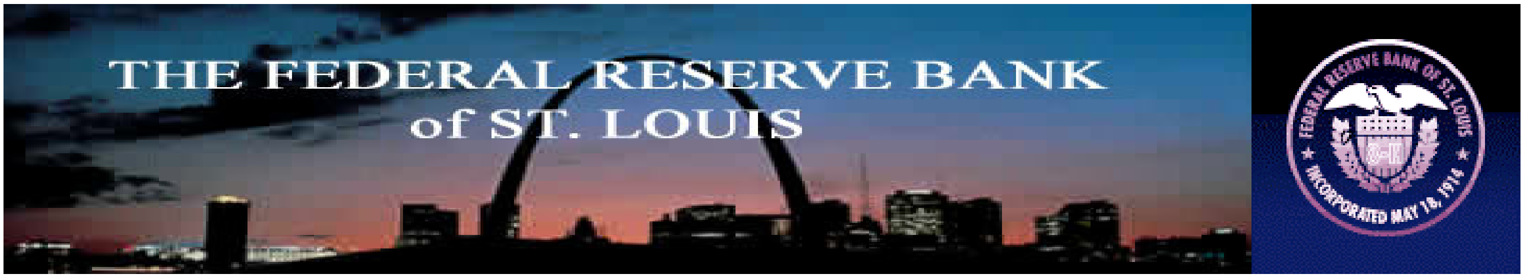

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Private and Public Circulating Liabilities

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Private and Public Circulating Liabilities

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Abstract. Changes in the legal and technological environment in the U.S. have created the possibility of private banknote issue, or its electronic equivalent. We wish to understand the implications of this possibility for economic performance. Accordingly, we construct and analyze a dynamic general equilibrium model in which privately-issued liabilities may circulate, either by themselves, or alongside a stock of outside money. In each case we provide results on the existence and multiplicity of equilibria, and we characterize local dynamics in a neighborhood of a steady state. Our results support Friedman's (1960) idea that circulating private liabilities are associated with endogenous (or "excess") volatility. But implementing Friedman's (1960) advice—the government should ban private issuance of close currency substitutes—causes significant inefficiency in our model. And implementing the polar opposite advice of Hayek (1976) and Fama (1980)—that the government should withdraw from currency issuance altogether in the presence of circulating private liabilities—also is often constrained-suboptimal in our economies. Instead, our economies have both public and private circulating liabilities as part of an optimal monetary arrangement. *JEL classification numbers: E3, E4, E5.*

*Any views expressed are those of the authors and do not necessarily reflect the views of the Federal Reserve Bank of St. Louis or the Federal Reserve System. This paper is available on the world wide web at www.stls.frb.org/research/econ/bullard/.

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1. Introduction

1.1. Overview. Two of the most basic questions in monetary economics are:

(1) Do we "need" the government to provide money, or can we rely on "the market" to produce a well-functioning monetary arrangement?

(2) Does an efficient monetary system require a mix of private and government money, or should the government be a monopoly provider of currency and close currency substitutes?

Implicit in the wording of these questions are some very different views about the private provision of currency. Clearly many economists believe that market mechanisms work well, *except* where the creation of money is concerned. But some—perhaps most prominently Hayek (1976) and Fama (1980)—have argued that, even with respect to money, private market provision can produce desirable outcomes.

On the other hand, many economists have argued that not only should the government print money, but that the government should have a monopoly on currency creation. Friedman (1960), for example, argued passionately that allowing private provision of close currency substitutes is a recipe for generating indeterminacy of equilibrium and "excessive" volatility. Therefore, the government should be the sole issuer of circulating liabilities, and the creation of currency substitutes should be carefully segregated from all private credit market activity.

Somewhere between these views lies the real bills doctrine. This theory does not claim that the government should necessarily leave all currency creation to the market. But it does assert that the existence of safe, privately-issued circulating liabilities poses no "threat" to economic well-being, and may actually be beneficial.

These three points of view form the core of most received monetary theory. Despite centuries of debate among their adherents, however, a resolution of the differences among them has never been achieved.[1] In addition, the monetary theory literature seems short of formal frameworks for fully evaluating all of these issues.

Recent developments make it particularly important to rectify this situation. One is

[1]See Mints (1945) for a summary of the historical debates on these topics. Smith (1994) describes some more modern exchanges.

that all legal impediments to the creation of private banknotes—or their equivalents—have been removed in the U.S. Another is that improvements in communication and transactions technologies have made it feasible to issue electronic equivalents of private banknotes. How will these developments affect the determination of the price level and rates of return? Will they lead to increased economic volatility or generate indeterminacies? It seems important to have an apparatus for evaluating these issues.

Finally, it is now common to hear arguments that "e-cash," the electronic equivalent of private circulating liabilities, will ultimately displace outside money.[2] The views of Hayek (1976) and his followers suggest that this would be a positive development. We propose to provide a framework for evaluating all of these assertions.[3]

1.2. Model features. What features should such a framework have? First, in keeping with discussions dating back to Smith (1776), monetary exchange is most interesting in environments where trade is imperfectly coordinated. Thus, we should allow trade to take place in a number of distinct markets, and also for the possibility that—at any moment in time—there may be only limited scope for interaction among various markets. Second, private liability issue is typically associated with credit transactions. There should therefore be some scope for agents to borrow and lend. Finally, for agents' liabilities to circulate, the original recipients of these liabilities must want to consume at times when they are unable to redeem them from the original issuers. This can easily be captured by having agents trade with at least some other agents whom they will not meet again.

In keeping with these requirements, we construct a pure exchange economy with three-period lived, overlapping generations. Following Townsend (1980, 1987), we assume there are multiple locations where trade occurs. Spatial separation and limited communication between these locations creates the decentralized setting that makes monetary exchange most interesting. We also put a very specific structure on the pattern of agents' meetings.

[2]See Schreft (1997) for a discussion of current and historical thinking on this point.

[3]There is a substantial literature on private money creation. Examples include Townsend and Wallace (1987), Williamson (1992, 1999), Champ, Smith, and Williamson (1996), Cavalcanti and Wallace (1999), Burdett, Trejos and Wright (2000), and Smith and Weber (1999). But few of these papers analyze whether government and private issuers "should" coexist, or whether the presence of private issuers leads to a multiplicity of equilibria or to endogenous volatility. A model related to ours in an international context is Fisher (1996).

At each date, some young lenders are together with some young borrowers in each market. There is an incentive to transfer resources from lenders to borrowers. However, our assumptions on agents' patterns of movement imply that, if a young lender and a young borrower are together today, they will never meet again. Nevertheless, young lenders will—in the future—meet other agents who will—even further in the future—meet the borrowers with whom they are currently paired.[4] Thus, as in Townsend and Wallace (1982, 1987), young lenders can buy the liabilities of borrowers in exchange for goods, and sell these liabilities to others in the future. It is in this sense that private liabilities circulate. But in contrast to Townsend and Wallace (1982, 1987), private liabilities may coexist with a stock of outside money. And, unlike Townsend and Wallace, we study an infinite horizon economy and pay particular attention to dynamic equilibria.

The specific questions we investigate are: How are private liabilities priced by the market? Do they support efficient allocations? Could systems with private circulating liabilities give rise to a profusion of equilibria, or what Friedman might call "excessive volatility"?

1.3. Main results and historical perspective. American history has produced a close approximation to a system of private money creation. According to Temin (1969), almost 90 percent of the U.S. money supply[5] in 1830 consisted of private banknotes. During this period of history, different notes circulated at different discounts and premia (outside their location of origin). In addition, these discounts and premia fluctuated. Thus holders of different banknotes—privately-issued currencies—potentially faced substantially different and fluctuating rates of return on their real balances.

What explains the discounts and premia on notes? And what explains the fluctuations in them? Commonly proposed candidates are transportation costs (costs of offering the notes for redemption) and the risk of default. But Gorton (1989) has shown that note discounts reflected transportation costs only to a very limited extent, and Rolnick, Smith

[4]This is an important dimension on which our model differs from Kiyotaki and Wright (1989, 1993).

[5]That is, coins plus privately-issued banknotes. This calculation ignores deposits, whose monetary role was very limited at the time. A relatively large fraction of banknotes in 1830 were issued by the Second Bank of the United States, at that time the nation's sole federally chartered bank. But, even the Second Bank was 80 percent privately owned.

and Weber (1996) show that default risks were often minimal. Moreover, neither fluctuated enough in "normal" times to explain the magnitude of fluctuations in note discounts or premia.

In a system of purely private currency issue, our model gives rise to discounts and premia on notes under weak conditions. Some notes will be discounted and some notes will bear premia, even though their redemption costs are identical, and even though there is no default risk. Moreover, these discounts and premia will be reflected in agents facing different rates of return on currency. This situation *cannot* be corrected by a purely private monetary arrangement. Moreover, note discounts and premia should generally be expected to fluctuate, even in the absence of any fluctuations in "fundamentals." Thus a purely private monetary system will give rise to endogenous volatility. We also show that the intertemporal allocation of resources is *constrained-inefficient* in economies of the Samuelson type with sufficiently "patient" traders.

We next ask, what happens if circulating private liabilities coexist with a stock of outside money? Is this mixed public/private monetary system capable of supporting an optimal allocation of resources? And, will such systems display indeterminacies or endogenous volatility? First, we show that, with zero default risk, a system with outside money and private currency does not permit discounts or premia on notes. As a result, agents face identical rates of return and allocations are Pareto optimal in a steady state where outside money has value. However, fluctuations in price levels and rates of return will be observed outside this steady state. And, along dynamical equilibrium paths, these fluctuations will normally dampen only very slowly. Thus excessive volatility continues to be a feature of an economy in which outside money and private currency coexist. But, the scope for indeterminacy here is no greater than in a conventional overlapping generations model with only outside money.

Finally, we consider an economy where private circulating liabilities are prohibited, and the government has a monopoly of currency issue. In our environment, a prohibition of private currency creation prevents households from smoothing consumption and generates very serious inefficiencies. It does not eliminate volatility, nor does it reduce the degree of indeterminacy.

What do we conclude from this? First, a system of purely private currency issue is generally not going to produce even a constrained-efficient allocation of resources. Constrained efficiency requires lump-sum taxes and transfers; full efficiency requires both inside and outside money. Second, the existence of safe, privately-issued liabilities does not threaten economic efficiency. In fact, a mix of private and government currency issues promotes efficiency. Thus our analysis does not support arguments that a purely private monetary system is to be desired. It does support a version of the real bills doctrine.

As Friedman conjectured, economies that need private circulating liabilities to achieve efficiency will be volatile. But this volatility cannot typically be eliminated simply by prohibiting private issue of notes, and prohibiting circulating private liabilities has potentially large welfare costs for borrowers.

2. AN ECONOMY WITH INSIDE MONEY ONLY

2.1. The environment. We consider a pure exchange economy consisting of an infinite sequence of three-period lived, overlapping generations. In each period agents inhabit two locations, which are indexed by $h = 1, 2$. We let $t = 0, 1, ...$ index time. All young generations are identical in size and composition, and all young agents are divided into two groups, which we label as borrowers and lenders for reasons that will become clear below. In each period agents are endowed with some of a single nonstorable consumption good. Let e_j, $j = 1, 2, 3$, denote the age j endowment of this good for a lender and let w_j, $j = 1, 2, 3$, denote the age j endowment of this good for a borrower. For simplicity we assume that $e_1 > 0$, while $e_2 = e_3 = 0$, and that $w_2 > 0$, while $w_1 = w_3 = 0$. Let $c_{\ell,h,t}(t+j)$, with $j = 0, 1, 2$ and $h = 1, 2$, denote the time $t+j$ consumption of a lender born at time t in location h, and let $c_{b,h,t}(t+j)$, with $j = 0, 1, 2$ and $h = 1, 2$, denote the time $t+j$ consumption of a borrower born at time t in location h.[6] We assume that the lifetime utility of a lender born at an arbitrary date t is given by

$$u\left[c_{\ell,h,t}(t), c_{\ell,h,t}(t+1), c_{\ell,h,t}(t+2)\right] = \ln c_{\ell,h,t}(t) + \beta \ln c_{\ell,h,t}(t+1), \tag{1}$$

[6]Under this convention for agent-level variables, the subscript represents the type, location of birth, and date of birth of the agent in question, respectively, while the date in parentheses represents the actual time at which an activity occurs.

with $\beta > 0$, while the lifetime utility of a borrower born at an arbitrary date t is given by[7]

$$v\left[c_{b,h,t}\left(t\right), c_{b,h,t}\left(t+1\right), c_{b,h,t}\left(t+2\right)\right] = \ln c_{b,h,t}\left(t\right) + \beta \ln c_{b,h,t}\left(t+1\right). \tag{2}$$

Note that, in the absence of our assumptions about spatial separation and limited communication, the third period of agents' lives would involve no economic activity. Thus, in the absence of these assumptions, our economy would collapse to a standard two period overlapping generations model.[8]

We assume that in each period in location 1 there are $N/2$ lenders and γN borrowers in the current young generation, with $\gamma \in [1/2, 1)$. In location 2 there are $N/2$ lenders and $(1-\gamma)N$ borrowers. Thus there is a total population of $2N$ in each generation, which is not necessarily equally divided across locations.[9] By setting $\gamma \geq 1/2$, we have (i) location 1 is relatively large, and (ii) the "excess demand" for credit in location 1 is relatively large at any rate of interest.

Some of the agents in this economy may move between locations. In particular, lenders who are born in location 1 (2) move to location 2 (1) in middle age and then remain in their new location when old. We assume that borrowers spend their entire lives in the location of their birth. These assumptions mean that borrowers and lenders who interact in youth will never directly interact with each other again.[10]

Finally, we assume that—within a period—there is no communication across islands. Thus spatial separation and limited communication across locations implies that trade can occur only between agents who are in direct contact with each other.

Together our assumptions imply the following. Young borrowers in any given location at t would like to acquire resources from young lenders. However, a borrower and a lender will never meet each other again. Thus trade must be accomplished as follows. Young borrowers acquire resources at t from young lenders. In exchange the borrowers issue a

[7]There would be no generality gained by allowing borrowers and lenders to have different discount factors, although it could be accomplished without difficulty.

[8]For expositions of the standard two-period model, see Azariadis (1993) and Sargent (1987, Chapter 7).

[9]Note that we have assumed that, in the aggregate, there are equal numbers of borrowers and lenders. Given our freedom to vary e_1 and w_2, this assumption entails no real loss of generality.

[10]Our results would be unaltered if lenders returned to their location of birth when old.

claim against themselves. Lenders take these claims to their new location and, at $t+1$, trade these claims to the next generation of lenders in exchange for goods. At $t+2$ these now newly middle-aged lenders bring these claims back to their location of origin, where they are redeemed by the original issuers. Thus at least some trade must occur through the use of circulating, privately-issued liabilities.

Our assumptions imply that two kinds of liabilities can be issued: Liabilities with a maturity of one period, and liabilities with a maturity of two periods. At each date there will be newly-issued liabilities of each type in each location. In addition, at time t there will be liabilities circulating in location 1(2) with a maturity of two periods that were issued in location $2\,(1)$ at $t-1$. These liabilities will mature at $t+1$ in location $2\,(1)$.

2.2. Lenders. Our assumptions about the itineraries of agents imply that a young lender will acquire only long-maturity liabilities, although possibly previously-issued ones that will mature next period. Let $\tilde{x}_{1,h}(t)$ denote the quantity of previously-issued liabilities acquired by a young lender in location h at time t that will mature *in the other location* at $t+1$. In this notation the subscript "1" represents the time until the liability "matures" (one period), the subscript h denotes the location of acquisition, and the date in parentheses represents the date of acquisition. Let $1/R_{h,k}(t)$; $h=1,2$; $k=1,2$; $h\neq k$; denote the price of these liabilities, in units of current consumption, in location h at t. Then the one period gross rate of return received by a lender moving from location h at t to location k at $t+1$ is $R_{h,k}(t)$. In addition, let $\tilde{x}_{2,h}(t)$ denote the quantity of newly-issued two-period liabilities acquired in location h at t by the same agent. These liabilities will be sold in location k $(k\neq h)$ at $t+1$ for $1/R_{k,h}(t+1)$; hence their current price (discounted present value) in location h is $1/R_{h,k}(t)\,R_{k,h}(t+1)$. It follows that the budget constraints confronting a young lender born at an arbitrary date t take the form

$$c_{\ell,h,t}(t)+\frac{\tilde{x}_{1,h}(t)}{R_{h,k}(t)}+\frac{\tilde{x}_{2,h}(t)}{R_{h,k}(t)\,R_{k,h}(t+1)}\leq e_1 \tag{3}$$

and

$$c_{\ell,h,t}(t+1)\leq\tilde{x}_{1,h}(t)+\frac{\tilde{x}_{2,h}(t)}{R_{k,h}(t+1)} \tag{4}$$

for $t\geq 0$, $h=1,2$, and $k\neq h$.

A young lender in location h at date t chooses values $c_{\ell,h,t}(t)$, $c_{\ell,h,t}(t+1)$, $\tilde{x}_{1,h}(t)$ and $\tilde{x}_{2,h}(t)$ to maximize $\ln c_{\ell,h,t}(t) + \beta \ln c_{\ell,h,t}(t+1)$, subject to (3), (4), $\tilde{x}_{1,h}(t) \geq 0$, and $\tilde{x}_{2,h}(t) \geq 0$. The solution to this problem sets $c_{\ell,h,t}(t) = e_1/(1+\beta)$, $c_{\ell,h,t}(t+1) = [\beta e_1/(1+\beta)] R_{h,k}(t)$, and

$$\frac{\beta e_1}{1+\beta} = \frac{\tilde{x}_{1,h}(t)}{R_{h,k}(t)} + \frac{\tilde{x}_{2,h}(t)}{R_{h,k}(t) R_{k,h}(t+1)} \tag{5}$$

where $t \geq 0$, $h = 1, 2$, and $h \neq k$. Finally, to simplify the notation, let $e \equiv \beta e_1/(1+\beta)$.

