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Working Paper Number	1989-006A
Creation Date	January 1989
Citable Link	https://doi.org/10.20955/wp.1989.006
Suggested Citation	de Vries, C., Jansen, D.W., 1989; On the Frequency of Large Stock Returns: Putting Booms and Busts into Perspective, Federal Reserve Bank of St. Louis Working Paper 1989-006. URL https://doi.org/10.20955/wp.1989.006

ON THE FREQUENCY OF LARGE STOCK RETURNS: PUTTING BOOMS AND BUSTS INTO PERSPECTIVE

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89-006

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Fall 1988

Abstract

Numerous articles have investigated the distribution of share prices, and find that the yields are leptokurtic. There is still controversy about the amount of leptokurtosis, and hence about the most appropriate distribution to use in modeling returns. This controversy has proven hard to resolve, as the alternatives are non nested. We propose to employ extreme value theory focusing exclusively on the larger observations, in order to assess the leptokurtosis within a unified framework. This enables one to generate robust probabilities on large changes, which put the recent stock market swings into historical perspective.

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We are grateful to Kees Koedijk for his comments.

1. Introduction

The often large and rapidly realized gains on the stock markets over the past decade were abruptly haltéd by a plunge in October 1987 which only has its equals in pre World War II years. During the aftermath, the stock market exhibited a volatility which seemed unsurpassed. These events have led to numerous investigations in order to understand and evaluate the performance of the financial markets. Several commissions were dedicated to investigate the intricacies of the crash and to come up with recommendations for improving the functioning of the financial markets. However, from the viewpoint of finance, several researchers have come to view the dramatic events of October 1987 as a necessary, though sudden, correction to the "overheated" markets, see e.g. Arbel, Carwell and Postnieks [1988]. With a wider scope, in macro economics, several theories offer competing explanations for how the markets became overheated and why, the plunge notwithstanding, the economy is basically sound. The explanations range from bursting bubbles to exploding budget deficits that are finally tamed through legislation.

Given all these competing views and as of yet untested theories, we propose a somewhat different tack to put the events of October 1988 into perspective. Suppose one possesses an estimate of the distribution of stock returns that is constructed from a sample excluding the recent past. The idea is then to compare the tail of this distribution to the frequency with which the larger changes in the share prices occured during 1987. Fortunately, several estimates for the distribution of the yields are available in the literature. The first widely used hypothesis held that the share prices followed a Brownian motion. However, starting with Mandelbrot [1963] and Fama [1965], it was found that the innovations exhibited a higher kurtosis than the normal distribution could account for. Mandelbrot [ibid.] proposed the stable distribution for modelling the yield distribution, as it captures the fatness in the tail of the empirical distribution and preserves the additivity property of the normal distribution (because yields are additive, it is thought desirable that the distribution exhibits this as well). Under the maintained hypothesis of the stable model, parameter estimates may be found by using the Fama and Roll [1971] procedure or by using the Fourier transform and numerical integration.

Later, Praetz [1972], and Blattberg and Gonedes [1974] advanced the β_3-class of distributions as an alternative to the stable model. The motive for advancing the β_3-class was the sometimes observed tendency towards normality for longer term yields. As is well known, the β_3 distribution, with degrees of freedom v above 2, have a finite variance and therefore satisfy the central limit theorem. (Note, the Student-t distributions , when v is an integer, are a special case of the β_3-class). Such is not the case for the stable model, which have themselves as the limit laws of normed sums. At the same time with v finite, not all moments do exist and hence the tails are leptokurtic. A disadvantage of the β_3-class is that the additivity property is lost; however, see below, where we show that the tails remain invariant under addition. Another alternative sometimes considered is a discrete mixture of normal distributions, see Kon [1984]. Such a distribution adequately describes the case when distinct types of events, i.e. that have different normal distributions, may occur with certain probability. But the mixtures seem not to accord well with the observed higher than normal leptokurtosis, see below. For both, the β_3 and mixture models, parameter estimates can be obtained by employing approximate maximum likelihood methods using non linear optimization techniques.

Hence, estimates for the distribution of the yields are readily available. However, while most estimated distributions try to capture the higher than normal leptokurtosis, there appears to be considerably controversy over the exact amount of the leptokurtosis. In particular, the schism is about whether or not the second moment is finite, i.e. about the applicability of the central limit theorem. The stable hypothesis maintains an infinite variance, while the β_3- and mixture hypotheses maintain a finite variance. As our aim is to put the large changes in the share prices into perspective, the controversy is not immaterial, because the amount of leptokurtosis is related one to one with the frequency with which larger yields occur. Therefore, it seems desirable to settle the issue about the leptokurtosis first.

A comparison between the competing hypotheses is hampered by the fact that the alternative models are non nested. Sometimes, c.f. Blattberg and Gonedes [ibid.] and Kon [ibid.], a likelihood ratio is used. However, this procedure is not appropriate as neither of the models is nested. The Cox procedure is not applicable either, as the second moment may be infinite; see White [1982] for conditions under which the Cox procedure is appropriate. It appears that comparing different distributions is not a feasible way for resolving the controversy.

However, there is a more direct way to tackle the problem. Instead of comparing entire distributions, we may compare the tails only. The idea is as follows. Because the leptokurtosis is primarily a tail phenomenon, the distribution of the maxima (or minima) is quite informative in this respect. As it turns out, the tail behavior of the competing models can be parametrized by the so called <u>tail index α</u> from the limit law of the distribution of the maxima. For the stable distribution hypothesis $\alpha < 2$ equals the characteristic exponent. In case of the β_3-class, $\alpha \geq 2$ represents the degrees of freedom. Finally, for the mixture hypothesis α is infinite. Hence, the different distributions follow the 'same' limit law but with different values of the tail index. In this way α represents the amount of leptokurtosis. Below, we present ways in which the tail index α can be estimated directly, i.e. in such a way that the different hypotheses appear as nested alternatives. The advantage is that the estimates do not rely on one of the alternatives as a maintained hypothesis. A disadvantage is, of course, that α is not a sufficient statistic for the entire distribution of the yields. But for our purposes, i.e. inferring the distribution of large yields, α is a sufficient statistic. Thus, concentrating on the tail of the distribution is an application of Occam's razor.

The estimation procedure also implies an asymptotic confidence interval for the estimate $\hat{\alpha}$. This gives the present procedure a clear advantage over the traditional methods such as Fama and Roll's [ibid.] method. As the β_3-class with finite variance requires $\alpha > 2$, and the leptokurtic stable alternative has $\alpha < 2$, the confidence interval may be used to directly discriminate between these alternatives. Moreover, the current procedure allows one to test for parameter stability as well. For example, we will test how the distribution of the extreme yields was affected by the introduction of the options market in 1973.

