A Note on the Relative Efficiency of the Cochrane-Orcutt and OLS Estimators when the Autocorrelation Process has a Finite Past

Authors	Daniel L. Thornton
Working Paper Number	1984-002A
Creation Date	January 1984
Citable Link	https://doi.org/10.20955/wp.1984.002
Suggested Citation	Thornton, D.L., 1984; A Note on the Relative Efficiency of the Cochrane-Orcutt and OLS Estimators when the Autocorrelation Process has a Finite Past, Federal Reserve Bank of St. Louis Working Paper 1984-002. URL https://doi.org/10.20955/wp.1984.002

A NOTE ON THE RELATIVE EFFICIENCY OF THE COCHRANE-ORCUTT AND OLS ESTIMATORS WHEN THE AUTOCORRELATION PROCESS HAS A FINITE PAST

Daniel L. Thornton
Federal Reserve Bank of St. Louis

84-002

This note shows that the ordinary least squares estimator of a first-order autoregressive model is always more efficient relative to the Cochrane-Orcutt estimator if the autocorrelation process has a finite past than if its past is infinite. This result cast doubt on the usual suggestion that it might be better to delete the initial observation rather than weight it if the autocorrelation process has a finite past.

ID 4704R
DLT:cla
Disk 76R
Draft 15
12/6/83

A NOTE ON THE RELATIVE EFFICIENCY OF THE COCHRANE-ORCUTT AND OLS ESTIMATORS WHEN THE AUTOCORRELATION PROCESS HAS A FINITE PAST

1. Introduction

A number of studies have noted the relative inefficiency of the Cochrane-Orcutt (C-O) transformation to circumvent the problem of first-order autocorrelation. Chipman (1979) has shown that the C-O estimator is less efficient than ordinary least squares (OLS) in a simple linear time trend model, while Kadiyala (1968) has shown that is is less efficient than the least squares estimator for the mean model.

Furthermore, Maeshiro (1979), Doran (1979), Doran and Griffiths (1983) and Fomby and Guilkey (1983) have shown that the Prais-Winsten estimator which includes the weighted first observation is considerably more efficient relative to the C-O estimator. These comparisons are usually made, however, in terms of an AR(1) process which is assumed to have an infinite past. If there is reason to suspect that the autocorrelation process has a finite past, it is usually suggested that the C-O estimator is preferable to weighting the initial observation, [e.g. Judge, et. al. (1980, p. 182) and Theil (1971, p. 253)].

Maeshiro (1976, 1980) has shown, however, that the inefficiency of the C-O estimator relative to OLS is due to increased colinearity of the transformed variables. He argues that the gain efficiency associated with the reduction or elimination of autocorrelation is not enough to offset the loss in efficiency due to the increased colinearity of the data are smoothly trended. Maeshiro (1979) shows that much of the advantage to weighting the initial observation results from the reduction in colinearity.

If the suggestion to use the C-O estimator when there is no reason to assume that the AR(1) process has an infinite past is correct, Maeshiro's results would suggest that the efficiency of OLS relative to the C-O estimator should decline if the autoregressive process has a finite past. The result obtained here, however, are not consistent with this conjecture. It is shown that the OLS estimator is even more efficient relative to the C-O estimator when the autocorrelation process has a finite past than when it is infinite.[1/] Calculated values of the efficiency of C-O relative to the OLS estimator are presented for the simple mean model.

2. The Model

Consider the regression model

(1) $Y = X\beta + \varepsilon$,

where Y is a nx1 vector of successive observations on the dependent variable, X is a nxk matrix of full rank on successive observations of fixed regressor variables, β is a kx1 vector of unknown coefficients and ε is a nx1 vector of disturbances. Assume that

$$(2)\quad \varepsilon_t = \begin{cases} \rho\varepsilon_{t-1} + u_t, & t > -q \\ u_t & t \leq -q, \end{cases}$$

where u_t is nid $(0, \sigma_u^2)$.[2/] The parameter q is assumed to be known and is continuous for $q \geq 1$; q-1 denotes the number of periods since the process began.