2.3. Borrowers. For a young borrower in location h at t matters are as follows. The borrower can issue a claim to a unit of consumption maturing at $t+1$. Such a claim can be held only by agents staying in location h, and its time t price is $1/R_{h,h}(t)$. Thus $R_{h,h}(t)$ is the gross one period rate of return between t and $t+1$ for agents (borrowers) remaining in location h. Let $y_h(t)$ be the quantity of one-period (non-circulating) liabilities issued by a representative young borrower in location h at t.

A young borrower could also issue (circulating) liabilities that are claims to consumption at $t+2$. These claims will be sold to young lenders at t, who will then sell them at $t+1$ in their new location. Thus, if a borrower issues $x_h(t)$ two-period claims in location h at t, these will sell for $x_h(t)/R_{h,k}(t) R_{k,h}(t+1)$, with $h = 1, 2$, $k = 1, 2$, and $k \neq h$.

How do borrowers redeem two period claims when old when they have no old age endowment? The answer is that they must acquire claims on resources at $t+1$ when they are middle-aged. Let $\tilde{y}_h(t+1)$ be the quantity of one-period liabilities acquired at $t+1$ by middle-aged borrowers in location h to redeem their two-period liabilities that will come due at $t+2$. Clearly middle-aged borrowers must pay $\tilde{y}_h(t+1)/R_{h,h}(t+1)$ for these claims. In order to redeem their liabilities,

$$x_h(t) \leq \tilde{y}_h(t+1) \tag{6}$$

for $t \geq 0$, $h = 1, 2$, must hold.

Notice that transactions mediated through circulating liabilities require middle-aged "borrowers" to save in order to redeem their two-period (circulating) liabilities. Thus our "borrowers" label only accurately reflects the behavior of these agents in their first period of life. One-period maturities sold by young borrowers are purchased by middle-aged borrowers who inhabit the same location. We can also deduce that some such claims must be traded; otherwise borrowers could not redeem their circulating liabilities when old.[11]

A young borrower in location h at t, then, chooses values $c_{b,h,t}(t)$, $c_{b,h,t}(t+1)$, $y_h(t)$, $\tilde{y}_h(t+1)$, and $x_h(t)$ to maximize $\ln c_{b,h,t}(t) + \beta \ln c_{b,h,t}(t+1)$, subject to (6), $x_h(t) \geq 0$,

$$c_{b,h,t}(t) \leq \frac{y_h(t)}{R_{h,h}(t)} + \frac{x_h(t)}{R_{h,k}(t)\, R_{k,h}(t+1)} \tag{7}$$

and

$$c_{b,h,t}(t+1) + \frac{\tilde{y}_h(t+1)}{R_{h,h}(t+1)} \leq w_2 - y_h(t). \tag{8}$$

It is easy to verify that an absence of arbitrage opportunities requires that

$$R_{h,h}(t)\, R_{h,h}(t+1) = R_{h,k}(t)\, R_{k,h}(t+1) \tag{9}$$

where $t \geq 0$; $h = 1, 2$; and $k = 1, 2$. The solution to the borrower's problem therefore sets $c_{b,h,t}(t) = w_2/(1+\beta)\, R_{h,h}(t)$, and $c_{b,h,t}(t+1) = \beta w_2/(1+\beta)$. In addition,

$$x_h(t) = R_{h,h}(t+1) \left[\left(\frac{w_2}{1+\beta} \right) - y_h(t) \right] \tag{10}$$

for $t \geq 0$ and $h = 1, 2$ holds. To simplify notation, we henceforth let $w \equiv w_2/(1+\beta)$.

To summarize, then, trade in this model requires that young borrowers issue some long-maturity liabilities and some short-maturity liabilities. Long-maturity liabilities are sold to young lenders who take them elsewhere and trade them. Thus these liabilities circulate. Short-maturity liabilities are sold to middle-aged borrowers, who acquire them as a method of honoring their own liability issues. Short-maturity liabilities do not circulate. We now investigate what is required in order for markets in these liabilities to clear.

2.4. Equilibrium. In equilibrium three kinds of liabilities are traded: Newly-issued one-period liabilities, newly-issued two-period liabilities, and previously-issued two-period

[11] Inequality (6) forces all liabilities to be default-free. In our view this is a natural "first case" to consider. First, it seems appropriate to understand how an economy without default risk operates. This can be regarded as a prelude to understanding how the possibility of default risk might change matters. Second, to understand the real bills doctrine, we must understand how an economy works when safe private liabilities are traded. And third, as discussed by Rolnick, Smith, and Weber (1996), "small" default risks on notes are the historically most relevant situation.

liabilities. The markets for these liabilities are linked together in the following way. At date t, young borrowers in location h issue one-period liabilities with a current value of

$$\frac{y_h(t)}{R_{h,h}(t)} = \frac{w}{R_{h,h}(t)} - \frac{x_h(t)}{R_{h,h}(t)\,R_{h,h}(t+1)}. \tag{11}$$

These liabilities are sold to middle-aged borrowers, who wish to purchase one-period liabilities (maturing in location h) with a value of $x_h(t-1)/R_{h,h}(t)$. (The purchase of one-period liabilities with this value enables these borrowers to pay off their liabilities that mature at $t+1$.) Hence an equality between the demand for newly-issued one-period liabilities and their supply requires that

$$x_h(t-1) = w - \frac{x_h(t)}{R_{h,h}(t+1)} \tag{12}$$

for $h = 1, 2$, and $t \geq 0$. Rearranging terms in (12) yields the following law of motion for circulating liabilities issued by each borrower:

$$x_h(t) = [w - x_h(t-1)]\,R_{h,h}(t+1) \tag{13}$$

for $h = 1, 2$, and $t \geq 0$.

In location 1 there are γN borrowers, while in location 2 there are $(1-\gamma)N$ borrowers. There are $N/2$ lenders in each location. In location 1 at t, there are $(1-\gamma)Nx_2(t-1)$ one-period old liabilities circulating that were issued in location 2 in the previous period. Similarly, in location 2 at t, there are $\gamma N x_1(t-1)$ one-period old liabilities circulating that were issued in location 1 in the previous period. Clearly, then, in equilibrium, $\tilde{x}_{1,1}(t) = 2(1-\gamma)\,x_2(t-1)$ and $\tilde{x}_{1,2}(t) = 2\gamma x_1(t-1)$ must hold. Similarly, $\tilde{x}_{2,1}(t) = 2\gamma x_1(t)$ and $\tilde{x}_{2,2}(t) = 2(1-\gamma)\,x_2(t)$ must be satisfied. Using these conditions in (5), we obtain the condition for the market in circulating liabilities to clear in location 1:

$$\frac{e}{2} = (1-\gamma)\,\frac{x_2(t-1)}{R_{1,2}(t)} + \gamma\frac{x_1(t)}{R_{1,2}(t)\,R_{2,1}(t+1)} \tag{14}$$

for $t \geq 0$. Similarly, in location 2 we must have

$$\frac{e}{2} = \gamma\frac{x_1(t-1)}{R_{2,1}(t)} + (1-\gamma)\,\frac{x_2(t)}{R_{2,1}(t)\,R_{1,2}(t+1)} \tag{15}$$

for $t \geq 0$.

Equations (14) and (15) guarantee that the market for two-period (circulating) liabilities clears at each date and in each location. Equation (13) is implied by clearing of the market in one-period liabilities in each location. These equations, and the no-arbitrage conditions (9) are the equilibrium conditions for an economy with no outside money. Finally, it is easy to show that these conditions, along with (6)-(8) and (10), imply the satisfaction of the following goods market clearing conditions:

$$\frac{e}{2} + \gamma w = \frac{\gamma w}{R_{1,1}(t)} + \left(\frac{e}{2}\right) R_{2,1}(t-1) \tag{16}$$

$$\frac{e}{2} + (1-\gamma) w = (1-\gamma) \frac{w}{R_{2,2}(t)} + \left(\frac{e}{2}\right) R_{1,2}(t-1) \tag{17}$$

for $t \geq 1$. Equation (16) [(17)] implies that goods demand equals goods supply in location 1 (2) at each date after the initial period.

In the initial period, $t = 0$, we assume that middle-aged borrowers in location h have an inherited stock, $x_h(-1)$, $h = 1, 2$, of previously-issued liabilities. Hence $x_1(-1)$ and $x_2(-1)$ are given as initial conditions.

2.5. Steady state equilibria. We begin our analysis of equilibria with privately-issued circulating liabilities by considering steady states. In a steady state equilibrium we have $R_{1,1}(t) = R_{11}$, $R_{2,2}(t) = R_{22}$, $R_{1,2}(t) = R_{12}$, $R_{2,1}(t) = R_{21}$, $x_1(t) = x_1$, and $x_2(t) = x_2$. In a steady state the no-arbitrage conditions $R_{1,1}(t) R_{1,1}(t+1) = R_{1,2}(t) R_{2,1}(t+1)$ and $R_{2,2}(t) R_{2,2}(t+1) = R_{2,1}(t) R_{1,2}(t+1)$ immediately imply that $R_{11} = R_{22} = R$, and that

$$R_{12} R_{21} = R^2. \tag{18}$$

Moreover, in a steady state, (13) yields

$$x_1 = x_2 = \left(\frac{R}{1+R}\right) w. \tag{19}$$

Defining $\pi = 1/R$, we can also write the steady state versions of (14) and (15) as

$$R_{12} = \frac{2(1-\gamma) x_2}{e - 2\gamma \pi^2 x_1} \tag{20}$$

and

$$R_{21} = \frac{2\gamma x_1}{e - 2(1-\gamma) \pi^2 x_2}. \tag{21}$$

Using $R_{12}R_{21} = 1/\pi^2$, $x_1 = x_2 = w/(1+\pi)$, and equations (20) and (21), we obtain the single equilibrium condition

$$4\gamma(1-\gamma)w^2\pi^2 = \left[(1+\pi)e - 2(1-\gamma)w\pi^2\right]\left[(1+\pi)e - 2\gamma w\pi^2\right]. \tag{22}$$

Clearly any steady state equilibrium value of π must satisfy (22). In addition, it must imply positive values for R_{12} and R_{21}. Equations (19), (20), and (21), along with the condition $\gamma \geq 1/2$, imply that $R_{12} > 0$ and $R_{21} > 0$ hold iff π satisfies

$$\frac{\pi^2}{1+\pi} < \frac{e}{2\gamma w}. \tag{23}$$

Thus any solution to (22) that also obeys (23) is a legitimate steady state equilibrium value of π. We now state a result about such solutions.

Proposition 1. *There is a unique steady state equilibrium value, denoted by $\pi^\star$, that satisfies (22) and (23). The steady state equilibrium interest rates R_{11} and R_{22} are equal to $R^\star = 1/\pi^\star$.*

Proof. See Appendix A. ■

2.6. Properties of steady state equilibria: interest rates. In order to characterize the steady state equilibrium values R_{11} and R_{22} ($R^\star$), it is useful to proceed as follows. Imagine that all agents in our economy inhabited the same location, or equivalently, that inter-location trade was possible. Call this analog to our environment the centralized (trading) economy, and call our actual economy the decentralized economy. In the centralized economy, the unique equilibrium real rate of interest (steady state or otherwise) is $R = w/e = 1/\pi$. If $w/e \geq (<) \, 1$ the equilibrium of the centralized economy is dynamically efficient (inefficient), and there is no role (there is a role) for outside money. Gale (1973) refers to economies with $w > (<) \, e$ as "classical" ("Samuelson") economies. The next result shows what happens in decentralized economies when endowment patterns make their centralized analogs Samuelsonian or classical.

Proposition 2. *Part (a). If $\gamma = 1/2$ (locations are symmetric), then $1/\pi^\star = R^\star = w/e$. Part (b). If $\gamma > 1/2$ and $w/e < 1$, then $1/\pi^\star = R^\star \in (w/e, 1)$. Part (c). If $\gamma > 1/2$ and $w/e > 1$, then $1/\pi^\star = R^\star \in (1, w/e)$.*

Proof. See Appendix B. ■

Proposition 2 asserts that if the locations are symmetric, then spatial separation and limited communication have no effect on the steady state allocations received by borrowers (and, as we show below, on lenders as well). In addition, these features do not affect the conditions under which an economy admits a role for outside money. Finally, if the two locations are asymmetric ($\gamma > 1/2$), then spatial separation and limited communication put upward (downward) pressure on real rates of interest faced by borrowers in Samuelson (classical) economies.

What about steady state rates of return faced by lenders? The next proposition states a result in this regard.

Proposition 3. *Part (a). Suppose that* $\gamma = 1/2$. *Then* $R_{12} = R_{21} = R^{\star} < (>)\, 1$ *in a classical (Samuelson) economy. Part (b). Suppose that* $\gamma > 1/2$. *Then* $R^{\star} > R_{12}$ *if* $R^{\star} > 1$, *while* $R^{\star} > R_{21}$ *if* $R^{\star} < 1$.

Proof. See Appendix C. ■

Proposition 3 makes two important claims. First, in an inside-money-only economy, rates of interest faced by borrowers and rates of interest faced by lenders can never be equal (unless $\gamma = 1/2$ or $e = w$). Thus, since the marginal rates of substitution between borrowers and lenders are not equated, resource allocations are statically inefficient (relative to what is possible in a centralized economy). Second, if $\gamma > 1/2$, then lenders in different locations will also face different rates of return. Hence their marginal rates of substitution are unequal. Note also that the sign of $R^{\star} - 1$ depends only on the sign of $e - w$; hence endowment patterns alone determine which lenders earn high returns on savings.

Proposition 3 also has implications for the discounts and premia observed on different circulating liabilities. When $1/R_{hk} < (>)\, 1/R_{hh}$, previously-issued circulating liabilities sell at a discount (premium) in location h—relative to noncirculating liabilities. Thus, as implied by Proposition 3, if $\gamma > 1/2$ such liabilities sell at a premium (discount) in location 1 in a classical (Samuelson) economy. The converse is true in location 2. Notice that whose liabilities bear a discount or premium depends only on the endowment patterns of borrowers and lenders, not on the relative magnitudes of the excess demand for credit in the different locations.

The discount or premium on newly-issued liabilities in location h—relative to noncirculating liabilities—is $R_{hh}/R_{hk}R_{kh} = 1/R_{hh}$. In a steady state, this discount/premium is identical across locations. This observation has an immediate corollary: Previously-issued two-period circulating liabilities and newly-issued two-period liabilities will bear different discounts. As a consequence, the discount or premium on any given "note" or liability must fluctuate over time. In addition, perfectly safe liabilities of different issuers will not have the same discount in the same location. Both observations seem consistent with the facts mentioned in the introduction concerning discounts and premia on privately-issued banknotes in the antebellum U.S.

2.7. Constrained efficiency of stationary equilibria. We examine next whether the stationary allocations described by equations (18)-(22) can be improved upon by a central planner who is able to reallocate endowments among individuals inhabiting the *same* location, but cannot transfer resources between different locations. The planner thus maximizes a convex combination of the stationary ordinal utilities for agents (j, h) where $j = b, \ell$ and $h = 1, 2$, subject to two distinct resource constraints for locations 1 and 2.

A *constrained-efficient stationary allocation with equal treatment* is a list $\{c_{y,j,h}, c_{m,j,h}\}$ of stationary consumptions for young ($= y$) agents of type (j, h) and middle-aged ($= m$) agents of the same type, which maximizes the social welfare function

$$W = \sum_{j\in\{b,\ell\}} \sum_{h\in\{1,2\}} \left[\alpha_{j,h} \left(\ln c_{y,j,h} + \beta \ln c_{m,j,h}\right) + \left(1 - \alpha_{j,h}\right) c_{m,j,h}\right] \tag{24}$$

subject to the resource constraints

$$\frac{1}{2}\left[c_{y,\ell,1} + c_{m,\ell,2}\right] + \gamma\left[c_{y,b,1} + c_{m,b,1}\right] \leq \frac{e_1}{2} + \gamma w_2 \tag{25}$$

for location 1, and

$$\frac{1}{2}\left[c_{y,\ell,2} + c_{m,\ell,1}\right] + (1-\gamma)\left[c_{y,b,2} + c_{m,b,2}\right] \leq \frac{e_1}{2} + (1-\gamma) w_2 \tag{26}$$

for location 2. Here $\alpha_{j,h} \in (0,1]$ is the weight assigned by the planner to a type (j,h) member of a normal generation, and $1-\alpha_{j,h}$ is the weight assigned to the same type of the transitional generation, born middle-aged at time 1.

The first-order conditions for this problem reduce to:

$$[MRS\,(b,h)]^2 = MRS\,(\ell,1)\,MRS\,(\ell,2) \geq 1 \tag{27}$$

for $h=1,2$, where $MRS\,(j,h)$ is the absolute value of the marginal rate of substitution for household (j,h). These conditions require that all borrowers face a common marginal rate of substitution which should be no less than "one plus the growth rate of resources," while lenders in different locations may face different marginal rates of substitution. However, the lenders' marginal rates of substitution must have a geometric average equal to the common MRS of borrowers.