Once an estimate of the tail index α is available, one can construct probabilites of observing extremely large yields (positive and negative). All this work would be of little extra use if one were only interested in the frequency of yield sizes that are observed within the sample. In the latter case employing the empirical distribution function is a good procedure due to its mean squared error consistency, c.f. Mood et al. [1974, p. 507]. However, for constructing probabilities on yields that exceed the maximum size observed within the sample, this procedure is of no avail. We employ a new method for constructing such probabilities which relies on the limit law for extremes and uses the higher order statistics. In this way we generate tables with 'exceedance probabilities' which give an idea about the likelihood of the occurance of yet unseen crashes and put the events of 1987 into perspective.

Hitherto, exploiting the shape of the tail of the distribution function has found few applications in economics. In the area of labor economics, this device is sometimes used to study unemployment durations, see Kiefer [1988]. In the area of finance, Akgiray and Booth [1988] propose to focus explicitly on the tail shape and use a maximum likelihood procedure for estimation of the tail index. This study follows the same philosophy, but uses methods that are based on order statistics to estimate α. These latter methods have the advantage that the extremes do not have to follow the limit law exactly. With respect to the exceedance probabilities, McCulloch [1981] provides an example of how these may be used for calculating bankruptcy probabilities under the maintained hypothesis of a stable distribution. The methods used below seem more robust, as they only presuppose a particular limit law, allowing for a larger class of particular distributions.

Before we present the empirical results, we cover some of the necessary theoretical background, without going into all the technical details. The next section is devoted to this latter task, and section 3 gives the empirical findings.

2. Theory

Consider a stationary sequence X_1, X_2,... of independent and identically distributed (i.i.d.) random variables with a distribution function F. Suppose one is interested in the probability that the maximum

(1) $$M_n = \max(X_1, X_2, \ldots, X_n)$$

of the first n variables is below a certain level x. As is well known, this probability is given by

(2) $$P\{M_n \leq s\} = F^n(x).$$

Extreme value theory studies the limiting distribution of the order statistic M_n (appropriately scaled). That is, one is interested under what conditions there exists suitable normalizing constants $a_n > 0$, b_n, such that

(3) $$P\{a_n(M_n - b_n) \leq x\} \overset{w}{\to} G(x),$$

i.e.

$$F^n(x/a_n + b_n) \overset{w}{\to} G(x),$$

where G(x) is one of the three asymptotic distributions, and w stands for weak convergence. If (3) holds, we shall say that F belongs to the domain of attraction of G, and write $F \in D(G)$. Define the class of limiting laws which may appear in (3) as follows:

Definition 1

A nondegenerate distribution function (further d.f.) G is called max-stable if there exist real constants $A_n > 0$ and B_n such that for all real x and $n = 1,2,\ldots$

$$G^n(A_n x + B_n) = G(x).$$

One can show that if (3) obtains, then G is max-stable. The main result is the Extremal Types Theorem:

Theorem 1

The max-stable distributions can be represented by

$$
\begin{aligned}
(4) \qquad & \text{Type I} : G(x) = \exp(-e^{-x}) && -\infty < x < \infty; \\
& \text{Type II} : G(x) = 0 && x \le 0, \\
& \qquad\qquad = \exp(-x^{-\alpha}), && x > 0; \\
& \text{Type III} : G(x) = \exp(-(-x)^{\alpha}) && x < 0, \\
& \qquad\qquad = 1 && x \ge 0;
\end{aligned}
$$

with the index $\alpha > 0$.

The index α is called the tail index, and for convenience we sometimes use its inverse $\gamma = 1/\alpha$. Mood et al. [1974, p. 261] provide an introductory account to this result. Leadbetter, Lindgren and Rootzen [1983] give a comprehensive treatment, with relatively straightforward proofs. The latter reference also treats the case of dependency, when the scaling parameter a_n has to be modified by a constant multiplicative factor θ, $0 < \theta < 1$. Most of the results below do carry over to dependent variates, and is therefore not treated explicitly for space considersations. The limit in (3) explicated in (4) is most easily interpreted by analogy with the central limit theorem. The difference is the focus on order statistics rather than averages, but its usefulness is the same as no detailed knowledge of F(x) is needed to apply the asymptotic theory. A complication is the fact that there are three limit laws. Usually economic theory is not informative about the specific distribution F(x) that applies. However, the qualitative characteristics of the economic process may point to the relevant limit law. Consider the following two necessary conditions from De Haan [1976].

<u>Condition 1</u>

If $F \in D(\text{Type I } G(x))$ and $F(x) < 1$ for all x, then $\int_1^\infty t^\beta \, dF(t)$ is finite for all β.

<u>Condition 2</u>

If $F \in D(\text{Type II } G(x))$, then $F(x) < 1$ for all x and $\int_1^\infty t^\beta \, dF(t)$ is finite for $\beta < \alpha$ and infinite for $\beta > \alpha$.

The intuition behind these conditions is as follows. Loosely speaking, the tail of the distribution is either declining exponentially or by a power. In the first case all moments exist, but in the second case the higher moments do not decay rapidly enough when "weighted" by the tail probabilities to be integrable, i.e. the d.f. F(x) is leptokurtic. This explains the appearance of the double exponential in the Type I limit law and the exponential format of the Type II law as well. The third limit law is characterized by the fact that it has a finite upper endpoint. Anticipating on the next section, given that stock returns are strongly leptokurtic, and unbounded in principle, the type II limit law is the relevant one, if the maximum yield distribution converges at all.

A sufficient condition on F(x) for the type II limit to obtain is:

<u>Condition 3</u>

It is sufficient for $F \in D(\text{Type II } G(x))$ that it has no finite upper endpoint, and for each $x > 0$ and some $\gamma > 0$

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = x^{-1/\gamma}.$$

The latter condition boils down to regular variation at infinity, see Feller [1971, ch. VIII. 8]. If the density exists, by l'Hospital

$$\lim_{t\to\infty} \frac{1-F(tx)}{1-F(t)} = \lim_{t\to\infty} \frac{xf(tx)}{f(t)} = x^{-1/\gamma}.$$

See also Mood et. al. [1974, p. 261, th. 16].

