Open Science Repository Mathematics

Raab, W. (2013). The Truth of the Riemann Hypothesis. Open Science Repository Mathematics, Online(open-access), e70081985.
doi:10.7392/openaccess.70081985

The truth of the Riemann hypothesis

Werner Raab

Abstract

It is proposed to show that the reciprocal of the Riemann zeta function is holomorphic in the complex half-plane on the right of the abscissa 1/2.

Keyword Riemann hypothesis
Mathematics Subject Classification 11M26

1 Summary

We describe a way to prove that the Mellin transform

$$v(s) = \frac{\pi}{\sin(\pi s)(1/2 - s)\zeta(3/2 - s)} = \int_0^\infty t^{s-1} w(t)\,\mathrm{d}t \tag{1}$$

of the function

$$w(t) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} t^{-s} v(s)\,\mathrm{d}s = \frac{1}{\sqrt{t}}\int_0^\infty \frac{1}{\cosh(\pi y)y}\Im\frac{t^{iy}}{\zeta(1+iy)}\,\mathrm{d}y \tag{2}$$

is holomorphic in the strip: $0 < \Re(s) < 1$ of the complex variable s. This result would imply the truth of the Riemann hypothesis, since it says that the zeta-function $\zeta(s)$ has no zero s with $1/2 < \Re(s) < 3/2$, and, for reasons of symmetry, no zero s with $0 < \Re(s) < 1/2$.

In proof of this we derive the estimates

$$w(t) = O(1) \quad \text{when} \quad t \to 0 \tag{3}$$

and

$$w(t) = O(1/t) \quad \text{when} \quad t \to \infty\,. \tag{4}$$

2 Ramanujan's master theorem

In order to show that a given function $v(s)$ is holomorphic within a strip: $\alpha < \Re(s) < \beta$ of the complex variable s, it suffices to represent it as the Mellin transform

$$v(s) = \int_0^\infty t^{s-1} w(t)\,\mathrm{dt}$$

of a real function $w(t)$ of the positive real variable t with the properties $w(t) = O(t^{-\alpha})$ when $t \to 0$ and $w(t) = O(t^{-\beta})$ when $t \to \infty$. Sometimes it happens that a method of Ramanujan is applicable to construct a Mellin transform. This method, nowadays called 'Ramanujan's master theorem' [6], is discussed in great detail by Hardy in chapter XI of his book on Ramanujan's life and work [1]. The two formulas

$$\text{(A)} \quad \int_0^\infty x^{s-1} \left\{ \phi(0) - x\phi(1) + x^2\phi(2) - ... \right\} dx = \frac{\pi}{\sin s\pi}\phi(-s)$$

and

$$\text{(B)} \quad \int_0^\infty x^{s-1} \left\{ \lambda(0) - \frac{x}{1!}\lambda(1) + \frac{x^2}{2!}\lambda(2) - ... \right\} dx = \Gamma(s)\lambda(-s)$$

are commented as follows. "In this lecture I propose to speak about some theorems of Ramanujan which have not attracted very much attention, which are, as I said in my opening lecture, 'inevitably less impressive' than much of his work, but which are still very interesting and will repay a careful analysis."

If we apply the calculus of residues to the integral (2), then we obtain the series

$$w(t) = \sum_{k=0}^\infty \mathrm{Res}_{s=-k}\, t^{-s} v(s)$$

$$= \sum_{k=0}^\infty \frac{t^k}{(1/2+k)\zeta(3/2+k)} \mathrm{Res}_{s=-k} \frac{\pi}{\sin(\pi s)} = \sum_{k=0}^\infty \frac{(-t)^k}{(1/2+k)\zeta(3/2+k)}.$$

Introducing the function

$$u(s) = \frac{1}{(s-1)\zeta(s)} \tag{5}$$

we can write

$$v(s) = \frac{\pi}{\sin(\pi s)} u(3/2 - s) \tag{6}$$

and

$$w(t) = \sum_{k=0}^{\infty} u(3/2+k)(-t)^k. \tag{7}$$

This power series has the radius 1 of convergence. With $\phi(s) = u(3/2+s)$ Ramanujan's formula (A) yields our formula (1).