From equation (18) and Proposition 2, it is easy to check that stationary equilibria are constrained optima only in classical economies in which the yields, and the corresponding marginal rates of substitution, exceed unity. Stationary yields in Samuelson economies with circulating private liabilities are below the natural growth rate; hence allocations can be improved by a system of lump-sum taxes and transfers among agents living in the *same* location.

2.8. Equilibrium dynamics. We now take up the topic of dynamical equilibria with circulating liabilities. To do so, use as state variables the quantities $x_h\,(t)$ of two-period claims issued by young borrowers at time t in location h. We also define the lagged values $v_h\,(t) = x_h\,(t-1)$ and the auxiliary functions

$$f\left[x_1\,(t-1)\,, x_2\,(t-1)\,, x_2\,(t-2)\right] \equiv \tag{28}$$

$$\frac{ex_2\,(t-1) - 2(1-\gamma)\left[w - x_2\,(t-1)\right]\left[w - x_2\,(t-2)\right]}{2\gamma x_1\,(t-1)\,x_2\,(t-1)},$$

$$Q\left[x_1\,(t-1)\,, x_2\,(t-1)\,, x_2\,(t-2)\right] \equiv \tag{29}$$

$$\frac{2\gamma w\left[w - x_1\,(t-1)\right] f\left[x_1\,(t-1)\,, x_2\,(t-1)\,, x_2\,(t-2)\right]}{(e+2\gamma w) f\left[x_1\,(t-1)\,, x_2\,(t-1)\,, x_2\,(t-2)\right] - e},$$

$$g\left[x_1\,(t-1)\,, x_2\,(t-1)\,, x_1\,(t-2)\right] \equiv \tag{30}$$

$$\frac{ex_1\left(t-1\right)-2\gamma\left[w-x_1\left(t-1\right)\right]\left[w-x_1\left(t-2\right)\right]}{2(1-\gamma)x_1\left(t-1\right)x_2\left(t-1\right)},$$

and

$$Z\left[x_1\left(t-1\right),x_2\left(t-1\right),x_1\left(t-2\right)\right]\equiv \tag{31}$$

$$\frac{2(1-\gamma)w\left[w-x_2\left(t-1\right)\right]g\left[x_1\left(t-1\right),x_2\left(t-1\right),x_1\left(t-2\right)\right]}{\left[e+2(1-\gamma)w\right]g\left[x_1\left(t-1\right),x_2\left(t-1\right),x_1\left(t-2\right)\right]-e}.$$

Then we show in Appendix D that dynamical equilibria are solutions to the dynamical system

$$x_1\left(t\right) = Q\left[x_1\left(t-1\right),x_2\left(t-1\right),v_2\left(t-1\right)\right], \tag{32}$$

$$x_2\left(t\right) = Z\left[x_1\left(t-1\right),x_2\left(t-1\right),v_1\left(t-1\right)\right], \tag{33}$$

$$v_1\left(t\right) = x_1\left(t-1\right), \tag{34}$$

$$v_2\left(t\right) = x_2\left(t-1\right). \tag{35}$$

We now proceed to analyze local dynamics in a neighborhood of the unique steady state. This steady state has

$$x_1 = x_2 = x = \frac{w}{1+\pi^\star}, \tag{36}$$

where $\pi^\star$ is the steady state inverse gross real return derived above. Linearizing the system (32) - (35) in a neighborhood of the steady state yields

$$\begin{pmatrix} x_1\left(t\right)-x \\ x_2\left(t\right)-x \\ v_1\left(t\right)-x \\ v_2\left(t\right)-x \end{pmatrix} = J\begin{pmatrix} x_1\left(t-1\right)-x \\ x_2\left(t-1\right)-x \\ v_1\left(t-1\right)-x \\ v_2\left(t-1\right)-x \end{pmatrix} \tag{37}$$

where the Jacobian matrix is given by

$$J = \begin{pmatrix} Q_1 & Q_2 & 0 & Q_3 \\ Z_1 & Z_2 & Z_3 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \tag{38}$$

and where the partial derivatives $(Q_i, Z_i)_{i=1}^3$ are evaluated at the steady state.

The eigenvalues of J are the roots of the following polynomial:

$$P(\lambda) \equiv \lambda^4 - (Q_1 + Z_2)\lambda^3 + (Q_1 Z_2 - Q_2 Z_1)\lambda^2 - (Q_2 Z_3 + Q_3 Z_1)\lambda - Q_3 Z_3 = 0. \quad (39)$$

Proposition 4 gives a characterization of these eigenvalues.

Proposition 4. *Suppose that $\gamma < 1$. Then (a) J has four real eigenvalues; (b) $-R^\star = -1/\pi^\star$ is an eigenvalue of J; (c) J has an eigenvalue in the interval $(-1/(1+\pi^\star), 0)$; (d) -1 is an eigenvalue of J; (e) if*

$$(\pi^\star)^3 \geq \frac{e\left[(\pi^\star)^2 - 1\right]}{2(1-\gamma)w}, \quad (40)$$

then J has an eigenvalue in the interval $(1, \infty)$; (f) a sufficient condition for equation (40) to be satisfied is that $w \geq e$, or that

$$\left[\frac{2(1-\gamma)w}{e}\right]^2 \geq 2\gamma - 1. \quad (41)$$

Proof. See Appendix E. ■

We should note that extensive numerical analysis revealed that the positive eigenvalue of J always exceeds unity, even if (40) fails to hold. The nature of this analysis is described in the next subsection.

Proposition 4 states that, in a Samuelson economy ($\pi^\star > 1$), J has *at least* two eigenvalues in the interval $(-1, 0)$. Thus, it is possible to approach the steady state; any paths that do approach the steady state will display slowly damped oscillation. Second, in a classical economy ($\pi^\star < 1$), equation (40) necessarily holds. Then J has only one eigenvalue in the interior of the unit interval. Third, -1 is always an eigenvalue of J. Thus the dynamical system consisting of equations (32)-(35) always possesses a center manifold. Dynamics along the center manifold are investigated in the next subsection.

The results just described indicate the following fact. Along any equilibrium path that converges to the steady state, privately-issued circulating liabilities are necessarily a source of endogenous oscillations.

2.9. Motion on the center manifold. As we have noted, the steady state possesses a center manifold. We now follow the procedure outlined by Wiggins (1990) to characterize the stability of the dynamical system along the center manifold in a neighborhood of the steady state. The essential idea is that the first-order approximation to the dynamical system does not yield sufficient information to fully characterize local dynamics in a neighborhood of the steady state, and so the next step is to construct a second-order approximation. Naturally it is not possible to do so analytically. Therefore, we construct a numerical approximation to the center manifold across a set that essentially "spans the space" of possible parameter value. The procedures underlying the construction of the center manifold are described in Appendix F.

We begin with Samuelson ($w < e$) economies. In our numerical exercise, we sampled 1,000 economies with $e = 1$ and with w selected using a uniform distribution over the interval $(0, 1)$.[12] This procedure collects a sample over all possible ratios of e/w that are consistent with the Samuelson case. The parameter γ was selected using a uniform distribution over the interval $(1/2, 1)$. Proposition 4 stops short of showing that the positive eigenvalue associated with the linearization at the steady state is always greater than one. But numerical calculations for these 1,000 economies indicated that the positive eigenvalue always exceeded unity. We therefore are left with two stable, negative eigenvalues and an eigenvalue equal to -1. Our numerical calculation of the motion on the center manifold for this case indicated that the dynamics along the center manifold are always stable (that is, for all $1,000$ economies we sampled). Since we have two initial conditions, we conclude that equilibrium is *indeterminate* with one degree of freedom. Thus there are many equilibrium paths for given initial conditions. In addition, along any equilibrium path, the economy will display oscillatory motion en route to the steady state. These oscillations are damped, but may decay only very slowly with time, because of the presence of a center manifold.

We now turn to classical economies. Here we sampled 1,000 economies with $w = 1$ and with e selected using a uniform distribution over the interval $(0, 1)$, and we chose γ as before. Again, this procedure collects a sample over all possible ratios of w/e that are consistent with the classical case. And as before, the positive eigenvalue discussed

[12] An examination of the equilibrium conditions will indicate that the values e and w do not matter individually for the properties of equilibrium. Only the ratio e/w matters.

Figure 1

Center manifold dynamics, classical case economies

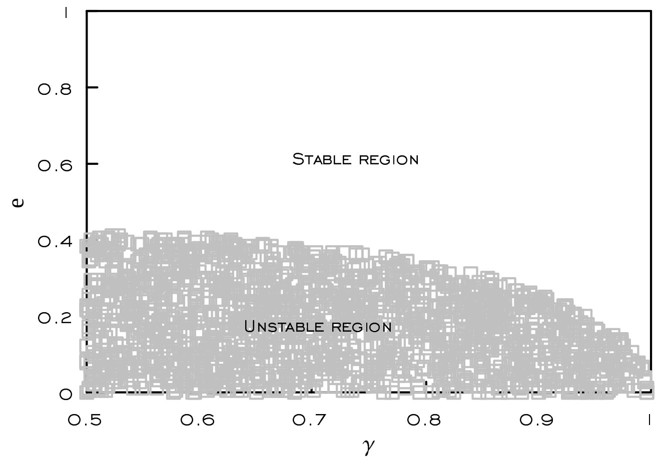

Figure 1: Center manifold dynamics for classical case economies. A square indicates a randomly selected economy which displayed unstable center manifold dynamics. The blank area corresponds to economies which had stable center manifold dynamics. The stable region is associated with determinacy of equilibrium.

in Proposition 4 was always greater than unity. This leaves us with one negative, stable eigenvalue as well as the eigenvalue equal to -1. The calculation of the economy's motion along the center manifold indicated that the dynamics can be either stable or unstable. Figure 1 displays regions of the parameter space that deliver stability or instability. The essential message of the figure is that for large enough values of e, the dynamics along the center manifold are stable, and hence, combined with the one stable eigenvalue and the two initial conditions, equilibrium is determinate. If the economy is relatively asymmetric ($\gamma \to 1$), then the minimum value of e consistent with determinacy falls. For relatively low values of e, and γ not too large, the dynamics on the center manifold are unstable, and hence the steady state cannot generally be approached.[13]

To summarize, any equilibrium paths that do approach the steady state display oscil-

[13] A characterization of the global dynamics is not straightforward, and we know little about the properties of equilibrium in this situation.

lation en route. This is true of both Samuelson and classical economies. These oscillations are damped, but due to the presence of -1 as an eigenvalue, they may dampen only very slowly. Samuelson economies support Friedman's (1960) conjecture that privately-issued, circulating liabilities can give rise to an indeterminacy of equilibrium. We conclude that all economies with privately-issued circulating liabilities will display endogenously arising—or "excessive"—volatility, just as Friedman (1960) asserted.

3. An economy with both inside and outside money

3.1. Changes to the environment. We now allow for the possibility that a stock of fiat (outside) money coexists alongside private circulating liabilities. We assume that these government-issued liabilities are never redeemed. Except for the presence of outside liabilities, all other features of the economy are unaltered. Thus, in this economy, as in the previous economy, borrowers can issue both one- and two-period liabilities. The latter will circulate, as described above, prior to being redeemed. Lenders can hold the circulating liabilities of borrowers and, in addition, they can now hold government-issued fiat currency. We let M be the constant per capita stock of outside money, and we let $p_h(t)$ be the time t dollar price of a unit of consumption in location h. Finally, as before, inside liabilities are (indexed) claims to future consumption.[14]

3.2. Lenders. Let $m_h(t)$ be the real value of outside money balances acquired by a young lender at date t in location h. Then this agent chooses values $c_{\ell,h,t}(t)$, $c_{\ell,h,t}(t+1)$, $\tilde{x}_{1,h}(t)$, $\tilde{x}_{2,h}(t)$, and $m_h(t)$ to maximize $\ln c_{\ell,h,t}(t) + \beta \ln c_{\ell,h,t}(t+1)$, subject to

$$c_{\ell,h,t}(t) + \frac{\tilde{x}_{1,h}(t)}{R_{h,k}(t)} + \frac{\tilde{x}_{2,h}(t)}{R_{h,k}(t)\, R_{k,h}(t+1)} \leq e_1 - m_h(t) \tag{42}$$

and

$$c_{\ell,h,2}(t) \leq \tilde{x}_{1,h}(t) + \frac{\tilde{x}_{2,h}(t)}{R_{k,h}(t+1)} + m_h(t)\frac{p_h(t)}{p_k(t+1)} \tag{43}$$

for $h = 1, 2$, $k = 1, 2$, and $k \neq h$. In order for both inside and outside money to be held in this economy, both must carry the same one period gross rate of return. Since young lenders who carry outside money from location 1 (2) to location 2 (1) between t and $t+1$

[14] Nothing changes in the analysis if inside liabilities are a claim to delivery of outside money at some future date.

earn the gross real return $p_1(t)/p_2(t+1)$ $[p_2(t)/p_1(t+1)]$, it follows that

$$R_{1,2}(t) = p_1(t)/p_2(t+1) \tag{44}$$

and

$$R_{2,1}(t) = p_2(t)/p_1(t+1) \tag{45}$$

for all $t \geq 0$. The solution to the lender's problem is to set $c_{\ell,h,t}(t) = e_1/(1+\beta)$ and

$$c_{\ell,h,t}(t+1) = \frac{\beta e_1 R_{h,k}(t)}{1+\beta}. \tag{46}$$

Again, we let $e \equiv e_1\beta/(1+\beta)$.

3.3. Borrowers. For young borrowers matters are essentially the same as before, except that they too can carry fiat money between periods. For middle-aged "borrowers" an obvious motive for doing so exists: Holding outside money enables them to redeem the circulating liabilities they issued when young. Borrowers in their first period of life have no obvious reason to hold government liabilities. However, it is innocuous to allow them to do so. And, as we will see, allowing them to do so facilitates a particular interpretation of the model. Thus we allow both young and middle-aged borrowers to accumulate real balances in every period.

Let $z_h(t)$ $[\tilde{z}_h(t+1)]$ be the quantity of real balances held by a young (middle-aged) borrower between t and $t+1$ ($t+1$ and $t+2$). Then a borrower born at t faces the budget constraints

$$c_{b,h,t}(t) \leq \frac{y_h(t)}{R_{h,h}(t)} + \frac{x_h(t)}{R_{h,k}(t)\,R_{k,h}(t+1)} - z_h(t) \tag{47}$$

and

$$c_{b,h,t}(t+1) + \frac{\tilde{y}_h(t+1)}{R_{h,h}(t+1)} + \tilde{z}_h(t+1) \leq w_2 - y_h(t) + z_h(t)\left(\frac{p_h(t)}{p_h(t+1)}\right). \tag{48}$$

Finally, a middle-aged borrower born at t must hold enough claims on period $t+2$ consumption to enable him to redeem the circulating liabilities he issued at t. Given the ability of such an agent to hold outside money, the appropriate analog of equation (6) is

$$x_h(t) \leq \tilde{y}_h(t+1) + \tilde{z}_h(t+1)\frac{p_h(t+1)}{p_h(t+2)} \tag{49}$$

for $h = 1, 2$.

A young borrower in location h at t chooses values $c_{b,1,t}(t)$, $c_{b,2,t}(t+1)$, $x_h(t)$, $y_h(t)$, $\tilde{y}_h(t+1)$, $z_h(t)$, and $\tilde{z}_h(t+1)$ to maximize $\ln c_{b,h,t}(t) + \beta \ln c_{b,h,t}(t+1)$, subject to (47)-(49), $x_h(t) \geq 0$, and $y_h(t) \geq 0$. As before, in order for this problem to have a solution, the no arbitrage conditions (9) must be satisfied. In addition, the return on real balances held between t and $t+1$ by middle-aged borrowers is $p_h(t)/p_h(t+1)$. The return on one-period liabilities is $R_{h,h}(t)$. Thus, a lack of arbitrage opportunities for middle-aged borrowers requires that

$$R_{h,h}(t) = \frac{p_h(t)}{p_h(t+1)} \tag{50}$$

for $t \geq 0$, and $h = 1, 2$. When the no arbitrage conditions hold, the solution to a borrower's problem sets $c_{b,h,t}(t) = w_2/(1+\beta) R_{h,h}(t) = w/R_{h,h}(t)$ and $c_{b,h,t}(t+1) = \beta w$, where as above $w = w_2/(1+\beta)$.

3.4. Equilibrium. In the inside money only economy, it was most convenient to describe dynamical equilibria in terms of the sequences $\{x_1(t)\}$ and $\{x_2(t)\}$. However, in the presence of outside money, the quantity of circulating private liabilities issued will generally be indeterminate. It is therefore most convenient to work directly with the goods market clearing and the no-arbitrage conditions.

The per capita supply of available goods in location 1 is $\frac{e_1}{2} + \gamma w_2 = \frac{e(1+\beta)}{2\beta} + \gamma(1+\beta) w$. The per capita demand of young borrowers in this location at t is $\gamma w_2/(1+\beta) R_{1,1}(t) = \gamma w/R_{1,1}(t)$, while that of borrowers in the second period of life is $\gamma \beta w = \gamma \beta w_2/(1+\beta)$. The per capita demand for goods by young lenders is $e/2\beta = e_1/2(1+\beta)$ while the per capita demand of middle-aged lenders is $\frac{\beta e_1 R_{2,1}(t-1)}{1+\beta} = e R_{2,1}(t-1)$. Hence, the goods market clears in location 1 at t iff

$$\frac{e_1}{2} + \gamma w_2 = \frac{\gamma w_2}{(1+\beta) R_{1,1}(t)} + \frac{\gamma \beta w_2}{(1+\beta)} + \frac{e_1}{2(1+\beta)} + \frac{\beta e_1}{2(1+\beta)} R_{2,1}(t-1) \tag{51}$$

for $t \geq 1$. This simplifies to

$$\frac{e}{2} + \gamma w = \frac{\gamma w}{R_{1,1}(t)} + \frac{e}{2} R_{2,1}(t-1). \tag{52}$$

Similarly, the goods market clears in location 2 at t iff

$$\frac{e}{2}+(1-\gamma)\,w=\frac{(1-\gamma)\,w}{R_{2,2}\,(t)}+\frac{e}{2}R_{1,2}\,(t-1) \tag{53}$$

for $t\geq 1$. In addition, the no-arbitrage conditions (9) and (50) must obtain in each location at each date.