The following discussion shows how the above conditions may be employed in specific cases. Return to the d.f.'s F(x) that have received most of the attention in the literature on stock returns. First, the normal and mixtures of the normal possess all moments, and hence Condition 1 applies. Given such a lack of leptokurtosis, these alternatives seem unfit for modelling stock returns. Second, the β_3 and the stable d.f. both satisfy Condition 2, because not all moments are finite. To verify that the β_3 class satisfies Condition 3 is straightforward. See for example Mood et. al. [1974, p. 262] for a proof in case of the Student-t distribution. Note that the degrees of freedom v are equal to the tail index α in (4). The proof that the stable distribution (which is not to be confused with the max-stable distribution G(x)), satisfies Condition 3, takes a little bit more effort, as only in specific cases does a closed form solution for its density $f_\alpha(t)$ exist (i.e. the normal, the Cauchy, and the inverted chi-square, for which the so called characteristic exponent α is respectively 2,1 and 1/2).

However, Ibragimov and Linnik [1971] provide the following asymptotic formula in the case $0 < \alpha < 2$, $\alpha \neq 1$:

$$(5) \qquad f_\alpha(t) = \frac{-1}{\pi t} \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \sin(\tfrac{1}{2}n\pi\alpha)\ \Gamma(n\alpha+1)\ t^{-n\alpha}.$$

Again, the α in (5) equals the α in (4). By Condition 3 we have

$$\lim_{t\to\infty} \frac{\Sigma\ (n!)^{-1}(-1)^n \sin(\tfrac{1}{2}n\pi\alpha)\ \Gamma(n\alpha+1)\ (tu)^{-n\alpha}}{\Sigma\ (n!)^{-1}(-1)^n \sin(\tfrac{1}{2}n\pi\alpha)\ \Gamma(n\alpha+1)\ (t)^{-n\alpha}}\ \frac{t^\alpha}{t^\alpha} = u^{-\alpha}.$$

From this discussion it is immediate that the competing F(x)'s are nested within their limit law G(x), and are distinguished by different values for α. Specifically, the leptokurtic stable hypothesis requires $\alpha < 2$ and the β_3-class allows for $\alpha \geq 2$. The idea is now to estimate α directly without a prior commitment to either hypothesis.

Broadly speaking, the estimation procedures for α fall into two categories. A traditional approach uses "yearly maxima" and assumes that each period's maximum exactly follows one of the three limit laws. If the type II limit law applies, direct estimation by maximum likelihood is consistent. A drawback of this method is that the excesses are assumed to follow the limit law exactly, whereas this is only approximately the case, c.f. Akgiray and Booth [ibid.]. Recently, some estimators have been proposed based on the largest order statistics, which require only that the distribution generating these observations is in a sense well behaved. This implies that the remaining estimation error can be solely attributed to the use of finite samples. For example, regular variation at infinity is often a sufficient condition. For this reason the focus here is on these methods.

Let X_1, X_2, ..., X_n be a sequence of stationary i.i.d. observations from some distribution function $F \in D(\text{Type II } G(x))$. We are interested in obtaining an estimate for γ, given that the type II limit applies. Define $X_{(1)} \le X_{(2)} \dots, \le X_{(n)}$ as the ascending order statistics from a sample X_1, X_2, ..., X_n of n consecutive exchange rate yields X_i. The proposed estimator reads:

$$(6) \qquad \hat{\gamma} = 1/\hat{\alpha} = \frac{1}{m} \sum_{i=1}^{m} \left[\log X_{(n+1-i)} - \log X_{(n-m)} \right]$$

The statistic $\hat{\gamma}$ first appears in Hill [1975]. Mason [1982] proves that if condition 3 is satisfied, $\hat{\gamma}$ is a consistent estimator for γ. Consistency obtains as well for a nonindependent sequence of X_i's, if the dependency is not too strong. By a result in Goldie and Smith [1987], it follows that $(\hat{\gamma}-\gamma)m^{1/2}$ is asymptotically normal with mean zero and variance γ^2.

The estimation procedure requires $m(n) \to \infty$, but for a finite sample it is not known how to choose m optimally. A heuristic procedure is to compute $\hat{\alpha}$ for different m and to select an m in the region over which $\hat{\alpha}$ is more or less constant. The existence of such a region is plausible by the following argument. In case one uses too few order statistics, i.e. m small, then α will vary heavily with m due to insufficient and imprecise information. In the opposite case, if one uses too many order statistics, the curvature of the distribution F generating the data weighs too heavily, i.e. only the tail probabilities are well approximated by the limit distribution G.

In the empirical section we follow a somewhat less arbitrary approach by first conducting a Monte Carlo study. Due to the asymptotic normality of $\hat{\gamma}$, the MSE criterion may be used for selecting an optimal m for given sample size n and distribution function. Subsequently, this m is used for computing $\hat{\gamma}$ in equation (6).

The implied asymptotic confidence interval also allows one to test directly for the two competing hypothesis about F(x), i.e. the stable and β_3 distributions. The former requires $0 < \alpha < 2$ and the latter allows for $\alpha \geq 2$. As noted, discrimination between the two hypotheses is hampered by their non nestedness. However, as our estimate of α is not conditional upon one of the two hypotheses being true, the asymptotic confidence interval may be used to test for H_0: $\alpha < 2$ against H_1: $\alpha \geq 2$. The asymptotic normality of $1/\hat{\alpha}$ may also be exploited to compare α estimates from different samples. The following statistic Q:

(7) $$Q = \left[\frac{\alpha_1}{\hat{\alpha}_1} - 1\right]^2 m_1 + \left[\frac{\alpha_2}{\hat{\alpha}_2} - 1\right]^2 m_2,$$

where the α and m are as in (6) and the subindexes refer to two independent samples, is asymptotically $\chi^2(2)$ distributed. It can be used to test for stability over different subsamples.

The data we use consist of daily stock returns. However, in view of the question concerning the time additivity of the yield distribution, the yields over extended periods of time are relevant as well. In case of a sum-stable distribution for daily returns, it follows immediately that monthly, yearly, etc. returns follow the same distribution, as this class of distributions is invariant under addition. However, for other leptokurtic alternatives like the Student-t this is not the case. What has not been realized in the economics literature, though, is the fact that the tail behavior of alternatives like the Student-t is unaffected by aggregation. That is, M_n generated from a Student-t or any finite sum of Student-t variates all tend to follow the type II limit law, with the same α! A sufficient condition for this invariance is given in Theorem 2.

<u>Theorem 2</u>

If 1-F(x) varies regularly at infinity, i.e. satisfies Condition 3, then the M_n from F(x) or any finite convolution of F(x) tend to follow the same limit law.

