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Trend-Reverting Fluctuations in the Life-Cycle Model.

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Trend-Reverting Fluctuations in the Life-Cycle Model

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Abstract. Aggregate time series provide evidence of short term dynamic adjustment that appears to be governed by complex or negative real eigenvalues. This finding is at odds with the predictions of reasonably parameterized, convex one-sector growth models with complete markets. We study life cycle economies in which aggregate saving depends non-trivially on the distribution of wealth among cohorts. If consumption goods are weak gross substitutes near the steady state price vector, we prove that the unique equilibrium of a life cycle exchange economy converges to the unique non-monetary steady state via damped oscillations. We also discuss examples and extensions. *Journal of Economic Literature* Classification Numbers: E0, E3.

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1. Introduction

Persistence and trend reversion are two widely-documented properties of the United States output growth rate. Reduced-form representations of time series data suggest, with a fair degree of confidence, that real per capita GDP and related measures of economic activity undergo damped oscillations in response to temporary external shocks, as documented for instance by Taylor [32, 33], Nelson and Plosser [22], Blanchard and Quah [7], Cogley and Nason [14], and Rotemberg and Woodford [25]. This pattern appears in both the autocorrelation and impulse-response functions of simple GDP autoregressions as well as in vector autoregressions that include investment, employment, interest rates and other variables.

Trend-reverting motion poses a serious challenge for standard, neoclassical economic theory. The damped oscillations documented in the empirical macroeconomics literature suggest that complex or negative real eigenvalues may be governing the short-term dynamics of U.S. output and other aggregate variables. But all one-sector, complete-market models of convex growth that we know of—keeping parameter values within empirically plausible ranges—are incapable of reversing their equilibrium motion because all their stable eigenvalues are positive and real. Vector autoregressions, in short, turn out to imply adjustments to transitory disturbances which differ strikingly from the predictions of all widely-used growth models.

One way out of this quandary is to complicate or discard some of the basic assumptions underlying convex, neoclassical models of one-sector growth with complete markets. For example, negative stable eigenvalues occur in one-sector overlapping generations models of endogenous fluctuations if we assume certain types of large income effects, as in Grandmont [16], highly elastic labor supply as in Reichlin [23], or low elasticities of capital-labor substitution as in Benhabib and Laroque [6]. Boldrin and Montrucchio [9] demonstrate that a multisector optimum growth model has a negative stable eigenvalue when the rate of time preference is sufficiently high. Complex eigenvalues conversely appear in representative agent models with large nonconvexities due to monopolistic competition, as in Gali [15], or technological increasing returns to scale, as in Benhabib and Farmer [5], and also in non-classical growth models with investment, gestation or production lags, such as Samuelson [26].

Another way out is to take seriously market incompleteness, especially frictions in financial markets which would compromise the ability of these markets to smooth external shocks. Endogenous fluctuations emerge naturally as non-monetary equilibria in both closed and open economies and also as monetary equilibria, as shown in Azariadis and Smith [2], Boyd and Smith [10, 11], and Cass and Shell [13]. Potential sources of endogenous fluctuations also include activist monetary policies that target interest rates, inflation rates, or exchange rates, as in Bencivenga, Huybens and Smith [4], Smith [30, 31], Shell [27], Sims [29], or Woodford [34].

Simple growth models of complete markets do not do justice to trend reversion because they do not seem to be able to predict negative or complex eigenvalues for empirically plausible choices of taste and technology parameters. Overlapping generations models with a two-period lifecycle typically need an activist monetary authority, or else elasticities of substitution in consumption or production far below one in order to generate endogenous fluctuations. Multisector optimum growth economies experience cyclical and chaotic dynamics at rates of time preference corresponding to annual interest rates of 100% or more, according to Boldrin [8]. And estimates of returns to scale in U.S. production due to Basu and Fernald [3] put in question the scale effects required to extract complex eigenvalues from nonconvex economies with a representative household.

It is hard to resist the conclusion that economic theory has not yet come up with a simple and compelling explanation for output trend reversion. What mechanisms allow an economy with complete markets to convert a temporary external shock into a self-reversing motion? In this paper, we propose to help answer this question by taking a careful look at the dynamics of a relatively unfamiliar class of one-sector growth models—overlapping generations economies with finite lifecycles of three or more periods. This set of economies is understudied relative to lifecycle models with lifespans $L = 2$ or $L = \infty$ whose dynamic behavior is straightforward and typically monotone. One reason economic theory has fixed attention on these simpler lifecycle structures is the expectation of gaining some insight into more complicated and more realistic economies with $L = 55$ years or $L = 220$ quarters.

This hope turns out to be partly incorrect: The dynamic behavior of economies with finite $L > 2$ is qualitatively different from the cases $L = 2$ or $L = \infty$. As we will show, economies with finite $L \geq 3$ have two properties that are unique among growth models: (1) equilibrium dynamics depend nontrivially on the intergenerational distribution of wealth, and (2) our calculations show that many of the stable eigenvalues turn out to be complex (or real and negative). These facts suggest that overlapping generations models with realistic lifecycles and standard parameter configurations may improve our ability to duplicate the short-run business cycle movements of output and other macroeconomic time series.

We proceed as follows. In Section 2 we provide evidence on the eigenvalues of the simplest autoregressive and vector autoregressive representations of quarterly aggregate time series. We show that complex and/or negative characteristic roots are a fairly robust feature of the data. In Section 3 we prove a theorem that rules out monotone convergence to the unique steady state for endowment lifecycle economies with and without money, and certain classes of production economies, whenever consumption goods are weak gross substitutes at price vectors close to the steady state price vector. We then turn to providing some intuition for the ubiquity of negative or complex eigenvalues in overlapping generations economies with nontrivial lifecycles. Accordingly, in Section 4 we provide examples of overlapping generations economies, with and without production, for lifecycles of three periods as well as 55 periods, in which there are complex or negative real eigenvalues. A summary and a list of possible extensions make up the final section.

2. Eigenvalues in autoregressive aggregate time series models

2.1. Overview. There is a large literature that analyzes the time series properties of U.S. GDP and other aggregate variables, as for example in Sims [28] and subsequent research. A widely-documented finding is that most variables are not well characterized as first-order linear univariate processes. Instead, the data indicate a richer form of autocorrelation than is consistent with a first-order autoregressive (AR(1)) model, and thus raise the possibility of complex or negative eigenvalues in empirical reduced forms. While the autocorrelation properties of macroeconomic time series have been analyzed in detail in the literature, less is known about the corresponding eigenvalues in these reduced form models.

In this section, we study the eigenvalues in estimated AR models of GDP, and VAR models of GDP and other aggregate time series. We focus on the likelihood that there are complex or negative eigenvalues in the AR and VAR representations of aggregate time series that are routinely used in empirical macroeconomics. We start by estimating simple univariate autoregressions for real GDP and report the eigenvalues based on the point estimates from the autoregression. To assess sampling uncertainty in the eigenvalues, we use a nonparametric bootstrap technique to build up an empirical distribution. Following the univariate analysis, we consider VARs between real GDP and other variables. We use postwar annual U.S. data, ranging from 1948 to 1999. The data include real GDP, consumption, fixed investment, and the interest rate on three-month U.S. Treasury Bills.