It is easy to show that

$$E(\epsilon\epsilon') = \frac{\sigma_u^2}{1-\rho^2} \phi,$$

where

$$\phi^{-1} = \frac{1}{1-\rho^2} \begin{bmatrix} \frac{1-\rho^{2(q+1)}}{1-\rho^{2q}} & -\rho & 0 & 0 & \cdots & 0 & 0 \\ -\rho & 1+\rho^2 & -\rho & 0 & \cdots & 0 & 0 \\ 0 & -\rho & 1+\rho^2 & -\rho & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot \\ 0 & 0 & 0 & 0 & \cdots & -\rho & 1 \end{bmatrix}$$

Furthermore, equation (1) can be transformed to the classical model with a scalar covariance matrix by premultiplying (1) by the transformation matrix C,

$$C = \begin{bmatrix} (1-\rho^2/1-\rho^{2q})^{1/2} & 0 & 0 & \cdots & 0 & 0 & 0 \\ -\rho & 1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & -\rho & 1 & \cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & -\rho & 1 \end{bmatrix}$$

The C-O transformation premultiplies equation (1) by a transformation matrix Q, where Q is a (n-1) x n matrix obtained by deleting the top row of C. If q is known, the only advantage to the C-O approach is computational.

If the customary stationarity condition is assumed (i.e., $|\rho|<1$), then the usual covariance matrix A is

$$\lim_{q\to\infty} \phi = A,$$

and the usual P-W transformation matrix, M, is

$$\lim_{q\to\infty} \phi = M.$$ 3/

3. The Relative Efficiency of OLS and Cochrane-Orcutt Estimates

Denote the least squares and C-O estimators by $\hat{\beta}$ and $\tilde{\beta}$, respectively. The relative efficiency, E, of $\tilde{\beta}$ with respect to $\hat{\beta}$ can be expressed as

$$E = \frac{|\Sigma_{\hat{\beta}}|}{|\Sigma_{\tilde{\beta}}|},$$

where $\Sigma_{\hat{\beta}}$ and $\Sigma_{\tilde{\beta}}$ are the covariance matrices for $\hat{\beta}$ and $\tilde{\beta}$, respectively, i.e.,

$$\Sigma_{\hat{\beta}} = \frac{\sigma_u^2}{1-\rho^2} \; (X'X)^{-1} X' \; \phi X \; (X'X)^{-1}$$

and

$$\Sigma_{\tilde{\beta}} = \frac{\sigma_u^2}{1-\rho^2} \; (X'Q'QX)^{-1} X'Q'Q\phi Q'QX(X'Q'QX)^{-1}.$$

It is easy to show that

$$Q'Q = (1-\rho^2) \; (A^{-1} - \theta)$$

and

$$Q\phi Q' = (1-\rho^2) \; I,$$

where θ is a nxn matrix whose (1, 1) element is one and all other elements are zero. Making use of these results, the efficiency ratio reduces to

$$E = \frac{|X'\phi X||X'(A^{-1}-\theta)X|}{|X'X|^2}.$$

Note that

$$\lim_{q\to\infty} E = \frac{|X'AX|\ |X'(A^{-1}-\theta)X|}{|X'X|^2} = E^*$$

By the binominal inverse theorem,

$$\phi = A - \frac{Auu'A}{1+u'Au},$$

where $u' = (\rho^{2q}/(1-\rho^{2q})\ 0\ 0\ .\ .\ .\ 0)$. Using this result, it is easy to show that $|X'AX| > |X'\theta X|$ and, thus, $E < E^*$.4/

4. Results for the Mean Model

Consider the mean model, where $X = \ell = (1\ 1\ 1\ ...\ 1)'$. In this case,

$$X'\phi X = \frac{n(1-\rho^2) - 2\rho(1-\rho^n)}{(1-\rho)^2} - \rho^{2q}\left(\frac{1-\rho^n}{1-\rho}\right)^2.$$