A lucid proof may be found in Feller [1974, ch. VIII.8]. Hence focussing on the limit law has the advantage that the amount of leptokurtosis is invariant to the chosen period length between observations for "all" leptokurtic alternatives. It guarantees a robustness to our methods that is not present in the alternative procedures. In particular, we can use the estimates for the tail index from daily returns in order to compute probabilities on exceedances for both daily and, say, monthly returns, and thereby exploit the relative efficiency of the daily estimates. Such probabilities on exceedances can be computed in different ways. Again, the procedures for deriving tail probabilities can be classified as to whether or not it is assumed that the extremes exactly follow one of the limit laws. A straightforward device is to assume that the exceedances Y over some threshold u exactly obey (4). Here, one needs in addition to an estimate for α, an estimate for the norming constant $1/a_n = F^{\leftarrow}(1 - \frac{1}{n})$, where the arrow stands for the inverse function, c.f. McCulloch [ibid.].

However, it is more elegant not to assume that the extreme observations exactly follow an extreme value distribution. This would be in the same spirit as the estimator given above. The following procedure only requires regular variation, and uses order statistics to estimate the exceedance levels $\hat{x}_p$

$$P\{X_1 \leq \hat{x}_p , \ldots, X_k \leq \hat{x}_p\} = F^k(\hat{x}_p) = 1 - p, \tag{8}$$

for small p and given k. The following is a consistent estimator of the exceedance levels:

$$(9) \qquad \hat{x}_p = \frac{(kr/pn)^{\hat{\gamma}} - 1}{1-2^{-\hat{\gamma}}} (X_{(n-r)} - X_{(n-2r)}) + X_{(n-r)},$$

where n is the number of observations, k is the time period considered, r = m/2, and p is the probability of exceedance. The proof of consistency of $\hat{x}_p$ is given in Dekkers and De Haan [1987].

A heuristic interpretation of the estimator in (9) is as follows. The pattern of the empirical d.f. $F_n(x)$ as signified by the level $X_{(n-r)}$ and step size $(X_{(n-r)} - X_{(n-2r)})$ is extrapolated outside its domain by using the way the limit law extends. The latter is represented through the multiplication factor in front of the stepsize. For p > 1/n the empirical d.f. is a good estimator for $\hat{x}_p$, due to its unbiased mean squared error consistency, see Mood et. al. [1974, p. 507]. But for p < 1/n, $F_n(x)$ is of no avail, and the above is a device to extend $F_n(x)$ beyond 1/n and use it for estimating $\hat{x}_p$. Alternatively, given an x_p level, we can invert equation (9) to estimate the associated probability on exceedance $\hat{p}$. Such is our aim in the empirical analysis.

3. Empirical Analysis

In this section we endeavor to evaluate the amount of leptokurtosis in stock returns and generate tables with probabilities on observing excessively high and low returns. The data consist of daily stock returns for ten stocks from the S&P

100 list, and two market indices: the S&P 500 index and the UMI which is an unweighted market index (i.e. the sum of all the shares listed on the New York Stock Exchange). The sample starts February 1962 and ends December 1986. Details about the data are given in the Data Sources section below.

By now, there is little disagreement about the qualitative properties of stock returns. Typically, daily returns are a stationary series that are strongly leptokurtic and possibly exhibit some low order serial dependence.[1] One can formally test for these properties by means of test statistics like the Dickey-Fuller test against unit roots. However, because the literature abounds with such tests and as there is little controversy, we prefer to revert to a more intuitive method for data description previously introduced by Mandelbrot [1963b]. In Figures 1 and 2 we computed the first four sequential moments for the S&P 500 index and three simulated date series. The simulated series are the Student-t distribution with 1 degree of freedom, i.e. the Cauchy distribution, and the Student-t with 2 and 3 degrees of freedom v. With v = 2, the mean is finite, and for v = 3 the mean and variance are finite.

For a distribution with a finite n-th moment, one expects the sequential plot of the n-th moment to settle down at its

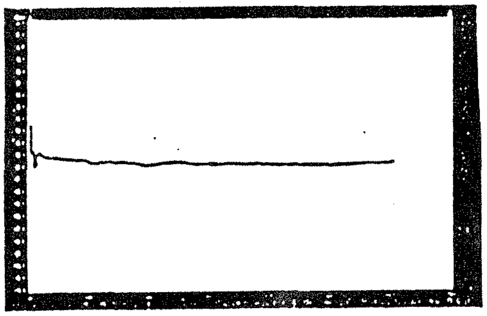
[1] Thus, the data meet the criteria for application of the theorems of the previous section, except for the serial dependence. However, as long as this dependence is not too strong the above results carry over, see for example Watson [1954].

theoretical level, as the sample gets large. Vice versa, if the n-th moment is infinite, it should eventually wander away. The idea is to compare the moment plots of observed data to simulated data, in order to get a rough indication about which moments might exist and hence infer the amount of leptokurtosis. From the Figures 1 and 2 it appears that the first two moments of the S&P index exist, while the existence of the fourth moment is doubtful. (The former conclusion is confirmed by Akgiray and Booth [1988, Fig. 5], but as they only plot the first two moments the latter result cannot be compared). The moment plots of the index correspond best to the pictures of the simulated Student-t with about 3 degrees of freedom.

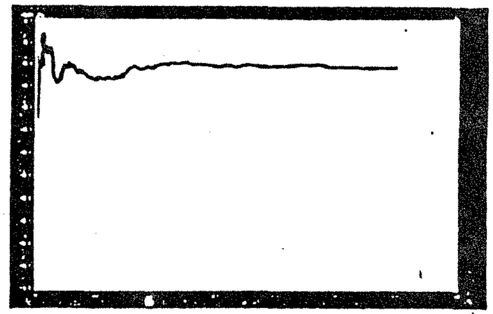
Although the above procedure for determining the amount of leptokurtosis is far from being precise, it clearly conveys the message that the return data are leptokurtic. In order to provide a higher precision, we propose to estimate the tail index α by using the estimator (6). A problem with this estimator is that it is conditional upon the portion m/n of the sample used for calculating the statistic $\hat{\gamma}$. As it is not known how to choose m in finite samples, we conducted a Monte Carlo experiment in order to find the m-level, conditional upon a sample size n, for which the MSE is minimal. This criterion seems appropriate given the asymptotic normality of $\hat{\gamma}$. As our sample return data comprise about 6000 observations, we set n = 6000. We simulated with four different distributions: the Student-t distribution with 1, 2 and 3 degrees of freedom that were used above, and the inverted chi square distribution. The latter distribution is known as the