2.2. Eigenvalues in univariate autoregressions of GDP. We first consider univariate models of GDP:

$$A(L)Y_t = \varepsilon_t \tag{1}$$

where Y is the natural log of real GDP and A is a vector of coefficients in the lag operator L. Since there is no consensus view on how to decompose economic time series into trend and cyclical components, we use three different approaches in modeling trends that have been used in the literature: (1) linear time trend, (2) linear and quadratic time trends, and (3) the Hodrick-Prescott (HP) filter. For each model of the underlying trend, the cyclical component (y_t) is defined as the difference between the raw data (Y_t) and the estimated trend component $\left(\hat{Y}_t\right)$:

$$y_t \equiv Y_t - \hat{Y}_t. \tag{2}$$

Once the cyclical component of the data has been extracted, it is necessary to choose the lag order in the autoregressions. Since several researchers have reported that AR(1) models are rejected in favor of higher-order models, we begin by estimating an AR(2) and an AR(3) model of the cyclical component of real GDP. We estimate the coefficients using OLS. The two eigenvalues (λ_1 and λ_2) based on the estimated AR(2) coefficients are presented in Table 1, and the three eigenvalues based on the estimated AR(3) coefficients are presented in Table 2.

[TABLE 1 ABOUT HERE]

For the AR(2) process, we find that both eigenvalues are real and positive for the two deterministic trends, and complex for the HP detrending. For the AR(3) process, we find that all eigenvalues are either negative or complex for the quadratic and HP trends, while one eigenvalue is negative for linear detrending.

Since the eigenvalues are functions of the estimated autoregressive coefficients, they are subject to sampling uncertainty. To assess this uncertainty, we use a non-parametric bootstrap procedure to construct an empirical distribution of the eigenvalues. This involves the following steps. First, we take the residuals from the fitted equations, and shuffle their position using randomly generated numbers from a uniform density. Second, we construct pseudo-data, $\{\tilde{y}_t\}_{t=1}^{T}$, using the reordered innovations and the originally estimated autoregressive parameters. We then re-estimate the autoregressive parameters from the pseudo-data, and use those new parameters to calculate new eigenvalues. By repeating this resampling procedure many times, one can construct a histogram of the empirical eigenvalues.

[TABLE 2 ABOUT HERE.]

Based on 500 replications, we found that the eigenvalues were real and positive for the linear and quadratic trend cases for the AR(2) specification in 78 percent, and 43 percent of the bootstrap replications, respectively. For the HP filtered data, however, eigenvalues were complex in 97 percent of the trials. For the AR(3) specification, negative and/or complex eigenvalues were a common feature across all three trend specifications. For the linear trend, 98 percent of the trials had at least one real negative eigenvalue or one complex conjugate pair of eigenvalues. For the quadratic trend, 96 percent of the trials had complex eigenvalues, and for the HP trend, 97 percent of the trials had at least one real negative eigenvalue, and one complex conjugate pair of eigenvalues. To summarize, we found that for univariate representations, complex or negative eigenvalues are a robust feature if HP detrending is used, and are robust for the other detrending methods for AR(3) or higher order processes.[1]

[TABLE 3 ABOUT HERE.]

[1]We also analyzed an AR(4) and an AR(5) process, and the results were very similar to the AR(3) in terms of the characteristics of the eigenvalues.

2.3. Eigenvalues in VAR representations of GDP. We next analyze eigenvalues in VARs using two variable systems with one lag. The cyclical components of the variables are extracted using the same procedure as in the univariate analysis. The variables in the model are GDP with either (1) consumption, (2) investment, or (3) the three-month Treasury bill rate. The eigenvalues based on the VAR coefficients are presented in Tables 3, 4, and 5. The two variables systems are denoted as Y/C, for the output-consumption system, Y/I, for the output-investment system, and Y/R, for the output-interest rate system. We use one lag in each of the VARs.

[TABLE 4 ABOUT HERE.]

Tables 3, 4, and 5 present the eigenvalues for the three VARs. These results show that the signs of the eigenvalues depends on both the specification of the VARs and the detrending method. We find real and positive eigenvalues for the Y/C VAR across all detrending procedures. For the Y/I VAR, we find real and positive eigenvalues for the two deterministic trends, but complex eigenvalues for the HP trend. For the Y/R VAR, we find complex eigenvalues across all detrending specifications. This last finding is of considerable interest, given that it has become very common to specify output-interest rate VARs.[2]

As in the univariate analysis, we used the nonparametric bootstrap to assess sampling uncertainty of the eigenvalues from the VARs. Table 6 summarizes these results by presenting the fraction of draws in which there were either complex or negative eigenvalues in the three systems we studied using the three different detrending procedures. We performed 500 draws in the bootstrap analysis. The table shows that negative or complex roots occurred quite frequently in the bootstrap analysis for HP detrending, and for the Y/R VAR. In particular, there were complex or negative roots in the output-interest rate VAR between 76 and 96 percent of the bootstrap trials. The analysis presented in this section thus suggests that some commonly-used reduced-form models of major macroeconomic time series, detrended with con-

[2] Since the model economy we develop below has only real interest rates, we have also conducted this VAR experiment with ex-post real interest rates (using the GDP deflator). The results concerning the frequency of negative/complex eigenvalues are similar to those presented here with nominal rates. We focus on the nominal rate specification, however, given that it is used almost exclusively in the empirical literature.

ventional procedures, produce characteristic roots that are frequently negative or complex-valued.

[TABLE 5 ABOUT HERE.]

3. CONVERGENCE THEOREMS FOR LIFECYCLE ECONOMIES

3.1. Overview. To understand the facts summarized in the previous section, we now turn to an analysis of a fairly large class of standard lifecycle economies, with and without outside assets like fiat money. In every economy we consider there is one consumption good, the trend rate of growth is exactly zero, the lifecycle is $L \geq 3$ periods, and agents are identical within each generation.

[TABLE 6 ABOUT HERE.]

If we denote total household assets held at time t by $A(t)$, then the equilibrium condition for endowment economies

$$A(t) = 0, \tag{3}$$

when government liabilities are unvalued or nonexistent, and

$$A(t) - M(t) = 0 \tag{4}$$

when government liabilities have real value $M(t)$.

Each economy contains L asset-trading generations, from the youngest at age $i = 0$, to the oldest at age $i = L - 1$. Trading plans at time t are conditional on the $2L - 1$ dimensional price vector $p^t = (p_{t-L+1}, ..., p_t, ..., p_{t+L-1})$ of accounting prices for dated goods. Without loss of generality, we set $p_1 = 1$ which means that p_t is the price of date-t goods in terms of date-1 goods. Then Walras' law and the zero-degree homogeneity of excess demands mean that trading plans depend on just $2L - 3$ price ratios. Hence the equilibrium conditions equations (3) and (4) are nonlinear difference equations of order $2L - 4$ and $2L - 3$ respectively. These are supplemented by $L - 2$ initial conditions in the endowment economy describing the pre-existing asset and liability positions of households in the initial period $t = 0$.

3.2. Convergence in exchange economies with and without money. In a large class of endowment lifecycle economies we have briefly described (and which are described in more detail below), gross substitutes turns out to be inconsistent with monotone convergence. Specifically, if all pairs of dated consumption goods are weak gross substitutes in the neighborhood of the steady state price vector, then equilibria near either a non-monetary or a monetary steady state exhibit damped oscillations for all exchange economies with a lifecycle $L \geq 3$. Our main results, Theorems 4 and 5 below, extend the findings of Kehoe, Levine, Mas-Colell and Woodford [20] who show that every endowment economy in this class admits no limit cycles; it has instead a unique real steady state, and a unique equilibrium sequence converging to that steady state.