Likewise,

$$X'(A^{-1}-\theta)X = \frac{(n-1)(1-\rho)}{1+\rho}.$$

Therefore, E reduces to

$$E = \frac{1}{n^2}\left[\frac{n(1-\rho^2) - 2\rho(1-\rho^n) - \rho^{2q}(1-\rho^n)^2}{(1-\rho)^2}\right]\left[\frac{(n-1)(1-\rho)}{(1+\rho)}\right].$$

Note that

$$\lim_{q\to\infty} E = \frac{n(1-\rho^2) - 2\rho(1-\rho^n)}{n^2 (1-\rho)^2} \cdot \frac{(n-1)(1-\rho)}{1+\rho} \quad .$$

This is the expression obtained by Kadiyala. Calculated values of E and E* for the mean model are presented in table 1. The results indicate that the C-O estimator is inefficient relative to the OLS estimator, especially for large positive values of ρ and small values of q.

4. Conclusions

It has been shown that the efficiency of the Cochrane-Orcutt estimator relative to the least squares estimator is strictly smaller when the autocorrelation process has a finite past than when its past is infinite. Thus, the usual suggestion that the C-O transformation be employed in situations where it is not reasonable to assume the autocorrelation process has an infinite past is questioned.

FOOTNOTES

1/Actually, this result is consistent with Maeshiro's result since the initial observation gets more weight for the finite past model than for the infinite past model. Compare matrices C and M below.

2/There are of course an infinite number of finite past assumptions that could be made. For example, let εt be nid(o, Σ_ε^2) for $t \leq - q$ and let u_t be nid(o, σ_u^2) for $t > q$, and further assume that $\sigma_\varepsilon^2 = \sigma_u^2/(1-\rho^2)$. This would eliminate the heteroskedasticity in (2), but would also eliminate the distinguish between finite and infinite pasts. Since this distinction has been an important characteristic of nearly all discussions of the first-order autoregression model, attention was limited to the generalization of the usual model.

3/The reader unfamiliar with these matrix forms should see Kadiyala (1969) or Judge, et. al. (1981, p. 181).

4/See Graybill (169, p. 211 and 330).

REFERENCES

Chipman, J. S., "The Efficiency of Least Squares Estimation of Linear Trend When Residuals are Autocorrelated," Econometrica 47 (1979), pp. 115-128.

Doran, H. E., Omission of Weighted First Observation in an Autocorrelated Regression Model: A Discussion on Loss of Efficiency," Working Papers in Econometrics and Applied Statistics No. 6, University of New England, Armidale, Australia, 1979.

__________, and W. E. Griffiths, "On the Relative Efficiency of Estimators Which Include the Initial Observations in the Estimation of Seemingly Unrelated Regressions with First-Order Autoregressive Disturbances" Journal of Econometrics 23 (October 1983), pp. 165-91.

Fomby, T. B. and D. K. Guilkey, "An Examination of Two-Step Estimators for Models with Lagged Dependent Variables and Autocorrelated Errors," Journal of Econometrics 22 (August 1983), pp. 291-300.

Graybill, F. A., Introduction to Matrices with Application in Statistics, Belmont California: Wadsworth Publishing Co., Inc. 1969.

Judge, G. G., W. E. Griffiths, R. C. Hill, and T. Lee, The Theory and Practice of Econometrics, New York: John Wiley and Sons, 1980.

Kadiyala, K. R., "A Transformation Used to Circumvent the Problem of Autocorrelation," Econometrica 36 (1968), pp. 93-96.

Maeshiro, A., "Autoregression Transformation, Trended Independent Variables and Autocorrelated Disturbance Terms," The Review Of Economics and Statistics 58 (1976), pp. 497-504.

__________, "Autocorrelation and Trended Explanatory Variables: A Reply," Review of Economics and Statistics, 62 (1980), pp. 487-89.