The formulas (A) and (B) are logically equivalent because of the identity

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

The idea of our proof is due in substance to Marcel Riesz [5] who in 1916 tried to show that

$$\frac{\Gamma(1-s/2)}{\zeta(s)} = \int_0^\infty \frac{F(x)}{x} x^{-s/2}\,\mathrm{d}x \tag{8}$$

with the Mellin inversion

$$F(x) = \frac{1}{2\pi i}\int_{1/2-i\infty}^{1/2+i\infty} \frac{\Gamma(1-z)x^z}{\zeta(2z)}\,\mathrm{d}z = x\sum_{k=0}^{\infty}\frac{(-x)^k}{k!\zeta(2+2k)} \tag{9}$$

is holomorphic in the strip: $1/2 < \Re(s) < 2$, because $F(x) = O(x)$ when $x \to 0$ and $F(x) = O(x^{1/4})$ when $x \to \infty$, the latter leaving unproven. The Riesz function

$$F(x) = x\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}e^{-x/n^2}$$

with the Möbius numbers $\mu(n)$ may be interesting in itself, but it seems to be 'inestimable' at infinity, in contrast to our function $w(t)$, as we shall show.

In order to apply Ramanujan's formula (B) to the attempt of Riesz, we transform it into the standard shape of a Mellin transformation:

$$\frac{\Gamma(s)}{\zeta(2-2s)} = \int_0^\infty \frac{F(x)}{x} x^{s-1}\,\mathrm{d}x \tag{10}$$

is the same as formula (8). With $\lambda(s) = 1/\zeta(2+2s)$ we have

$$\frac{F(x)}{x} = \sum_{k=0}^{\infty}\frac{\lambda(k)}{k!}(-x)^k.$$

Under the impression of Riesz's memoir Hardy and Littlewood [2] made a similar attempt with $\lambda(s) = 1/\zeta(1+2s)$. Thus they obtained the integral

$$\int_0^\infty x^{s-1} P(x)\,\mathrm{d}x = \frac{\Gamma(s)}{\zeta(1-2s)} \tag{11}$$

with the function

$$P(x) = \sum_{k=1}^\infty \frac{(-x)^k}{k!\zeta(1+2k)} = \sum_{n=1}^\infty \frac{\mu(n)}{n} e^{-x/n^2}.$$

However, they werre not able to derive the desired estimate $P(x) = O(x^{-1/4})$ when x tends to infinity.

In our opinion the 'mistake' that Riesz as well as Hardy and Littlewood made consists in the 'false' choice of the functions to be Mellin transformed.

3 Euler series transformation

If we please we can consider formula (7) as a definition of the function $w(t)$. But this definition restricts $w(t)$ to the disk: $|t| < 1$ and is therefore not suitable for an immediate application of Ramanujan's formula (A).

We consider the generalization

$$f(s,t) = \sum_{k=0}^\infty u(s+k)(-t)^k \tag{12}$$

of the function $w(t) = f(3/2, t)$ and replace the powers t^k in this series by the series

$$t^k = \frac{1}{1+t} \sum_{m=k}^\infty \binom{m}{k} \left(\frac{t}{1+t}\right)^m, \tag{13}$$

which emerge from the relations

$$\sum_{m=k}^\infty \binom{m}{k} z^m = \frac{z^k}{k!} \left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^k \frac{1}{1-z} = \frac{1}{1-z} \left(\frac{z}{1-z}\right)^k$$

with

$$z = \frac{t}{1+t} \quad \text{or} \quad t = \frac{z}{1-z}.$$

We assume that $|t| < 1$ as well as $\Re(t) > -1/2$, i. e. $|t| < |1+t|$. Thus we obtain the double series

$$f(s,t) = \frac{1}{1+t}\sum_{k=0}^{\infty} u(s+k)(-1)^k \sum_{m=k}^{\infty}\binom{m}{k}\left(\frac{t}{1+t}\right)^m$$

$$= \frac{1}{1+t}\sum_{m=0}^{\infty}\left(\frac{t}{1+t}\right)^m \sum_{k=0}^{m}\binom{m}{k}u(s+k)(-1)^k.$$