More relevant for business cycle analysis is that local uniqueness holds under very weak assumptions. Kehoe, *et al.*, [20] prove that the equilibrium sequence is unique near the steady state if dated consumption goods are gross substitutes at all price ratios close to the steady state. Uniqueness means that the number of stable eigenvalues exactly equals the number of initial conditions describing the distribution of wealth among generations at the beginning of time. This convergence result means that lifecycle economies eventually dissipate external impulses, and cannot deviate from their steady state for any prolonged period of time unless they are periodically shocked from outside. Kehoe, *et al.*, [20] however, do not examine whether convergence is monotone or oscillatory.

Formal results come from studying a double-ended deterministic exchange economy with finite lifecycles of length $L = T+1 \geq 3$, constant population, and one agent in each cohort. Our economy is very similar to the one studied in Kehoe, *et al.*, [20]. We denote time by $t = 0, 1, ...$ and cohorts by $\upsilon = -T-1, -T, ..., 0, 1, ...$ Each normal cohort $\upsilon \geq 0$ has a stationary, non-negative endowment vector $e^{\nu} = \left\{e^{\nu}_{\nu+i}\right\}_{i=0}^{T} = \{e_i\}_{i=0}^{T}$, a consumption vector $c^{\nu} = \{c_{\nu}(\nu+i)\}_{i=0}^{T}$, and an additive utility function $U^{\nu} = \sum_{i=0}^{L-1} \beta^i u\left[c_{\nu}(\nu+i)\right]$, $\beta > 0$, satisfying standard smoothness, monotonicity and convexity properties.

The price vector relevant to the non-transitional cohort ν is $p^{\nu} = \{p_t\}_{t=\nu}^{\nu+T}$ which implies price ratios or interest rates $\{R_t\}_{t=\nu}^{\nu+T}$ with $R_t = p_t/p_{t+1}$. Maximizing utility subject to the budget constraint $p^{\nu}(c^{\nu} - e^{\nu}) \leq 0$ leads to asset demand schedules of the form

$$\begin{aligned}\alpha_\nu(t) &= \sum_{i=\nu}^{t}\left(\frac{p_i}{p_t}\right)\left[e_{i-\nu}-c_\nu\left(i\right)\right], &(5)\\ &\equiv z\left(p^\nu/p_t;t-\nu\right). &(6)\end{aligned}$$

These schedules satisfy standard accounting and budget constraints, that is,

$$\alpha_\nu(t) = R_{t-1}\alpha_\nu\left(t-1\right)+e_{t-\nu}-c_\nu(t), \tag{7}$$

where $t=\nu+1,...,\nu+T-1$, and

$$\alpha_\nu(\nu-1)=0=\alpha_\nu\left(\nu+T\right). \tag{8}$$

For each stage of the lifecycle $t-\nu=0,...,T$, the asset demands are homogeneous of degree zero in the vector p^ν, depending only on the vector $p^\nu/p_t=(p_\nu/p_t,...,p_{\nu+T}/p_t)$ for any cohort ν and any time period $t=\nu,...,\nu+T$. Since price ratios are products of interest factors, we may rewrite real asset demands as functions of these interest factors, that is,

$$\alpha_\nu(t)=z^\star\left(R_\nu,...,R_{\nu+T-1};t-\nu\right). \tag{9}$$

Gross substitutability implies that each asset demand schedule is monotone in the vector of interest factors. Specifically:

Lemma 1. *(Monotonicity.) If all pairs of dated consumption goods are weak gross substitutes for cohort ν, then $\alpha_\nu(t)$ is an increasing function of the vector $(R_\nu$, ..., $R_{\nu+T-1})$ for each $t=\nu,...,\nu+T-1$.*

Proof. We show that asset demand $\alpha_\nu(t)$ by cohort ν at time $t=\nu,...,\nu+T$ is increasing in the interest factor R_t. To do this we ask how $\alpha_\nu(t)$ reacts when we replace the price vector $(p^\nu)=(p_\nu,...,p_{\nu+T})$ with $(\hat{p}^\nu)=(\lambda p_\nu,...,\lambda p_t,p_{t+1},...,p_{\nu+T})$ or with $(\tilde{p}^\nu)=(p_\nu,...,p_t,p_{t+1}/\lambda,...,p_{\nu+T}/\lambda)$ for some fixed $\lambda>1$. Each of these two substitutions raises the price ratio $R_t=p_t/p_{t+1}$ while keeping all other price ratios constant. Weak gross substitutability means that the consumption vectors $(c^\nu,\hat{c}^\nu,\tilde{c}^\nu)$ corresponding to these price systems will satisfy

$$\hat{c}_\nu(s)\geq c_\nu(s) \tag{10}$$

for $s = t+1, ..., \nu + T$, and

$$\tilde{c}_\nu(s) \leq c_\nu(s) \tag{11}$$

for $s = 1, ..., t$. From these inequalities and the asset accumulation identity in equation (7) we obtain

$$\hat{c}_\nu(s) - c_\nu(s) = R_{s-1}\left[\hat{\alpha}_\nu(s-1) - \alpha_\nu(s-1)\right] - \left[\hat{\alpha}_\nu(s) - \alpha_\nu(s)\right] \geq 0 \tag{12}$$

for $s = t+1, ..., \nu + T$, and

$$\tilde{c}_\nu(s) - c_\nu(s) = R_{s-1}\left[\tilde{\alpha}_\nu(s-1) - \alpha_\nu(s-1)\right] - \left[\tilde{\alpha}_\nu(s) - \alpha_\nu(s)\right] \leq 0 \tag{13}$$

for $s = \nu, ..., t$. Initial wealth is zero, and rational consumption requires terminal wealth to be zero as well, that is, equation (8) holds and so

$$\alpha_\nu(\nu + T) = \hat{\alpha}_\nu(\nu + T) = \alpha_\nu(\nu - 1) = \tilde{\alpha}_\nu(\nu - 1) = 0. \tag{14}$$

Inserting (14) into (12) and (13) leads to

$$\hat{\alpha}_\nu(s) \geq \alpha_\nu(s) \tag{15}$$

for $s = t+1, ..., \nu + T$, and

$$\tilde{\alpha}_\nu(s) \geq \alpha_\nu(s) \tag{16}$$

for $s = \nu, ..., t$. Taken together, these two inequalities show that an increase in R_t raises the *entire* asset profile of cohort $\nu = t - T, ..., t$. ∎

Monotonicity easily extends to the aggregate real asset demand schedule at time t,

$$A(t) = \sum_{\nu=t-T+1}^{t} \alpha_\nu(t) \equiv z^\star\left(R_{t-T+1}, ..., R_{t+T-1}\right) \tag{17}$$

which depends positively on a $2T - 2$ dimensional vector of interest factors.