Figure 1

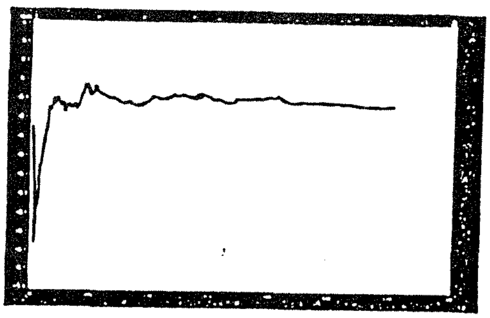
Sequential Moments: Mean and Variance



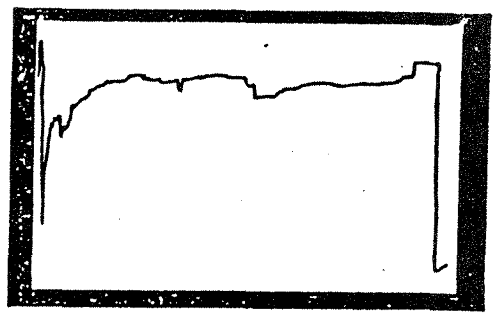
Mean of S&P 500



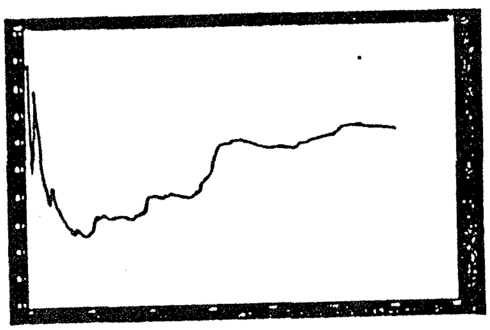
Mean of T with v = 3



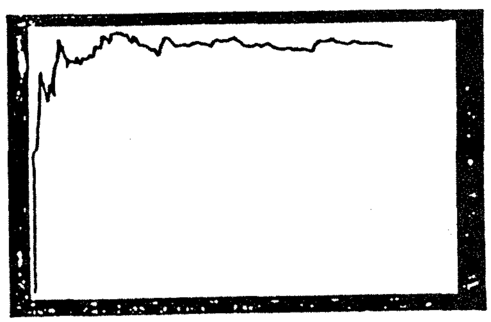
Mean of T with v = 3



Mean of Cauchy

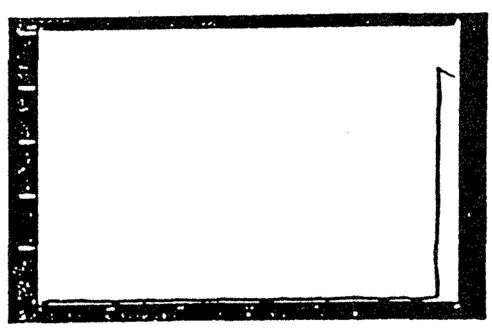


Variance of S&P 500



Variance of T with v = 3

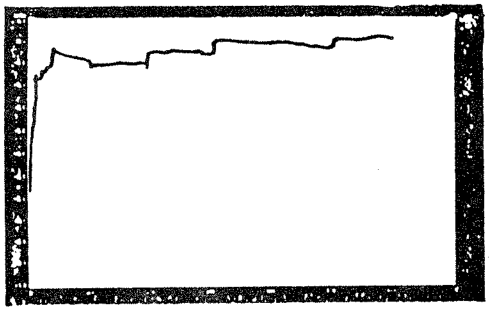
Variance of T with v = 2



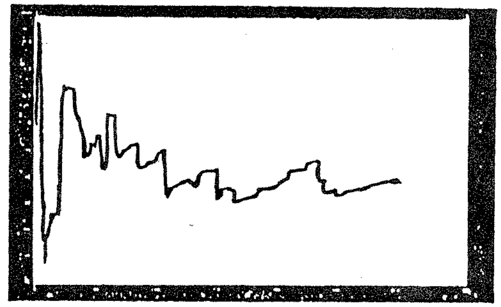
Variance of Cauchy

Figure 2

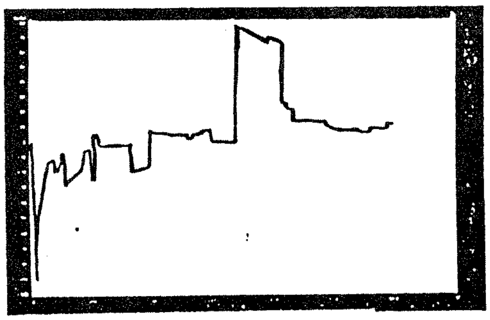
Sequential Moments: Skewness and Leptokurtosis