Introducing the finite differences

$$\Delta^m u(s) = \sum_{k=0}^{m}\binom{m}{k}u(s+k)(-1)^k \tag{14}$$

of the function $u(s)$ we have the series

$$f(s,t) = \frac{1}{1+t}\sum_{m=0}^{\infty}\Delta^m u(s)\left(\frac{t}{1+t}\right)^m, \tag{15}$$

which is usually called the Euler transform of the series (12) and effects its holomorphic continuation into the half-plane: $\Re(t) > -1/2$.

We dispense with a complete derivation of formula (15). Instead of this we refer to the discussion of the subject by Pólya and Szegö [4] in their *Aufgaben und Lehrsätze* I on the pages 128 and 306.

4 Finite differences

We begin with the extended Cauchy integral formula

$$u(s) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{u(z)}{s-z}\,\mathrm{d}z \tag{16}$$

as a valid representation of the function $u(s)$ in the complex half-plane: $\Re(s) > 1$. Each derivation of this formula is based on well-known estimates of the zeta-function which may be found, for instance, in Landau's classical *Handbuch* [3]. This hint was already sufficient for M. Riesz, who wrote in his paper on the Riemann hypothesis cited above: "En ce qui concerne l'ordre de grandeur de $1/\zeta(z)$ cf. LANDAU: Handbuch der Lehre der Verteilung der Primzahlen (Teubner, 1909) t. I, p. 177-178."

We proceed with the differences

$$\Delta^m u(s) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} u(z)\sum_{k=0}^{m}\binom{m}{k}\frac{(-1)^k}{s+k-z}\,\mathrm{d}z$$

$$= \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} u(z)\sum_{k=0}^{m}\binom{m}{k}(-1)^k\int_0^1 r^{s+k-z-1}\,\mathrm{d}r\,\mathrm{d}z$$

$$= \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} u(z)\int_0^1 r^{s-z-1}(1-r)^m\,\mathrm{d}r\,\mathrm{d}z$$

with the Beta-integrals

$$\int_0^1 r^{s-z-1}(1-r)^m\,\mathrm{d}r = B(s-z, m+1) = \frac{\Gamma(s-z)\Gamma(m+1)}{\Gamma(s-z+m+1)}.$$

Thus we obtain the formula

$$\Delta^m u(s) = \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} u(z)B(s-z, m+1)\,\mathrm{d}z \tag{17}$$

for $\Re(s) > 1$.

In order to derive a formula which is valid in the strip: $0 < \Re(s) < 1$ we consider the difference

$$u(s) - \Delta^m u(s) = \sum_{k=1}^{m}\binom{m}{k}u(s+k)(-1)^{k-1}$$

$$= \frac{1}{2\pi i}\int_{1-i\infty}^{1+i\infty} u(z)\int_0^1 r^{s-z-1}(1-(1-r)^m)\,\mathrm{d}r\,\mathrm{d}z$$

for each complex number s of this strip. The partial integration

$$\int_0^1 r^{s-z-1}(1-(1-r)^m)\,\mathrm{d}r = \frac{1}{s-z}\int_0^1 \frac{\mathrm{d}r^{s-z}}{\mathrm{d}r}(1-(1-r)^m)\,\mathrm{d}r$$

$$= \frac{1}{s-z}\int_0^1 \mathrm{d}\frac{r^{s-z}(1-(1-r)^m)}{\mathrm{d}r}\,\mathrm{d}r - \frac{1}{s-z}\int_0^1 r^{s-z}\mathrm{d}\frac{1-(1-r)^m}{\mathrm{d}r}\,\mathrm{d}r$$

$$= \frac{1}{s-z} - \frac{m}{s-z}\int_0^1 r^{s-z}(1-r)^{m-1}\,\mathrm{d}r$$

is correct when $m > 0$ and $\Re(z) = 1$, as in our case, and leads us to the formula

$$\Delta^m u(s) - u(s) = \frac{m}{2\pi i}\int_{1-i\infty}^{1+i\infty}\frac{u(z)}{s-z}B(s+1-z, m)\,\mathrm{d}z \tag{18}$$

because of

$$\int_{1-i\infty}^{1+i\infty} \frac{u(z)}{s-z}\,\mathrm{d}z = 0$$

when $0 < \Re(s) < 1$.