This economy admits a unique stationary real equilibrium but no periodic real equilibria whatsoever. Stationary equilibria are constant interest rate sequences $R_t = R^\star > 0$, or geometric price sequences $p_t = p_0\left(R^\star\right)^{-t}$, satisfying $A(t) = 0$ for $t \geq 0$. The increasingness of the schedule $z^\star$ rules out multiple steady states, and Kehoe, *et al.,* [20, pp. 13-15] demonstrate that one such state exists by bounding $z^\star$ from above and below. We sum up in

Lemma 2. *(Uniqueness of the steady state.) Weak gross substitutes implies the existence of a unique non-monetary steady state.*

Of immediate concern to us are high-frequency adjustments to temporary external shocks at, say, $t = 0$. To study the adjustment process, we look at how the economy evolves from $t = 1$ onward, after either an unexpected shock that disturbs a stationary equilibrium at $t = 0$, or after an anticipated policy intervention that fixes wealth for all generations at the end of period $t = 0$. In either case, a nonstationary equilibrium sequence $\{R_t\}_{t=1}^{\infty}$ satisfies $A(t) = 0$, $t = 1, 2, ...$ plus $T - 1$ independent initial conditions which fix the wealth of all pre-existing generations at the end of period $t = 0$. These initial conditions fix $\{\alpha_{1-T}(0), ..., \alpha_0(0)\}$ and, in addition, constrain $\sum_{\nu=1-T}^{0} \alpha_\nu(0) = 0$. The last equation means that transitory generations $\nu = 1 - T, ..., 0$ hold claims on each other only, not against cohorts born at $t = 1$ or later.

Non-stationary equilibria are solutions to the difference equation $A(t) = 0$, which has order $2T - 2$, or $2L - 4$, in the vector $\{R_{t-T+1}, ..., R_{t+T-1}\}$, subject to the $T - 1$ initial conditions. Kehoe, *et al.,* [20, pp. 6-7 and 18] prove that there can be no more than one such solution for each economy:

Lemma 3. *(Uniqueness of equilibrium.) Under weak gross substitutability, there is at most one equilibrium price sequence which, if it exists, converges to the steady state.*

Two corollaries of this result are that limit cycles cannot exist when consumption goods are gross substitutes at all price ratios, and limit cycles cannot exist near the steady state if gross substitutability obtains at prices near the steady state price vector. More relevant for our purposes is the added implication that the steady state $R^\star$ has $T - 1$ unstable eigenvalues with modulus larger than one, and $T - 1$ stable eigenvalues with modulus less than one. Convergence to $R^\star$ will take place on the stable manifold of this economy, a $T - 1$ dimensional subspace defined near $R^\star$ by the eigenvectors which correspond to the stable roots.

The phenomenon of trend reversion requires that deviations from the steady state should die out as damped oscillations, not as monotonically decaying motion. None of the preceding results say whether the $T-1$ stable eigenvalues are negative or complex, and it would seem very hard to extract such information about the nature of these roots from first principles. The main theoretical result of this section is that convergence to $R^\star$ involves some damped oscillatory motion. Formally, we have

Theorem 4. *(Convergence to the real steady state.) If dated consumption goods are weak gross substitutes at price ratios near the stationary real yield $R^\star$, then the unique equilibrium price sequence $\{R_t^\star\}_{t=1}^\infty$ of the non-monetary economy cannot be monotone.*

Proof. Monotone convergence implies that, for all $t \geq \tau$, either $R_t^\star > R^\star$ or $R_t^\star < R^\star$. In view of equation (17), the first alternative means $A(s) > 0$ for all $s \geq \tau + T - 1$; the second alternative means $A(s) < 0$ for all $s \geq \tau + T - 1$. Both implications violate the equilibrium condition $A(t) = 0$. ■

We conclude that some of the $T-1$ stable roots associated with the steady state $R^\star$ must be negative or complex, and that adjustment in the neighborhood of $R^\star$ is dominated by these eigenvalues.

Intuition supporting the conjecture of damped oscillations is easy to conjure up in an exchange economy with the typical single-peaked endowment pattern. Consumption smoothing in this environment produces a steady state in which the young borrow from the middle-aged, repay loans and build up assets in middle age, and draw down these assets in old age. If the interest rate at t is above its steady state value $R^\star$, then young agents postpone consumption and reduce their liabilities. Middle-aged agents, according to equation (22), must correspondingly *reduce* their asset holdings even as they are shifting consumption away from the current period t. This is a consistent course of action for them only if the middle-aged reduce current consumption to repay unusually high debts carried from the previous period $t-1$. Unusually high consumption in youth is rational only if the interest rate was unusually *low* at $t-1$. Hence, $R_t > R^\star$ implies $R_{t-1} < R^\star$.

Theorem 4 extends to dynamically inefficient exchange economies which permit equilibria with positively-valued government liabilities. Suppose, in particular, that there is a constant stock of fiat money per household, and zero government purchases or taxes. The central bank budget constraint is

$$H(t+1) = R_t H(t) \tag{18}$$

where $H(t)$ is per capita nominal balances, and $R_t = p_t/p_{t+1}$ is the real yield of money.

From equations (4) and (18) we obtain the equilibrium condition

$$A(t+1) = R_t A(t) \tag{19}$$

in which $A(t)$ is the community demand for real balances. The monetary stationary equilibrium of this economy is $R_t = 1$ at which money demand is assumed positive, that is,

$$A^\star = z^\star(1, ..., 1) > 0. \tag{20}$$

We know that this equilibrium is a saddle which implies a unique equilibrium sequence $\{R_t\}$ converging to 1. If this sequence were to converge monotonically from above (below), then the demand for real balances would be falling (rising) over time. In other words monotone convergence implies, for each t, either $R_t > R_{t+1} > 1$ and $A(t) > A(t+1)$ or $R_t < R_{t+1} < 1$ and $A(t) < A(t+1)$. Each of these events contradicts the market clearing condition (19), unless $A(t) < 0$ for each t which in turn violates our dynamic inefficiency assumption in (20). This proves the following theorem:

Theorem 5. *(Convergence to the monetary steady state.) If the non-monetary economy is dynamically inefficient and dated consumption goods are weak gross substitutes at yields near the golden rule, then there is a unique monetary equilibrium price sequence* $\{R_t\}_{t=1}^{\infty}$ *which converges non-monotonically to the golden rule.*

3.3. Convergence in production economies. The equilibrium of a production economy with a lifecycle of $L \geq 2$ periods satisfies a difference equation of order $2L-3$ which represents zero aggregate excess demand for assets,

$$A(t) - k(t+1) = 0, \tag{21}$$

where $k(t+1)$ represents the capital-labor ratio at $t+1$, plus $L-1$ independent initial conditions fixing the wealth of transitional generations, that is, $\{\alpha_{1-T}(0), ..., \alpha_0(0)\}$

given $\sum_{\nu=1-T}^{0} \alpha_\nu(0) \equiv k_1 > 0$. Despite the similarities of this economy with the previous one, Lemma 3 and Theorem 4 do not extend directly to economies with production. Calvo [12] and Kehoe [19], in particular, provide examples of non-unique equilibria in economies with gross substitutability in consumption and a high degree of complementarity in production. Calculations conducted later in the paper suggest that the stable eigenvalue with the largest modulus could be either positive or negative, even for utility functions in which consumption goods are gross substitutes at *all* price ratios. Hence gross substitutability is not sufficient to rule out monotone convergence in overlapping generations growth models with non-trivial lifecycles.

We do not have a general result. What we offer instead is informed speculation backed by numerical results reported later in the paper. The logic of these counterexamples to uniqueness and trend reversion seems to rest on the correlation of prices with incomes as an economy adjusts toward its steady state. For endowment economies, incomes are fixed and uncorrelated with prices or interest rates; in production economies, however, interest rates and wage incomes are negatively correlated by the factor-price frontier. Deviations from the steady state affect interest rates and wages in opposite directions, and exert two conflicting forces on savings plans: a higher-than-normal interest rate tends to postpone current consumption and raise asset holdings, while a lower-than-normal wage rate (and wage income) will lower assets as the household borrows against future earnings to smooth out its consumption path. Dynamic adjustment in production economies seems to depend on the balance of these two conflicting forces, that is, on the steepness of the factor-price frontier near the steady state. We conjecture that production economies with relatively flat factor-price frontiers will behave like endowment economies.