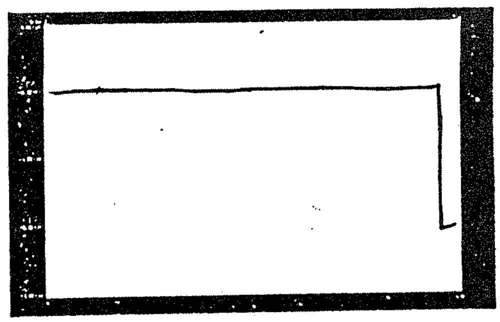
Skewness S&P 500



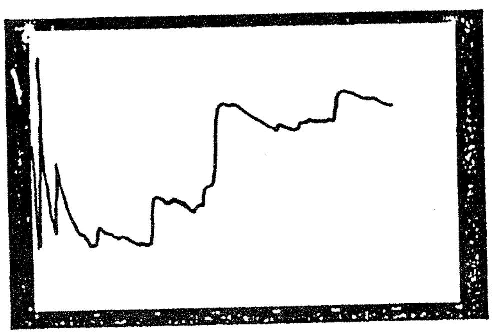
Skewness of T with v = 3



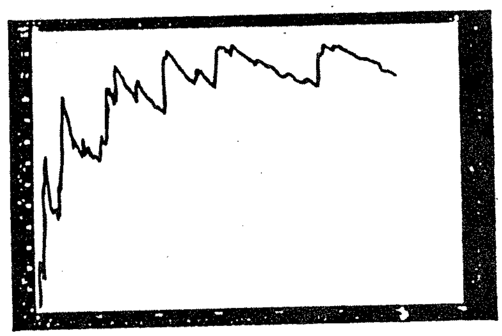
Skewness of T with v = 2



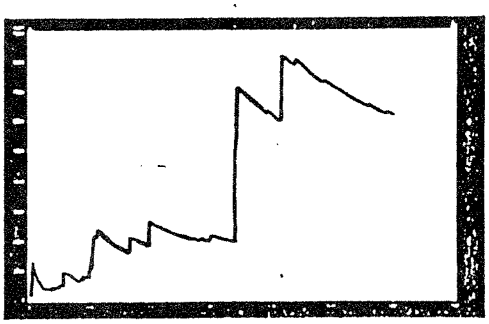
Skewness of Cauchy



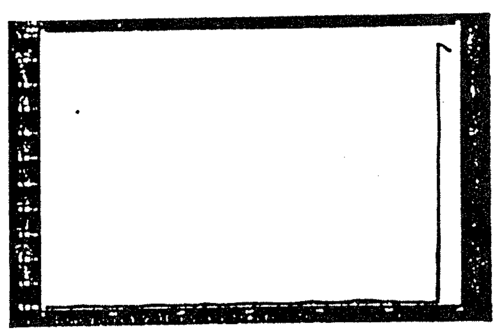
Kurtosis S&P 500



Kurtosis of T with v = 3



Kurtosis of T with v = 2



Kurtosis of Cauchy

Table 1

Minimizing MSE m-level*)

	$\alpha = 1/2$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
m	1680	470	170	100

*) The Monte Carlo experiment consisted of 100 replications of n = 6000 draws from four distributions with tail index α. From each replication α was estimated for m = 10, ..., 2000 with stepsizes of 10 by the estimator in (6). Subsequently the MSE was computed for each m.

Table 2

MSE*)

m	$\alpha = 1/2$	$\alpha = 1$	$\alpha = 2$	$\alpha = 3$
1680	.0028	.0491	.1443	.2079
	(.0024)			
470	.0081	.0020	.0048	.0138
		(.0021)		
170	.0206	.0054	.0014	.0031
			(.0015)	
100	.0349	.0092	.0019	.0023
				(.0011)

*) The table reports MSE's for different m-levels and distributions. Theoretical variances are reported between brackets for the diagonal elements (the diagonal elements, in a sense, represent the correct estimates).

stable distribution with characteristic exponent $\alpha = 1/2$, and is more leptokurtic than the other distributions. For each distribution 100 replications were conducted.

In Table 1 the MSE minimizing m-levels are reported. Clearly, these m-levels vary inversely with the true tail index α. The reason is that the lower is α, the fatter the tails of the distribution, and hence the more 'outliers' are available to estimate the tail index. The Table 1 also indicates that using 10% of the upper tail data, as suggested in Booth and Akgiray [ibid. p. 54], may be too much. Especially because Booth and Akgiray conclude that α is likely above 2. Table 2 gives some of the MSE's found in the simulation. Along the diagonal one finds the minimal MSE's for given α. Here we also calculated the theoretical variances of $(1/\hat{\alpha} - 1/\alpha)$ as $1/m\alpha^2$. These variances come quite close to the computed MSE's. Note that Table 2 exhibits an asymmetry. For high m, increasing α deterioriates the MSE's more than lowering α for low m. This suggests that using too many observations such that some do not belong to the tail, but rather to the center of the distribution, is more harmful than not using all the available information, i.e. all the observations from the tail. Thus the bias part of the MSE, due to inclusion of the center characteristics, rapidly dominates the variance part, which is due to the inefficient use of the available information. This conjecture is confirmed by the evidence from Table 2A.

Table 2A

The Bias Squared*)

m	α = 1/2	α = 1	α = 2	α = 3
1680	.0005763	.0483681	.143877	.207522
	(.21)	(.99)	(1.00)	(1.00)
470	.0000006	.000355	.0042288	.0119498
	(.00)	(.17)	(.88)	(.98)
170	.0000754	.0000605	.0004055	.0021182
	(.00)	(.01)	(.29)	(.68)
100	.0002305	.0000070	.0002949	.0010157
	(.01)	(.00)	(.15)	(.44)

*) The Table reports the bias (squared) part of the MSE of Table 2. The percentage of the MSE explained by the bias is given between brackets.

On the basis of these results we decided to be conservative and not usetoo many observations in the actual estimation. The estimates for the tail index in Table 3 are conditioned on m = 100. All estimates hoover between 3 and 4 and are significantly above 2 according to the 95% asymptotic confidence intervals. This resolves the long standing issue about the appropriateness of the β_3-class vis a vis the stable class. The latter hypothesis is clearly rejected against the former. Thus, our results confirm the findings of Praetz [ibid.] and Blattberg and Gonedes [ibid.]. The

Table 3

Tail Index Estimates*)

Stock	Upper Tail $\hat{\alpha}$	Maximum	Lower Tail $\hat{\alpha}$	Minimum
1	3.72	.10	4.65	-.09
	(2.99-4.45)		(3.74-5.54)	
2	3.64	.11	3.68	-.09
	(2.93-4.35)		(2.96-4.40)	
3	4.36	.09	5.22	-.10
	(3.51-5.21)		(4.20-6.24)	
4	3.76	.12	3.60	-.20
	(3.02-4.50)		(2.89-4.31)	
5	4.28	.11	4.41	-.15
	(3.44-5.12)		(3.54-5.28)	
6	3.34	.13	3.53	-.16
	(2.69-3.99)		(2.84-4.22)	
7	4.45	.12	4.53	-.13
	(3.58-5.32)		(3.64-5.42)	
8	4.62	.21	4.11	-.11
	(3.71-5.53)		(3.31-4.91)	
9	3.79	.15	4.17	-.10
	(3.05-4.53)		(3.35-4.82)	
10	4.56	.26	3.71	-.19
	(3.66-5.46)		(2.98-4.43)	
Index				
UMI	3.27	.07	3.37	-.05
	(2.63-3.91)		(2.71-4.03)	
S&P	3.96	.05	4.30	-.05
	(3.18-4.74)		(3.46-5.14)	

*) Reported are the α estimates and their 95% confidence intervals for both the upper and lower tail; the maximum and minimum sample returns are given as well. The estimates were based on the estimator in (6) with m = 100, c.f. Table 1.

hypothesis that the returns follow a discrete mixture of normal distributions, as in Kon [ibid.], is not tenable either, as the tail index is too small (significantly below 30, say). We conclude that stock returns are leptokurtic in comparison with the normal distribution, but still possess a finite variance such that the central limit theorem for addenda is applicable.