At $t=1$, middle-aged agents hold the initial outside money supply, plus any previously-issued circulating private liabilities. Let $M_h\,(t)$ be the stock of outside money[15] in location h at t (clearly $M_1\,(t)+M_2\,(t)\equiv M$), and let $x_h\,(-1)$ for $h=1,2$, denote the initial stock of circulating private liabilities. Then middle-aged agents in location 1 at $t=1$ hold circulating liabilities with a real value $M_1\,(0)\,/p_1\,(1)+x_2\,(-1)\,/R_{1,2}\,(0)$. It follows that the goods market clears in location 1 at $t=0$ if

$$\frac{e}{2}=\frac{\gamma w}{R_{1,1}\,(0)}+\frac{M_1\,(0)}{p_1\,(0)}+\frac{(1-\gamma)\,x_2\,(-1)}{R_{1,2}\,(0)}. \tag{54}$$

Similarly, the goods market clears in location 2 at $t=0$ if

$$\frac{e}{2}=\frac{(1-\gamma)\,w}{R_{2,2}\,(0)}+\frac{M_2\,(0)}{p_2\,(0)}+\frac{\gamma x_1\,(-1)}{R_{2,1}\,(0)}. \tag{55}$$

Here $M_1\,(0)$, $M_2\,(0)$, $x_1\,(-1)$, and $x_2\,(-1)$ are given initial conditions. Moreover, the no-arbitrage conditions imply that

$$R_{1,2}\,(0)=R_{2,2}\,(0)\,[p_1\,(0)\,/p_2\,(0)] \tag{56}$$

and

$$R_{2,1}\,(0)=R_{1,1}\,(0)\,[p_2\,(0)\,/p_1\,(0)]\,. \tag{57}$$

Substituting these conditions into (54) and (55) allows us (generically) to determine $p_1\,(0)$ and $p_2\,(0)$ (and, therefore, $R_{1,2}\,(0)$ and $R_{2,1}\,(0)$), once $R_{1,1}\,(0)$ and $R_{2,2}\,(0)$ are specified. It follows that our economy has two equations connecting the initial-period yields $\{R_{1,1}\,(0)\,,R_{1,2}\,(0)\,,R_{2,1}\,(0)\,,R_{2,2}\,(0)\}$ and, hence, two initial degrees of freedom. Once these are specified equations (54) and (55)—along with the no-arbitrage conditions for $t=1$—determine the initial price levels. Equations (52) and (53), along with

[15] This represents liabilities of the central bank or, equivalently, never-to-be-redeemed liabilities of an initial middle-aged generation that "invents" fiat money.

the no-arbitrage conditions (9), then determine the sequence $\{R_{1,1}(t), R_{1,2}(t), R_{2,1}(t), R_{2,2}(t)\}_{t=1}^{\infty}$. And the sequence $\{p_1(t), p_2(t)\}_{t=1}^{\infty}$ can be derived from the initial price levels and the relations $R_{h,k}(t) = p_h(t)/p_k(t+1)$.

Prior to describing equilibria of our economy characterized by spatial separation and limited communication, it will be useful to describe briefly its counterpart economy where all agents inhabit a common location at all dates. We now turn our attention to this task.

3.5. The centralized economy. Suppose that at each date all agents inhabit the same location. Then there is a common price level, $p(t)$, and a common one period rate of return, $R(t)$, between t and $t+1$. In addition, there is no need for private circulating liabilities. The goods market clears at $t \geq 1$ if

$$e + w = \frac{w}{R(t)} + eR(t-1). \tag{58}$$

At $t = 1$ the goods market clears if

$$e = \frac{w}{R(1)} + \frac{M}{p(1)}. \tag{59}$$

There are two possible steady state equilibria. One has $R(t) = 1$ for all t and $M/p(t) = e - w/R(t) = e - w$. Real balances are positive iff

$$e > w, \tag{60}$$

as we henceforth assume.[16] The other steady state is nonmonetary ($lim_{t\to\infty} p(t) = \infty$) and has $R = w/e < 1$. It is easy to verify from equation (58) that the monetary steady state is unstable, while the nonmonetary steady state is asymptotically stable. Moreover, $R(1)$ is a free initial condition. For all $R(1) \in (w/e, 1)$ there is an equilibrium with $R(t) \downarrow w/e$. Clearly this implies that $p(t) \uparrow \infty$. There is, then, a one-dimensional indeterminacy of equilibrium. There is also a unique equilibrium where the value of money is bounded away from zero.

[16] Of course, (60) just implies that, in order for outside money to have value, the centralized economy must be "Samuelsonian," in Gale's (1973) sense.

3.6. The decentralized economy.

Overview. We now characterize equilibria of our economy in the presence of spatial separation and limited communication. To do so, we note that equations (52)-(53) and the arbitrage conditions described by equation (9) can be rewritten as the following dynamical system, which obtains for $t \geq 1$:

$$R_{1,1}(t+1) = \frac{2\gamma w}{e + 2\gamma w - eR_{2,1}(t)}, \tag{61}$$

$$R_{2,2}(t+1) = \frac{2(1-\gamma)w}{e + 2(1-\gamma)w - eR_{1,2}(t)}, \tag{62}$$

$$R_{1,2}(t+1) = \frac{2(1-\gamma)wR_{2,2}(t)}{[e + 2(1-\gamma)w - eR_{1,2}(t)]R_{2,1}(t)}, \tag{63}$$

and

$$R_{2,1}(t+1) = \frac{2\gamma wR_{1,1}(t)}{[e + 2\gamma w - eR_{2,1}(t)]R_{1,2}(t)}. \tag{64}$$

As we have noted previously, there are two free initial conditions. We begin with an analysis of steady state equilibria.

A monetary[17] steady state. When the values $R_{1,1}(t)$, $R_{1,2}(t)$, $R_{2,1}(t)$, and $R_{2,2}(t)$ are constant, it is apparent that $R_{11} = R_{22}$. It is also apparent that setting $R_{1,1}(t) = R_{2,2}(t) = R_{2,1}(t) = R_{1,2}(t) = 1$ $\forall t$ constitutes a constant solution to the system (61)-(64). Notice that, in a monetary steady state, all agents face the same rate of return. Thus the marginal rates of substitution of all agents are equated. Moreover, the allocation of resources that obtains in a monetary steady state of the decentralized economy duplicates that which obtains in the steady state of the centralized economy. It follows from Balasko and Shell (1981) that this allocation is Pareto optimal. We therefore conclude that, at least with respect to steady states, a combination of a government-issued fiat currency and circulating private liabilities allows the economy to completely overcome the problems of spatial separation and limited communication. As we have seen, this is not true if outside money is absent. And, as we will show presently, it is not true if inside

[17]By "monetary" we mean a steady state where the outside money has positive value. Privately-issued liabilities will be circulating as well in this steady state.

money is absent. Both types of money are required to support the centralized resource allocation. It is therefore possible to conclude that, in this economy, the Hayek proposal for eliminating outside money prevents the attainment of an optimal allocation of resources.

Price level determination. When $R_{1,1}(t) = p_1(t)/p_1(t+1) = 1$ and $R_{2,2}(t) = p_2(t)/p_2(t+1) = 1$, the price level in each location is constant. And, when $R_{1,2}(t) = p_1(t)/p_2(t+1) = 1$, the price level is identical in each location as well. We now turn to the problem of determining the steady state price level, which we denote simply by p.

Let x_h be the (constant) per capita steady state value of circulating liability issues in location h, let y_h be the (constant) per capita steady state value of one-period (non-circulating) liabilities issued in location h, and let M_h be the (constant) quantity of fiat currency circulating in location h in a steady state. Then, since all gross rates of return equal one, and since young lenders in location h must hold (a) all new issues of circulating liabilities in location h, plus (b) all circulating liabilities issued last period in the other location, plus (c) all outside money circulating in location h that is not absorbed by borrowers, in location 1 we have that

$$\begin{aligned} \frac{e}{2} &= \gamma x_1 + (1-\gamma)x_2 + \frac{M_1}{p} - \gamma(x_1 - y_1) \\ &= (1-\gamma)x_2 + \frac{M_1}{p} + \gamma w - \gamma x_1. \end{aligned} \tag{65}$$

Similarly, in location 2 we have

$$\begin{aligned} \frac{e}{2} &= \gamma x_1 + (1-\gamma)x_2 + \frac{M_2}{p} - (1-\gamma)(x_2 - y_2) \\ &= \gamma x_1 - (1-\gamma)x_2 + \frac{(M - M_1)}{p} + (1-\gamma)w. \end{aligned} \tag{66}$$

Adding (65) and (66) yields

$$p = \frac{M}{e - w} \tag{67}$$

which coincides with the steady state price level of the centralized economy. Our maintained assumption that $w/e < 1$ implies that $p > 0$.

Discounts or premia on private "notes". It is also worthy of mention that, when $R_{11} = R_{12} = R_{21} = R_{22}$, there are no discounts or premia on privately-issued liabilities.

This is true both with respect to currency, and with respect to noncirculating liabilities. Hence, at least with respect to steady states, the introduction of outside money eliminates any “lack of uniformity” of the currency.

3.7. Local dynamics.

Overview. We now analyze local dynamics in a neighborhood of both the monetary and the nonmonetary steady state. To do so, we construct a linear approximation at a steady state of the dynamical system consisting of equations (61)-(64):

$$\begin{bmatrix} R_{1,1}(t+1) - R_{11} \\ R_{2,2}(t+1) - R_{22} \\ R_{1,2}(t+1) - R_{12} \\ R_{2,1}(t+1) - R_{21} \end{bmatrix} = J \begin{bmatrix} R_{1,1}(t) - R_{11} \\ R_{2,2}(t) - R_{22} \\ R_{1,2}(t) - R_{12} \\ R_{2,1}(t) - R_{21} \end{bmatrix} \tag{68}$$

where J is the Jacobian matrix

$$J = \begin{bmatrix} \frac{\partial R_{1,1}(t+1)}{\partial R_{1,1}(t)} & \frac{\partial R_{1,1}(t+1)}{\partial R_{2,2}(t)} & \frac{\partial R_{1,1}(t+1)}{\partial R_{1,2}(t)} & \frac{\partial R_{1,1}(t+1)}{\partial R_{2,1}(t)} \\ \frac{\partial R_{2,2}(t+1)}{\partial R_{1,1}(t)} & \frac{\partial R_{2,2}(t+1)}{\partial R_{2,2}(t)} & \frac{\partial R_{2,2}(t+1)}{\partial R_{1,2}(t)} & \frac{\partial R_{2,2}(t+1)}{\partial R_{2,1}(t)} \\ \frac{\partial R_{1,2}(t+1)}{\partial R_{1,1}(t)} & \frac{\partial R_{1,2}(t+1)}{\partial R_{2,2}(t)} & \frac{\partial R_{1,2}(t+1)}{\partial R_{1,2}(t)} & \frac{\partial R_{1,2}(t+1)}{\partial R_{2,1}(t)} \\ \frac{\partial R_{2,1}(t+1)}{\partial R_{1,1}(t)} & \frac{\partial R_{2,1}(t+1)}{\partial R_{2,2}(t)} & \frac{\partial R_{2,1}(t+1)}{\partial R_{1,2}(t)} & \frac{\partial R_{2,1}(t+1)}{\partial R_{2,1}(t)} \end{bmatrix} \tag{69}$$

with all partial derivatives evaluated at the steady state in question, and where $[R_{11}, R_{22}, R_{12}, R_{21}]'$ denotes the vector of steady state interest rates. In a monetary steady state $R_{11} = R_{22} = R_{12} = R_{21} = 1$; in a nonmonetary steady state the vector $[R_{11}, R_{22}, R_{12}, R_{21}]'$ is described by Proposition 1. The elements of the matrix J are described in Appendix G.

The dynamics of equation (68) are governed by the eigenvalues of J. We now characterize these eigenvalues at each steady state.

The monetary steady state. Define the polynomials $H(\lambda)$ and $P(\lambda)$ by

$$H(\lambda) \equiv \lambda^3 - \left\{1 + \frac{e}{2\gamma(1-\gamma)w}\right\}\lambda^2 + \frac{(1+\lambda)e^2}{4\gamma(1-\gamma)w^2} \tag{70}$$

and

$$P(\lambda) \equiv (1+\lambda)H(\lambda). \tag{71}$$

Then we have the following result:

Proposition 5. *(a) The eigenvalues of J, at the monetary steady state, are the roots of the equation $P(\lambda) = 0$. (b) The value -1 is an eigenvalue of J. In addition, J has one eigenvalue in the interval $(-1, 0)$, and two real eigenvalues that exceed one.*

Proof. See Appendix H. ■

Proposition 5 establishes that the monetary steady state possesses a one-dimensional stable manifold, a two-dimensional unstable manifold, and a center manifold. Below we establish stability of the monetary steady state along the center manifold. Thus, since there are two free initial conditions, there is a *unique* dynamical equilibrium path that approaches the monetary steady state. This result is an analog of the uniqueness of an equilibrium where money asymptotically retains value in the centralized economy. Note that, in particular, the coexistence of private circulating liabilities with outside money is *not* a source of equilibrium indeterminacy. Along this dimension, then, quantity theoretic criticisms of this "real bills" arrangement are incorrect.

It is true, however, that any paths that approach the monetary steady state must display endogenous—or "excess"—volatility. Moreover, since -1 is an eigenvalue of J, this volatility may dampen only very slowly. Thus, a coexistence of private and governmental liabilities—or the need to trade with both kinds of liabilities—will lead to fluctuations. On this dimension quantity-theoretic criticism of the "real bills" doctrine are justified.

The nonmonetary steady state. Define the polynomials $\psi(\lambda)$ and $\rho(\lambda)$ by

$$\psi(\lambda) \equiv \lambda^3 - \lambda^2 \left\{1 + \left(\frac{eR_{11}R_{21}}{2\gamma w}\right) + \left[\frac{eR_{11}R_{12}}{2(1-\gamma)w}\right]\right\} + \frac{(1+\lambda)e^2(R_{11})^4}{4\gamma(1-\gamma)w^2} \tag{72}$$

and

$$\rho(\lambda) \equiv (1+\lambda)\psi(\lambda). \tag{73}$$

Then we obtain the following result.

Proposition 6. *(a) The eigenvalues of J, at the nonmonetary steady state, are the roots of $\rho(\lambda) = 0$. (b) The value -1 is an eigenvalue of J. J also has one eigenvalue in the interval $(-1, 0)$, one eigenvalue in the interval $(0, 1)$, and one eigenvalue in the interval $(1, \infty)$.*

Proof. See Appendix I. ■

Proposition 6 suggests that the nonmonetary steady state possesses a two-dimensional stable manifold, a one-dimensional unstable manifold, and a center manifold. We now discuss equilibrium dynamics along the center manifold.

Motion on the center manifold. As we have noted, the economy with outside money has two steady states. And, at both, the Jacobian matrix has -1 as an eigenvalue. We therefore evaluate equilibrium dynamics along the center manifold in a neighborhood of each steady state.

In order to do so, we follow the procedure outlined in Appendix F. In addition, in order for money to be valued, $e > w$ must hold. Moreover, the properties of equilibrium dynamics depend only on the ratio e/w. Thus, we construct a numerical approximation to the center manifold—at each steady state—choosing parameters as follows. First, we set $e = 1$ and choose values for w using a uniform distribution over the interval $(0, 1)$. As before, we choose γ using a uniform distribution on the interval $(1/2, 1)$. We choose 1,000 economies, randomly, defined in this way, and calculated the center manifold approximation in each case.

For the monetary steady state, we have one negative stable eigenvalue and two initial conditions. Our calculations indicate that the dynamics along the center manifold are always stable in a neighborhood of this steady state. We conclude that equilibrium is determinate in this case. The unique equilibrium trajectory for given initial conditions approaches the steady state with oscillatory motion, but these oscillations may only dampen very slowly over time, due to the presence of -1 as an eigenvalue.

In a neighborhood of the nonmonetary steady state, we have two stable eigenvalues and two initial conditions. Our approximations of the center manifold dynamics indicated again that local dynamics are stable along the center manifold. We conclude that there is a one-dimensional indeterminacy of equilibria that approach the nonmonetary steady state, just as in the centralized analog of our economy. However, except for economies with special configurations of initial conditions, all equilibrium paths display oscillatory dynamics en route to the steady state. This is *not* true when trade is centralized.

We again conclude in favor of Friedman's (1960) argument that endogenous volatility naturally arises in an economy with circulating private liabilities. Generically, all equilibrium paths display fluctuations, even in the absence of other shocks to the economy. However, in contrast to Friedman's assertions, the co-existence of publicly and privately-issued liabilities is *not* a source of indeterminacy, at least near the monetary steady state. Indeterminacies do arise in a neighborhood of the nonmonetary steady state. However, this situation is no different than that in a conventional overlapping generations economy in which private liabilities do not circulate.