4. Example economies

4.1. Overview. We now turn to a series of examples to illustrate our main finding with respect to endowment economies, and to explore our conjecture with respect to economies with production. We begin with the simplest example, and then continue to more complicated cases, including many-period models with production. In each case, we calculate eigenvalues, analytically in the simplest cases, and numerically for the more complicated economies.

4.2. Logarithmic endowment economies with three-period lives.

Without government liabilities. As a simple starting example that illustrates Theorem 4, we use the logarithmic utility function $u^t = \ln c_t(t) + \beta \ln c_t(t+1) + \beta^2 \ln c_t(t+2)$ for each generation $t = 1, 2, ...$, where $\beta > 0$ is a discount factor, and the endowment vector $(e_0, e_1, e_2) \in \mathbb{R}^3_+$ such that $e_0 + e_1 + e_2 = 1$. Equilibrium in these economies is any solution to the second order difference equation

$$\alpha_t(t) + \alpha_{t-1}(t) = 0, \tag{22}$$

for $t = 1, 2, ...$, which satisfies the initial condition

$$\alpha_0(0) + \alpha_{-1}(0) = 0. \tag{23}$$

In each of these equations $\alpha_t(t+i)$ denotes claims on other households held by a household of generation t at the end of period $t+i$, for $i = 0, 1, 2, ...$ Equation (23) is an initial condition that specifies what the two transitional generations owe each other; it is equivalent to fixing the interest rate at $t = 0$. The asset accumulation identities are given by

$$\alpha_t(t) = e_0 - c_t(t) \tag{24}$$

$$\alpha_t(t+1) = R_t \alpha_t(t) + e_1 - c_t(t+1) \tag{25}$$

where $R_t = p_t/p_{t+1}$ is the real interest factor on loans made at t and repaid at $t+1$. The consumer's first-order conditions yield

$$c_t(t+1) = \beta R_t c_t(t) \tag{26}$$

and

$$c_t(t) = \frac{e_0 + \frac{e_1}{R_t} + \frac{e_2}{R_t R_{t+1}}}{1 + \beta + \beta^2}. \tag{27}$$

Substitution into (22) and (23) yields the second-order equation

$$x_{t+1} = R_t \tag{28}$$

$$R_{t+1} = \frac{\gamma e_2}{f(x_t, R_t)} \tag{29}$$

where $\gamma = \frac{1}{1+\beta+\beta^2}$, and

$$f(x,R) = (1-\gamma)\,e_0 R + [1-(1+\beta)\,\gamma]\,e_1 R + e_0\,[1-(1+\beta)\,\gamma]\,xR - \gamma\,[e_1 + (1+\beta)\,e_2]\,. \quad (30)$$

Equation (30) turns out to have a unique steady state $R^\star$, just as predicted by Kehoe, *et al.*, [20]. Uniqueness of the steady state $R^\star$ follows from the monotonicity of the expression $R - \gamma e_2 / f(R,R)$, which is derived from equation (28). It also has a characteristic polynomial

$$\pi(\lambda) = \lambda^2 - T\lambda + D \quad (31)$$

with

$$T = -\left(\frac{AR^2}{\gamma e_2} + D\right) < 0, \quad (32)$$

$$D = \frac{R^3 e_0\,[1-(1+\beta)\,\gamma]}{\gamma e_2}, \quad (33)$$

$$A = (1-\gamma)\,e_0 + [1-(1+\beta)\,\gamma]\,e_1. \quad (34)$$

Since $(R^\star)^2 > \gamma e_2/A$, $\pi(\lambda)$ has a positive discriminant, $\pi(-1) = 1 + T + D < 0$, and there are two negative eigenvalues which straddle -1:

$$\lambda_2 < -1 < \lambda_1 < 0. \quad (35)$$

Combining the initial condition (23) with the single stable eigenvalue (35) we conclude that real equilibrium is unique in this economy. If the initial distribution of claims $\{\alpha_0(0), \alpha_{-1}(0)\}$ happens to deviate slightly from its steady state configuration, the interest rate will also differ from its stationary value $R^\star$. Dynamic adjustment to $R^\star$ can be locally approximated by the linear equation

$$R_t - R^\star = \lambda_1 (R_{t-1} - R^\star) \quad (36)$$

in which $-1 < \lambda_1 < 0$. Therefore, for a given initial condition $R_0 \neq R^\star$, the equilibrium follows a damped oscillatory path toward the steady state.

With valued government liabilities. To illustrate Theorem 5, we again use a logarithmic preferences three-period endowment economy, but this time with a valued outside asset in fixed supply. For this example, we set the discount factor $\beta = 1$ so that utility is given by $u^t = \ln c_t(t) + \ln c_t(t+1) + \ln c_t(t+2)$. The endowment vector is again $(e_0, e_1, e_2) \in \mathbb{R}^3_+$ such that $e_0 + e_1 + e_2 = 1$. Solving the households' problem indicates that aggregate holdings of the outside asset is given by

$$A(t) = e_0 - \frac{1}{3}\left[e_0 + \frac{e_1}{R_t} + \frac{e_2}{R_t R_{t+1}}\right] + e_1 + R_{t-1}e_0 - R_{t-1}\frac{2}{3}\left[e_0 + \frac{e_1}{R_{t-1}} + \frac{e_2}{R_{t-1}R_t}\right]. \quad (37)$$

At the monetary steady state value $R^\star = 1$, inspection reveals that the condition for $A > 0$ is $e_0 > e_2$. The equilibrium condition is given by (19), and we can write the implied dynamic system as a third-order difference equation

$$R_{t+2} = \frac{e_2}{-(e_1 + e_2) + 2R_{t+1}(e_0 + e_1 + e_2) - R_{t+1}R_t(e_0 + e_1) - R_{t+1}R_t R_{t-1} e_0}. \quad (38)$$

The eigenvalues associated with this equation at $R_t = R^\star \; \forall t$ are the solutions to

$$P(\lambda) = \lambda^3 + \left(\frac{e_1 + 2e_2}{e_2}\right)\lambda^2 - \left(\frac{2e_0 + e_1}{e_2}\right)\lambda - \frac{e_0}{e_2} = 0. \quad (39)$$

We deduce that $P(0) = -\frac{e_0}{e_2} < 0$, $P(-1) > 0$, and $P(1) = \frac{3(e_2 - e_0)}{e_3} < 0$ under the maintained hypothesis that the demand for the outside asset is positive at $R^\star = 1$. Since $\lim_{\lambda \to +\infty} P(\lambda) = \infty$ and $\lim_{\lambda \to -\infty} P(\lambda) = -\infty$, we conclude that the three eigenvalues are real and are arranged as $\lambda_2 < -1 < \lambda_1 < 0 < 1 < \lambda_0$. There is one initial condition corresponding to the stable negative real eigenvalue, hence equilibrium is unique and adjustment toward the steady state is nonmonotonic.