Given the estimates of the tail index for the individual stocks, the value of $\hat{\alpha}$ for the UMI is not too surprising. Recall Theorem 2, which says that the distribution of the maximum of i.i.d. variates or i.i.d. sums of these variates has the same tail index, conditional upon the distribution being leptokurtic. Because the UMI is just a sum of individual stock returns, and all individual stocks (approximately) have the same tail index, one expects the UMI to have the same α. There is one caveat, and that is the independence assumption, which does not apply as individual stock returns are usually assumed to be partially correlated. The appropriateness of extreme value theory for the S&P index is even more questionable, as this index is not just the sum of individual stocks, but a weighted sum. It is unclear what the weighing does to the distubional properties of the constituent parts, as the weights do vary over time.

A question is how robust our estimates for the tail index are. One might argue for example that institutional changes on financial markets affect the α's, c.f. Akgiray and Booth [ibid., p. 52]. In order to investigate this issue we split the sample into two parts, using April 26 of 1973 as the dividing day. At this day the

Table 4

Stability of the Tail Index*)

Stock	Upper Tail Index $\hat{\alpha}$ pre 73	post 73	Interval	Lower Tail Index $\hat{\alpha}$ pre 73	post 73	Interval
1	3.52	3.04	2.69-3.85	3.22	3.82	2.84-4.10
2	3.12	3.33	2.58-3.85	3.52	2.79	2.57-3.58
3	3.79	3.38	2.88-4.25	3.37	4.06	3.00-4.30
4	3.21	3.26	2.59-3.88	3.18	3.19	2.55-3.82
5	3.85	3.10	2.82-3.97	3.21	3.21	2.57-3.85
6	3.28	2.90	2.48-3.65	3.31	3.03	2.54-3.77
7	3.07	4.09*	2.95-3.93	3.28	3.75	2.83-4.14
8	3.26	2.91	2.48-3.65	3.62	3.36	2.80-4.16
9	3.32	3.09	2.57-3.83	3.10	3.39	2.60-3.86
10	3.82*	2.85	2.75-3.65	3.06	3.25	2.53-3.77
Index						
UMI	2.84	3.32	2.48-3.61	3.73*	2.90	2.71-3.72
S&P	2.52	3.33*	2.40-3.23	3.71	3.65	2.95-4.14

*) The subsample α estimates are conditoned upon m = 75. This number is also used in computing the Q-level in formula (7), i.e. $m_1 = m_2 = 75$. The 'Interval' columns provide the interval for which H_0: pre α = post α is not rejected at the 5% significance level. Whenever equality is rejected for a particular α, this is indicated by a superindex *.

Chicago Board Option Exchange was organized. Moreover, it was the year of the final demise of the Bretton Woods agreement and the economy was hit by the aggregate shock of the jump in the price of oil. Table 4 investigates the stability of α by means of the Q-test in equation (7). For almost all the cases the stability of α cannot be rejected, at the 5% significance level. Hence, the tail properties of the stock returns probably have not been affected by the above mentioned changes.

Now that we have a fair idea about the leptokurtosis of the stock returns in terms of the tail index, it is insightful to translate this into probabilities on large changes. Table 5 was generated by formula (9), using the $\hat{\alpha}$ from Table 3. It gives probabilities on observing daily yields above (or below) certain extremely high (or low) returns, over the time span of one year. For example, the probability that <u>within a given year</u> stock 6 experiences a drop in its share price of more than 20% is 0.02475 from Table 5B. Stated differently, about once in every 1/.025 = 40 years, the share price of stock 6 will fall by more than 20%. From Table 3 we know that the largest daily drop observed within the sample of 24 years was 16%. Hence, Table 5 uses our knowledge about the tails of the distribution to extend our knowledge of stock returns over longer time spans and lower probabilities than empirically observed. By comparing the probabilities for different stocks, the table provides an alternative indicator for the amount of leptokurtosis. In this respect it can be used as a device for portfolio selection. Suppose an investor is interested in selecting the stock which minimizes the probability of extreme losses (i.e. the

Table 5A

Upper Tail Probabilities on Daily Returns*)

Returns	Probabilities					
	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5	Stock 6
. 1	.08183	.15470	.07085	.20125	.18893	.38445
.14	.02447	.04516	.01766	.05940	.04602	.13267
.18	.00985	.01802	.00617	.02368	.01594	.05927
.20	.00671	.01227	.00395	.01607	.01021	.04218
.25	.00297	.00543	.00153	.00706	.00397	.02045
.30	.00152	.00279	.00070	.00359	.00183	.01128
	Stock 7	Stock 8	Stock 9	Stock 10	UMI	S&P
. 1	.14772	.40489	.12767	.94938	.01604	.00718
.14	.03126	.08157	.03582	.22617	.00538	.00191
.18	.00990	.02488	.01385	.07609	.00237	.00071
.20	.00613	.01515	.00930	.04801	.00168	.00047
.25	.00222	.00532	.00399	.01799	.00081	.00019
.30	.00097	.00226	.00200	.00803	.00045	.00009

Table 5B

Lower Tail Probabilities on Daily Returns*)

Returns	Probalities					
	Stock 1	Stock 2	Stock 3	Stock 4	Stock 5	Stock 6
. 1	.01952	.14225	.01092	.16479	.10972	.25174
.14	.00433	.04508	.00192	.05129	.02599	.08250
.18	.00139	.01880	.00052	.02127	.00879	.03539
.20	.00086	.01299	.00030	.01469	.00557	.02475
.25	.00031	.00590	.00009	.00668	.00211	.01156
.30	.00013	.00308	.00003	.00350	.00095	.00618

	Stock 7	Stock 8	Stock 9	Stock 10	UMI	S&P
. 1	.12805	.10826	.07519	.51892	.02165	.00361
.14	.02910	.02529	.01948	.16111	.00720	.00087
.18	.00955	.00865	.00703	.06629	.00314	.00029
.20	.00597	.00553	.00458	.04555	.00222	.00019
.25	.00220	.00216	.00184	.02047	.00105	.00007
.30	.00097	.00100	.00087	.01061	.00057	.00003

*) The table is constructed by using formula (9), with $k = 260$, i.e. approximately covering one year, $r = 50$, n is given in the data sources and the α's are from Table 2.

minimax strategy). Fixing the extreme losses at 30% or higher, the investor should select stock 3. Note that this stock also has the highest tail index estimate, $\hat{\alpha} = 5.22$, i.e. is the least leptokurtic.