Prohibition of private circulating liabilities. Suppose that as some (for instance Friedman) have advocated, the use of private circulating liabilities is completely prohibited in this economy. Then this prevents borrowers and lenders from trading, so that borrowers cannot smooth their life-cycle consumption; their utility is driven to minus infinity. Lenders can still trade using outside money. In this case the demand for real balances in each location is $e/2$, while the supply of real balances is $M_h(t)/p_h(t)$. Thus, the equilibrium conditions of an economy with a prohibition on private circulating liabilities are

$$M_h(t)/p_h(t) = e/2 \tag{74}$$

for $h = 1, 2$, and $t \geq 0$.

In addition, since young lenders in location 1 (2) at t carry the outstanding stock of currency in their location to location 2 (1) at $t+1$, we have

$$M_1(t+1) = M_2(t) = M - M_1(t) \tag{75}$$

for $t \geq 0$.

Thus, unless $M_1(0) = M_2(0) = M/2$, a prohibition on circulating private liabilities does *not* eliminate price level volatility, although it does eliminate any indeterminacies. And, in contrast to the situation along dynamical equilibrium paths that approach a steady state in an economy with inside money, any volatility here does not dampen asymptotically.

As a historical matter, regional imbalances in the distribution of outside money were alleged to be a continuing problem—at least in U.S. monetary history—from the colonial era well into the 1830s or later. As equations (74) and (75) indicate, such imbalances would create economic volatility when private circulating liabilities are prohibited. This fact, along with the negative welfare consequences of such a prohibition for borrowers, makes it clear that both inside and outside money are required to attain efficient outcomes in the decentralized economy.

4. Conclusion

Because of recent technological and legal developments in the U.S., private banknote issue is feasible. This possibility is rekindling the debate about an old topic in monetary economics, namely, whether the provision of currency should be an activity left strictly to the government, or, at the other extreme, whether market provision of close money substitutes would possibly eliminate any need for government currency issue. While plausible arguments have been made on both sides of this issue, there are few formal frameworks sufficiently rich to analyze this topic and evaluate competing policy prescriptions. Our intent in this paper has been to propose such a formal framework. Our approach has been to use an environment with heterogeneous agents, spatial separation, and limited communication, in which potential borrowers and potential lenders meet only once in their lifetimes. These features of the environment force intertemporal trading to be intermediated in part by private liabilities issued by borrowers, which are taken to another location and exchanged for goods before they are brought back to the issuer for redemption.

Economies without outside money possess a unique steady state. If the economy's centralized analog is dynamically inefficient—that is, if the steady state interest rate is sufficiently low—then the equilibrium of the decentralized economy is locally indeterminate. In addition, all equilibria converge to the steady state through slowly damped oscillations in consumption and interest rates.[18] This provides some theoretical support for Friedman's (1960) contention that an economy with only private currency issuance would display endogenous (or "excess") volatility, and possibly indeterminacies. If the economy's centralized analog is dynamically efficient—the steady state interest rate is sufficiently high—then the equilibrium is either locally determinate, or, if incomes of borrowers and lenders are sufficiently dissimilar, the steady state is unapproachable from generic initial conditions.

In economies with both circulating private liabilities and government-issued fiat currency, two steady states exist.[19] In the monetary steady state, private liabilities circulate, and the agents willingly hold the outside money as well. This steady state is in fact Pareto optimal and equilibrium is determinate in its immediate neighborhood. Equilibrium paths, however, again approach the steady state via very slowly damped oscillations. We again interpret the nature of this equilibrium motion as consistent with Friedman's (1960) conjecture that systems with both private and public circulating liabilities are subject to endogenous volatility. There is also a nonmonetary steady state in this economy. It has agents who trade privately-issued liabilities, but not government-issued liabilities. There are hyperinflationary equilibrium paths that approach this steady state; that is, equilibrium paths in which money is used at all dates, but with the property that $\lim_{t\to\infty} p(t) = \infty$. Such equilibria are indeterminate. And, generically speaking, any equilibrium paths approaching the nonmonetary steady state display oscillatory motion.

Obviously our results strongly support the notion that the use of privately-issued, circulating liabilities is a formula for generating endogenous volatility. This is true whether or not outside money is present. Our results do *not* support the idea that the use of such liabilities is conducive to the indeterminacy of equilibrium. Finally, our analysis supports

[18] Convergence via damped oscillations also occurs in overlapping generations economies with no circulating liabilities and life cycles of three or more periods [Azariadis, Bullard, and Ohanian (1998)]. Non-monetary equilibria in these economies, however, are unique under standard gross substitutability assumptions.

We might also note that, if we allow our agents to consume in their third period of life (and, possibly, to have more general endowment patterns), equilibrium laws of motion are governed by dynamical systems of higher order than those displayed here.

[19] Under our maintained assumption for this portion of the paper that $e > w$.

neither the Hayek (1976) and Fama (1980) position that currency provision can safely be "left to the market," nor the Friedman (1960) proposal for regulating privately-issued circulating liabilities out of existence. As we already know from standard overlapping generations environments with (within-generation) heterogeneity, both private and public liabilities are generally needed if an efficient allocation of resources is to be attained in an economy of the Samuelson type. Of course in those environments private liabilities need not circulate; here we can discuss prohibitions on the issue of private liabilities that circulate in exactly the same way as currency.

There are, of course, many important questions regarding the private creation of circulating liabilities that our analysis has not addressed. One concerns the determination of rates of return, currency premia, and price levels in an economy where there is some risk of default attached to private liabilities. When such default risk exists, one can also consider the possibility of "fraudulent" creation of private currency. This possibility greatly concerned Friedman (1960). Finally, it would be interesting to consider the circulation of private liabilities whose redemption date is not known by the issuer. We hope to explore the generality of our results in future research incorporating these features.

References

[1] Azariadis, C. 1993. *Intertemporal Macroeconomics*. Oxford: Blackwell Publishing.

[2] Azariadis, C., J. Bullard, and L. Ohanian. 1998. "Complex Eigenvalues and Trend-Reverting Fluctuations." Working paper, Federal Reserve Bank of St. Louis.

[3] Balasko, Y., and K. Shell. 1981. "The Overlapping-Generations Model. II: The Case of Pure Exchange with Money." *Journal of Economic Theory* 24(1): 112-42.

[4] Benveniste, L. 1987. "Incomplete Market Participation and the Optimal Exchange of Credit." In E. C. Prescott and N. Wallace, eds., *Contractual Arrangements for Intertemporal Trade*, University of Minnesota Press, Minneapolis.

[5] Burdett, K., A. Trejos, and R. Wright. 2000. "Cigarette Money." Manuscript, University of Pennsylvania, Department of Economics.

[6] Cavalcanti, R., and N. Wallace. 1999. "Inside and Outside Money as Alternative Media of Exchange." *Journal of Money, Credit, and Banking* 31: 443-457.

[7] Champ, B., B. Smith, and S. Williamson. 1996. "Currency Elasticity and Banking Panics: Theory and Evidence." *Canadian Journal of Economics* 29: 828-864.

[8] Fama, E. 1980. "Banking in the Theory of Finance." *Journal of Monetary Economics* 6(1): 39-57.

[9] Fisher, Eric O'N. 1996. "On Exchange Rates and Efficiency." *Economic Theory* 7: 267-281.

[10] Friedman, M. 1960. *A Program for Monetary Stability.* New York: Fordham University Press.

[11] Gale, D. 1973. "Pure Exchange Equilibrium of Dynamic Economic Models." *Journal of Economic Theory* 6: 12-36.

[12] Gorton, G. 1989. "An Introduction to Van Court's Note Reporter and Counterfeit Detector." Manuscript, University of Pennsylvania, Wharton School.

[13] Hayek, F. 1976. *The Denationalization of Money.* London: Institute of Economic Affairs.

[14] Kiyotaki, N., and R. Wright. 1989. "On Money as a Medium of Exchange." *Journal of Political Economy* 97: 927-955.

[15] Kiyotaki, N., and R. Wright. 1993. "A Search Theoretic Approach to Monetary Economics." *American Economic Review* 83: 63-77.

[16] Mints, L. 1945. *A History of Banking Theory in Great Britain and the United States.* Chicago: University of Chicago Press.

[17] Rolnick, A., B. Smith, and W. Weber. 1996. "The United States' Experience with State Bank Notes." Manuscript, Federal Reserve Bank of Minneapolis.

[18] Samuelson, P. 1958. "An Exact Consumption-Loan Model of Interest With or Without the Social Contrivance of Money." *Journal of Political Economy* 66: 467-482.

[19] Sargent, T. 1987. *Dynamic Macroeconomic Theory*. Harvard University Press.

[20] Schreft, S. 1997. "Looking Forward: the Role for Government in Regulating Electronic Cash." *FRB-Kansas City Review*. Fourth Quarter, pp. 59-84.

[21] Smith, A. [1776]. 1937. *An Inquiry into the Wealth of Nations.* New York: Modern Library.

[22] Smith, B. 1994. "Mischief and Monetary History: Friedman and Schwartz Thirty Years Later." *Journal of Monetary Economics* 34(1): 27-45.

[23] Smith, B., and W. Weber. 1999. "Private Money Creation and the Suffolk Banking System." *Journal of Money, Credit, and Banking* 31: 624-659.

[24] Temin, P. 1969. *The Jacksonian Economy*. New York: Norton.

[25] Townsend, R. 1980. "Models of Money with Spatially Separated Agents." In J. Kareken and N. Wallace, eds., *Models of Monetary Economies*. Minneapolis: Federal Reserve Bank of Minneapolis.

[26] Townsend, R. 1987. "Economic Organization with Limited Communication." *American Economic Review* 77: 954-71.

[27] Townsend, R., and N. Wallace. 1982. "A Model of Circulating Private Debt." Manuscript, Federal Reserve Bank of Minneapolis.

[28] Townsend, R., and N. Wallace. 1987. "Circulating Private Debt: An Example with a Coordination Problem." In E. Prescott and N. Wallace, eds., *Contractual Arrangements for Intertemporal Trade*. Minneapolis: University of Minnesota Press.

[29] Wiggins, S. 1990. *Introduction to Applied Nonlinear Dynamical Systems and Chaos.* Springer-Verlag.

[30] Williamson, S. 1992. "Laissez-faire Banking and Circulating Media of Exchange." *Journal of Financial Intermediation* 2: 134-167.

[31] Williamson, S. 1999. "Private Money." *Journal of Money, Credit, and Banking* 31: 469-491.

A. PROOF OF PROPOSITION 1

To begin, we define the function

$$G(\pi) \equiv e^2(1+\pi) - 2ew\pi^2 - 4\gamma(1-\gamma)w^2\pi^2(1-\pi). \tag{76}$$

Then we have the following result.

Lemma 7. *π constitutes a solution to equation (22) iff π satisfies*

$$G(\pi) = 0. \tag{77}$$

Proof. Define the function $H(\pi)$ by

$$H(\pi) \equiv e^2(1+\pi)^2 - 2\pi^2(1+\pi)ew - 4\gamma(1-\gamma)\pi^2 w^2(1-\pi)(1+\pi). \tag{78}$$

Then it is easily verified that π satisfies (22) iff $H(\pi) = 0$. It is also easily verified that $H(\pi) = (1+\pi)G(\pi)$. This establishes the claim. ■

Lemma 7 implies that it suffices to find the roots of a cubic, as in equation (77).

Evidently it will be useful to know more about the properties of the function G. These are stated in the following lemma.

Lemma 8. *$G(\pi)$ is a continuous function satisfying the following conditions. Part (a): $lim_{\pi\to-\infty}G(\pi) = -\infty$. Part (b): $lim_{\pi\to\infty}G(\pi) = \infty$. Part (c): $G(0) = e^2 > 0$. Part (d): $G(1) = 2e(e-w)$. Part (e): Define π' to be the unique solution to*

$$\frac{\pi^2}{1+\pi} = \frac{e}{2\gamma w}. \tag{79}$$

Then $G(\pi') \leq 0$.

Proof. Parts (a)-(d) of the lemma are obvious, as is the continuity of G. For part (e), note that

$$G(\pi) \equiv e(1+\pi)\left[e - \frac{2\gamma w\pi^2}{1+\pi}\right] - 2(1-\gamma)w\pi^2\left[e - 2\gamma w(\pi-1)\right]. \tag{80}$$

Then

$$G(\pi') = 2(1-\gamma)\, w\pi^2 \left[2\gamma w (\pi' - 1) - e\right]. \tag{81}$$

Moreover, $2\gamma w(\pi'-1)-e \leq \frac{2\gamma w(\pi')^2}{1+\pi'} - e \equiv 0$ obviously holds. Thus $G(\pi') \leq 0$ as claimed.

■

Lemma 8 implies that equation (77) has exactly one solution in the interval $(0, \pi']$. Since $\pi \leq \pi'$ must hold in order for $R_{12} \geq 0$, only solutions to (77) lying in the interval $(0, \pi']$ are economically meaningful. Thus Proposition 1 is proved.

B. Proof of Proposition 2

From the definition of G we have

$$\begin{aligned} G\left(\frac{e}{w}\right) &= (e)^2 \left[1 + \left(\frac{e}{w}\right)\right] - 2ew\left(\frac{e}{w}\right)^2 \\ &\quad -4\gamma(1-\gamma)(w)^2\left(\frac{e}{w}\right)^2\left[1-\left(\frac{e}{w}\right)\right] \\ &= \left[1-4\gamma(1-\gamma)\right](e)^2\left[1-\left(\frac{e}{w}\right)\right]. \end{aligned} \tag{82}$$

If $\gamma = 1/2$ holds, clearly $G(e/w) = 0$. Moreover, it is easy to show that when $\gamma = 1/2$, $e/w < \pi'$. Thus, if $\gamma = 1/2$, $\pi^\star = e/w$, establishing part (a) of the proposition. If $\gamma > 1/2$, then $1 > 4\gamma(1-\gamma)$ holds. Hence, $G\left(\frac{e}{w}\right) > (<)\, 0$ iff $1 > (<) \left(\frac{e}{w}\right)$. In addition $e/w \leq \pi'$ holds iff [see equation (76)]

$$\begin{aligned} \left(\frac{e}{w}\right)^2 &\leq \left(\frac{e}{2\gamma w}\right)\left[1+\left(\frac{e}{w}\right)\right] \\ &= \left(\frac{e}{2\gamma w}\right) + \left(\frac{e}{w}\right)^2\left(\frac{1}{2\gamma}\right). \end{aligned} \tag{83}$$

It follows from these observations that, if $\gamma > 1/2$ and $w < e$, $G(e/w) < 0$ holds. In addition, part (d) of Lemma 8 implies that $G(1) > 0$. Finally, $e/w \leq \pi'$ necessarily holds. Hence $\pi^\star \in (1, e/w)$, establishing part (b). For part (c), if $\gamma > 1/2$ and $w > e$ we have $G(e/w) > 0$, while part (d) of Lemma 8 implies $G(1) < 0$. Thus $\pi^\star \in (e/w, \min\{1, \pi'\})$.

C. Proof of Proposition 3

First we state and prove three useful lemmas.

Lemma 9. $\pi' > \frac{e}{2\gamma w}$ *holds.*

Proof. The proof is by contradiction. Suppose to the contrary that $\pi' \leq \frac{e}{2\gamma w}$. Then equation (76) implies that

$$\frac{e^2}{4\gamma^2 w} \geq \left(\frac{e}{2\gamma w}\right)\left[1+\frac{e}{2\gamma w}\right]. \tag{84}$$

But this is obviously false, yielding the desired contradiction. ■

Lemma 10. *If* $\gamma > 1/2$, $G\left[\frac{e}{2\gamma w}\right] > 0$ *holds.*

Proof. Using the definition of G, we have

$$\begin{aligned} G\left(\frac{e}{2\gamma w}\right) &= (e)^2\left[1+\left(\frac{e}{2\gamma w}\right)\right] - 2we\left(\frac{e}{2\gamma w}\right)^2 \\ &\quad -4\gamma(1-\gamma)(w)^2\left(\frac{e}{2\gamma w}\right)^2\left[1-\left(\frac{e}{2\gamma w}\right)\right] \\ &= \frac{(2\gamma-1)(e)^2}{\gamma} > 0. \end{aligned}$$

This establishes the result. ■

Lemma 11. *If* $\gamma > 1/2$, $G\left[\frac{e}{2(1-\gamma)w}\right] < 0$ *holds.*

Proof. Using the definition of G we have

$$\begin{aligned} G\left[\frac{e}{2(1-\gamma)w}\right] &= (e)^2\left\{1+\left[\frac{e}{2(1-\gamma)w}\right]\right\} - 2we\left[\frac{e}{2(1-\gamma)w}\right]^2 \\ &\quad -4\gamma(1-\gamma)(w)^2\left[\frac{e}{2(1-\gamma)w}\right]^2\left\{1-\left[\frac{e}{2(1-\gamma)w}\right]\right\} \\ &= \frac{-(2\gamma-1)(e)^2}{(1-\gamma)} < 0 \end{aligned}$$

This establishes the result. ■

It is now possible to prove Proposition 3.