4.3. CRRA utility endowment economies with three period lifetimes. If we allow significant curvature in preferences, the gross substitutes property does not necessarily hold and the uniqueness of steady state equilibrium can be lost. However, we would like to explore how our results from the previous subsection would change under CRRA preferences. We use numerical methods in this case. Accordingly, we view an economy in the set E_1 as a list $(\beta, \gamma, e_0, e_1, e_2) \in E_1$, in which (e_0, e_1, e_2) is the endowment vector, $u[c_t(t)]+\beta u[c_t(t+1)]+\beta^2 u[c_t(t+2)]$ is the utility function, with $u(c)=\frac{c^{1-\gamma}}{1-\gamma}$ for $\gamma\neq 1$ and $u(c)=\ln c$ for $\gamma=1$. We chose the coefficient of relative risk aversion $\gamma\in(0,3)$ from a uniform distribution, which implies intertemporal elasticities of substitution as small as $\frac{1}{3}$. The discount factor is chosen from a uniform distribution on $(.5,2)$, corresponding to rates of time preference that might be viewed as realistic given the length of the time period. The endowment sequence is restricted to be hump-shaped, with $e_0=e_2$, and the middle endowment chosen from a uniform distribution with $e_1\in(e_0,3e_0)$. This pattern corresponds roughly to data sometimes used to calibrate larger models, in which the peak endowment is about 1.7 times the first endowment and the final endowments are of about equal magnitude as the initial endowments. We maintain the restriction that $e_0+e_1+e_2=1$. We randomly select 1,000 economies from E_1, and calculate the steady state as well as the associated eigenvalues.

[FIGURE 1 ABOUT HERE.]

The results are summarized in Figure 1. These economies turned out to always have a unique steady state, and they continued to possess a single stable eigenvalue. This stable eigenvalue, however, can now sometimes be positive. In the Figure, we plot the value of the stable eigenvalue for each of our 1,000 sample economies against four characteristics of the economy. In panel A, the value of the stable eigenvalue is plotted against e_1, which can be interpreted as the peakedness of the endowment pattern. It is clear that flatter endowment sequences tend to preserve the negative sign on the stable eigenvalue. Similarly, panels B and D show that discount factors less than unity and curvature parameters less than 2 tend to preserve the negative sign on the stable eigenvalue, regardless of other parameters. Panel C relates the value of the stable eigenvalue to the steady state gross interest rate. Here, we see that it is the inefficient economies, those with gross interest rates less than unity, which may be characterized by a positive stable eigenvalue. These results suggest that oscillatory adjustment to transitory shocks may characterize efficient economies with empirically plausible features: positive rates of time preference, relatively flat endowment profiles, and relatively high elasticities of intertemporal substitution (above $\frac{1}{2}$).

4.4. Production economies with three-period lifecycles. In economies with three period lifecycles and production, the dimension of the associated dynamic system increases by one. Initial conditions are now the holdings of capital owned by the agents who have been alive for one and two periods respectively. For determinacy to hold, we expect two stable eigenvalues in these systems. Labor supply is inelastic and normalized to unity. We use the same CRRA utility function, along with a standard CES production function given by

$$y(t) = \theta \left[\left\{ \alpha k(t)^{-\rho} + (1-\alpha) \right\}^{-1/\rho} \right], \tag{40}$$

for $\rho \neq 0$, and

$$y(t) = \theta k(t)^{\alpha} \tag{41}$$

for $\rho = 0$, where $y(t)$ is output, α is capital share when $\rho = 0$,[3] ρ governs the inverse of the elasticity of substitution between capital and labor, and θ is a scale factor. An economy in the set E_2 is a list $(\beta, \gamma, e_0, e_1, e_2, \alpha, \rho, \delta)$ in which δ is the net rate of depreciation for physical capital. For the preference parameters we choose $\beta \in (.5, 2)$ and $\gamma \in (.5, 3)$ from uniform distributions. We use the hump-shaped (interpreted as labor productivity) endowment pattern described in the previous section. For the production parameters, we choose, again from uniform distributions, $\alpha \in (.25, .4)$, $\rho \in (-.5, .5)$, and $\delta \in (.8, 1)$. We set the scale parameter θ to 10, which is sufficient to guarantee existence. We calculate the steady state and the associated eigenvalues.

[FIGURE 2 ABOUT HERE.]

The 1,000 economies in E_2 always possessed a unique, determinate steady state. In each case, we found that one stable eigenvalue was real and positive, and the other was real and negative. The modulus of the two eigenvalues was of similar magnitude in most cases, but the positive eigenvalue was larger in absolute value in about 85% of the economies. Figure 2 summarizes our findings, and relates the characteristics of the economies to the magnitude of the two stable eigenvalues. The results here are much less ambiguous than those for the economies in E_1. Here, the negative

[3]When $\rho \neq 0$, capital share is given by $\alpha k^{-\rho} \left[ak^{-\rho} + (1+\alpha)\right]^{-1}$, where k is the steady state capital-labor ratio.

eigenvalue is always present, and can be dominant under many different parameter configurations.

4.5. Production economies with 55-period lifecycles. We ultimately wish to understand the local dynamics of lifecycle economies with many periods, in which agents are allowed to make decisions, and thus react to shocks, at many points in their lifetimes. Accordingly, we study economies with $L = 55$, an "annual" model.[4] Here we calibrate the economies much more sharply in order to reduce the number of cases we need to calculate. Our calibration proceeds as follows. We use a productivity profile based on Hansen [17].[5] We set the following parameters at annualized values: $\beta = .98$, $\delta = .065$. We set $\alpha = .33$. That leaves two parameters, curvature in preferences and capital-labor substitutability, which might be viewed as the most interesting ones for dynamic adjustment. We explored nine cases based on the following (ρ, γ) pairs: $A = (-.5, 1.1)$, $B = (-.5, 2)$, $C = (-.5, 5)$, $D = (0, 1.1)$, $E = (0, 2)$, $F = (0, 5)$, $G = (.5, 1.1)$, $H = (.5, 2)$, and $I = (.5, 5)$. Accordingly, case D is close to a log-log specification, where preferences are logarithmic and production is Cobb-Douglas.[6] For each case $A, ..., I$, we calculated the steady state of the system. In principle, uniqueness of the steady state is not guaranteed, but multiple steady states did not occur for these parameter configurations. There is also the question of whether the calculated steady states are efficient or inefficient. The gross steady state growth rate in these economies is one, so any gross interest rate greater than one indicates the

[4]Calibrated examples of production economies with adjustment costs are studied by Auerbach and Kotlikoff [1], and Huang, Imrohoroğlu, and Sargent [18]. Rios-Rull [24], on the other hand, examines the sample moments of a calibrated stochastic overlapping generations economy with a 55-period lifecycle and compares them with the corresponding predictions of real business cycle models.

[5]The Hansen data is collected from samples taken in 1979 and 1987. The data separate males from females. We average the data from the two years, and we also average the data across males and females using weights of 0.6 and 0.4. The resulting profile is a step function, because the data are collected for age groupings. We fit a fifth-order polynomial to this step function. This yields the smooth profile $e_{i-20} = m_0 + m_1 i + m_2 i^2 + m_3 i^3 + m_4 i^4 + m_5 i^5$ for $i = 21, ..., 76$, with the vector of coefficients $m = [-4.34, 0.613, -0.0274, 0.0063, -0.717 \times 10^{-5}, 0.314 \times 10^{-7}]$. This profile peaks at agent age 28 (figurative age 48), when productivity is about 1.6 times its level at agent age 1 (figurative age 21). Productivity in the final year of life is virtually the same as in the first year of life.