5 A crucial estimate

We conclude from the formulas (15) and (18) that

$$\frac{1}{1+t} f\left(s, \frac{-t}{1+t}\right) = \sum_{m=0}^{\infty} \Delta^m u(s)(-t)^m$$

$$= \frac{u(s)}{1+t} + \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{u(z)}{s-z} \sum_{m=1}^{\infty} m(-t)^m B(s+1-z, m)\,\mathrm{d}z$$

with

$$\sum_{m=1}^{\infty} m(-t)^m B(s+1-z, m) = -t \int_0^1 r^{s-z} \sum_{m=1}^{\infty} m(tr-t)^{m-1}\,\mathrm{d}r$$

$$= -t \int_0^1 \frac{r^{s-z}}{(1+t-tr)^2}\,\mathrm{d}r$$

when $0 < t \leq 1$. The term-by-term integrations are easily to be justified by Lebesgue's principle of dominated convergence. Thus we obtain

$$g(s,t) = f\left(s, \frac{-t}{1+t}\right) - u(s)$$

$$= -\frac{(1+t)t}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{u(z)}{s-z} \int_0^1 \frac{r^{s-z}}{(1+t-tr)^2}\,\mathrm{d}r\,\mathrm{d}z \tag{19}$$

for $0 < \Re(s) < 1$ and $0 < t \leq 1$. In order to identify the integral

$$\int_0^1 \frac{r^{s-z}}{(1+t-tr)^2}\,\mathrm{d}r = \frac{1}{(1+t)^2} \int_0^1 \frac{r^{s-z}}{(1-\frac{t}{1+t}r)^2}\,\mathrm{d}r$$

$$= \frac{1}{(1+t)^2} \sum_{k=0}^{\infty} (1+k) \left(\frac{t}{1+t}\right)^k \int_0^1 r^{s+k-z}\,\mathrm{d}r$$

$$= \frac{1}{(1+t)^2} \sum_{k=0}^{\infty} \frac{1+k}{s+1+k-z} \left(\frac{t}{1+t}\right)^k$$

with a standard function we consider the hypergeometric series

$$F(a,b,c;z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{k=0}^{\infty} \frac{\Gamma(a+k)\Gamma(b+k)}{\Gamma(c+k)} \frac{z^k}{k!}$$

of Gauss in case that $a = 2$ and $c = b + 1$:

$$F(2,b,b+1;z) = b \sum_{k=0}^{\infty} \frac{1+k}{b+k} z^k.$$

With $b = s + 1 - z$ and $z = t/(1+t)$ we obtain the formula

$$\int_0^1 \frac{r^{s-z}}{(1+t-tr)^2}\,\mathrm{d}r = \frac{F(2, s+1-z, s+2-z; \frac{t}{1+t})}{(1+t)^2(s+1-z)}. \tag{20}$$

Thus we have

$$g(s,t) = \frac{ti}{2\pi(1+t)} \int_{1-i\infty}^{1+i\infty} \frac{u(z)F(2, s+1-z, s+2-z; \frac{t}{1+t})}{(s-z)(s+1-z)}\,\mathrm{d}z \tag{21}$$

for $0 < \Re(s) < 1$ and $0 < t \leq 1$. It is clear that this integral converges still in the boarderline case $t = 1$.