For part (a), it is immediate from (20) and (21) that $R_{12} = R_{21}$. In addition, using (20), it is easy to show that $R_{12} < R^\star$ holds in this case iff $R^\star < w/e$. For part (b), equations (19) and (20) imply that $1/\pi^\star = R^\star > R_{12}$ holds iff

$$e(1+\pi^\star) - 2w(\pi^\star)^2 > 2w(1-\gamma)\pi^\star(1-\pi^\star). \tag{85}$$

Equations (19) and (21) imply that $1/\pi^\star = R^\star > R_{21}$ holds iff

$$e(1+\pi^\star) - 2w\left(\pi^\star\right)^2 > 2w\gamma\pi^\star\left(1-\pi^\star\right). \tag{86}$$

Moreover, equation (77) and the definition of G imply that

$$e\left(1+\pi^\star\right) - 2w\left(\pi^\star\right)^2 = 4\gamma\left(1-\gamma\right)\left(\frac{w^2}{e}\right)\left(\pi^\star\right)^2\left(1-\pi^\star\right). \tag{87}$$

Thus the condition (85) is equivalent to

$$2\gamma\left(\frac{w}{e}\right)\left(1-\pi^\star\right)\left[\pi^\star - \left(\frac{e}{2\gamma w}\right)\right] \geq 0, \tag{88}$$

while condition (86) is equivalent to

$$2(1-\gamma)\left(\frac{w}{e}\right)\left(1-\pi^\star\right)\left[\pi^\star - \frac{e}{2\left(1-\gamma\right)w}\right] \geq 0. \tag{89}$$

In addition, Lemmas 10 and 11 establish that

$$\frac{e}{2\gamma w} < \pi^\star < \frac{e}{2\left(1-\gamma\right)w}. \tag{90}$$

Part (b) then follows, establishing the proposition.

D. The dynamical system with no outside money

From equation (14) we have

$$R_{1,2}\left(t\right) = \frac{2\left(1-\gamma\right)x_2\left(t-1\right)}{e - \left[\frac{2\gamma x_1(t)}{R_{1,2}(t)R_{2,1}(t+1)}\right]}. \tag{91}$$

Similarly, equation (15) can be rearranged as

$$R_{2,1}\left(t\right) = \frac{2\gamma x_1\left(t-1\right)}{e - \left[\frac{2(1-\gamma)x_2(t)}{R_{2,1}(t)R_{1,2}(t+1)}\right]}. \tag{92}$$

Moreover, equations (10), (16), and (17) imply that

$$\frac{1}{R_{1,2}\left(t\right)R_{2,1}\left(t+1\right)} = \frac{\left[w_2 - x_1\left(t-1\right)\right]\left[w_2 - x_1\left(t-2\right)\right]}{x_1\left(t\right)x_1\left(t-1\right)} \tag{93}$$

and

$$\frac{1}{R_{2,1}\left(t\right)R_{1,2}\left(t+1\right)} = \frac{\left[w_2 - x_2\left(t-1\right)\right]\left[w_2 - x_2\left(t-2\right)\right]}{x_2\left(t\right)x_2\left(t-1\right)}. \tag{94}$$

Substituting (93) into (91) yields

$$\frac{1}{R_{1,2}(t)} = \frac{ex_1(t-1) - 2\gamma[w - x_1(t-1)][w - x_1(t-2)]}{2(1-\gamma)x_1(t-1)\,x_2(t-1)}. \tag{95}$$

And substituting (94) into (92) yields

$$\frac{1}{R_{2,1}(t)} = \frac{ex_2(t-1) - 2(1-\gamma)[w - x_2(t-1)][w - x_2(t-2)]}{2\gamma x_1(t-1)\,x_2(t-1)}. \tag{96}$$

In addition, solving the location 1 goods market clearing condition (16) for $R_{2,1}(t-1)$, leading the result one period, and using equation (13) to eliminate $R_{1,1}(t+1)$, we obtain

$$R_{2,1}(t) = \frac{ex_1(t) - 2\gamma w[w - x_1(t) - x_1(t-1)]}{ex_1(t)}. \tag{97}$$

Applying the same sequence of steps to the location 2 goods market clearing condition (17) yields

$$R_{1,2}(t) = \frac{ex_2(t) - 2(1-\gamma)w[w - x_2(t) - x_2(t-1)]}{ex_2(t)}. \tag{98}$$

We now proceed as follows. Equations (95) and (98) imply that

$$\frac{ex_1(t)}{ex_1(t) - 2\gamma w[w - x_1(t) - x_1(t-1)]} = \frac{ex_2(t-1) - 2(1-\gamma)[w - x_2(t-1)][w - x_2(t-2)]}{2\gamma x_1(t-1)\,x_2(t-1)}. \tag{99}$$

Using the definition of the function $f[x_1(t-1), x_2(t-1), x_2(t-2)]$, equation (99) can be rearranged to yield

$$\begin{aligned} x_1(t) &= \frac{2\gamma w[w - x_1(t-1)]\,f[x_1(t-1), x_2(t-1), x_2(t-2)]}{(e + 2\gamma w)f[x_1(t-1), x_2(t-1), x_2(t-2)] - e} \\ &\equiv Q[x_1(t-1), x_2(t-1), x_2(t-2)]. \end{aligned} \tag{100}$$

Similarly, equations (95) and (96) imply that

$$\frac{ex_2(t)}{ex_2(t) - 2(1-\gamma)w(w - x_2(t) - x_2(t-1))} = \frac{ex_1(t-1) - 2\gamma(w - x_1(t-1))(w - x_1(t-2))}{2(1-\gamma)x_1(t-1)\,x_2(t-1)}. \tag{101}$$

Using the definition of the function $g[x_1(t-1), x_2(t-1), x_1(t-2)]$, we may rewrite (101) as

$$\begin{aligned} x_2(t) &= \frac{2(1-\gamma)w[w - x_2(t-1)]\,g[x_1(t-1), x_2(t-1), x_1(t-2)]}{[e + 2(1-\gamma)w]\,g[x_1(t-1), x_2(t-1), x_1(t-2)] - e} \\ &\equiv Z[x_1(t-1), x_2(t-1), x_1(t-2)]. \end{aligned} \tag{102}$$

Equations (100) and (102) constitute a system of two second order difference equations that govern the evolution of the equilibrium sequence $\{x_1(t), x_2(t)\}$. This system reduces readily to equations (32)-(35) given in the text.

E. PROOF OF PROPOSITION 4

We begin by stating several lemmas that will be needed for the proof of the proposition.

Lemma 12. *In a steady state equilibrium,*

$$4\gamma(1-\gamma)(w-x)^2 x^2 = \left[ex - 2\gamma(w-x)^2\right]\left[ex - 2(1-\gamma)(w-x)^2\right] \tag{103}$$

Proof. *Equations (95) and (96) imply that*

$$\frac{1}{R_{1,2}(t)\,R_{2,1}(t+1)} = \tag{104}$$

$$\frac{\left[ex_1(t-1) - 2\gamma(w-x_1(t-1))(w-x_1(t-2))\right]\left[ex_2(t) - 2(1-\gamma)(w-x_2(t))(w-x_2(t-1))\right]}{4\gamma(1-\gamma)\,x_1(t)\,x_2(t)\,x_1(t-1)\,x_2(t-1)}.$$

Equation (97), along with (104), implies that

$$(w-x_1(t-1))(w-x_1(t-2)) = \tag{105}$$

$$\frac{\left[ex_1(t-1) - 2\gamma(w-x_1(t-1))(w-x_1(t-2))\right]\left[ex_2(t) - 2(1-\gamma)(w-x_2(t))(w-x_2(t-1))\right]}{4\gamma(1-\gamma)\,x_2(t)\,x_2(t-1)}$$

which holds $\forall t \geq 1$. Imposing $x_1(t) = x_1(t-1) = x_1(t-2) = x_2(t) = x_2(t-1) = x_2(t-2) = x$ in (105) and rearranging terms we obtain (103). This establishes the lemma. ■

Lemma 13. *The partial derivatives of $f(\cdot)$ and $g(\cdot)$, evaluated at a steady state, satisfy*

Part (i)

$$\frac{xf_1(x,x,x)}{f(x,x,x)} = -1 \tag{106}$$

Part (ii)

$$\frac{xg_2(x,x,x)}{g(x,x,x)} = -1 \tag{107}$$

Part (iii)

$$\frac{xf_2(x,x,x)}{f(x,x,x)} = \frac{2(1-\gamma)(w-x)w}{ex - 2(1-\gamma)(w-x)^2} \tag{108}$$

Part (iv)

$$\frac{xg_1(x,x,x)}{g(x,x,x)} = \frac{2\gamma w(w-x)}{ex - 2\gamma(w-x)^2} \tag{109}$$

Part (v)

$$\frac{xf_3(x,x,x)}{f(x,x,x)} = \frac{2(1-\gamma)(w-x)x}{ex - 2(1-\gamma)(w-x)^2} \tag{110}$$

Part (vi)

$$\frac{xg_3(x,x,x)}{g(x,x,x)} = \frac{2\gamma(w-x)x}{ex - 2\gamma(w-x)^2} \tag{111}$$

Part (vii)

$$\begin{aligned}\left[\frac{xf_2(x,x,x)}{f(x,x,x)}\right]\left[\frac{xg_1(x,x,x)}{g(x,x,x)}\right] &= \left(\frac{w}{x}\right)^2 \\ &= (1+\pi^\star)^2\end{aligned} \tag{112}$$

Part (viii)

$$\begin{aligned}\left[\frac{xf_2(x,x,x)}{f(x,x,x)}\right]\left[\frac{xg_3(x,x,x)}{g(x,x,x)}\right] &= 1+\pi^\star \\ &= \left[\frac{xf_3(x,x,x)}{f(x,x,x)}\right]\left[\frac{xg_1(x,x,x)}{g(x,x,x)}\right]\end{aligned} \tag{113}$$

Part (ix)

$$\left[\frac{xf_3(x,x,x)}{f(x,x,x)}\right]\left[\frac{xg_3(x,x,x)}{g(x,x,x)}\right] = 1. \tag{114}$$

Proof. *Parts (i) and (ii) of the lemma follow from (128) and (131) below. For Part (iii), differentiation of (28) implies that, at a steady state*

$$\frac{xf_2(x,x,x)}{f(x,x,x)} = \frac{2(1-\gamma)(w-x)w}{ex - 2(1-\gamma)(w-x)^2}. \tag{115}$$

For Part (iv) differentiation of (102) implies that, at a steady state,

$$\frac{xg_1(x,x,x)}{g(x,x,x)} = \frac{2\gamma w(w-x)}{ex - 2\gamma(w-x)^2}. \tag{116}$$

Parts (v) and (vi) of the lemma follow similarly. For Part (vii), note that at a steady state

$$\left[\frac{xf_2(\cdot)}{f(\cdot)}\right]\left[\frac{xg_1(\cdot)}{g(\cdot)}\right] = \frac{4\gamma(1-\gamma)(w-x)^2(w)^2}{\left[ex - 2\gamma(w-x)^2\right]\left[ex - 2(1-\gamma)(w-x)^2\right]}. \tag{117}$$

It follows from Lemma 12 that $[xf_2(\cdot)/f(\cdot)][xg_1(\cdot)/g(\cdot)] = (w/x)^2$. *Moreover,* $x = w/(1+\pi^\star)$. *Using these facts establishes Part (vii). For Part (viii), we have*

$$\left[\frac{xf_2(\cdot)}{f(\cdot)}\right]\left[\frac{xg_3(\cdot)}{g(\cdot)}\right] = \frac{4\gamma(1-\gamma)(w-x)^2 wx}{\left[ex-2(1-\gamma)(w-x)^2\right]\left[ex-2\gamma(w-x)^2\right]} = \frac{w}{x} = 1+\pi^\star, \tag{118}$$

where the second equality follows from (103). Similarly, we have that

$$\left[\frac{xf_3(x,x,x)}{f(x,x,x)}\right]\left[\frac{xg_1(x,x,x)}{g(x,x,x)}\right] = \frac{4\gamma(1-\gamma)(w-x)^2 wx}{\left[ex-2(1-\gamma)(w-x)^2\right]\left[ex-2\gamma(w-x)^2\right]} = \frac{w}{x} = 1+\pi^\star \tag{119}$$

Finally, for part (ix) of the lemma, we have that

$$\left[\frac{xf_3(\cdot)}{f(\cdot)}\right]\left[\frac{xg_3(\cdot)}{g(\cdot)}\right] = \frac{4\gamma(1-\gamma)(w-x)^2(x)^2}{\left[ex-2\gamma(w-x)^2\right]\left[ex-2(1-\gamma)(w-x)^2\right]} = 1, \tag{120}$$

where the last equality follows from (103). This completes the proof of the lemma. ■

Lemma 14. *Part (a):*

$$Q_1(x,x,x) = -\left(\frac{x}{w-x}\right)+\left[\frac{e}{(e+2\gamma w)f(x,x,x)-e}\right] = -\left(\frac{1}{\pi^\star}\right)+\left[\frac{e}{(e+2\gamma w)f(\cdot)-e}\right]. \tag{121}$$

Part (b):

$$Z_2(x,x,x) = -\left(\frac{x}{w-x}\right)+\left[\frac{e}{[e+2(1-\gamma)w]g(x,x,x)-e}\right] = -\left(\frac{1}{\pi^\star}\right)+\left[\frac{e}{[e+2(1-\gamma)w]g(\cdot)-e}\right]. \tag{122}$$

Part (c):

$$Q_2(x,x,x) = -\left[\frac{xf_2(x,x,x)}{f(x,x,x)}\right]\left[\frac{e}{(e+2\gamma w)f(x,x,x)-e}\right]. \tag{123}$$

Part (d):

$$Z_1(x,x,x) = -\left[\frac{xg_1(x,x,x)}{g(x,x,x)}\right]\left[\frac{e}{(e+2(1-\gamma)w)g(x,x,x)-e}\right]. \tag{124}$$

Part (e):

$$Q_3(x,x,x) = -\left[\frac{xf_3(x,x,x)}{f(x,x,x)}\right]\left[\frac{e}{(e+2\gamma w) f(x,x,x) - e}\right]. \tag{125}$$

Part (f):

$$Z_3(x,x,x) = -\left[\frac{xg_3(x,x,x)}{g(x,x,x)}\right]\left[\frac{e}{(e+2(1-\gamma) w) g(x,x,x) - e}\right]. \tag{126}$$

Proof. *Part (a). The definition of $Q(\cdot)$ in equation (100) implies that*

$$\frac{x_1(t-1) Q_1[x_1(t-1), x_2(t-1), x_2(t-2)]}{Q[x_1(t-1), x_2(t-1), x_2(t-2)]} =$$

$$\begin{aligned} & -\left(\frac{x_1(t-1)}{w - x_1(t-1)}\right) + \left[\frac{x_1(t-1) f_1[x_1(t-1), x_2(t-1), x_2(t-2)]}{f(\cdot)}\right] \\ & -\left[\frac{x_1(t-1) f_1(\cdot)}{f(\cdot)}\right]\left[\frac{(e+2\gamma w) f(\cdot)}{(e+2\gamma w) f(\cdot) - e}\right] \\ = & -\left(\frac{x_1(t-1)}{w - x_1(t-1)}\right) - \left[\frac{x_1(t-1) f_1(\cdot)}{f(\cdot)}\right]\left[\frac{e}{(e+2\gamma w) f(\cdot) - e}\right]. \end{aligned} \tag{127}$$

Moreover, equation (28) implies that

$$\frac{x_1(t-1) f_1[x_1(t-1), x_2(t-1), x_2(t-2)]}{f[x_1(t-1), x_2(t-1), x_2(t-2)]} = -1. \tag{128}$$

Using (127) and (128), imposing $x_1(t) = Q[x_1(t-1), x_2(t-1), x_2(t-2)] = x_1(t-1) = x_2(t-1) = x_2(t-2) = x$, and rearranging terms yields the first equality in Part (a). The second equality follows from the fact that

$$\pi^\star = \frac{w-x}{x}. \tag{129}$$

Part (b). The definition of $Z(\cdot)$ in equation (102) implies that

$$\frac{x_2(t-1) Z_2[x_1(t-1), x_2(t-1), x_1(t-2)]}{Z[x_1(t-1), x_2(t-1), x_1(t-2)]} =$$

$$-\left(\frac{x_2(t-1)}{w - x_2(t-1)}\right) + \left\{\frac{x_2(t-1) g_2[x_1(t-1), x_2(t-1), x_1(t-2)]}{g[x_1(t-1), x_2(t-1), x_1(t-2)]}\right\} \times \left\{1 - \left[\frac{[e+2(1-\gamma) w] g(\cdot)}{[e+2(1-\gamma) w] g(\cdot) - e}\right]\right\}. \tag{130}$$

Moreover, equation (30) implies that

$$\frac{x_2(t-1) g_2(\cdot)}{g(\cdot)} = -1. \tag{131}$$

Substituting (131) into (130), imposing $x_2(t) = Z[x_1(t-1), x_2(t-1), x_1(t-2)] = x_2(t-1) = x_1(t-1) = x_1(t-2) = x$, *and rearranging terms yields the first equality in Part (b). The second follows from (129).*

Parts (c) and (d). The definition of $Q(\cdot)$ *in equation (100) implies that*

$$\frac{x_2(t-1)\,Q_2[x_1(t-1), x_2(t-1), x_2(t-2)]}{Q[x_1(t-1), x_2(t-1), x_2(t-2)]} =$$

$$\left[\frac{x_2(t-1)\,f_2[x_1(t-1), x_2(t-1), x_2(t-2)]}{f[x_1(t-1), x_2(t-1), x_2(t-2)]}\right]\left\{1 - \left[\frac{(e+2\gamma w)\,f_2(\cdot)}{(e+2\gamma w)\,f(\cdot) - e}\right]\right\} \tag{132}$$

$$= -\left[\frac{x_2(t-1)\,f_2(\cdot)}{f(\cdot)}\right]\left[\frac{e}{(e+2\gamma w)\,f(\cdot) - e}\right].$$

In a steady state, equation (132) reduces to

$$Q_2(x,x,x) = -\left[\frac{x f_2(x,x,x)}{f(x,x,x)}\right]\left\{\frac{e}{(e+2\gamma w)\,f(\cdot) - e}\right\}. \tag{133}$$

By the same token, the definition of $Z(\cdot)$ *in equation (102) implies that*

$$\frac{x_1(t-1)\,Z_1[x_1(t-1), x_2(t-1), x_1(t-2)]}{Z[x_1(t-1), x_2(t-1), x_1(t-2)]} =$$

$$\left[\frac{x_1(t-1)\,g_1[x_1(t-1), x_2(t-1), x_1(t-2)]}{g[x_1(t-1), x_2(t-1), x_1(t-2)]}\right]\left\{1 - \left[\frac{[e+2(1-\gamma)w]\,g(\cdot)}{[e+2(1-\gamma)w]\,g(\cdot) - e}\right]\right\}. \tag{134}$$

In steady state this expression reduces to that in Part (d) of the lemma.