[6]To avoid special programming code we did not allow the case where γ is exactly equal to unity. This makes little difference for the results.

equilibrium is efficient. Steady state interest rates were always greater than one for each of the nine cases A, ..., I, and so we are only looking at efficient economies.

[FIGURE 3 ABOUT HERE.]

We study the local dynamics of these economies. In Figure 3 we plot for case H the associated eigenvalues in the complex plane. Each square plotted represents an eigenvalue associated with the unique steady state. There are exactly 54 stable eigenvalues indicating that equilibrium is determinate. Perhaps more importantly for our purposes, nearly all the roots are complex and of about equal modulus.[7] Thus one expects the local dynamics of these systems to be characterized by fluctuating motion. The qualitative features of this figure were the same for the other nine cases we studied—we always found two groups of eigenvalues, one group lying roughly evenly spaced on an ellipse outside the unit circle, and the other lying roughly evenly spaced on an ellipse inside the unit circle. The idea that the eigenvalues are roughly evenly spaced can be documented if we translate to polar coordinates and measure the distance between eigenvalues in degrees; in this metric, the stable eigenvalues are all about 360/54 degrees apart, the only substantial exceptions occurring near the point $(1, 0)$ in the diagram, where the roots are slightly farther apart. Of the two stable real roots, one has modulus comparable to the complex roots, and the other is relatively small and negative. We summarize the results for the nine cases in Table 7.

[TABLE 7 ABOUT HERE.]

5. CONCLUSIONS

Convex lifecycle economies with pure exchange, or with production under constant returns to scale are the simplest class of models consistent with the trend-reverting behavior of U.S. output. This behavior is documented by reduced-form VAR's in their response to temporary impulses and in their characteristic roots, which tend to be overwhelmingly negative or complex. Complex eigenvalues also occur in overlapping

[7] Laitner [21] calculates eigenvalues for a small sample of alternatively parameterized large lifecycle economies with taxes and transfers. Eigenvalues were typically complex in these economies.

generations economies with pure exchange when dated consumption goods are gross substitutes near the steady state, and aggregate saving depends non-trivially on the distribution of household wealth among successive cohorts of individuals. We conjecture that an array of complex eigenvalues is a likely feature of all lifecycle economies with a reasonably large number of decision points in the lifecycle. These complex eigenvalues are of comparable modulus to the largest real root.

The qualitative similarity between lifecycle economies and vector autoregressions in their dynamic adjustment paths naturally brings to the fore the question of quantitative fit. Are there empirically plausible parameterizations of overlapping generations economies whose eigenvalues, autocorrelation functions, and responses to temporary productivity or liquidity shocks match quantitatively those of fitted vector autoregressions? We think this question clearly deserves further research.

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Table 1
Eigenvalues, GDP AR(2) Model

	Detrending Method		
Eigenvalue	Linear	Quadratic	Hodrick-Prescott
λ_1	0.806	0.544	$0.201 + 0.506i$
λ_2	0.107	0.285	$0.201 - 0.506i$

Table 1: Eigenvalues for the AR(2) univariate model for GDP.

Table 2
Eigenvalues, GDP AR(3) Model

	Detrending Method		
Eigenvalue	Linear	Quadratic	Hodrick-Prescott
λ_1	0.727	$0.484 + 0.210i$	$0.363 + 0.586i$
λ_2	0.217	$0.484 - 0.210i$	$0.363 - 0.586i$
λ_3	-0.041	-0.151	-0.370

Table 2: Eigenvalues for the AR(3) univariate model for GDP.

Table 3
Eigenvalues, Y/C VAR Model

	Detrending Method		
Eigenvalue	Linear	Quadratic	Hodrick-Prescott
λ_1	0.953	0.819	0.507
λ_2	0.583	0.558	0.203

Table 3: Eigenvalues for the VAR with GDP and consumption.

Table 4
Eigenvalues, Y/I VAR Model

	Detrending Method		
Eigenvalue	Linear	Quadratic	Hodrick-Prescott
λ_1	0.884	0.789	$0.212 + .0174i$
λ_2	0.228	0.252	$0.212 - 0.174i$

Table 4: Eigenvalues for the VAR with GDP and investment.

Table 5
Eigenvalues, Y/R VAR Model

Eigenvalue	Detrending Method		
	Linear	Quadratic	Hodrick-Prescott
λ_1	$0.843 + 0.156i$	$0.677 + 0.135i$	$0.404 + 0.427i$
λ_2	$0.843 - 0.156i$	$0.677 - 0.135i$	$0.404 - 0.427i$

Table 5: Eigenvalues for the VAR with GDP and an interest rate.

Table 6
Frequency of Negative or Complex Eigenvalues in VARs

	Detrending Method		
System	Linear	Quadratic	Hodrick-Prescott
Y/C	14.6%	21.8%	45.0%
Y/I	3.6%	2.8%	67.0%
Y/R	92.0%	76.4%	95.8%

Table 6: Frequency of negative or complex eigenvalues in VAR estimates, based on bootstrap estimates of sampling uncertainty.

Table 7
Comparison of cases $A, ..., I$.

Economy (ρ, γ)	Modulus Smallest	2^{nd} Smallest	Largest
$A = (-.5, 1.1)$	0.153	0.856	0.944
$B = (-.5, 2)$	0.049	0.881	0.956
$C = (-.5, 5)$	0.003	0.833	0.993
$D = (0, 1.1)$	0.163	0.837	0.930
$E = (0, 2)$	0.104	0.855	0.939
$F = (0, 5)$	0.017	0.777	0.981
$G = (.5, 1.1)$	0.164	0.759	0.932
$H = (.5, 2)$	0.122	0.797	0.937
$I = (.5, 5)$	0.033	0.744	0.973

Table 7: Summary of eigenvalues for cases A, ..., I.

Figure 1. How the negative stable eigenvalue relates to properties of endowment economies.

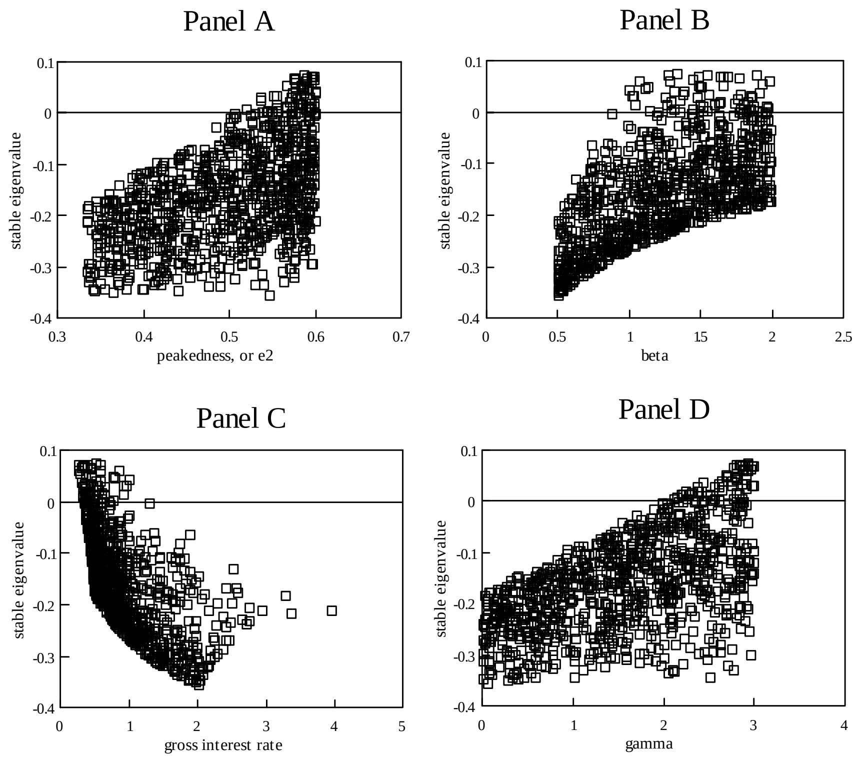

Figure 1: How the negative stable eigenvalue relates to properties of endowment economies. The stable eigenvalue for the 1,000 economies in E_1 is typically negative. The four panels relate the stable eigenvalue to characteristics of these economies.