On the other side we have

$$g(s,t) - u(s) = f\left(s, \frac{-t}{1+t}\right)$$

$$= (1+t) \sum_{m=0}^{\infty} \Delta^m u(s)(-t)^m = \sum_{k=0}^{\infty} u(s+k) \left(\frac{t}{1+t}\right)^k.$$

From the convergence of the series

$$2 \sum_{m=0}^{\infty} \Delta^m u(s)(-1)^m = \sum_{k=0}^{\infty} \frac{u(s+k)}{2^k} = f(s, -1/2) = g(s,1) - u(s)$$

we conclude that

$$\lim_{m\to\infty} \Delta^m u(s) = 0 \tag{22}$$

especially for each real number s between 0 and 1.

This is the most important step of our invstigation, as we shall show in the next section.

6 Abel's theorem of continuity

From the recursions

$$\Delta^{m+1}u(s) = \Delta^m u(s) - \Delta^m u(s+1) \tag{23}$$

we conclude that the partial sums

$$\sum_{m=0}^{n-1} \Delta^m u(s+1) = \sum_{m=0}^{n-1} \Big(\Delta^m u(s) - \Delta^{m+1}u(s)\Big) = u(s) - \Delta^n u(s)$$

tend to the limit

$$\sum_{m=0}^{\infty} \Delta^m u(s+1) = u(s) - \lim_{n\to\infty} \Delta^n u(s) = u(s)$$

due to equation (22). In particular we obtain the limit

$$u(1/2) = \sum_{m=0}^{\infty} \Delta^m u(3/2) = \sum_{m=0}^{\infty} \Delta^m u(3/2) \lim_{t\to\infty} \left(\frac{t}{1+t}\right)^{m+1}$$

$$= \lim_{t\to\infty} \sum_{m=0}^{\infty} \Delta^m u(3/2) \left(\frac{t}{1+t}\right)^{m+1} = \lim_{t\to\infty} f(3/2,t)t = \lim_{t\to\infty} w(t)t$$

according to Abel's famous theorem on the continuity of power series. Thus we have indicated the truth of the estimate (4) and herewith the truth of the Riemann hypothesis, since the estimate (3) is trival after formula (7).

7 Some additional remarks

The Möbius numbers $\mu(\nu)$ may be defined by the Dirichlet series

$$\frac{1}{\zeta(s)} = \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu^s} \tag{24}$$

which converges at least in the half-plane: $\Re(s) > 1$ and vanishes at $s = 1$:

$$\sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu} = 0\,. \tag{25}$$

Now we look once more at the power series (7) and see that

$$w(t) = \sum_{k=0}^{\infty} \frac{(-t)^k}{1/2+k} \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu^{3/2+k}} = \frac{2}{\sqrt{t}} \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu} \sum_{k=0}^{\infty} \frac{(-1)^k}{1+2k} \left(\frac{t}{\nu}\right)^{1/2+k}$$

when $0 \leq t < 1$. Thus we obtain the representation

$$w(t) = \frac{2}{\sqrt{t}} \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu} \arctan \sqrt{\frac{t}{\nu}} \tag{26}$$

of the function $w(t)$ which converges for each positive real number t due to Abel's test, since the series (25) converges and the numbers $\arctan \sqrt{t/\nu}$ form a monotone and bounded sequence. This is a very nice example of Abel's criterion.

Another way to prove the convergence of the series (26), not less beautiful than that by Abel's test, is that by Dirichlet's test: the series converges because

$$\left| \sum_{\nu=1}^{n} \frac{\mu(\nu)}{\nu} \right| \leq 1 \tag{27}$$

holds for each natural number n and because the positive real numbers $\arctan \sqrt{t/\nu}$ decrease monotonically toward zero.

Concerning the formulas (25) and (27) we refer to the paragraph 153 of Landau's *Handbuch*.

From formula (25) and the identity

$$\arctan \sqrt{\frac{t}{\nu}} = \frac{\pi}{2} - \arctan \sqrt{\frac{\nu}{t}}$$

we deduce the representation

$$w(t) = -\frac{2}{\sqrt{t}} \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu} \arctan \sqrt{\frac{\nu}{t}} \tag{28}$$

of $w(t)$, wich lets us conjecture that

$$\lim_{t\to\infty} w(t)t = -2 \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\nu} \lim_{t\to\infty} \sqrt{t} \arctan \sqrt{\frac{\nu}{t}}$$

$$= -2 \sum_{\nu=1}^{\infty} \frac{\mu(\nu)}{\sqrt{\nu}} = \frac{-2}{\zeta(1/2)} = u(1/2)$$

according to a theorem of Littlewood, who demonstrated that the Dirichlet series (24) converges in the complex half-plane: $\Re(s) > 1/2$, if the Riemann hypothesis is true.