Parts (e) and (f). From the definition of $Q(\cdot)$ *in equation (100) we have*

$$\frac{x_2(t-2)\,Q_3[x_1(t-1), x_2(t-1), x_2(t-2)]}{Q[x_1(t-1), x_2(t-1), x_2(t-2)]} =$$

$$\left[\frac{x_2(t-2)\,f_3[x_1(t-1), x_2(t-1), x_2(t-2)]}{f[x_1(t-1), x_2(t-1), x_2(t-2)]}\right]\left\{1 - \left[\frac{(e+2\gamma w)\,f(\cdot)}{(e+2\gamma w)\,f(\cdot) - e}\right]\right\}. \tag{135}$$

In steady state, this reduces to the expression in Part (e) of the lemma. And, similarly, the definition of $Z(\cdot)$ *in equation (102) implies that*

$$\frac{x_1(t-2)\,Z_3[x_1(t-1), x_2(t-1), x_1(t-2)]}{Z[x_1(t-1), x_2(t-1), x_1(t-2)]} =$$

$$\left[\frac{x_1(t-2)\,g_3[x_1(t-1), x_2(t-1), x_1(t-2)]}{g[x_1(t-1), x_2(t-1), x_1(t-2)]}\right]\left\{1 - \left[\frac{[e+2(1-\gamma)w]\,g(\cdot)}{[e+2(1-\gamma)w]\,g(\cdot) - e}\right]\right\}. \tag{136}$$

It follows that, in steady state, the expression in Part (f) of the lemma obtains. ■

We next establish some results about the polynomial $P(\lambda)$.

Lemma 15. *The polynomial $P(\lambda)$ satisfies (a) $\lim_{\lambda\to-\infty} P(\lambda) = \infty$, (b) $\lim_{\lambda\to\infty} P(\lambda) = \infty$, and (c) $P(0) = -Q_3(x,x,x)\,Z_3(x,x,x) < 0$.*

Proof. Parts (a) and (b) are obvious. Part (c) follows from the definition of $P(\lambda)$, along with Parts (e) and (f) of Lemma 14, and Part (ix) of Lemma 13. ■

Lemma 16. *The characteristic polynomial $P(\lambda)$ has the following representation:*

$$P(\lambda) = \lambda^2(\lambda - Q_1)(\lambda - Z_2) - \left(Z_2 + \frac{1}{\pi^\star}\right)\left(Q_1 + \frac{1}{\pi^\star}\right)[\lambda(1+\pi^\star)+1]^2. \tag{137}$$

Proof. It is straightforward to verify that

$$P(\lambda) = \lambda^2(\lambda - Q_1)(\lambda - Z_2) - Q_2 Z_1\left[\lambda + \left(\frac{Q_3}{Q_2}\right)\right]\left[\lambda + \left(\frac{Z_3}{Z_1}\right)\right]. \tag{138}$$

In addition, Lemmas 13 and 14 imply that

$$\frac{Q_3}{Q_2} = \frac{Z_3}{Z_1} = \frac{x}{w} = \frac{1}{(1+\pi^\star)}. \tag{139}$$

Finally, Lemmas 13 and 14 also imply that

$$Q_2 Z_1 = (1+\pi^\star)^2\left(Z_2 + \frac{1}{\pi^\star}\right)\left(Q_1 + \frac{1}{\pi^\star}\right). \tag{140}$$

Then (137) follows from (138)-(140). ■

Lemma 17. *If $\gamma < 1$, we have*

$$\frac{-1}{1+\pi^\star} < Q_1 < 0 < Z_2. \tag{141}$$

Proof. *We have already shown that the first inequality in (141) obtains. The inequality $Q_1 < 0$ holds iff*

$$e(1+\pi^\star) > (e + 2\gamma w)\,f(x,x,x). \tag{142}$$

Equation (142) is easily shown to be equivalent to

$$\begin{aligned} 4\gamma(1-\gamma)(w)^2(\pi^\star)^2 &< (e)^2(1+\pi^\star) - 2(1-\gamma)\,ew(\pi^\star)^2 \\ &= 2\gamma ew(\pi^\star)^2 + 4\gamma(1-\gamma)(w)^2(\pi^\star)^2(1-\pi^\star), \end{aligned} \tag{143}$$

where the second equality follows from the definition of $G(\pi^\star)$ *along with* $G(\pi^\star) = 0$. *Finally, equation (143) reduces to*

$$\pi^\star < \frac{e}{2(1-\gamma)ew} \tag{144}$$

which has already been shown [see (90)]. Thus $Q_1 < 0$. *To demonstrate that* $Z_2 > 0$, *note that* $Z_2 > 0$ *holds iff*

$$\begin{aligned} e(1+\pi^\star) &> [e + 2(1-\gamma)w]\, g(x,x,x) \\ &= \tfrac{(e)^2(1+\pi^\star)}{2(1-\gamma)w} - \tfrac{\gamma}{(1-\gamma)} e(\pi^\star)^2 + e(1+\pi^\star) - 2\gamma w(\pi^\star)^2 . \end{aligned} \tag{145}$$

Rearranging terms in (145), we have that $Z_2 > 0$ *iff*

$$\begin{aligned} 4\gamma(1-\gamma)(w)^2(\pi^\star)^2 &> (e)^2(1+\pi^\star) - 2\gamma ew(\pi^\star)^2 \\ &= 2(1-\gamma)ew(\pi^\star)^2 + 4\gamma(1-\gamma)(w)^2(\pi^\star)^2(1-\pi^\star), \end{aligned} \tag{146}$$

where, as before, the second equality follows from the definition of $G(\pi^\star)$ *and* $G(\pi^\star) = 0$. *Rearranging terms in (146), we have that* $Z_2 > 0$ *iff*

$$\pi^\star > \frac{e}{2\gamma w}, \tag{147}$$

which, again, has been shown to hold [see (90)]. ■

With these lemmas in hand, we are now prepared to provide a proof of Proposition 4. Clearly

$$P(Z_2) < 0 < \lim_{\lambda\to\infty} P(\lambda). \tag{148}$$

It follows that J has an eigenvalue in the interval (Z_2, ∞). It is also easy to verify that $P(-1/\pi^\star) = 0$ which establishes Part (b) of the proposition. In order to establish Part (c), it suffices to show that

$$P\left[\frac{-1}{(1+\pi^\star)}\right] \geq 0 \tag{149}$$

holds. Part (c) then follows from the fact that $P(0) < 0$. It is clear from (137) that (149) holds if

$$0 \geq -\left[\frac{1}{(1+\pi^\star)}\right] - Q_1 \tag{150}$$

and

$$0 \geq -\left[\frac{1}{(1+\pi^{\star})}\right] - Z_2 \tag{151}$$

are both satisfied. Condition (150) is readily shown to be equivalent to

$$e\pi^{\star}(1+\pi^{\star}) \geq (e+2\gamma w) f(x,x,x) - e, \tag{152}$$

and condition (151) is equivalent to

$$e\pi^{\star}(1+\pi^{\star}) \geq (e+2(1-\gamma)w) g(x,x,x) - e. \tag{153}$$

It is straightforward but tedious to show that

$$\begin{aligned}(e+2\gamma w) f(x,x,x) &= \frac{(e)^2(1+\pi^{\star})}{2\gamma w} - \frac{(1-\gamma)}{\gamma} e(\pi^{\star})^2 + \\ &\quad e(1+\pi^{\star}) - 2(1-\gamma) w (\pi^{\star})^2\end{aligned} \tag{154}$$

and that

$$\begin{aligned}[e+2(1-\gamma)w] g(x,x,x) &= \frac{(e)^2(1+\pi^{\star})}{2(1-\gamma)w} - \frac{\gamma}{(1-\gamma)} e(\pi^{\star})^2 + \\ &\quad e(1+\pi^{\star}) - 2\gamma w(\pi^{\star})^2.\end{aligned} \tag{155}$$

Substituting (154) into (152), it follows that (150) holds if

$$4\gamma(1-\gamma)(w)^2(\pi^{\star})^2 \geq (e)^2(1+\pi^{\star}) - 2ew(\pi^{\star})^2. \tag{156}$$

But satisfaction of (156) is guaranteed by the definition of $G(\pi^{\star})$ and the fact that $G(\pi^{\star}) = 0$. (Indeed, (156) must hold as a strict inequality if $\gamma < 1$). Similarly, substituting (155) into (153), it follows that (151) is satisfied if (156) holds. But this establishes that (149) is satisfied (and that it holds as a strict inequality if $\gamma < 1$.) This completes the proof of Part (c) of the proposition.

To establish Part (e) of the proposition, we proceed as follows. Suppose, for the purpose of deriving a contradiction, that (40) holds, and so does $P(1) \geq 0$. For all $\lambda \geq Q_1$, we have

$$P(\lambda) < \lambda^2(\lambda - Q_1)^2 - \left(Z_2 + \frac{1}{\pi^{\star}}\right)\left(Q_1 + \frac{1}{\pi^{\star}}\right)[\lambda(1+\pi^{\star})+1]^2. \tag{157}$$

Then, since $P(1) \geq 0$, it follows that

$$1 \geq \left(Z_2 + \frac{1}{\pi^\star}\right)\left(Q_1 + \frac{1}{\pi^\star}\right)(1+\pi^\star)^2 \left[\frac{1+\left(\frac{1}{(1+\pi^\star)}\right)}{1-Q_1}\right]^2 \tag{158}$$

necessarily holds. Moreover, Lemma 17 implies that

$$\frac{\left[1+\left(\frac{1}{(1+\pi^\star)}\right)\right]}{(1-Q_1)} \geq 1. \tag{159}$$

We then have our desired contradiction if

$$\left(Q_1 + \frac{1}{\pi^\star}\right)(1+\pi^\star) \geq 1, \tag{160}$$

since $\left(Z_2 + \frac{1}{\pi^\star}\right) > \left(Q_1 + \frac{1}{\pi^\star}\right)$ holds. Moreover, equation (160) is readily shown to be equivalent to

$$e(1+\pi^\star) \geq (e + 2\gamma w) f(x,x,x) - e \tag{161}$$

or to

$$e \geq \frac{(e)^2(1+\pi^\star)}{2\gamma w} - \left(\frac{1-\gamma}{\gamma}\right) e (\pi^\star)^2 - 2(1-\gamma) w (\pi^\star)^2. \tag{162}$$

Rearranging terms in (160) yields condition (40). Thus (40) implies that inequality (160) holds, yielding the desired contradiction. Thus $P(1) < 0$ when (40) holds. Part (e) of the proposition then follows from $\lim_{\lambda\to\infty} P(\lambda) = \infty$.

To establish Part (f) of the proposition, recall that

$$\frac{e}{2\gamma w} < \pi^\star < \frac{e}{2(1-\gamma)w}. \tag{163}$$

It follows that (40) holds if $\pi^\star \leq 1$, or if $\pi^\star > 1$ and

$$\frac{e}{2\gamma w} \geq \left[\frac{e}{2(1-\gamma)w}\right] \left\{ \frac{\left[\frac{e}{2(1-\gamma)w}\right]^2 - 1}{\left[\frac{e}{2(1-\gamma)w}\right]^2} \right\}. \tag{164}$$

Rearranging terms in (164) yields condition (41) in Part (f) of the proposition.

It remains to establish part (d) of the proposition. To do so, note that

$$\begin{aligned} P(-1) &= (1+Q_1)(1+Z_2) - (\pi^\star)^2\left(Z_2+\frac{1}{\pi^\star}\right)\left(Q_1+\frac{1}{\pi^\star}\right) \\ &= (1+Q_1)(1+Z_2) - (1+\pi^\star Q_1)(1+\pi^\star Z_2) \\ &= (1-\pi^\star)\left[Q_1 + Z_2 + (1+\pi^\star) Q_1 Z_2\right]. \end{aligned} \tag{165}$$

We now claim that

$$Q_1 + Z_2 + (1 + \pi^\star) Q_1 Z_2 = 0 \tag{166}$$

always holds. Part (d) of the proposition then follows.

To demonstrate that (166) is necessarily satisfied, rewrite it as

$$-\frac{1}{Z_2} - \frac{1}{Q_1} = (1 + \pi^\star). \tag{167}$$

It follows from Lemma 14 that equation (167) is equivalent to the condition

$$\begin{aligned} \frac{1+\pi^\star}{\pi^\star} &= \frac{[e + 2(1-\gamma) w] g(x,x,x) - e}{[e + 2(1-\gamma) w] g(x,x,x) - e(1+\pi^\star)} + \\ & \quad \frac{(e + 2\gamma w) f(x,x,x) - e}{(e + 2\gamma w) f(x,x,x) - e(1+\pi^\star)}. \end{aligned} \tag{168}$$

Using equations (154) and (155) in (168), we have that (166) holds iff

$$\begin{aligned} \frac{1+\pi^\star}{\pi^\star} &= 2 + \frac{2\gamma e w \pi^\star}{2\gamma e w (\pi^\star)^2 - 4\gamma(1-\gamma) w^2 (\pi^\star)^3} + \\ & \quad \frac{2(1-\gamma) e w \pi^\star}{2(1-\gamma) e w (\pi^\star)^2 - 4\gamma (1-\gamma) w^2 (\pi^\star)^3}, \end{aligned} \tag{169}$$

where we have used the definition of $G(\pi)$ to obtain (169). Rearranging terms in (169), it follows that $P(-1) = 0$ holds if

$$1 - \pi^\star = \left[1 - 2(1-\gamma)\left(\frac{w}{e}\right)\pi^\star\right]^{-1} + \left[1 - 2\gamma\left(\frac{w}{e}\right)\pi^\star\right]^{-1}. \tag{170}$$

Moreover, it is straightforward to verify that (170) is equivalent to the condition

$$e^2 (1+\pi^\star) - 2ew(\pi^\star)^2 - 4\gamma(1-\gamma) w^2 (\pi^\star)^2 (1-\pi^\star) = 0. \tag{171}$$

But (171) necessarily holds, as is clear from the definition of G and the fact that $G(\pi^\star) = 0$. Thus $P(-1) = 0$, completing the proof of Part (d).

F. Dynamics along the center manifold

In this appendix we describe our calculations to determine the nature of motion along a center manifold. We illustrate these calculations by studying the dynamic system described by equations (61)-(64) in more detail in a neighborhood of the monetary steady state. However, the same procedure can be applied to the system without money, or the system with money but at the nonmonetary steady state. In all of these cases, our linear approximation of equilibrium dynamics in a neighborhood of the steady state left us with an eigenvalue equal to -1. The main idea is then straightforward: In order to deduce the nature of motion along the center manifold near the steady state, we need to construct a second order Taylor's series approximation to the original dynamical system. We let $\delta = \frac{e}{2\gamma w}$ and $\xi = \frac{e}{2(1-\gamma)w}$. Then the approximated system can be written as

$$\begin{pmatrix} R_{11}(t+1) - R_{11} \\ R_{22}(t+1) - R_{22} \\ R_{21}(t+1) - R_{21} \\ R_{12}(t+1) - R_{12} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \xi \\ 1 & 0 & \delta & -1 \\ 0 & 1 & -1 & \xi \end{pmatrix} \begin{pmatrix} R_{11}(t) - R_{11} \\ R_{22}(t) - R_{22} \\ R_{21}(t) - R_{21} \\ R_{12}(t) - R_{12} \end{pmatrix} + S \tag{172}$$

where the steady state interest rates are given by $R_{11} = R_{22} = R_{21} = R_{12} = 1$, and where S is a 4×1 vector of second order terms defined by

$$S = \begin{pmatrix} \delta^2 \left[R_{21}(t) - 1\right]^2 \\ \xi^2 \left[R_{12}(t) - 1\right]^2 \\ \begin{array}{c} \left[R_{12}(t) - 1\right]^2 + \delta^2 \left[R_{21}(t) - 1\right]^2 - \left[R_{11}(t) - 1\right]\left[R_{12}(t) - 1\right] - \\ \delta \left[R_{21}(t) - 1\right]\left[R_{12}(t) - 1\right] + \delta \left[R_{11}(t) - 1\right]\left[R_{21}(t) - 1\right] \end{array} \\ \begin{array}{c} \left[R_{21}(t) - 1\right]^2 + \xi^2 \left[R_{12}(t) - 1\right]^2 - \left[R_{22}(t) - 1\right]\left[R_{21}(t) - 1\right] - \\ \xi \left[R_{21}(t) - 1\right]\left[R_{12}(t) - 1\right] + \xi \left[R_{22}(t) - 1\right]\left[R_{12}(t) - 1\right] \end{array} \end{pmatrix} \tag{173}$$

This same vector of second order terms applies for the system with money, but with all terms evaluated at the nonmonetary steady state. However, for the system without money (equations (32)-(35) in the text), the vector is considerably more complicated, and we do not display it here.