Table 1:
Calculated Values for E for the Mean Model

		n				
q	ρ	10	20	50	100	250
	-0.99	1.71571	1.73302	1.59969	1.41851	1.19457
	-0.50	0.98994	0.99750	0.99960	0.99990	0.99998
	-0.20	0.93375	0.96781	0.98735	0.99371	0.99749
	0.20	0.85875	0.92823	0.97102	0.98546	0.99417
	0.50	0.75018	0.87083	0.94733	0.97350	0.98936
1	0.75	0.50576	0.72697	0.88760	0.94333	0.97722
	0.90	0.18190	0.39845	0.71257	0.85401	0.94127
	0.95	0.06211	0.17524	0.47286	0.70766	0.88149
	0.97	0.02548	0.08129	0.28613	0.53750	0.80192
	0.98	0.01210	0.04127	0.17101	0.38434	0.70469
	0.99	0.00324	0.01183	0.05909	0.16956	0.46604
	-0.99	1.72242	1.74586	1.62461	1.45094	1.22199
	-0.50	1.01988	1.01333	1.00613	1.00320	1.00131
	-0.20	0.93750	0.96979	0.98817	0.99412	0.99766
	0.20	0.86250	0.93021	0.97183	0.98587	0.99434
	0.50	0.78012	0.88667	0.95387	0.97680	0.99069
10	0.75	0.60822	0.78732	0.91266	0.95599	0.98231
	0.90	0.32023	0.53126	0.78286	0.88988	0.95571
	0.95	0.14297	0.28431	0.56604	0.76225	0.90372
	0.97	0.06717	0.14771	0.36665	0.59784	0.82867
	0.98	0.03437	0.08007	0.22959	0.43942	0.73377
	0.99	0.00994	0.02467	0.08402	0.20199	0.49346
	-0.99	1.72858	1.75765	1.64750	1.48071	1.24717
	-0.50	1.01988	1.01333	1.00613	1.00320	1.00131
	-0.20	0.93750	0.96979	0.98817	0.99412	0.99766
	0.20	0.86250	0.93021	0.97183	0.98587	0.99434
	0.50	0.78012	0.88667	0.95387	0.97680	0.99069
20	0.75	0.60880	0.78766	0.91280	0.95606	0.98234
	0.90	0.34169	0.55186	0.79376	0.89544	0.95795
	0.95	0.17715	0.33042	0.60543	0.78532	0.91311
	0.97	0.09322	0.18921	0.41695	0.63554	0.84537
	0.98	0.05125	0.10948	0.27400	0.48115	0.75581
	0.99	0.01610	0.03646	0.10690	0.23177	0.51864
	-0.99	1.74110	1.78163	1.69405	1.54129	1.29838
	-0.50	1.01988	1.01333	1.00613	1.00320	1.00131
	-0.20	0.93750	0.96979	0.98817	0.99412	0.99766
	0.20	0.86250	0.93021	0.97183	0.98587	0.99434
	0.50	0.78012	0.88667	0.95387	0.97680	0.99069
50	0.75	0.60881	0.78766	0.91280	0.95606	0.98234
	0.90	0.34466	0.55470	0.79527	0.89621	0.95826
	0.95	0.19537	0.35500	0.62643	0.79763	0.91812
	0.97	0.11928	0.23072	0.46727	0.67325	0.86208
	0.98	0.07506	0.15097	0.33664	0.54004	0.78690
	0.99	0.02863	0.06043	0.15345	0.29234	0.56985
	-0.99	1.75624	1.81060	1.75030	1.61447	1.36027
	-0.50	1.01988	1.01333	1.00613	1.00320	1.00131
	-0.20	0.93750	0.96979	0.98817	0.99412	0.99766
	0.20	0.86250	0.93021	0.97183	0.98587	0.99434
	0.50	0.78012	0.88667	0.95387	0.97680	0.99069
∞	0.75	0.60881	0.78766	0.91280	0.95606	0.98234
	0.90	0.34466	0.55471	0.79527	0.89621	0.95826
	0.95	0.19625	0.35619	0.62744	0.79822	0.91836
	0.97	0.12427	0.23867	0.47692	0.68048	0.86529
	0.98	0.08514	0.16854	0.36318	0.56498	0.80008
	0.99	0.04376	0.08940	0.20970	0.36553	0.63173