Let us mention a final observation concerning the Mellin transformation of the function $w(t)$ in order to complete our 'symphony' of themes around the truth of the Riemann hypothesis.

The integral

$$\int_0^\infty t^{s-3/2} \arctan\sqrt{\frac{t}{\nu}}\,\mathrm{d}t = \int_0^\infty t^{s-3/2} \int_0^{\sqrt{\frac{t}{\nu}}} \frac{\mathrm{d}x}{1+x^2}\,\mathrm{d}t$$

$$= \frac{1}{\sqrt{\nu}} \int_0^\infty t^{s-1} \int_0^1 \frac{\mathrm{d}x}{1+x^2t/\nu}\,\mathrm{d}t = \frac{1}{\sqrt{\nu}} \int_0^1 \int_0^\infty \frac{t^{s-1}}{1+x^2t/\nu}\,\mathrm{d}t\,\mathrm{d}x$$

$$= \nu^{s-1/2} \int_0^1 x^{-2s}\,\mathrm{d}x \int_0^\infty \frac{t^{s-1}}{1+t}\,\mathrm{d}t = \frac{\pi\,\nu^{s-1/2}}{\sin(\pi s)(1-2s)}$$

exists in the complex strip: $0 < \Re(s) < 1/2$ and leads us once more to the Mellin transform

$$\int_0^\infty t^{s-1} w(t)\,\mathrm{d}t = 2\sum_{\nu=1}^\infty \frac{\mu(\nu)}{\nu} \int_0^\infty t^{s-3/2} \arctan\sqrt{\frac{t}{\nu}}\,\mathrm{d}t$$

$$= \frac{\pi}{\sin(\pi s)(1/2-s)} \sum_{\nu=1}^\infty \frac{\mu(\nu)}{\nu^{3/2-s}} = \frac{\pi}{\sin(\pi s)(1/2-s)\zeta(3/2-s)} = v(s)$$

at least for $0 < \Re(s) < 1/2$. Again we can state that the term-by-term integration is allowed due to dominated convergence.

Finally we wish to venture our opinion that we succeeded in indicating the truth of the Riemann hyothesis, despite the warning that "there are always billions of rational reasons not to look at a problem which has been unsuccessfully looked at by generations of mathematicians" (Allain Connes).

A.M.D.G.

References

[1] G. H. Hardy: *Ramanujan. Twelve Lectures on Subjects Suggested by His Life and Work.* Chelsea Publishing, New York, originally pubished by Cambridge University Press 1940.

[2] G. H. Hardy and J. E. Littlewood: Contributions to the theory of the Riemann zeta-function and the theory of the distribution of primes. Acta Math. 41 (1918), 119-196.

[3] E. Landau: *Handbuch der Lehre von der Verteilung der Primzahlen.* Third (corrected) edition, two volumes in one, Chelsea Publishing, New York 1974. (First edition, in two volumes, by Teubner, Leipzig 1909.)

[4] G. Pólya und G. Szegö: *Aufgaben und Lehrsätze aus der Analysis* I, Verlag von Julius Springer, Berlin 1925.

[5] M. Riesz: Sur l'hypothèse de Riemann. Acta Math. 40 (1916), 185-190. Collected papers, Springer-Verlag, Berlin, Heidelberg, New York 1988, 165-170.

[6] http://en.wikipedia.org/wiki/ Ramanujan's_master_theorem.

Werner Raab
Professor, Dr. phil., retired member of the
Mathematical Institute of the University of Bonn, Germany
Residence: Anton-Klieber-Str. 14, 6410 Telfs, Austria
E-mail: werner.raab@hotmail.com