We label the eigenvalues of the Jacobian matrix as $\lambda_c = -1$, $-1 < \lambda_s < 0$, and $\lambda_{u_1}, \lambda_{u_2} > 1$ for the monetary steady state (and analogously for the nonmonetary steady state or the steady state of the system without money). We have complicated analytic expressions for these eigenvalues, as functions of γ, e, and w, which we do not display here.

The Jordan decomposition of the Jacobian matrix J can be written as

$$J = T \begin{pmatrix} \lambda_c & 0 & 0 & 0 \\ 0 & \lambda_s & 0 & 0 \\ 0 & 0 & \lambda_{u_1} & 0 \\ 0 & 0 & 0 & \lambda_{u_2} \end{pmatrix} T^{-1} \tag{174}$$

where the columns of the matrix T are the eigenvectors associated with the system (68). The matrix T^{-1} is not available analytically. In order to write equations outside of matrix notation, we will label the elements of T and of T^{-1} as follows:

$$T = \begin{pmatrix} t_{11} & t_{12} & t_{13} & t_{14} \\ t_{21} & t_{22} & t_{23} & t_{24} \\ t_{31} & t_{32} & t_{33} & t_{34} \\ t_{41} & t_{42} & t_{43} & t_{44} \end{pmatrix} \text{ and } T^{-1} = \begin{pmatrix} i_{11} & i_{12} & i_{13} & i_{14} \\ i_{21} & i_{22} & i_{23} & i_{24} \\ i_{31} & i_{32} & i_{33} & i_{34} \\ i_{41} & i_{42} & i_{43} & i_{44} \end{pmatrix}. \tag{175}$$

We now create a new coordinate system according to

$$\begin{pmatrix} u \\ v \\ x \\ y \end{pmatrix} = T^{-1} \begin{pmatrix} R_{11} - 1 \\ R_{22} - 1 \\ R_{21} - 1 \\ R_{12} - 1 \end{pmatrix}. \tag{176}$$

This implies

$$R_{11} = 1 + t_{11}u + t_{12}v + t_{13}x + t_{14}y, \tag{177}$$

$$R_{22} = 1 + t_{21}u + t_{22}v + t_{23}x + t_{24}y, \tag{178}$$

$$R_{21} = 1 + t_{31}u + t_{32}v + t_{33}x + t_{34}y, \tag{179}$$

and

$$R_{12} = 1 + t_{41}u + t_{42}v + t_{43}x + t_{44}y. \tag{180}$$

We will use the shorthand

$$z_1 = t_{11}u + t_{12}v + t_{13}x + t_{14}y, \tag{181}$$

with z_2, z_3, and z_4 defined similarly. By substituting appropriately into S, we find that

$$S = \begin{pmatrix} \delta^2 z_3^2 \\ \xi^2 z_4^2 \\ z_4^2 + \delta^2 z_3^2 - z_1 z_4 - \delta z_3 z_4 + \delta z_1 z_3 \\ z_3^2 + \xi^2 z_4^2 - z_2 z_3 - \xi z_3 z_4 + \xi z_2 z_4 \end{pmatrix}. \tag{182}$$

In this notation, we have to keep in mind that the $z's$ are functions of u, v, x, and y. We let $N = T^{-1}S$, and we label the four rows of N as n_1, n_2, n_3, and n_4. The map is now

$$\begin{pmatrix} u \\ v \\ x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} \lambda_c & 0 & 0 & 0 \\ 0 & \lambda_s & 0 & 0 \\ 0 & 0 & \lambda_{u_1} & 0 \\ 0 & 0 & 0 & \lambda_{u_2} \end{pmatrix} \begin{pmatrix} u \\ v \\ x \\ y \end{pmatrix} + N(u, v, x, y). \tag{183}$$

We now wish to find an approximation to the center manifold. With our transformed coordinates, the steady state takes the form $(u, v, x, y) = (0, 0, 0, 0)$, and the center manifold can be locally represented by a graph as follows:

$$W^c(0) = \left\{ (u, v, x, y) \in \mathbb{R}^4 \;\middle|\; \begin{array}{c} v = h_1(u), x = h_2(u), y = h_3(u), \\ h_i(0) = 0, Dh_i(0) = 0, i = 1, 2, 3. \end{array} \right\} \tag{184}$$

for u sufficiently small. We work with equation (2.1.56) in Wiggins (1990, p. 206). The center manifold obeys

$$h(Au + f(u, h(u)) - Bh(u) - g(u, h(u)) = 0 \tag{185}$$

where

$$\begin{aligned} h &= (h_1, h_2, h_3), \qquad (186) \\ f(u, v, x, y) &= n_1, \\ g(u, v, x, y) &= \begin{pmatrix} n_2 \\ n_3 \\ n_4 \end{pmatrix}, \\ A &= -1 = \lambda_c, \end{aligned}$$

$$B = \begin{pmatrix} \lambda_s & 0 & 0 \\ 0 & \lambda_{u_1} & 0 \\ 0 & 0 & \lambda_{u_2} \end{pmatrix}.$$

Following Wiggins, we assume a center manifold of the form

$$h(u) = \begin{pmatrix} h_1(u) \\ h_2(u) \\ h_3(u) \end{pmatrix} = \begin{pmatrix} a_1u^2 + b_1u^3 + \mathcal{O}(u^4) \\ a_2u^2 + b_2u^3 + \mathcal{O}(u^4) \\ a_3u^2 + b_3u^3 + \mathcal{O}(u^4) \end{pmatrix}. \tag{187}$$

We would like to rewrite equation (185) in light of this approximation. We do this in parts. First, consider the portion of equation (185) $h(Au+f(u,h(u)))$. Using the definitions above to substitute into this expression, we find that

$$h(Au + f(u,h(u))) = \begin{pmatrix} a_1u^2 + (2a_1c - b_1)u^3 + \mathcal{O}(u^4) \\ a_2u^2 + (2a_2c - b_2)u^3 + \mathcal{O}(u^4) \\ a_3u^2 + (2a_3c - b_3)u^3 + \mathcal{O}(u^4) \end{pmatrix}. \tag{188}$$

where

$$\begin{aligned} c \;=\; & i_{14}t_{21}t_{31} - \delta i_{13}t_{11}t_{31} - \delta^2 i_{11}\,(t_{31})^2 - \delta^2 i_{13}\,(t_{31})^2 \\ & - i_{14}\,(t_{31})^2 + i_{13}t_{11}t_{41} - i_{14}\xi t_{21}t_{41} + \delta i_{13}t_{31}t_{41} \\ & + i_{14}\xi t_{31}t_{41} - i_{13}\,(t_{41})^2 - i_{12}\xi^2\,(t_{41})^2 - i_{14}\xi^2\,(t_{41})^2 . \end{aligned} \tag{189}$$

Thus equation (185) can be written as

$$\begin{aligned} & \begin{pmatrix} a_1u^2 + (2a_1c - b_1)u^3 + \mathcal{O}(u^4) \\ a_2u^2 + (2a_2c - b_2)u^3 + \mathcal{O}(u^4) \\ a_3u^2 + (2a_3c - b_3)u^3 + \mathcal{O}(u^4) \end{pmatrix} - \\ & \begin{pmatrix} \lambda_s & 0 & 0 \\ 0 & \lambda_{u_1} & 0 \\ 0 & 0 & \lambda_{u_2} \end{pmatrix} \begin{pmatrix} a_1u^2 + b_1u^3 + \mathcal{O}(u^4) \\ a_2u^2 + b_2u^3 + \mathcal{O}(u^4) \\ a_3u^2 + b_3u^3 + \mathcal{O}(u^4) \end{pmatrix} = \begin{pmatrix} n_2 \\ n_3 \\ n_4 \end{pmatrix} \end{aligned} \tag{190}$$

By expanding the term on the right hand side and equating coefficients, we can create a system of six equations in six unknowns, a_1, a_2, a_3, b_1, b_2, and b_3. These equations have a unique solution which we label $a_1^\star$, $a_2^\star$, $a_3^\star$, $b_1^\star$, $b_2^\star$, $b_3^\star$.

We now have all the pieces to approximate the center manifold. The map restricted to the center manifold can be written as follows:

$$u(t+1) = -u(t) + n_1 \tag{191}$$

where $n_1 = i_{11}s_1 + i_{12}s_2 + i_{13}s_3 + i_{14}s_4$; $s_1 = \delta^2 z_3^2$, $s_2 = \xi^2 z_4^2$, $s_3 = z_4^2 + \delta^2 z_3^2 - z_1 z_4 - \delta z_3 z_4 + \delta z_1 z_3$, $s_4 = z_3^2 + \xi^2 z_4^2 - z_2 z_3 - \xi z_3 z_4 + \xi z_2 z_4$; $z_1 = t_{11}u + t_{12}v + t_{13}x + t_{14}y$, $z_2 = t_{21}u + t_{22}v + t_{23}x + t_{24}y$, $z_3 = t_{31}u + t_{32}v + t_{33}x + t_{34}y$, $z_4 = t_{11}u + t_{12}v + t_{13}x + t_{14}y$, and $v = h_1(u) = a_1^\star u^2 + b_1^\star u^3$, $x = h_2(u) = a_2^\star u^2 + b_2^\star u^3$, $y = h_3(u) = a_3^\star u^2 + b_3^\star u^3$. In this expression, the i's, t's, a's, and b's, as well as δ and ξ, are all functions of the underlying primitive parameters of the model, γ, e, and w. Because we do not have analytic expressions for the elements of the T^{-1} matrix, we have to examine the motion on the center manifold numerically.

The results of this calculation at the monetary steady state of the system with money, as well as at the nonmonetary steady state and the steady state of the system without money, are reported in the text.

G. Elements of the Jacobian matrix

In either steady state we have

$$\frac{\partial R_{1,1}(t+1)}{\partial R_{1,1}(t)} = \frac{\partial R_{1,1}(t+1)}{\partial R_{2,2}(t)} = \frac{\partial R_{1,1}(t+1)}{\partial R_{1,2}(t)} = 0 \tag{192}$$

and

$$\frac{\partial R_{1,1}(t+1)}{\partial R_{2,1}(t)} = \frac{eR_{11}}{(e + 2\gamma w - eR_{21})}. \tag{193}$$

In addition,

$$\frac{\partial R_{2,2}(t+1)}{\partial R_{2,2}(t)} = \frac{\partial R_{2,2}(t+1)}{\partial R_{1,1}(t)} = \frac{\partial R_{2,2}(t+1)}{\partial R_{2,1}(t)} = 0 \tag{194}$$

and

$$\frac{\partial R_{2,2}(t+1)}{\partial R_{1,2}(t)} = \frac{eR_{22}}{[e + 2(1-\gamma)w - eR_{12}]}. \tag{195}$$

Also,

$$\frac{\partial R_{1,2}(t+1)}{\partial R_{1,1}(t)} = 0, \tag{196}$$

$$\frac{\partial R_{1,2}(t+1)}{\partial R_{2,2}(t)} = \frac{R_{22}}{R_{21}}, \tag{197}$$

$$\frac{\partial R_{1,2}(t+1)}{\partial R_{1,2}(t)} = \frac{eR_{12}}{[e + 2(1-\gamma)w - eR_{12}]}, \tag{198}$$

and

$$\frac{\partial R_{1,2}(t+1)}{\partial R_{2,1}(t)} = \frac{-R_{12}}{R_{21}}. \tag{199}$$

Finally,

$$\frac{\partial R_{2,1}(t+1)}{\partial R_{1,1}(t)} = \frac{R_{11}}{R_{12}}, \tag{200}$$

$$\frac{\partial R_{2,1}(t+1)}{\partial R_{2,2}(t)} = 0, \tag{201}$$

$$\frac{\partial R_{2,1}(t+1)}{\partial R_{1,2}(t)} = \frac{-R_{21}}{R_{12}}, \tag{202}$$

and

$$\frac{\partial R_{2,1}(t+1)}{\partial R_{2,1}(t)} = \frac{eR_{21}}{e + 2\gamma w - eR_{21}}. \tag{203}$$

H. Proof of Proposition 5

We begin by noting that, at a monetary steady state, $R_{11} = R_{22} = R_{21} = R_{12} = 1$. In addition, it is straightforward to verify that the characteristic polynomial of J is given by

$$\begin{aligned} P(\lambda) \quad = \quad & \lambda^4 - \left\{\frac{e}{2\gamma(1-\gamma)w}\right\}\lambda^3 + \\ & \left\{\frac{e^2}{4\gamma(1-\gamma)w^2} - 1 - \frac{e}{2\gamma(1-\gamma)w}\right\}\lambda^2 + \\ & \left\{\frac{e^2}{2\gamma(1-\gamma)w^2}\right\}\lambda + \frac{e^2}{4\gamma(1-\gamma)w^2}. \end{aligned} \tag{204}$$

It is then immediate that -1 is an eigenvalue of J. To establish the remainder of the proposition, observe that

$$H(-1) = -2 - \frac{e}{2\gamma(1-\gamma)w} < 0 \tag{205}$$

and

$$H(0) = \frac{e^2}{4\gamma(1-\gamma)w^2} > 0. \tag{206}$$

Thus, there is an eigenvalue in the interval $(-1, 0)$. Moreover, a comparison of the polynomials G and H will indicate that $H(\pi^\star) = 0$, so that $\lambda = \pi^\star > 1$ is an eigenvalue of J. Finally, as above, define π' to be the unique solution to

$$\frac{\pi^2}{1+\pi} = \frac{e}{2\gamma w}. \tag{207}$$

Then $\pi' > \pi^\star$ and $H(\pi') \leq 0$ hold. (See Lemma 8). The existence of an eigenvalue in the interval $[\pi', \infty)$ then follows from

$$\lim_{\lambda \to \infty} H(\lambda) = \infty. \tag{208}$$

I. Proof of Proposition 6

We begin by proving a useful lemma.

Lemma 18. *If $w/e < 1$, then $\psi(1) < 0$.*

Proof. To prove the lemma, observe that

$$\psi(1) = \frac{e(R_{11})^4}{2\gamma(1-\gamma)w^2} - \left(\frac{eR_{11}}{2w}\right)\left(\frac{(1-\gamma)R_{21} + \gamma R_{12}}{\gamma(1-\gamma)}\right). \tag{209}$$

Therefore $\psi(1) < 0$ holds iff

$$\frac{e(R_{11})^3}{w} < (1-\gamma)R_{21} + \gamma R_{12}. \tag{210}$$

It follows from (61), (62), and the definition of $\pi^\star$ that

$$(1-\gamma)R_{21} + \gamma R_{12} = \left[\frac{e(1+\pi^\star)}{w(\pi^\star)^2}\right] - 1. \tag{211}$$

Therefore, $\psi(1) < 0$ is satisfied iff

$$\left(\frac{w}{e}\right)(\pi^\star)^3 < (\pi^\star)^2 + \pi^\star - 1. \tag{212}$$

However, $w/e < 1$ is satisfied. And, in addition, $\pi^\star > 1$ implies that $(\pi^\star)^3 < (\pi^\star)^2 + \pi^\star - 1$ holds. Therefore, (212) is satisfied, and $\psi(1) < 0$ holds. ■

We can now prove the proposition.

At a nonmonetary steady state, the following relations obtain:

$$\frac{eR_{12}}{e + 2(1-\gamma)w - eR_{12}} = \frac{eR_{11}R_{12}}{2(1-\gamma)w}, \tag{213}$$

$$\frac{eR_{11}}{e + 2\gamma w - eR_{21}} = \frac{e(R_{11})^2}{2\gamma w}, \tag{214}$$

$$\frac{eR_{22}}{e + 2(1-\gamma)w - eR_{12}} = \frac{e(R_{11})^2}{2(1-\gamma)w}, \tag{215}$$

and

$$\frac{eR_{21}}{e+2\gamma w-eR_{21}}=\frac{eR_{11}R_{21}}{2\gamma w}. \tag{216}$$

Using (213)-(216), it is straightforward to verify that the characteristic polynomial of J is given by

$$\begin{aligned}\rho(\lambda) \quad = \quad & \lambda^4-\lambda^3\left\{\frac{eR_{11}R_{21}}{2\gamma w}+\frac{eR_{11}R_{12}}{2(1-\gamma)w}\right\}- \\ & \lambda^2\left\{1+\frac{eR_{11}R_{21}}{2\gamma w}+\frac{eR_{11}R_{12}}{2(1-\gamma)w}-\frac{e(R_{11})^4}{4\gamma(1-\gamma)w}\right\}+ \\ & \lambda\left[\frac{e^2(R_{11})^4}{2\gamma(1-\gamma)w^2}\right]+ \\ & \frac{e^2(R_{11})^4}{4\gamma(1-\gamma)w^2}.\end{aligned} \tag{217}$$

It is then immediate that -1 is an eigenvalue of J. For the remainder of the proposition, observe that $\psi(-1)<0<\psi(0)$ holds, so that there is an eigenvalue in the interval $(-1,0)$. In addition, we have

$$\lim_{\lambda\to\infty}\psi(\lambda)=\infty \tag{218}$$

and $\psi(1)<0$ from Lemma 18. These last two observations, along with $\psi(0)>0$, imply that J has an eigenvalue in the interval $(0,1)$, and an eigenvalue in the interval $(1,\infty)$.