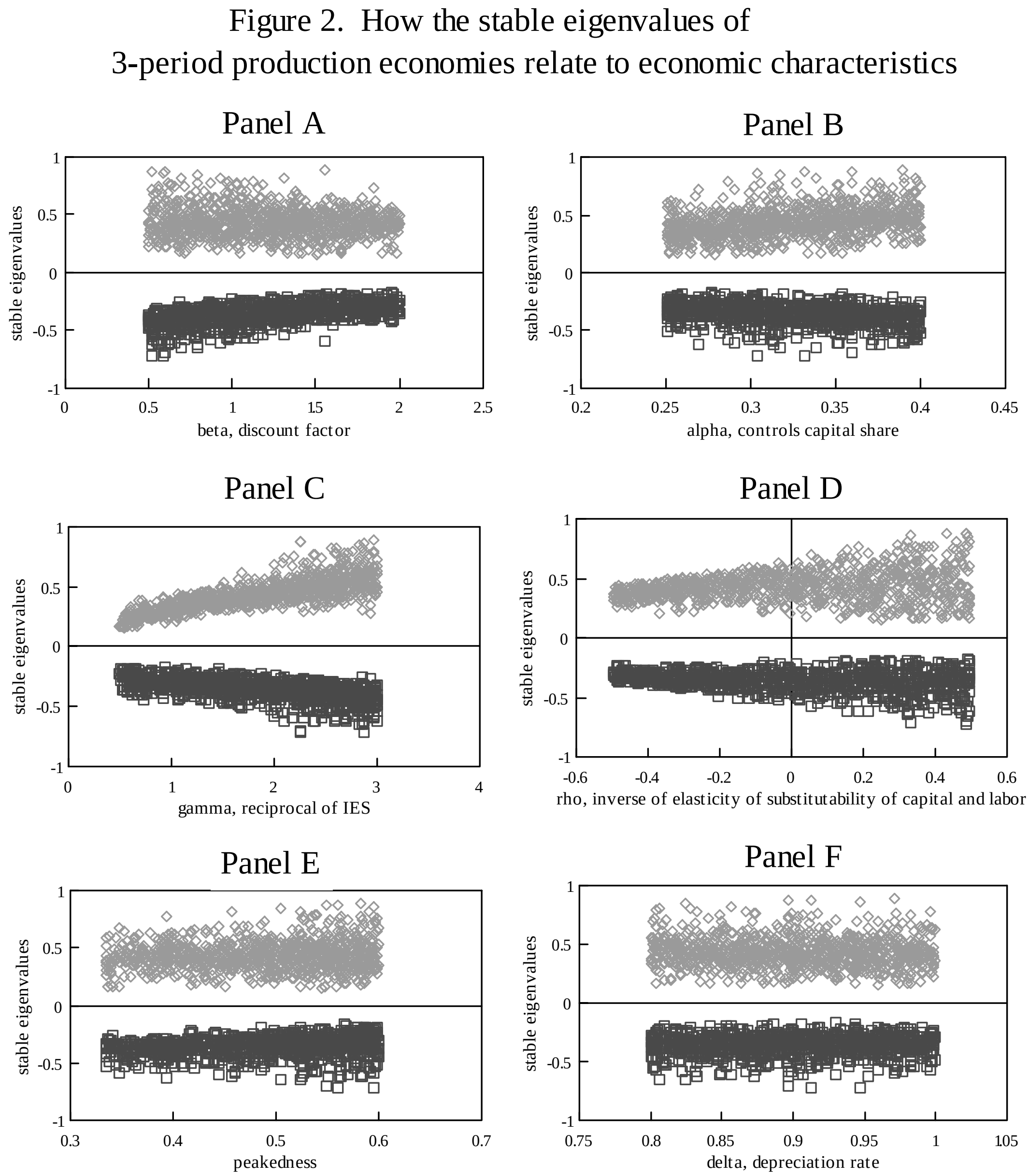

Figure 2: How the stable eigenvalues of three-period production economies relate to economy characteristics.

Figure 3

Eigenvalues, 55-period model

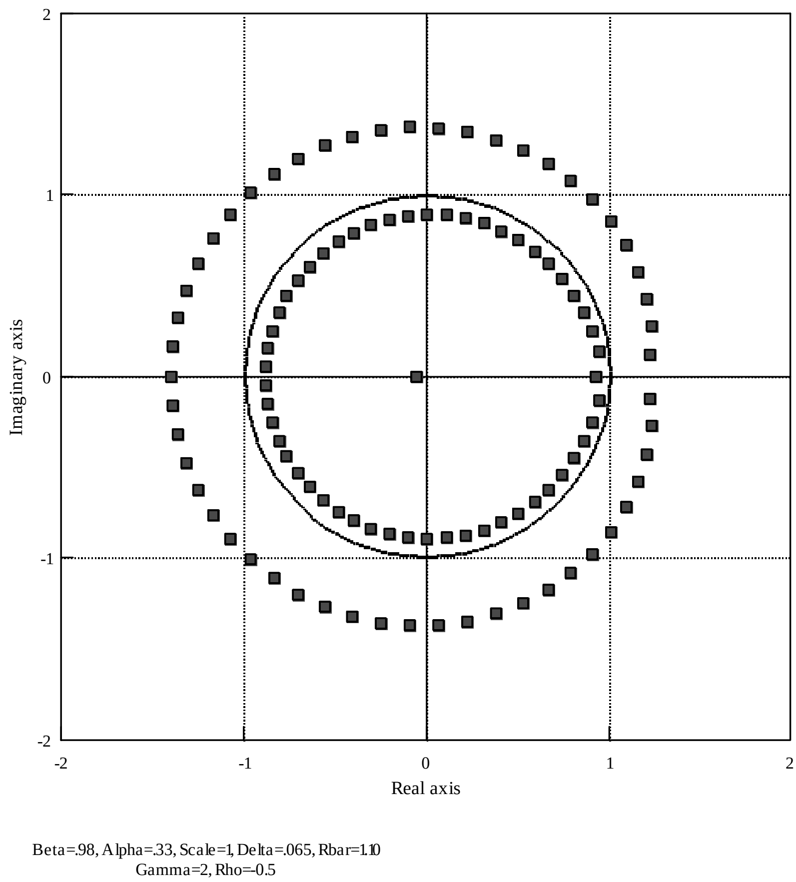

Figure 3: The eigenvalues for a 55-period production economy. The heavy black line denotes the unit circle. There are 54 eigenvalues inside the unit circle, so that equilibrium is determinate. The cases $A, ..., I$ all produced qualitatively similar diagrams.