The table also gives some perspective to the events of October 1987. On October the 19th many stocks and the market indices fell by more than 20%. From the Table 5 we see this was not an unlikely event for most of the individual stocks, and is corroborated by the minima reported in Table 3. However, what made this into a rare event, was that all stocks dropped simultaneously by such a large amount. This market crash should be and was also indicated by the market indices. However, according to our Table 5, this was a highly unlikely event: Probability {UMI falls by more than 0.2 in any given year} = .0022. This suggests that the crash of October 19 occurs once in about every 450 years. Therefore, one might conclude that the crash was quite accidental.

However, this analysis is not wholly convincing for the following reason. Since 1899 until 1987 there have been two days with drops (in the Dow Jones industrial average) in excess of 20% and four days with drops exceeding the 10%, generating tail probability estimates of the empirical distribution function of about .023 and .045. Hence, the probabilities in Table 5 may underestimate the true probabilities on market crashes. A problem with our sample is that the largest observed changes in the indices was about 5%, c.f. Table 3. These maxima and minima are well below the ones for the individual stocks. This makes economic sense as returns on particular stocks are only partially correlated with the other assets, and hence a portfolio should diversify the risk on individual stock specific shocks. Hence, one expects the indices to exhibit large changes less frequently than the individual stocks. Moreover, during the sample period aggregate shocks, like the oil shock, were observed. But, apparently, these did not lead to an instantaneous market plunge. Therefore, the evidence in Table 5 is understandable, but in view of a longer time span may be too optimistic. It would be an useful exercise to extend the sample and repeat the above. For the moment we like to take a somewhat different route.

Although the aggregate shocks covered by our sample period did not imply excessive daily plunges, they did leed to sustained declines. Hence, instead of focussing on daily returns, we use returns over longer time spans in order to capture the probabilities on market crashes. Table 6 provides probabilities on observing <u>monthly yields</u> in excess of a given return over the time span of <u>one year</u>. Note that the table was generated by using the daily $\hat{\alpha}$'s. Such is permissible in view of Theorem 2.

Table 6 goes some way towards resolving the probable downward bias present in Table 5. From the Table 6 we see that the probability that within any year a monthly market drop exceeds 20% or 30% is only .16 or .06 respectively, i.e. occurs once in about every 6 or 15 years respectively. On the other hand, in the 42 years following World War II, there were eight periods in which the Dow fell by more than 20% and only 2 periods in which the Dow fell by

Table 6

Probabilities on Monthly Returns for the UMI*)

Returns

.1	.14	.18	.20	.25	.30	.50	1.00	2.00
Probabilities in Upper Tail								
.7740	.3493	.1842	.1395	.0760	.0456	.0102	.0012	.0001
Probabilities in Lower Tail								
.5653	.3209	.1977	.1589	.0972	.0633	.0170	.0023	.0003

*) The table was constructed by forming monthly yields from the daily yields through addition and then applying formula (9) with $k = 12$, i.e. covering a year, $r = 30$, $n = 294$ and the α's from Table 2.

Table 7

LARGEST CORRECTIONS OF POSTWAR PERIOD*

Following are the 10 biggest percent corrections of the Dow Jones industrial average since the end of World War II.

Date	Beginning Price	End Price	Percent Change
Dec. 3,1968 - May 26,1970	985.20	630.15	-35.94%
March 13 - Oct. 4,1974	891.57	584.65	-34.44
Dec. 13,1961 - June 26,1962	734.90	535.75	-27.10
Sept. 21,1976 - Feb. 28,1978	1014.78	742.11	-26.87
Feb. 9 - Oct. 7,1966	995.14	744.31	-25.21
April 27,1981 - Aug. 12,1982	1024.04	776.91	-24.13
May 29 - Oct. 9,1946	212.50	163.12	-23.24
Oct. 26 - Dec. 5,1973	987.05	788.30	-20.14
July 12 - Oct. 22,1957	520.76	419.78	-19.39
Jan. 11 - Aug. 22,1973	1051.69	851.89	-19.00

*Includes corrections in which the average dropped more than 10 percent and then rose as much or more.

Source: New York Times, Friday October 16, 1987.

more than 30%, c.f. Table 7. This corresponds to empirical exceedance probabilities of .19 and .05 respectively. These appear to be reasonably close to the ones reported in Table 6, although one hasto bear in mind that the periods covered in Table 7 are in excess of one month. (which makes the bias go in the oppositite direction).

4. Conclusion

The literature on the distribution of stock returns unanimously agrees that the returns are leptokurtic. There is, however, disagreement about the amount of leptokurtosis. Three types of alternative distributions for stock returns have been widely considered, varying in their amount of leptokurtosis. As these different models are non nested, the controversy lingers on. In this paper we take another route to investigate this leptokurtosis, by investigating only the tail behavior of the returns, instead of looking at the entire distribution. This gives the following trade off. One loses the possibility to say something about the center of the distribution, but one gains the ability to nest the different models via the limit law for maxima. Empirical estimates of this encompassing model point towards the existence of a finite mean and variance but infinite higher moments, lending support to the β_3-class distributions vis a vis the stable class and discrete mixtures of normal distributions.

The tail estimates were in turn used to generate probabilities on exceedances. Such tables may be useful to investors who want to select a conservative portfolio. The tables also indicate the difference between investing in a specific stock or in a market portfolio. Not surprisingly, the risk on an extremely large or small yield is much higher for the former strategy than the latter. In this light, the events of October 1987 seem more a rare event than a drama that occurs repeatedly during a person's life.

While we do not claim to have resolved all problems with respect to the distribution of share prices, it is hoped that these new views and techniques may prove useful for further research in this area. For example, one might employ the above method to generate trading limits for the much discussed circuit breakers. More generally, as many economic problems can be phrased in terms of dealing with maxima or minima, extreme value theory seems a useful but as of yet insufficiently exploited device for empirical analysis.

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Data Sources

Data were obtained from the CRSP Tape compiled by the Graduate School of Business of the University of Chicago. We used 6000 daily dividend compensated returns starting February 14, 1962. Specifically, for stocks 2, 3, 4, 6, 7, 8, 9 and the S&P 500 index the sample size was 6156, stocks 5 and 10 comprised 6155 returns, 1 consisted of 6153 and the UMI index of 6000 observations. For the individual stocks every tenth stock from the S&P 100 list was chosen, provided data was available over the entire sample period. The stocks are listed with the following ticker symbol: (1) IBM, (2) MOB, (3) MBK, (4) KM, (5) AMP, (6) HON, (7) NCR, (8) GW, (9) OI, and (10) BC. Furthermore two market indexes were used: the S&P 500 and the Unweighted Market Index, abbreviated to UMI. Further details about the data are available from the first author upon request.