aR}{\theta_b w}\right]^{\frac{1}{2}}\right\}\right] = 0. \tag{178}$$

The proposition is then immediate.■